Capital allocation à la Aumann–Shapley for non-differentiable risk measures

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Abstract

We introduce a family of capital allocation rules based on the dual representation for risk measures and inspired by the Aumann–Shapley allocation principle. These rules extend some well known methods of capital allocation for coherent and convex risk measures to the case of non-Gateaux-differentiable risk measures. We also study their properties and discuss their suitability in the quasi-convex context.

Key words: Risk management, Capital allocation rules, Convex/quasi-convex risk measures, Aumann–Shapley value, Gateaux differential.

1 Introduction

To face future uncertainty about their net worth, firms, insurance companies, and portfolio managers often have to satisfy capital requirements, that is, to hold an amount of riskless assets to hedge themselves. This fact then raises the issue of how to share all this immobilized capital in an *a priori* fair way among the different lines or business units (full allocation).

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As risk capital is commonly accepted in the literature to be modeled through the use of risk measures [1], [12], [19], [25] ; capital allocation problems in risk management and the theory of risk measures are naturally linked. On the other hand, fair allocation methods are also widely used in cooperative game theoretic contexts, so borrowing methodologies and desirable properties from this field appears to be a sensible choice. A first analysis in this sense appears in [12].

Instead, an axiomatic approach based on the classical concept of the Aumann–Shapley value [3] has been proposed in the work of Denault [14] who models a capital allocation problem as a fuzzy cost game (see [2]), defines a coherent allocation principle, and studies its link with coherent risk measures as well as some game theoretic properties that are assumed to define fairness in this case. All these papers (see also [7]) are based on some assumptions of differentiability for the risk measure under consideration (with respect to the weights of the sub-portfolios of a fixed portfolio or with respect to the direction given by a portfolio with respect to another portfolio). In the case of coherent and differentiable risk measures, there is a diversification advantage (the risk of pooling is never greater than the sum of the risks related to the single positions) that, thanks to Euler's theorem, can be fully subdivided among the business units based on each unit's marginal contribution to the overall risk. When, instead, there is a lack of positive homogeneity, there is a need to find an alternative; this, among others, is one of the issues addressed in the work of Tsanakas [35] (see also [27] for a dynamic approach). Starting from Deprez and Gerber's [15] work on convex risk premiums, Tsanakas defines a capital allocation rule (CAR) for Gateaux-differentiable risk measures inspired by the Aumann–Shapley value [3], and studies its properties for some widely used classes of convex risk measures, also providing explicit formulas. His analysis leaves open the case of general non-Gateaux-differentiable risk measures (although he treats some cases of distortion exponential risk measures, but it is easy to find other meaningful examples of convex and quasi-convex non-Gateaux-differentiable risk measures) as well as the study of quasi-convex risk measures, the importance of which has been well recognized quite recently in the literature (see [8], [17], and [23]). Another work that deals with non-differentiability, but is limited to the coherent case, is [5]. The reader is also advised to refer to [14] and [34] for further considerations on the topic.

The purpose of the present work is to try to plug these gaps, though not in full generality. To this aim, we first consider the existence of capital allocation schemes satisfying certain desirable properties in full generality for the quasi-convex case, and then we define a family of CARs based on the representation theorems for risk measures, study their properties, and show that they reduce to Denault's and Tsanakas' known allocation principles under Gateaux differentiability. In the meantime, we discuss the suitability of the use of quasi-convex risk measures for capital allocation purposes. Throughout our analysis, we make use of tools of convex and quasi-convex analysis, as those appearing in [9] and [36].

The paper is organized as follows: in Section 2 we define our setting and recall some known facts about risk measures; in Section 3 we consider general capital allocation schemes, while Section 4 is devoted to the introduction and the study of our capital allocation principle à la Aumann–Shapley–Tsanakas. Some examples are also discussed. Short conclusions are presented at the end of the paper.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space and assume that the space of all risky positions' profits and losses to be analyzed is given by $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$, that is the space of all essentially bounded random variables on $(\Omega, \mathcal{F}, P)$. Equalities and inequalities must be understood to hold P-almost surely. In the following, we use the dual pair (L^∞, L^1) with the $\sigma(L^\infty, L^1)$-topology.

A convex risk measure $\rho : L^\infty \to \mathbb{R}$ is a functional satisfying the following properties.

Monotonicity: if $X \leq Y$, P-almost surely, then $\rho(X) \geq \rho(Y)$.

Convexity: $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$, for any $\alpha \in [0, 1]$ and $X, Y \in L^\infty$.

Translation-invariance: $\rho(X + c) = \rho(X) - c$, for any $X \in L^\infty$ and $c \in \mathbb{R}$.

Normalization: $\rho(0) = 0$.

A *convex* risk measure ρ also satisfying the following property is called a *coherent* risk measure.

Positive homogeneity: $\rho(bX) = b\rho(X)$, for any $X \in L^\infty$ and $b \geq 0$.

Both translation-invariance and convexity have been discussed in the recent literature. By means of discounting arguments, El Karoui and Ravanelli [18] proposed to replace translation-invariance with the weaker axiom of cash subadditivity, thus introducing a wider class of risk measures.

In addition, Cerreia-Vioglio et al. [8] argued that, when translation-invariance is no longer assumed, the correct formulation of diversification is in terms of quasi-convexity instead of convexity. Hence, they introduced the so-called *quasi-convex* risk measures, i.e. risk measures $\rho : L^\infty \to \overline{\mathbb{R}}$ satisfying monotonicity, normalization, and

$$\rho(\alpha X + (1-\alpha)Y) \leq \max\{\rho(X); \rho(Y)\},$$

for any $X, Y \in L^\infty$ and $\alpha \in [0,1]$.

We now recall the dual representation of convex and quasi-convex risk measures. In the following, $\mathcal{P}$ will denote the set of all probability measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to P. Any probability measure $Q \in \mathcal{P}$ will be identified with its Radon–Nikodym density $\frac{dQ}{dP}$.

- Convex case (see [19], [20], and [25]): any convex risk measure $\rho : L^\infty \to \mathbb{R}$ that is continuous from below[1] can be represented as

  $$\rho(X) = \max_{Q \in \mathcal{P}} \{E_Q[-X] - F(Q)\} \tag{1}$$

  for some lower semi-continuous and convex penalty functional $F : L^1 \to [0; +\infty]$ with $\inf_Q F(Q) = 0$ where, with an abuse of notation, $F(Q)$ will stand for $F\left(\frac{dQ}{dP}\right)$.

