Applications of Symmetries and Conservation Laws to the Study of Nonlinear Elasticity Equations

A Thesis Submitted to the

College of Graduate Studies and Research

in Partial Fulfillment of the Requirements

for the degree of Master of Science

in the Department of Mathematics and Statistics

University of Saskatchewan

Saskatoon

By

Simon St. Jean

Permission to Use

In presenting this thesis in partial fulfilment of the requirements for a Postgraduate degree from the University of Saskatchewan, I agree that the Libraries of this University may make it freely available for inspection. I further agree that permission for copying of this thesis in any manner, in whole or in part, for scholarly purposes may be granted by the professor or professors who supervised my thesis work or, in their absence, by the Head of the Department or the Dean of the College in which my thesis work was done. It is understood that any copying or publication or use of this thesis or parts thereof for financial gain shall not be allowed without my written permission. It is also understood that due recognition shall be given to me and to the University of Saskatchewan in any scholarly use which may be made of any material in my thesis.

Requests for permission to copy or to make other use of material in this thesis in whole or part should be addressed to:

Head of the Department of Mathematics and Statistics

Room 142 McLean Hall

University of Saskatchewan

Saskatoon, Saskatchewan

Canada

S7N 5E6

Abstract

Mooney-Rivlin hyperelasticity equations are nonlinear coupled partial differential equations (PDEs) that are used to model various elastic materials. These models have been extended to account for fiber reinforced solids with applications in modeling biological materials. As such, it is important to obtain solutions to these physical systems. One approach is to study the admitted Lie symmetries of the PDE system, which allows one to seek invariant solutions by the invariant form method. Furthermore, knowledge of conservation laws for a PDE provides insight into conserved physical quantities, and can be used in the development of stable numerical methods.

The current Thesis is dedicated to presenting the methodology of Lie symmetry and conservation law analysis, as well as applying it to fiber reinforced Mooney-Rivlin models. In particular, an outline of Lie symmetry and conservation law analysis is provided, and the partial differential equations describing the dynamics of a hyperelastic solid are presented. A detailed example of Lie symmetry and conservation law analysis is done for the PDE system describing plane strain in a Mooney-Rivlin solid. Lastly, Lie symmetries and conservation laws are studied in one and two dimensional models of fiber reinforced Mooney-Rivlin materials.

Acknowledgements

I would like to thank my supervisor Professor Alexey F. Shevyakov for his encouragement, support, and insights into applied mathematics. I am also grateful to my colleagues for many interesting and helpful discussions. I am indebted to my family and previous teachers for their patience and encouragement that has gotten me to this point. I would also like to thank the University of Saskatchewan and the Mathematics and Statistics Department for financial support during my program.

Simon St. Jean

Saskatoon, October 2014

For my family.

CONTENTS

List of Tables

List of Figures

Chapter 1

Introduction to Lie Symmetry and Conservation Law Analysis

1.1 Introduction

Differential equations (DEs) are an important class of equations which appear in many models of physical processes. The primary goal of analyzing differential equations is to seek solutions to these equations. One possible approach is to utilize properties of the equations to better understand, and sometimes solve, the problem. Two properties useful in studying DEs are Lie symmetries [21, 23, 68, 96, 98] and conservation laws [3–5, 22, 95].

A symmetry of a system of partial differential equations (PDEs) is a group of transformations which maps solutions of the system into other solutions of the same system. A symmetry transformation acts on the space of independent and dependent variables, and does not alter the form of the differential equation (i.e. the equation is invariant under the transformation). Knowledge of symmetries of a PDE system allow for the construction of new solutions from known solutions of the system. In general, symmetry transformations of a PDE system are determined by a nonlinear invariance criteria, which are difficult to solve in the case of complicated PDEs.

A particular class of symmetries which can be determined in a (relatively) simple manner are continuous symmetry transformations, named Lie symmetries after their founder Sophus Lie [79–81].Lie symmetries are desirable to study since they can be sought algorithmically (by Lie's algorithm). This is achieved by studying invariance under the equivalent local infinitesimal transformation, for which the invariance criteria reduces to solving a system of

linear partial differential equations.

Of additional importance are applications of Lie symmetries. Invariance of an ODE under a one-parameter Lie symmetry allows one to constructively reduce the order of the ODE by one while preserving the solution set. Furthermore, for both ODEs and systems of PDEs, one may construct particular solutions invariant under an admitted Lie symmetry.

Seeking conservation laws of a physical system are important for understanding the governing physical processes, as well as to prove existence of solutions, stability of shockwave solutions, and developing stable numerical methods (see [96] for a detailed discussion with references). The study of conservation laws is tied closely to that of Lie symmetries due to the work of Emmy Noether [95]. Noether proved that, for a system of Euler-Lagrange equations (a variational system of PDEs), Lie symmetries of the Lagrangian correspond to conservation laws of the system. This result has been extended in the work of Bessel-Hagen to include Lie symmetries for which the Lagrangian is invariant up to a divergence expression [12].

An alternative to Noether's theorem in constructing conservation laws of PDE systems is the algorithmic 'direct method' developed by Anco and Bluman [3–5]. The direct method works by seeking particular functions that the PDE system can be multiplied by which can then be manipulated into a divergence expression. This method supersedes Noether's theorem since it can be applied to both variational and non-variational systems alike, and yields all the conservation laws of Noether's method [22].

In this Chapter, we present the basic theory and methodology of Lie symmetry analysis and conservation law analysis based on the work of Bluman, Cheviakov, and Anco [22], Bluman and Anco [21], Bluman and Kumei [23], and Hydon [65]. We begin in Section 1.2 by introducing the concept of symmetry of algebraic equations with the geometric picture as an intuitive motivation. In addition, we define Lie point symmetries of algebraic equations, and discuss several important concepts fundamental to Lie symmetry analysis. In Section 1.3, we present the extension of Lie symmetries to differential equations, and several important applications therein. Then, in Section 1.4 we introduce the notions of conservation laws of PDEs, and briefly present the mathematical formulation. Finally, the connection between Lie symmetries and conservation laws for PDE systems is discussed in Section 1.5, with additional notes on the construction of nonlocally related PDE systems through known conservation

laws.

The notation used within the current Thesis is as follows. Vectors are indicated with an overhead arrow, and superscript notation refers to vector components (e.g. $\vec{x} = (x^1, x^2, \ldots, x^n)$). Subscripts on scalar functions and vector quantities denote partial differentiation (e.g. $\partial v^i / \partial x^j \equiv v^i_{x^j} \equiv v^i_j$ denotes partial differentiation with respect to x^j). In the case of ordinary differential equations, we will use superscript numerals in brackets to represent differentiation (i.e. $y = y(x)$, then $y^{(k)}$ is the kth-derivative of y). Operators are denoted by capital, non-italic letters (e.g. X).

It is assumed all functions discussed in the current Thesis are infinitely differentiable on their domain (i.e. $f : D \to \mathbb{R}$ a scalar valued function, then $f \in C^\infty(D)$). The domain of any function is assumed an open interval of Euclidean space $\mathbb{R}^n$ such that the standard definition of differentiability within the domain holds.

1.2 Lie Symmetries

We introduce the notions of symmetry and Lie point symmetries. In particular, we focus on the application to equations of real variables.

1.2.1 Geometric Example of Symmetries

We begin by heuristically presenting the notions of symmetry of an object under a transformation, as well as invariance of an object under a transformation.

Consider a circle of radius one centered at the origin in Figure 1.1. Clearly, the circle is symmetric about the y-axis. This reflection transformation can be thought of as a mapping of points on the left half of the circle to points on the right half, and vice versa. The points at coordinate $(x, y) = (0, 1)$ and $(x, y) = (0, -1)$ are called *invariant points* since they are mapped by the reflection transformation into themselves. We say the circle is *invariant* under the reflection transformation since it is mapped from itself into itself. In general, any transformation of under which a geometric object is invariant is called a *symmetry transformation*, and we say the object admits this type of symmetry.
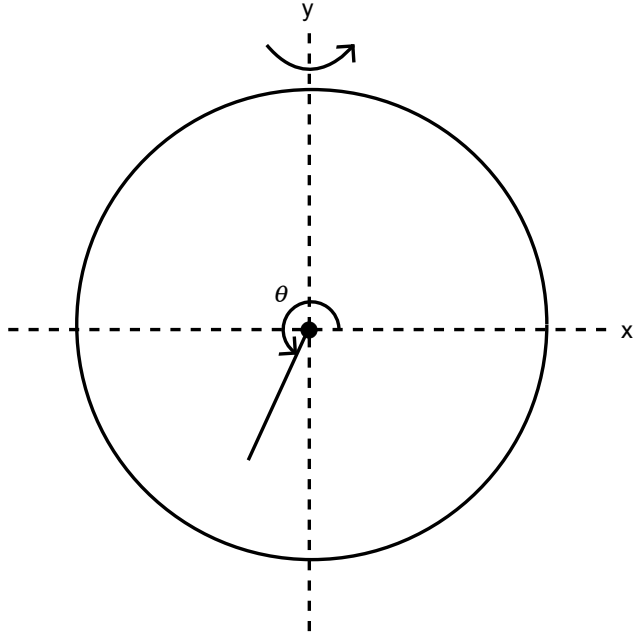
Figure 1.1: Circle with reflection and rotation symmetries shown.

The circle in Figure (1.1) is also invariant under rotations by an angle θ about the origin, and so admits a rotation symmetry. This rotation transformation forms a *continuous symmetry* of the circle since the rotation angle θ is continuous (i.e. $\theta \in \mathbb{R}$). On the other hand, the reflection symmetry of the circle about the y-axis is called a *discrete symmetry*.

1.2.2 Algebraic Formulation of Symmetries

It is important to quantitatively describe a symmetry transformation of an algebraic equation as a precursor to symmetries of differential equations. To do so, we formally define a point transformation, as well as invariance of an algebraic equation under a symmetry transformation.

Definition 1.2.1. A *point transformation* $\vec{X} : D \to D$ is a set of one-to-one functions mapping points $\vec{x} \in D$ to points $\vec{x}^* \in D$. In particular, we write

$$\vec{x}^* = \vec{X}(\vec{x}). \tag{1.1}$$

Definition 1.2.2. A hypersurface $f(\vec{x}) = 0$ is *invariant* under a point transformation (1.1)

if

$$f(\vec{x}^*) = 0 \quad \text{when} \quad f(\vec{x}) = 0. \tag{1.2}$$

A point transformation which leaves a hypersurface invariant is called a *symmetry transformation* of the hypersurface, and we say the hypersurface admits the particular symmetry.

Consider the following examples.

Example 1.2.3. Consider the equation of the circle in Figure 1.1,

$$x^2 + y^2 - 1 = 0. \tag{1.3}$$

We wish to verify reflections about the y-axis is a symmetry transformation.

Reflection of the xy-lane about the y-axis can be written as

$$x^* = X^1(x, y) = -x, \tag{1.4a}$$

$$y^* = X^2(x, y) = y, \tag{1.4b}$$

which is a point transformation since each point (x, y) is mapped uniquely into the point $(x^*, y^*) = (-x, y)$.

Substituting (1.4) into equation (1.3) yields

$$(x^*)^2 + (y^*)^2 - 1 = (-x)^2 + (y)^2 - 1 \equiv 0.$$

As such, the circle (1.3) is invariant under the reflection transformation (1.4).

Example 1.2.4. We wish to verify that rotations about the center of the circle (1.3) in the xy-plane is a symmetry transformation of the circle.

First, rotations of the xy-plane about the origin can be parametrized as the point transformation

$$x^* = x \cos \epsilon - y \sin \epsilon \tag{1.5a}$$

$$y^* = x \sin \epsilon + y \cos \epsilon, \tag{1.5b}$$

where $\epsilon \in \mathbb{R}$. For fixed ϵ, it can be shown (1.5) is a point transformation.

Substituting (1.5) into the equation of the circle (1.3) and simplifying yields:

$$(x^*)^2 + (y^*)^2 - 1 = (\cos \epsilon \, x - \sin \epsilon \, y)^2 + (\sin \epsilon \, x + \cos \epsilon \, y)^2 - 1$$

$$= x^2 \cos^2 \epsilon - 2xy \cos \epsilon \sin \epsilon + y^2 \sin^2 \epsilon + x^2 \sin^2 \epsilon + 2xy \cos \epsilon \sin \epsilon + y^2 \cos^2 \epsilon - 1$$

$$= (\cos^2 \epsilon + \sin^2 \epsilon)(x^2 + y^2) - 1 = x^2 + y^2 - 1 \equiv 0.$$

Thus, $(x^*)^2 + (y^*)^2 - 1 = 0$ when $(x)^2 + (y)^2 - 1 = 0$. As such, the transformation (1.5) is a symmetry of the circle.

1.2.3 Lie Groups of Transformations

With the idea of symmetry for a algebraic equations in tow, we wish to restrict our attention to Lie point symmetries. To accomplish this, we first recall the definition of a group, group of point transformations, and a Lie group of point transformations [23].

Definition 1.2.5. A *group* (G, ϕ) is a set G with an operation $\phi : G \times G \to G$ such that for all $a, b, c \in G$:

- The group is closed: $\phi(a, b) \in G$;

- The operation ϕ is associative: $\phi(\phi(a, b), c) = \phi(a, \phi(b, c))$;

- There exists an identity element $e \in G$ such that $\phi(e, a) = \phi(a, e) = a$; and,

- There exists an inverse element $a^{-1} \in G$ such that $\phi(a^{-1}, a) = \phi(a, a^{-1}) = e$.

Definition 1.2.6. Let (S, ϕ) be a group on $S \subset \mathbb{R}$ with operation ϕ. Consider some $\vec{x} \in D \subset \mathbb{R}^n$ acted on by a continuous point transformation $\vec{X} : D \times S \to D$ such that

$$\vec{x}^* = \vec{X}(\vec{x}, \epsilon), \tag{1.6}$$

where ϵ is the transformation parameter. Then, $\vec{X}$ forms a *group of transformations* on D if for every $\epsilon, \delta \in S$:

- $\vec{X}(\vec{x}, \epsilon)$ is bijective (one to one and onto) on D;

- $\vec{X}(\vec{x}, \epsilon_0) = \vec{x}$, where ϵ_0 is the identity element; and,

- If $\vec{x}^* = \vec{X}(\vec{x}, \epsilon)$ and $\vec{x}^{**} = \vec{X}(\vec{x}^*, \delta)$, then $\vec{x}^{**} = \vec{X}(\vec{x}, \phi(\epsilon, \delta))$.

Definition 1.2.7. A group of transformations is a *one-parameter Lie group of point trans-formations* if:

- $\epsilon \in S$ is a continuous parameter (i.e. S is an interval in $\mathbb{R}$);

- the transformation $\vec{X}$ is infinitely differentiable with respect to $\vec{x} \in D$ and is an analytic function of ϵ in S; and,

- $\phi(\epsilon, \delta)$ is analytic in ϵ and δ, $\epsilon, \delta \in S$.

Definition 1.2.8. For a Lie group of transformations, a smooth function $f(\vec{x})$ is *invariant* under the group of transformations if and only if $f(\vec{x}^*) = f(\vec{x})$. Furthermore, this Lie group of point transformations is a *Lie point symmetry* of $f(\vec{x})$.

For the remainder of the current thesis, we restrict our attention to Lie point symmetries. In addition, we assume the transformations act on the space $D = \mathbb{R}^n$ unless otherwise specified.

Example 1.2.9. We wish to verify the rotation symmetry of the circle (1.5) is a Lie group of point transformations with transformation (1.6), and $\phi(\epsilon, \delta) = (\epsilon + \delta) \bmod 2\pi$ the law of composition for $\epsilon, \delta \in S = [0, 2\pi)$.

First, it is clear that (S, ϕ) is a group.

Second, rotations of the circle (up to $\bmod 2\pi$) form a group of point transformations because: the transformation is bijective on $\mathbb{R}^2$; taking $\epsilon = 0$ yields the identity transformation (i.e. $\vec{X}(\vec{x}, 0) = \vec{x}$); and, it is clear that rotating the circle an angle θ followed by an angle ω is equivalent to rotating the circle an angle $(\theta + \omega) \bmod 2\pi$.

Finally, we must verify the three properties in Definition 1.2.7. By observation, the trans-formation parameter ϵ is continuous since S is an interval in $\mathbb{R}$. As well, the transformation (1.5) is linear in x and y, and so is smooth in $\vec{x}$. Additionally, the transformation is ana-lytic in ϵ because sine and cosine are analytic functions. Lastly, ϕ is analytic in both of its arguments.

Thus, we have verified rotations of the circle about its center forms a Lie group of point transformations.

Without a loss of generality, we may assume each Lie group of point transformations to have the operation $\phi(a, b) = a + b$ for $a, b \in \mathbb{R}$, as done in [21–23].

1.2.4 Infinitesimal Generators

A one-parameter Lie group of point transformations can be uniquely characterized by its infinitesimal generator. Infinitesimal generators are important because they allow one to algorithmically determine the admitted Lie symmetries of a given equation. This naturally allows for implementation in computer algebra systems, which makes feasible the computation of Lie symmetries admitted by large systems of equations. In the following, we present the infinitesimal generator and its connection to one parameter Lie groups of point transformations.

We now expand a Lie group of point transformations $\vec{X}(\vec{x}, \epsilon)$ about $\epsilon = 0$ for small ϵ in a Taylor series:

$$\vec{x}^* = \vec{x} + \epsilon \left. \frac{\partial \vec{X}}{\partial \epsilon}(\vec{x}, \epsilon) \right|_{\epsilon=0} + O(\epsilon^2). \tag{1.7}$$

Let

$$\vec{\xi}(\vec{x}) = \left. \frac{\partial \vec{X}}{\partial \epsilon} \right|_{\epsilon=0}, \tag{1.8}$$

which is the *infinitesimal* of the *infinitesimal transformation* $\vec{x}^* \approx \vec{x} + \epsilon \vec{\xi}$. A Lie group of point transformations $\vec{X}(\vec{x}, \epsilon)$ is characterized locally by its infinitesimal transformation $\vec{x}^* \approx \vec{x} + \epsilon \vec{\xi}$, which is shown in the following theorem.

Theorem 1.2.10. *A one parameter Lie group of transformations with the parametrization* $\phi(a, b) = a + b$ *is equivalent to the solution of the initial value problem*

$$\frac{d\vec{x}^*}{d\epsilon} = \vec{\xi}(\vec{x}^*), \qquad \vec{x}^*|_{\epsilon=0} = \vec{x}. \tag{1.9}$$

The proof, under more general conditions, can be found in [23].

Of importance, Theorem 1.2.10 provides a method with which to obtain the global Lie group of point transformations exactly from a known infinitesimal transformation. With this in mind, we present the infinitesimal generator and theorems showing how it characterizes a Lie group of point transformations.

Definition 1.2.11. For a one-parameter Lie group of transformations, the associated *infinitesimal generator* is given by the linear differential operator

$$\mathrm{X} = \mathrm{X}(\vec{x}) = \vec{\xi}(\vec{x}) \cdot \nabla = \sum_{i=1}^{n} \xi^i(\vec{x}) \frac{\partial}{\partial(x^i)}, \qquad (1.10)$$

where

$$\nabla = \left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n} \right).$$

Note X acts on differentiable functions $f(\vec{x})$. In particular, $\vec{\xi}(\vec{x}) = \mathrm{X}(\vec{x})\vec{x}$.

Theorem 1.2.12. *A one-parameter Lie group of transformations* (1.6) *with infinitesimal generator*

$$\mathrm{X} = \mathrm{X}(\vec{x}) = \sum_{i=1}^{n} \xi^i(\vec{x}) \frac{\partial}{\partial(x^i)}$$

is equivalent to

$$\vec{x}^* = e^{\epsilon \mathrm{X}} \vec{x}, \qquad (1.11)$$

where

$$e^{\epsilon \mathrm{X}} \equiv \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \mathrm{X}^i \vec{x},$$

and X^i *is* X *applied* i *times.*

Proof. Let

$$\mathrm{X}(\vec{x}^*) = \sum_{i=1}^{n} \xi^i(\vec{x}^*) \frac{\partial}{\partial(x^i)^*}.$$

The Taylor expansion of (1.6) about $\epsilon = 0$ is

$$\vec{x}^* = \sum_{i=1}^{\infty} \frac{\epsilon^i}{i!} \left(\frac{\partial^i \vec{X}(\vec{x}, \epsilon)}{\partial \epsilon^i} \bigg|_{\epsilon=0} \right) = \sum_{i=1}^{\infty} \frac{\epsilon^i}{i!} \left(\frac{d^i \vec{x}^*}{d\epsilon^i} \bigg|_{\epsilon=0} \right). \qquad (1.12)$$

Consider that, for any differentiable function $F(\vec{x})$,

$$\frac{d}{d\epsilon} F(\vec{x}^*) = \sum_{i=1}^{n} \frac{\partial F(\vec{x}^*)}{\partial(x^i)^*} \frac{d(x^i)^*}{d\epsilon} \qquad \bigg| \text{ substitute } \quad (1.9)$$

$$= \sum_{i=1}^{n} \xi^i(\vec{x}^*) \frac{\partial F(\vec{x}^*)}{\partial(x^i)^*}$$

$$\equiv \mathrm{X}(\vec{x}^*) F(\vec{x}^*).$$

Hence,

$$\frac{d\vec{x}^*}{d\epsilon} = X(\vec{x}^*)\,\vec{x}^*,$$

$$\frac{d^2\vec{x}^*}{d\epsilon^2} = \frac{d}{d\epsilon}\left(\frac{d\vec{x}^*}{d\epsilon}\right) = X(\vec{x}^*)(X(\vec{x}^*)\,\vec{x}^*) = X^2(\vec{x}^*)\,\vec{x}^*.$$

Proceeding by induction, one has

$$\frac{d^i\vec{x}^*}{d\epsilon^i} = X^i(\vec{x}^*)\,\vec{x}^*, \quad i = 1, 2, \ldots.$$

As such, one has

$$\left.\frac{d^i\vec{x}^*}{d\epsilon^i}\right|_{\epsilon=0} = X^i(\vec{x})\,\vec{x} = X^i\,\vec{x}, \quad i = 1, 2, \ldots. \tag{1.13}$$

Substituting (1.13) into (1.12) yields

$$\vec{x}^* = \left(\sum_{i=1}^{\infty} \frac{\epsilon^i}{i!} X^i\right)\vec{x}, \tag{1.14}$$

which is the identity (1.11). $\qquad\square$

Corollary 1.2.13. *Given $F(\vec{x})$ an infinitely differentiable function, then for a Lie group of transformations (1.6) with infinitesimal generator* X,

$$F(\vec{x}^*) = F(e^{\epsilon X}\vec{x}) = e^{\epsilon X}F(\vec{x}). \tag{1.15}$$

The proof appears in [21].

