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Dynamic credit investment in partially observed markets

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Abstract We consider the problem of maximizing the expected utility for a power investor who can allocate his wealth in a stock, a defaultable security, and a money market account. The dynamics of these security prices are governed by geometric Brownian motions modulated by a hidden continuous-time finite-state Markov chain. We reduce the partially observed stochastic control problem to a complete observation risk-sensitive control problem via the filtered regime switching probabilities. We separate the latter into predefault and postdefault dynamic optimization subproblems and obtain two coupled Hamilton–Jacobi–Bellman (HJB) partial differential equations. We prove the existence and uniqueness of a globally bounded classical solution to each HJB equation and give the corresponding verification theorem. We provide a numerical analysis showing that the investor increases his holdings in stock as the filter probability of being in high-growth regimes increases, and decreases his credit risk exposure as the filter probability of being in high default risk regimes gets larger.

Keywords Partial information · Filtering · Risk-sensitive control · Default risk · Hidden Markov chain

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1 Introduction

Regime-switching models constitute an appealing framework, stemming from their ability to capture the relevant features of asset price dynamics that behave differently depending on the specific phase of the business cycle in place. In the context of continuous-time utility maximization, some studies have considered observable regimes, whereas others have accounted for the possibility that they are not directly visible. In the case of observable regimes, [39] considers an infinite-horizon investment–consumption model where the agent can invest her wealth in a stock and risk-free bond, with borrowing and stock short-selling constraints. In a similar regime-switching framework, [36] studies the infinite-horizon problem of a risk-averse investor maximizing regime-dependent utility from terminal wealth and consumption. A different branch of literature has considered the case where regimes are hidden and need to be estimated from publicly available market information. [29] considers a finite-horizon portfolio optimization problem, where a power investor allocates his wealth across money market account and stocks whose price dynamics follow a diffusion process modulated by a hidden finite-state Markov process. [37] extends the analysis to the case where the time horizon is infinite. [11] studies the optimal investment problem of an insurer when the model uncertainty is governed by a hidden Markov chain. [35] considers optimal investment problems in general non-Markovian hidden regime-switching models. [34] considers a multistock market model with stochastic interest rates and drift modulated by a hidden Markov chain. Combining appropriate Malliavin calculus and filtering results from hidden Markov models, they derive explicit representations of the optimal strategies. In a series of two papers, [16] and [17] consider a regime-switching framework where logarithmic and power investors optimize terminal utility by rebalancing their stock portfolios only at discrete random points in time. Those correspond to time instants when changes in stock prices are observed.

The literature surveyed above has considered markets consisting of securities carrying market, but not default, risk. In recent years, a few studies have considered a portfolio optimization framework inclusive of defaultable securities. [24] studies optimal portfolio problems with defaultable assets within a Black–Cox framework. [26] considers an investor who can allocate her wealth across multiple defaultable bonds in a model where simultaneous defaults are allowed. In the same market model, [25] defines default as the beginning of financial distress and discuss contagion effects on prices of defaultable bonds. [2] derives optimal investment strategies for a CRRA investor, allocating her wealth among a defaultable bond, risk-free bank account and a stock. [5] considers a portfolio optimization problem where a logarithmic investor can choose a consumption rate and invest her wealth across a defaultable perpetual bond, a stock, and a money market account. [21] combines duality theory and dynamic programming to optimize the utility of a CRRA investor in a market consisting of a riskless bond and a stock subject to counterparty risk. Optimal investment under contagion risk has been considered by [4], who construct an empirically motivated framework based on interacting intensity models and analyze how contagion effects impact optimal allocation decisions due to abrupt changes in prices. A related study by [20] develops a portfolio framework where multiple default events can occur and some of the securities may still be traded after default.

The first attempt at using regime switching within a portfolio optimization framework consisting of defaultable securities was done by [6]. Such a modeling choice is also empirically supported by a study of [18], which identifies three credit regimes characterized by different levels of default intensity and recovery rates via a historical analysis of the corporate bond market. [6] considers an investor trading in a stock and defaultable security whose price dynamics are modulated by an observable Markov chain. Using the HJB approach, they recover the optimal investment strategies as the unique solution to a coupled system of partial differential equations.

The present paper considers the case where regimes are hidden, so that the power investor must decide on the optimal allocation policy using only the observed market prices. This improves upon the realism of the model in [6], given that in several circumstances market regimes such as inflation and recession, or credit regimes characterized by high or low credit spreads, are typically not observed by investors. Moreover, the hidden regime feature requires a completely different analysis and leads us to solving a partially observed stochastic control problem where regime information must be inferred from an enlarged market filtration. The latter is composed *both* of a reference filtration generated by the observable security prices *and* of a credit filtration tracking the occurrence of the default event. To the best of our knowledge, ours represents the first study in this direction.

We next describe our main contributions. First, by considering a portfolio optimization problem in a context of partial information with possibility of default, we advance earlier literature that has so far considered either one or the other aspect, but never both simultaneously. We construct an equivalent fully observed risk-sensitive control problem where the new state is given by the regime filtered probabilities, generalizing the approach of [29] who do not deal with default event information. We use the filter probabilities to obtain the Hamilton–Jacobi–Bellman (HJB) equation for the dynamic optimization problem, which we separate into coupled predefault and postdefault optimization subproblems. This is done using the projected filter process. We remark that the decomposition of a global optimal investment problem into control subproblems in a progressively enlarged filtration has also been considered by [20] and [31]. Their approach consists in first defining the control subproblems in the reference market filtration exclusive of default event information and then connecting them by assuming the existence of a conditional density on the default times, previously introduced in [9]. Despite some similarities between ours and their approach arising from the fact that both consider a pre- and postdefault decomposition and solve backwards, there are also significant differences between the two approaches. We consider the wealth dynamics under the enlarged market filtration inclusive of default events and do not perform any pre-/postdefault decomposition of the control problem at the level of the stochastic differential equation. It is only after deriving the HJB partial differential equations that the decomposition into pre- and postdefault PDEs naturally arises. Their approach instead exploits the exponential utility preference function of the investor to reduce the optimal investment problem to solving a recursive system of backward stochastic differential equations with respect to the default-free market filtration. A detailed analysis of these BSDEs including the possibility of jump times driven by Brownian motion is provided in [22].

Secondly, the presence of default risk makes the HJB PDE satisfied by the predefault value function nonlinear. There are two sources of nonlinearity, namely quadratic growth of the gradient and exponential nonlinearity. We first perform a suitable transformation yielding a parabolic PDE whose associated operator is linear in the gradient and matrix of second derivatives, but nonlinear in the solution. We then provide a rigorous analysis of the transformed PDE and prove the existence of a classical solution via a monotone iterative method. Since the nonlinear term is only locally, but not globally, Lipschitz-continuous because the derivative explodes at zero, we also need to prove that the solution is bounded away from zero. In particular, we establish both lower and upper bounds for the solution and prove $\mathcal{C}_P^{2,\alpha}$ regularity. We then use this result to prove a verification theorem establishing the correspondence between the solution to the PDE and the value function of the control problem. The proof of the theorem requires the development of a number of technical results, such as the guaranteed positivity of the filtering process. By contrast, the HJB PDE corresponding to the postdefault optimization problem can be linearized using a similar transformation to the one adopted by [29], and a unique classical solution can be guaranteed as shown, for instance, in [37].

Thirdly, we provide a thorough comparative statics analysis to illustrate the impact of partial information on the optimal allocation decisions. We consider a square-root investor and a two-state Markov chain. We find that the fraction of wealth invested in the stock increases as the filter probability of being in the regime with the higher growth rate increases. In order to be hedged against default, the investor shorts a higher number of units of defaultable security if the filter probability of staying in the higher default risk regime increases. Vice versa, when the probability of being in the safer regime increases, the investor increases his exposure to credit risk by shorting smaller amount of units of the defaultable security. If the regime is characterized by a sufficiently low level of default intensity, then the square-root investor may even go long credit and purchase units of the defaultable security. We find that lower values of price volatility induce the investor to increase the fraction of wealth invested in the risky asset. More specifically, if the stock volatility is low, then the filter gain coming from received observations is higher, and the investor purchases increasingly more units as the stock volatility decreases. Similarly, for a sufficiently high probability of being in the high default risk regime, the investor shorts increasingly larger number of units of defaultable security as the volatility of the latter decreases. This reflects the risk-averse nature of the investor, who wants to reduce his credit risk exposure more if the filter estimate becomes more accurate due to the higher informational gain from price observations. We also find that as observations become less informative due to higher price volatilities, the investor deposits a significant fraction of his wealth in the money market account. All this suggests that partial information tends to push the investor toward strategies reducing both market and credit risk exposure.

The rest of the paper is organized as follows. Section 2 defines the market model. Section 3 sets up the utility maximization problem. Section 4 derives the HJB equations corresponding to the risk sensitive control problem. Section 5 analyzes the solutions of the HJB PDE equations. Section 6 develops a numerical analysis. Section 7 summarizes our main conclusions. Finally, two appendices present the main proofs of the paper.

2 The market model

Let $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ be a complete filtered probability space, where $\mathbb{P}$ is the historical probability measure, $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ is an enlarged filtration given by $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$, where $(\mathcal{H}_t)$ is a filtration to be introduced further. We take the right-continuous version of $\mathbb{G}$, that is, $(\mathcal{G}_t)$ is the smallest right-continuous filtration containing $\mathbb{F}$ and $\mathbb{H}$, with $\mathcal{G}_t := \bigcap_{\epsilon>0}(\mathcal{F}_{t+\epsilon} \vee \mathcal{H}_{t+\epsilon})$ (see also [1]).

Here, $\mathbb{F} := (\mathcal{F}_t)$ is a suitable filtration supporting a two-dimensional Brownian motion $W = (W^{(1)}, W^{(2)})^\top$, where $^\top$ denotes the transpose. We also assume that the *hidden* states of the economy are modeled by a finite-state continuous-time Markov chain $X := (X_t)_{t\geq 0}$, which is adapted to $\mathbb{G}$ and assumed to be independent of $W := (W_t)_{t\geq 0}$. Without loss of generality, the state space is identified with the set of unit vectors $\{e_1, e_2, \dots, e_N\}$, where $e_i = (0, \dots, 1, \dots, 0)^\top \in \mathbb{R}^N$. The following semimartingale representation is well known (see [10]):

$$X_t = X_0 + \int_0^t A(s)^\top X_s \, ds + \varphi(t), \tag{2.1}$$

where $\varphi(t) = (\varphi_1(t), \dots, \varphi_N(t))^\top$, $t \geq 0$, is an $\mathbb{R}^N$-valued martingale under $\mathbb{P}$, and $A(t) := [\varpi_{i,j}(t)]_{i,j=1,\dots,N}$ is the so-called generator of the Markov process. Specifically, denoting $p_{i,j}(t,s) := \mathbb{P}[X_s = e_j \mid X_t = e_i]$ for $s \geq t$ and $\delta_{i,j} = \mathbf{1}_{\{i=j\}}$, we have (see [3, Chap. 11.2]) that

$$\varpi_{i,j}(t) = \lim_{h\to 0} \frac{p_{i,j}(t, t+h) - \delta_{i,j}}{h}.$$

In particular, $\varpi_{i,i}(t) = -\sum_{j\neq i} \varpi_{i,j}(t)$. We also impose the mild condition

$$\sup_{t\geq 0} \max_{i,j} \varpi_{i,j}(t) < \infty.$$

We denote by $p^\circ = (p^{\circ,1}, \dots, p^{\circ,N})$ the initial distribution of the Markov chain and **throughout the paper assume that** $p^{\circ,i} > 0$.

We consider a frictionless financial market consisting of three instruments: a risk-free bank account, a defaultable security, and a stock.

Risk-free bank account. The instantaneous market interest rate is assumed to be constant. The dynamics of the price process (B_t) that describes the risk-free bank account is given by

$$dB_t = r B_t \, dt, \qquad B_0 = 1.$$

Stock security. We assume that the appreciation rate μ_t of the stock depends on the economic regime X_t via

$$\mu_t := \langle \mu, X_t \rangle,$$

where $\mu = (\mu_1, \mu_2, \dots, \mu_N)$ is a vector with constant components denoting the values of the drift associated to the different economic regimes, and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^N$. Under the historical measure, the stock dynamics are given by

$$dS_t = \mu_t S_t\,dt + \sigma S_t\,dW_t^{(1)}, \quad S_0 = s^\circ. \tag{2.2}$$

Defaultable security. Before defining the vulnerable security considered in the present paper, we need to introduce the default process. Let τ be a nonnegative random variable defined on $(\Omega, \mathcal{G}, \mathbb{P})$, representing the default time of the counterparty selling the security. Let $\mathcal{H}_t = \sigma(H_u : u \le t)$ be the filtration generated by the *default process* $H_t := \mathbf{1}_{\{\tau \le t\}}$. We use the canonical construction of the default time τ in terms of a given hazard process. The latter is defined by $\Gamma_t := -\ln(1 - \mathbb{E}^{\mathbb{P}}[H_t \mid \mathcal{F}_t])$ (see also Definition 9.2.1 in [3]) and postulated to have absolutely continuous sample paths with respect to Lebesgue measure on $\mathbb{R}_+$. In other words, it is assumed to admit the integral representation

$$\Gamma_t = \int_0^t h_u\,du$$

for an $\mathbb{F}$-progressively measurable nonnegative stochastic process $(h_t)_{t\ge 0}$ with integrable sample paths. The process (h_t) is referred to as the $\mathbb{F}$-hazard rate of τ and will be specified later. We next give the details of the construction of the random time τ. We assume the existence of an exponential random variable χ defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, independent of the process (X_t). We define τ by setting

$$\tau := \inf\left\{t \in \mathbb{R}_+ : \int_0^t h_u\,du \ge \chi\right\}$$

with the convention $\inf\emptyset = +\infty$. It can be proved (see [3], Sect. 6.5) that

$$\xi_t := H_t - \int_0^t \bar{H}_{u-} h_u\,du = H_t - \int_0^{t\wedge\tau} h_u\,du \tag{2.3}$$

is a $\mathbb{G}$-martingale under $\mathbb{P}$, where $\bar{H}_u := 1 - H_u$ and $\bar{H}_{u-} := \lim_{s\uparrow u} \bar{H}_s = \mathbf{1}_{\{\tau \ge u\}}$. Intuitively, (2.3) says that the single jump process needs to be compensated for default, prior to the occurrence of the event. As with the appreciation rate, we assume that the process h is driven by the hidden Markov chain via

$$h_t := \langle h, X_t\rangle,$$

where $h = (h_1, \dots, h_N) \in (0,\infty)^N$ denotes the possible values that the default rate process can take depending on the economic regime in place. We model the predefault dynamics of the defaultable security as

$$\frac{dP_t}{P_t} = a(t, X_t)\,dt + \upsilon\,dW_t^{(2)}, \quad t < \tau,\ P_0 = P^\circ, \tag{2.4}$$

where $P^\circ \in \mathbb{R}_+$, and $a : \mathbb{R}_+ \times \{e_1, \dots, e_N\} \to \mathbb{R}_+$ is a deterministic function. After default the security becomes worthless, that is, $P_t := 0$ for any $t \ge \tau$, and thus $(P_t)_{t\ge 0}$ follows the dynamics

$$dP_t = P_{t-}\big(a(t, X_t)\,dt + \upsilon\,dW_t^{(2)} - dH_t\big). \tag{2.5}$$

For future reference, we also impose the mild technical assumption

$$\int_0^T a^2(t, e_i)\, dt < \infty, \quad \text{for any } T > 0 \text{ and } i \in \{1, \dots, N\}. \tag{2.6}$$

Remark 2.1 As usual, when dealing with hidden Markov models, the volatility components are assumed to be constant; see, for instance, [29]. If σ and υ were not constant but consisting of distinct components depending on X, then the Markov chain $(X_t)_{t\geq 0}$ would become observable. This is because the quadratic variation of X would converge almost surely to the integrated volatility; see [19, Sect. I.4]. Consequently, by inversion, the regime in place at time t would become known. Further, we notice that the choice of constant volatility might also provide a fairly good fit to market data when calibrating the hidden regime-switching model to market prices. This has been empirically shown by [27] on data from the New York Merchantile stock exchange using Markov chain Monte Carlo methods.

Remark 2.2 The specification given in (2.4) captures several relevant market models considered in the literature:

(1) First, the model (2.4) may be specialized to capture the predefault dynamics of a defaultable stock. The latter is a widely used instrument in hybrid models of equity and credit. For instance, [28] and [7] model the pre-bankruptcy risk-neutral dynamics of a defaultable stock as

$$dS_t = (r + h_t) S_t\, dt + \upsilon S_t\, dW_t,$$

where $(W_t)_{t\geq 0}$ is a Brownian driver, and $(h_t)_{t\geq 0}$ is a stochastic (adapted) default intensity process. Clearly, such a specification is a special instance of (2.4), where we set $a(t, X_t) = r + h(X_t)$. The addition of the hazard rate in the drift ensures that the discounted stock price process is a martingale.

(2) Secondly, the dynamics in (2.4) may be used to model the time evolution of prices of credit sensitive securities when an additive type of "microstructure or market friction" noise is taken into account. In general, secondary market investors only observe market quotes for traded credit derivatives, such as spreads, at discrete points in time, for example, at times $t_k = k\Delta$, $k = 0, \dots, N$, for a certain fixed time mesh $\Delta > 0$. The corresponding observed yield spreads z_{t_k} are then often modeled as $z_{t_k} = a_{t_k}(t_k, X_{t_k}) + \epsilon_k$ with an i.i.d. sequence (ϵ_k), independent of X, capturing microstructure noise due to illiquidity, transaction costs, or transmission "errors." In that case, $a(t_k, X_{t_k})$ represents the underlying unobserved yield spread whose dynamics is modeled by a suitably chosen arbitrage-free dynamic term structure model. [13] argues that as the interarrival time $t_k - t_{k-1}$ between consecutive observations gets smaller, the cumulative log return process $z_t := \log(P_t/P_0)$ of the defaultable security converges in law to

$$\int_0^t a(s, X_s)\, ds + \upsilon W_t^{(2)}. \tag{2.7}$$

Again, the dynamics of (2.7) is in the form of our dynamics (2.4).

For future convenience, we introduce the two-dimensional observed pre-default log-price process $Y_t = (\log S_t, \log P_t)^\top$, whose dynamics is given by

$$dY_t = \vartheta(t, X_t)\,dt + \Sigma_Y\,dW_t, \tag{2.8}$$

where

$$\Sigma_Y := \begin{pmatrix} \sigma & 0 \\ 0 & \upsilon \end{pmatrix}, \tag{2.9}$$

$$\vartheta(t, X_t) := \left(\mu_t - \frac{\sigma^2}{2}, a(t, X_t) - \frac{\upsilon^2}{2}\right)^\top = \left(\langle \mu, X_t \rangle - \frac{\sigma^2}{2}, a(t, X_t) - \frac{\upsilon^2}{2}\right)^\top.$$

We also define two subfiltrations of $\mathbb{G}$, the market filtration $\mathbb{G}^I := (\mathcal{G}_t^I)_{t\geq 0}$ where

$$\mathcal{G}_t^I := \mathcal{F}_t^I \vee \mathcal{H}_t, \qquad \mathcal{F}_t^I := \sigma(S_u, P_u; u \leq t),$$

and the subfiltration $\mathbb{F}^X := (\mathcal{F}_t^X)_{t\geq 0}$ generated by the Markov chain (X_t),

$$\mathcal{F}_t^X = \sigma(X_u; u \leq t).$$

Therefore, we may also write $\mathcal{G}_t = \mathcal{F}_t^X \vee \mathcal{G}_t^I$. From this it is evident that although $(X_t)_{t\geq 0}$ is $(\mathcal{G}_t)$-adapted, it is not $(\mathcal{G}_t^I)$-adapted.

3 The utility maximization problem

We consider an investor who wants to maximize her expected final utility during a trading period $[0, T]$ by dynamically allocating her financial wealth into (1) the risk-free bank account, (2) the stock, and (3) the defaultable security, as defined in the previous section. Let us denote by ν_t^B the number of shares that the investor holds in the bank account at time t. Similarly, ν_t^S and ν_t^P denote the investor's portfolio positions in the stock and defaultable security at time t, respectively. The process $\nu := (\nu_t^B, \nu_t^S, \nu_t^P)_{t\geq 0}$ is called a *portfolio process*. We denote $V_t(\nu)$ the wealth of the portfolio process $\nu = (\nu^B, \nu^S, \nu^P)$ at time t, that is,

$$V_t(\nu) = \nu_t^B B_t + \nu_t^S S_t + \nu_t^P \mathbf{1}_{\{\tau > t\}} P_t.$$

We require the processes ν^B, ν^S, and ν^P to be $\mathcal{G}^I$-predictable. The investor does not have intermediate consumption nor capital income to support her trading of financial assets, and hence we also assume the self-financing condition

$$dV_t = \nu_t^B\,dB_t + \nu_t^S\,dS_t + \nu_t^P \mathbf{1}_{\{\tau > t\}}\,dP_t.$$

Let

$$\pi_t^B := \frac{\nu_t^B B_t}{V_{t-}(\nu)}, \qquad \pi_t^S := \frac{\nu_t^S S_t}{V_{t-}(\nu)}, \qquad \pi_t^P = \frac{\nu_t^P P_t}{V_{t-}(\nu)} \mathbf{1}_{\{\tau > t\}} \tag{3.1}$$

if $V_{t-}(\nu) > 0$, whereas $\pi_t^B = \pi_t^P = \pi_t^S = 0$ when $V_{t-}(\nu) = 0$. The vector $\bar{\pi} := (\pi^B, \pi^S, \pi^P)$, called a *trading strategy*, represents the corresponding fractions of wealth invested in each asset at time t. Note that if $\bar{\pi} := (\bar{\pi}_t)$ is admissible (the precise definition will be given later), then the dynamics of the resulting wealth process in terms of $\bar{\pi}$ can be written as

$$dV_t = V_{t-}\left(\pi_t^B \frac{dB_t}{B_t} + \pi_t^S \frac{dS_t}{S_t} + \pi_t^P \frac{dP_t}{P_t}\right) \tag{3.2}$$

with the convention that $0/0 = 0$. The latter convention is needed to deal with the case where default has occurred ($t \geq \tau$), so that $P_t = 0$, and we have $\pi_t^P = 0$. Using that $\pi^B + \pi^P + \pi^S = 1$ and the corresponding dynamics of B, S, and P, we can further rewrite the dynamics (3.2) as

$$\begin{aligned} \frac{dV_t^\pi}{V_{t-}^\pi} &= r\,dt + \pi_t^S(\mu_t - r)\,dt + \pi_t^S \sigma\,dW_t^{(1)} + \pi_t^P\big(a(t,X_t) - r\big)\,dt + \pi_t^P \upsilon\,dW_t^{(2)}, \\ V_0^\pi &= v, \end{aligned} \tag{3.3}$$

for a given initial budget $v \in (0,\infty)$. Here we use $\pi := (\pi^S, \pi^P)^\top$ to denote the investment strategy only consisting of positions in the stock and defaultable security and write V^π to emphasize the dependence of the wealth process on the strategy π. The objective of the power investor is then to choose $\pi = (\pi^S, \pi^P)^\top$ to maximize the expected utility from terminal wealth,

$$J(v,\pi,T) := \frac{1}{\gamma}\mathbb{E}^{\mathbb{P}}[(V_T^\pi)^\gamma], \tag{3.4}$$

for a given fixed value of $\gamma \in (0,1)$. By Itô's formula and (3.3) we readily obtain that $V^\gamma := (V^\pi)^\gamma$ follows the dynamics

$$\begin{aligned} dV_t^\gamma = \gamma V_{t-}^\gamma\Big(&r\,dt + \pi_t^S(\mu_t - r)\,dt + \pi_t^S\sigma\,dW_t^{(1)} \\ &+ \pi_t^P\big(a(t,X_t) - r\big)\,dt + \pi_t^P \upsilon\,dW_t^{(2)}\Big) \\ &+ \frac{1}{2}\gamma(\gamma-1)V_{t-}^\gamma\big((\pi_t^S)^2\sigma^2\,dt + (\pi_t^P)^2\upsilon^2\,dt\big). \end{aligned}$$

Next, recalling that $W := (W^{(1)}, W^{(2)})^\top$ and $\pi = (\pi^S, \pi^P)^\top$ and the definition of Σ_Y given in (2.9), we may rewrite the last SDE as

$$dV_t^\gamma = V_{t-}^\gamma\big(-\gamma\eta(t,X_t,\pi_t)\,dt + \gamma\pi_t^\top\Sigma_Y\,dW_t\big), \tag{3.5}$$

where

$$\begin{aligned} \eta(t,X_t,\pi_t) = -r + \pi_t^S(r - \langle\mu, X_t\rangle) + \pi_t^P\big(r - a(t,X_t)\big) \\ + \frac{1-\gamma}{2}\pi_t^\top\Sigma_Y^\top\Sigma_Y\pi_t. \end{aligned} \tag{3.6}$$

It is then clear that the solution to the stochastic differential equation (3.5) with initial condition $V_0 = v$ is given by

$$V_t^\gamma = v^\gamma \exp\left(\gamma \int_0^t \pi_s^\top \Sigma_Y \, dW_s - \gamma \int_0^t \eta(s, X_s, \pi_s) \, ds - \frac{\gamma^2}{2} \int_0^T \pi_s^\top \Sigma_Y \Sigma_Y^\top \pi_s \, ds\right). \tag{3.7}$$

From (3.4) and (3.7) we can see that we need to solve a maximization problem with partial information since the regime X_t is not directly observable and investment strategies can only be based on past information of security prices. Our approach is to transform it into a fully observed risk-sensitive control problem. Such a reduction is accomplished through two main steps. First, in Sect. 3.1, we show equivalence to a complete observation control problem with finite-dimensional Markovian state. Then, in Sect. 3.2, we transform the complete observation control problem into a risk-sensitive stochastic control problem.