- Quasi-convex case (see [31], [8], and [17]): any quasi-convex risk measure $\rho : L^\infty \to \overline{\mathbb{R}}$ satisfying continuity from below can be represented as

  $$\rho(X) = \max_{Q \in \mathcal{P}} K\left(E_Q[-X], Q\right) \tag{2}$$

  where $K : \mathbb{R} \times \mathcal{P} \to \overline{\mathbb{R}}$ is a function that is upper semi-continuous, increasing, and quasi-concave in the first variable.

3 Capital allocation

We now recall the classical definition of a CAR (see [14] and [26]).

Definition 1 Given a risk measure ρ on a linear space $\mathcal{X}$, a *capital allocation rule (CAR)* is a map $\Lambda : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$ such that $\Lambda(X; X) = \rho(X)$ for every $X \in \mathcal{X}$.

When $\Lambda(X; X) \leq \rho(X)$ for any $X \in \mathcal{X}$ we call Λ an *audacious* CAR; while when $\Lambda(X; X) \geq \rho(X)$ for any $X \in \mathcal{X}$ we call it a *prudential* CAR.

[1] We recall that a risk measure ρ is said to be continuous from below if for any sequence $(X_n)_{n \geq 0}$ such that $X_n \uparrow X$ it holds that $\rho(X_n) \to \rho(X)$.

When equality holds, this means that the capital allocated to X when considered as a standalone portfolio is exactly $\rho(X)$. As pointed out by Brunnermeier and Cheridito [6] this might not always be needed, for example when the capital is collected for monitoring purposes.

Given any portfolio $X \in \mathcal{X}$, a portfolio $Y \in \mathcal{X}$ is called a *sub-portfolio* of X, if X can be written as

$$X = Y + Z, \tag{3}$$

with $Z \in \mathcal{X}$.

The capital allocation problem deals with establishing how to assign "fairly" the required capital to subunits (sub-portfolios from now on) $X_1, \ldots, X_n$ of a business unit (portfolio) X. In other words, given X and its capital requirement $\rho(X)$, the aim is to find a suitable rule Λ such that $\rho(X)$ is entirely and "optimally" split into $k_i = \Lambda(X_i; X)$, $i = 1, \ldots, n$.

In the literature, it is usually required that the rule is linear with respect to the first variable, that is, it satisfies the *full allocation* property:

$$\rho(X) = \Lambda(X; X) = \sum_{i=1}^{n} \Lambda(X_i; X), \quad \text{when } X_1 + \cdots + X_n = X.$$

Following the approaches of Delbaen [13], Denault [14], Buch and Dorfleitner [7], Kalkbrener [26], and Tsanakas [35], we aim to:

- define a CAR that extends the known marginal contribution and Aumann–Shapley CARs to cover also quasi-convex risk measures, while maintaining the game theoretic interpretation;
- include the case of non-Gateaux-differentiable risk measures.

Kalkbrener [26] also imposes what we here call *no undercut*: for every $X, Y \in \mathcal{X}$,

$$\Lambda(Y; X) \leq \rho(Y).$$

Since $\Lambda(Y; X)$ can be interpreted as the capital allocated to the sub-portfolio Y to hedge/cover the global portfolio X, the no undercut axiom just requires that the capital allocated to Y to cover X should not exceed the capital allocated in Y to cover Y itself. In the terminology of Tsanakas [35], no undercut corresponds to the *non-split* requirement of Y from X.

In [26, Theorems 4.2 and 4.3], Kalkbrener shows the equivalence between positive homogeneity and subadditivity of a risk measure and linearity and no undercut properties of the corresponding CAR. Moreover, under the additional assumption of Gateaux differentiability for the risk measure, he proves that Λ is exactly the gradient allocation rule (see, for example, [33]). Thus, if we wish to investigate CARs for convex and quasi-convex (but not coherent) risk measures maintaining the no undercut property, we must relinquish linearity and replace it with a weaker axiom. For this purpose, we introduce the following axiom:

diversifying: for every $X, Y, Z \in \mathcal{X}$ and every $\lambda \in [0, 1]$

$$\Lambda(\lambda X + (1 - \lambda)Y; Z) \leq \max\{\Lambda(X; Z), \Lambda(Y; Z)\};$$

or the stronger axiom of:

s-diversifying: for every $X, Y, Z \in \mathcal{X}$ and $\lambda \in [0, 1]$

$$\Lambda(\lambda X + (1 - \lambda)Y; Z) = \lambda\Lambda(X; Z) + (1 - \lambda)\Lambda(Y; Z).$$

We observe that, in the case of a convex risk measure ρ, given for example a portfolio $X = X_1 + X_2$, it could happen that $\rho(X) > \rho(X_1) + \rho(X_2)$. In this case, given a principle Λ that does not fully allocate the risk capital, under the hypothesis of no undercut it turns out $\Lambda(X; X) = \rho(X) > \rho(X_1) + \rho(X_2) \geq \Lambda(X_1; X) + \Lambda(X_2; X)$. We can imagine that the difference $\rho(X) - (\Lambda(X_1; X) + \Lambda(X_2; X))$ constitutes an indistinct and undivided deposit/cost. This justifies the introduction of the following axiom.

Definition 2 A CAR is said to have the *sub allocation* property if

$$\Lambda(X; X) \geq \sum_{i=1}^{n} \Lambda(X_i; X)$$

for every X and every $X_1, \ldots, X_n$ such that $X = X_1 + \cdots + X_n$.

As we shall see in the following, this can happen in the case of a CAR based on the representation of a convex risk measure.

Proposition 3 *If $\Lambda : L^\infty \times L^\infty \to \mathbb{R}$ is a CAR that satisfies the axioms of no undercut and diversifying, then the associated risk measure ρ defined on L^∞ is quasi-convex. In contrast, if ρ is a convex and continuous from below risk measure on L^∞, then there exists a CAR Λ_ρ satisfying the properties of sub-allocation, no undercut, and diversifying.*

Proof. If Λ satisfies the properties of no undercut and diversifying, then, for $\lambda \in [0,1]$ and $X_1, X_2 \in L^\infty$, we have

$$\begin{aligned}
\rho(\lambda X_1 + (1-\lambda)X_2) &= \Lambda(\lambda X_1 + (1-\lambda)X_2; \lambda X_1 + (1-\lambda)X_2) \\
&\leq \max\{\Lambda(X_1, \lambda X_1 + (1-\lambda)X_2), \Lambda(X_2; \lambda X_1 + (1-\lambda)X_2)\} \\
&\leq \max\{\Lambda(X_1; X_1), \Lambda(X_2; X_2)\} \leq \max\{\rho(X_1), \rho(X_2)\}.
\end{aligned}$$

Hence, ρ is quasi-convex.