In light of the previous theorem and corollary, invariance under a Lie group of point transformations can be determined with the associated infinitesimal generator, given explicitly in the following theorem.

Theorem 1.2.14. *For a Lie group of point transformations with infinitesimal generator* X, *then:*

1. *A hypersurface $F(\vec{x}) = 0$ is invariant under the group of transformations (as in Definition (1.2.2) if and only if*

$$XF(\vec{x}) = 0 \quad when \quad F(\vec{x}) = 0. \tag{1.16}$$

2. A function $f(\vec{x})$ is invariant under the group of transformations (as in Definition (1.2.8)) if and only if

$$\mathrm{X}f(\vec{x}) \equiv 0. \tag{1.17}$$

Proof. From Theorem (1.2.12) and Corollary (1.2.13), it follows that

$$F(\vec{x}^*) \equiv F(x) + \epsilon \mathrm{X}F(\vec{x}) + \epsilon^2 \mathrm{X}^2 F(\vec{x}) + \dots. \tag{1.18}$$

To prove part 1, first let $F(\vec{x}^*) = 0$ when $F(\vec{x}) = 0$. Then, from equation (1.18), we have that

$$0 = \epsilon \mathrm{X}F(\vec{x}) + \epsilon^2 \mathrm{X}^2 F(\vec{x}) + \dots,$$

where ϵ is arbitrary. Thus, $\mathrm{X}F(\vec{x}) = 0$.

Conversely, if $\mathrm{X}F(\vec{x}) = 0$ when $F(\vec{x}) = 0$, it follows from (1.18) that $F(\vec{x}^*) = 0$.

Similarly, to prove part 2, suppose that $f(\vec{x}^*) = f(\vec{x})$. Then, from (1.18) with $f = F$, it follows that

$$0 = \epsilon \mathrm{X}f(\vec{x}) + \epsilon^2 \mathrm{X}^2 f(\vec{x}) + \dots,$$

where ϵ is arbitrary. As such, $\mathrm{X}f(\vec{x}) = 0$.

Conversely, assume $\mathrm{X}f(\vec{x}) = 0$. Then, from (1.18) with $f = F$, it follows that $f(\vec{x}^*) = f(\vec{x})$.

$\square$

As such, the action of a one parameter Lie group of point transformations on a hypersurface or function is characterized by the associated infinitesimal generator.

Consider the following examples.

Example 1.2.15. We wish to determine the infinitesimal generator

$$\mathrm{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

associated with the rotation Lie group of point transformations

$$x^* = x\cos\epsilon - y\sin\epsilon \tag{1.19a}$$

$$y^* = x\sin\epsilon + y\cos\epsilon. \tag{1.19b}$$

11

By Theorem 1.2.10, the infinitesimals ξ and η are

$$\left.\frac{dx^*}{d\epsilon}\right|_{\epsilon=0} = -y, \tag{1.20a}$$

$$\left.\frac{dy^*}{d\epsilon}\right|_{\epsilon=0} = x. \tag{1.20b}$$

As such, the infinitesimal generator X associated with (1.19) is

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \tag{1.21}$$

Example 1.2.16. We wish to recover the global Lie group of point transformations, as well as the invariant functions, of the infinitesimal generator

$$X = x\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}. \tag{1.22}$$

The global transformation on (x, u)-space is obtained by solving the initial value problem (IVP) from Theorem 1.2.10,

$$\frac{dx^*}{d\epsilon} = x^*, \quad x^*|_{\epsilon=0} = x, \tag{1.23a}$$

$$\frac{du^*}{d\epsilon} = x^*, \quad u^*|_{\epsilon=0} = u. \tag{1.23b}$$

The IVP (1.23a) admits the solution

$$x^* = e^\epsilon x.$$

Substituting this solution into (1.23b) and noting x is constant with respect to ϵ, we obtain u^* as

$$u^* = e^\epsilon x + u - x.$$

As such, the global group of transformations associated with (1.22) is

$$x^* = e^\epsilon x,$$

$$u^* = u + (e^\epsilon - 1)x.$$

For a function $f(x, u)$ to be an invariant of (1.22), it is required that $Xf \equiv 0$. Hence, f is defined by the linear advection PDE

$$x\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial u} = 0,$$

which is solved by the method of characteristics as

$$f(x, u) = f(u - x). \tag{1.24}$$

1.2.5 Canonical Coordinates

In the reduction of ODEs, it is convenient to express the equations in terms of *canonical coordinates* of a Lie group of point transformations. In the current section, the notions of canonical coordinates of a Lie group of point transformations are developed.

Theorem 1.2.17. *Given a Lie group of point transformations* (1.6) *with infinitesimal generator* X, *then a function* $f(\vec{x})$ *undergoes the transformation*

$$f(\vec{x}^*) \equiv f(\vec{x}) + \epsilon,$$

if and only if

$$\mathrm{X}f(\vec{x}) \equiv 1. \tag{1.25}$$

Suppose we make one-to-one, continuously differentiable change of coordinates from $\vec{x} = (x^1, \ldots, x^n)$ to $\vec{y} = (y^1(\vec{x}), \ldots, y^n(\vec{x}))$. A Lie group of point transformations with the infinitesimal generator X in coordinates $\vec{x}$ then becomes the infinitesimal generator Y in the coordinates $\vec{y}$, where

$$\mathrm{Y} = \vec{\eta} \cdot \left(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right). \tag{1.26}$$

Note that $\vec{\eta}(\vec{y}) = \mathrm{Y}\vec{y}(\vec{x})$.

Canonical coordinates of a Lie group of point transformations are defined in the following Theorem.

Theorem 1.2.18. *For any Lie group of point transformations on coordinates* $\vec{x}$ *with infinitesimal generator* X, *there exists canonical coordinates* $\vec{y} = \vec{y}(\vec{x})$ *with infinitesimal generator*

$$\mathrm{Y} = \vec{\eta}(\vec{y}) \cdot \left(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right)$$

such that

$$\mathrm{Y}y^i = 0 \quad i = 1, 2, \ldots, n-1,$$
$$\mathrm{Y}y^n = 1.$$

Proof of the above theorem appears in [23].

Example 1.2.19. Consider the group of rotations with infinitesimal generator

$$\mathrm{X} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \tag{1.27}$$

We seek the canonical coordinates $(r(x, y), s(x, y))$ such that $\mathrm{X}r = 0$ and $\mathrm{X}s = 1$.

The equation $\mathrm{X}r = 0$ yields the characteristic system

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dr}{0},$$

which admits the particular solution

$$r = \sqrt{x^2 + y^2}. \tag{1.28}$$

The equation $\mathrm{X}s = 1$

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{ds}{1}.$$

As such, a candidate for s is a particular solution to

$$\frac{ds}{dy} = \frac{1}{x} = \frac{1}{\sqrt{r^2 - y^2}}.$$

As such,

$$s = \sin^{-1}\left(\frac{y}{r}\right). \tag{1.29}$$

Thus, the canonical coordinates of (1.27) are polar coordinates

$$(r, s) = \left(\sqrt{x^2 + y^2}, \sin^{-1}\left(\frac{y}{r}\right)\right). \tag{1.30}$$

1.2.6 Multi-Parameter Lie Groups of Point Transformations

When a given function or equation admits multiple one-parameter Lie groups of point transformations, they may be combined into a multi-parameter Lie group of point transformations. The collection of infinitesimal generators for the multi-parameter Lie group of point transformations forms a Lie algebra with the additional operation of the commutator. Lie algebras play an important role in multiple reductions of order of ordinary differential equations (discussed in Section 1.3.4). Of importance is if the Lie algebra, or a subalgebra thereof, is solvable.

For our purposes, we do not discuss multi-parameter Lie groups, but rather focus on finite dimensional Lie algebras.

Definition 1.2.20. Consider two infinitesimal generators X and Y which act on a function $f(\vec{x})$, $\vec{x} \in \mathbb{R}^n$. Then, the *commutator* of the two infinitesimal generators is

$$[X, Y] = XY - YX.$$

Definition 1.2.21. An r-dimensional *Lie Algebra* $\mathfrak{L}^{(r)}$ is a vector space over the field $\mathbb{R}$ spanned by the infinitesimal generators of an r-parameter Lie group of transformations with the additional law of composition of its elements given by the commutator. In particular, for infinitesimal generators X, Y, Z $\in \mathfrak{L}^{(r)}$ and constants α, $\beta \in \mathbb{R}$, the following properties hold [23]:

- Closure: $[X, Y] \in \mathfrak{L}^{(r)}$;

- Anticommutivity: $[X, Y] = -[Y, X]$;

- Linearity: $[\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]$; and,

- Jacobi Identity: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

Definition 1.2.22. A *subalgebra* is a subspace $\mathfrak{K}$ of the Lie algebra such that for any X, Y $\in$ $\mathfrak{K}$, then $[X, Y] \in \mathfrak{K}$.

Definition 1.2.23. An *ideal* subalgebra $\mathfrak{K}$ of $\mathfrak{L}^{(r)}$ is a subset of infinitesimal generators in $\mathfrak{L}^{(r)}$ such that for any X $\in \mathfrak{K}$ and Y $\in \mathfrak{L}^{(r)}$, then $[X, Y] \in \mathfrak{K}$.

Definition 1.2.24. An r-dimensional Lie algebra $\mathfrak{L}^{(r)}$ is *solvable* if there exists a chain of subalgebras

$$\mathfrak{L}^{(1)} \subset \mathfrak{L}^{(2)} \subset \cdots \subset \mathfrak{L}^{(r)}$$

where $\mathfrak{L}^{(k-1)}$ is a $(k-1)$-dimensional ideal subalgebra of $\mathfrak{L}^{(k)}$, $k = 1, \ldots, r$.

Definition 1.2.25. A commutator table of an r-parameter Lie algebra $\mathfrak{L}^r$ with basis $\{X^i\}_{i=1}^r$ is a table of the form

	X^1	X^2	...	X^r
X^1	0	$[X^1, X^2]$	...	$[X^1, X^r]$
X^2	$[X^2, X^1]$	0	...	$[X^2, X^r]$
⋮	⋮	0	⋱	⋮
X^r	$[X^r, X^1]$	$[X^r, X^2]$	...	0

where the $(i,j)^{\text{th}}$-entry corresponds to the commutator $[X^i, X^j]$.

Note the diagonal of zeros is due to $[X^i, X^i] \equiv 0$ for $i = 1, 2, \ldots, r$. Also, it is important to note the commutator table is antisymmetric about the diagonal of zeros.

The structure of a Lie algebra can be viewed more easily through its commutator table. Consider this in the following example.

Example 1.2.26. Consider the three-parameter Lie Algebra $\mathfrak{L} = \{X^1, X^2, X^3\}$ on (x, y)-space, where

$$X^1 = \frac{\partial}{\partial x}, \quad X^2 = \frac{\partial}{\partial y}, \quad X^3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

The commutator table of $\mathfrak{L}$ is:

	X^1	X^2	X^3
X^1	0	0	$-X^2$
X^2	0	0	X^1
X^3	X^2	$-X^1$	0

Clearly, X^1 and X^2 commute. For infinitesimal $Y \in \mathfrak{L}^2 = \{X^1, X^2\}$ a subalgebra, we have that $[X^3, Y] \in \mathfrak{L}^2$. From these observations, we have the chain of subalgebras

$$\{X^1\} \subset \{X^1, X^2\} \subset \{X^1, X^2, X^3\} = \mathfrak{L}.$$

As such, the Lie algebra $\mathfrak{L}$ is solvable.

16

1.3 Lie Point Symmetries of Differential Equations

We now focus on the application of Lie symmetries to differential equations. We first present the extension of a Lie groups of point transformations to derivatives, and how to algorithmically derive the admitted Lie point symmetries of a PDE system. Afterwards, we discuss how algorithmically construct solutions to a scalar ODE from an admitted Lie symmetry. In particular, it is shown that the order of an ODE can be reduced by one through canonical coordinates of an admitted Lie symmetry, or through the differential invariants the symmetry. We also comment upon considerations to be taken in multiple reductions of order of an ODE admitting multiple one-parameter Lie symmetries. Lastly, construction of invariant solutions for a PDE system through invariant forms is discussed.

After, we present how to reduce the number of independent variables of partial differential equations that are invariant under Lie point symmetries.

From now on, we consider Lie groups of point transformations on the space of n independent variables $\vec{x}$ and m dependent variables $\vec{u}$. Here, each $u^i = u^i(\vec{x})$ are smooth (C^∞) functions on $\mathbb{R}^n$. Additionally, we assume equality of mixed partials (i.e. $u^i_{jk} = u^i_{kj}$, where $u^i_j = \partial u^i / \partial x^j$). As such, in any sum where mixed partials occur, it is understood that only one permutation is counted; that is,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} u_{ij} = u_{11} + \cdots + u_{1n} + u_{22} + \cdots + u_{2n} + u_{33} + \cdots + u_{3n} + \cdots + u_{nn}$$

We first establish some additional notation.

Definition 1.3.1. Define the ordered set of all partial derivatives of u^i as

$$\partial u^i = \left\{ u^i_1, \ldots, u^i_n \right\}.$$

Similarly, we denote the ordered set of all k^{th} partial derivatives of u^i as

$$\partial^k u^i = \left\{ u^i_{i^1,\ldots,i^k} \mid\ i^1, \ldots, i^k = 1, \ldots, n \right\}.$$

Additionally, the ordered set of all k^{th} partial derivatives of $\vec{u}$ are denoted as

$$\partial^k \vec{u} = \left\{ u^\mu_{i^1,\ldots,i^k} \mid\ \mu = 1, \ldots, m;\ \ i^1, \ldots, i^k = 1, \ldots, n \right\}.$$

By $\vec{R} = \vec{R}(\vec{x}, \partial\vec{u}, \ldots, \partial^k \vec{u}) = 0$, we mean a system of σ PDEs with n-independent variables, m-dependent variables, and with derivatives of at most k^{th}-order. As well, we assume each PDE in the system is written in *solved form* with respect to some highest partial derivative of $(l^j)^{\text{th}}$-order; that is,

$$R^j = u^{p^j}_{i^1, \ldots, i^{l^j}} - f^j(\vec{x}, \vec{u}, \partial\vec{u}, \ldots, \partial^k \vec{u}) = 0, \quad j = 1, \ldots, \sigma, \tag{1.31}$$

where each $f^j(\vec{x}, \vec{u}, \partial\vec{u}, \ldots, \partial^k \vec{u})$ does not depend on $u^{p^j}_{i^1, \ldots, i^{l^j}}$, $p^j \in \{1, \ldots, m\}$, and each $i^\nu \in \{1, \ldots, k\}$ for $\nu = 1, \ldots, j$.

Definition 1.3.2. Define the *differential consequences* of a PDE system $\vec{R} = 0$ as all derivatives (including higher order derivatives) of the system with respect to independent variables; that is, the differential consequences of a PDE system refers to the set

$$\partial\vec{R} = \left\{ \frac{\partial^k}{x^{i^1} \ldots x^{i^k}} R^j \;\middle|\; j = 1, \ldots, \sigma, \;\; i^1, \ldots, i^k = 1, \ldots, n, \;\; k = 1, 2, 3, \ldots \right\}. \tag{1.32}$$

Note each element of $\partial\vec{R}$ is identically zero since $\vec{R} = 0$. In an abuse of notation, we write $\partial\vec{R} = 0$ as a shorthand to indicate each element of $\partial\vec{R}$ is zero.

1.3.1 Prolongation of Lie Groups of Point Transformations

For a given Lie group of point transformations on $(\vec{x}, \vec{u})$-space with $\vec{u} = \vec{u}(\vec{x})$, one can extend (i.e. prolong) the transformation to act on derivatives $\partial^j \vec{u}$ for $j = 1, 2, 3, \ldots$ such that contact conditions are preserved. In the current section, the prolongation of the infinitesimal transformation and its associated infinitesimal generator are presented.

First, we introduce the following operator.

Definition 1.3.3. Consider a function $F(\vec{x}, \vec{u}, \partial\vec{u}, \ldots, \partial^k \vec{u})$ with n independent variables $\vec{x} = (x^1, \ldots, x^n)$ and m dependent variables $\vec{u}(\vec{x}) = (u^1(\vec{x}), \ldots, u^n(\vec{x}))$. Then, the *total derivative* D_{x^i} (alternatively D_i) of F with respect to x^i is

$$\mathrm{D}_i F = \frac{\partial F}{\partial x^i} + u^j_i \frac{\partial F}{\partial u^j} + \cdots + u^j_{i\, l^1\, l^2 \ldots l^k} \frac{\partial F}{\partial u^j_{l^1\, l^2 \ldots l^k}}, \tag{1.33}$$

where we have assumed summation across repeated indices, $j = 1, \ldots, m$, and $l^p = 1, \ldots, n$ for $p = 1, \ldots, k$. It is assumed that no mixed partial derivative occurs more than once due to equivalence of mixed partial derivatives.

The *extended* (or *prolonged*) infinitesimal transformation associated with a one-parameter Lie group of point transformations is defined as follows.

Definition 1.3.4. Consider a one-parameter Lie group of point transformations on n independent variables $\vec{x}$ and m dependent variables $\vec{u}(\vec{x})$

$$(x^i)^* = X^i(\vec{x}, \vec{u}, \epsilon) = x^i + \epsilon\xi^i(\vec{x}, \vec{u}) + O(\epsilon^2), \tag{1.34a}$$

$$(u^j)^* = U^j(\vec{x}, \vec{u}, \epsilon) = u^j + \epsilon\eta^j(\vec{x}, \vec{u}) + O(\epsilon^2) \tag{1.34b}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The corresponding infinitesimal generator is

$$X = \sum_{i=1}^{n} \xi^i(\vec{x}, \vec{u})\frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \eta^j(\vec{x}, \vec{u})\frac{\partial}{\partial u^j}. \tag{1.35}$$

Then, the pth *prolonged infinitesimal transformation* on $(\vec{x}, \vec{u}, \partial\vec{u}, \ldots, \partial^p\vec{u})$-space is

$$(x^i)^* = X^i(\vec{x}, \vec{u}, \epsilon) = x^i + \epsilon\xi^i(\vec{x}, \vec{u}) + O(\epsilon^2), \tag{1.36a}$$

$$(u^j)^* = U^j(\vec{x}, \vec{u}, \epsilon) = u^j + \epsilon\eta^j(\vec{x}, \vec{u}) + O(\epsilon^2) \tag{1.36b}$$

$$(u^j_{i^1})^* = U^j_{i^1}(\vec{x}, \vec{u}, \partial\vec{u}, \epsilon) = u^j_{i^1} + \epsilon\eta^{(1)\,j}_{i^1}(\vec{x}, \vec{u}, \partial\vec{u}) + O(\epsilon^2), \tag{1.36c}$$

$$\vdots \tag{1.36d}$$

$$(u^j_{l^1 l^2 \ldots l^k})^* = U^j_{l^1 l^2 \ldots l^k}(\vec{x}, \vec{u}, \partial\vec{u}, \ldots, \partial^k\vec{u}, \epsilon)$$

$$= u^j_i + \epsilon\eta^{(k)\,j}_{l^1 l^2 \ldots l^k}(\vec{x}, \vec{u}, \partial\vec{u}, \ldots, \partial^k\vec{u}) + O(\epsilon^2) \tag{1.36e}$$

where $i^q = 1, \ldots, n$ for $q = 1, \ldots, k$, $j = 1, \ldots, m$, $k = 1, \ldots, p$. Here, subscripts on $U^j_{l^1 l^2 \ldots l^k}$ and the *prolonged infinitesimals* $\eta^{j\,(k)}$ indicate different quantities, not partial derivatives. Each infinitesimal $\eta^{(k)\,j}_{i_1 i_2 \ldots i_k}$ can be found as

$$\eta^{(1)\,j}_i = \left(D_i\eta^j\right) - \left(D_i\xi^l\right)u_l, \tag{1.37a}$$

$$\vdots \tag{1.37b}$$

$$\eta^{(k)\,j}_{i^1 \ldots i^k} = \left(D_{i^k}\eta^{(k-1)\,j}_{i^1 \ldots i^{k-1}}\right) - \left(D_{i^k}\xi^l\right)u_{i^1 \ldots i^{k-1}\,l}, \tag{1.37c}$$

where summation across $l = 1, \ldots, n$ is assumed. Note $\eta^{(0)\,j} \equiv \eta^j$.

The pth *prolonged infinitesimal generator* is then

$$X^{(p)} = X + \eta^{(1)\,j}_{i^1}\frac{\partial}{\partial u^j_i} + \cdots + \eta^{(k)\,j}_{i^1 i^2 \ldots i^k}\frac{\partial}{\partial u^j_{i^1 i^2 \ldots i^k}} + \cdots + \eta^{(p)\,j}_{i^1 i^2 \ldots i^p}\frac{\partial}{\partial u^j_{i^1 i^2 \ldots i^p}}, \tag{1.38}$$

19

where summation is assumed across $j = 1, \ldots, m$, and $i^q = 1, \ldots, n$ for $q = 1, \ldots, k$ and $k = 1, \ldots, p$. Note order matters, and that no mixed partial appears more than once in the sum (i.e. only $\eta_{12}^{(1)\,1}$ appears in X^p, but not $\eta_{21}^{(1)\,1}$).