3.1 An equivalent formulation as a complete observation control problem

The objective of this section is to show how the partially observed control problem in (3.4) may be reduced to a complete observation control problem. This is accomplished by showing that the criterion (3.4) may be equivalently rewritten as an expectation, taken with respect to a suitably chosen probability measure, of an exponential function of the (observable) regime filtered probabilities. Next, we start developing the change to the new measure, chosen so that the underlying chain (X_t) becomes independent of the investor filtration $\mathbb{G}^I$ under such a measure. First, we introduce some needed notation and terminology. Given two semimartingales L and M, we denote by $[L]$ and $[L, M]$ the quadratic variation of L and the quadratic covariation of L and M, respectively. We also denote the stochastic exponential of L by $\mathcal{E}(L)$. If L is of the form $L_t = \int_0^t \theta_s^\top \, dY_s$, where Y is an $\mathbb{R}^d$-valued continuous Itô process and $(\theta_s)_{s\geq 0}$ is $\mathbb{G}$-predictable, then

$$\mathcal{E}_t(L) = \exp\left(\int_0^t \theta_u^\top \, dY_u - \frac{1}{2} \int_0^t \theta_u^\top \, d[Y]_u \theta_u\right). \tag{3.8}$$

If Z is of the form $Z_t = \int_0^t \iota_s \, d\xi_s$, where ξ has been defined in (2.3) and $(\iota_s)_{s\geq 0}$ is $\mathbb{G}$-predictable with $\iota > -1$, then

$$\mathcal{E}_t(Z) = \exp\left(\int_0^t \log(1 + \iota_s) \, dH_s - \int_0^{t\wedge\tau} \iota_s h_s \, ds\right). \tag{3.9}$$

It is well known (see [3], Sect. 4.3) that the process $R := \mathcal{E}(L)\mathcal{E}(Z)$ follows the SDE

$$R_t = 1 + \int_0^t R_{s-}(\theta_s^\top \, dY_s + \iota_s \, d\xi_s). \tag{3.10}$$

We now proceed to introduce the new measure $\hat{\mathbb{P}}$ on $(\Omega, \mathbb{G})$. It is defined in terms of its density process by

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\bigg|_{\mathcal{G}_t} := \mathcal{E}_t\left(\int_0^\cdot -\vartheta(s, X_s)^\top \Sigma_Y^{-1}\, dW_s\right) \mathcal{E}_t\left(\int_0^\cdot \frac{1-h_{s-}}{h_{s-}}\, d\xi_s\right) =: \rho_t^{(1)} \rho_t^{(2)}.$$

In particular, using (3.8) and (3.9), we have that $\rho^{(1)}$ and $\rho^{(2)}$ are given by

$$\rho_t^{(1)} = \exp\left(-\int_0^t \vartheta(s, X_s)^\top \Sigma_Y^{-1}\, dW_s - \frac{1}{2}\int_0^T \vartheta(s, X_s)^\top (\Sigma_Y \Sigma_Y^\top)^{-1} \vartheta(s, X_s)\, ds\right),$$

$$\begin{aligned}\rho_t^{(2)} &= \exp\left(-\int_0^t \log h_{u-}\, dH_u - \int_0^{t\wedge\tau} (1-h_u)\, du\right)\\ &= h_{\tau-}^{-\mathbf{1}_{\{\tau \le t\}}} \exp\left(-\int_0^{t\wedge\tau} (1-h_u)\, du\right).\end{aligned}$$

Moreover, from (3.10), the density process $\rho = \rho^{(1)}\rho^{(2)}$ admits the representation

$$\rho_t = 1 + \int_0^t \rho_{s-}\left(-\vartheta(s, X_s)^\top \Sigma_Y^{-1}\, dW_s + \frac{1-h_{s-}}{h_{s-}}\, d\xi_s\right). \tag{3.11}$$

In order to show that $\hat{\mathbb{P}}$ is well defined, we must verify that $\mathbb{E}^{\mathbb{P}}[\rho_T] = 1$. To this end, we use a general version of Novikov's condition, as proved in [32, Thm. 9], which states that the stochastic exponential $\mathcal{E}(M)$ of a locally square-integrable martingale M is a martingale on $[0, T]$ if

$$\mathbb{E}^{\mathbb{P}}\left[e^{\frac{1}{2}\langle M^c, M^c\rangle_T + \langle M^d, M^d\rangle_T}\right] < \infty, \tag{3.12}$$

where M^c and M^d are the continuous and purely discontinuous martingale parts of M. From (3.11), $\rho = \mathcal{E}(M)$ with

$$M_t = -\int_0^t \vartheta(s, X_s)^\top \Sigma_Y^{-1}\, dW_s + \int_0^t \frac{1-h_{s-}}{h_{s-}}\, d\xi_s.$$

Therefore, we have

$$\begin{aligned}\langle M^c, M^c\rangle_T &= \int_0^T \vartheta^\top(s, X_s)(\Sigma_Y \Sigma_Y)^{-1}\vartheta(s, X_s)\, ds,\\ \langle M^d, M^d\rangle_T &= \int_0^T \frac{(1-h_s)^2}{h_s} \bar{H}_s\, ds.\end{aligned}$$

Clearly, $\langle M^c, M^c\rangle_T$ is bounded in view of the condition (2.6). It remains to prove that $\langle M^d, M^d\rangle_T$ is also bounded. We have

$$\langle M^d, M^d\rangle_T = \int_0^T \left(\frac{1}{h_s} + h_s - 2\right)\bar{H}_s\, ds \le \int_0^T \left(\frac{1}{h_s} + h_s\right) ds.$$

Since $h_i > 0$ for all $i \in \{1, \dots, N\}$ and $h_s = \langle h, X_s \rangle$, we obtain that

$$\max_{i \in \{1,\dots,N\}} \left(\frac{1}{h_i} + h_i \right) < C$$

for some constant $C > 0$. Thus, we conclude that $\langle M^d, M^d \rangle_T$ is also bounded.

Under the probability measure $\hat{\mathbb{P}}$, by Girsanov's theorem (see [3], Sect. 5.3),

$$\hat{W}_t = W_t + \int_0^t \Sigma_Y^{-1} \vartheta(s, X_s)\, ds$$

is a Brownian motion, and

$$\hat{\xi}_t = \xi_t - \int_0^{t \wedge \tau} (1 - h_u)\, du = H_t - \int_0^{t \wedge \tau} du = H_t - \int_0^t \bar{H}_{u-}\, du$$

is a $\mathbb{G}^I$-martingale. Note also that by (2.8) the observed predefault log-price process $Y = (\log S, \log P)^\top$ possesses the dynamics $dY_t = \Sigma_Y d\hat{W}_t$ under $\hat{\mathbb{P}}$. Furthermore, the inverse density process

$$U_t := \frac{d\mathbb{P}}{d\hat{\mathbb{P}}}\bigg|_{\mathcal{G}_t}$$

can be written as $U_t = U_t^{(1)} U_t^{(2)}$, where

$$U_t^{(1)} := \exp\left(\int_0^t \vartheta(s, X_s)^\top \Sigma_Y^{-1}\, dW_s + \frac{1}{2} \int_0^t \vartheta(s, X_s)^\top (\Sigma_Y \Sigma_Y^\top)^{-1} \vartheta(s, X_s)\, ds \right),$$

$$U_t^{(2)} := h_{\tau-}^{\mathbf{1}_{\{\tau \le t\}}} \exp\left(\int_0^{t \wedge \tau} (1 - h_u)\, du \right) = \mathcal{E}_t \left(\int_0^\cdot (h_{s-} - 1)\, d\hat{\xi}_s \right).$$

Using $\hat{\mathbb{P}}$ together with the representation (3.7), we may rewrite (3.4) as

$$\begin{aligned} \frac{1}{\gamma} \mathbb{E}^{\mathbb{P}}[V_T^\gamma] &= \frac{\upsilon^\gamma}{\gamma} \mathbb{E}^{\hat{\mathbb{P}}}\Big[e^{-\gamma \int_0^T \eta(s, X_s, \pi_s)\, ds + \gamma \int_0^T \pi_s^\top \Sigma_Y\, dW_s - \frac{\gamma^2}{2} \int_0^T \pi_s^\top \Sigma_Y \Sigma_Y^\top \pi_s\, ds} U_T \Big] \\ &= \frac{\upsilon^\gamma}{\gamma} \mathbb{E}^{\hat{\mathbb{P}}}[L_T], \end{aligned} \tag{3.13}$$

where

$$L_t := \mathcal{E}_t \left(\int_0^\cdot Q(s, X_s, \pi_s)^\top \Sigma_Y\, d\hat{W}_s \right) U_t^{(2)} \exp\left(-\gamma \int_0^t \eta(s, X_s, \pi_s)\, ds \right) \tag{3.14}$$

and

$$\begin{aligned} Q(s, e_i, \pi_s) &:= (\Sigma_Y \Sigma_Y^\top)^{-1} \vartheta(s, e_i) + \gamma \pi_s \\ &= \left(\frac{1}{\sigma^2} \Big(\mu_i - \frac{\sigma^2}{2} \Big) + \gamma \pi_s^S, \frac{1}{\upsilon^2} \Big(a(t, e_i) - \frac{\upsilon^2}{2} \Big) + \gamma \pi_s^P \right)^\top. \end{aligned} \tag{3.15}$$

Next, we proceed to give the filter probabilities and some useful related relationships, for which we first need to introduce some notation. Throughout, the unit simplex in $\mathbb{R}^N$ is denoted by

$$\Delta_{N-1} = \{(d^1, d^2, \ldots, d^N) : d^1 + d^2 + \cdots + d^N = 1,\ d^i \geq 0,\ i = 1, \ldots, N\}.$$

Let $g : D \to \mathbb{R}$, where $D = D_1 \times \{e_1, \ldots, e_N\} \times D_2$ with arbitrary (possibly empty) domains D_1 and D_2. The mapping $\hat{g} : D_1 \times \Delta_{N-1} \times D_2 \to \mathbb{R}$ is defined as

$$\hat{g}(y, d, z) = \sum_{i=1}^{N} g(y, e_i, z) d^i \tag{3.16}$$

for $y \in D_1$, $d \in \Delta_{N-1}$, and $z \in D_2$. Given a vector $\ell = (\ell_1, \ldots, \ell_N) \in \mathbb{R}^N$, we define the associated mapping $\hat{\ell} : \Delta_{N-1} \to \mathbb{R}$ as

$$\hat{\ell}(d) := \sum_{i=1}^{N} \ell_i d^i. \tag{3.17}$$

The filter probability that the regime X_t is e_i at time t, conditional on the filtration $\mathcal{G}_t^I$, is denoted by

$$p_t^i := \mathbb{P}[X_t = e_i \mid \mathcal{G}_t^I], \quad i = 1, \ldots, N. \tag{3.18}$$

In particular, in terms of the transformations introduced in (3.16) and (3.17) and with $p_t := (p_t^1, \ldots, p_t^N)^\top$, we have the useful relationships

$$\mathbb{E}^{\mathbb{P}}[g(y, X_t, z) \mid \mathcal{G}_t^I] = \hat{g}(y, p_t, z), \qquad \mathbb{E}^{\mathbb{P}}[\langle \ell, X_t \rangle \mid \mathcal{G}_t^I] = \hat{\ell}(p_t).$$

We then have the following fundamental result, proved in [14].

Proposition 3.1 (Proposition 3.6 in [14]) *The normalized filter probabilities p_t are governed by the SDE*

$$\begin{aligned} dp_t^i &= \sum_{\ell=1}^{N} \varpi_{\ell,i}(t) p_t^\ell \, dt \\ &\quad + p_{t-}^i \big(\vartheta(t, e_i)^\top - \hat{\vartheta}(t, p_{t-})^\top\big)(\Sigma_Y \Sigma_Y^\top)^{-1}\big(dY_t - \hat{\vartheta}(t, p_t)\, dt\big) \\ &\quad + p_{t-}^i \frac{h_i - \hat{h}(p_{t-})}{\hat{h}(p_{t-})}\big(dH_t - \hat{h}(p_{t-})\bar{H}_{t-}\, dt\big) \end{aligned} \tag{3.19}$$

with initial condition $p_0^i = p^{\circ,i}$.

Note that since $\hat{h}(p_t) \geq \min_i h_i > 0$, there is no singularity arising in the filtering equation (3.19). We remark that [12, Sect. 4.1] also consider filter equations for finite-state Markov chains in the presence of multiple default events. However, they provide the dynamics of the unnormalized filter probabilities using a Zakai-type SDE and then construct an algorithm to compute the filter probabilities.

Remark 3.2 The existence and uniqueness of a strong solution to the system (3.19) was established in [23, Thm. 2.2].

We are now ready to give the main result of this section. Define

$$\hat{L}_t = \mathcal{E}_t\left(\int_0^\cdot \hat{Q}(s, p_s, \pi_s)^\top \Sigma_Y \, d\hat{W}_s\right) \times \mathcal{E}_t\left(\int_0^\cdot \left(\hat{h}(p_{s-}) - 1\right) d\hat{\xi}_s\right) e^{-\gamma \int_0^t \hat{\eta}(s, p_s, \pi_s)\, ds}, \tag{3.20}$$

where we use (3.16) and recall the definitions of η and Q in (3.6) and (3.15) to write

$$\begin{aligned}
\hat{\eta}(t, p_t, \pi_t) &= \sum_{i=1}^N \eta(t, e_i, \pi_t) p_t^i \\
&= -r + \pi_t^S\left(r - \hat{\mu}(p_t)\right) + \pi_t^P\left(r - \hat{a}(t, p_t)\right) \\
&\quad + \frac{1-\gamma}{2}\left(\sigma^2 (\pi_t^S)^2 + \upsilon^2 (\pi_t^P)^2\right), \\
\hat{Q}(t, p_t, \pi_t) &= \sum_{i=1}^N Q(t, e_i, \pi_t) p_t^i \\
&= \left(\frac{1}{\sigma^2}\left(\hat{\mu}(p_t) - \frac{\sigma^2}{2}\right) + \gamma \pi_t^S, \frac{1}{\upsilon^2}\left(\hat{a}(t, p_t) - \frac{\upsilon^2}{2}\right) + \gamma \pi_t^P\right)^\top.
\end{aligned}$$

Using (3.8) and (3.9), it follows directly from Itô's formula that

$$d\hat{L}_t = \hat{L}_{t-}\left(\hat{Q}(t, p_t, \pi_t)^\top dY_t + \left(\hat{h}(p_{t-}) - 1\right) d\hat{\xi}_t\right) - \gamma \hat{\eta}(t, p_t, \pi_t) \hat{L}_t \, dt. \tag{3.21}$$

Then, we have the following crucial result whose proof is reported in Appendix A.

Proposition 3.3 *We have*

$$J(v, \pi, T) = \frac{v^\gamma}{\gamma} \mathbb{E}^{\hat{\mathbb{P}}}[\hat{L}_T]. \tag{3.22}$$

Representation (3.22) establishes the correspondence between the original partially observed control problem (criterion (3.4) depending on the hidden state X_t) and a complete observation control problem (criterion (3.22) depending on the observed (filter) probability vector p_t).

3.2 The risk-sensitive control problem

The objective of this section is to show how the complete observation control problem (3.22) may be reduced to a risk-sensitive stochastic control problem. Such a representation proves to be useful for analyzing the control problem via the HJB approach in the next section. The reduction is obtained by building on the approach of [29], who do not consider the defaultable security. Next, we develop the change to the new measure $\tilde{\mathbb{P}}$, to write the criterion (3.22) in the risk-sensitive form. The measure change from $\hat{\mathbb{P}}$ to $\tilde{\mathbb{P}}$ is defined via its Radon–Nikodým density by

$$\frac{d\tilde{\mathbb{P}}}{d\hat{\mathbb{P}}}\bigg|_{\mathcal{G}_t^I} := \zeta_t := \mathcal{E}_t\left(\int_0^{\cdot} \hat{Q}(s, p_s, \pi_s)^\top \Sigma_Y \, d\hat{W}_s\right) \mathcal{E}_t\left(\int_0^{\cdot} \big(\hat{h}(p_{s-}) - 1\big)\, d\hat{\xi}_s\right). \quad (3.23)$$

Note that the probability measure $\tilde{\mathbb{P}}$ depends, through ζ, on the strategy π. Hence, in order for $\tilde{\mathbb{P}}$ to be a probability measure, we need to require that the set of admissible strategies satisfies the condition

$$\mathbb{E}^{\hat{\mathbb{P}}}[\zeta_T] = \mathbb{E}^{\mathbb{P}}[\rho_T \zeta_T] = 1. \quad (3.24)$$

In order to verify (3.24), we again use the general Novikov condition (3.12). In this case, it is easy to check that $\rho\zeta = \mathcal{E}(M)$ with

$$M_t = \int_0^t \big(\hat{Q}(s, p_s, \pi_s)^\top \Sigma_Y - \vartheta(s, X_s)^\top \Sigma_Y^{-1}\big)\, dW_s + \int_0^t \frac{\hat{h}(p_{s-}) - h_{s-}}{h_{s-}}\, d\xi_s,$$

and thus

$$\begin{aligned}\langle M^c, M^c\rangle_T &\le 2\int_0^T \hat{Q}(s, p_s, \pi_s)^\top \Sigma_Y \Sigma_Y^\top \hat{Q}(s, p_s, \pi_s)\, ds \\ &\quad + 2\int_0^T \vartheta(s, X_s)^\top (\Sigma_Y \Sigma_Y)^{-1} \vartheta(s, X_s)\, ds, \\ \langle M^d, M^d\rangle_T &= \int_0^{\tau\wedge T} \left(\frac{\hat{h}(p_s) - h_s}{h_s}\right)^2 h_s\, ds.\end{aligned}$$

The second summand in the expression of $\langle M^c, M^c\rangle_T$ is uniformly bounded in view of the condition (2.6), whereas $\langle M^d, M^d\rangle_T$ is also bounded because the integrand therein is bounded by $2\max_i h_i^2/\min_i h_i$. Therefore, we only need to require that $\mathbb{E}^{\mathbb{P}}[e^{\frac{1}{2}\int_0^T \hat{Q}(s,p_s,\pi_s)^\top \Sigma_Y \Sigma_Y^\top \hat{Q}(s,p_s,\pi_s)\, ds}] < \infty$, for which it suffices that $\pi = (\pi^S, \pi^P)^\top$ meets the integrability condition

$$\mathbb{E}^{\mathbb{P}}\Big[e^{\frac{\sigma^2\gamma^2}{2}\int_0^T (\pi_s^S)^2\, ds + \frac{\upsilon^2\gamma^2}{2}\int_0^T (\pi_s^P)^2\, ds}\Big] < \infty. \quad (3.25)$$

Once we have established conditions for the validity of the probability transformation (3.23), we can then apply Girsanov's theorem to conclude that

$$\tilde{W}_t = \hat{W}_t - \int_0^t \Sigma_Y^\top \hat{Q}(s, p_s, \pi_s)\, ds$$

is a $\mathbb{G}^I$-Brownian motion under $\tilde{\mathbb{P}}$, whereas

$$\begin{aligned}\tilde{\xi}_t = \hat{\xi}_t - \int_0^{t\wedge\tau} \big(\hat{h}(p_s) - 1\big)\, ds &= H_t - \int_0^{t\wedge\tau} \hat{h}(p_s)\, ds \\ &= H_t - \int_0^t \hat{h}(p_s)\bar{H}_{s-}\, ds \quad (3.26)\end{aligned}$$

is a $\mathbb{G}^I$-martingale under $\tilde{\mathbb{P}}$. It then follows immediately that

$$J(v,\pi,T)=\frac{v^\gamma}{\gamma}\mathbb{E}^{\tilde{\mathbb{P}}}\big[\hat{L}_T\zeta_T^{-1}\big]=\frac{v^\gamma}{\gamma}\mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{-\gamma\int_0^T\hat{\eta}(s,p_s,\pi_s)\,ds}\right], \tag{3.27}$$

where the dynamics of p^i in (3.19) may be rewritten under the measure $\tilde{\mathbb{P}}$ as

$$\begin{aligned}dp_t^i&=p_{t-}^i\big(\vartheta(t,e_i)^\top-\hat{\vartheta}(t,p_{t-})^\top\big)\Sigma_Y^{-1}\,d\tilde{W}_t\\&\quad+\left(\sum_{\ell=1}^N\varpi_{\ell,i}(t)p_t^\ell+\gamma p_t^i\big(\vartheta(t,e_i)^\top-\hat{\vartheta}(t,p_t)^\top\big)\pi_t\right)dt\\&\quad+p_{t-}^i\frac{h_i-\hat{h}(p_{t-})}{\hat{h}(p_{t-})}\,d\tilde{\xi}_t,\quad t\geq 0.\end{aligned} \tag{3.28}$$

Hence, the overall conclusion is that the original problem is reduced to a risk-sensitive control problem of the form

$$\sup_\pi J(v;\pi;T)=\frac{v^\gamma}{\gamma}\sup_\pi\mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{-\gamma\int_0^T\hat{\eta}(s,p_s,\pi_s)\,ds}\right], \tag{3.29}$$

where the maximization is done across suitable strategies (π_t). We shall specify later the precise class of trading strategies π on which the portfolio optimization problem is defined.