Vice versa, let ρ be a convex and continuous from below risk measure on L^∞.

For any $X \in L^\infty$, there exists $Q = Q_X$ such that $\rho(X) = E_{Q_X}[-X] - F(Q_X) = \max_Q\{E_Q[-X] - F(Q)\}$. Set now

$$\Lambda_\rho(Y; X) \triangleq E_{Q_X}[-Y] - F(Q_X). \tag{4}$$

It follows immediately that Λ_ρ is a CAR.

To prove the sub-allocation property of Λ_ρ take any $X_1, \ldots, X_n \in L^\infty$ such that $\sum_{i=1}^n X_i = X$. Since $F(Q_X) \geq 0$,

$$\begin{aligned}
\Lambda_\rho(X; X) &= \Lambda_\rho\left(\sum_{i=1}^n X_i; X\right) = E_{Q_X}\left[-\sum_{i=1}^n X_i\right] - F(Q_X) \\
&\geq \sum_{i=1}^n [E_{Q_X}[-X_i] - F(Q_X)] \\
&= \sum_{i=1}^n \Lambda_\rho(X_i; X).
\end{aligned}$$

The no undercut property of Λ_ρ is due to the representation of ρ and to the optimality of the scenario Q_Y. Indeed:

$$\begin{aligned}
\Lambda_\rho(Y; Y) &= E_{Q_Y}[-Y] - F(Q_Y) = \max_Q\{E_Q[-Y] - F(Q)\} \\
&\geq E_{Q_X}[-Y] - F(Q_X) = \Lambda_\rho(Y; X).
\end{aligned}$$

Moreover, for any $X, Y, Z \in L^\infty$ and $\lambda \in [0,1]$, it holds

$$\begin{aligned}
\Lambda_\rho(\lambda X + (1-\lambda)Y; Z) &= E_{Q_Z}[-\lambda X - (1-\lambda)Y] - F(Q_Z) \\
&= \lambda(E_{Q_Z}[-X] - F(Q_Z)) + (1-\lambda)(E_{Q_Z}[-Y] - F(Q_Z)) \\
&= \lambda\Lambda_\rho(X; Z) + (1-\lambda)\Lambda_\rho(Y; Z) \\
&\leq \max\{\Lambda_\rho(X; Z); \Lambda_\rho(Y; Z)\},
\end{aligned}$$

so Λ_ρ is diversifying. ■

Note that a CAR similar to (4) can be found in Kromer, Overbeck, and Zilch [29], where they consider $\Lambda_\rho(X_i; X) = E_{Q_X}[-X_i] - \gamma_i F(Q_X)$, with γ_i weights summing up to one.

Proposition 4 *If $\rho : L^\infty \to \overline{\mathbb{R}}$ is a quasi-convex and continuous from below risk measure, then there exists a CAR $\Lambda_\rho : L^\infty \times L^\infty \to \overline{\mathbb{R}}$ satisfying no undercut and, for any $X, X_1, \ldots, X_n \in L^\infty$ such that $\sum_{i=1}^n X_i = X$,*

$$\Lambda_\rho \left(\sum_{i=1}^n X_i; X \right) \geq \min_{i=1,2,\ldots,n} \Lambda_\rho(nX_i; X).$$

Hence, $\rho(X) \geq \min_{i=1,2,\ldots,n} \Lambda_\rho(nX_i; X)$ for any X and X_i as above.

Proof. Let ρ be a quasi-convex and continuous from below risk measure on L^∞. For any $X \in L^\infty$, there exists $Q = Q_X$ such that $\rho(X) = K(E_{Q_X}[-X], Q_X) = \max_Q K(E_Q[-X], Q)$. Set now

$$\Lambda_\rho(Y; X) \triangleq K(E_{Q_X}[-Y], Q_X). \tag{5}$$

It follows immediately that Λ_ρ is a CAR.

No undercut of Λ_ρ can be checked as in the convex case.

Take now any $X_1, \ldots, X_n \in L^\infty$ such that $\sum_{i=1}^n X_i = X$. By quasi-concavity of K in its first variable, it follows that

$$\begin{aligned} \Lambda_\rho (X; X) &= \Lambda_\rho \left(\sum_{i=1}^n X_i; X \right) = K \left(E_{Q_X} \left[- \sum_{i=1}^n X_i \right], Q_X \right) \\ &= K \left(E_{Q_X} \left[-\frac{1}{n}(nX_1) - \cdots - \frac{1}{n}(nX_n) \right], Q_X \right) \\ &\geq \min_{i=1,2,\ldots,n} K(E_{Q_X}[-nX_i], Q_X) \\ &= \min_{i=1,2,\ldots,n} \Lambda_\rho(nX_i; X). \end{aligned}$$

■

4 Capital allocation à la Aumann–Shapley–Tsanakas

Given a portfolio X, most CARs subsume linearity: unfortunately, in the absence of coherence and differentiability of the underlying risk measure, the principles of marginality satisfying the full allocation property are not applicable. Thus, when going beyond the above-mentioned cases, the problem of sensible CARs is still open. To the best of the authors' knowledge, only the works of Biagini et al.[2] [4], Boonen et al. [5], and Cheridito and Kromer [10] propose capital allocation principles suitable for frameworks of non-differentiability. A natural way to follow would be to rely on dual representation theorems for risk measures, keeping in mind that both in the convex and in the quasi-convex cases, these are nonlinear. Thus, the problem is that of combining the property of being a CAR with that of being linear.

With this as a starting point, assume that $\rho : L^\infty \to \mathbb{R}$ is a convex risk measure or $\rho : L^\infty \to \overline{\mathbb{R}}$ is a quasi-convex risk measure. In both cases, continuity from below is assumed. Hence, in the convex case we have

$$\rho\left(X\right) = \max_{Q \in \mathcal{P}} \left\{ E_Q[-X] - F(Q) \right\} \tag{6}$$

while in the quasi-convex case it holds

$$\rho\left(X\right) = \max_{Q \in \mathcal{P}} K\left(E_Q\left[-X\right], Q\right). \tag{7}$$

Given $X \in L^\infty$, set

$$Q_X = Q_{X,F} \in \operatorname{argmax}_{Q \in \mathcal{P}} \left\{ E_Q[-X] - F(Q) \right\} \tag{8}$$

or

$$Q_X = Q_{X,K} \in \operatorname{argmax}_{Q \in \mathcal{P}} K\left(E_Q\left[-X\right], Q\right). \tag{9}$$

We define the following family of maps $\Lambda_\rho^{AS} : L^\infty \times L^\infty \to \mathbb{R}$

$$\Lambda_\rho^{AS}(Y;X) \triangleq \int_0^1 E_{Q_{\gamma X}}[-Y]\, d\gamma, \tag{10}$$

for every $X, Y \in L^\infty$.