We now derive equation (1.37a) as done in [22, 23].

To "naturally" extend the transformation (1.34) on $(\vec{x}, \vec{u}, \partial\vec{u})$-space, we require the contact conditions to be preserved. That is,

$$du^j = \sum_{i=1}^{n} u_i^j \, dx^i, \quad j = 1, \ldots, m, \tag{1.39}$$

if and only if

$$d(u^j)^* = \sum_{i=1}^{n} (u_i^j)^* \, d(x^i)^*, \quad j = 1, \ldots, m. \tag{1.40}$$

From (1.36), we obtain

$$d(u^j)^* = dU^j(\vec{x}, \vec{u}, \epsilon) = \sum_{i=1}^{n} \left(\frac{\partial U^j}{\partial x^i} \, dx^i \right) + \sum_{l=1}^{n} \frac{\partial U^j}{\partial u^l} \, du^l \quad \Bigg| \text{ substitute (1.39)}$$

$$= \sum_{i=1}^{n} \left(\frac{\partial U^j}{\partial x^i} \, dx^i + \sum_{l=1}^{n} \frac{\partial U^j}{\partial u^l} \, u_i^l dx^i \right),$$

which can be written as

$$d(u^j)^* = \sum_{i=1}^{n} D_i U^j \, dx^i, \tag{1.41}$$

where D_i is the total derivative (1.33).

Similarly, from (1.36) we obtain

$$d(x^j)^* = \sum_{i=1}^{n} D_i X^j \, dx^i. \tag{1.42}$$

Let $\mathbf{A}$ be the $n \times n$ matrix with entries

$$\mathbf{A}_{ij} = D_i X^j. \tag{1.43}$$

Assuming $\mathbf{A}$ is invertible, one may substitute (1.41) and (1.42) into (1.40) and solve for $(u_i^j)^*$ as

$$\begin{bmatrix} (u_1^j)^* \\ \vdots \\ (u_n^j)^* \end{bmatrix} = \mathbf{A}^{-1} \begin{bmatrix} D_1 U^k \\ \vdots \\ D_n U^k \end{bmatrix}. \tag{1.44}$$

For the infinitesimal transformation (1.36), the matrix $\mathbf{A}$ becomes $\mathbf{A} = I + \epsilon \mathbf{B} + O(\epsilon^2)$ where

$$\mathbf{B}_{ij} = D_i \xi^j.$$

As such, for small ϵ, the matrix $\mathbf{A}^{-1}$ is

$$\mathbf{A}^{-1} = \mathbf{I} - \epsilon \mathbf{B} + O(\epsilon^2). \tag{1.45}$$

Now, substituting the infinitesimal transformation (1.36) and (1.45) into (1.44) yields

$$\begin{bmatrix} u_1^j + \epsilon \eta_1^{(1)j} + O(\epsilon^2) \\ \vdots \\ u_1^j + \epsilon \eta_1^{(1)j} + O(\epsilon^2) \end{bmatrix} = (\mathbf{I} - \epsilon \mathbf{B} + O(\epsilon^2)) \begin{bmatrix} u_1^j + \epsilon D_1 \eta^j + O(\epsilon^2) \\ \vdots \\ u_n^j + \epsilon D_n \eta^j + O(\epsilon^2) \end{bmatrix}. \tag{1.46}$$

Hence, the order ϵ terms equate as

$$\begin{bmatrix} \eta_1^{(1)j} \\ \vdots \\ \eta_1^{(1)j} \end{bmatrix} = \begin{bmatrix} D_1 \eta^j \\ \vdots \\ D_n \eta^j \end{bmatrix} - \begin{bmatrix} \sum_{i=1}^{n} u_i^j D_1 \xi^i \\ \vdots \\ \sum_{i=1}^{n} u_i^j D_n \xi^i \end{bmatrix}. \tag{1.47}$$

The components of (1.47) are exactly the prolonged infinitesimals given in (1.37a).

An example of calculating the prolongation of an infinitesimal generator is given in Section 1.3.2 for the heat equation.

1.3.2 Symmetries of Differential Equations

We present the essential theorems for invariance of a differential equation under a Lie group of point transformation, as well as the algorithm to find Lie point symmetries.

Definition 1.3.5. A kth-order PDE system $\vec{R} = 0$ is invariant under a one-parameter Lie group of transformations if and only if the PDE system is invariant under the kth-extension of the Lie group.

Theorem 1.3.6. *A kth-order PDE system $\vec{R} = 0$ (1.31) is invariant under a one-parameter Lie group of point transformations with infinitesimal generator* X *and k^{th}-extension* $X^{(k)}$ *(given by Definition 1.3.4) if and only if*

$$X^{(k)} R^j = 0 \quad \text{when} \quad \vec{R} = 0 \text{ and } \partial \vec{R} = 0, \quad \forall\, j = 1, \ldots, \sigma. \tag{1.48}$$

The proof appears in [96].

Algorithm to Find Lie Symmetries

Theorem 1.3.6 leads to an algorithm to determine Lie symmetries of a PDE system $\vec{R} = 0$ in solved form (1.31). This algorithm is as follows:

1. Construct the general k^{th} extended infinitesimal generator $X^{(k)}$ in terms of arbitrary infinitesimals $\xi(\vec{x}, \vec{u})$ and $\eta(\vec{x}, \vec{u})$ using Definition 1.3.4.

2. Apply the prolonged infinitesimal generator to the PDE system $\vec{R} = 0$ and simplify. Then, substitute the PDE system $\vec{R} = 0$ and its differential consequences $\partial \vec{R} = 0$. By Theorem 1.3.6, the infinitesimals ξ^i and η^j for which this equation holds are infinitesimals of a Lie group of point transformations.

3. Simplify the resulting equation while collecting with respect to derivatives of $\vec{u}$. Since ξ and η are functions of $\vec{x}$ and $\vec{u}$, the coefficients of each independent function of derivatives $\vec{u}$ is identically zero. This yields a system of differential equations in ξ^i and η^j called the *system of determining equations* for the Lie symmetries of the PDE system $\vec{R} = 0$.

4. Solve the system of determining equations for ξ^i and η^j.

It is important to note that for ODEs of order $n \geq 2$, as well as for PDEs, the system of determining equations for Lie point symmetries is an over-determined system of linear homogeneous PDEs [23]. In addition, the general solution of the determining equations will be one of three cases: if $\xi^i = 0$ and $\eta^j = 0$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, then the general solution is the trivial solution; if there are a finite number of parameters in the general solution, say r, then the PDE system admits r one-parameter Lie symmetries;

or, if the general solution contains an infinite number of parameters, or contains arbitrary functions, then we say it contains an infinite number of one-parameter Lie symmetries.

Consider a scalar linear homogeneous PDE $F(\vec{x}, u, \partial u, \ldots, \partial^k u) = 0$ with solution $u = g(\vec{x})$. Then, a linear non-homogeneous scalar PDE of the form $F(\vec{x}, u, \partial u, \ldots, \partial^k u) = f(\vec{x})$ always admits the Lie group of point transformations [23]

$$\vec{x}^* = \vec{x}, \quad u^* = u + \epsilon g(\vec{x})$$

with infinitesimal generator $\mathrm{X} = g(\vec{x})\partial/\partial u$.

Consider now the following example of the above algorithm.

Example 1.3.7. We now derive the symmetries of the heat equation,

$$u_t - u_{xx} = 0, \tag{1.49}$$

where $u = u(x, t)$.

An arbitrary infinitesimal generator has the form

$$\mathrm{X} = \xi^x(x, t, u)\frac{\partial}{\partial x} + \xi^t(x, t, u)\frac{\partial}{\partial t} + \eta(x, t, u)\frac{\partial}{\partial u},$$

with corresponding second prolongation

$$\mathrm{X}^{(2)} = \mathrm{X} + \eta_x^{(1)}\frac{\partial}{\partial u_x} + \eta_t^{(1)}\frac{\partial}{\partial u_t} + \eta_{xx}^{(2)}\frac{\partial}{\partial u_{xx}} + \eta_{xt}^{(2)}\frac{\partial}{\partial u_{xt}} + \eta_{tt}^{(2)}\frac{\partial}{\partial u_{tt}}, \tag{1.50}$$

where the prolonged infinitesimals are given in terms of ξ^x, ξ^t, and η by (1.37c).

The particular ξ^x, ξ^t, and η for which X is an infinitesimal generator corresponding to a Lie symmetry of (1.49) must satisfy the invariance criteria (1.48). After simplifying, (1.49) with (1.49), one obtains the equations

$$0 = \eta_t - \eta_{xx} + (\xi_{xx}^x - \xi_t^x - 2\eta_{xu})u_x + (2\xi_x^x + \xi_{xx}^t - \xi_t^t)u_{xx} + 2\xi_x^t u_{xxx}$$

$$+ (2\xi_{xu}^x - \eta_{uu})(u_x)^2 + (\xi_{uu}^t)(u_x)^2 u_{xx} + (\xi_{uu}^x)(u_x)^3 + 2(\xi_u^x + \xi_{xu}^t)u_x u_{xx} + 2\xi_u^t u_x u_{xxx}.$$

23

As such, the system of determining equations is

$$\xi_u^t = 0, \tag{1.51a}$$

$$\xi_x^t = 0, \tag{1.51b}$$

$$\xi_u^x = 0, \tag{1.51c}$$

$$\eta_{uu} = 0, \tag{1.51d}$$

$$2\xi_x^x + \xi_{xx}^t - \xi_t^t = 0, \tag{1.51e}$$

$$\xi_{xx}^x - \xi_t^x - 2\eta_{xu} = 0, \tag{1.51f}$$

$$\eta_t - \eta_{xx} = 0. \tag{1.51g}$$

Solving the system of determining equations yields

$$\xi^x = \epsilon^1 + \epsilon^4 x + 2\epsilon^5 t + 4\epsilon^6 xt$$

$$\xi^t = \epsilon^2 + 2\epsilon^4 t + 4\epsilon^6 t^2$$

$$\eta = \epsilon^3 u - \epsilon^5 xu - \epsilon^6 \left(x^2 + 2t\right) u + g(x, t).$$

where ϵ^i are constants, and $g(x, t)$ is such that $g_t - g_{xx} = 0$.

Thus, the heat equation admits the trivial symmetry with infinitesimal generator $X^0 = g\partial/\partial u$, and six non-trivial one-parameter symmetries with infinitesimal generators

$$X^1 = \frac{\partial}{\partial x}, \ X^2 = \frac{\partial}{\partial t}, \ X^3 = u\frac{\partial}{\partial u}, \ X^4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \ X^5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u},$$

$$X^6 = 4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2 + 2t)u\frac{\partial}{\partial u}.$$

We now determine the prolonged infinitesimals of the Lie group of point transformations with infinitesimal generator $X^5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}$. The first prolonged infinitesimals are obtained from (1.37a) as

$$\eta_t^{(1)} = D_t\eta - u_x(D_t\xi^x) - u_t(D_t\xi^t)$$

$$= -xu_t - 2u_x,$$

$$\eta_x^{(1)} = -u - xu_x.$$

Similarly, from (1.37c), the second prolonged infinitesimals are found as

$$\eta_{tt}^{(2)} = D_t\eta_t^{(1)} - u_{tx}(D_t\xi^x) - u_{tt}(D_t\xi^t)$$

$$= -xu_{tt} - 2u_{xt} - 2u_{tx},$$

$$\eta_{xt}^{(2)} = -u_t - xu_{xt} - 2u_{xx},$$

$$\eta_{xx}^{(2)} = -2u_x - xu_{xx}.$$

Note one can verify $\eta_{xt}^{(2)} = \eta_{tx}^{(2)}$.

The corresponding global group of transformations is found from solving the IVP (1.9) in Theorem 1.2.10. We explicitly obtain the global group for X^5, and present the global group for the remaining infinitesimal generators.

Consider $X^5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}$, where $\xi^x = 2t$, $\xi^t = 0$, $\eta = -xu$. The IVP for the global transformation from (1.9) is

$$\frac{dx^*}{d\epsilon} = 2t^*, \tag{1.52a}$$

$$\frac{dt^*}{d\epsilon} = 0, \tag{1.52b}$$

$$\frac{du^*}{d\epsilon} = -x^*u^*, \tag{1.52c}$$

with $x^*|_{\epsilon=0} = x$, $t^*|_{\epsilon=0} = t$, and $u^*|_{\epsilon=0} = u$.

Solving (1.52b) yields $t^* = t$. Hence, (1.52a) becomes

$$\frac{dx^*}{d\epsilon} = 2t, \quad x^*|_{\epsilon=0} = x,$$

which admits solution $x^* = x + 2\epsilon t$. As such, (1.52c) is

$$\frac{du^*}{d\epsilon} = -(x + 2\epsilon t)u^*, \quad u^*|_{\epsilon=0} = u.$$

As such, the global group equivalent to (1.52) is

$$x^* = x + 2\epsilon t, \quad t^* = t, \quad u^* = ue^{-\epsilon(x+\epsilon t)}. \tag{1.53}$$

The remaining global groups are as follows.

$$\mathrm{X}^1: \quad x^* = x + \epsilon, \qquad t^* = t, \qquad u^* = u; \tag{1.54a}$$

$$\mathrm{X}^2: \quad x^* = x, \qquad t^* = t + \epsilon, \qquad u^* = u; \tag{1.54b}$$

$$\mathrm{X}^3: \quad x^* = x, \qquad t^* = t, \qquad u^* = e^\epsilon u; \tag{1.54c}$$

$$\mathrm{X}^4: \quad x^* = e^\epsilon x, \qquad t^* = e^{2\epsilon}t, \qquad u^* = u; \tag{1.54d}$$

$$\mathrm{X}^6: \quad x^* = \frac{x}{1 - 4\epsilon t}, \quad t^* = \frac{t}{1 - 4\epsilon t}, \quad u^* = u\sqrt{1 - 4\epsilon t}\,\exp\left(\frac{-\epsilon x^2}{(1 - 4\epsilon t)}\right). \tag{1.54e}$$

In the above example, applying the prolonged infinitesimal generator (1.38) to (1.49) results in the equation $\eta_t^{(1)} - \eta_{xx}^{(2)} = 0$, the coefficients of $\partial/\partial t$ and $\partial^2/\partial x^2$. As such, the invariance criteria (1.48) for the heat equation could have been constructed more efficiently by making this observation prior to calculations. Various theorems regarding the form of the invariance criteria (1.48) based on the form of the PDE under study can be found in [21, 23].

1.3.3 Reduction of Order of Ordinary Differential Equations

An important application of Lie groups of point transformations is their utility in solving a scalar ODE. In particular, the order of an ODE can be reduced using an admitted Lie symmetry while preserving the solution set [21]. A kth-order ODE that admits r-symmetries can be reduced, at best, to a $(k-r)^{\text{th}}$-order ODE plus r quadratures. For the case of $r = k$, then one can obtain the general solution of the ODE through r quadratures. We outline below two reduction methodologies for an ODE: reduction by canonical coordinates (for ODEs of any order), and reduction by differential invariants (applied to ODEs $n \geq 2$).

In the following, we consider a kth-order ODE

$$G(x, y, y^{(1)}, \ldots, y^{(k)}) = y^{(k)} - F(x, y, y^{(1)}, \ldots, y^{(k-1)}) = 0, \tag{1.55}$$

where $y = y(x)$, and superscript $y^{(j)}$ is the jth-derivative of y with respect to x. In addition, we assume the ODE admits a Lie symmetry

$$x^* = x + \epsilon\xi(x, y) \tag{1.56a}$$

$$y^* = y + \epsilon\eta(x, y) \tag{1.56b}$$

26

with infinitesimal generator

$$\mathrm{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}. \tag{1.57}$$

Reduction of Order by Canonical Coordinates

Theorem 1.3.8. *Assume that a nontrivial one-parameter Lie group of transformations with infinitesimal generator* (1.57) *is admitted by the kth-order ODE* (1.55), $k \geq 2$. *Let $r(x,y)$ and $s(x,y)$ be corresponding canonical coordinates such that $\mathrm{X}r = 0$ and $\mathrm{X}s = 1$. Then solving the kth-order ODE* (1.55) *reduces to solving a $(k-1)^{\text{th}}$-order ODE*

$$\frac{\partial^{k-1}z}{\partial r^{k-1}} = H(r, z, z^{(1)}, \ldots, z^{(k-2)}) \tag{1.58}$$

where

$$\frac{ds}{dr} = z. \tag{1.59}$$

The proof appears in [21, 23], which leads to the following reduction algorithm.

1. From equations (1.17) and (1.25), one determines the canonical coordinates r and s such that

$$\mathrm{X}r = 0, \quad \mathrm{X}s = 1.$$

   Note the infinitesimal generator in canonical coordinates is

$$\mathrm{X} = \frac{\partial}{\partial s}.$$

2. Set $s = s(r)$, and calculate all derivatives of s with respect to r up to $(k)^{\text{th}}$-order through

$$\frac{d^{(i+1)}s}{dr^{(i+1)}} = \frac{\mathrm{D}_x s^{(i)}}{\mathrm{D}_x r} \tag{1.60}$$

   where D_x is the total derivative (1.33) with respect to x. For (1.60) to be nonsingular, we assume $\mathrm{D}_x r = r_x + r_y y^{(1)} \neq 0$.

3. Substitute the canonical coordinates r and $s(r)$, and all derivatives up to x^{th}-order, into the ODE (1.55) and simplify until all variables $\{x, y(x), y^{(1)}, \ldots, y^{(k)}\}$ are eliminated. It can be shown the resulting ODE is of the form

$$s^{(k)} = F(r,\ s,\ s^{(1)},\ \ldots,\ s^{(k-1)}). \tag{1.61}$$

Furthermore, this function is independent of s due to invariance under the translation symmetry transformation $r^* = r,\ s^* = s + \epsilon$.

4. Setting $z = s^{(1)}$, the reduced ODE is thus of the form

$$z^{(n-1)} = F(r,\ z,\ z^{(1)},\ \ldots,\ z^{(n-2)}). \tag{1.62}$$

After z is found, one needs to integrate $z = ds/dr$, yielding an algorithmic equation in x and y.

The above algorithm is now illustrated in the following example [65].

Example 1.3.9. We wish to reduce the order of the Riccati equation

$$y' = xy^2 - 2\frac{y}{x} - \frac{1}{x^3}, \quad (x \neq 0). \tag{1.63}$$

using canonical coordinates. It can be verified that equation (1.63) admits the Lie group of point transformations with infinitesimal generator

$$X = x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}.$$

Canonical coordinates of X are found as follows. $Xr = 0$ yields the linear advection PDE

$$x\frac{\partial r}{\partial x} - 2y\frac{\partial r}{\partial y} = 0$$

with solution

$$r = yx^2. \tag{1.64}$$

$Xs = 1$ yields the PDE

$$x\frac{\partial s}{\partial x} - 2y\frac{\partial s}{\partial y} = 1$$

28

with solution

$$s = \ln |x| . \tag{1.65}$$

The first derivative of s with respect to r is found as

$$
\begin{aligned}
\frac{ds}{dr} &= \frac{\mathrm{D}_x s}{\mathrm{D}_x r} \\
&= \frac{1/|x|}{2xy + y'x^2} \quad | \text{ substitute (1.63)} \\
&= \frac{1}{\mathrm{sgn}(x)\left((x^4 y^2 - 2x^2 y - 1) + 2x^2 y\right)} \quad | \text{ substitute } r = yx^2 \\
&= \frac{\mathrm{sgn}(x)}{r^2 - 1},
\end{aligned}
$$

where sgn is the signum function, and $r \neq \pm 1$ (we consider $r = \pm 1$ after). Solving this ODE for s yields

$$s = \ln \sqrt{\left|\frac{r-1}{r+1}\right|} + c, \tag{1.66}$$

where c is a constant. Substituting (1.64) and (1.65) into (1.66) then solving for y, the general solution is found as

$$y = \frac{\hat{c} + x^2}{x^2(\hat{c} - x^2)}, \tag{1.67}$$

where

$$
\hat{c} =
\begin{cases}
e^{2c} & yx^2 < -1 \text{ or } yx^2 > 1 \\
-e^{2c} & -1 < yx^2 < 1.
\end{cases}
$$

Consider now if $r = \pm 1$. Hence,

$$y = \pm \frac{1}{x^2},$$

which can be verified as a solution to (1.63). Note that $y = \frac{1}{x^2}$ corresponds to the limit of (1.67) as $\hat{c} \to \pm\infty$, and $y = -\frac{1}{x^2}$ to (1.67) for $\hat{c} = 0$ [65].

29

Reduction of Order by Differential Invariants

The kth-order ODE (1.55) admits the Lie symmetry with infinitesimal generator X (1.57) if and only if the invariance criteria (1.48) holds; that is,

$$\mathrm{X}^{(k)}\left(G(x,y,y^{(1)},\ldots,y^{(k)})\right) = 0 \quad \text{when} \quad G(x,y,y^{(1)},\ldots,y^{(k)}) = 0. \tag{1.68}$$

Consider functions $(I,V_1,V_2,\ldots,V_k)$ (note: not partial derivatives) invariant under the kth-prolongation of (1.56) such that

$$\mathrm{X}I = 0, \quad \mathrm{X}^{(k)}V_i \quad \text{with} \quad \frac{\partial V_i}{\partial y^{(i)}} \neq 0, \quad \text{where} \quad i = 1,\ldots,k.$$

Note that $(I,V_1,V_2,\ldots,V_k)$ are found as constants of integration from solving the characteristic system

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} = \frac{dy^{(1)}}{\eta^{(1)}(x,y,y^{(1)})} = \cdots = \frac{dy^{(k)}}{\eta^{(k)}(x,y,y^{(1)},\ldots,y^{(k)})}.$$

It follows from (1.68) that the ODE (1.55) is a function of the invariants,

$$G(x,y,y^{(1)},\ldots,y^{(k)}) = H(I,V_1,\ldots,V_k) = 0. \tag{1.69}$$

It can be shown that (1.69) is equivalent to a $(k-1)^{\text{th}}$-order ODE, which is summarized as follows.