Remark 3.4 As customary with Markovian optimal control problems, we solve the risk-sensitive control problem starting at time $t=0$ by embedding it into a dynamic control problem starting at any time $t\in[0,T]$. Roughly speaking, the latter can be seen as the original problem (3.29), but starting at time $t\in(0,T]$ instead of 0. In order to formally define the dynamic problem, we consider a family of SDEs indexed by t of the form

$$\begin{aligned}dp_s^{t,i}&=p_{s-}^{t,i}\big(\vartheta(s,e_i)^\top-\hat{\vartheta}(s,p_{s-}^t)^\top\big)\Sigma_Y^{-1}\,d\widetilde{W}_s^t\\&\quad+\left(\sum_{\ell=1}^N\varpi_{\ell,i}(s)p_s^{t,\ell}+\gamma p_s^{t,i}\big(\vartheta(s,e_i)^\top-\hat{\vartheta}(s,p_s^t)^\top\big)\pi_s^t\right)ds\\&\quad+p_{s-}^{t,i}\frac{h_i-\hat{h}(p_{s-}^t)}{\hat{h}(p_{s-}^t)}\,d\widetilde{\xi}_s^t,\quad s\in(t,T],\end{aligned} \tag{3.30}$$

with initial condition $p_t^{t,i}=p^{\circ,i}$, defined on a suitable space $(\Omega^t,\mathcal{G}^t,\mathbb{G}^t,\widetilde{\mathbb{P}}^t)$ equipped with a Wiener process $(\widetilde{W}_s^t)_{s\geq t}$ starting at t and an independent one-point counting process $(H_s^t)_{s\geq t}$ such that $H_t^t=z^\circ\in\{0,1\}$ and

$$\tilde{\xi}_s^t:=H_s^t-\int_t^s\hat{h}(p_u^t)\bar{H}_{u-}^t\,du,\quad s\geq t,$$

is a $\widetilde{\mathbb{P}}^t$-martingale.

The construction of the process $(p_s^t)_{s\in[t,T]} := (p_s^{t,1}, \dots, p_s^{t,N})^\top_{s\in[t,T]}$ is carried out in a similar way as the construction of the solution to (3.28). Concretely, start defining the process $(p_s^t)_{s\ge 0} := (p_s^{t,1}, \dots, p_s^{t,N})^\top_{s\ge 0}$ via the representation (3.18), which, analogously to the fact that $(p_s)_{s\ge 0}$ follows the SDE (3.19), is the solution of a system of SDEs of the form

$$\begin{aligned} dp_s^{t,i} &= \sum_{\ell=1}^N \varpi_{\ell,i}(s) p_s^{t,\ell}\,ds \\ &\quad + p_{s-}^{t,i}\big(\vartheta(s,e_i)^\top - \hat{\vartheta}(s,p_{s-}^t)^\top\big)(\Sigma_Y\Sigma_Y^\top)^{-1}\big(dY_s^t - \hat{\vartheta}(s,p_s^t)\,ds\big) \\ &\quad + p_{s-}^{t,i}\frac{h_i - \hat{h}(p_{s-}^t)}{\hat{h}(p_{s-}^t)}\big(dH_s^t - \hat{h}(p_{s-}^t)\bar{H}_{s-}^t\,ds\big), \quad s \ge t, \end{aligned} \tag{3.31}$$

on a probability space $(\Omega^t, \mathcal{G}^t, \mathbb{G}^t, \mathbb{P}^t)$. Here we set, $Y_s^t := (\log S_s^t, \log P_s^t)^\top$, $s \ge t$, with $(S_s^t)_{s\ge t}$ and $(P_s^t)_{s\ge t}$ defined analogously to (2.2) and (2.5). Specifically, we let

$$\begin{aligned} dS_s^t &= \langle \mu, X_s^t\rangle S_s^t\,ds + \sigma S_s^t\,dW_s^{(1,t)}, \quad s > t,\ S_t^t = s^\circ, \\ dP_s^t &= P_{s-}^t\big(a(s,X_s^t)\,ds + \upsilon\,dW_s^{(2,t)} - dH_s^t\big), \quad s > t,\ P_t^t = P^\circ. \end{aligned}$$

The hidden chain $(X_s^t)_{s\ge t}$ has initial distribution $p_t^{t,i} = \mathbb{P}^t[X_t^t = e_i] = p^{\circ,i}$ and generator $A(s) := (\varpi_{i,j}(s))_{i,j=1,\dots,N}$, $s \ge t$. Once we have defined the process $(p_s^t)_{s\ge t}$, we define $\tilde{\mathbb{P}}^t$ in terms of a suitable trading strategy $(\pi_s^t)_{s\in[t,T]} := (\pi_s^{t,S}, \pi_s^{t,P})^\top_{s\in[t,T]}$ analogously to $\tilde{\mathbb{P}}$, and processes $(\widetilde{W}^t)_{s\ge t}$ and $(\widetilde{\xi}_s^t)_{s\ge t}$ analogously to $\tilde{W}$ and $\tilde{\xi}$, so that under $\tilde{\mathbb{P}}^t$, the process $p^t = (p^{t,1}, \dots, p^{t,N})^\top$ satisfies (3.30). Note that the existence of the measure transformation $\widetilde{\mathbb{P}}^t$ and hence of the solution to the SDE (3.30) is guaranteed provided that π^t satisfies the analogue of (3.25), that is,

$$\mathbb{E}^{\mathbb{P}^t}\Big[e^{\frac{\sigma^2\gamma^2}{2}\int_t^T (\pi_s^{t,S})^2\,ds + \frac{\upsilon^2\gamma^2}{2}\int_t^T (\pi_s^{t,P})^2\,ds}\Big] < \infty. \tag{3.32}$$

4 HJB formulation

This section is devoted to formulating the HJB equation. Given that the filter probability process $p = (p^1, \dots, p^N)$ is degenerate in $\mathbb{R}^N$, we consider the projected $(N-1)$-dimensional process

$$\tilde{p} := (\tilde{p}^1, \dots, \tilde{p}^{N-1})^\top := (p^1, \dots, p^{N-1})^\top,$$

as opposed to the actual filtering process. Next, we rewrite the problem (3.27) in terms of this process, which in view of Lemma B.1 in Appendix B now lies in the space

$$\tilde{\Delta}_{N-1} = \{(d^1, \dots, d^{N-1}) : d^1 + \dots + d^{N-1} < 1,\ d^i > 0\}.$$

We start with some notation needed to write the SDE of $\tilde{p}$ in matrix form. First, similarly to (3.16) and (3.17), for a vector $\ell = (\ell_1, \dots, \ell_N) \in \mathbb{R}^N$ and a function

$g : D \to \mathbb{R}$, where $D = D_1 \times \{e_1, \dots, e_N\} \times D_2$ with arbitrary (possibly empty) domains D_1 and D_2, define the mappings $\tilde{g} : D_1 \times \tilde{\Delta}_{N-1} \times D_2 \to \mathbb{R}$ and $\tilde{\ell} : \tilde{\Delta}_{N-1} \to \mathbb{R}$ by

$$\tilde{g}(y, d, z) = g(y, e_N, z) + \sum_{i=1}^{N-1} \big(g(y, e_i, z) - g(y, e_N, z)\big) d^i,$$

$$\tilde{\ell}(d) := \ell_N + \sum_{i=1}^{N-1} (\ell_i - \ell_N) d^i, \tag{4.1}$$

for $d = (d^1, \dots, d^{N-1}) \in \tilde{\Delta}_{N-1}$ and $y \in D_1$, $z \in D_2$. The following relationships are useful in what follows: for $y \in D_1$ and $z \in D_2$,

$$\hat{g}(y, p_t, z) = \tilde{g}(y, \tilde{p}_t, z), \qquad \hat{\ell}(p_t) = \tilde{\ell}(\tilde{p}_t). \tag{4.2}$$

Throughout, the projection of a vector $\ell = (\ell_1, \dots, \ell_N)$ on the first $N-1$ coordinates is denoted by $\ell' := (\ell_1, \dots, \ell_{N-1})$. Similarly, for a given matrix B, we use B' to denote its projection on the submatrix consisting of the first $N-1$ columns. Hence,

$$\vartheta(t)' := \big(\vartheta(t, e_1), \dots, \vartheta(t, e_{N-1})\big) = \begin{pmatrix} \mu_1 - \frac{\sigma^2}{2}, & \dots & , \mu_{N-1} - \frac{\sigma^2}{2} \\ a(t, e_1) - \frac{\upsilon^2}{2}, & \dots & , a(t, e_{N-1}) - \frac{\upsilon^2}{2} \end{pmatrix}.$$

We denote by Diag($\mathbf{b}$) the diagonal matrix whose diagonal element is the ith component of the vector $\mathbf{b}$. Further, let $\beta_{\varpi}(t, \tilde{p}_t)$ be the $(N-1) \times 1$ vector defined by

$$\beta_{\varpi}(t, \tilde{p}_t) = \Bigg(\varpi_{N,1}(t) + \sum_{\ell=1}^{N-1} \big(\varpi_{\ell,1}(t) - \varpi_{N,1}(t)\big) \tilde{p}_t^{\ell}, \dots, \\ \varpi_{N,N-1}(t) + \sum_{\ell=1}^{N-1} \big(\varpi_{\ell,N-1}(t) - \varpi_{N,N-1}(t)\big) \tilde{p}_t^{\ell}\Bigg)^{\top}. \tag{4.3}$$

Finally, write $\mathbf{1}$ for the $(N-1)$-dimensional column vector whose entries are all 1.

We work with the collection $(p_s^t)_{t \le s \le T} = (p_s^{t,1}, \dots, p_s^{t,N})_{t \le s \le T}^{\top}$ constructed on a probability space $(\Omega^t, \mathcal{G}^t, \mathbb{G}^t, \mathbb{P}^t)$ as in Remark 3.4. Using (3.30), we may rewrite the dynamics of $\tilde{p}^t := (\tilde{p}^{t,1}, \dots, \tilde{p}^{t,N-1}) := (p^{t,1}, \dots, p^{t,N-1})$ under $\widetilde{\mathbb{P}}^t$ as

$$\begin{aligned} d\tilde{p}_s^t &= \mathrm{Diag}(\tilde{p}_{s-}^t) \Big(\vartheta(s)' - \mathbf{1}\tilde{\vartheta}(s, \tilde{p}_{s-}^t)\Big)^{\top} \Sigma_Y^{-1} \, d\tilde{W}_s^t + \beta_{\varpi}(s, \tilde{p}_s^t) \, ds \\ &\quad + \gamma \, \mathrm{Diag}(\tilde{p}_s^t) \Big(\vartheta(s)' - \mathbf{1}\tilde{\vartheta}(s, \tilde{p}_s^t)\Big)^{\top} \pi_s^t \, ds \\ &\quad + \mathrm{Diag}(\tilde{p}_{s-}^t) \frac{1}{\tilde{h}(\tilde{p}_{s-}^t)} \Big(h' - \mathbf{1}\tilde{h}(\tilde{p}_{s-}^t)\Big) \, d\tilde{\xi}_s^t, \quad t < s \le T, \\ \tilde{p}_t^t &= \tilde{p}^{\circ}, \end{aligned}$$

with initial value $\tilde{p}^\circ := (\tilde{p}^{\circ,1}, \dots, \tilde{p}^{\circ,N-1})^\top = (p^{\circ,1}, \dots, p^{\circ,N-1})^\top$. Next, define

$$\begin{aligned}
\kappa(s, \tilde{p}^t_s) &:= \mathrm{Diag}(\tilde{p}^t_s)\big(\vartheta(s)' - \mathbf{1}\tilde{\vartheta}(s, \tilde{p}^t_s)\big)^\top \Sigma_Y^{-1}, \\
\beta_\gamma(s, \tilde{p}^t_s, \pi^t_s) &:= \beta_\varpi(s, \tilde{p}^t_s) + \gamma\kappa(s, \tilde{p}^t_s)\Sigma_Y^\top \pi^t_s, \\
\varrho(\tilde{p}^t_{s-}) &:= \mathrm{Diag}(\tilde{p}^t_{s-})\frac{1}{\tilde{h}(\tilde{p}^t_{s-})}\big(h' - \mathbf{1}\tilde{h}(\tilde{p}^t_{s-})\big).
\end{aligned}$$

Then the dynamics of $\tilde{p}^t_s = (\tilde{p}^{t,1}_s, \dots, \tilde{p}^{t,N-1}_s)^\top$ for $s \in [t, T]$, under the probability measure $\tilde{\mathbb{P}}^t$, are given by

$$\begin{aligned}
d\tilde{p}^t_s &= \beta_\gamma(s, \tilde{p}^t_s, \pi^t_s)\,ds + \kappa(s, \tilde{p}^t_{s-})\,d\tilde{W}^t_s + \varrho(\tilde{p}^t_{s-})\,d\tilde{\xi}^t_s, \quad t < s \le T, \\
\tilde{p}^t_t &= \tilde{p}^\circ \in \tilde{\Delta}_{N-1}.
\end{aligned} \tag{4.4}$$

For the vector $\tilde{p}_s := (\tilde{p}^1_s, \dots, \tilde{p}^{N-1}_s)^\top := (p^1_s, \dots, p^{N-1}_s)^\top$, we may write a similar expression, which lives in the "real-world" space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ and starts at time 0.

Note that π^t affects the evolution of $(\tilde{p}^t_s)_{t\le s\le T}$ through the drift β_γ and also through the measure $\tilde{\mathbb{P}}^t$ due to an admissibility constraint analogous to (3.24). As explained in Remark 3.4, the condition (3.24) is satisfied, provided that (3.32) holds. In light of these observations, the following class of admissible controls is natural.

Definition 4.1 The class of *admissible strategies* $\mathcal{A}(t, T; \tilde{p}^\circ, z^\circ)$ consists of locally bounded feedback trading strategies such that for $t < s \le T$,

$$\pi^t_s := (\pi^{t,S}_s, \pi^{t,P}_s) = \big(\pi^{t,S}(s, \tilde{p}^t_{s-}, H^t_{s-}), \pi^{t,P}(s, \tilde{p}^t_{s-}, H^t_{s-})\big)$$

and

$$\begin{aligned}
\pi^t_t := (\pi^{t,S}_t, \pi^{t,P}_t) &= \big(\pi^{t,S}(t, \tilde{p}^t_t, H^t_t), \pi^{t,P}(t, \tilde{p}^t_t, H^t_t)\big) \\
&= \big(\pi^S(t, \tilde{p}^\circ, z^\circ), \pi^P(t, \tilde{p}^\circ, z^\circ)\big).
\end{aligned}$$

Moreover, for each strategy in this class, we impose that

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^t}\Bigg[\exp\bigg(&\frac{\sigma^2\gamma^2}{2}\int_t^T \big(\pi^{t,S}(s, \tilde{p}^t_{s-}, H^t_{s-})\big)^2\,ds \\
&+ \frac{\upsilon^2\gamma^2}{2}\int_t^T \big(\pi^{t,P}(s, \tilde{p}^t_{s-}, H^t_{s-})\big)^2\,ds\bigg)\Bigg] < \infty,
\end{aligned} \tag{4.5}$$

so that the measure change defined by (3.23) is well defined.

Let us now define the dynamic programming problem associated with our original utility maximization problem. For any $t \in [0, T)$, $\tilde{p}^\circ \in \tilde{\Delta}_{N-1}$, $z^\circ \in \{0, 1\}$, and feedback trading strategy $\pi^t \in \mathcal{A}(t, T; \tilde{p}^\circ, z^\circ)$, we set

$$G(t, \tilde{p}^\circ, z^\circ, \pi^t) := \mathbb{E}^{\tilde{\mathbb{P}}^t}\left[e^{-\gamma\int_t^T \tilde{\eta}(s, \tilde{p}^t_s, \pi^t_s)\,ds}\right],$$

where we recall that by construction $H_t^t = z^\circ$, $\tilde{p}^t$ is given as in (4.4), and $\tilde{\eta}$ is defined from η in accordance to (4.1) as

$$\tilde{\eta}(s, \tilde{p}_s^t, \pi_s^t) = \eta(s, e_N, \pi_s^t) + \sum_{i=1}^{N-1} \big(\eta(s, e_i, \pi_s^t) - \eta(s, e_N, \pi_s^t)\big)\tilde{p}_s^{t,i}.$$

Next, we define the value function

$$w(t, \tilde{p}^\circ, z^\circ) := \sup_{\pi^t \in \mathcal{A}(t,T;\tilde{p}^\circ,z^\circ)} \log G(t, \tilde{p}^\circ, z^\circ, \pi^t). \tag{4.6}$$

The crucial step to link the stated dynamic programming problem with our original problem is given by the fact that

$$\begin{aligned}
\sup_{\pi \in \mathcal{A}(0,T;\tilde{p}^\circ,z^\circ)} J(v; \pi; T) &= \frac{v^\gamma}{\gamma} \sup_{\pi \in \mathcal{A}(0,T;\tilde{p}^\circ,z^\circ)} \mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{-\gamma \int_0^T \hat{\eta}(s,p_s,\pi_s)\,ds}\right] \\
&= \frac{v^\gamma}{\gamma} \sup_{\pi \in \mathcal{A}(0,T;\tilde{p}^\circ,z^\circ)} \mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{-\gamma \int_0^T \tilde{\eta}(s,\tilde{p}_s,\pi_s)\,ds}\right] \\
&= \frac{v^\gamma}{\gamma} \sup_{\pi^0 \in \mathcal{A}(0,T;\tilde{p}^\circ,z^\circ)} \mathbb{E}^{\tilde{\mathbb{P}}^0}\left[e^{-\gamma \int_0^T \tilde{\eta}(s,\tilde{p}_s^0,\pi_s^0)\,ds}\right] \\
&= \frac{v^\gamma}{\gamma} e^{w(0,\tilde{p}^{\circ,1},\ldots,\tilde{p}^{\circ,N-1},z^\circ)},
\end{aligned}$$

where the first and second equalities follow from (3.27) and (4.2), whereas the third equality follows from the uniqueness of the strong solution to the system (3.31), which can be established as noticed in Remark 3.2.

We now proceed to derive the HJB equation corresponding to the value function in (4.6). Before doing so, we need to compute the generator $\mathcal{L}$ of the Markov process $s \in [t, T] \mapsto (s, \tilde{p}_s^t, H_s^t)$. This is done in the following lemma. We further use $y \cdot h$ to denote componentwise multiplication of two vectors y and h.

Lemma 4.2 *Let $(\tilde{p}_s^t)_{s\in[t,T]}$ be the process in (4.4) with π^t of the form*

$$\pi_s^t := \pi(s, \tilde{p}_{s-}^t, H_{s-}^t), \quad t < s \le T, \qquad \pi_t^t := \pi(t, \tilde{p}^\circ, z^\circ),$$

for a suitable function $\pi(s, \tilde{p}, z)$ such that (4.4) admits a unique strong solution. Then, for any $f(t, \tilde{p}, z)$ such that $f(t, \tilde{p}, 1)$ and $f(t, \tilde{p}, 0)$ are both $C^{1,2}$-functions, we have

$$f(s, \tilde{p}_s^t, H_s^t) = f(t, \tilde{p}^\circ, z^\circ) + \int_t^s \tilde{\mathcal{L}} f(u, \tilde{p}_u^t, H_u^t)\,du + \tilde{M}_s(f), \quad s \in (t, T], \tag{4.7}$$

where, denoting $\nabla_{\tilde{p}} f(t, \tilde{p}, z) := (\frac{\partial f}{\partial \tilde{p}^1}, \ldots, \frac{\partial f}{\partial \tilde{p}^{N-1}})$, $D^2 f := [\frac{\partial^2 f}{\partial \tilde{p}^i \partial \tilde{p}^j}]_{i,j=1}^{N-1}$, and $f_t(t, \tilde{p}, z) := \frac{\partial f}{\partial t}$ and recalling the notation $\tilde{h}(\tilde{p}) := h^N + \sum_{i=1}^{N-1}(h_i - h_N)\tilde{p}^i$ and

$$h' := (h_1, \dots, h_{N-1})^\top,$$

$$\tilde{\mathcal{L}} f(t,\tilde{p},z) := f_t(t,\tilde{p},z) + (\nabla_{\tilde{p}} f)\beta_\gamma\big(t,\tilde{p},\pi(t,\tilde{p},z)\big) + \frac{1}{2}\operatorname{tr}\big(\kappa\kappa^\top D^2 f(t,\tilde{p},z)\big)$$
$$+ (1-z)\bigg(f\Big(t, \frac{1}{\tilde{h}(\tilde{p})}(\tilde{p}\cdot h'), 1\Big) - f\,(t,\tilde{p},0)\bigg)\tilde{h}(\tilde{p}).$$

Moreover, the $\widetilde{\mathbb{P}}^t$*-local martingale component is*

$$\tilde{M}_s(f) = \int_t^s (\nabla_{\tilde{p}} f)\kappa(u,\tilde{p}^t_{u-})\, d\tilde{W}^t_u$$
$$+ \int_t^s \bigg(f\Big(u, \frac{1}{\tilde{h}(\tilde{p}^t_{u-})}(\tilde{p}^t_{u-}\cdot h'), 1\Big) - f\,\big(u, \tilde{p}^t_{u-}, 0\big)\bigg)\, d\tilde{\xi}^t_u.$$

Proof For simplicity, throughout the proof we drop the superscript t in the processes $\tilde{p}^t$, π^t, $\tilde{W}^t$, $\tilde{\xi}^t$, and H^t. Let $\tilde{p}^{c,i}$ denote the continuous component of $\tilde{p}^i$, determined by the first two terms on the right-hand side of (4.4). Using Itô's formula, we have

$$f(s,\tilde{p}_s,H_s) = f(t,\tilde{p}_t,H_t) + \int_t^s f_u(u,\tilde{p}_u,H_u)\,du + \sum_{i=1}^{N-1}\int_t^s \frac{\partial f}{\partial \tilde{p}^i}\, d\tilde{p}^{c,i}_u$$
$$+ \frac{1}{2}\sum_{i,j=1}^{N-1}\int_t^s \frac{\partial^2 f}{\partial \tilde{p}^i \tilde{p}^j}\, d\langle \tilde{p}^{c,i}, \tilde{p}^{c,j}\rangle_u$$
$$+ \sum_{t<u\le s}\big(f(u,\tilde{p}_u,H_u) - f(u,\tilde{p}_{u-},H_{u-})\big). \tag{4.8}$$