If no ambiguity over the chosen risk measure ρ occurs, occasionally we will simply write Λ^{AS} instead of Λ_ρ^{AS}.

As we will show shortly, this approach is similar to that of Tsanakas [35] who extended the Aumann–Shapley value [3] CAR under the assumption of Gateaux differentiability. More precisely, for Gateaux-differentiable

[2] Although the paper of Biagini et al. [4] considers both quasi-convex and non-differentiable risk measures, the authors' approach is based on acceptance sets that are completely different from that considered here.

coherent and convex risk measures, Tsanakas [35] considered the following CAR

$$\Lambda^T(Y;X) \triangleq \int_0^1 D\rho(Y;\gamma X)\, d\gamma \tag{11}$$

(the definitions of directional derivative $D\rho(Y;\gamma X)$ and of Gateaux differentiability will be recalled shortly). Differently from the approach above, however, our approach is much more general for two reasons: first, because it can be applied to a wider class of risk measures; second, it is also valid for not necessarily Gateaux-differentiable risk measures.

In the following, we study the properties of Λ, with respect to those requested for a CAR. To analyze our approach and its relation to that of Tsanakas [35] in detail, for the reader's convenience we recall some results of convex analysis (here formulated on L^∞ even if they are valid in more general spaces) that will be useful in a while. For further details please refer to Zălinescu [36], Ruszcynski and Shapiro [32], and Cheridito and Li [11].

Given a functional ρ (in our case a risk measure), the sub-differential and the Greenberg–Pierskalla sub-differential of ρ at $X \in L^\infty$ are defined, respectively, as

$$\partial\rho(X) \triangleq \left\{\xi \in L^1 : \rho(Y) - \rho(X) \geq E[\xi(Y-X)] \quad \text{for any } Y \in L^\infty\right\}$$
$$\partial^{GP}\rho(X) \triangleq \left\{\xi \in L^1 : E[\xi(Y-X)] < 0 \quad \text{for any } Y \in L^\infty \text{ s.t. } \rho(Y) < \rho(X)\right\},$$

while the directional derivative of ρ at $X \in L^\infty$ in the direction $Y \in L^\infty$, as

$$D\rho(Y;X) \triangleq \lim_{t \downarrow 0} \frac{\rho(X+tY)-\rho(X)}{t}.$$

Furthermore, ρ is said to be Gateaux differentiable at X with Gateaux derivative $\nabla\rho(X) \in L^1$, if $D\rho(Y;X) = E_P[Y\nabla\rho(X)]$ for any $Y \in L^\infty$. As a consequence, Gateaux differentiability implies the existence of $\lim_{t\to 0} \frac{\rho(X+tY)-\rho(X)}{t}$. Moreover, if $\nabla\rho(X)$ exists, then it is unique.

It is well known in convex analysis (see, for example, [32, Corollary 3.1]) that any convex and proper risk measure $\rho : L^p \to \bar{\mathbb{R}}$ (with $p \in [1,+\infty)$) is continuous and its sub-differential $\partial\rho(X)$ is non-empty for every X in the interior of the domain of ρ. This result does not hold true in general for the dual pair (L^∞, L^1). However, for convex risk measures $\rho : L^\infty \to \mathbb{R}$ and $X \in L^\infty$ it holds that the sub-differential $\partial\rho(X)$ has the following representation (see [36, Theorem 2.4.2]):

$$\partial\rho(X) = -\text{argmax}_{Q\in\mathcal{P}}\{E_Q[-X] - F(Q)\}.$$

Also, under continuity of ρ in X, the Gateaux differentiability of ρ in X is equivalent to the fact the $\partial\rho(X)$ is a singleton $\{-Q^*\}$ (see [36, Corollary 2.4.10]). Since, by hypothesis, ρ is continuous from below, for any given $X \in L^\infty$ the maximum in (1) is attained at some Q^*, hence $\partial\rho(X)$ is not empty. Thus, the Gateaux-differentiable case yields a unique Λ^{AS}.

If instead the risk measure ρ is quasi-convex, lower semi-continuous, and continuous from below, by [31, Theorem 5.6], [9, Proposition 2], and [30, Proposition 3], it holds:

$$\partial_N^{GP}\rho(X) = -\mathrm{argmax}_Q K(E_Q[-X], Q),$$

where $\partial_N^{GP}\rho(X)$ stands for the normalized Greenberg–Pierskalla sub-differential, that is $\partial^{GP}\rho(X) \cap \{\xi \in L^1 : E[\xi X] = 1\}$. Under the assumptions of [9, Corollary 2], it follows[3] that $\partial^{GP}\rho(X) = \{-\lambda\nabla\rho(X); \lambda > 0\}$, hence the normalized Greenberg–Pierskalla sub-differential is again a singleton.

Thus, without any further assumptions on ρ apart from convexity/quasi-convexity and continuity from below, also in the non-differentiable case our definition of capital allocation in (10) gives a variety of different possible allocations all based on the same principle (for an illustrative example, see Example 12). The choice of Q to be used can be made following different criteria, e.g. as for pricing in incomplete markets: see, among many others, Föllmer and Schweizer [21] and Frittelli [22].

Let us now study in detail the properties satisfied by our allocation rule and its relation to that of Tsanakas [35].

Proposition 5 *Let $\rho : L^\infty \to \overline{\mathbb{R}}$ be a quasi-convex risk measure (in particular, a convex risk measure) satisfying continuity from below. Then the principles Λ_ρ^{AS} are well-defined, diversifying, and linear.*

Proof. The principles Λ_ρ^{AS} are well-defined since both in the quasi-convex and in the convex case it holds that for any $X \in L^\infty$ and $\gamma \in [0,1]$, there exist at least one $Q_{\gamma X}$ where the maximum in (6) and (7) is attained. Furthermore,

$$\begin{aligned}
\Lambda_\rho^{AS}(\lambda X + (1-\lambda)Y; Z) &= \int_0^1 E_{Q_{\gamma Z}}[-\lambda X - (1-\lambda)Y]\, d\gamma \\
&= \lambda \int_0^1 E_{Q_{\gamma Z}}[-X]\, d\gamma + (1-\lambda)\int_0^1 E_{Q_{\gamma Z}}[-Y]\, d\gamma \\
&\leq \max\{\Lambda_\rho^{AS}(X;Z), \Lambda_\rho^{AS}(Y;Z)\}.
\end{aligned}$$

[3]Note that the definition of Greenberg–Pierskalla ∂^{GP} given by Cerreia-Vioglio et al. [9] refers to super-differentials. Let us denote it by ∂_{GP}^{super} so to distinguish it from that defined here. It is easy to check that $-\xi \in \partial^{GP}\rho(X) \Leftrightarrow \xi \in \partial_{GP}^{super}(-\rho(X))$.