1. Consider $I(x,y)$ and $V(x,y,y^{(1)})$.

2. Calculate all derivatives of V with respect to I up to $(k-1)^{\text{th}}$-order. This is achieved by taking

$$\frac{d^{(i+1)}V}{dI^{(i+1)}} = \frac{\mathrm{D}_x V^{(i)}}{\mathrm{D}_x I}, \tag{1.70}$$

where D_x is the total derivative (1.33) with respect to x. From invariance of I and V under the $(k)^{\text{th}}$ prolongation of (1.56), it follows that $\frac{dV}{dI}$ is invariant under the $(k)^{\text{th}}$ prolongation of (1.57) since

$$\left(\frac{dV}{dI}\right)^* = \frac{dV^*}{dI^*} = \frac{dV}{dI}. \tag{1.71}$$

By induction, $\{\frac{d^{(i)}V}{dI^{(i)}}\}_{i=1}^{k-1}$ are invariant under the kth-prolongation of (1.56), called *differential invariants* of (1.56).

3. It can be shown that

$$\frac{dV}{dI} = \frac{D_x v}{D_x I} = V_2(x, y, y^{(1)}, y^{(2)}),$$

and inductively that

$$\frac{d^i V}{dI^i} = V_{i+1}(x, y, y^{(1)}, \ldots, y^{(i+1)}), \quad \text{for} \quad i = 1, 2, \ldots, k - 1.$$

4. Substituting the invariants $(I, V, V^{(1)}, \ldots, V^{(k-1)})$ for $(x, y, y^{(1)}, \ldots, y^{(k)})$ into the ODE (1.55) yields a $(k-1)^{\text{th}}$-order ODE

$$H(I, V, V^{(1)}, \ldots, V^{(k-1)}). \tag{1.72}$$

Consider an example of reduction of order by differential invariants.

Example 1.3.10. Consider the second order ODE

$$0 = \frac{y''}{1 + (y')^2} + xy' - y. \tag{1.73}$$

It can be shown that (1.73) admits Lie symmetry with infinitesimal generator

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}. \tag{1.74}$$

The second prolongation of (1.74) is

$$X^{(2)} = X + \eta^{(1)}(x, y, y^{(1)}) \frac{\partial}{\partial y^{(1)}} + \eta^{(2)}(x, y, y^{(1)}) \frac{\partial}{\partial y^{(2)}}$$

$$= X + (1 + (y^{(1)})^2) \frac{\partial}{\partial y^{(1)}} + ((y^{(1)})^2 + 2y^{(1)} y^{(2)}) \frac{\partial}{\partial y^{(2)}}. \tag{1.75}$$

Invariants (I, V) of (1.75) such that $XI = 0$, $XV = 0$, and $V_{y^{(1)}} \neq 0$, are

$$I = \sqrt{x^2 + y^2}, \quad V = \frac{xy' - y}{x + yy'},$$

The derivative $V^{(1)}$ invariant under the second prolongation of (1.75) is

$$\frac{dV}{dI} = \frac{D_x V}{D_x I} = \frac{(y - xy^{(1)})(1 + (y^{(1)})^2)(1 + x^2 + y^2)}{(x + yy^{(1)})^3}.$$

Substituting $(I, V, V^{(1)})$ into (1.73) results in the reduced ODE

$$0 = V\left(1 + \frac{1}{I^2}\right) + \left(\frac{1}{1 + V^2}\right) \frac{dV}{dI}. \tag{1.76}$$

As such, we have reduced (1.73) to a first order ODE.

1.3.4 Reduction of Order of ODE by Multi-parameter Lie Algebra

One may reduce an ODE multiple times using admitted Lie symmetries. Naively, one may proceed iteratively by first using an admitted Lie symmetry to reduce the equation, then calculate the symmetries of the reduced equation and perform another reduction, repeating the second step until no more reductions can be done. However, it is not guaranteed that a kth-order ODE with r admitted Lie symmetries can be reduced to a $(k-r)^{\text{th}}$-order ODE. On the contrary, this can be achieved if the Lie algebra of the infinitesimal generators is *solvable* [23]. We provide a rough outline of the method for multiple reductions in the following with an example.

Consider an r-dimensional Lie algebra $\mathfrak{L}^r$ composed of infinitesimal generators for Lie symmetries admitted by a kth-order ODE in solved form $y^{(k)} = F(x, y, y^{(1)}, \ldots, y^{(k-1)})$. Suppose the Lie algebra is solvable with ordering $\mathfrak{L}^{(1)} \subset \mathfrak{L}^{(2)} \subset \cdots \subset \mathfrak{L}^{(r)}$ such that $X^j \in \mathfrak{L}^{(i)}$ for $j = 1, 2 \ldots, i$. Then, one may perform subsequent reductions starting with generators in $\mathfrak{L}^{(r)}$ not in $\mathfrak{L}^{(r-1)}$, followed by generators in $\mathfrak{L}^{(r-1)}$ not in $\mathfrak{L}^{(r-2)}$, and so on until $\mathfrak{L}^{(1)}$. The result will be a $(k-r)^{\text{th}}$-order ODE plus r quadratures.

Consider the following example of a multiple reduction of order of an ODE (exercise 1 in Section 6 of [65]).

Example 1.3.11. We wish to perform multiple reduction to the ODE

$$y'' = \frac{y'(1-y')}{y},$$

which admits Lie groups of point transformations

$$X^1 = \frac{\partial}{\partial x}, \quad X^2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Since $[X^1, X^2] = X^1$, we may do a reduction first by X^1 and then by X^2. We find the invariants of X_1 as

$$I^1 = y, \quad V^1 = y', \quad \frac{dV^1}{d(I^1)} = \frac{y''}{y'},$$

where superscript indicates these are invariants of X^1, and prime notation indicates differentiation.

Using the commutation relation $[X^1, X^2] = X^1$, we can express the first prolongation of X^2 in terms of the invariants of X^1 as [23]

$$X^{2(1)} = \left(X^2 I^1(x,y)\right) \frac{\partial}{\partial(I^1)} + \left(X^{2(1)} V^1(x,y,y')\right) \frac{\partial}{\partial(V^1)} = I^1 \frac{\partial}{\partial(I^1)}, \qquad (1.77)$$

where $X^2 I^1(x,y)$ indicates X^2 in terms of x and y applied to $I^1(x,y)$, and $X^{2(1)}$ is the first prolongation of X^2 given by (1.3.4).

The canonical coordinates of (1.77) are

$$r^2 = V^1, \quad s^2 = \ln I^1.$$

Taking the derivative of s^2 with respect to r^2 yields the ODE

$$\frac{ds^2}{d(r^2)} = \frac{1}{1 - r^2},$$

which admits the solution

$$s^2 = -\ln(1 - r^2) + c. \qquad (1.78)$$

where c is a constant. Reintroducing x, y, and y' into (1.78) yields the ODE

$$y(1 - y') = \bar{c}, \qquad (1.79)$$

where $\bar{c} = e^c$. Now, the canonical coordinates for X_1 are

$$r^1 = y, \quad s^1 = x,$$

which, when substituted into the current reduced ODE (1.79), yields

$$\frac{ds^1}{d(r^1)} = \frac{1}{V^1} = 1 + \frac{\bar{c}}{r^1 - \bar{c}}.$$

Solving this ODE and switching to x and y coordinates, we have the general solution

$$x = y + \bar{c}\ln(y - \bar{c}) + k,$$

where k is a constant. As such, we have solved the original ODE via two reductions using a combination of differential invariants and canonical coordinates.

We note that if the Lie symmetries of an ODE do not admit a solvable Lie algebra, then one may consider a solvable subalgebra to perform as many reductions as possible. In particular, if a kth-order ODE admits $r \geq k$ Lie symmetries for which k infinitesimal generators form a solvable subalgebra, then one may reduce the ODE to an algebraic equation with k quadratures.

1.3.5 Invariant Solutions of PDEs

Special classes of solutions to PDEs can be constructed which from admitted Lie point symmetries, called *invariant* (or *similarity*) *solutions*. Invariant solutions are obtained by use of *invariant forms* derived from the invariants of an admitted Lie group of point transformations. In particular, the original variables of the PDE system are replaced by the invariant forms of the admitted Lie symmetry, which results in the number of dependent variables being reduced by one.

Consider a kth-order system of PDEs $\vec{R}(\vec{x}, \vec{u}, \partial \vec{u}, \ldots, \partial^k \vec{u}) = 0$ in solved form (1.31) with an admitted Lie symmetry with infinitesimal generator

$$\mathrm{X} = \sum_{i=1}^{n} \xi^i(\vec{x}, \vec{u}) + \sum_{j=1}^{m} \eta^j(\vec{x}, \vec{u}), \tag{1.80}$$

where we assume $\vec{\xi}(\vec{x}, \vec{u}) \not\equiv 0$.

Definition 1.3.12. A solution $\vec{u} = \vec{\Theta}(\vec{x})$ to the PDE system $\vec{R} = 0$ is an *invariant solution* corresponding to the Lie group of point transformations (1.80) if and only if:

1. $\vec{u} = \vec{\Theta}(\vec{x})$ is a solution of $\vec{R} = 0$; and,

2. $\mathrm{X}(u^j - \Theta^j(\vec{x})) = 0$ when $u^j - \Theta^j(\vec{x}) = 0$, $j = 1, \ldots, m$; that is,

$$\sum_{i=1}^{n} \xi^i(\vec{x}, \vec{\Theta}(\vec{x})) \frac{\partial \Theta^j(\vec{x})}{\partial x^i} = \eta^j(\vec{x}, \vec{\Theta}(\vec{x})), \quad j = 1, \ldots, m. \tag{1.81}$$

The equations in (1.81) are called the *invariant surface conditions*.

Invariant solutions of a PDE system are constructed by the following algorithm:

1. Solve equation (1.81) by solving the characteristic system

$$\frac{d\,x^1}{\xi^1} = \cdots = \frac{d\,x^n}{\xi^n} = \frac{d\,u^1}{\eta^1} = \cdots = \frac{d\,u^m}{\eta^m}. \tag{1.82}$$

2. For $n-1$ invariants $\vec{I} = (I^1, \ldots, I^{n-1})$, and m-invariants $\vec{V} = (V^1, \ldots, V^m)$ such that the Jacobian determinant $|\partial \vec{v}/\partial \vec{u}| \neq 0$, the general solution $\vec{u} = \vec{\Theta}(\vec{x})$ of (1.81) is given implicitly by the *invariant forms*

$$\vec{V}(\vec{x}, \vec{u}) = \vec{\theta}(\vec{I}(\vec{x}, \vec{u})), \tag{1.83}$$

where each θ^j, $j = 1, \ldots, m$, is a function of its arguments.

3. Solve $\vec{V} = \vec{V}(\vec{x}, \vec{u})$ for $\vec{u}$ as $\vec{u} = \vec{u}(\vec{x}, \vec{V})$, and set $\vec{V} = \vec{V}(\vec{I}(\vec{x}, \vec{u}))$.

4. Substitute $\vec{I}$ and $\vec{V}(\vec{I}(\vec{x}))$ into the PDE system $\vec{R} = 0$ to eliminate $\vec{x}$ and $\vec{u}$. Simplify the resulting PDE system, applying the chain rule where necessary; that is, differentiation of V^j with respect to x^i is carried out as

$$\frac{\partial}{\partial x^i}\left(V^j(\vec{I}(\vec{x}))\right) = \sum_{p=1}^{n-1} \frac{\partial V^j}{\partial (I^p)}\frac{\partial (I^p)}{\partial x^i}.$$

The resulting PDE system will be of the form

$$\vec{S}(\vec{I}, \vec{V}, \partial\vec{V}, \dots, \partial^k\vec{V}) = 0, \tag{1.84}$$

which has $n-1$ independent variables $\vec{I}$ and m dependent variables $\vec{V}$.

Example 1.3.13. Consider the heat equation

$$u_t = u_{xx}, \tag{1.85}$$

which admits Lie symmetries (see Example 1.3.7)

$$X^1 = \frac{\partial}{\partial x}, \quad X^2 = \frac{\partial}{\partial t}, \quad X^1 = \frac{\partial}{\partial x}.$$

As such, $\bar{X} = cX^1 + X^2 = c\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$ for $c \in \mathbb{R}$ is an admitted Lie symmetry. We now construct the solution invariant to $\bar{X}$ for $c \neq 0$. From (1.17), one constructs the characteristic system

$$\frac{dx}{c} = \frac{dt}{1} = \frac{du}{0}. \tag{1.86}$$

Solving (1.86) yields invariants

$$I = x - ct, \quad V = u. \tag{1.87}$$

As such, the invariant form for $\bar{X}$ is $u = V(I) = V(x - ct)$, which is the traveling wave ansatz.

Substituting $u = V(I) = V(x - ct)$ into (1.85), applying the chain rule, and simplifying yields the linear ODE

$$-cV' = V'', \tag{1.88}$$

35

which admits the general solution

$$V = \frac{1}{\bar{c}} e^{\bar{c}I + k^1} + k^2,$$

where k^1 and k^2 are constants, and $\bar{c} = -c$. As such, the solution to the heat equation invariant under $\bar{X}$ is

$$u = \frac{1}{\bar{c}} e^{\bar{c}(x + \bar{c}t) + k^1} + k^2. \tag{1.89}$$

Note that invariant solutions of ODEs are a special case in the above discussion.

1.3.6 Equivalence Transformations

Of importance in studying differential equations is to reduce redundancy in the work done. It is common practice when studying differential equations with arbitrary constants to first non-dimensionalize the equation, as demonstrated in the following example.

Example 1.3.14. The Kortweg-de Vries (KdV) equation with a and b constant

$$u_t + auu_x + bu_{xxx} = 0 \tag{1.90}$$

can be mapped into the equivalent form (assuming $a \neq 0$)

$$\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} = 0, \tag{1.91}$$

where $\tilde{x} = x$, $\tilde{t} = (b/a)t$, and $\tilde{u} = (b/a)u$. As such, one may study (1.91) instead of (1.90) without a loss of generality.

Consider now if a and b are arbitrary functions of x, t, and u in (1.90); we now have a classification problem (as in, e.g., [25]). We would like to eliminate redundancy in our work by finding equivalence classes of the arbitrary functions. As well, we want to make our work simpler by (possibly) mapping our equation into one with fewer/simpler arbitrary functions. One way to do this is by using Lie groups of point transformations to find a set of *equivalence transformations*, as outlined in the following definition [22].

Definition 1.3.15. Consider a family of PDE systems $\mathfrak{F}_{\vec{K}}$ with L-arbitrary functions and/or parameters $\vec{K} = (K^1, \ldots, K^L)$. Then, a *one-parameter Lie group of equivalence transformations* of $\mathfrak{F}_{\vec{K}}$ is a one-parameter Lie group of transformations given by

$$\vec{x}^* = \vec{f}(\vec{x}, \vec{u}; \epsilon),$$
$$\vec{u}^* = \vec{g}(\vec{x}, \vec{u}; \epsilon),$$
$$\vec{K}^* = \vec{h}(\vec{x}, \vec{u}, \vec{K}; \epsilon),$$

such that a PDE system $(\vec{R}(\vec{x}, \vec{u}, \vec{K}) = 0) \in \mathfrak{F}_{\vec{K}}$ is mapped into another PDE system $(\vec{R}(\vec{x}^*, \vec{u}^*, \vec{K}^*) = 0) \in \mathfrak{F}_{\vec{K}}$.

Consider the following example of finding equivalence classes, which is originally done in [16].

Example 1.3.16. Consider the nonlinear wave equation

$$u_{tt} = \left(c^2(u)u_x\right)_x, \tag{1.92}$$

where $c(u)$ is the wave speed. In [16], the group of equivalence transformations is found as a six parameter group.

$$x^* = \epsilon^1 x + \epsilon^4, \tag{1.93a}$$
$$t^* = \epsilon^2 t + \epsilon^5, \tag{1.93b}$$
$$u^* = \epsilon^3 u + \epsilon^6, \tag{1.93c}$$
$$c^*(u^*) = \epsilon^1 (\epsilon^2)^{-1} c(u), \tag{1.93d}$$

where $\epsilon^1 \epsilon^2 \epsilon^3 \neq 0$.

For example, if $c(u) = Au^n$, one may equivalently study Lie symmetries of (1.92) for $c(u) = u^n$ due to the scaling by ϵ^3 in (1.93c).

Of note, symmetry classifications for equations with arbitrary functions are typically presented up to equivalence transformations as is done in [16], thus reducing the number of cases to consider [22].

1.4 Conservation Laws

A conservation law is the statement that a quantity of a physical system does not dissipate with time; that is, the quantity is conserved. Equations of motion governing physical systems are typically derived from particular conservation laws, such as conservation of momentum. However, it may be that other physical quantities are conserved by the system, which are important to know in order to better understand underlying physical processes governing the motion. Additionally, knowledge of conservation laws play a crucial role to solution methods, such as in the class of conservative numerical methods which rely on equations being in conservation law form (see, e.g., [77]).

The current section is dedicated to presenting the fundamental notions of conservation law analysis of PDE systems based on [22]. We first define the conservation law form of a PDE system, then discuss how it is obtained from conservation law multipliers (i.e. the direct method). Afterwards, we provide several notes pertaining to trivial conservation laws, equivalent conservation laws, and the correspondence between conservation laws and conservation law multipliers.

1.4.1 Conservation Law Form

Definition 1.4.1. A *local divergence conservation law* of the PDE $\vec{R}(\vec{x}, \vec{u}, \partial\vec{u}, \ldots, \partial^k\vec{u}) = 0$ is a divergence expression

$$\sum_{i=1}^{n} \mathrm{D}_i \Phi^i(\vec{x}, \vec{u}, \ldots, \partial^k\vec{u}) = 0, \tag{1.94}$$

hold for all solutions of the PDE system $\vec{R} = 0$. In (1.94), D_i is the total derivative with respect to x^i (as in (1.33)), Φ^i are called the *fluxes* of the conservation law, and the highest-order derivative (r) present in the fluxes Φ^i is called the (differential) *order of a conservation law*.

Note that if one of the independent variables is time (say $x^1 = t$), then the flux $\Phi^1 = \Psi$ is called the *conserved density*.

The meaning of a conservation law is derived from the conserved density for time dependent systems. For example, consider the linear advection equation for a mass density

$u(t, x)$,

$$u_t + cu_x = 0,$$

where c is a constant. The equivalent divergence conservation law form is

$$\mathrm{D}_t(\rho_0 u) + \mathrm{D}_x(\rho_0 u) = 0. \tag{1.95}$$

Here, mass density u is the conserved density, and we say mass is conserved.

1.4.2 The Direct Method

It may be possible to bring an equation into conservation law form by multiplying the equation by a specific function. As an example, consider again (1.95), which can be multiplied by u and manipulated to yield:

$$u\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0,$$

which is the expansion of the chain rule applied to the equation

$$\frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0.$$

Here, we have the original equation in a different conservation law form.

In general, it may be possible to bring a system of partial differential equations $\vec{R} = 0$ into a divergence form through multiplication by a particular set of functions $\{\Lambda^i\}_{i=1}^{\sigma}$ such that $\Lambda^i(\vec{x}, \vec{u}, \ldots, \partial^k\vec{u})$.

Definition 1.4.2. For a kth-order PDE system $\vec{R} = 0$ consisting of σ PDEs, a set of functions $\{\Lambda^i(\vec{x}, \vec{u}, \ldots, \partial^r\vec{u})\}_{i=1}^{\sigma}$ are called r^{th}-*order conservation law multipliers* if and only if $\sum_{i=1}^{\sigma} \Lambda^i R^i = 0$ can be expressed in a divergence conservation law form; that is:

$$\sum_{i=1}^{\sigma} \Lambda^i R^i = \sum_{i=1}^{\sigma} \mathrm{D}_i \Phi^i = 0, \quad i = 1, \ldots, \sigma.$$

Additionally, a multiplier Λ^i that is a singular function on solutions to the PDE system is called a *singular conservation law multiplier*.

Of importance, singular multipliers of a PDE system are not of interest since they can lead to arbitrary divergence expressions unrelated to the conservation laws of the PDE system [22]. A particular example of a singular conservation law multiplier is $\Lambda^i = D_i \Phi^i / R^i$ for some $i \in \{1, \ldots, k\}$, which is singular on solutions to $\vec{R} = 0$.

Conservation laws, like Lie symmetries, can be obtained by inspection (e.g. $\Lambda = u$ for (1.95)). More complicated systems require a systematic way in which to find conservation laws. This is accomplished through use of the Euler operator of mechanics [22], which will be seen in the following.

Definition 1.4.3. The *Euler operator* with respect to the dependent variable u^j is given as

$$E_{u^j} = \frac{\partial}{\partial u^j} + \sum_{l=1}^{\infty} (-1)^l D_{i^1} \ldots D_{i^l} \frac{\partial}{\partial u^j_{i^1 \ldots i^l}}, \tag{1.96}$$

where $i^p = 1, \ldots, n$, and $p = 1, \ldots, l$.

In mechanics, the equations of motion of a physical system may be derived by application of the Euler operator to the Lagrangian (see, e.g., [22, 54, 96]). However, the Euler operator is fundamental to conservation law analysis due to the following theorem.

Theorem 1.4.4. *The equations* $E_{u^j} F(\vec{x}, \vec{u}, \partial \vec{u}, \ldots \partial^k \vec{u}) \equiv 0$, $j = 1, \ldots, m$, *hold for all* $(\vec{x}, \vec{u}(\vec{x}))$ *if and only if* $F \equiv \sum_{i=1}^{\sigma} D_i \Phi^i$ *for some* $\Phi^i = \Phi^i \left(\vec{x}, \vec{u}, \partial \vec{u}, \ldots, \partial^{k-1} \vec{u} \right)$.