Note that the size of the jump of $\tilde{p}^i_t$ at the default time τ is given by

$$\tilde{p}^i_\tau - \tilde{p}^i_{\tau-} = \tilde{p}^i_{\tau-}\frac{h_i - \tilde{h}(\tilde{p}_{\tau-})}{\tilde{h}(\tilde{p}_{\tau-})},$$

implying that $\tilde{p}^i_\tau = \tilde{p}^i_{\tau-}h_i/\tilde{h}(\tilde{p}_{\tau-})$ and $\tilde{p}_\tau = (1/\tilde{h}(\tilde{p}_{\tau-}))(\tilde{p}_{\tau-}\cdot h')$. For $t<\tau\le s$, this leads to

$$\sum_{t<u\le s}\big(f(u,\tilde{p}_u,H_u) - f(u,\tilde{p}_{u-},H_{u-})\big)$$
$$= \bigg(f\Big(\tau, \frac{1}{\tilde{h}(\tilde{p}_{\tau-})}(\tilde{p}_{\tau-}\cdot h'), 1\Big) - f\,(\tau,\tilde{p}_{\tau-},0)\bigg)(H_s - H_t)$$
$$= \int_t^s \bigg(f\Big(u, \frac{1}{\tilde{h}(\tilde{p}_{u-})}(\tilde{p}_{u-}\cdot h'), 1\Big) - f\,(u,\tilde{p}_{u-},0)\bigg)\, dH_u$$
$$= \int_t^s \bigg(f\Big(u, \frac{1}{\tilde{h}(\tilde{p}_{u-})}(\tilde{p}_{u-}\cdot h'), 1\Big) - f\,(u,\tilde{p}_{u-},0)\bigg)\Big(d\tilde{\xi}_u + \bar{H}_{u-}\tilde{h}(\tilde{p}_{u-})\,du\Big),$$

where in the last equality we have used (3.26) along with the fact that

$$\hat{h}_u = \sum_{i=1}^{N} h_i p_u^i = h_N + \sum_{i=1}^{N-1} (h_i - h_N) p_u^i = \tilde{h}(\tilde{p}_u).$$

From this we deduce that (4.8) may be rewritten as

$$\begin{aligned}
&f(s, \tilde{p}_s, H_s) \\
&\quad = f(t, \tilde{p}_t, H_t) + \int_t^s f_u(u, \tilde{p}_u, H_u)\, du + \int_t^s (\nabla_{\tilde{p}} f)\beta_\gamma(u, \tilde{p}_u, \pi_u)\, du \\
&\qquad + \frac{1}{2}\sum_{i,j=1}^{N-1} \int_t^s (\kappa\kappa^\top)_{ij} \frac{\partial^2 f}{\partial \tilde{p}^i \tilde{p}^j}(u, \tilde{p}_u, H_u)\, du + \int_t^s (\nabla_{\tilde{p}} f)\kappa(u, \tilde{p}_{u-})\, d\tilde{W}_u \\
&\qquad + \int_s^t \left(f\left(u, \frac{1}{\tilde{h}(\tilde{p}_{u-})}(\tilde{p}_{u-} \cdot h'), 1\right) - f(u, \tilde{p}_{u-}, 0) \right) d\tilde{\xi}_u \\
&\qquad + \int_t^s \left(f\left(u, \frac{1}{\tilde{h}(\tilde{p}_u)}(\tilde{p}_u \cdot h'), 1\right) - f(u, \tilde{p}_u, 0) \right)(1 - H_u)\tilde{h}(\tilde{p}_u)\, du,
\end{aligned}$$

which proves the lemma. □

We are now ready to derive the HJB equation associated to the control problem. We first obtain it based on standard heuristic arguments, and then we provide rigorous verification theorems for the solution in the subsequent section. In light of the dynamic programming principle, we expect that for any $s \in (t, T]$,

$$w(t, \tilde{p}^\circ, z^\circ) = \sup_{\pi^t \in \mathcal{A}(t,T;\tilde{p}^\circ,z^\circ)} \log \mathbb{E}^{\widetilde{\mathbb{P}}^t} \left[e^{w(s, \tilde{p}_s^t, H_s^t) - \gamma \int_t^s \tilde{\eta}(u, \tilde{p}_u^t, \pi_u^t)\, du} \right] \tag{4.9}$$

with $\tilde{p}_t^t = \tilde{p}^\circ$ and $H_t^t = z^\circ$. Next, define $\varepsilon(s, \tilde{p}, z) = e^{w(s,\tilde{p},z)}$ and note that in light of Lemma 4.2,

$$\varepsilon(s, \tilde{p}_s^t, H_s^t) = \varepsilon(t, \tilde{p}^\circ, z^\circ) + \int_t^s \tilde{\mathcal{L}}\varepsilon(u, \tilde{p}_u^t, H_u^t)\, du + \tilde{M}_s(\varepsilon),$$

where the last term $\tilde{M}_s(\varepsilon)$ is the local martingale component of $\varepsilon(s, \tilde{p}_s^t, H_s^t)$. Plugging the previous equation into (4.9), we expect the relation

$$\begin{aligned}
0 = \sup_{\pi^t \in \mathcal{A}(t,T;\tilde{p}^\circ,z^\circ)} \mathbb{E}^{\widetilde{\mathbb{P}}^t} \Bigg[& \varepsilon(t, \tilde{p}^\circ, H^\circ) \left(e^{-\gamma \int_t^s \tilde{\eta}(u, \tilde{p}_u^t, \pi_u^t)\, du} - 1 \right) \\
& + e^{-\gamma \int_t^s \tilde{\eta}(u, \tilde{p}_u^t, \pi_u^t) du} \int_t^s \tilde{\mathcal{L}}\varepsilon(u, \tilde{p}_u^t, H_u^t)\, du \Bigg],
\end{aligned}$$

assuming that the local martingale component is a true martingale. Dividing by $s - t$ and taking the limit of the last expression as $s \to t$ leads us to the HJB equation

$$0 = \sup_\pi \left(\tilde{\mathcal{L}} - \gamma \tilde{\eta}(t, \tilde{p}^\circ, \pi) \right) \varepsilon(t, \tilde{p}^\circ, z^\circ). \tag{4.10}$$

Let us write (4.10) in terms of w. To this end, denote the differential component of $\tilde{\mathcal{L}}$ as $\tilde{\mathcal{D}}$, that is,

$$\tilde{\mathcal{D}} f(t,\tilde{p},z) := f_t(t,\tilde{p},z) + (\nabla_{\tilde{p}} f)(t,\tilde{p},z)\beta_\gamma\big(t,\tilde{p},\pi(t,\tilde{p},z)\big) + \frac{1}{2}\operatorname{tr}\big(\kappa\kappa^\top D^2 f(t,\tilde{p},z)\big).$$

Then, we note that

$$\begin{aligned}\tilde{\mathcal{L}}\varepsilon(t,\tilde{p},z) &= \tilde{\mathcal{D}}\varepsilon(t,\tilde{p},z) + (1-z)\tilde{h}(\tilde{p})\Big(e^{w(t,\frac{1}{\tilde{h}(\tilde{p})}\tilde{p}\cdot h',1)} - e^{w(t,\tilde{p},0)}\Big)\\ &= e^{w(t,\tilde{p},z)}\bigg(\tilde{\mathcal{D}}w + \frac{1}{2}\|(\nabla_{\tilde{p}} w)\kappa\|^2\\ &\quad + (1-z)\tilde{h}(\tilde{p})\Big(e^{w(t,\frac{1}{\tilde{h}(\tilde{p})}\tilde{p}\cdot h',1)-w(t,\tilde{p},0)} - 1\Big)\bigg).\end{aligned} \tag{4.11}$$

Thus, (4.10) takes the form

$$0 = \sup_\pi\bigg(\varepsilon(t,\tilde{p}^\circ,z^\circ)\bigg(\tilde{\mathcal{D}}w + \frac{1}{2}\|\nabla_{\tilde{p}} w\kappa\|^2 + (1-z^\circ)\tilde{h}(\tilde{p}^\circ)\Big(e^{w(t,\frac{1}{\tilde{h}(\tilde{p}^\circ)}\tilde{p}^\circ\cdot h',1)-w(t,\tilde{p}^\circ,0)} - 1\Big) - \gamma\tilde{\eta}(t,\tilde{p}^\circ,z^\circ,\pi)\bigg)\bigg). \tag{4.12}$$

In order to get a more explicit form, let us recall that

$$\eta(t,e_i,\pi) = -r + \pi^S(r - \langle\mu,e_i\rangle) + \pi^P\big(r - a(t,e_i)\big) + \frac{1-\gamma}{2}\pi^\top\Sigma_Y^\top\Sigma_Y\pi$$

and note that

$$\begin{aligned}\tilde{\eta}(t,\tilde{p},\pi) &= \eta\big(t,e_N,\pi(t,\tilde{p},z)\big)\\ &\quad + \sum_{i=1}^{N-1}\Big(\eta\big(t,e_i,\pi(t,\tilde{p},z)\big) - \eta\big(t,e_N,\pi(t,\tilde{p},z)\big)\Big)\tilde{p}^i \qquad (4.13)\\ &= -r + \pi^S\big(r - \tilde{\mu}(\tilde{p})\big) + \pi^P\big(r - \tilde{a}(t,\tilde{p})\big) + \frac{1-\gamma}{2}\pi^\top\Sigma_Y^\top\Sigma_Y\pi.\end{aligned}$$

We can now rewrite (4.12) as

$$\begin{aligned}&\frac{\partial w}{\partial t} + \frac{1}{2}\operatorname{tr}(\kappa\kappa^\top D^2 w) + \frac{1}{2}(\nabla_{\tilde{p}} w)\kappa\kappa^\top(\nabla_{\tilde{p}} w)^\top + \gamma r\\ &\quad + (1-z^\circ)\tilde{h}(\tilde{p}^\circ)\Big(e^{w(t,\frac{1}{\tilde{h}(\tilde{p}^\circ)}\tilde{p}^\circ\cdot h',1)-w(t,\tilde{p}^\circ,0)} - 1\Big)\\ &\quad + \sup_\pi\bigg((\nabla_{\tilde{p}} w)\beta_\gamma - \gamma\pi^S\big(r - \tilde{\mu}(\tilde{p}^\circ)\big) - (1-z^\circ)\gamma\pi^P\big(r - \tilde{a}(t,\tilde{p}^\circ)\big)\\ &\quad - \frac{1}{2}\gamma(1-\gamma)\pi^\top\Sigma_Y^\top\Sigma_Y\pi\bigg) = 0\end{aligned} \tag{4.14}$$

with terminal condition $w(T, \tilde{p}^\circ, z^\circ) = 0$ and all the derivatives of w evaluated at $(t, \tilde{p}^\circ, z^\circ)$.

Depending on whether or not default has occurred, we have two separate optimization problems to solve. Indeed, after default has occurred, the investor cannot invest in the defaultable security and only allocates his wealth in the stock and risk-free asset. The next section analyzes in detail the two cases.

5 Solution to the optimal control problem

We analyze the control problem developed in the previous section. We first decompose it into two related optimization subproblems, the post- and predefault problems. As we shall demonstrate, in order to solve the predefault optimization subproblem, we need the solution of the postdefault one. Next, we recall some function spaces needed for the following proofs. We denote by $\mathcal{C}_P^{2,\alpha}$ the set of functions locally in $C_P^{2,\alpha}((0,T) \times \tilde{\Delta}_{N-1}) \cap C([0,T] \times \tilde{\Delta}_{N-1})$, where we recall that for a given domain D of $\mathbb{R}^{N-1}$ and $\alpha \in (0,1)$, the parabolic Hölder space $C_P^{2,\alpha}(D)$ is defined by the norms

$$\begin{aligned}\|\psi\|_{C_P^{2,\alpha}(D)} &:= \|\psi\|_{C_P^\alpha(D)} + \|\partial_t \psi\|_{C_P^\alpha(D)} \\ &\quad + \sum_{i=1}^{N-1} \|\partial_{\tilde{p}_i} \psi\|_{C_P^\alpha(D)} + \sum_{i,j=1}^{N-1} \|\partial_{\tilde{p}_i \tilde{p}_j} \psi\|_{C_P^\alpha(D)} \end{aligned} \tag{5.1}$$

with

$$\|\psi\|_{C_P^\alpha(D)} := \sup_{(t,\tilde{p}) \in D} |\psi(t,\tilde{p})| + \sup_{\substack{(t,\tilde{p}),(t',\tilde{p}') \in D \\ (t,\tilde{p}) \neq (t',\tilde{p}')}} \frac{|\psi(t,\tilde{p}) - \psi(t',\tilde{p}')|}{(|\tilde{p} - \tilde{p}'|^2 + |t - t'|)^{\frac{\alpha}{2}}}.$$

Further, we denote

$$\|\psi\|_{C_P^{1,\alpha}(D)} := \|\psi\|_{C_P^\alpha(D)} + \sum_{i=1}^{N-1} \|\partial_{\tilde{p}_i} \psi\|_{C_P^\alpha(D)}.$$

5.1 Postdefault optimization problem

Assume that default has already occurred, that is, we are at a time t such that $\tau < t$. In particular, this means that $\pi_t^P = 0$. Let us denote by $\underline{w}(t, \tilde{p}) := w(t, \tilde{p}, 1)$ the value function in the postdefault optimization problem. Then, we may rewrite (4.14) as

$$\begin{aligned} 0 &= \underline{w}_t + \frac{1}{2} \operatorname{tr}(\underline{\kappa\kappa}^\top D^2 \underline{w}) + \frac{1}{2} (\nabla_{\tilde{p}} \underline{w}) \underline{\kappa\kappa}^\top (\nabla_{\tilde{p}} \underline{w})^\top + \gamma r \\ &\quad + \sup_{\pi^S} \Big((\nabla_{\tilde{p}} w) \underline{\beta}_\gamma - \gamma \pi^S \big(r - \tilde{\mu}(\tilde{p})\big) - \frac{\sigma^2}{2} \gamma (1-\gamma)(\pi^S)^2 \Big), \end{aligned} \tag{5.2}$$

$$\underline{w}(T, \tilde{p}) = 0.$$

Here, $\underline{\kappa}(\tilde{p})$ is an $(N-1)\times 1$ vector determined by the first column of $\kappa(t,\tilde{p})$ (the second column of $\kappa(t,\tilde{p})$ consists of zeros). Concretely,

$$\underline{\kappa}(\tilde{p}) := \mathrm{Diag}(\tilde{p})\big(\underline{\vartheta}^\top - \mathbf{1}\underline{\vartheta}(\tilde{p})\big)\frac{1}{\sigma}, \tag{5.3}$$

where $\underline{\vartheta} = (\mu_1 - \frac{1}{2}\sigma^2, \dots, \mu_{N-1} - \frac{1}{2}\sigma^2)$ is the first row of $\vartheta(t)'$ (the second row consists of zeros), and, correspondingly, $\underline{\vartheta}(\tilde{p}) = \tilde{\mu}(\tilde{p}) - \frac{1}{2}\sigma^2$ is a scalar with $\tilde{\mu}(\tilde{p}) = \mu_N + \sum_{i=1}^{N-1}(\mu_i - \mu_N)\tilde{p}^i$. Similarly, $\underline{\beta}_\gamma(t,\tilde{p},\pi)$ in (5.2) is defined as

$$\underline{\beta}_\gamma(t,\tilde{p},\pi) := \beta_\varpi(t,\tilde{p}) + \gamma\sigma\pi^S\underline{\kappa}(\tilde{p}),$$

where we recall that $\beta_\varpi(t,\tilde{p})$ has been defined in (4.3). It can easily be checked that the maximizer of (5.2) is given by

$$\pi^S(t,\tilde{p}) = \frac{1}{\sigma^2(1-\gamma)}\Big(\tilde{\mu}(\tilde{p}) - r + \sigma\big(\nabla_{\tilde{p}}\underline{w}(t,\tilde{p})\big)\underline{\kappa}(\tilde{p})\Big). \tag{5.4}$$

Plugging the maximizer (5.4) into (5.2), we obtain on the domain $(0,T)\times\tilde{\Delta}^{N-1}$ the PDE

$$\underline{w}_t + \frac{1}{2}\mathrm{tr}(\underline{\kappa\kappa}^\top D^2\underline{w}) + \frac{1}{2(1-\gamma)}(\nabla_{\tilde{p}}\underline{w})\underline{\kappa\kappa}^\top(\nabla_{\tilde{p}}\underline{w})^\top + (\nabla_{\tilde{p}}\underline{w})\underline{\Phi} + \underline{\Psi} = 0, \tag{5.5}$$

$$\underline{w}(T,\tilde{p}) = 0, \tag{5.6}$$

where

$$\underline{\Phi}(t,\tilde{p}) = \beta_\varpi(t,\tilde{p}) + \frac{\gamma}{1-\gamma}\frac{\tilde{\mu}(\tilde{p}) - r}{\sigma}\underline{\kappa}(\tilde{p}),$$

$$\underline{\Psi}(\tilde{p}) = \gamma r + \frac{\gamma}{2(1-\gamma)}\left(\frac{\tilde{\mu}(\tilde{p}) - r}{\sigma}\right)^2.$$

Next we state, without proof, a useful result as a lemma.

Lemma 5.1 (Theorem 3.1 in [37]) *For any $T \geq 0$, there exists a classical solution $\underline{w}$ that solves the Cauchy problem* (5.5), (5.6).

Remark 5.2 To show the existence of a classical solution to (5.5), (5.6), [37] extends the domain of this problem to $(0,T)\times\mathbb{R}^{N-1}$ and then solve the related Cauchy problem. It is well known from standard results (see [15, Thm. 4.1]) that such a solution is in $C_P^{2,\alpha}$. We use this fact in our subsequent proofs.

We then have the following:

Theorem 5.3 *The following assertions hold*:

(1) *The solution $\underline{w}(t,\tilde{p})$ coincides with the value function $w(t,\tilde{p},1)$ introduced in* (4.6).

(2) *The optimal feedback control, denoted by* $\widetilde{\pi}_s^S = \widetilde{\pi}^S(s, \tilde{p}_s^t)$, *is given by*

$$\widetilde{\pi}^S(s, \tilde{p}) := \frac{1}{\sigma^2(1-\gamma)} \big(\tilde{\mu}(\tilde{p}) - r + \sigma \nabla_{\tilde{p}} \underline{w}(s, \tilde{p}) \underline{\kappa}(\tilde{p})\big). \tag{5.7}$$

Moreover, the feedback trading strategy $\widetilde{\pi}_s := (\widetilde{\pi}_s^S, \widetilde{\pi}_s^P)^\top$, $\widetilde{\pi}_s^P := 0$, *is admissible, that is, it satisfies the conditions of Definition 4.1.*

The proof of Theorem 5.3 is reported in Appendix B. Next, we mention a few useful remarks about the technique used to establish the existence of a solution $\underline{w}$ of (5.5), (5.6) in Theorem 5.3. A similar technique will be used in the subsequent section to analyze the solution of the PDE associated with the predefault optimization problem. The idea is to transform the problem into a linear PDE via the Cole–Hopf transformation (see also, e.g., the proof of Theorem 3.1 in [37] and [29]). This leads to

$$\underline{\psi}(t, \tilde{p}) = e^{\frac{1}{1-\gamma} \underline{w}(t, \tilde{p})}. \tag{5.8}$$

Then it follows that $\underline{w}(t, \tilde{p})$ solves (5.5), (5.6) if and only if $\underline{\psi}(t, \tilde{p})$ solves on the domain $(0, T) \times \tilde{\Delta}^{N-1}$ the linear PDE

$$\frac{\partial \underline{\psi}}{\partial t} + \frac{1}{2} \operatorname{tr}(\underline{\kappa\kappa}^\top D^2 \underline{\psi}) + \underline{\Phi}^\top \nabla_{\tilde{p}} \underline{\psi} + \frac{\underline{\Psi}}{1-\gamma} \underline{\psi} = 0, \tag{5.9}$$

$$\underline{\psi}(T, \tilde{p}) = 1. \tag{5.10}$$

5.2 Predefault optimization problem

Assume that $\tau > t$, that is, default has not occurred by time t. Let us denote by $\bar{w}(t, \tilde{p}) := w(t, \tilde{p}, 0)$ the value function corresponding to the predefault optimization problem. Then, we may rewrite (4.14) as

$$\begin{aligned}
&\bar{w}_t + \frac{1}{2} \operatorname{tr}(\bar{\kappa}\bar{\kappa}^\top D^2 \bar{w}) + \frac{1}{2} (\nabla_{\tilde{p}} \bar{w}) \bar{\kappa}\bar{\kappa}^\top (\nabla_{\tilde{p}} \bar{w})^\top \\
&\quad + \sup_{\pi = (\pi^S, \pi^P)} \Big((\nabla_{\tilde{p}} \bar{w}) \bar{\beta}_\gamma - \gamma \pi^S \big(r - \tilde{\mu}(\tilde{p})\big) - \gamma \pi^P \big(r - \tilde{a}(t, \tilde{p})\big) \\
&\quad - \frac{1}{2} \gamma (1-\gamma) \pi^\top \Sigma_Y^\top \Sigma_Y \pi \Big) + \tilde{h}(\tilde{p}) \Big(e^{\underline{w}(t, \frac{1}{\tilde{h}(\tilde{p})} \tilde{p} \cdot h') - \bar{w}(t, \tilde{p})} - 1 \Big) + \gamma r = 0, \qquad (5.11) \\
&\bar{w}(T, \tilde{p}) = 0.
\end{aligned}$$

Here, $\bar{\kappa}(t, \tilde{p})$ is an $(N-1) \times 2$ matrix given by

$$\bar{\kappa}(t, \tilde{p}) := \kappa(t, \tilde{p}) = \operatorname{Diag}(\tilde{p}) \big(\bar{\vartheta}(t)^\top - \mathbf{1} \bar{\vartheta}(t, \tilde{p})^\top \big) (\Sigma_Y \Sigma_Y^\top)^{-1} \Sigma_Y$$

with $\bar{\vartheta}(t) = \vartheta(t)'$ being a $2 \times (N-1)$ matrix and $\bar{\vartheta}(t, \tilde{p}) := \tilde{\vartheta}(t, \tilde{p})$. Further,

$$\bar{\beta}_\gamma(t, \tilde{p}, \pi) := \beta_\gamma(t, \tilde{p}, \pi) = \beta_\varpi(t, \tilde{p}) + \gamma \bar{\kappa}(t, \tilde{p}) \Sigma_Y^\top \pi.$$

It is important to point out the explicit appearance of the solution $\underline{w}$ to the HJB postdefault equation in the PDE (5.11) satisfied by the predefault HJB equation $\bar{w}$. This establishes the required relationship between pre- and postdefault optimization subproblems.