Linearity is straightforward. ■

The following results underline that, under the Gateaux-differentiability assumption, our allocation principles reduce to that of Tsanakas [35] for coherent and convex risk measures.

Proposition 6 (Coherent risk measures) *If $\rho : L^\infty \to \mathbb{R}$ is a coherent risk measure satisfying continuity from below, then every Λ_ρ^{AS} is a CAR satisfying the no undercut property.*

If ρ is Gateaux differentiable, then

$$\Lambda_\rho^{AS}(Y;X) = \int_0^1 E_{Q_{\gamma X}}[-Y]\, d\gamma = \int_0^1 D\rho(Y;\gamma X)\, d\gamma,$$

that is, our allocation principles coincide with that of Tsanakas [35].

Proof. By the assumptions on ρ it holds that $\rho(X) = \max_{Q \in \mathcal{P}} E_Q[-X]$ for every $X \in L^\infty$ (see [12]). Since $\mathrm{argmax}_Q E_Q[-X] \equiv \mathrm{argmax}_Q E_Q[-\gamma X]$ for any $\gamma \in [0,1]$ and $X \in L^\infty$, it is easy to check that

$$\Lambda_\rho^{AS}(X;X) = \int_0^1 E_{Q_{\gamma X}}[-X]\, d\gamma = \int_0^1 E_{Q_X}[-X]\, d\gamma = \rho(X).$$

Hence, Λ_ρ^{AS} is a CAR. Furthermore, for any Y it holds

$$\Lambda_\rho^{AS}(Y;X) = \int_0^1 E_{Q_{\gamma X}}[-Y]\, d\gamma \leq \int_0^1 \rho(Y)\, d\gamma = \rho(Y).$$

If ρ is also Gateaux differentiable, then it is well known (see [12]) that the directional derivative of ρ at X in the direction Y is given by

$$D\rho(Y;X) = E_{Q_X}[-Y].$$

Hence,

$$\Lambda_\rho^{AS}(Y;X) = \int_0^1 E_{Q_{\gamma X}}[-Y]\, d\gamma = \int_0^1 D\rho(Y;\gamma X)\, d\gamma,$$

that is, our allocation principles coincide with that of Tsanakas. ■

Remark 7 *Note that when the coherent risk measure ρ is continuous from below and Gateaux differentiable, then our CARs reduce also to the gradient allocation of Kalkbrener [26]. Indeed,*

$$\Lambda_\rho^{AS}(Y;X) = D\rho(Y;X) = \int_0^1 D\rho(Y;\gamma X)\, d\gamma$$

(see [26, Theorem 3.1]).

Proposition 6 underlines that the interpretation of the capital allocation principle via games remains valid also with our approach.

Note that the game theoretic interpretation of the principle in the spirit of Aumann and Shapley, which is evident in the differentiable case, can anyway be extended: indeed, for a fixed $\gamma \in [0,1]$, the weight (or participation degree) of the sub-coalition Y in the portfolio, or fuzzy coalition [2], [14] γX is $dQ_{\gamma X}$, so coalition Y is rewarded by the principle with the "sum"of its expected contribution to all possible initial portfolios γX.

The following result emphasizes that our principles extend that of Tsanakas [35] to more general risk measures.

Proposition 8 (Convex and quasi-convex risk measures) *(a) (Convex case) Let $\rho : L^\infty \to \mathbb{R}$ be a convex risk measure satisfying continuity from below.*

Then every Λ_ρ^{AS} is a CAR satisfying

$$\Lambda_\rho^{AS}(Y;X) \leq \rho(Y) + A_F(X) \tag{12}$$

for any $X, Y \in L^\infty$, where $A_F(X) \triangleq \int_0^1 F\left(Q_{\gamma X}\right) d\gamma$. Furthermore, $\Lambda_\rho^{AS}(c, X) = -c$, for $c \in \mathbb{R}$ (riskless allocation).

Moreover, if ρ is also Gateaux differentiable, then Λ_ρ^{AS} reduce to the allocation principle of Tsanakas [35].

(b) (Quasi-convex case) Let $\rho : L^\infty \to \overline{\mathbb{R}}$ be a quasi-convex risk measure satisfying continuity from below and such that $\rho(X) \in \mathbb{R}$ for every $X \in L^\infty$.

Then

$$\Lambda_\rho^{AS}(Y;X) \leq \rho(Y) + M_K(X;Y) \tag{13}$$

for any Y, where $M = M_K(X;Y) = \int_0^1 \left(E_{Q_{\gamma X}}[-Y] - K(E_{Q_{\gamma X}}[-Y]; Q_{\gamma X})\right) d\gamma$.

Proof. (a) Convex case: by the assumptions on ρ and by Proposition 5, each Λ_ρ^{AS} is well-defined. To verify that it is also a capital allocation we only need to check that $\Lambda_\rho^{AS}(X;X) = \rho(X)$.

Let $X \in L^\infty$ be arbitrarily fixed. By the assumptions on ρ, it follows that $G(\gamma) \triangleq \rho(\gamma X)$ is a function $G : [0,1] \to \mathbb{R}$ that is convex in γ. Hence, there exist $G'_-(\gamma), G'_+(\gamma)$ for any $\gamma \in [0,1]$ and $G'_-(\gamma) \neq G'_+(\gamma)$ at most for countably many points.