The proof can be found in [96].

As such, any divergence expression will be annihilated by an Euler operator. This leads naturally to the following theorem regarding conservation law multipliers:

Theorem 1.4.5. *A set of* k^{th}*-order non-singular local multipliers* $\{\Lambda^i\}_{i=1}^{\sigma}$ *yields a local conservation law for the system* $\vec{R} = 0$ *if and only if* $\forall \vec{u}(\vec{x})$,

$$E_{u^j}(\Lambda^i R^i) \equiv 0, \quad j = 1, \ldots, m.$$

Algorithm to Find Conservation Law Multipliers

Theorem 1.4.5 leads naturally to an algorithm to find conservation law multipliers:

1. Define a set of arbitrary conservation law multipliers $\{\Lambda^i\}_{i=1}^{\sigma}$ up to a desired order (say r^{th}-order).

2. Multiply the PDE system $\vec{R} = 0$ by the arbitrary conservation law multipliers to obtain the equation $\sum_{i=1}^{\sigma} \Lambda^i R^i = 0$.

3. For each dependent variable, apply the associated Euler operator to obtain a system of m equations.

4. Simplify each of the resulting equations, collecting terms with respect to $(\partial^{r+1}\vec{u}, \partial^{r+2}\vec{u}, \ldots, \partial^k \vec{u})$. Setting the resulting coefficients of $(\partial^{r+1}\vec{u}, \ldots, \partial^k \vec{u})$ to zero yields a linear system of *determining equations* for $\{\Lambda^i\}_{i=1}^{\sigma}$ (called the *multiplier determining equations*).

5. Solve the system of determining equations for $\{\Lambda^i\}_{i=1}^{\sigma}$ to obtain the conservation law multipliers.

An example of the above algorithm appears in Section 1.4.4.

Once the conservation laws are known for a PDE system $\vec{R} = 0$, $\sum_{i=1}^{\sigma} \Lambda^i R^i = 0$ can be manipulated into a divergence conservation law form using the chain rule. General methods of this process have been developed and are presented in [22], and have been algorithmically applied in Maple [34]. These methods are not discussed so as to not detract from the main content of the current Thesis.

1.4.3 Notes About The Direct Method

The following notes are important to conservation law analysis, and should be kept in mind when performing calculations through the direct method.

Trivial Conservation Laws

Definition 1.4.6. For a PDE system, a *trivial conservation law* is one such that the flux is of the form $\Phi^i = M^i + H^i$, where each M^i vanish on solutions of the system and H^i is such that $\sum_{i=1}^{n} \mathrm{D}_{x^i} H^i \equiv 0$ (i.e. the conservation law is identically divergence free).

Trivial conservation laws do not provide information about the system [22]. They come in two forms:

1. The fluxes vanish identically on solutions to the system.

2. The conservation law vanishes identically as a differential identity (e.g. a differential consequence).

Example 1.4.7. Consider the PDE system

$$v_x - u_y = 0, \tag{1.97a}$$

$$v_y + u_t + uu_x + u_{xxx} = 0, \tag{1.97b}$$

where $u = u(t, x, y)$ and $v = v(t, x, y)$. Upon eliminating v, this system becomes the Kadomtsev-Petviashvili (KP) equation.

The conservation law

$$D_t(v^2(v_x - u_y)) + D_x(v_y + u_t + uu_x + u_{xxx} - 4y)$$
$$+ D_y(v_y + u_t + uu_x + u_{xxx} - 4v_x + 4u_y) = 0$$

is identically zero on solutions, making it a trivial conservation law of first type.

The conservation law

$$D_t(u_x) - D_x(u_t) = 0$$

is a trivial conservation law of the second type since it is a differential consequence.

Recall that a PDE system $\vec{R} = 0$ in solved form (1.31) is written as

$$R^j = u^{p^j}_{i^1, \ldots, i^{l^j}} - f^j(\vec{x}, \vec{u}, \partial\vec{u}, \ldots, \partial^k\vec{u}) = 0, \quad j = 1, \ldots, \sigma, \tag{1.98}$$

where each $f^j(\vec{x}, \vec{u}, \partial\vec{u}, \ldots, \partial^k\vec{u})$ does not depend on $u^{p^j}_{i^1, \ldots, i^{l^j}}$, $p^j \in \{1, \ldots, m\}$, and each $i^\nu \in \{1, \ldots, k\}$ for $\nu = 1, \ldots, j$. For such a PDE system, one may avoid trivial conservation laws by excluding each $u^{p^j}_{i^1, \ldots, i^{l^j}}$ and its differential consequences in the multiplier dependence [22].

Equivalent Conservation Laws

Another important fact is that conservation laws admitted by a PDE system $\vec{R} = 0$ come in equivalence classes, which is defined as follows.

Definition 1.4.8. Given two conservation laws $D_i \Phi^i = 0$ and $D_i \Theta^i = 0$, they are *equivalent* if $D_i (\Phi^i - \Theta^i) = 0$ is a trivial conservation law. In addition, the two conservation laws are said to be in an *equivalence class of conservation laws.*

Example 1.4.9. Consider the linear wave equation

$$u_{tt} = u_{xx},$$

which admits the conservation law (from the multiplier $\Lambda = u$)

$$D_t \left(\frac{1}{2}(u_t)^2 + \frac{1}{2}(u_x)^2 \right) - D_x(u_t u_x) = 0.$$

An equivalent conservation law is

$$D_t \left(\frac{1}{2}(u_t)^2 + \frac{1}{2}(u_x)^2 + u_x \right) - D_x(u_t u_x + u_t) = 0$$

since

$$D_t \left(\frac{1}{2}(u_t)^2 + \frac{1}{2}(u_x)^2 + u_x \right) - D_x(u_t u_x + u_t)$$

$$- \left(D_t \left(\frac{1}{2}(u_t)^2 + \frac{1}{2}(u_x)^2 \right) - D_x(u_t u_x) \right)$$

$$= D_t(u_x) - D_x(u_t) = 0,$$

which is a trivial conservation law of the second type.

Correspondence Between Multipliers and Conservation Laws

A question of fundamental importance is if two conservation law multipliers of a PDE system yield unique conservation laws. The answer is that a PDE system must be admit a Cauchy-Kovalevskaya form in order for there to exist a one-to-one correspondence between conservation law multipliers and admitted conservation laws. This is presented in the following.

Definition 1.4.10. A PDE system $\vec{R} = 0$ is in *Cauchy-Kovalevskaya form* (CK-form) with respect to a variable x^i if the system, possibly after a point transformation, can be written

in solved form for the highest derivative of each dependent variable with respect to x^i after a point transformation; that is,

$$\frac{\partial^{s^\sigma}}{\partial (x^i)^{s^\sigma}} = G^\sigma(\vec{x}, \vec{u}, \partial \vec{u}, \dots, \partial^k \vec{u}), \tag{1.99}$$

where $1 \le s^\sigma \le k$ for $\sigma = 1, \dots, m$.

Consider the following examples of differential equations which admit a Cauchy-Kovalevskaya form.

Example 1.4.11. The linear wave equation

$$u_{tx} = 0$$

admits the CK-form $u_{tt} = u_{xx}$ upon the point transformation $t \to t - x$, $x \to x + t$.

Example 1.4.12. The Kadomstev-Petviashvili (KP) equation (as in [1]) is an extension of the KdV equation, and is given as

$$\text{D}_x(u_t + 6uu_x + u_{xxx}) + u_{yy} = 0,$$

where $u = u(x, y, t)$. The KP equation is in CK form with respect to y, and can be brought into CK form with respect to x by expanding the total derivative with respect to x and solving for u_{xxxx}.

Uniqueness of the correspondence between conservation laws of a PDE system and conservation law multipliers is given due to the following theorem by Anco and Bluman [5].

Theorem 1.4.13. *Suppose a PDE system admits a Cauchy-Kovalevskaya form* (1.99) *with respect to x^j. Then all of its nontrivial (up to equivalence) local conservation laws arise from multipliers. Moreover, there is a one-to-one correspondence between equivalence classes of conservation laws and sets of conservation law multipliers with no dependence on derivatives of $\vec{u}$ with respect to x^j*

The proof appears in [5].

1.4.4   Example

We now wish to show a sample calculation of finding the conservation laws, as done in [22].

Example 1.4.14. We wish to find the local zeroth order conservation laws of the KdV equation,

$$u_t + uu_x + u_{xxx} = 0. \tag{1.100}$$

Note the equation is in Cauchy-Kovalevskaya form with respect to x, and so each conservation law multiplier yields a unique conservation law.

We apply the algorithm in Section 1.4.2 based on Theorem 1.4.5 to seek zeroth order multipliers,

$$\Lambda = \Lambda(x, t, u). \tag{1.101}$$

The Euler operator with respect to u up to third order is constructed from (1.96). In the current calculation, we only require the terms in the Euler operator containing derivatives up to first order in t and up to third order in x since the KdV equation (1.100) is of this form. As such, we have the restricted Euler operator as

$$\mathrm{E}_u = \frac{\partial}{\partial u} - \mathrm{D}_t \frac{\partial}{\partial u_t} - \mathrm{D}_x \frac{\partial}{\partial u_x} + (\mathrm{D}_x)^2 \frac{\partial}{\partial u_{xx}} - (\mathrm{D}_x)^3 \frac{\partial}{\partial u_{xxx}}. \tag{1.102}$$

Applying (1.102) to (1.100) multiplied by Λ yields

$$0 \equiv \mathrm{E}_u(\Lambda(x, t, u)(u_t + uu_x + u_{xxx}))$$

$$= \Lambda_u(u_t + uu_x + u_{xxx}) + \Lambda u_x - \mathrm{D}_t(\Lambda) - \mathrm{D}_x(\Lambda u) - (\mathrm{D}_x)^3(\Lambda)$$

$$= -(\Lambda_t + u\Lambda_x + \Lambda_{xxx}) - 3\Lambda_{xxu}u_x - 3\Lambda_{xu}u_{xx} - 3\Lambda_{uu}u_x u_{xx} - 3\Lambda_{xuu}(u_x)^2 - \Lambda_{uuu}(u_x)^3.$$

As such, the determining equations for Λ are

$$\Lambda_t + u\Lambda_x + \Lambda_{xxx} = 0, \quad \Lambda_{xu} = 0, \quad \Lambda_{uu} = 0, \tag{1.103}$$

which admits the solution

$$\Lambda = C^1 + C^2 u + C^3(x - ut), \tag{1.104}$$

where C^1, C^2, and C^3 are constants. Furthermore, equation (1.104) is a linear combination of three nontrivial conservation law multipliers,

$$\Lambda^1 = 1, \qquad \Lambda^2 = u, \qquad \Lambda^1 = x - ut. \qquad (1.105)$$

The conservation law forms of (1.100) corresponding to (1.105) are found, respective to the order in (1.105), as

$$D_t(u) + D_x\left(\frac{1}{2}u^2 + u_{xx}\right) = 0,$$

$$D_t\left(\frac{1}{2}u^2\right) + D_x\left(\frac{1}{3}u^3 - \frac{1}{2}u_x2\right) = 0,$$

$$D_t\left(\frac{1}{6}u^3 - \frac{1}{2}u^2\right) + D_x\left(\frac{1}{8}u^4 - uu_x + \frac{1}{2}u^2u_{xx} + \frac{1}{2}u_{xx}^2 - u_xu_{xxx}\right) = 0.$$

In later Chapters, we report conservation law multipliers as in (1.105) with the understanding independent nontrivial multipliers can be separated when necessary.

1.4.5 Conservation Laws and Potential Systems

We now briefly remark on potential systems, which are important in further extensions to Lie symmetry and conservation law analysis. Potential systems are constructed from conservation laws of a PDE system as follows.

Definition 1.4.15. Consider a PDE $R(x, t, u, \partial u, \ldots, \partial^k u) = 0$ with independent variables (x, t) and dependent variable $u(x, t)$. Suppose $\vec{R} = 0$ admits the conservation law $D_t\Psi + D_x\Phi = 0$. Then, one can construct the *potential system*

$$\vec{R} = 0, \quad \frac{\partial v}{\partial x} = \Psi, \quad \frac{\partial v}{\partial t} = -\Phi,$$

where $v = v(t, x)$ is called the *potential variable*.

The potential variable v in the above potential system cannot be expressed as a function of (x, t, u), and so is a *nonlocal* variable. As such, the potential system is nonlocally related to the original PDE. Methods to obtain further nonlocally related systems and subsystems are presented in [15], and outlined in further detail in [22].

1.5 Connections Between Lie Point Symmetries and Conservation Laws

We now discuss the connection between Lie symmetries and conservation laws, known as Noether's theorem. Before, we provide an overview of the evolutionary form of a Lie symmetry, which describes an 'equivalent' transformation of the hypersurface in the dependent variables alone.

1.5.1 Evolutionary Form of Infinitesimal Generators

Consider a Lie group of point transformations

$$(x^i)^* = x^i + \epsilon \xi^i(\vec{x}, \vec{u}) + O(\epsilon^2), \quad i = 1, \ldots, n, \tag{1.106a}$$

$$(u^j)^* = u^j + \epsilon \eta^j(\vec{x}, \vec{u}) + O(\epsilon^2), \quad j = 1, \ldots, m. \tag{1.106b}$$

which acts on $(\vec{x}, \vec{u})$-space.

Consider a hypersurface $\vec{u} = \vec{\Theta}(\vec{x})$ which is not invariant under (1.106). To present Noether's theorem, it is important to understand how the transformation (1.106) acts on $\vec{u} = \vec{\Theta}(\vec{x})$ explicitly. In particular, we are interested in deriving an 'equivalent' transformation which in which $\vec{x}$ is invariant.

For an arbitrary fixed ϵ, the transformation (1.106) maps a point $(\vec{x}, \vec{u})$ on the hypersurface to a point $(\vec{x}^*, \vec{u}^*)$ as

$$(x^i)^* = x^i + \epsilon \xi^i(\vec{x}, \vec{\Theta}(\vec{x})) + O(\epsilon^2), \quad i = 1, \ldots, n, \tag{1.107a}$$

$$(u^j)^*(\vec{x}^*) = \Theta^j(\vec{x}) + \epsilon \eta^j(\vec{x}, \vec{\Theta}(\vec{x})) + O(\epsilon^2), \quad j = 1, \ldots, m. \tag{1.107b}$$

To obtain a transformation in $\vec{u}$ alone, we must apply the inverse transformation of (1.107a) to (1.107); that is, we substitute

$$\vec{x} = \vec{x}^* - \epsilon \vec{\xi}(\vec{x}, \vec{\Theta}(\vec{x})) + O(\epsilon^2) \tag{1.108}$$

into (1.107) to yield (upon expanding about $\epsilon = 0$)

$$(x^i)^* = x^i, \quad i = 1, \ldots, n, \tag{1.109a}$$

$$(u^j)^* = \Theta^i(\vec{x}^*) + \epsilon \left(\eta^j(\vec{x}, \vec{\Theta}(\vec{x})) - \sum_{i=1}^{n} \xi^i(\vec{x}, \vec{\Theta}(\vec{x})) \frac{\partial \Theta^i(\vec{x}^*)}{\partial(x^i)^*} \right) + O(\epsilon^2), \quad j = 1, \ldots, m. \tag{1.109b}$$

Replacing $(\vec{x}^*, \vec{u}^*, \epsilon)$ with $(\vec{x}, \vec{u}, -\epsilon)$ in (1.109b) yields

$$(u^j) = \Theta^i(\vec{x}) + \epsilon \left(\eta^j(\vec{x}, \vec{\Theta}(\vec{x})) - \sum_{i=1}^{n} \xi^i(\vec{x}, \vec{\Theta}(\vec{x})) \frac{\partial \Theta^i(\vec{x})}{\partial(x^i)} \right) + O(\epsilon^2), \quad j = 1, \ldots, m. \tag{1.110}$$

Theorem 1.5.1. *Suppose that $\vec{u} = \vec{\Theta}(\vec{x})$ is not an invariant hypersurface of (1.107). Then (1.110) implicitly defines a mapping of $\vec{u} = \vec{\Theta}(\vec{x})$ into a family of surfaces $\vec{u} = \vec{\phi}(\vec{x}, \epsilon)$.*

Furthermore, observe that the image of $u = \Theta(\vec{x})$ can also be obtained from the transformation

$$(x^i)^* = x^i, \quad i = 1, \ldots, n, \tag{1.111a}$$

$$(u^j)^* = u^j + \epsilon \hat{\eta}^j(\vec{x}, \vec{u}, \partial \vec{u}) + O(\epsilon^2), \quad j = 1, \ldots, m, \tag{1.111b}$$

where

$$\hat{\eta}^j = \eta^j(\vec{x}, \vec{u}) - \sum_{i=1}^{n} \xi^i(\vec{x}, \vec{u}) \frac{\partial u^j}{\partial x^i}, \quad j = 1, \ldots, m. \tag{1.112}$$

Theorem 1.5.2. *Let $\vec{u} = \vec{\theta}(\vec{x})$ be a hypersurface not invariant under the Lie group of point transformations (1.107) with infinitesimal generator*

$$\mathrm{X} = \sum_{i=1}^{n} \xi^i(\vec{x}, \vec{u}(\vec{x})) \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \eta^j(\vec{x}, \vec{u}(\vec{x})) \frac{\partial}{\partial u^j}.$$

Then, the local transformation (1.111) with infinitesimal generator

$$\hat{\mathrm{X}} = \sum_{j=1}^{m} \hat{\eta}^j(\vec{x}, \vec{u}, \partial \vec{u}) \frac{\partial}{\partial u^j} = \sum_{i=1}^{n} \sum_{j=1}^{m} \left[\eta^j(\vec{x}, \vec{u}) - u_i^j \xi^i(\vec{x}, \vec{u}) \right] \frac{\partial}{\partial u^j} \tag{1.113}$$

is equivalent to (1.107) in the sense that both transformations map $\vec{u} = \vec{\Theta}(\vec{x})$ into the same hypersurface $\vec{u}^* = \vec{\phi}(\vec{x}, \epsilon)$. The infinitesimal generator $\hat{\mathrm{X}}$ is referred to as the characteristic (or evolutionary) form of X.

Example 1.5.3. Consider the heat equation

$$u_t = u_{xx}, \tag{1.114}$$

which admits symmetries (see Example 1.3.7)

$$X^1 = \frac{\partial}{\partial x}, \quad X^2 = \frac{\partial}{\partial t}, \quad X^4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}. \tag{1.115}$$

One can verify

$$\Theta(x,t) = Ae^{-cx+c^2t}. \tag{1.116}$$

is a solution of the heat equation which is invariant under $\bar{X} = cX^1 + X^2$ for c a nonzero constant. Furthermore, (1.116) is not invariant under X^4.

We wish to construct a one-parameter family of solutions from the infinitesimal transformation defined by X^4, and to verify the characteristic form of the transformation defined by (1.111) yields the same family of solutions.

The transformation corresponding to X^4 is

$$x^* = x + \epsilon x + O(\epsilon^2), \quad t^* = t + 2\epsilon t + O(\epsilon^2), \quad u^* = u. \tag{1.117}$$

Therefore, (1.116) is transformed by (1.117) as

$$
\begin{aligned}
u^* = \Theta(x^*, t^*) &= Ae^{-cx^*+c^2t^*} \\
&= Ae^{-c(x+\epsilon x)+c^2(t+2\epsilon t)+O(\epsilon^2)} \\
&= Ae^{-cx+c^2t} + \epsilon(-cx + 2c^2t)Ae^{-cx+c^2t} + O(\epsilon^2).
\end{aligned} \tag{1.118}
$$

Now for the characteristic transformations. From (1.113), one obtains $\hat{X}^4$ as

$$
\begin{aligned}
\hat{X}^4 &= (\eta - \xi^x u_x - \xi^t u_t)\frac{\partial}{\partial u} \\
&= (-xu_x - 2tu_t)\frac{\partial}{\partial u}.
\end{aligned} \tag{1.119}
$$

The associated infinitesimal transformation of (1.119) is

$$x^* = x, \quad t^* = t, \quad u^* = u - \epsilon(xu_x + 2tu_t) + O(\epsilon^2). \tag{1.120}$$

The first derivatives of (1.116) are

$$\Theta_x = -c\Theta, \quad \Theta_t = c^2\Theta.$$

Therefore, (1.116) is transformed by (1.120) as (suppressing (x, t) on Θ)

$$u^* = \phi(x, t, \epsilon) = \Theta + \epsilon \left(\eta(x, t, \Theta) - \xi^x(x, t, \Theta)\Theta_x - \xi^t(x, t, \Theta)\Theta_t \right) + O(\epsilon^2),$$
$$= Ae^{-cx+c^2t} - \epsilon(-cx + 2c^2t)Ae^{-cx+c^2t} + O(\epsilon^2), \tag{1.121}$$

which is identical to (1.118) upon setting ϵ to negative ϵ.

1.5.2 Noether's Theorem

Recall that in mechanics, a *Lagrangian L* is a function arising from a variational principle such that a system of equations (the Euler-Lagrange equations) are derived by applying the Euler operator to the Lagrangian (see, for example, [22, 54, 84]).

Definition 1.5.4. A system of equations is called *variational* if there exists a Lagrangian from which the equations may be derived in the Lagrangian mechanical framework.

Noether's theorem only applies to variational systems of PDEs. As such, it is important to be able to determine if a given PDE system is variational. This can be achieved by determining if the linearizing operator of a system of PDEs is self-adjoint, which is presented in the following.