Next, define

$$\Upsilon(t,\tilde{p}) = \big(r - \tilde{\mu}(\tilde{p}), r - \tilde{a}(t,\tilde{p})\big)^\top.$$

Then we can rewrite (5.11) as

$$\begin{aligned}
&\bar{w}_t + \frac{1}{2}\operatorname{tr}(\bar{\kappa}\bar{\kappa}^\top D^2\bar{w}) + \frac{1}{2}(\nabla_{\tilde{p}}\bar{w})\bar{\kappa}\bar{\kappa}^\top(\nabla_{\tilde{p}}\bar{w})^\top \\
&\quad + \sup_{\pi}\Big((\nabla_{\tilde{p}}\bar{w})\bar{\beta}_\gamma - \gamma\pi^\top\Upsilon - \frac{1}{2}\gamma(1-\gamma)\pi^\top\Sigma_Y^\top\Sigma_Y\pi\Big) \\
&\quad + \tilde{h}(\tilde{p})\Big(e^{\underline{w}(t,\frac{1}{\tilde{h}(\tilde{p})}\tilde{p}\cdot h') - \bar{w}(t,\tilde{p})} - 1\Big) + \gamma r = 0, \qquad (5.12)\\
&\bar{w}(T,\tilde{p}) = 0.
\end{aligned}$$

Using the first-order condition, we obtain that the maximal point π^* is the solution of the equation

$$\gamma\Sigma_Y\bar{\kappa}^\top(\nabla_{\tilde{p}}\bar{w})^\top - \gamma\Upsilon - \gamma(1-\gamma)\Sigma_Y^\top\Sigma_Y\pi^* = 0.$$

Solving the previous equation for π^* yields

$$\pi^* = \frac{1}{1-\gamma}(\Sigma_Y^\top\Sigma_Y)^{-1}\big(-\Upsilon + \Sigma_Y\bar{\kappa}^\top(\nabla_{\tilde{p}}\bar{w})^\top\big). \qquad (5.13)$$

After plugging π^* into (5.12) and performing algebraic simplifications (see Appendix B for details), we obtain

$$\begin{aligned}
&\bar{w}_t + \frac{1}{2}\operatorname{tr}(\bar{\kappa}\bar{\kappa}^\top D^2\bar{w}) + \frac{1}{2(1-\gamma)}(\nabla_{\tilde{p}}\bar{w})\bar{\kappa}\bar{\kappa}^\top(\nabla_{\tilde{p}}\bar{w})^\top \\
&\quad + (\nabla_{\tilde{p}}\bar{w})\bar{\Phi} + \tilde{h}(\tilde{p})e^{\underline{w}(t,\frac{\tilde{p}\cdot h'}{\tilde{h}(\tilde{p})})}e^{-\bar{w}(t,\tilde{p})} + \bar{\Psi} = 0, \qquad (5.14)\\
&\bar{w}(T,\tilde{p}) = 0, \qquad (5.15)
\end{aligned}$$

where

$$\begin{aligned}
\bar{\Phi}(t,\tilde{p}) &= \beta_{\varpi}(t,\tilde{p}) - \frac{\gamma}{1-\gamma}\bar{\kappa}\Sigma_Y^{-1}\Upsilon(t,\tilde{p}),\\
\bar{\Psi}(t,\tilde{p}) &= \frac{1}{2}\frac{\gamma}{1-\gamma}\Upsilon^\top(\Sigma_Y^\top\Sigma_Y)^{-1}\Upsilon(t,\tilde{p}) + \gamma r - \tilde{h}(\tilde{p}).
\end{aligned}$$

Next, we prove the existence of a classical solution to this Cauchy problem. We first perform a similar transformation as in the postdefault case and obtain that the function $\bar{w}(t,\tilde{p})$ solves the problem (5.14), (5.15) if and only if the function

$$\bar{\psi}(t,\tilde{p}) = e^{\frac{1}{1-\gamma}\bar{w}(t,\tilde{p})} \qquad (5.16)$$

solves the Cauchy problem

$$\bar{\psi}_t(t,\tilde{p}) + \frac{1}{2}\mathrm{tr}(\bar{\kappa}\bar{\kappa}^\top D^2\bar{\psi}(t,\tilde{p})) + \bar{\Phi}(t,\tilde{p})\nabla_{\tilde{p}}\bar{\psi}(t,\tilde{p})$$
$$+ \bar{\Psi}(t,\tilde{p})\frac{\bar{\psi}(t,\tilde{p})}{1-\gamma} + \tilde{h}(\tilde{p})e^{\underline{w}\left(t,\frac{\tilde{p}\cdot h'}{\tilde{h}(\tilde{p})}\right)}\frac{\bar{\psi}^\gamma(t,\tilde{p})}{1-\gamma} = 0, \quad (5.17)$$

$$\bar{\psi}(T,\tilde{p}) = 1. \quad (5.18)$$

Notice that the problem (5.17), (5.18) is *nonlinear*. Hence, proving the existence of a classical solution is not as direct as in the postdefault case, where the transformed HJB PDE given by (5.9), (5.10) turned out to be linear. We establish this result by applying a monotone iterative method used in [8] and in [30] (see Chap. 8 therein) for the study of obstacle problems for American options. There are, however, significant differences between the two problems, mainly arising from the appearance of the nonlinear term $\bar{\psi}^\gamma$ in our PDE (5.17). This term is not globally Lipschitz-continuous, whereas all PDE coefficients in [8] satisfy this condition. For this reason, it is crucial for us to prove that $\bar{\psi}$ is bounded away from zero, whereas for their obstacle problem, [8] only need to show that the solution is bounded, that is, $|\bar{\psi}(t,\tilde{p})| \le ce^{ct}$ for some positive constant c. We further make use of the parabolic Hölder space $C_P^{2,\alpha}$ defined by the norm (5.1). We then have the following:

Theorem 5.4 *Problem* (5.17), (5.18) *admits a classical solution* $\bar{\psi} \in C_P^{2,\alpha}$ *for any* $\alpha \in (0,1)$. *Moreover, there exists a constant* $C \ge 1$, *only depending on the* L^∞*-norms of the coefficients of the PDE, such that for each* $(t,\tilde{p})$,

$$\frac{1}{C} \le \bar{\psi}(t,\tilde{p}) \le e^{CT}. \quad (5.19)$$

The proof of Theorem 5.4 is reported in Appendix B. Let us remark that also

$$\bar{w}(t,\tilde{p}) \in C_P^{2,\alpha}$$

in view of relation (5.16) and estimate (5.19), yielding that $\bar{w}$ has the same properties as $\bar{\psi}$ in the previous theorem. The following result shows a verification theorem for the predefault optimization problem.

Theorem 5.5 *Suppose that the conditions of Theorem* 5.3 *are satisfied and, in particular, let* $\underline{w} \in C_P^{2,\alpha}$ *be the solution of* (5.5) *with terminal condition* (5.6). *Additionally, let* $\bar{w} \in C_P^{2,\alpha}$ *be the solution to the Cauchy problem* (5.14), (5.15) *established in Theorem* 5.4. *Then the following assertions hold true*:

(1) *The solution* $\bar{w}(t,\tilde{p})$ *coincides with the value function* $w(t,\tilde{p},0)$ *introduced in* (4.6).
(2) *The optimal feedback controls, denoted by* $\widetilde{\pi} := (\widetilde{\pi}^S, \widetilde{\pi}^P)^\top$, *can be written as* $\widetilde{\pi}_s^S = \widetilde{\pi}^S(s,\tilde{p}_{s-}^t, H_{s-}^t)$ *and* $\widetilde{\pi}_s^P = \widetilde{\pi}^P(s,\tilde{p}_{s-}^t, H_{s-}^t)$ *with*

$$
\begin{aligned}
&\big(\widetilde{\pi}^S(s,\tilde{p},0),\widetilde{\pi}^P(s,\tilde{p},0)\big)^\top \\
&\quad := \frac{(\Sigma_Y^\top\Sigma_Y)^{-1}}{1-\gamma}\big(-\Upsilon(s,\tilde{p})+\Sigma_Y\bar{\kappa}^\top\nabla_{\tilde{p}}\bar{w}(s,\tilde{p})^\top\big),
\end{aligned}
\tag{5.20}
$$

$$
\begin{aligned}
&\big(\widetilde{\pi}^S(s,\tilde{p},1),\widetilde{\pi}^P(s,\tilde{p},1)\big)^\top \\
&\quad := \left(\frac{(\tilde{\mu}(\tilde{p})-r+\sigma\underline{\kappa}^\top\nabla_{\tilde{p}}\underline{w}(s,\tilde{p})^\top)}{\sigma^2(1-\gamma)},0\right)^\top.
\end{aligned}
\tag{5.21}
$$

The proof of Theorem 5.5 is reported in Appendix B.

6 Numerical analysis

We provide a numerical analysis of the optimal strategies and value functions derived in the previous sections. We set $N=2$, that is, we consider two regimes; thus, the vector $\tilde{p}:=p$ becomes one-dimensional with p denoting the filter probability that the Markov chain is in regime "1". Unless otherwise specified, throughout the section, we use the following benchmark parameters: $\sigma=0.4$, $\upsilon=0.5$, $r=0.03$, $\mu_1=0.5$, $\mu_2=0.2$, $h_1=1$, $h_2=0.2$, $\varpi_{11}=-0.5$, and $\varpi_{22}=-1$. We fix the time horizon $T=10$. We set $\gamma=0.5$, that is, we consider a square-root investor. We describe the numerical setup in Sect. 6.1 and give an economic analysis of the strategies in Sect. 6.2.

6.1 Setup

Since the solution to the predefault depends on the solution to the postdefault HJB PDE, we first need to solve (5.9), (5.10). In the case of two regimes, that PDE becomes two-dimensional with $t\in\mathbb{R}_+$ and $0\le p\le 1$. More specifically, it reduces to

$$
\begin{aligned}
&\frac{\partial\underline{\psi}(t,p)}{\partial t}+\frac{1}{2}\underline{\kappa}(t,p)^2\frac{\partial^2\underline{\psi}(t,p)}{\partial p^2}+\underline{\Phi}(t,p)\frac{\partial\underline{\psi}(t,p)}{\partial p}+\underline{\Psi}(t,p)\frac{\underline{\psi}(t,p)}{1-\gamma}=0,\\
&\underline{\psi}(T,p)=1,
\end{aligned}
$$

where

$$
\begin{aligned}
\underline{\kappa}(t,p)&=\sigma^{-1}p\Big(\mu_1-\big(\mu_1p+\mu_2(1-p)\big)\Big),\\
\beta_\varpi(t,p)&=\varpi_{11}p+\varpi_{21}(1-p)=\varpi_{11}p-\varpi_{22}(1-p),\\
\underline{\Phi}(t,p)&=\beta_\varpi(t,p)+\frac{\gamma}{1-\gamma}\frac{\mu_1p+(1-p)\mu_2-r}{\sigma}\underline{\kappa}(t,p),\\
\underline{\Psi}(t,p)&=\gamma r+\frac{\gamma}{2(1-\gamma)}\Big(\frac{\mu_1p+(1-p)\mu_2-r}{\sigma}\Big)^2.
\end{aligned}
$$

We numerically solve the above PDE using a standard Crank–Nicolson method. From the transformation (5.8) we obtain that the postdefault value function is given by

$\underline{w}(t,p) = (1-\gamma)\log \underline{\psi}(t,p)$. The latter is then used in the predefault PDE, computed as described next. On (t,p) with $t \in \mathbb{R}_+$ and $0 \le p \le 1$, the PDE (5.17) satisfied by $\bar{\psi}(t,p)$ reduces to

$$\frac{\partial \bar{\psi}(t,p)}{\partial t} + \frac{1}{2}\bar{\kappa}(t,p)\bar{\kappa}(t,p)^\top \frac{\partial^2 \bar{\psi}(t,p)}{\partial p^2} + \bar{\Phi}(t,p)\frac{\partial \bar{\psi}}{\partial p} + \bar{\Psi}(t,p)\frac{\bar{\psi}(t,p)}{1-\gamma}$$
$$+ (h_2 + (h_1 - h_2)p)e^{\underline{w}(t, \frac{ph_1}{h_2+(h_1-h_2)p})}\frac{\bar{\psi}(t,p)^\gamma}{1-\gamma} = 0,$$
$$\bar{\psi}(T,p) = 1,$$

where

$$\bar{\kappa}(t,p) = p(1-p)\left(\frac{\mu_1-\mu_2}{\sigma}, \frac{a(t,e_1)-a(t,e_2)}{\upsilon}\right) =: \big(\bar{\kappa}_{11}(t,p),\ \bar{\kappa}_{12}(t,p)\big),$$
$$\bar{\Phi}(t,p) = \varpi_{21} + (\varpi_{11} - \varpi_{21})p + \frac{\gamma}{1-\gamma}\left(\frac{\bar{\kappa}_{11}(t,p)}{\sigma}\big(\tilde{\mu}(p) - r\big) + \frac{\bar{\kappa}_{12}(t,p)}{\upsilon}\big(\tilde{a}(t,p) - r\big)\right)$$
$$\bar{\Psi}(t,p) = \frac{1}{2}\frac{\gamma}{1-\gamma}\left(\frac{(\tilde{\mu}(p)-r)^2}{\sigma^2} + \frac{(\tilde{a}(t,p)-r)^2}{\upsilon^2}\right) + \gamma r - h_2 + (h_1 - h_2)\,p.$$

In the following analysis, we set $a(t,e_1) = -(r+h_1)$ and $a(t,e_2) = -(r+h_2)$. In agreement with the notation in Sect. 2, $a(t,e_1)$ and $a(t,e_2)$ denote the risk-adjusted returns of the defaultable security when the current regime is "1", respectively "2". Similarly to the postdefault case, we employ a standard Crank–Nicolson method to solve the previous nonlinear PDE. The solution to the predefault PDE is then obtained as $\bar{w}(t,p) = (1-\gamma)\log \bar{\psi}(t,p)$.

6.2 Analysis of strategies

Figure 1 shows that the stock investment strategy is increasing in the filter probability of the hidden chain being in the first regime. This happens because under our parameter choices, the growth rate of the stock is higher in regime "1", whereas the volatility stays unchanged. Consequently, if the filter estimate of being in the more profitable regime gets higher, the risk-averse investor would prefer to shift a larger amount of wealth into the stock. On the other hand, as the probability of being in regime "1" increases, the risk-averse investor shorts a higher amount of the defaultable security. This happens because the default intensity in regime "1" is higher, and thus the risk-averse investor wishing to decrease his exposure to default risk goes short in the vulnerable security. Notice also the key role played by the stock volatility σ. As the volatility gets lower, the investor shorts more units of the vulnerable security and invests the resulting proceeds in the stock security. Indeed, for a fixed level of default risk, when the stock volatility is low, the risk-averse investor prefers to invest a larger fraction of wealth in the stock security. He does so by raising cash via short-selling of the vulnerable security. The latter action also results in him having a reduced exposure to credit risk. Since all model parameters depend on time only through the

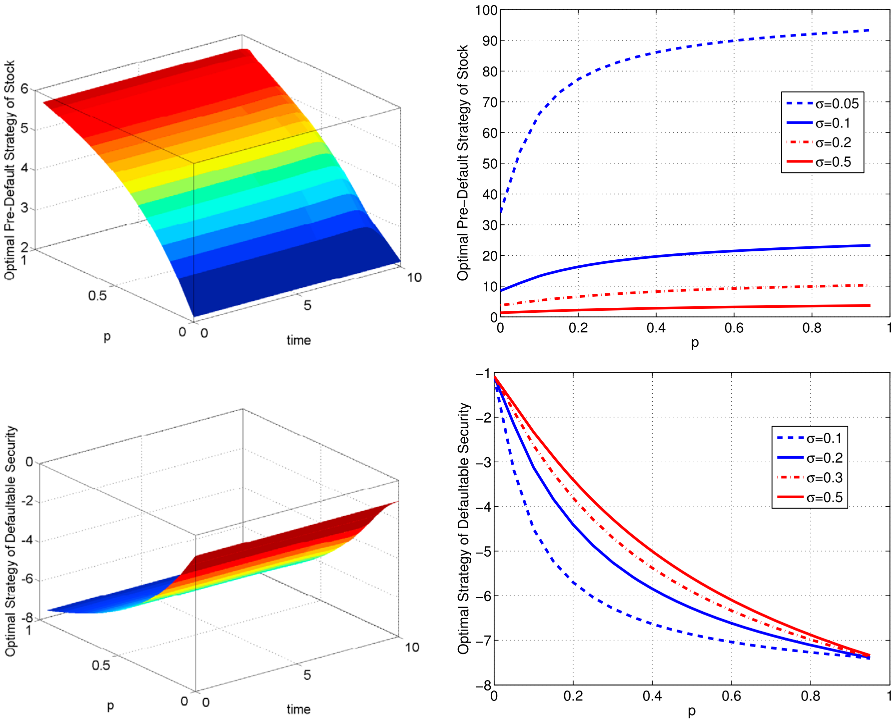

Fig. 1 The *top panels* report the stock investment strategy. The *bottom panels* report the investment strategy in the vulnerable security. In the *right panels*, we set $t = 0$

underlying hidden regime, investment strategies are not very sensitive to the passage of time.

From Fig. 2 we can also see that both the predefault and postdefault value functions are decreasing in time and increasing in the filter probability p. Moreover, as the filter probability of being in regime "1" increases, the investor extracts more utility given that he realizes higher gains by simultaneously shorting the vulnerable security and purchasing the stock security.

The investor increases the amount of shorted units of the vulnerable security if the filter probability of staying in the regime characterized by high default intensity gets larger. By contrast, if the filter probability of staying in the regime with low default intensity increases, then he takes higher credit risk exposure by purchasing more bond units. This is clearly illustrated in the left panel of Fig. 3. Moreover, when $h_1 < h_2$, smaller amounts of the vulnerable security are shorted if the probability p of staying in the low-risk regime gets higher. However, when the default intensity h_1 in regime "1" exceeds the default intensity h_2 in regime "2" ($h_2 = 0.2$), the opposite effect is observed. As h_1 gets higher, a larger amount of the vulnerable security is shorted if the filter probability of staying in regime "1" increases. We also notice that, ceteris paribus, the predefault value function is increasing in h_1 for a fixed p, and also, for a given value of h_1, the predefault value function increases in p.

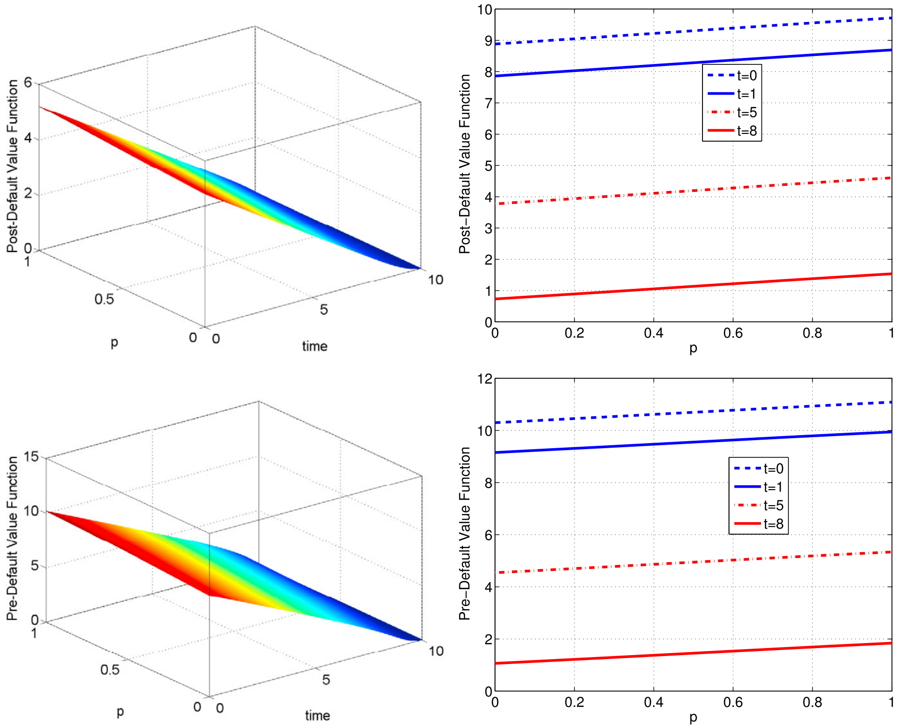

Fig. 2 The predefault and postdefault value functions plotted versus time t and filter probability p

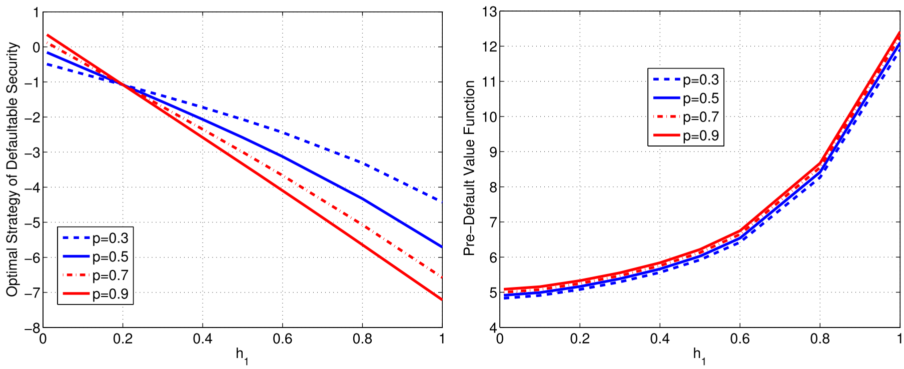

Fig. 3 The *left panel* reports the optimal investment strategy in the defaultable security. The *right panel* reports the predefault value function. We set $t = 0$

We next analyze the dependence of the optimal stock and vulnerable security investment strategies on the volatilities σ and υ. Figure 4 shows that when the stock volatility is low, the investor puts a large fraction of wealth in the stock security. This happens because the investor is risk-averse, and hence he deposits a larger fraction of wealth in the stock when the volatility risk is lower. This fraction will be higher if

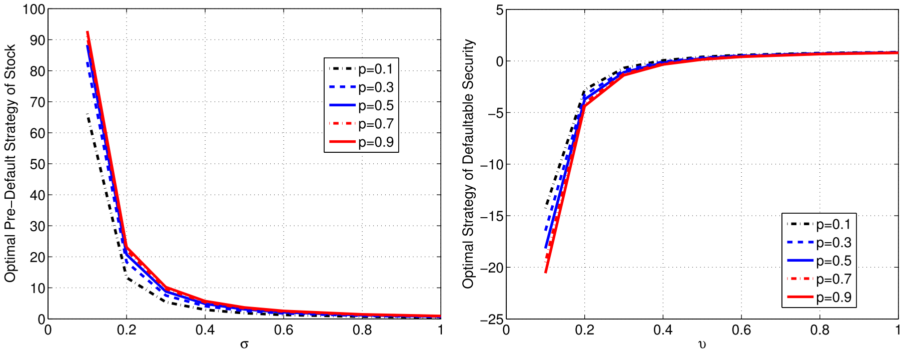

Fig. 4 The *left panel* reports the dependence of the optimal stock investment strategy on σ. The *right panel* reports the dependence on υ of the optimal investment strategy in the vulnerable security. We set $t=0$. In the *right panel*, we set $h_1=0.05$ and $h_2=0.01$

the filter probability of being in the high-growth regime (regime "1" with $\mu_1=0.5$) is high. As the volatility gets larger, stock price observations become less informative, leading the investor to decrease the wealth fraction invested in the stock. When the volatility exceeds a certain threshold, regardless of the filter probability, the investor always puts a small amount of wealth in the stock. We also notice that a similar role is played by the volatility υ of the defaultable security. From the right panel of Fig. 4 we notice that when υ is low, that is, price observations of the vulnerable security are very informative, the investor wants to reduce more his exposure to default risk. Hence, he shorts more units of the vulnerable security, especially if the filter probability of being in the higher credit risk regime (regime "1" with $h_1=0.05$) is large. This reflects the risk-averse nature of the investor who dislikes default risk and uncertainty. As for the stock, when υ gets larger, the investment strategy in the defaultable security becomes less sensitive to the filter probability, and for large values of υ, the investor may even find it optimal to purchase the defaultable security. This happens when the potential loss incurred by the investor when he is long credit and default occurs (hence making the vulnerable security worthless) is outweighed by the risk-adjusted return resulting from holding the defaultable security.