By the hypothesis on ρ there exist $Q_{\gamma X} \in$ argmax $\{E_Q[-\gamma X] - F(Q)\}$, hence $-Q_{\gamma X} \in \partial\rho(\gamma X)$. It follows that $\rho(\gamma X + tX) - \rho(\gamma X) \geq E_{Q_{\gamma X}}[-tX]$, so

$$\begin{aligned} G'_-(\gamma) &= \lim_{t\uparrow 0} \frac{G(\gamma+t)-G(\gamma)}{t} \leq E_{Q_{\gamma X}}[-X], \\ G'_+(\gamma) &= \lim_{t\downarrow 0} \frac{G(\gamma+t)-G(\gamma)}{t} \geq E_{Q_{\gamma X}}[-X]. \end{aligned}$$

Hence,

$$\int_0^1 G'_-(\gamma) d\gamma \leq \int_0^1 E_{Q_{\gamma X}}[-X]\, d\gamma \leq \int_0^1 G'_+(\gamma)\, d\gamma$$

Since the set $\{\gamma : G'_-(\gamma) \neq G'_+(\gamma)\}$ has Lebesgue measure equal to zero, then, by applying an argument similar to that of Kromer and Overbeck [27] (see[27, Corollary 4.1] or [35, equation (1)]), we deduce that

$$\int_0^1 E_{Q_{\gamma X}}[-X]\, d\gamma = \int_0^1 G'_+(\gamma)\, d\gamma = \int_0^1 G'_-(\gamma)\, d\gamma = \int_0^1 \frac{dG}{d\gamma}\, d\gamma = \rho(X).$$

Furthermore, it holds that

$$\Lambda_\rho^{AS}(Y;X) = \int_0^1 E_{Q_{\gamma X}}[-Y]\, d\gamma \leq \int_0^1 \left(\rho(Y) + F(Q_{\gamma X})\right)\, d\gamma = \rho(Y) + A(X)$$

for any $Y \in L^\infty$.

Moreover, if ρ is also Gateaux differentiable, it follows that $\partial\rho(X) = \{-Q_X\}$ and the Gateaux differential of ρ in X in the direction of Y is precisely

$$D\rho(Y;X) = E_{Q_X}[-Y],$$

Hence,

$$\rho(X) = \int_0^1 D\rho(X;\gamma X)\, d\gamma = \int_0^1 E_{Q_{\gamma X}}[-X]\, d\gamma = \Lambda_\rho^{AS}(X;X),$$

where the first equality can be found in Kromer and Overbeck [27] and Tsanakas [35]. Thus, in this case, the allocation principle is precisely that of Tsanakas [35]. The riskless allocation property is trivial.

(b) Quasi-convex case: by the assumptions on ρ and by Proposition 5, Λ_ρ^{AS} is well-defined. Moreover, it holds that

$$\begin{aligned} \Lambda_\rho^{AS}(Y;X) &= \int_0^1 \left(E_{Q_{\gamma X}}[-Y] - K(E_{Q_{\gamma X}}[-Y];Q_{\gamma X}) + K(E_{Q_{\gamma X}}[-Y];Q_{\gamma X})\right)\, d\gamma \\ &\leq \int_0^1 \left(E_{Q_{\gamma X}}[-Y] - K(E_{Q_{\gamma X}}[-Y];Q_{\gamma X}) + \rho(Y)\right)\, d\gamma \\ &= \rho(Y) + M_K(X;Y) \end{aligned}$$

for any $Y \in L^\infty$.

■

Note that, in the purely convex case, our allocation principle cannot satisfy the no undercut property for every $X, Y \in L^\infty$ (see [26, Theorem 4.2 (a)] and also our Example 9). In the quasi-convex case, in contrast, the maps need not necessarily define a CAR ($\rho(X) \neq \Lambda(X,X)$). As a consequence, in this case the no undercut property can be fulfilled. In particular, this happens when $M \leq 0$, which is satisfied for instance when $K(t;Q) \geq t$ for any Q and any $t \in \mathbb{R}$.

Although the no undercut property is replaced by (12) and (13), the interpretation of the aforementioned inequalities and of the related bounds seems to be quite reasonable when one thinks about the motivation for the introduction of convex risk measures. The financial key issue was indeed that coherent risk measures do not take into account liquidity because of the positive homogeneity axiom while convex risk measures do (see Föllmer and Schied [19] and Frittelli and Rosazza Gianin [25]). Moreover, at the level of dual representation the main difference between coherent and convex risk measures consists of a penalty functional F, while between convex and quasi-convex measures it consists of a more general functional K.

Going back to (12) and (13), on the one hand, it seems financially reasonable that the upper bound of the capital allocation needed for Y as a sub-portfolio of X should depend not only on Y but also on the whole X and its size. This issue can possibly be avoided in the coherent case (because of scaling invariance), but definitely not in the convex and quasi-convex cases where the scale is important. On the other hand, it is not surprising that such an additional dependence incorporates also the penalty functional F or the more general K that distinguish convex or quasi-convex risk measures from coherent risk measures.

4.1 Examples

We now illustrate through examples the behavior of our CAR for convex and quasi-convex risk measures, as well as for non-Gateaux-differentiable risk measures.

Notice that the key assumption of our approach is that the maximum in (6) and (7) is attained. The assumption on continuity from below has been made to guarantee that property. In the following examples, however, the key assumption above is always automatically fulfilled.

Example 9 (Convex risk measures not satisfying the no undercut property) *Let $\Omega = \{\omega_1; \omega_2\}$ and*

$$\rho(Z) = \max_{Q \in \mathcal{Q}} \{E_Q[-Z] - F(Q)\}$$

with $\mathcal{Q} = \{P; Q_1\}$, $F(P) = 0$, $F(Q_1) = 1$, $P(\omega_1) = P(\omega_2) = \frac{1}{2}$, and $Q_1(\omega_1) = \frac{1}{4}$.

Take now

$$X = \begin{cases} 0; & \omega_1, \\ -8; & \omega_2. \end{cases}$$

Hence, $\rho(X) = 5$ and $Q_{\gamma X} = \begin{cases} P; & 0 \leq \gamma < \frac{1}{2}, \\ Q_1; & \frac{1}{2} < \gamma \leq 1. \end{cases}$. Consequently, $\Lambda_\rho^{AS}(X; X) = 5 = \rho(X)$.