Definition 1.5.5. For $\vec{U} = \vec{U}(\vec{x}) = (U^1(\vec{x}), \ldots, U^m(\vec{x}))$ an arbitrary vector function, the *linearizing operator* $\mathrm{L}[\vec{U}]$ of a system of PDEs $\vec{R}[\vec{u}] = \vec{R}(\vec{x}, \vec{u}, \partial\vec{u}, \ldots, \partial^k\vec{u}) = 0$ is given by

$$\mathrm{L}_q^p[\vec{U}]V^q = \left[\frac{\partial R^p[\vec{U}]}{\partial U^q} + \frac{\partial R^p[\vec{U}]}{\partial U_{i^1}^q}\mathrm{D}_{i^1} + \cdots + \frac{\partial R^p[\vec{U}]}{\partial U_{i^1\ldots i^k}^q}\mathrm{D}_{i^1}\ldots\mathrm{D}_{i^k} \right] V^q, \quad p = 1, \ldots, \sigma, \tag{1.122}$$

where V^q are entries of the arbitrary vector function $\vec{V}(\vec{x}) = (V^1(\vec{x}), \ldots, V^m(\vec{x}))$, summation it is assumed for $q = 1, \ldots, m$ and for $i^r = 1, \ldots, n$ where $r = 1, \ldots, k$.

Definition 1.5.6. For $\vec{U} = \vec{U}(\vec{x}) = (U^1(\vec{x}), \ldots, U^m(\vec{x}))$ an arbitrary vector function, the *adjoint operator* $\mathrm{L}^*[U]$ of a PDE system $\vec{R}[\vec{u}] = \vec{R}(\vec{x}, \vec{u}, \partial\vec{u}, \ldots, \partial^k\vec{u}) = 0$ takes the form of

an $m \times \sigma$ matrix with elements given by

$$L_q^{*p}[\vec{U}]W^p = \frac{\partial R^p[\vec{U}]}{\partial U^q}W^p - D_{i^1}\left(\frac{\partial R^p[\vec{U}]}{\partial U_{i^1}^q}W^p\right) + \ldots$$

$$+(-1)^k D_{i^1}\ldots D_{i^k}\left(\frac{\partial R^p[\vec{U}]}{\partial U_{i^1\ldots i^k}^q}W^p\right), \quad q = 1,\ldots,m \tag{1.123}$$

where W^p are entries of an arbitrary vector function $\vec{W}(\vec{x}) = (W^1(\vec{x}),\ldots,W^\sigma(\vec{x}))$, and summation is assumed for $p = 1,\ldots,\sigma$ and for $i^r = 1,\ldots,n$ where $r = 1,\ldots,k$.

Definition 1.5.7. A system of PDEs $\vec{R} = 0$ is *self-adjoint* if its linearizing operator is self-adjoint (i.e. $L = L^*$).

Note it is implicit in the above definition that a PDE system can be self-adjoint only if the number of dependent variables m is equal to the number of differential equations σ.

Theorem 1.5.8. *A PDE system is variational if and only if its linearizing operator is self-adjoint.*

The proof is given in [96].

Theorem 1.5.8 provides a criteria to check if Noether's theorem applies to a PDE system without explicitly knowing the Lagrangian. We now describe how the symmetry framework of Noether's theorem applies to variational systems of PDEs where the Lagrangian is known, and comment afterwards on how to proceed for systems where the Lagrangian is not known.

Definition 1.5.9. A Lie group of transformations with infinitesimal generator in evolutionary form

$$\hat{X} = \sum_{i=1}^{n}\sum_{j=1}^{m}\left[\eta^j(\vec{x},\vec{u}) - u_i^j\xi^i(\vec{x},\vec{u})\right]\frac{\partial}{\partial u^j}$$

is a *variational symmetry* of the Lagrangian $L = L(\vec{x},\vec{u},\partial\vec{u},\ldots,\partial^k\vec{u})$ if for any $u(x)$ there exists some set of functions $\Phi^i = \Phi^i(\vec{x},\vec{u},\partial\vec{u},\ldots,\partial^r\vec{u})$, $i = 1,2,\ldots,n$, such that

$$\hat{X}^{(k)}L \equiv \sum_{i}^{n} D_i\Phi^i. \tag{1.124}$$

Theorem 1.5.10. *Consider a variational PDE system $\vec{R} = 0$ with a variational symmetry $\hat{X}$ of the associated Lagrangian L. Then, the set of infinitesimals $\{\hat{\eta}^j\}_{j=1}^{\sigma}$ form a conservation law multiplier of the PDE system.*

The proof can be found in [22].

Conservation laws of a system of Euler-Lagrange PDEs can be sough from admitted variational symmetries of the Lagrangian. However, for a variational system for which the Lagrangian is not known, the following theorem is useful.

Theorem 1.5.11. *A variational symmetry of the Lagrangian L yields a local symmetry of the corresponding Euler-Lagrange system.*

The proof is shown in [22].

As such, for a self-adjoint PDE system where the Lagrangian is not known, one may seek conservation law multipliers as the infinitesimals of local symmetries (in evolutionary form) to the equations of motion. However, the converse to Theorem 1.5.11 is not true; a local symmetry of the Euler-Lagrange system may not necessarily be a variational symmetry of the associated Lagrangian. As such, it may be that not all local symmetries of a variational PDE system lead to conservations of the system.

Consider now the following examples of Noether's theorem.

Example 1.5.12. Consider a simple harmonic oscillator (HO) with a single spring attaching a mass m to a wall. The Lagrangian for the HO system is given as the difference between kinetic and potential energy,

$$L = K - U = m\frac{(x')^2}{2} - k\frac{x^2}{2}, \tag{1.125}$$

where $x = x(t)$ the displacement of the mass from equilibrium, k is the spring constant, and prime denotes differentiation.

The Lagrangian (1.125) has a time translation symmetry, given in characteristic form as

$$\hat{X} = x'\frac{\partial}{\partial t}.$$

Applying the Euler operator (1.96) yields the equation of motion

$$mx'' = kx. \tag{1.126}$$

Multiplying by the coefficient of the symmetry, $\hat{\eta} = x'$, yields the local conservation law

$$\mathrm{D}_t\left(m\frac{(x')^2}{2} - k\frac{x^2}{2}\right) = 0.$$

Example 1.5.13. Consider the linear wave equation governing the transverse motion of a string [56].

$$u_{tt} = c^2 u_{xx}, \tag{1.127}$$

where $u = u(x, t)$ is the displacement from equilibrium, and c a constant is the wavespeed.

A Lagrangian for (1.127) is

$$L = \frac{1}{2}(u_t)^2 - \frac{1}{2}c^2(u_x)^2. \tag{1.128}$$

By observation, L admits the u-translation symmetry $\mathrm{X}^1 = \frac{\partial}{\partial u}$ and time translation symmetry $\mathrm{X}^2 = -\frac{\partial}{\partial t}$ with corresponding evolutionary form

$$\hat{\mathrm{X}}^1 = \frac{\partial}{\partial u},$$

$$\hat{\mathrm{X}}^2 = u_t\frac{\partial}{\partial u}.$$

As such, from Noether's theorem, the linear wave equation (1.127) admits conservation law multipliers $\Lambda^1 = 1$ and $\Lambda^2 = u_t$. The resulting divergence forms of (1.127) are, respectively,

$$\mathrm{D}_t(u) - \mathrm{D}_x(c^2 u_x) = 0, \tag{1.129}$$

$$\mathrm{D}_t\left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2 u_x^2\right) - \mathrm{D}_x(c^2 u_x u_t) = 0. \tag{1.130}$$

Here, (1.129) corresponds to the conservation of linear momentum, and (1.130) to the conservation of energy.

Example 1.5.14. We wish to determine if the KdV equation

$$u_t + uu_x + u_{xxx} = 0 \tag{1.131}$$

is self-adjoint, where $u = u(x, t)$.

The linearizing operator of (1.131) is found from (1.122) as

$$\mathrm{L}[U] = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + U_x + \frac{\partial^3}{\partial x^3}. \tag{1.132}$$

The adjoint operator of (1.131) is found from (1.123) as

$$L^*[U] = -\frac{\partial}{\partial t} - U\frac{\partial}{\partial x} - \frac{\partial^3}{\partial x^3}. \tag{1.133}$$

Clearly, $L \neq L^*$, and so the KdV equation is not self-adjoint as written.

1.6 Discussion

In this Chapter, we presented the basic theory and methodology of Lie symmetry and conservation law analysis as it applies to studying differential equations. We began in Section 1.2 by introducing the concept of symmetry, precisely defining Lie point symmetries, and discussing several important concepts fundamental to Lie symmetry analysis. In Section 1.3, we presented the extension of Lie symmetries to differential equations, and several important applications therein. Then, in Section 1.4, we introduced the notions of conservation laws and briefly presented the mathematical formulation for them. Finally, Section 1.5 discussed the connection between Lie symmetries and conservation laws.

We now remark on the body of results for Lie symmetry and conservation law analysis, as well as further extensions of the theory not discussed in the current thesis.

Lie symmetries have been calculated for many systems of equations, of which a large number of results have been compiled into handbooks [66, 67]. While these books are not exhaustive, they provide a good starting place to seek Lie symmetries for many equations (e.g. various wave equations, equations of gas dynamics, equations of incompressible fluid motion, etc.). Examples of equivalence transformations applied to classification problems can be found in [13, 16, 25, 50, 70].

In addition, there are numerous works in which solutions of differential equations are constructed from admitted Lie point symmetries (see, for example, [21, 23, 43, 65–67, 96], as well as references therein). More recent examples include calculation of invariant solutions of a system of Kadomstev-Petviashvili equations [1], the Benjamin-Bona-Mahony-Burger equation [74], and the SIR model of epidemiology [97].

Solutions of DEs can be constructed from admitted Lie point symmetries by other methods than those discussed in the current thesis. Both Bluman and Kumei [23] and Hydon [65]

present how to construct an integrating factor (i.e. conservation law multiplier) for first order ODEs from admitted Lie point symmetries. Bluman and Anco [21] outline a general method of constructing integrating factors for higher order ODEs from the linearizing and adjoint operators of the ODE.

Abraham-Shrauner discusses 'hidden symmetries', which arise as a result of increasing the order of an ODE by a nonlocal transformation, or by reducing the order of an ODE by a symmetry reduction.

Of further interest is the application of Lie symmetries in numerical simulations. Symmetry preserving numerical schemes for ODEs are discussed by Levi and Winternitz in [78], and for PDEs by Rebelo and Valiquette in [99].

There are many extensions to Lie symmetries that we do not present in the current work. Bluman and Kumei [23], Bluman and Anco [21], and Bluman, Cheviakov, and Anco all discuss contact symmetries and higher order symmetries. Contact symmetries are symmetry transformations which depend upon the independent variables, dependent variables, and all first order derivatives of the dependent variables. Higher order symmetries (also known as Lie-Bäcklund symmetries) are symmetry transformations in which the infinitesimals depend upon a finite number of higher order derivatives.

Bluman [19] outlines a methodology to seek non-classical solutions of a PDE system. In particular, non-classical solutions are constructed by seeking Lie symmetries of an augmented PDE system consisting of the original PDE system with the invariant surface condition and its differential consequences [22, 65]. This methodology is applied in [37] to the Boussinesq equation. Murata obtains non-classical solutions for a hyperbolic PDE in [91], and for a nonlinear diffusion equation in [92].

Another extension is to seek approximate Lie symmetries of Baikov type [6] and Fuschich type [48]. Baikov type approximate symmetries are obtained by seeking 'approximate symmetry generators' for a PDE system expanded in a small parameter. Fuschich type approximate symmetries are sought as symmetries to a PDE system in which the dependent variables are perturbed from equilibrium. Examples of approximate symmetries can be found in [29, 103, 115].

Under more recent study are nonlocal symmetries of a PDE system, which are obtained

as local symmetries of a nonlocally related PDE system. A general discussion on seeking nonlocal symmetries appears in [2]. Bluman [15] uses potential systems (as discussed in Section 1.4.5) to seek nonlocal symmetries, of which the methodology is outlined in detail in Bluman, Cheviakov, and Anco [22]. Applications of nonlocal symmetries appear in, for example, [16, 17, 29, 49].

In addition to Lie symmetries, discrete symmetries of a PDE system may also be sought. Discrete symmetries are symmetry transformations that are not continuous [96]. Applications of discrete symmetries to differential equations, as well as an algorithm to seek discrete symmetries from the Lie algebra of a PDE system, are outlined in [65].

We now outline the body of results for conservation laws, as well as extensions to conservation law analysis.

The handbooks of symmetry results [66, 67] contain conservation laws for variational PDE systems, in which the Lagrangian of particular systems stated explicitly. The KdV equation has been well studied (see [51, 52, 73, 89, 90, 104]), and was shown to have an infinite sequence of conservation laws [90].

Methods to construct conservation law fluxes of a PDE system from known conservation law multiplers appears in [22]. Additionally, many results and references for conservation laws of PDE systems are given in [22].

Noether's theorem has been extended by Boyer [27], in which higher order conservation law multipliers are given by the infinitesimals of higher order symmetry transformations. See [22] for a concise statement and proof.

An extension to conservation law analysis is the study of nonlocal conservation laws of a PDE system, which arise from local conservation laws of a nonlocally related PDE system [22]. Bluman, Cheviakov, and Ivanova [18] present a method to seek nonlocal conservation laws of a PDE system from local conservation laws of potential systems. Further discussion can be found in [22].

For readers interested in a more general group theoretic approach to studying differential equations of physics, I direct you to the texts by Dresselhaus, Dresselhaus, and Jorio [44], and Marsden and Ratiu [84].

With the preliminary mathematical theory in tow, we proceed to introduce some basic

notions in the theory elastodynamics in Chapter 2. The remaining Chapters, 3 and 4, are each dedicated to applying Lie symmetry and conservation law analysis to study particular nonlinear systems of elastodynamics.

CHAPTER 2

EQUATIONS OF NONLINEAR ELASTODYNAMICS

2.1   Introduction

Elasticity theory is the study of properties and characteristics of elastic solids, and has a rich history in mathematics studying both static and dynamic systems (see, e.g., [105, 107, 110], and the preface by Antman to the third edition of [108]). The applications of elasticity in the physical sciences are numerous, and have a prominent role in both engineering and material science (see [26] for examples). Of particular interest are hyperelastic solids, a class of materials that act as 'ideal' elastic solids (i.e., stress depends only on deformation, and not on other factors such as the history or rate of deformation). In particular, the stress within a hyperelastic solid is related to the deformation through a strain energy density. Hyperelastic materials are modelled by a system of nonlinear coupled partial differential equations. The study of these equations has been extensive, with much study through use of analytical methods (see, e.g., [7, 36, 83, 108] and references therein). In application, solutions to this PDE system are typically sought by finite element methods (see, e.g., [9, 72, 113]).

In the last twenty five years, fiber reinforced hyperelastic materials have come under study with a prime application in modeling fibrous biological tissues, such as arteries [10, 59, 60] and skin [11, 100]. The fundamental notion in these models is to add an anisotropic component to the strain energy density which accounts for the fiber behaviour. Currently, solutions to these models are primarily sought through numerical methods due to the inherent complexity of the models (see, e.g., [10, 40, 100, 102, 112]).

Our interest in Chapter 3 is to study the Lie symmetries and conservation laws of an isotropic hyperelastic model, and a simple model of a fiber reinforced hyperelastic solid in Chapter 4. As such, in the current Chapter, we discuss the basic elements of the theory of

elastodynamics. We begin by introducing essential notions and quantities from continuum mechanics in Section 2.2. Afterwards, we derive the equations of motion for an elastic material in Section 2.3. Then, in Section 2.4, we discuss the constitutive relationships between stress and strain for models of elastic and hyperelastic materials. After, we discuss important models of isotropic hyperelasticity. Finally, we present the extension of the theory to fiber reinforced hyperelastic materials, and discuss several constitutive models prominent in the literature.

It should be noted that our primary interest in later Chapters is to study the Lie symmetries and conservation laws of equations of motion of hyperelasticity expressed in fixed coordinates, as is done in [35]. As such, in our brief presentation of mathematical elasticity, we make use of the term 'tensor' to be consistent with the terminology of Marsden and Hughes [83] and Ciarlet [36]. In actuality, we treat second order tensors as matrices with function-valued entries. As well, we informally use the term 'coordinates' to refer to a particular frame of reference fixed with Cartesian coordinates.

In addition to the notation established in Chapter 1, tensors and matrices are denoted in boldface.

2.2 Notions of Continuum Mechanics

Consider a solid body which, at time $t = 0$, occupies an open connected subset $\Omega_0 \subset \mathbb{R}^3$ with a continuous boundary. Ω_0 is called the *Lagrangian configuration* (or *reference configuration*). The *Eulerian configuration* (or *actual configuration*) is the region $\Omega \subset \mathbb{R}^3$ the body occupies at a time t. A *deformation* $\vec{\phi} : \Omega_0 \to \Omega$ is a smooth, invertible, orientation preserving transformation which maps *Lagrangian points* (or *material points*) $\vec{X} = (X^1, X^2, X^3) \in \Omega_0$ to *Eulerian points* (or *spatial points*) $\vec{x} = (x^1, x^2, x^3) \in \Omega$ at a fixed time t (see Figure 2.1); that is,

$$\vec{x} = \vec{\phi}(\vec{X}).$$

A *motion* of the body is a time dependent family of deformations, written as $\vec{x} = \vec{\phi}(\vec{X}, t)$ (alternatively $\vec{x} = \vec{x}(\vec{X}, t)$).
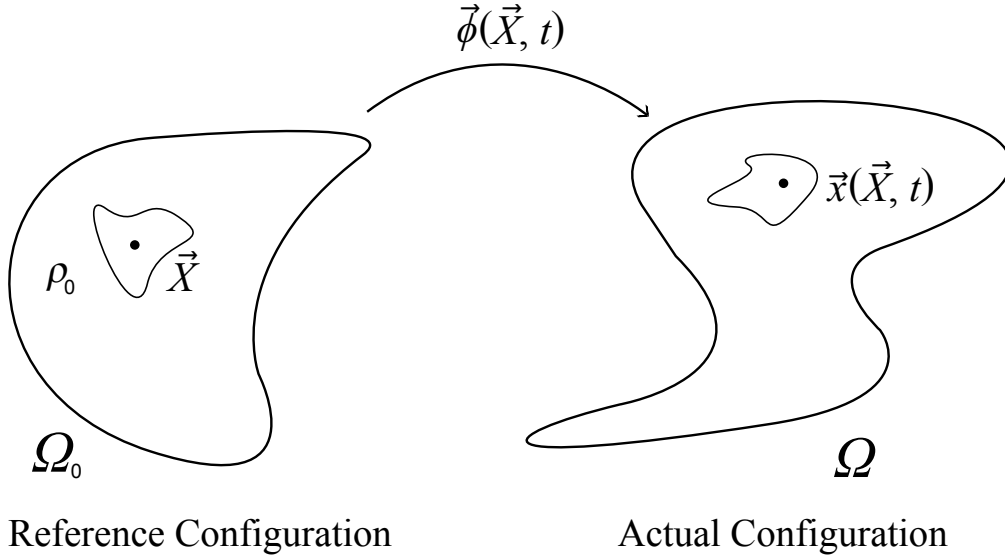
Figure 2.1: Lagrangian (Reference) and Eulerian (Actual) Configuration.

The *velocity* $V(\vec{X},\,t)$ of a material point is the time rate of change of motion

$$\vec{V}(\vec{X},\,t) = \frac{\partial \vec{\phi}}{\partial t}(\vec{X},\,t),$$

which is tangent to the motion at the Eulerian point $\vec{x}$. The *spatial velocity* $\vec{v}(\vec{x},\,t)$ is the time rate of change of the motion measured with respect to points in the Eulerian frame, and is equivalent to the material velocity (i.e. $\vec{V}(\vec{X},\,t) = \vec{v}(\vec{x},\,t)$ for $\vec{x} = \vec{\phi}(\vec{X},t)$).

For a material quantity $Q(\vec{X},t)$ with the equivalent spatial quantity $q(\vec{x},t) = Q(\vec{X},t)$, the *material derivative* is the time derivative with respect to t for $\vec{X}$ fixed; that is,

$$\frac{\partial Q}{\partial t}(\vec{X},t) = \mathrm{D}_t\, q(\vec{x}(\vec{X},t),t) = \frac{\partial q}{\partial t}(\vec{X},t) + (\vec{v}\cdot\nabla)q,$$

where ∇ is the gradient with respect to $\vec{X}$.

The *material acceleration* $\vec{A}(\vec{X},t)$ of a motion is

$$\vec{A}(\vec{X},t) \equiv \frac{\partial \vec{V}}{\partial t}(\vec{X},t) = \frac{\partial^2 \vec{x}}{\partial t^2}(\vec{X},t). \tag{2.1}$$

Using the material derivative, the material acceleration is given in terms of the spatial velocity $\vec{v} = \vec{v}(\vec{x}(\vec{X},t),t)$ as

$$\vec{A} = \mathrm{D}_t\vec{v} = \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v},$$

where ∇ is the gradient with respect to $\vec{X}$.

An important quantity in the subsequent theory is the *deformation gradient* $\mathbf{F}(\vec{X}, t)$, which is defined (in coordinates) as the Jacobian matrix of the motion $\vec{\phi}(\vec{X}, t)$; that is,

$$\mathbf{F}(\vec{X}, t) = \nabla \vec{\phi}(\vec{X}, t). \tag{2.2}$$

The determinant of the deformation gradient satisfies $J = \det \mathbf{F} > 0$ for all $\vec{X} \in \Omega_0$ since $\vec{\phi}(\vec{X}, t)$ preserves orientation.

2.3 Equations of Motion in Continuum Mechanics

We now derive the system of partial differential equations governing the motion of a solid. Traditionally, these PDEs are written in terms of the Eulerian configuration, or equivalently in terms of the Lagrangian configuration. We present both systems of PDEs, noting that we will study the system in terms of the Lagrangian configuration in later sections.