We conclude by relating partial to full information settings. As price volatilities become smaller, the regime switching model becomes more observable. This is because price observations become more informative and allow the investor to build more accurate estimates of the regime in place. Consequently, the above analysis outlines the important role played by regime uncertainty in determining the optimal strategies of risk-averse investors. Compared to the case of fully observed regimes studied in [6], the presence of incomplete information induces the risk-averse investor to decrease the wealth amount invested in the risky securities. As clearly illustrated in Fig. 4, when the price volatilities are sufficiently high ($\sigma\approx 0.8$ for the stock and $\upsilon\approx 0.6$ for the defaultable security), the investor deposits almost his entire amount of wealth in the money market account.

7 Conclusions

We have studied the optimal investment problem of a power investor in an economy consisting of a defaultable security, a stock, and a money market account. The price processes of these securities are assumed to have drift coefficients and default intensities modulated by a hidden Markov chain. We have reduced the partially observed stochastic control problem to a risk-sensitive one, where the state is given by the filtered regime probabilities. The conditioning filtration, generated by the prices of the stock and of the defaultable security and by the indicator of default occurrence, is driven both by a Brownian component and by a pure jump martingale. The filter has been used to derive the HJB partial differential equation corresponding to the risk sensitive control problem. We have split the latter into a predefault and a postdefault dynamic programming subproblem. The HJB PDE corresponding to the postdefault value function can be transformed to a linear parabolic PDE, for which the existence and uniqueness of a classical solution can be guaranteed. By contrast, the HJB PDE corresponding to the predefault value function has exponential nonlinearity and quadratic gradient growth. We have provided a detailed mathematical analysis of this PDE and have established the existence of a classical solution with $\mathcal{C}_P^{2,\alpha}$ regularity. We have then proved verification theorems establishing the correspondence between the PDE solutions and the value functions of the control problem. Our study has been complemented with a thorough numerical analysis illustrating the role of regime uncertainty, default risk, and price volatilities on the optimal allocation decisions and value functions.

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Appendix A: Proofs related to Sect. 3

Lemma A.1 *Let*

$$q_t^i = \mathbb{E}^{\hat{\mathbb{P}}}\big[L_t \mathbf{1}_{\{X_t=e_i\}} \big| \mathcal{G}_t^I\big]. \tag{A.1}$$

Then the dynamics of $(q_t^i)_{t\geq 0}$, $i = 1, \ldots, N$, *under the measure* $\hat{\mathbb{P}}$ *is given by the system of stochastic differential equations* (*SDEs*)

$$\begin{aligned} dq_t^i &= \sum_{\ell=1}^N \varpi_{\ell,i}(t) q_t^\ell \, dt + q_{t-}^i Q^\top(t, e_i, \pi_t) \Sigma_Y \, d\hat{W}_t \\ &\quad + q_{t-}^i (h_i - 1)\, d\hat{\xi}_t - \gamma \eta(t, e_i, \pi_t) q_t^i \, dt, \\ q_0^i &= p_0^i. \end{aligned} \tag{A.2}$$

Proof Let us introduce the notation $H_t^i := \mathbf{1}_{\{X_t=e_i\}}$. Note that $X_t = (H_t^1, \dots, H_t^N)^\top$ and, by (2.1),

$$H_t^i = H_0^i + \int_0^t \sum_{\ell=1}^N \varpi_{\ell,i}(s) H_s^\ell \, ds + \varphi_i(t).$$

From (3.14) and (3.10) we deduce that, under $\hat{\mathbb{P}}$,

$$dL_t = L_{t-}(h_t - 1)\, d\hat{\xi}_t + L_{t-} Q^\top(t, X_t, \pi_t)\, dY_t - L_t \gamma \eta(t, X_t, \pi_t)\, dt,$$

which yields that

$$[L, H^i]_t = \int_0^t L_{s-} Q^\top(s, X_s, \pi_s)\, d[Y, H^i]_s + \int_0^t L_{s-}(h_s - 1)\, d[\hat{\xi}, H^i]_s.$$

Since $(Y_t)_{t\geq 0}$ and $(H_t)_{t\geq 0}$ are independent of $(X_t)_{t\geq 0}$ (and hence of H^i), under $\hat{\mathbb{P}}$, we have (see also [38, Lemma 7.3.1]) that, $\hat{\mathbb{P}}$-almost surely,

$$[Y, H^i]_s = [\hat{\xi}, H^i]_s = 0 \quad \text{for all } s \geq 0.$$

Thus, applying Itô's formula, we obtain

$$\begin{aligned} L_t H_t^i &= H_0^i + \int_0^t H_{s-}^i \, dL_s + \int_0^t L_{s-} \, dH_s^i \\ &= H_0^i + \int_0^t H_{s-}^i L_{s-} Q^\top(s, X_s, \pi_s)\, dY_s + \int_0^t H_{s-}^i L_{s-}(h_{s-} - 1)\, d\hat{\xi}_s \qquad \text{(A.3)} \\ &\quad - \int_0^t H_s^i L_s \gamma \eta(s, X_s, \pi_s)\, ds + \int_0^t L_s \sum_{\ell=1}^N \varpi_{\ell,i}(s) H_s^\ell \, ds + \int_0^t L_{s-} \, d\varphi_i(s). \end{aligned}$$

Since $(\varphi_i(t))_{t\geq 0}$ is an $((\mathcal{F}_t^X)_{t\geq 0}, \hat{\mathbb{P}})$-martingale and $\mathcal{G}_T^I$ is independent of $\mathcal{F}_T^X$ under $\hat{\mathbb{P}}$, we have that $\mathbb{E}^{\hat{\mathbb{P}}}[\int_0^t L_{s-} d\varphi_i(s) | \mathcal{G}_t^I] = 0$. Therefore, taking the $\mathcal{G}_t^I$-conditional expectations in (A.3), we obtain

$$\begin{aligned} \mathbb{E}^{\hat{\mathbb{P}}}\big[L_t H_t^i \big| \mathcal{G}_t^I\big] = 1 &+ \int_0^t \mathbb{E}^{\hat{\mathbb{P}}}\big[L_{s-} H_{s-}^i Q^\top(s, e_i, \pi_s) \big| \mathcal{G}_s^I\big]\, dY_s \\ &+ \int_0^t \mathbb{E}^{\hat{\mathbb{P}}}\big[L_{s-} H_{s-}^i (h_{s-} - 1) \big| \mathcal{G}_s^I\big]\, d\hat{\xi}_s \\ &- \int_0^t \mathbb{E}^{\hat{\mathbb{P}}}\big[H_s^i L_s \gamma \eta(s, e_i, \pi_s) \big| \mathcal{G}_s^I\big]\, ds \\ &+ \int_0^t \mathbb{E}^{\hat{\mathbb{P}}}\Bigg[\sum_{\ell=1}^N \varpi_{\ell,i}(s) L_s H_s^\ell \Bigg| \mathcal{G}_s^I\Bigg]\, ds, \qquad \text{(A.4)} \end{aligned}$$

where we have used that if ϕ is $\mathbb{G}$-predictable, then (see, e.g., [38, Lemma 7.3.2])

$$\mathbb{E}^{\hat{\mathbb{P}}}\left[\int_0^t \phi_s L_{s-}\, dY_s \middle| \mathcal{G}_t^I\right] = \int_0^t \mathbb{E}^{\hat{\mathbb{P}}}[\phi_s L_{s-}|\mathcal{G}_s^I]\, dY_s,$$
$$\mathbb{E}^{\hat{\mathbb{P}}}\left[\int_0^t \phi_s L_{s-}\, d\hat{\xi}_s \middle| \mathcal{G}_t^I\right] = \int_0^t \mathbb{E}^{\hat{\mathbb{P}}}[\phi_s L_{s-}|\mathcal{G}_s^I]\, d\hat{\xi}_s,$$
$$\mathbb{E}^{\hat{\mathbb{P}}}\left[\int_0^t \phi_s L_{s-}\, ds \middle| \mathcal{G}_t^I\right] = \int_0^t \mathbb{E}^{\hat{\mathbb{P}}}[\phi_s L_{s-}|\mathcal{G}_s^I]\, ds.$$

Observing that $dY_t = \Sigma_Y d\hat{W}_t$ under $\hat{\mathbb{P}}$, using that $Q(t, e_i, \pi_t)$ and $\eta(t, e_i, \pi_t)$ are $(\mathcal{G}_t^I)_{t\geq 0}$-adapted and that the Markov chain generator $A(t)$ is deterministic, we obtain (A.2) upon taking the differential of (A.4). □

Lemma A.2 *We have the following identities*:

$$q_t^i = \hat{L}_t p_t^i, \tag{A.5}$$

$$p_t^i = \frac{q_t^i}{\sum_{j=1}^N q_t^j}, \tag{A.6}$$

where q_t^i, $\hat{L}_t$, *and* p_t^i *are defined*, *respectively*, *by* (A.1), (3.20), *and* (3.18).

Proof We first establish relation (A.5) by comparing the dynamics of (q_t^i) and of $(\hat{L}_t p_t^i)$. The former is known from Lemma A.1 and given in (A.2). Next, we derive the latter. We have

$$d(\hat{L}_t p_t^i) = \hat{L}_{t-}\, dp_t^i + p_{t-}^i\, d\hat{L}_t + d[\hat{L}, p^i]_t.$$

From (3.21) and (3.19) we obtain

$$\begin{aligned} d[\hat{L}, p^i]_t &= p_t^i \hat{L}_t \hat{\vartheta}^\top(t, p_t) \Sigma_Y \Sigma_Y^{-1} \big(\vartheta(t, e_i) - \hat{\vartheta}(t, p_t)\big)\, dt \\ &\quad + p_t^i \hat{L}_t \gamma \pi_t^\top \big(\vartheta(t, e_i) - \hat{\vartheta}(t, p_t)\big)\, dt \\ &\quad + (\hat{h}_{t-} - 1) \frac{h_i - \hat{h}_{t-}}{\hat{h}_{t-}} \hat{L}_{t-} p_{t-}^i\, dH_t. \end{aligned}$$

Using these equations, along with (3.19), we obtain

$$
\begin{aligned}
d(\hat{L}_t p_t^i) = {} & \hat{L}_t\left(\sum_{\ell=1}^{N} \varpi_{\ell,i}(t) p_t^\ell\, dt\right) \\
& + \hat{L}_{t-} p_{t-}^i \big(\vartheta^\top(t,e_i) - \hat{\vartheta}^\top(t,p_{t-})\big)(\Sigma_Y \Sigma_Y^\top)^{-1}\big(dY_t - \hat{\vartheta}(t,p_t)\,dt\big) \\
& + \hat{L}_{t-} p_{t-}^i \frac{h_i - \hat{h}_{t-}}{\hat{h}_{t-}}\big(dH_t - \hat{h}_{t-}\bar{H}_{t-}\,dt\big) \\
& + p_{t-}^i \hat{L}_t \hat{Q}^\top(t,p_{t-},\pi_t)\,dY_t - p_t^i \hat{L}_t \gamma \hat{\eta}(t,p_t,\pi_t)\,dt \\
& + p_{t-}^i \hat{L}_{t-}(\hat{h}_{t-} - 1)(dH_t - \bar{H}_{t-}\,dt) + (\hat{h}_{t-} - 1)\frac{h_i - \hat{h}_{t-}}{\hat{h}_{t-}} \hat{L}_{t-} p_{t-}^i\, dH_t \\
& + p_t^i \hat{L}_t \hat{\vartheta}^\top(t,p_t)(\Sigma_Y \Sigma_Y^\top)^{-1}\big(\vartheta(t,e_i) - \hat{\vartheta}(t,p_t)\big)\,dt \\
& + \gamma p_t^i \hat{L}_t \pi_t^\top\big(\vartheta(t,e_i) - \hat{\vartheta}(t,p_t)\big)\,dt.
\end{aligned} \tag{A.7}
$$

Next, observe that

$$
\begin{aligned}
& \hat{L}_{t-} p_{t-}^i \big(\vartheta^\top(t,e_i) - \hat{\vartheta}^\top(t,p_{t-})\big)(\Sigma_Y \Sigma_Y^\top)^{-1}\big(dY_t - \hat{\vartheta}(t,p_t)\,dt\big) \\
& \quad + p_{t-}^i \hat{L}_{t-} \hat{Q}^\top(t,p_{t-},\pi_t)\,dY_t \\
& = \hat{L}_{t-} p_{t-}^i Q^\top(t,e_i,\pi_t)\,dY_t \\
& \quad - \hat{L}_t p_t^i \big(\vartheta^\top(t,e_i) - \hat{\vartheta}^\top(t,p_t)\big)(\Sigma_Y \Sigma_Y^\top)^{-1}\hat{\vartheta}(t,p_t)\,dt.
\end{aligned} \tag{A.8}
$$

Moreover,

$$
\eta(t,e_i,\pi_t) - \hat{\eta}(t,p_t,\pi_t) = \pi_t^\top\big(\hat{\vartheta}(t,p_t) - \vartheta(t,e_i)\big). \tag{A.9}
$$

Using (A.8) and (A.9) and straightforward simplifications, we may simplify (A.7) to

$$
\begin{aligned}
d(\hat{L}_t p_t^i) = {} & \left(\sum_{\ell=1}^{N} \varpi_{\ell,i}(t) \hat{L}_t p_t^\ell\, dt\right) + \hat{L}_{t-} p_{t-}^i Q^\top(t,e_i,\pi_t)\,dY_t \\
& - \gamma \hat{L}_t p_t^i \eta(t,e_i,\pi_t)\,dt + \hat{L}_{t-} p_{t-}^i (h_i - 1)\,d\hat{\xi}_t.
\end{aligned} \tag{A.10}
$$

Using that $dY_t = \Sigma_Y d\hat{W}_t$, we have that (A.5) holds via a direct comparison of (A.10) and (A.2). Next, we establish (A.6). Using (A.5) and $\sum_{i=1}^N p_t^i = 1$, we deduce that

$$
d\left(\sum_{i=1}^{N} q_t^i\right) = d\left(\sum_{i=1}^{N} \hat{L}_t p_t^i\right) = d\hat{L}_t,
$$

hence obtaining that $\sum_{i=1}^N q_t^i = \hat{L}_t$. Using again (A.5), this gives

$$
p_t^i = \frac{q_t^i}{\hat{L}_t} = \frac{q_t^i}{\sum_{j=1}^N q_t^j}.
$$

This completes the proof. □

Proof of Proposition 3.3 Using (3.13), (A.1), and relation (A.5) established in Lemma A.2, we have that

$$J(v,\pi,T)=\frac{1}{\gamma}\mathbb{E}^{\mathbb{P}}[V_T^\gamma]=\frac{v^\gamma}{\gamma}\mathbb{E}^{\hat{\mathbb{P}}}[L_T]=\frac{v^\gamma}{\gamma}\mathbb{E}^{\hat{\mathbb{P}}}\big[\mathbb{E}^{\hat{\mathbb{P}}}[L_T\mid\mathcal{G}_T^I]\big]$$

$$=\frac{v^\gamma}{\gamma}\sum_{i=1}^N\mathbb{E}^{\hat{\mathbb{P}}}\Big[\mathbb{E}^{\hat{\mathbb{P}}}\big[L_T\mathbf{1}_{\{X_T=e_i\}}\big|\mathcal{G}_T^I\big]\Big]=\frac{v^\gamma}{\gamma}\sum_{i=1}^N\mathbb{E}^{\hat{\mathbb{P}}}[q_T^i]$$

$$=\frac{v^\gamma}{\gamma}\sum_{i=1}^N\mathbb{E}^{\hat{\mathbb{P}}}[\hat{L}_T p_T^i]=\frac{v^\gamma}{\gamma}\mathbb{E}^{\hat{\mathbb{P}}}[\hat{L}_T],$$

thus proving the statement. □

Appendix B: Proofs related to Sect. 5

We start with a lemma that will be needed in the section where the verification theorem is proved.

Lemma B.1 *For any $T>0$ and $i\in\{1,\dots,N\}$, we have*:

(1) $\mathbb{P}[p_t^i>0 \textit{ for all } t\in[0,T)]=1$.
(2) $\mathbb{P}[p_t^i<1 \textit{ for all } t\in[0,T)]=1$.

Proof Define $\varsigma=\inf\{t: p_t^i=0\}\wedge T$. If p^i can hit zero, then $\mathbb{P}[p_\varsigma^i=0]>0$. Recall that $p_t^i=\frac{q_t^i}{\sum_j q_t^j}$ from (A.6); hence, $p_\varsigma^i=\frac{q_\varsigma^i}{\sum_j q_\varsigma^j}$, where

$$q_\varsigma^i=\mathbb{E}^{\hat{\mathbb{P}}}\big[L_\varsigma\mathbf{1}_{\{X_\varsigma=e_i\}}\big|\mathcal{G}_\varsigma^I\big]$$

by the optional projection property; see [33, Thm. VI.7.10]. Define the two-dimensional (observed) log-price process $Y_t=(\log S_t,\log P_t)^\top$. Because of $q_\varsigma^i=\mathbb{E}^{\hat{\mathbb{P}}}[L_\varsigma\mathbf{1}_{\{X_\varsigma=e_i\}}\mid\mathcal{G}_\varsigma^I]$ and using that $L_\varsigma>0$, we can choose a modification $g(Y,H,X_\varsigma)$ of $\mathbb{E}^{\hat{\mathbb{P}}}[L_\varsigma\mid\mathcal{G}_\varsigma^I,X_\varsigma]$ such that $g>0$ and for each e_i, $g(Y,H,e_i)$ is G_ς^I-measurable. By the tower property,

$$\begin{aligned}q_\varsigma^i&=\mathbb{E}^{\hat{\mathbb{P}}}\big[g(Y,H,X_\varsigma)\mathbf{1}_{\{X_\varsigma=e_i\}}\big|\mathcal{G}_\varsigma^I\big]\\&=g(Y,H,e_i)\hat{\mathbb{P}}[X_\varsigma=e_i\mid\mathcal{G}_\varsigma^I]\\&=g(Y,H,e_i)\mathbb{P}[X_t=e_i]\big|_{t=\varsigma},\end{aligned}$$

where the first equality follows because ς is $\mathcal{G}_\varsigma^I$-measurable, and the last two because X is independent of $\mathcal{G}^I$ under $\hat{\mathbb{P}}$. Since $\mathbb{P}[X_t=e_i]>0$ and $g>0$, we get that

$q^i_\varsigma > 0$ a.s., which contradicts that $\mathbb{P}[p^i_\varsigma = 0] > 0$. This proves the first statement in the lemma. Next, we notice that

$$\mathbb{P}[p^i_t = 0 \text{ for some } t \in [0,T)] = 1 - \mathbb{P}[p^i_t > 0 \text{ for all } t \in [0,T)] = 0,$$

where the last equality follows from the first statement. This immediately yields the second statement. □

Proof of (5.14) Let us first analyze the first term $\beta_\gamma^\top \nabla_{\tilde{p}} w$ in the sup of (5.12). For brevity, we use $\beta_\varpi := \beta_\varpi(t, \tilde{p}, 0)$. By definition of β_γ and using the maximizer $\pi := \pi^*$ in (5.13), we have

$$\begin{aligned}\beta_\gamma^\top &= \beta_\varpi^\top + \gamma \pi^\top \Sigma_Y \bar{\kappa}^\top \\ &= \beta_\varpi^\top + \frac{\gamma}{1-\gamma}\big(\Sigma_Y \bar{\kappa}^\top (\nabla_{\tilde{p}} \bar{w})^\top - \Upsilon\big)^\top (\Sigma_Y^\top \Sigma_Y)^{-1} \Sigma_Y \bar{\kappa}^\top \\ &= \beta_\varpi^\top + \frac{\gamma}{1-\gamma}(\nabla_{\tilde{p}} \bar{w}) \bar{\kappa} \Sigma_Y^\top (\Sigma_Y^\top \Sigma_Y)^{-1} \Sigma_Y \bar{\kappa}^\top \\ &\quad - \frac{\gamma}{1-\gamma} \Upsilon^\top (\Sigma_Y^\top \Sigma_Y)^{-1} \Sigma_Y \bar{\kappa}^\top. \end{aligned} \tag{B.1}$$

Further, again using the expression for $\pi = \pi^*$, the second term in the sup is equal to

$$\begin{aligned} -\gamma \pi^\top \Upsilon &= -\frac{\gamma}{1-\gamma}\big(-\Upsilon + \Sigma_Y \bar{\kappa}^\top (\nabla_{\tilde{p}} \bar{w})^\top\big)^\top (\Sigma_Y^\top \Sigma_Y)^{-1} \Upsilon \\ &= \frac{\gamma}{1-\gamma} \Upsilon^\top (\Sigma_Y^\top \Sigma_Y)^{-1} \Upsilon - \frac{\gamma}{1-\gamma} (\nabla_{\tilde{p}} \bar{w}) \bar{\kappa} \Sigma_Y^\top (\Sigma_Y^\top \Sigma_Y)^{-1} \Upsilon. \end{aligned} \tag{B.2}$$

The third term in the sup may be simplified to

$$\begin{aligned} &-\frac{1}{2}\frac{\gamma}{1-\gamma}\big(-\Upsilon + \Sigma_Y \bar{\kappa}^\top (\nabla_{\tilde{p}} \bar{w})^\top\big)^\top (\Sigma_Y^\top \Sigma_Y)^{-1} \big(-\Upsilon + \Sigma_Y \bar{\kappa}^\top (\nabla_{\tilde{p}} \bar{w})^\top\big) \\ &= -\frac{1}{2}\frac{\gamma}{1-\gamma} \Upsilon^\top (\Sigma_Y^\top \Sigma_Y)^{-1} \Upsilon + \frac{1}{2}\frac{\gamma}{1-\gamma} \Upsilon^\top (\Sigma_Y^\top \Sigma_Y)^{-1} \Sigma_Y \bar{\kappa}^\top (\nabla_{\tilde{p}} \bar{w})^\top \\ &\quad + \frac{1}{2}\frac{\gamma}{1-\gamma} (\nabla_{\tilde{p}} \bar{w}) \bar{\kappa} \Sigma_Y^\top (\Sigma_Y^\top \Sigma_Y)^{-1} \Upsilon \\ &\quad - \frac{1}{2}\frac{\gamma}{1-\gamma} (\nabla_{\tilde{p}} \bar{w}) \bar{\kappa} \Sigma_Y^\top (\Sigma_Y^\top \Sigma_Y)^{-1} \Sigma_Y \bar{\kappa}^\top (\nabla_{\tilde{p}} \bar{w})^\top. \end{aligned} \tag{B.3}$$

Using (B.1)–(B.3), we obtain that

$$\begin{aligned} &\sup_\pi \Big(\beta_\gamma^\top (\nabla_{\tilde{p}} \bar{w})^\top - \gamma \pi_t^\top \Upsilon - \frac{1}{2}\gamma(1-\gamma) \pi_t^\top \Sigma_Y^\top \Sigma_Y \pi_t\Big) \\ &= \beta_\varpi^\top (\nabla_{\tilde{p}} \bar{w})^\top + \frac{1}{2}\frac{\gamma}{1-\gamma} (\nabla_{\tilde{p}} \bar{w}) \bar{\kappa} \Sigma_Y^\top (\Sigma_Y^\top \Sigma_Y)^{-1} \Sigma_Y \bar{\kappa}^\top (\nabla_{\tilde{p}} \bar{w})^\top \\ &\quad + \frac{1}{2}\frac{\gamma}{1-\gamma} \Upsilon^\top (\Sigma_Y^\top \Sigma_Y)^{-1} \Upsilon - \frac{\gamma}{1-\gamma} (\nabla_{\tilde{p}} \bar{w}) \bar{\kappa} \Sigma_Y^{-1} \Upsilon, \end{aligned}$$

and therefore, after rearrangement, we obtain (5.14). □

Proof of Theorem 5.3 In order to ease the notational burden, throughout the proof, we write $\tilde{p}$ for $\tilde{p}^{\circ}$, $\tilde{p}_s$ for $\tilde{p}_s^t$, π for π^t, $\tilde{\mathbb{P}}$ for $\tilde{\mathbb{P}}^t$, $\mathbb{P}$ for $\mathbb{P}^t$, $\tilde{W}$ for $\tilde{W}^t$, X for X^t, and $\mathcal{G}_s^I$ for $\mathcal{G}_s^{t,I}$. Let us first remark that

$$\mathbb{P}[\tilde{p}_s \in \tilde{\Delta}_{N-1},\ t \le s \le T] = 1. \tag{B.4}$$

Indeed, set $\tilde{p}_s^N = 1 - \sum_{j=1}^{N-1} \tilde{p}_s^j$ and recall from Remark 3.4 that the process $\tilde{p}^i$ is given by

$$\tilde{p}_s^i := \mathbb{P}\big[X_s = e_i \big| \mathcal{G}_s^I\big], \quad t \le s \le T,\ i = 1, \dots, N.$$

Therefore, using Lemma B.1, we deduce that all the $\tilde{p}^i$, $i = 1, \dots, N$, remain positive in $[t, T]$ a.s., and hence (B.4) is satisfied.