Suppose that X has two subunits corresponding to the following sub-portfolios:

$$X_1 = \begin{cases} 2; & \omega_1, \\ -12; & \omega_2, \end{cases} \qquad X_2 = \begin{cases} -2; & \omega_1, \\ 4; & \omega_2. \end{cases}$$

Note that X_1 and X_2 are countermonotone with $X_1 + X_2 = X$. It is easy to check that $\rho(X_1) = \frac{15}{2}$ and $\rho(X_2) = -1$ while

$$\begin{aligned} \Lambda_\rho^{AS}(X_1; X) &= \int_0^1 E_{Q_{\gamma X}}[-X_1]\, d\gamma = \frac{27}{4} \leq \rho(X_1), \\ \Lambda_\rho^{AS}(X_2; X) &= \int_0^1 E_{Q_{\gamma X}}[-X_2]\, d\gamma = -\frac{7}{4} \leq \rho(X_2). \end{aligned}$$

Therefore, for both X_1 and X_2 there is no incentive to split, indeed the capital allocation required for X_i as a sub-portfolio is smaller than the capital requirement as a position alone. Note also that $\Lambda_\rho^{AS}(X_1; X) + \Lambda_\rho^{AS}(X_2; X) = \rho(X)$.

Let us consider now a comonotone decomposition of X in sub-portfolios by taking

$$Y_1 = \begin{cases} 2; & \omega_1, \\ -5; & \omega_2, \end{cases} \qquad Y_2 = \begin{cases} -2; & \omega_1, \\ -3; & \omega_2. \end{cases}$$

In the present case we get $\rho(Y_1) = \frac{9}{4}$ and $\rho(Y_2) = \frac{5}{2}$, while

$$\begin{aligned} \Lambda_\rho^{AS}(Y_1; X) &= \frac{19}{8} > \rho(Y_1), \\ \Lambda_\rho^{AS}(Y_2; X) &= \frac{21}{8} > \rho(Y_2). \end{aligned}$$

Hence, there is an incentive to split Y_1 and Y_2 from the whole portfolio X. In that case, Y_i alone (i.e. not as sub-portfolio of X) would require a smaller capital requirement. However, $\Lambda_\rho^{AS}(Y_i;X) \leq \rho(Y_i)+\int_0^1 F(Q_{\gamma X})\,d\gamma = \rho(Y_i)+\frac{1}{2}$ (for $i=1,2$). Note also that $\Lambda_\rho^{AS}(Y_1;X)+\Lambda_\rho^{AS}(Y_2;X)=\rho(X)$.

The incentive to split for at least a sub-portfolio was already guaranteed by [26, Theorem 4.2 (a)]. Moreover, such a situation seems to occur for comonotone decompositions. Investigations in this direction are beyond the scope of the present paper and will be the subject of future work.

Example 10 (Case of a purely quasi-convex risk measure) *Let $\Omega = \{\omega_1;\omega_2\}$ and*

$$\rho(Z) = \max_{Q\in\mathcal{Q}} K(E_Q[-Z],Q)$$

with $\mathcal{Q}=\{P;Q_1\}$, $P(\omega_1)=P(\omega_2)=\frac{1}{2}$, $Q_1(\omega_1)=\frac{1}{4}$, and K as below.

Take now

$$X = \begin{cases} 12; & \omega_1, \\ -8; & \omega_2. \end{cases}$$

Let us consider the following examples that induce different situations concerning $\Lambda(X;X)$ and its relation with $\rho(X)$.

1. *Case 1: $K(E_Q[-Z],Q) = g(E_Q[-Z])$ with $g(t) = \begin{cases} t+1; & t<-1, \\ 0; & -1\leq t<0, \\ t; & t\geq 0. \end{cases}$*

   Hence, $K(t,Q)\geq t$ for any $t\in\mathbb{R}$ and ρ is monotone, quasi-convex, continuous from above and such that $\rho(0)=0$. It is also easy to check that $\rho(X)=3$, $Q_{\gamma X}\equiv Q_1$ for any $\gamma\in[0,1]$ and $\Lambda(X;X) = \int_0^1 E_{Q_{\gamma X}}[-X]\,d\gamma = 3 = \rho(X)$.

2. *Case 2: $K(E_Q[-Z],Q) = g(E_Q[-Z]) - F(Q)$ with g as above and $F(P)=0$ and $F(Q_1)=\frac{1}{2}$.*

   In that case, $K(t,Q)\gtreqless t$ depending on t and Q and ρ is monotone, quasi-convex, continuous from above and such that $\rho(0)=0$. It is also easy to check that $\rho(X)=\frac{5}{2}$, $Q_{\gamma X} = \begin{cases} P; & 0<\gamma<\frac{1}{6} \\ Q_1; & \frac{1}{6}\leq\gamma<1 \end{cases}$, so

   $$\Lambda(X;X) = \int_0^{1/6} E_P[-X]\,d\gamma + \int_{1/6}^1 E_{Q_1}[-X]\,d\gamma = \frac{13}{6} < \rho(X).$$

3. *Case 3: $K(E_Q[-Z],Q) = h(E_Q[-Z])$ with $h(t) = \begin{cases} t; & t<0, \\ 0; & 0\leq t<1, \\ t-1; & t\geq 1. \end{cases}$*

Hence, $K(t,Q) \leq t$ for any $t \in \mathbb{R}$ and ρ is monotone, quasi-convex, continuous from above and such that $\rho(0) = 0$ and $\rho(c) \leq -c$ for any $c \in \mathbb{R}$. It is also easy to check that $\rho(X) = 2$, $Q_{\gamma X} \equiv Q_1$ for any $\gamma \in [0,1]$ and $\Lambda(X;X) = \int_0^1 E_{Q_\gamma}[-X]\, d\gamma = 3 > \rho(X)$.

In this last case Λ is "locally"prudential. The same phenomenon happens, for every X, in the work of Brunnermeier and Cheridito on systemic risk measures (see [6, Proposition 3.1]). In the case of prudential CAR, the contribution of any subunit can be diminished according to some exogenous principle, to obtain full allocation. When instead the allocation is audacious, $\Lambda_\rho^{AS}(X;X) = \sum_i \Lambda_\rho^{AS}(X_i;X) \leq \rho(X)$, we can imagine that $\rho(X)$ is allocated to the X_i according to Λ_ρ^{AS} and $\rho(X) - \sum_i \Lambda_\rho^{AS}(X_i;X)$ is an unshared deposit that remains at the whole firm's disposal.

Example 11 (Non-differentiable quasi-convex risk measure) *Consider the mean value premium principle (that is, the risk measure associated to the certainty equivalent)*

$$\rho(X) = \ell^{-1}\left(E_P\left[\ell(-X)\right]\right)$$

with ℓ strictly increasing and convex. It is well known (see, among others, Cerreia-Vioglio et al. [8] and Frittelli and Maggis [24]) that it is a quasi-convex and continuous from below risk measure.