There are four quantities which describe the motion of a solid in three dimensional space: mass density, and the three components of position. As such, we require (at least) four equations to completely determine the motion of the system. The equations governing elasticity theory are derived from conservation of mass, balance of linear momentum, and balance of angular momentum.

2.3.1 Equations of Motion in Eulerian Coordinates

The partial differential equations describing the motion of a solid (or fluid) in terms of the Eulerian configuration are derived from: conservation of mass, balance of linear momentum, balance of angular momentum, and the Euler-Cauchy stress principle.

Transport Theorem

Of importance in the subsequent theory is the transport theorem, which describes how the time derivative and spatial integral interchange. For $q = q(\vec{x}, t)$ a spatial quantity with $\vec{x} \in A \subset \Omega$ with smooth boundary, the transport theorem states that [83]

$$\mathrm{D}_t \int_A q \, d\vec{x} = \int_A \left(\mathrm{D}_t q + q \operatorname{div}_{\vec{x}} \vec{v} \right) d\vec{x} = \int_A \left(\frac{\partial q}{\partial t} + \operatorname{div}_{\vec{x}} (q\vec{v}) \right) d\vec{x}, \tag{2.3}$$

where D_t is the material derivative, $\vec{v}(\vec{x}, t)$ is the spatial velocity, $\mathrm{div}_{\vec{x}}$ is the divergence with respect to $\vec{x}$, and $d\vec{x}$ denotes an infinitesimal volume element.

Conservation of Mass

Consider the body with *mass density* $\rho(\vec{x},\, t)$ in the Eulerian configuration Ω. For every open set $A \subset \Omega$ with smooth boundary, conservation of mass is

$$D_t \int_A \rho(\vec{x}, t)d\vec{x} = 0. \tag{2.4}$$

By the transport theorem (2.3), conservation of mass has the equivalent form

$$\int_A \left(D_t\rho + \rho\,\mathrm{div}_{\vec{x}}(\vec{v})\right) dx = 0. \tag{2.5}$$

By definition, equation (2.5) holds for every open set $A \subset \Omega$, and so the integrand must be identically zero. As such, the conservation of mass in the Eulerian frame has the equivalent PDE form

$$D_t\rho + \rho\,\mathrm{div}_{\vec{x}}(\vec{v}) = 0, \tag{2.6}$$

where $\vec{x} \in \Omega$.

Cauchy Stress

The *Stress Principle of Euler and Cauchy* posits the existence of a vector $\vec{t} = \vec{t}(\vec{x}, t, \vec{n})$ for any smooth, closed surface $S \subset \Omega$ of a deformed solid Ω at time t [32, 83]. In particular, the *Cauchy stress vector* $\vec{t} = \vec{t}(\vec{x}, t, \vec{n})$ originates from the point $\vec{x} \in S$, and measures the force $d\vec{f}$ per unit area da applied across a surface element with unit normal $\vec{n}$; that is, $d\vec{f} = \vec{t}(\vec{x}, t, \vec{n})da$.

Balance of Linear Momentum

Consider now that forces within the solid must balance by Newton's Second Law. For *applied body forces* with density $\vec{b} : \Omega \to \mathbb{R}^3$ acting on the solid, Newton's second law is

$$D_t \int_A \rho\vec{v}\,d\vec{x} = \int_{\partial A} \vec{t}(\vec{x},\, t,\, \vec{n})\,d\vec{x} + \int_A \rho\vec{b}\,d\vec{x}, \tag{2.7}$$

$$\vec{df} = \vec{t}da$$

$$\vec{n}$$

$$da$$

$$\vec{x}(\vec{X},\ t)$$

$$\Omega$$
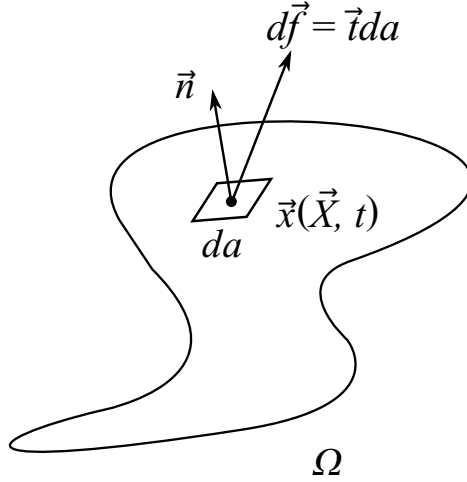
Actual Configuration

Figure 2.2: Cauchy stress vector $\vec{t}(\vec{x}, t, \vec{n})$ acting across an infinitesimal surface, where $\vec{x}$ is the point which $\vec{t}$ originates from, t is time, and $\vec{n}$ is normal to the infinitesimal surface.

where D_t is the material derivative, and $A \subset \Omega$ is an open subset with smooth boundary ∂A. Here, the Cauchy stress vector $\vec{t}(\vec{x}, t, \vec{n})$ is evaluated on the unit normal $\vec{n}$ to ∂A at $\vec{x} \in \partial A$. If (2.7) holds for every open subset $A \subset \Omega$ with a smooth boundary, then linear momentum is balanced [83].

In order to obtain the PDE system describing linear momentum, the Cauchy stress vector in equation (2.7) needs to be expressed in a form with explicit dependence on $\vec{n}$. As such, consider the following theorem.

Theorem 2.3.1. *If the Cauchy stress vector $\vec{t}(\vec{x}, \vec{n})$ is continuously differentiable with respect to $\vec{x} \in \Omega$ for each $\vec{n} \in S$ and continuous with respect to $\vec{n}$ for each $\vec{x}$, then balance of linear momentum (2.7) implies the existence of a unique tensor $\boldsymbol{\sigma}(\vec{x}, t)$ (the Cauchy Stress tensor) such that*

$$\vec{t}(\vec{x},\ \vec{n}) = \boldsymbol{\sigma}(\vec{x})\vec{n} \quad \forall\ \vec{x} \in \Omega,\ \vec{n} \in S \tag{2.8}$$

The proof can be found under more general conditions in [83].Of importance, equation (2.8) is one result of Cauchy's Theorem.

The PDE system equivalent to balance of linear momentum is derived as follows. Substi-

63

tuting (2.8) into (2.7) and applying the divergence theorem yields

$$D_t \int_A \rho \vec{v} \, d\vec{x} = \int_A \left(\mathrm{div}_{\vec{x}} \, \boldsymbol{\sigma} + \rho \vec{b} \right) d\vec{x}. \tag{2.9}$$

Applying the transport theorem to (2.9), and substituting the conservation of mass (2.6), we obtain

$$\int_A \left(\rho D_t \vec{v} - \mathrm{div}_{\vec{x}} \, \boldsymbol{\sigma} - \rho \vec{b} \right) d\vec{x} = 0. \tag{2.10}$$

The integral (2.10) must hold for any subset $A \in \Omega$. As such, since $D_t \vec{v}$ and $\boldsymbol{\sigma}$ are continuously differentiable and $\vec{b}$ is continuous, the integrand must be identically zero. Thus, balance of linear momentum is equivalent to

$$\rho D_t \vec{v} = \mathrm{div}_{\vec{x}} \, \boldsymbol{\sigma} + \rho \vec{b}. \tag{2.11}$$

Balance of Angular Momentum

The integral form of balance of angular momentum is

$$D_t \int_A \vec{x} \times \rho \vec{v} \, d\vec{x} = \int_{\partial A} \vec{x} \times \vec{t}(\vec{x}, \, t, \, \vec{n}) \, d\vec{a} + \int_A \vec{x} \times \rho \vec{b} \, d\vec{x}. \tag{2.12}$$

Assuming conservation of mass (2.6) and balance of linear momentum (2.14) hold, then balance of angular momentum (2.12) can be shown to be equivalent to symmetry of the Cauchy stress tensor (see Theorem 2.10 in [83]),

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{2.13}$$

In the current Thesis, we consider balance of angular momentum in the Eulerian configuration in the form of equation (2.13).

Equations of Motion

To summarize, the equations of motion for a solid in terms of the Eulerian configuration arise from conservation of mass (2.6), balance of linear momentum (2.11), and balance of angular momentum (2.13) as:

$$D_t \rho + \rho \, \mathrm{div}_{\vec{x}} \left(\vec{v} \right) = 0, \tag{2.14a}$$

$$\rho D_t \vec{v} = \mathrm{div}_{\vec{x}} \, \boldsymbol{\sigma} + \rho \vec{b}, \tag{2.14b}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T. \tag{2.14c}$$

Here, mass density $\rho(\vec{x}, t)$ and velocity $\vec{v}(\vec{x}, t)$ are unknown quantities. The stress $\boldsymbol{\sigma}$ is determined by its relationship to the strain within the solid (i.e. the deformation). We do not explicitly discuss the relationship between Cauchy stress and strain in the current thesis since we work with the equations of motion in terms of the Lagrangian configuration (derived in Section 2.3.2).

2.3.2   Equations of Motion in Lagrangian Coordinates

While the Eulerian picture is fruitful, it is convenient to express the equations of motion with respect to the Lagrangian frame of reference (see, e.g., [10, 59, 76, 83]). In the current section, we re-express the equations of motion (2.14) with respect to points $\vec{X}$ in the Lagrangian configuration.

Conservation of Mass

Define the mass density of the Lagrangian configuration as $\rho_0(\vec{X})$. Conservation of mass (2.4) can equivalently be stated as

$$\int_A \rho(\vec{x}, t) \, d\vec{x} = \int_{A_0} \rho_0(\vec{X}) \, d\vec{X}, \tag{2.15}$$

where $A_0 \in \Omega_0$ is the domain such that $\vec{\phi} : A_0 \to A$.

By a change of variables, we have that

$$\int_A \rho(\vec{x}) \, d\vec{x} = \int_{A_0} \rho(\vec{x}(\vec{X}, t), t) J(\vec{X}, t) \, d\vec{X}, \tag{2.16}$$

where $J = \det \mathbf{F}$ is the Jacobian determinant of the change of variables. Equating the right hand sides of (2.15) and (2.16), and noting this holds for all open subsets $A_0 \subset \Omega_0$, yields

$$\rho_0 = J\rho. \tag{2.17}$$

As such, mass conservation can be viewed as equivalent to (2.17).

Remark 2.3.2. The mass density in the Lagrangian configuration $\rho_0(\vec{X})$ is specified at the outset of the problem, and does not change with deformation. As such, mass conservation

does not yield an equation of motion with respect to the Lagrangian configuration. However, it will play a role when deriving the equations of motion in terms of Lagrangian coordinates, as will be seen in subsequent sections.

Stresses

In the Lagrangian frame, there is a similar notion to the Cauchy stress vector. The *first Piola-Kirchhoff stress vector* $\vec{T}(\vec{X}, t, \vec{N})$ is the force $d\vec{f}$ per unit undeformed area dA with unit normal vector $\vec{N}$ (see Figure 2.3) [36, 83]; that is, $d\vec{f} = \vec{T}(\vec{X}, t, \vec{N})dA$.
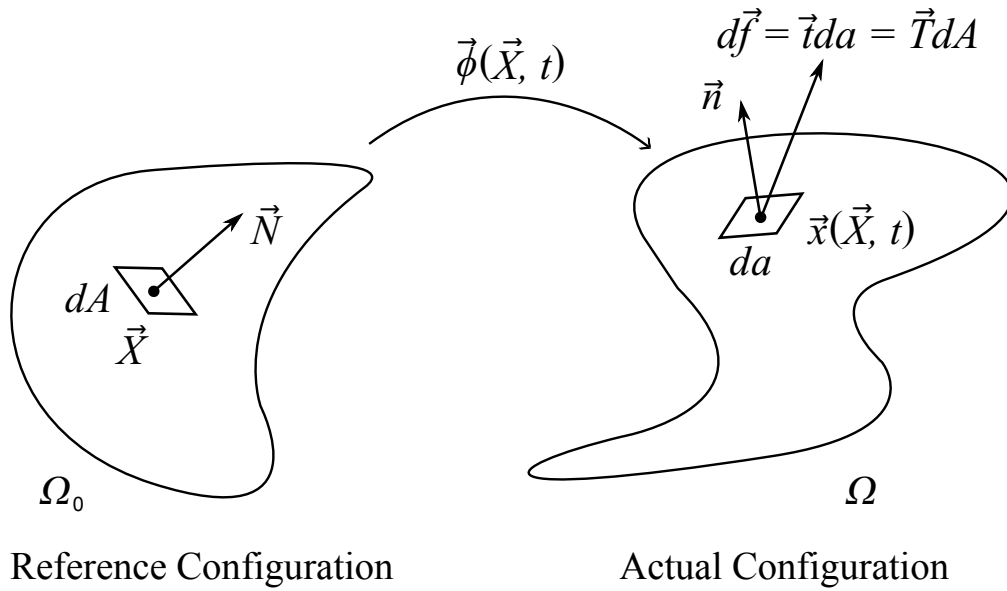
Figure 2.3: First Piola-Kirchhoff stress vector $\vec{T}(\vec{X}, t, \vec{N})$ acting across a surface.

The Piola-Kirchhoff stress vector, like the Cauchy stress vector, can be represented in terms of a stress tensor multiplied by the unit normal vector:

$$\vec{T} = \mathbf{P}\vec{N}, \tag{2.18}$$

where $\mathbf{P} = \mathbf{P}(\vec{X}, t)$ is the *first Piola-Kirchhoff stress tensor*.

Due to preservation of the force when changing between Lagrangian and Eulerian reference frames (i.e $\vec{T} dA = \vec{t} da$, where dA is an infinitesimal area in the Lagrangian configuration, and da is an infinitesimal area in the Eulerian configuration), the first Piola-Kirchhoff stress tensor can be shown to be related to the Cauchy stress tensor through the *Piola transform* [36, 83],

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}. \tag{2.19}$$

66

An important property is how the divergence of the stress tensors behaves under the Piola transform, given in the following theorem.

Theorem 2.3.3. *Consider the first Piola-Kirchhoff stress tensor* $\mathbf{P}$ *and and Cauchy stress tensor* $\boldsymbol{\sigma}$ *related by the Piola transform* (2.19). *Then,*

$$\mathrm{div}_{\vec{x}}\boldsymbol{\sigma} = J\mathrm{div}_{\vec{X}}\mathbf{P}, \tag{2.20}$$

where $\mathrm{div}_{\vec{X}}$ *is the divergence with respect to* $\vec{X}$, $\mathrm{div}_{\vec{x}}$ *is the divergence with respect to* $\vec{x}$.

The proof can be found in [36].

Balance of Linear Momentum

Balance of linear momentum with respect to the Eulerian configuration (2.14b) can be expressed in terms of the Lagrangian configuration as follows. Substituting (2.17), (2.19), (2.20), and $\mathrm{D}_t\vec{v} = \vec{V}_t \equiv \vec{x}_{tt}$ into (2.14b) and simplifying yields

$$\rho_0 \frac{\partial^2 \vec{x}}{\partial t^2} = \mathrm{div}_{\vec{X}}\mathbf{P} + \rho_0 \vec{B} \tag{2.21}$$

where $\vec{x} = \vec{x}(\vec{X}, t)$, $\mathrm{div}_{\vec{X}}$ is the material divergence, and $\vec{B}(\vec{X}, t) = \vec{b}(\vec{x}, t)$ is the body force with respect to the Lagrangian configuration.

Balance of Angular Momentum

Balance of angular momentum in the Eulerian configuration (2.14c) can be expressed with respect to the Lagrangian configuration by applying the Piola transform (2.19), which yields

$$\mathbf{F}\mathbf{P}^T = \mathbf{P}\mathbf{F}^T. \tag{2.22}$$

Equations of Motion

To summarize, the equations describing motion of a continuous object expressed in terms of the Lagrangian configuration are

$$\rho_0 \frac{\partial^2 \vec{x}}{\partial t^2} = \mathrm{div}_{\vec{X}}\mathbf{P} + \rho_0 \vec{B}, \tag{2.23a}$$

$$\mathbf{F}\mathbf{P}^T = \mathbf{P}\mathbf{F}^T. \tag{2.23b}$$

Here, the independent variables are the Lagrangian coordinates $\vec{X}$ and time t, and the dependent variables are the Eulerian coordinates $\vec{x}(\vec{X}, t)$. There are six equations governing the motion from balance of linear momentum and balance of angular momentum. The stress $\mathbf{P}$ depends on the motion of the solid, for which the specific relationship is determined from properties of the material under study. We discuss the particular form of the stress in terms of strain for elastic solids in the next section.

2.4 Constitutive Relationships for Elastic Materials

The equations of motion for continuum mechanics have been derived in (2.14) and (2.23) for a given mass density ρ_0. This PDE system cannot be studied until a relationship between stress and strain is specified.

In the current section, we present the relationship between the first Piola-Kirchhoff stress and strain for elastic and hyperelastic materials. Afterwards, we discuss several material properties which restrict the stress-strain relationship, such as, e.g., isotropy of the material. The exact PDEs governing motion of hyperelastic materials are then stated. Then, specific strain energy densities for isotropic hyperelastic materials appearing in the literature are discussed. Lastly, we discuss how the theory is extended to model fiber reinforced materials, as well as examples of forms of the strain energy density of these models.

2.4.1 Elasticity and Hyperelasticity

Consider Hooke's Law, which states that the restoring force acting on a mass attached to a spring is proportional to the displacement of the mass from equilibrium. In continuum mechanics, elastic solids follow broad strokes of Hooke's Law in that the stress is a function of the deformation. A material is said to be *elastic* if the first Piola-Kirchhoff stress can be expressed as a function $\hat{\mathbf{P}}$ of material points $\vec{X}$ and the deformation gradient $\mathbf{F}$ (i.e. $\mathbf{P}(\vec{X}, t) = \hat{\mathbf{P}}(\vec{X}, \mathbf{F}(\vec{X}, t)))$ [83].

A material is *hyperelastic* if there exists a *stored energy function* $W(\vec{X}, \mathbf{F})$ such that

$$\hat{\mathbf{P}}(\vec{X}, t) = \rho_0(\vec{X}) \frac{\partial W}{\partial \mathbf{F}}(\vec{X}, t). \tag{2.24}$$

Hyperelastic models assume that the stress is related only to the material points $\vec{X}$ and the deformation $\mathbf{F}$, and is independent of history and rate of loading [26].

2.4.2 Frame Indifference and Material Properties of Isotropy, Homogeneity, and Incompressiblity

We now present assumptions on the form of the first Piola-Kirchhoff stress based on the axiom of material frame indifference (i.e. objectivity), and the material properties of isotropy, homogeneity, and incompressibility. We state the form of the stress for each assumption, and the respective form of the strain energy density for hyperelastic materials.

The axiom of material frame indifference states an observable quantity must be independent of the orthogonal basis in which it is measured [36]. In elasticity, the observable quantity of interest is the stress.

Axiom 2.4.1. Let the deformation $\vec{\phi} : \Omega_0 \to \Omega$ be rotated into another deformation $\vec{\phi}' : \Omega_0 \to \Omega'$, i.e. $\vec{\phi}' = \mathbf{Q}\vec{\phi}$ for proper orthogonal matrix $\mathbf{Q}$. Then,

$$\hat{\mathbf{P}}(\vec{X}, \mathbf{QF}) = \hat{\mathbf{P}}(\vec{X}, \mathbf{F}) \quad \forall \vec{X} \in \Omega_0. \tag{2.25}$$

For hyperelastic materials with strain energy density $W(\vec{X}, \mathbf{F})$, material frame indifference is equivalent to

$$W(\vec{X}, \mathbf{QF}) = W(\vec{X}, \mathbf{F}) \quad \forall \vec{X} \in \Omega_0. \tag{2.26}$$

An important class of materials are *isotropic* materials, for which material properties are the same in each direction [36].

Definition 2.4.2. An elastic material is *isotropic* at a point $\vec{X} \in \Omega_0$ if, for any proper orthogonal matrix $\mathbf{Q}$,

$$\hat{\mathbf{P}}(\vec{X}, \mathbf{FQ}) = \hat{\mathbf{P}}(\vec{X}, \mathbf{F})\mathbf{Q}. \tag{2.27}$$

Furthermore, a material is isotropic if it is isotropic at every point $\vec{X} \in \Omega_0$.

For hyperelastic materials, the material is isotropic if

$$W(\vec{X}, \mathbf{FQ}) = W(\vec{X}, \mathbf{F})\mathbf{Q} \quad \forall \vec{X} \in \Omega_0. \tag{2.28}$$

Another important class of material are *homogeneous* materials [36, 83].

Definition 2.4.3. An elastic material is *homogeneous* if the first Piola-Kirchhoff stress does not depend explicitly on the material points $\vec{X} \in \Omega_0$; that is

$$\hat{\mathbf{P}}(\vec{X}, \mathbf{F}) = \hat{\mathbf{P}}(\mathbf{F}). \tag{2.29}$$

Similarly, a hyperelastic material is homogeneous if

$$W(\vec{X}, \mathbf{F}) = W(\mathbf{F}). \tag{2.30}$$

Lastly is the important class of *incompressible* (or *isochoric*) materials, for which the volume is preserved as the solid undergoes deformation [83].

Definition 2.4.4. An elastic material is *incompressible* if the deformation undergoes no change in volume; that is, for any open subset $A_0 \in \Omega$ with smooth boundary and A such that $\vec{\phi} : A_0 \to A$,

$$\int_A d\vec{x} = \int_{A_0} d\vec{X}. \tag{2.31}$$

Proposition 2.4.5. *The incompressibility condition* (2.31) *is equivalent to*

$$J = 1, \tag{2.32}$$

where $J = \det \mathbf{F}$.