Next, we prove that the feedback strategy $\tilde{\pi}_s := (\tilde{\pi}_s^S, \tilde{\pi}_s^P)^\top$, $\tilde{\pi}_s^P := 0$, is admissible, that is,

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{\sigma^2\gamma^2}{2}\int_t^T \big(\tilde{\pi}^S(s, \tilde{p}_s)\big)^2 ds\right)\right] < \infty. \tag{B.5}$$

We have that (B.5) follows from (B.4) and the fact that $(\tilde{\pi}^S(s, \tilde{p}))^2$ is uniformly bounded on $[0, T] \times \widetilde{\Delta}_{N-1}$. To see the latter property, note that

$$\begin{aligned} \sup_{(s,\tilde{p})\in[0,T]\times\widetilde{\Delta}_{N-1}} \big(\tilde{\pi}^S(s, \tilde{p})\big)^2 &\le \frac{2}{\sigma^4(1-\gamma)^2} \sup_{(s,\tilde{p})\in[0,T]\times\widetilde{\Delta}_{N-1}} \big(\tilde{\mu}(\tilde{p}) - r\big)^2 \\ &\quad + \frac{2}{\sigma^2(1-\gamma)^2} \sup_{(s,\tilde{p})\in[0,T]\times\widetilde{\Delta}_{N-1}} \big(\nabla_{\tilde{p}} \underline{w}(s, \tilde{p}) \underline{\kappa}(\tilde{p})\big)^2. \end{aligned}$$

The first term on the right-hand side is clearly bounded since $|\tilde{\mu}(\tilde{p})| \le \max_i |\mu_i|$ for any $\tilde{p} \in \widetilde{\Delta}_{N-1}$. For the second term, using the definition of $\underline{\kappa}$ given in (5.3), we have

$$\begin{aligned} &\sup_{(s,\tilde{p})\in[0,T]\times\widetilde{\Delta}_{N-1}} \big(\nabla_{\tilde{p}} \underline{w}(s, \tilde{p}) \underline{\kappa}(\tilde{p})\big)^2 \\ &= \frac{1}{\sigma^2} \sup_{(s,\tilde{p})\in[0,T]\times\widetilde{\Delta}_{N-1}} \left(\sum_{j=1}^{N-1} \partial_{\tilde{p}^j} \underline{w}(s, \tilde{p}) \tilde{p}^j \left(\mu_j - \sum_{i=1}^{N} \mu_i \tilde{p}^i\right)\right)^2, \end{aligned} \tag{B.6}$$

where $\tilde{p}^N := 1 - \sum_{i=1}^{N-1} \tilde{p}^i$. The last expression is bounded since each partial derivative term $\partial_{\tilde{p}^j} \underline{w}(s, \tilde{p})$, $j = 1, \dots, N-1$, is bounded on $[0, T] \times \widetilde{\Delta}_{N-1}$ by Lemma 5.1 and following Remark 5.2. Therein we have shown $\mathcal{C}_P^{2,\alpha}$ regularity for $\underline{w}(s, \tilde{p})$, which directly implies bounded first- and second-order space derivatives on $\widetilde{\Delta}_{N-1}$. Now fix an arbitrary feedback control $\pi_s^S := \pi^S(s, \tilde{p}_s)$ such that $(\pi^S, \pi^P) \in \mathcal{A}(t, T; \tilde{p}, 1)$, where $\pi_s^P \equiv 0$ and $\mathcal{A}(t, T; \tilde{p}, 1)$ is as in Definition 4.1, and define the process

$$M_s^{\pi^S} := e^{-\gamma \int_t^s \underline{\eta}(u, \tilde{p}_u, \pi_u^S) du} e^{\underline{w}(s, \tilde{p}_s)}, \quad t \le s \le T,$$

where

$$\underline{\eta}(u, \tilde{p}, \pi^S) = \eta\big(u, \tilde{p}, (\pi^S, 0)^\top\big) = -r + \pi^S\big(r - \tilde{\mu}(\tilde{p})\big) + \frac{1-\gamma}{2}\sigma^2(\pi^S)^2. \tag{B.7}$$

In what follows, we write for simplicity M^{π} for M^{π^S} and π for π^S. Note that the process $(M^{\pi}_s)_{t\le s\le T}$ is uniformly bounded. Indeed, (B.7) is convex in π^S, and by minimizing it over π^S, it follows that for any $\tilde{p}\in\tilde{\Delta}_{N-1}$,

$$-\underline{\eta}(t,\tilde{p},\pi)\le r+\frac{(\tilde{\mu}(\tilde{p})-r)^2}{2(1-\gamma)\sigma^2}\le r+\frac{(\max_i \mu_i^2+r^2)}{(1-\gamma)\sigma^2}<\infty.$$

Therefore, since $\underline{w}\in C([0,T]\times\tilde{\Delta}_{N-1})$, there exists a constant $K<\infty$ for which

$$M^{\pi}_s=e^{-\gamma\int_t^s \underline{\eta}(u,\tilde{p}_u,\pi_u)du}e^{\underline{w}(s,\tilde{p}_s)}\le Ke^{\gamma\|\underline{\eta}\|_\infty(T-t)}=:A<\infty. \tag{B.8}$$

We prove the two assertions of Theorem 5.3 through the following steps:

(i) Define the process $\mathcal{Y}_s=e^{\underline{w}(s,\tilde{p}_s)}$. By Itô's formula and the generator formula (4.7) with $f(s,\tilde{p})=e^{\underline{w}(s,\tilde{p})}$,

$$\begin{aligned}M^{\pi}_s&=M^{\pi}_t+\int_t^s e^{-\gamma\int_t^u\underline{\eta}(r,\tilde{p}_r,\pi_r)\,dr}\,d\mathcal{Y}_u-\gamma\int_t^s\underline{\eta}(u,\tilde{p}_u,\pi_u)e^{-\gamma\int_t^u\underline{\eta}(r,\tilde{p}_r,\pi_r)\,dr}\mathcal{Y}_u\,du\\&=M^{\pi}_t+\int_t^s M^{\pi}_u\left(\frac{\partial\underline{w}}{\partial u}+\frac{1}{2}\mathrm{tr}(\underline{\kappa\kappa}^\top D^2\underline{w})+\frac{1}{2}(\nabla_{\tilde{p}}\underline{w})\underline{\kappa\kappa}^\top(\nabla_{\tilde{p}}\underline{w})^\top\right)du\\&\quad+\int_t^s M^{\pi}_u\big((\nabla_{\tilde{p}}\underline{w})\underline{\beta}_\gamma-\gamma\underline{\eta}\big)\,du+\int_t^s M^{\pi}_u\nabla_{\tilde{p}}\underline{w\kappa}\,d\tilde{W}^{(1)}_u.\end{aligned}$$

Using the expression of $\underline{\eta}$ in (B.7) and some rearrangement, we may write M^{π} as

$$M^{\pi}_s=M^{\pi}_t+\int_t^s M^{\pi}_u R(u,\tilde{p}_u,\pi_u)\,du+\int_t^s M^{\pi}_u\nabla_{\tilde{p}}\underline{w\kappa}\,d\tilde{W}^{(1)}_u$$

with

$$\begin{aligned}R(u,\tilde{p},\pi)&=\underline{w}_u+\frac{1}{2}\mathrm{tr}(\underline{\kappa\kappa}^\top D^2\underline{w})+\frac{1}{2}(\nabla_{\tilde{p}}\underline{w})\underline{\kappa\kappa}^\top(\nabla_{\tilde{p}}\underline{w})^\top\\&\quad+\gamma r+(\nabla_{\tilde{p}}w)\underline{\beta}_\gamma-\gamma\pi\big(r-\tilde{\mu}(\tilde{p})\big)-\frac{\sigma^2}{2}\gamma(1-\gamma)\pi^2.\end{aligned}\tag{B.9}$$

Clearly, $R(u,\tilde{p},\pi)$ is a concave function in π since $R_{\pi\pi}=-\sigma^2\gamma(1-\gamma)<0$. If we maximize $R(u,\tilde{p},\pi)$ as a function of π for each $(u,\tilde{p})$, then we find that the optimum is given by (5.7). Upon substituting (5.7) into (B.9), we get that

$$\begin{aligned}R(u,\tilde{p},\pi)&\le R\big(u,\tilde{p},\tilde{\pi}^S(u,\tilde{p})\big)\\&=\underline{w}_u+\frac{1}{2}\mathrm{tr}(\underline{\kappa\kappa}^\top D^2\underline{w})+\frac{1}{2(1-\gamma)}(\nabla_{\tilde{p}}\underline{w})\underline{\kappa\kappa}^\top(\nabla_{\tilde{p}}\underline{w})^\top+(\nabla_{\tilde{p}}\underline{w})\underline{\Phi}+\underline{\Psi}\\&=0,\end{aligned}$$

where the last equality follows from (5.5). Therefore, we get the inequality

$$\mathbb{E}^{\tilde{\mathbb{P}}}[M^{\pi}_T]\le M^{\pi}_t+\mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_t^T M^{\pi}_u(\nabla_{\tilde{p}}\underline{w})\underline{\kappa}\,d\tilde{W}^{(1)}_u\right]$$

with equality if $\pi = \widetilde{\pi}^S$. From (5.3), $\sup_{\tilde{p}\in\tilde{\Delta}_{N-1}} \|\underline{\kappa}(\tilde{p})\|^2 \le 2\max_i\{\mu_i\}/\sigma$. Since the partial derivatives $\partial_{\tilde{p}^j}\underline{w}(s,\tilde{p})$ are uniformly bounded on $[0,T]\times\tilde{\Delta}_{N-1}$ (see also the argument after (B.6)), (B.8) implies that

$$\sup_{t\le u\le T} |M_u^\pi (\nabla_{\tilde{p}}\underline{w})\underline{\kappa}|^2 \le A \sup_{t\le u\le T} \|\underline{\kappa}(\tilde{p}_u)\|^2 \sup_{t\le u\le T} \|\nabla_{\tilde{p}}\underline{w}(u,\tilde{p}_u)\|^2 \le B$$

for some nonrandom constant $B<\infty$. We conclude that

$$\mathbb{E}^{\tilde{\mathbb{P}}}[M_T^\pi] \le M_t^\pi = e^{\underline{w}(t,\tilde{p}_t)} = e^{\underline{w}(t,\tilde{p})} \tag{B.10}$$

with equality if $\pi = \widetilde{\pi}^S$.

(ii) For simplicity, let us write $\widetilde{\pi}_s := \widetilde{\pi}^S(s,\tilde{p}_s)$. First, note that from the fact that we have equality in (B.10) when $\pi = \tilde{\pi}$ it follows that

$$\begin{aligned} e^{\underline{w}(t,\tilde{p})} &= \mathbb{E}^{\tilde{\mathbb{P}}}\big[M_T^{\tilde{\pi}}\big] \\ &= \mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{-\gamma\int_t^T \underline{\eta}(u,\tilde{p}_u,\tilde{\pi}_u)\,du} e^{\underline{w}(T,\tilde{p}_T)}\right] \\ &= \mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{-\gamma\int_t^T \underline{\eta}(u,\tilde{p}_u,\tilde{\pi}_u)\,du}\right]. \end{aligned} \tag{B.11}$$

Similarly, for every feedback control $\pi_s = \pi(s,\tilde{p}_s)$ such that $(\pi,0)\in\mathcal{A}(t,T;\tilde{p},1)$,

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{-\gamma\int_t^T \underline{\eta}(u,\tilde{p}_u,\pi_u)\,du}\right] = \mathbb{E}^{\tilde{\mathbb{P}}}[M_T^\pi] \le e^{\underline{w}(t,\tilde{p})} = \mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{-\gamma\int_t^T \underline{\eta}(u,\tilde{p}_u,\tilde{\pi}_u)\,du}\right],$$

where the inequality in the previous equation comes from (B.10), and the last equality follows from (B.11). The previous relationships show the optimality of $\tilde{\pi}$ and prove the assertions (1) and (2). □

Proof of Theorem 5.4 For brevity, define the operator

$$\mathcal{B} = \partial_t + \frac{1}{2}\operatorname{tr}(\bar{\kappa}\bar{\kappa}^\top D^2) + \nabla_{\tilde{p}}\bar{\Phi}$$

and denote by

$$H(t,\tilde{p},u) = -\tilde{h}(\tilde{p})e^{\underline{w}(t,\frac{1}{\tilde{h}(\tilde{p})}\tilde{p}\cdot h')}\frac{u^\gamma}{1-\gamma}, \quad u\in\mathbb{R}_+,$$

the nonlinear term of the PDE (5.17). Notice that since $\tilde{h}>0$ by construction, we have $H\le 0$. Moreover, $u\mapsto H(t,\tilde{p},u)$ is smooth and Lipschitz-continuous on $[\bar{c},+\infty)$ for any $\bar{c}>0$, uniformly with respect to $(t,\tilde{p})$. We set

$$\bar{\psi}_0(t,\tilde{p}) = e^{c(T-t)}, \quad t\in[0,T],$$

where c is a suitably large positive constant such that

$$cu + H(t,\tilde{p},u) - \frac{\bar{\Psi}(t,\tilde{p})}{1-\gamma}u \ge 0, \quad (t,\tilde{p})\in(0,T)\times\tilde{\Delta}_{N-1},\ u\ge 1. \tag{B.12}$$

Then we define recursively the sequence $(\bar{\psi}_j)_{j\in\mathbb{N}}$ by

$$\begin{cases}(\mathcal{B}+\frac{\bar{\Psi}}{1-\gamma})\bar{\psi}_j-\lambda\bar{\psi}_j=H(\cdot,\cdot,\bar{\psi}_{j-1})-\lambda\bar{\psi}_{j-1},\\ \bar{\psi}_j(T,\cdot)=1,\end{cases} \tag{B.13}$$

where λ is the Lipschitz constant of $u\mapsto H(\cdot,\cdot,u)$ on $[\bar{c},+\infty)$, and $\bar{c}$ is the strictly positive constant defined as

$$\bar{c}=e^{-\frac{T}{1-\gamma}\|\bar{\Psi}\|_\infty}. \tag{B.14}$$

Let us recall that the *linear problem* (B.13) has a classical solution in $C_P^{2,\alpha}$ whose existence can be proved as in Lemma 5.1; see also the following Remark 5.2. Next, we prove by induction that

(i) $(\bar{\psi}_j)$ is a decreasing sequence, that is,

$$\bar{\psi}_{j+1}\le\bar{\psi}_j,\quad j\ge 0; \tag{B.15}$$

(ii) $(\bar{\psi}_j)$ is uniformly strictly positive, and in particular

$$\bar{\psi}_{j+1}\ge\bar{c},\quad j\ge 0, \tag{B.16}$$

with $\bar{c}$ as in (B.14).

First, we observe that

$$\bar{\psi}_0\ge 1,\quad \text{and}\quad \Big(\mathcal{B}+\frac{\bar{\Psi}}{1-\gamma}\Big)\bar{\psi}_0=\Big(-c+\frac{\bar{\Psi}}{1-\gamma}\Big)\bar{\psi}_0. \tag{B.17}$$

Next, we prove (B.15) and (B.16) for $j=0$. By (B.17) and (B.12) we have

$$\begin{cases}(\mathcal{B}+\frac{\bar{\Psi}}{1-\gamma}-\lambda)(\bar{\psi}_1-\bar{\psi}_0)=H(\cdot,\cdot,\bar{\psi}_0)+c\bar{\psi}_0-\frac{\bar{\Psi}}{1-\gamma}\bar{\psi}_0\ge 0,\\ (\bar{\psi}_1-\bar{\psi}_0)(T,\tilde{p})=0,\end{cases} \tag{B.18}$$

where the inequality follows from the fact that c is chosen as in (B.12) and $\bar{\psi}_0\ge 1$ as observed in (B.17). Since the process $\tilde{p}$ never reaches the boundary of the simplex by Lemma B.1, it follows from the Feynman–Kac representation theorem (or, equivalently, the maximum principle) that $\bar{\psi}_1\le\bar{\psi}_0$. Indeed, we have

$$\begin{aligned}&(\bar{\psi}_0-\bar{\psi}_1)(t,\tilde{p})\\ &\quad=\mathbb{E}^{\tilde{\mathbb{P}}}\bigg[\int_t^T e^{-\int_t^s\frac{\bar{\Psi}(r,\tilde{p}_r)-\lambda}{1-\gamma}dr}\\ &\qquad\times\bigg(H(s,\tilde{p}_s,\bar{\psi}_0)+c\bar{\psi}_0(s,\tilde{p}_s)-\frac{\bar{\Psi}(s,\tilde{p}_s)\bar{\psi}_0(s,\tilde{p}_s)}{1-\gamma}\bigg)\bigg|\tilde{p}_t=\tilde{p}\bigg]\\ &\quad\ge 0,\end{aligned} \tag{B.19}$$

where the last inequality follows directly from (B.18). This proves (B.15) when $j=0$. Using the recursive definition (B.13) along with the facts that $H \le 0$ and $\lambda > 0$ and inequality (B.19), we obtain

$$\Big(\mathcal{B} + \frac{\bar{\Psi}}{1-\gamma}\Big)\bar{\psi}_1 = H(\cdot,\cdot,\bar{\psi}_0) + \lambda(\bar{\psi}_1 - \bar{\psi}_0) \le 0. \tag{B.20}$$

Then (B.16) with $j=0$ follows again from the Feynman–Kac theorem; indeed, by (B.20) we have

$$\begin{aligned}\bar{\psi}_1(t,\tilde{p}) &= \mathbb{E}^{\tilde{\mathbb{P}}}\bigg[-\int_t^T e^{-\int_t^s \frac{\bar{\Psi}(r,\tilde{p}_r)}{1-\gamma}dr}\Big(H(s,\tilde{p}_s,\bar{\psi}_0) + \lambda(\bar{\psi}_1 - \bar{\psi}_0)(s,\tilde{p}_s)\Big)\bigg|\tilde{p}_t = \tilde{p}\bigg] \\ &\quad + \mathbb{E}^{\tilde{\mathbb{P}}}\Big[e^{\frac{1}{1-\gamma}\int_t^T \bar{\Psi}(s,\tilde{p}_s)\,ds}\Big|\tilde{p}_t = \tilde{p}\Big] \\ &\ge e^{-\frac{T}{1-\gamma}\|\bar{\Psi}\|_\infty},\end{aligned} \tag{B.21}$$

where the last inequality follows from the positivity of the first expectation guaranteed by (B.20).

Next, we assume the inductive hypothesis, that is,

$$\bar{c} \le \bar{\psi}_j \le \bar{\psi}_{j-1}, \tag{B.22}$$

and prove (B.15), (B.16). Recalling that λ is the Lipschitz constant of the function $u \mapsto H(\cdot,\cdot,u)$ on $[\bar{c},+\infty)$, by (B.22) we have

$$\begin{cases}(\mathcal{B} + \frac{\bar{\Psi}}{1-\gamma} - \lambda)(\bar{\psi}_{j+1} - \bar{\psi}_j) = H(\cdot,\cdot,\bar{\psi}_j) - H(\cdot,\cdot,\bar{\psi}_{j-1}) - \lambda(\bar{\psi}_j - \bar{\psi}_{j-1}) \ge 0,\\ (\bar{\psi}_{j+1} - \bar{\psi}_j)(T,\tilde{p}) = 0.\end{cases}$$

Thus, (B.15) follows from the Feynman–Kac theorem using the same procedure as in (B.18) and (B.19). Moreover, we have

$$\Big(\mathcal{B} + \frac{\bar{\Psi}}{1-\gamma}\Big)\bar{\psi}_{j+1} = H(\cdot,\cdot,\bar{\psi}_j) + \lambda(\bar{\psi}_{j+1} - \bar{\psi}_j) \le 0,$$

where the inequality above follows by (B.15) and using that $H \le 0$ and $\lambda > 0$. Then, as in (B.21), we have that (B.16) follows from the Feynman–Kac theorem.