Take now $\Omega = \{\omega_1;\omega_2\}$, $P(\omega_i) = \frac{1}{2}$ for $i = 1,2$, $X = 0$, $\ell(x) = \begin{cases} x/2; & x < 0 \\ x; & x \geq 0 \end{cases}$ and a direction $Y = \begin{cases} -1; & \omega_1, \\ 1; & \omega_2. \end{cases}$ In that case, we obtain

$$\frac{\rho(X+tY)-\rho(X)}{t} = \frac{\rho(tY)-\rho(0)}{t} = \begin{cases} -1/4; & t < 0, \\ 1/4; & t > 0. \end{cases}$$

Consequently, ρ is not Gateaux differentiable since $\lim_{t\to 0} \frac{\rho(X+tY)-\rho(X)}{t}$ does not exist.

Example 12 (Capital allocation for a non-differentiable convex risk measure) *Let $\Omega = \{\omega_1;\omega_2;\omega_3\}$ and*

$$\rho(Z) = \max_{Q\in\mathcal{Q}}\{E_Q[-Z] - F(Q)\}$$

with $\mathcal{Q}$ as follows:

	P	Q_1	Q_2
ω_1	*1/3*	*1/2*	*1/4*
ω_2	*1/3*	*1/3*	*2/3*
ω_3	*1/3*	*1/6*	*1/12*
$F(\cdot)$	*0*	*1*	*1*

Take now

$$X = \begin{cases} -3; & \omega_1, \\ 0; & \omega_2, \\ 9; & \omega_3. \end{cases}$$

Let us consider $\mathcal{Q} = \{P, Q_1, Q_2\}$*. It is easy to check that* $\rho(X) = -1$ *and that* $\partial\rho(X) = \left\{ -\frac{dQ_1}{dP}; -\frac{dQ_2}{dP} \right\}$*, so* ρ *is not Gateaux differentiable. Furthermore,*

$$Q_{\gamma X} = \begin{cases} P; & 0 < \gamma < \frac{1}{2}, \\ Q_1, Q_2; & \frac{1}{2} < \gamma < 1. \end{cases}$$

Hence, for the CAR Λ^{Q_1} *obtained by choosing* Q_1 *for* $\gamma \in (1/2; 1)$ *we have*

$$\Lambda^{Q_1}(X; X) = \int_0^{1/2} E_P[-X]\, d\gamma + \int_{1/2}^1 E_{Q_1}[-X]\, d\gamma = -1 = \rho(X).$$

The same holds for Λ^{Q_2} *obtained by choosing* Q_2*.*

Note that $R, S \in Q_{\gamma X}$ *on an interval* $(\gamma_0; \gamma_1) \subset [0, 1]$ *with* $R \neq S$ *and* $\gamma_0 \neq \gamma_1$ *holds if and only if both* $E_R[-X] = E_S[-X]$ *and* $F(R) = F(S)$ *hold true.*

For the previous argument, we may conclude that whenever

$$Q_{\gamma X} = \begin{cases} \mathcal{Q}_1^*; & on\ I_1 \\ \mathcal{Q}_2^*; & on\ I_2 \\ \vdots & \vdots \\ \mathcal{Q}_n^*; & on\ I_n \end{cases}$$

with I_i *disjoint and not empty subsets of* $[0, 1]$*, then* $\Lambda^{Q_1^*, \ldots, Q_n^*}(X; X)$ *is invariant with respect to the choice of* $(Q_1^*, \ldots, Q_n^*) \in \mathcal{Q}_1^* \times \cdots \times \mathcal{Q}_n^*$*. the same does not hold true in general for* $\Lambda^{Q_1^*, \ldots, Q_n^*}(Y; X)$ *with* Y *a sub-portfolio of* X*.*

Consider, for instance,

$$Y_1 = \begin{cases} -3; & \omega_1, \\ 1; & \omega_2, \\ 3; & \omega_3, \end{cases} \qquad Y_2 = \begin{cases} 0; & \omega_1, \\ -1; & \omega_2, \\ 6; & \omega_3, \end{cases}$$

with $Y_1 + Y_2 = X$. *Hence,*

$$\Lambda^{Q_1}(Y_1;X) = \int_0^{1/2} E_P[-Y_1]\,d\gamma + \int_{1/2}^1 E_{Q_1}[-Y_1]\,d\gamma = \frac{1}{6},$$
$$\Lambda^{Q_2}(Y_1;X) = \int_0^{1/2} E_P[-Y_1]\,d\gamma + \int_{1/2}^1 E_{Q_2}[-Y_1]\,d\gamma = -\frac{1}{4},$$

while

$$\Lambda^{Q_1}(Y_2;X) = -\frac{7}{6}; \quad \Lambda^{Q_2}(Y_2;X) = -\frac{3}{4}.$$

As expected, $\Lambda^{Q_i}(Y_1;X) + \Lambda^{Q_i}(Y_2;X) = \Lambda^{Q_i}(X;X)$ *for* $i = 1,2$. *Furthermore, it is easy to check that* $\rho(Y_1) = -\frac{1}{3}$ *and* $\rho(Y_2) = -\frac{5}{6}$, *hence* $\Lambda^{Q_2}(Y_1;X) < \rho(Y_1) < \Lambda^{Q_1}(Y_1;X)$ *and* $\Lambda^{Q_1}(Y_2;X) < \rho(Y_2) < \Lambda^{Q_2}(Y_2;X)$.

5 Conclusions

In this paper, we have faced the problem of capital allocation using risk measures that are convex/quasi-convex and possibly non-differentiable in the sense of Gateaux.

The interest of treating the non-differentiable case was already evident in the works of Boonen et al. [5], Denault [14], Tsanakas and Barnett [34], and Tsanakas [35]. On the other hand, to the best of the authors' knowledge, ours is the first attempt to define a CAR with quasi-convex risk measures.

We have defined a family of principles based just on the dual representation theorems [8], [12], [20], [25] and that extend those of Denault [14] and Tsanakas [35], by also maintaining the game theoretic interpretation linked to the Aumann–Shapley value [3]. The non-uniqueness of the allocation rule, due to the fact that we have dropped the Gateaux-differentiability assumption, leaves open the question of how to choose the "most suitable"one.

From another standpoint, while the principles define a true CAR in the convex case (in the sense that they all allocate to each portfolio X, when considered as a sub-portfolio of itself, exactly its risk capital $\rho(X)$), the same does not always happen in the quasi-convex case. In this case, we have shown that the principles can behave in different ways, sometimes not allocating the whole capital. This feature can possibly represent an element of flexibility, which makes these schemes adoptable in various frameworks such as that of capital allocation for systemic risk measures [6].

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