In conclusion, for $j \in \mathbb{N}$, we have

$$\bar{c} \le \bar{\psi}_{j+1} \le \bar{\psi}_j \le \bar{\psi}_0. \tag{B.23}$$

Now the thesis follows by proceeding as in the proof of Theorem 3.3 in [8]. Indeed, let us denote by $\bar{\psi}$ the pointwise limit of $(\bar{\psi}_j)$ as $j \to +\infty$. Since $\bar{\psi}_j$ is a solution of (B.13) and, by the uniform estimate (B.23), we can apply standard a priori Morrey–Sobolev-type estimates (see Theorems 2.1 and 2.2 in [8]) to conclude that for any $\alpha \in (0,1)$, $\|\bar{\psi}_j\|_{C_P^{1,\alpha}((0,T)\times\tilde{\Delta}_{N-1})}$ is bounded by a constant only dependent on $\mathcal{B}$, α, and λ. Hence, by the classical Schauder interior estimate (see, e.g., Theorem 2.3

in [8]), we deduce that $\|\bar{\psi}_j\|_{C_P^{2,\alpha}((0,T)\times\tilde{\Delta}_{N-1})}$ is bounded uniformly in $j\in\mathbb{N}$. It follows that $(\bar{\psi}_j)_{j\in\mathbb{N}}$ admits a subsequence (denoted by itself) that converges in $C^{2,\alpha}$. Thus passing to the limit in (B.13) as $j\to\infty$, we have

$$\Big(\mathcal{B}+\frac{\bar{\Psi}}{1-\gamma}\Big)\bar{\psi}=H(\cdot,\cdot,\bar{\psi})\quad\text{in }(0,T)\times\tilde{\Delta}_{N-1}$$

and $\bar{\psi}(T,\cdot)=1$.

Finally, in order to prove that $\bar{\psi}\in C((0,T]\times\tilde{\Delta}_{N-1})$, we use the standard argument of barrier functions. We recall that w is a barrier function for the operator $(\mathcal{B}+\frac{\bar{\Psi}}{1-\gamma})$ on the domain $(0,T]\times\tilde{\Delta}_{N-1}$ at the point $(T,\bar{p})$ if $w\in C^2(V\cap((0,T]\times\tilde{\Delta}_{N-1}))$, where V is a neighborhood of $(T,\bar{p})$, and we have

(i) $(\mathcal{B}+\frac{\bar{\Psi}}{1-\gamma})w\le -1$ in $V\cap((0,T)\times\tilde{\Delta}_{N-1})$;
(ii) $w>0$ in $V\cap((0,T)\times\tilde{\Delta}_{N-1})\setminus\{(T,\bar{p})\}$ and $w(T,\bar{p})=0$.

Next, we fix $\bar{p}\in\tilde{\Delta}_{N-1}$. Following [15, Sect. 3.4], it is not difficult to check that

$$w(t,\tilde{p})=\big(|\tilde{p}-\bar{x}|^2+c_1(T-t)\big)e^{c_2(T-t)}$$

is a barrier at $(T,\bar{p})$, provided that c_1,c_2 are sufficiently large. Then we set

$$v^{\pm}(t,\tilde{p})=1\pm kw(t,\tilde{p}),$$

where k is a suitably large positive constant, independent of j, such that

$$\begin{aligned}\Big(\mathcal{B}+\frac{\bar{\Psi}}{1-\gamma}\Big)(\bar{\psi}_j-v^+)&\ge H(\cdot,\cdot,\bar{\psi}_{j-1})-\lambda(\bar{\psi}_{j-1}-\bar{\psi}_j)\\&\qquad-\frac{\bar{\Psi}}{1-\gamma}-k\Big(\mathcal{B}+\frac{\bar{\Psi}}{1-\gamma}\Big)w\\&\ge 0,\end{aligned}$$

and $\bar{\psi}_j\le v^+$ on $\partial(V\cap((0,T)\times\tilde{\Delta}_{N-1}))$. The maximum principle yields $\bar{\psi}_j\le v^+$ on $V\cap((0,T)\times\tilde{\Delta}_{N-1})$; analogously, $\bar{\psi}_j\ge v^-$ on the domain $V\cap((0,T)\times\tilde{\Delta}_{N-1})$, and letting $j\to\infty$, we get

$$1-kw(t,\tilde{p})\le\bar{\psi}(t,\tilde{p})\le 1+kw(t,\tilde{p}),\quad (t,\tilde{p})\in V\cap\Big((0,T)\times\tilde{\Delta}_{N-1}\Big).$$

Therefore, we deduce that

$$\lim_{(t,\tilde{p})\to(T,\bar{p})}\bar{\psi}(t,\tilde{p})=1,$$

which concludes the proof. □

Proof of Theorem 5.5 As in the proof of Theorem 5.3, to ease the notational burden, we write $\tilde{p}$ for $\tilde{p}^{\circ}$, $\tilde{p}_s$ for $\tilde{p}_s^t$, π for π^t, $\tilde{W}$ for $\tilde{W}^t$, $\tilde{\mathbb{P}}$ for $\tilde{\mathbb{P}}^t$, $\mathbb{P}$ for $\mathbb{P}^t$, and

$\mathcal{G}_s^I$ for $\mathcal{G}_s^{t,I}$. Similarly to the proof of Theorem 5.3, it is easy to see that the strategy $\widetilde{\pi}_s := (\widetilde{\pi}_s^S, \widetilde{\pi}_s^P)^\top = (\widetilde{\pi}^S(s, \tilde{p}_{s-}, H_{s-}^t), \widetilde{\pi}^P(s, \tilde{p}_{s-}, H_{s-}))^\top$ defined from (5.20), (5.21) is admissible, that is, satisfies (4.5). This essentially follows from condition (2.6) and the fact that both $\underline{w}(s, \tilde{p})$ and $\bar{w}(s, \tilde{p})$ belong to $\mathcal{C}_P^{2\alpha}$; hence, their first- and second-order space derivatives are bounded on $[0, T] \times \widetilde{\Delta}_{N-1}$. Here, it is also useful to recall that $\mathbb{P}[\tilde{p}_s \in \tilde{\Delta}_{N-1},\ t \le s \le T] = 1$ as shown in the proof of Theorem 5.3.

For a feedback control $\pi_s := (\pi_s^S, \pi_s^P) := (\pi^S(s, \tilde{p}_{s-}, H_{s-}), \pi^P(s, \tilde{p}_{s-}, H_{s-}))$ such that $(\pi^S, \pi^P) \in \bar{\mathcal{A}}(t, T; \tilde{p}, 0)$, define the process

$$M_s^\pi := e^{-\gamma \int_t^s \tilde{\eta}(u, \tilde{p}_u, \pi_u)\, du} e^{w(s, \tilde{p}_s, H_s)}, \quad t \le s \le T,$$

where $w(s, \tilde{p}, z) := (1 - z)\bar{w}(s, \tilde{p}) + z\underline{w}(s, \tilde{p})$, and $\tilde{\eta}$ is defined as in (4.13). Note that $\tilde{\eta}$ can be written as

$$\begin{aligned}\tilde{\eta}(t, \tilde{p}, \pi) = -r + \pi^S\big(r - \tilde{\mu}(\tilde{p})\big) + \frac{1-\gamma}{2}\sigma^2(\pi^S)^2 \\ + \pi^P\big(r - \tilde{a}(t, \tilde{p})\big) + \frac{1-\gamma}{2}\upsilon^2(\pi^P)^2,\end{aligned}$$

and thus $-\tilde{\eta}$ is concave in π. This in turn implies that there exists a nonrandom constant $A < \infty$ such that

$$0 < M_s^\pi \le A < \infty, \quad t \le s \le T, \tag{B.24}$$

since $\underline{w}, \bar{w} \in C([0, T] \times \tilde{\Delta}_{N-1})$. We prove the two assertions of Theorem 5.5 through the following steps:

(i) Define the processes $\mathcal{Y}_s = e^{w(s, \tilde{p}, H_s)}$ and $\mathcal{U}_s = e^{-\gamma \int_t^s \tilde{\eta}(u, \tilde{p}_u, \pi_u)\, du}$. By Itô's formula, the generator formula (4.7) with $f(s, \tilde{p}, z) = e^{w(s, \tilde{p}, z)}$, and the same arguments as those used to derive (4.11),

$$\begin{aligned}M_s^\pi &= M_t^\pi + \int_t^s \mathcal{U}_{u-}\, d\mathcal{Y}_u - \gamma \int_t^s \tilde{\eta}(u, \tilde{p}_u, \pi_u)\mathcal{U}_u\mathcal{Y}_u\, du \\ &= M_t^\pi + \int_t^s M_u^\pi \bigg(\frac{\partial w}{\partial u} + \frac{1}{2}\operatorname{tr}(\kappa\kappa^\top D^2 w) + \frac{1}{2}(\nabla_{\tilde{p}} w)\kappa\kappa^\top(\nabla_{\tilde{p}} w)^\top + (\nabla_{\tilde{p}} w)\beta_\gamma \\ &\quad + (1 - H_u)\tilde{h}(\tilde{p}_u)\Big(e^{\underline{w}(u, \frac{1}{\tilde{h}(\tilde{p}_u)}\tilde{p}_u \cdot h') - \bar{w}(u, \tilde{p}_u)} - 1\Big) - \gamma\tilde{\eta}\bigg)\, du + \mathcal{M}_s^c + \mathcal{M}_s^d,\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}_s^c &:= \int_t^s M_u^\pi \nabla_{\tilde{p}} w\kappa(u, \tilde{p}_u)\, d\tilde{W}_u, \\ \mathcal{M}_s^d &:= \int_t^s \mathcal{U}_{u-}\Big(e^{\underline{w}(u, \frac{1}{\tilde{h}(\tilde{p}_{u-})}\tilde{p}_{u-} \cdot h')} - e^{\bar{w}(u, \tilde{p}_{u-})}\Big)\, d\tilde{\xi}_u.\end{aligned} \tag{B.25}$$

Using the expression of η in (4.13) and arguments similar to those used to derive (4.13), we may write M^π as

$$M_s^\pi = M_t^\pi + \int_t^s M_u^\pi R(u, \tilde{p}_u, \pi_u, H_u)\, du + \mathcal{M}_s^c + \mathcal{M}_s^d$$

with

$$\begin{aligned} &R(u, \tilde{p}, \pi, z) \qquad \text{(B.26)} \\ &= \frac{\partial w}{\partial u} + \frac{1}{2}\operatorname{tr}(\kappa\kappa^\top D^2 w) + \frac{1}{2}(\nabla_{\tilde{p}} w)\kappa\kappa^\top(\nabla_{\tilde{p}} w)^\top + \gamma r \\ &+ (1-z)\tilde{h}(\tilde{p})\Big(e^{\underline{w}(u, \frac{1}{\tilde{h}(\tilde{p})}\tilde{p}\cdot h') - \bar{w}(u,\tilde{p})} - 1\Big) \\ &+ z\left((\nabla_{\tilde{p}}\underline{w})\beta_\gamma - \gamma\pi^S\big(r - \tilde{\mu}(\tilde{p})\big) - \frac{\gamma(1-\gamma)}{2}\sigma^2(\pi^S)^2\right) \\ &+ (1-z)\left((\nabla_{\tilde{p}}\bar{w})\beta_\gamma - \gamma\pi^P\big(r - \tilde{a}(t,\tilde{p})\big) - \frac{\gamma(1-\gamma)}{2}\upsilon^2(\pi^P)^2\right). \qquad \text{(B.27)} \end{aligned}$$

Clearly, $R(u, \tilde{p}, \pi, z)$ is a concave function in π for each $(u, \tilde{p}, z)$, and this function reaches its maximum at $\tilde{\pi}(u, \tilde{p}, z) = (\tilde{\pi}^S(u, \tilde{p}, z), \tilde{\pi}^P(u, \tilde{p}, z))$ as defined in (5.19), (5.20). Upon substituting this maximum into (B.26) and rearrangements similar to those leading to (5.5) and (5.14) (depending on whether $z = 1$ or $z = 0$), we get

$$R(u, \tilde{p}, \pi, z) \le R\big(u, \tilde{p}, \widetilde{\pi}(u, \tilde{p}, z), z\big) = 0,$$

in light of (5.5) or (5.14), respectively. Therefore, we get the inequality

$$\mathbb{E}^{\tilde{\mathbb{P}}}[M_T^\pi] \le M_t^\pi + \mathbb{E}^{\tilde{\mathbb{P}}}[\mathcal{M}_T^c + \mathcal{M}_T^d]$$

with equality if $\pi = \widetilde{\pi}$. Note that $\mathbb{E}^{\tilde{\mathbb{P}}}[\mathcal{M}_T^c] = 0$ since it is possible to find a nonrandom constant B such that

$$\sup_{t\le u\le T} |M_u^\pi(\nabla_{\tilde{p}} w)\kappa(u, \tilde{p}_u)|^2 \le A \sup_{t\le u\le T} \|\kappa(u, \tilde{p}_u)\|^2 \sup_{t\le u\le T} \|\nabla_{\tilde{p}} w(u, \tilde{p}_u)\|^2 \le B,$$

in view of (B.24) and the fact that the partial derivatives of $\underline{w}$ and $\bar{w}$ are uniformly bounded on $[0, T] \times \widetilde{\Delta}_{N-1}$. The latter statement follows from the fact that both $\underline{w}$ and $\bar{w}$ are $\mathcal{C}_P^{2,\alpha}$ on $\widetilde{\Delta}_{N-1}$ in light of Lemma 5.1 and Theorem 5.4. To deal with $\mathcal{M}^d$, note that since $\underline{w}, \bar{w} \in C([0, T] \times \tilde{\Delta}_{N-1})$ and $(\mathcal{U}_s)_{t\le s\le T}$ is uniformly bounded (due to the fact that $-\tilde{\eta}$ is concave), we have that the integrand of the second integral in (B.25) is uniformly bounded, and thus $\mathbb{E}^{\tilde{\mathbb{P}}}[\mathcal{M}_T^d] = 0$ as well. The two previous facts, together with the initial conditions $H_t = 0$ and $\tilde{p}_t = \tilde{p}$, lead to

$$\mathbb{E}^{\tilde{\mathbb{P}}}[M_T^\pi] \le M_t^\pi = e^{w(t, \tilde{p}_t, H_t)} = e^{w(t, \tilde{p}, 0)} = e^{\bar{w}(t, \tilde{p})} \qquad \text{(B.28)}$$

with equality if $\pi = \widetilde{\pi}$.

(ii) The rest of the proof is similar to the postdefault case. Concretely, using the fact that we have equality in (B.28) when $\pi = \tilde{\pi}$, we get

$$\begin{aligned} e^{\bar{w}(t,\tilde{p})} &= \mathbb{E}^{\tilde{\mathbb{P}}}[M_T^{\tilde{\pi}}] = \mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{-\gamma \int_t^T \tilde{\eta}(u,\tilde{p}_u,\tilde{\pi}_u)\,du} e^{w(T,\tilde{p}_T,H_T)}\right] \\ &= \mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{-\gamma \int_t^T \tilde{\eta}(u,\tilde{p}_u,\tilde{\pi}_u)\,du}\right] \end{aligned} \tag{B.29}$$

since $w(T, \tilde{p}_T, H_T) := (1 - H_T)\bar{w}(T, \tilde{p}_T) + H_T \underline{w}(T, \tilde{p}_T) \equiv 0$. Also, by (B.28), for every feedback control $\pi_s = \pi(s, \tilde{p}_s, H_s) \in \mathcal{A}(t, T; \tilde{p}, 0)$, we have

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{-\gamma \int_t^T \tilde{\eta}(u,\tilde{p}_u,\pi_u)\,du}\right] = \mathbb{E}^{\tilde{\mathbb{P}}}[M_T^{\pi}] \le M_t^{\pi} = e^{\bar{w}(t,\tilde{p})} = \mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{-\gamma \int_t^T \tilde{\eta}(u,\tilde{p}_u,\tilde{\pi}_u)\,du}\right],$$

where the last equality follows from (B.29). This proves assertions (1) and (2). □

References

1. Bélanger, A., Shreve, S., Wong, D.: A general framework for pricing credit risk. Math. Finance **14**, 317–350 (2004)
2. Bielecki, T., Jang, I.: Portfolio optimization with a defaultable security. Asia-Pac. Financ. Mark. **13**, 113–127 (2006)
3. Bielecki, T., Rutkowski, M.: Credit Risk: Modelling, Valuation and Hedging. Springer, New York (2001)
4. Bo, L., Capponi, A.: Optimal investment in credit derivatives portfolio under contagion risk. Math. Financ. (2014, forthcoming). Available online http://onlinelibrary.wiley.com/doi/10.1111/mafi.12074/abstract. doi:10.1111/mafi.12074
5. Bo, L., Wang, Y., Yang, X.: An optimal portfolio problem in a defaultable market. Adv. Appl. Probab. **42**, 689–705 (2010)
6. Capponi, A., Figueroa-López, J.E.: Dynamic portfolio optimization with a defaultable security and regime-switching markets. Math. Finance **24**, 207–249 (2014)
7. Carr, P., Linetsky, V., Mendoza-Arriaga, R.: Time-changed Markov processes in unified credit-equity modeling. Math. Finance **20**, 527–569 (2010)
8. Di Francesco, M., Pascucci, A., Polidoro, S.: The obstacle problem for a class of hypoelliptic ultraparabolic equations. Proc. R. Soc., Math. Phys. Eng. Sci. **464**, 155–176 (2008)
9. El Karoui, N., Jeanblanc, M., Jiao, Y.: What happens after a default: the conditional density approach. Stoch. Process. Appl. **120**, 1011–1032 (2010)
10. Elliott, R.J., Aggoun, L., Moore, J.B.: Hidden Markov Models: Estimation and Control. Springer, Berlin (1994)
11. Elliott, R.J., Siu, T.K.: A hidden Markov model for optimal investment of an insurer with model uncertainty. Int. J. Robust Nonlinear Control **22**, 778–807 (2012)
12. Frey, R., Runggaldier, W.J.: Credit risk and incomplete information: a nonlinear-filtering approach. Finance Stoch. **14**, 495–526 (2010)
13. Frey, R., Runggaldier, W.J.: Nonlinear filtering in models for interest-rate and credit risk. In: Crisan, D., Rozovski, B. (eds.) Oxford Handbook on Nonlinear Filtering, pp. 923–959. Oxford University Press, London (2011)
14. Frey, R., Schmidt, T.: Pricing and hedging of credit derivatives via the innovations approach to nonlinear filtering. Finance Stoch. **16**, 105–133 (2012)
15. Friedman, A.: Partial Differential Equations of Parabolic Type. Prentice Hall, New York (1964)
16. Fujimoto, K., Nagai, H., Runggaldier, W.J.: Expected log-utility maximization under incomplete information and with Cox-process observations. Asia-Pac. Financ. Mark. **21**, 35–66 (2014)
17. Fujimoto, K., Nagai, H., Runggaldier, W.J.: Expected power-utility maximization under incomplete information and with Cox-process observations. Appl. Math. Optim. **67**, 33–72 (2013)
18. Giesecke, K., Longstaff, F., Schaefer, S., Strebulaev, I.: Corporate bond default risk: a 150-year perspective. J. Financ. Econ. **102**, 233–250 (2011)

19. Jacod, J., Shiryaev, A.: Limit Theorems for Stochastic Processes. Springer, New York (2003)
20. Jiao, Y., Kharroubi, I., Pham, H.: Optimal investment under multiple defaults risk: a BSDE-decomposition approach. Ann. Appl. Probab. **23**, 455–491 (2013)
21. Jiao, Y., Pham, H.: Optimal investment with counterparty risk: a default density approach. Finance Stoch. **15**, 725–753 (2011)
22. Kharroubi, I., Lim, T.: Progressive enlargement of filtrations and backward SDEs with jumps. J. Theor. Probab. **27**, 683–724 (2014)
23. Kliemann, W., Koch, G., Marchetti, F.: On the unnormalized solution of the filtering problem with counting process observations. IEEE Trans. Inf. Theory **36**, 1415–1425 (1990)
24. Kraft, H., Steffensen, M.: Portfolio problems stopping at first hitting time with application to default risk. Math. Methods Oper. Res. **63**, 123–150 (2005)
25. Kraft, H., Steffensen, M.: Asset allocation with contagion and explicit bankruptcy procedures. J. Math. Econ. **45**, 147–167 (2009)
26. Kraft, H., Steffensen, M.: How to invest optimally in corporate bonds. J. Econ. Dyn. Control **32**, 348–385 (2008)
27. Liechty, J., Roberts, G.: Markov chain Monte Carlo methods for switching diffusion models. Biometrika **88**, 299–315 (2001)
28. Linetsky, V.: Pricing equity derivatives subject to bankruptcy. Math. Finance **16**, 255–282 (2006)
29. Nagai, H., Runggaldier, W.: PDE approach to utility maximization for market models with hidden Markov factors. In: Dalang, R.C., Dozzi, M., Russo, F. (eds.) Seminar on Stochastic Analysis, Random Fields, and Applications V. Progress in Probability, vol. 59, pp. 493–506. Birkhäuser, Basel (2008)
30. Pascucci, A.: PDE and martingale methods in option pricing. Bocconi & Springer Series. Springer, Milan. Bocconi University Press, Milan (2011)
31. Pham, H.: Stochastic control under progressive enlargement of filtrations and applications to multiple defaults risk management. Stoch. Process. Appl. **120**, 1795–1820 (2010)
32. Protter, P., Shimbo, K.: No arbitrage and general semimartingales. In: Ethier, S.N., Feng, J., Stockbridge, R.H. (eds.) Markov Processes and Related Topics: A Festschrift for Thomas G. Kurtz. IMS Collections, vol. 4, pp. 267–283 (2008)
33. Rogers, C., Williams, D.: Diffusions, Markov Processes, and Martingales: Itô Calculus. Wiley, New York (1987)
34. Sass, J., Haussmann, U.: Optimizing the terminal wealth under partial information: the drift process as a continuous time Markov chain. Finance Stoch. **8**, 553–577 (2004)
35. Siu, T.K.: A BSDE approach to optimal investment of an insurer with hidden regime switching. Stoch. Anal. Appl. **31**, 1–18 (2013)
36. Sotomayor, L., Cadenillas, A.: Explicit solutions of consumption-investment problems in financial markets with regime switching. Math. Finance **19**, 251–279 (2009)
37. Tamura, T., Watanabe, Y.: Risk-sensitive portfolio optimization problems for hidden Markov factors on infinite time horizon. Asymptot. Anal. **75**, 169–209 (2011)
38. Wong, E., Hajek, B.: Stochastic Processes in Engineering Systems. Springer, Berlin (1985)
39. Zariphopoulou, T.: Investment–consumption models with transaction fees and Markov-chain parameters. SIAM J. Control Optim. **30**, 613–636 (1992)