Proof. By a change of variables, we have

$$\int_A d\vec{x} = \int_{A_0} J \, d\vec{X}, \tag{2.33}$$

where $J = \det \mathbf{F}$. As such, comparing (2.31) and (2.33) yields $J = 1$. $\qquad\square$

Remark 2.4.6. It can be shown that a Lagrange multiplier $p = p(\vec{X}, t)$ (*hydrostatic pressure*) arises in the first Piola-Kirchhoff stress for the equation of constraint (2.32) [83]. In particular, for incompressible materials, the first Piola-Kirchhoff stress takes the form

$$\mathbf{P}(\vec{X}, t) = \hat{\mathbf{P}}(\vec{X}, \mathbf{F}(\vec{X}, t)) - p(\vec{X}, t)\mathbf{F}^{-T}(\vec{X}, t). \tag{2.34}$$

2.4.3 Special form of Strain Energy Density for Isotropic, Frame Indifferent, Homogeneous Hyperelastic Materials

For isotropic, frame indifferent, homogeneous, hyperelastic materials, the strain energy density can be expressed as a function of the principal invariants of the tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. In the literature, many constitutive models of hyperelastic materials are presented in terms of these invariants. As such, we present the principal invariants of $\mathbf{C}$, as well as the representation of the strain energy density as a function of these invariants.

First consider the polar decomposition of deformation gradient $\mathbf{F}$. The determinant of the deformation gradient satisfies $J = \det \mathbf{F} > 0$ since the deformation $\vec{\phi}$ preserves orientation. As a consequence, $\mathbf{F}$ is invertible and can be represented uniquely in the polar decomposition [83]

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR},$$

where $\mathbf{R}$ is a proper orthogonal matrix, and $\mathbf{U}$ and $\mathbf{V}$ are symmetric positive-definite matrices.

Consider the *right Cauchy-Green tensor* $\mathbf{C} = \mathbf{F}^T\mathbf{F} = U^2$. The following quantities defined in terms of $\mathbf{C}$ can be shown to be invariant under an orthogonal transformation [83]:

$$I^1(\mathbf{C}) = \mathrm{Tr}\mathbf{C}, \tag{2.35a}$$

$$I^2(\mathbf{C}) = \frac{1}{2}\left(\mathrm{Tr}(\mathbf{C})^2 - \mathrm{Tr}(\mathbf{C}^2)\right), \tag{2.35b}$$

$$I^3(\mathbf{C}) = \det\mathbf{C}. \tag{2.35c}$$

Note that each invariant can be shown to be positive by expressing each invariant in terms of the eigenvalues of $\mathbf{C}$, and then noting that the eigenvalues of $\mathbf{C}$ are positive (since $\mathbf{C}$ is positive-definite) [36].

Theorem 2.4.7. *The strain energy density $W(\mathbf{F})$ of a homogeneous hyperelastic material is frame indifferent and isotropic if and only if*

$$W(\mathbf{F}) = W(I^1, I^2, I^3). \tag{2.36}$$

The proof appears in [36].

In Section 2.4.5, constitutive relations between the stress and strain will be stated in terms of the invariants (2.35a)-(2.35c).

Remark 2.4.8. For finite-deformation models of nearly incompressible materials, the following set of invariants are sometimes used [26, 59, 75, 112]:

$$\bar{I}^1 = (J)^{-2/3} I^1, \quad \bar{I}^2 = (J)^{-4/3} I^2, \quad \bar{I}^3 = J, \tag{2.37}$$

where $(J)^{-1} = 1/J$. The invariants (2.37) are used to circumvent computational problems that arise in simulations of nearly incompressible materials [75, 112].

2.4.4 Equations of Motion for Frame Indifferent, Homogeneous Hyperelastic Materials

The motion of a frame indifferent, homogeneous, hyperelastic solid in terms of the Lagrangian coordinates $\vec{X}$ is governed by equations (2.23) for the stress-strain relationship established in Sections 2.4.1-2.4.3:

$$\rho_0 \frac{\partial^2 \vec{x}}{\partial t^2} = \text{div}_{\vec{X}} \, \mathbf{P} + \rho_0 \vec{B}, \tag{2.38a}$$

$$\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T, \tag{2.38b}$$

$$\mathbf{P} = \rho_0 \frac{\partial W}{\partial \mathbf{F}}, \tag{2.38c}$$

where $\rho_0 = \rho_0(\vec{X})$ is the mass density, $\vec{x} = \vec{x}(\vec{X}, t)$ are Eulerian coordinates, $\mathbf{P} = \mathbf{P}(\vec{X}, t)$ is the first Piola-Kirchhoff stress, $\vec{B} = \vec{B}(\vec{X}, t)$ are the body forces, $\mathbf{F} = \mathbf{F}(\vec{X}, t)$ is the deformation gradient (2.2), $W = W(I^1, I^2, I^3)$ is the strain energy density, and $\{I^1, I^2, I^3\}$ are the invariants (2.35).

For incompressible materials, the equations of motion with respect to the Lagrangian coordinates are

$$\rho_0 \frac{\partial^2 \vec{x}}{\partial t^2} = \text{div}_{\vec{X}} \, \mathbf{P} + \rho_0 \vec{B}, \tag{2.39a}$$

$$\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T, \tag{2.39b}$$

$$\mathbf{P}(\vec{X}, t) = \frac{\partial W}{\partial \mathbf{F}} - p\mathbf{F}^{-T}, \tag{2.39c}$$

$$J \equiv \det \mathbf{F} = 1, \tag{2.39d}$$

where $p = p(\vec{X}, t)$ is the hydrostatic pressure.

As such, one needs only to specify the exact relationship between the strain energy density W and the invariants (2.35) of the material under study.

Remark 2.4.9. It can be shown that (2.38b) (equation (2.39b) for incompressible materials) is an identity for isotropic materials [35].

2.4.5   Constitutive Laws for Particular Nonlinear Hyperelastic Models

In the current section, we review particular models of isotropic, frame indifferent, homogeneous hyperelastic materials. Applications of each model are provided in the Chapter Discussion.

Saint Venant-Kirchhoff Model

The 'simplest' nonlinear model of a hyperelastic solid is the Saint Venant-Kirchhoff model [36], which has strain energy density

$$W = \frac{\lambda}{2}(\text{Tr}(\mathbf{E}))^2 + \mu\text{Tr}(\mathbf{E}^2), \tag{2.40}$$

where $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ is the Green-Saint Venant tensor, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, and λ and μ are the Lamé constants of a material.

Ogden Model

The strain energy density of an Ogden material is given by [36, 83]

$$W = \sum_{i=0}^{m} a_i(\text{Tr}(\mathbf{C}^{\gamma_i/2}) - 3) + \sum_{i=0}^{n} b_i(\text{Tr}(\mathbf{Cof}\,\mathbf{C}^{\delta_i/2}) - 3) + \Gamma(I^3), \tag{2.41}$$

where $a_i > 0$, $\gamma_i \geq 1$, $b_j > 0$, and $\delta_j \geq 1$ are model parameters; $\mathbf{C} = \mathbf{F}^T\mathbf{F}$; $\mathbf{Cof\,C}$ is the matrix of cofactors of $\mathbf{C}$; and, $\Gamma : (0, \infty) \to \mathbb{R}$ is a convex function such that $\lim_{\delta \to 0^+} \Gamma(\delta) \to +\infty$ and suitable growth conditions as $\delta \to +\infty$ [36]. Constants m and n are chosen to obtain an accurate fit of the model to experimental data.

Of note, the '-3' in (2.41) are normalization constants to ensure the strain energy density W is zero for the identity deformation (i.e. $\mathbf{F} = \mathbf{I}$). Without a loss of generality, these '-3'

can be dropped since they disappear upon differentiation in (2.38b) and (2.39b). As well, $\Gamma(I^3)$ is a constant in the incompressible materials, and so can be dropped [36].

Mooney-Rivlin Model

A sub-class of Ogden materials that model nonlinear dynamics are the compressible Mooney-Rivlin materials [36] (also called Hadamard materials [35, 47, 83]). Upon restricting (2.41) for $m = 0$, $n = 0$, $a_0 = a$, $b_0 = b$, and $\gamma_0 = \delta_0 = 2$, the Mooney-Rivlin strain-energy density can be represented in terms of the invariants (2.35a)-(2.35c) as

$$W = a(I^1 - 3) + b(I^2 - 3) + \Gamma(I^3), \tag{2.42}$$

where the constants $a > 0$ and $b > 0$ are material parameters to be determined experimentally. The '-3' are normalization constants so that $W = 0$ as in (2.41). Again, the '-3' terms can be ignored in general, as well as $\Gamma(I^3)$ in the incompressible case.

Of note, taking $b = 0$ yields the neo-Hookean strain-energy density

$$W = a(I^1 - 3) + \Gamma(I^3), \tag{2.43}$$

where $a > 0$.

Generalized Mooney Rivlin Model

An extension of the Mooney-Rivlin model is the generalized Mooney-Rivlin model, also called the generalized polynomial-type material [57], with strain energy density

$$W = \sum_{i=0}^{m} \sum_{j=0}^{n} c_{ij}(I^1 - 3)^i(I^2 - 3)^j + \Gamma(I^3), \tag{2.44}$$

where c_{ij} are material constants, and $\{I^1, I^2, I^3\}$ are the invariants (2.35). The '-3' can again be dropped, as well as $\Gamma(I^3)$ in the incompressible case. As in the Ogden model (2.41), constants m and n are chosen such that the model accurately fits experimental data.

Model of Limiting Polymeric Chain Extensibility

For incompressible materials composed of polymers (e.g. rubber bands), the polymeric chains are limited in in the maximum length they may stretch [64]. The isotropic strain energy

density proposed by Gent [53] to model an incompressible material composed of polymers is

$$W = -\frac{\mu}{2} k \log \left(1 - \frac{I^1 - 3}{k} \right), \tag{2.45}$$

where μ is the shear modulus, and k is a limiting value such that $I^1 - 3 < k$. Here, the strain energy density W takes into account limiting stretch length since it approaches infinity as I^1 approaches $k + 3$.

2.4.6 Extension to Hyperelastic Materials with Embedded Fibers

We now present the theory for fiber reinforced isotropic hyperelastic materials. Examples of materials which display this type of composition are arteries (see [10, 59, 60]) and skin (see [11, 100]). We limit our discussion to one fiber bundle, and comment on the multiple fiber bundles after.

A *fiber bundle* is a collection of fibers in Ω_0 oriented along a unit vector A^1. The reinforcement of an isotropic hyperelastic material by a fiber bundle is modeled by adding an anisotropic term to the strain energy density corresponding to the fiber bundle [10, 39, 59]; that is, the strain energy density takes the form

$$W = W_{iso} + W_{aniso}, \tag{2.46}$$

where W_{iso} corresponds to the isotropic matrix, and W_{aniso} captures the anisotropic effects of the fiber bundle.

The anisotropic strain energy is constructed from new invariants corresponding to the fiber behaviour,

$$I^4 = \left(A^1 \right)^T \mathbf{C} \left(A^1 \right), \tag{2.47a}$$

$$I^5 = \left(A^1 \right)^T \mathbf{C^2} \left(A^1 \right), \tag{2.47b}$$

where $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ is the right Cauchy stress tensor, and $\mathbf{C^2} = \mathbf{CC}$. Here, I^4 accounts for deformations that modify the length of the fiber, and I^5 to the effect of the fiber on shear response in the material [39, 88].

The anisotropic component of the strain energy density is then modeled as a functions of the invariants I^4 and I^5,

$$W_{aniso} = f \left(I^4 - 1, I^5 - 1 \right), \tag{2.48}$$

where the '-1' are normalization constants as in (2.41).

For this model, we obtain the same general equations of motion (2.38) (equations (2.39) for incompressible materials) in Section 2.4.4, noting $\mathbf{PF}^T = \mathbf{FP}^T$ may no longer hold identically due to the anisotropy of the material.

Note that multiple fiber bundles can be taken into account by letting the anisotropic strain energy density depend on invariants specific to each fiber bundle, and on invariants which capture fiber bundle interactions [59].

2.4.7 Particular Strain Energy Densities for Fiber Reinforced Materials

We now discuss particular strain energy densities applicable to modeling fiber reinforced materials. We again restrict our attention to one fiber family.

Note that many of the models discussed assume the material is incompressible. This is common practice in modeling fiber reinforced biological materials, as can be seen in the references for each model.

Standard Reinforcing Model

The standard reinforcing model of an incompressible material has neo-Hookean isotropic strain energy density (2.43) and the anisotropic strain energy density [40, 64]

$$W_{aniso} = c(I^4 - 1)^2, \tag{2.49}$$

where $c > 0$ is a material constant.

Arterial Models

The Gasser-Holzapfel-Ogden model is a model of an artery is proposed by Holzapfel, Gasser, and Ogden [59] (see also [60]), in which the isotropic strain energy density is neo-Hookean (2.43), and the anisotropic strain energy density for a single fiber family is of the form

$$W_{aniso} = \frac{k^1}{k^2} \left(e^{k^2(I^4 - 1)^2} - 1 \right), \tag{2.50}$$

where k^1 and k^2 are material parameters.

The choice of an exponential function is due to the strong stiffening of the arterial tissue at higher loads, which is thought to be due to the fiber bundles [59, 60].

An additional incompressible artery model can be found in [10], where the isotropic component is neo-Hookean (2.43), and the anisotropic component is a nonlinear polynomial.

$$W_{aniso} = \sum_{k=2}^{n} c^k \left(I^4 - 1 \right)^k, \tag{2.51}$$

where c^k are material constants to be determined, and n is chosen such that model accuracy fits the data. The polynomial form of (2.51) was taken with $n = 6$ in [10] to optimize finite element computations.

Model of Shear Response in Fiber Reinforced Materials

Merodio and Ogden [88] test the effect of a fiber family on shear response of an incompressible material by taking a strain energy density with neo-Hookean isotropic strain energy density (2.43) and anisotropic strain energy density of the form

$$W_{aniso} = c(I^5 - 1)^2, \tag{2.52}$$

where c is a material parameter.

Namani and Bayly [93] study at shear waves in anisotropic tissue, in which the isotropic strain energy density is of a compressible neo-Hookean material, while the anisotropic strain energy density is

$$W_{aniso} = c(I^5 - (I^4)^2), \tag{2.53}$$

for c a material parameter.

Limited Fiber Extension Models

Horgan and Saccomandi [64] derive two anisotropic strain energy densities to account for the limited extensibility of fibers as

$$W_{aniso} = -\mu k^1 \left((I^4 - 1) + k^1 \ln \left(1 - \frac{I^4 - 1}{k^1} \right) \right), \tag{2.54}$$

$$W_{aniso} = -\mu k^2 \ln \left(1 - \frac{(I^4 - 1)^2}{k^2} \right), \tag{2.55}$$

where μ is a shear modulus measuring the degree of anisotropy, and k^1 and k^2 are constants that measure rigidity of the fiber reinforcement. Note that $I^4 - 1 < k^i$, $i = 1, 2$, such that the stress $\mathbf{P}$ is finite.

Materials with limited fiber extensibility can be modelled by both (2.54) and (2.55). Horgan and Saccomandi comment that model (2.54) exhibits a stronger stiffening effects compared to (2.55) in a uniaxial load test [64]. However, due to the similarity between the models, they focused on (2.55) due to its simpler form.

2.5   Discussion

In this Chapter, we presented the relevant notions from the theory of elastodynamics to be used in later chapters. We discussed the coordinates of continuum mechanics, the material derivative, and the deformation gradient. Additionally, we derived the equations of motion in terms of the Cauchy stress (in Eulerian coordinates) and the first Piola-Kirchhoff stress (in Lagrangian coordinates) from conservation of mass, balance of linear momentum, and balance of angular momentum. The form of the Piola-Kirchhoff stress was presented for isotropic, homogeneous, hyperelastic solids under the assumption of material frame indifference. Afterwards, examples of the stress-strain relationship were provided for isotropic materials, including the Mooney-Rivlin and Ogden models. Lastly, we presented the form of the strain energy density of fiber reinforced materials, and gave examples of this density commonly used in the literature.

We now remark on results pertaining to existence and uniqueness of solutions in elastostatics and elastodynamics from the work of Marsden and Hughes [83]. The linear theory of elasticity requires the strong ellipticity of the first Piola-Kirchhoff stress for 'good existence and uniqueness' of solutions in elastostatics, and local existence and uniqueness for small times in elastodynamics. In nonlinear elastostatics of hyperelastic materials, global existence and uniqueness has been shown for a polyconvex strain energy density. Regarding nonlinear elastodynamics, conditions have been derived in [83] for global existence and uniqueness for a compressible semilinear system of equations (see Theorem 5.1, Corollary 5.4, and Example 5.7), and for local existence and uniqueness in time of solutions to compressible quasilinear

systems (see Theorems 5.9 and and 5.10, and Example 5.12). Ball provides a detailed discussion of further results with many references, including an important review of open problems in elastostatics and thermoelasticity [7].

We now discuss applications of, and extensions to, models of hyperelastic solids presented in the current Chapter, after which we discuss the same of fiber reinforced models.

The Saint-Venant Kirchhoff model has been applied primarily in numerical simulations. A finite element method tailored to the Saint Venant-Kirchhoff model for improved computational efficiency is presented in [72]. As well, the model is utilized in [94] to test a newly developed numerical method in scenarios of material compression and buckling.

The Ogden model can be simplified by excluding the cofactor terms in equation (2.41) (see, e.g., [9, 26, 109]). Applications of the Ogden model are as follows. Material parameters for the Ogden model have been found for internal organs in [109]. Basar and Itskov [9] develop a finite element numerical scheme for models of rubber-like shells as incompressible Ogden materials. Further discussion of the Ogden model and its applications appears in [26].

Lie symmetries and equivalence transformations of the compressible Mooney-Rivlin model have been found in [35] for $\Gamma(I^3) = -cI^3 - (d/2)\log(I^3)$ in (2.42), $c, d \geq 0$. Testing of a numerical method using the Mooney-Rivlin model is done in [94]. Further discussion of the Mooney-Rivlin model and its applications can be found in [26, 109].

Hartmann [57] tests the assumption of positive parameters for the generalized Mooney-Rivlin model for validity of modeling physical behaviour. Parameters for internal organs are found in [109] for biological tissues modeled as generalized Mooney-Rivlin materials.

Horgan and Saccomandi [62] test phenomenological models of incompressible isotropic solids accounting for limited chain extensibility against experimental data, including the Gent model (2.45). Generalizations of the Gent model for incompressible materials are studied in [61], and for compressible materials in [63].

In [33], continuum models are introduced that account for effects of micro-structures in the material, such as strain-softening behaviour and wave propagation as a result of dispersion effects.

The interaction of longitudinal and transverse waves in compressible hyperelastic solids is studied in [41].

Destrade and Saccomandi [42] discuss connections between the nonlinear equations of motion of hyperelastic isotropic solids and the Milne-Pinney ODE, as well as connections with Carroll waves.

A study of porous materials are modeled as incompressible hyperelastic materials with pressurized cavities is done by Idiart and Lopez-Pamies in [69].

Models of fiber reinforced materials are applied, or extended, in the following.

The standard reinforcing model and the Gasser-Ogden-Holzapfel model is tested against experimental data in [40].

Applications and extensions to the Gasser-Ogden-Holzapfel model (2.50) appear in [100, 102]. A general discussion of modelling arterial systems is given in [106]. A review of arterial mechanics can be found in [71], with open problems discussed in [8].

Limited fiber extension models are tested for mathematical and mechanical feasibility by Horgan and Saccomandi in [64]. Additionally, Horgan and Saccomandi comment that one may substitute I^5 for I^4 into (2.54) and (2.55) to obtain models which describe the shear response of the material.

In [111], a fiber reinforced model of an incompressible material is considered where the fiber family is not uniform and oriented along a single direction, but distributed across an angle with respect to a 'dominant' direction.

The interaction of the fiber bundle and underlying matrix in incompressible hyperelastic materials is examined in [55].

Models of damaging fiber reinforced solids are studied in [101], with further study on fiber re-assembly after damage has occurred in [39].

Chapter 3

Symmetry Properties of Planar Incompressible Mooney-Rivlin Materials

3.1 Introduction

With the methodology reviewed in Chapter 1 and necessary physical theory outlined in Chapter 2, we wish to provide a detailed example of Lie symmetry and conservation law analysis. We consider a two dimensional incompressible Mooney-Rivlin model of an isotropic, frame indifferent, homogeneous elastic solid with no external forces. We classify the Lie symmetries of the equations of motion, and reduce the equations for selected Lie symmetries. Afterwards, we study the conservation laws of the system, and compare them with the admitted Lie point symmetries.

In Section 3.2, we begin by presenting the incompressible planar Mooney-Rivlin model to be studied, which includes the statement of the equations of motion, as well as a discussion of certain properties of these equations. Section 3.3 is dedicated to studying the Lie symmetries of the equations of motion, which includes the reduction of order of the system for several admitted symmetries. In Section 3.4, we determine the conservation laws of the current model through the direct method and discuss the physical significance of the conserved densities. Additionally, we analyze the similarities and differences between the admitted Lie symmetries and conservation laws of the PDE system.

3.2 Incompressible Planar Mooney-Rivlin Model

We now outline important features of the incompressible planar Mooney-Rivlin model.

The model is restricted to planar motion by specifying Eulerian coordinates $\vec{x} \in \Omega = \mathbb{R}^3$ to depend on the Lagrangian coordinates $\vec{X} \in \Omega_0 = \mathbb{R}^3$ at a time t (without a loss of generality) as

$$\vec{x} = \begin{bmatrix} x^1\left(X^1, X^2, t\right) \\ x^2\left(X^1, X^2, t\right) \\ X^3 \end{bmatrix}. \tag{3.1}$$

This choice in coordinate dependency sets displacement from equilibrium of the third coordinate x^3 to zero, restricting the motion of the solid to the plane. For clarity, a sample planar deformation is provided in Figure 3.1. We assume that the hydrostatic pressure p has coordinate dependence $p = p(X^1, X^2, t)$.

We assume a constant mass density ρ_0 in the Lagrangian frame. Under the assumption of material frame indifference, the equations of motion for an isotropic, homogeneous, incompressible hyperelastic solid are given by (2.39). For the coordinate dependence (3.1), these equations simplify as

$$\frac{\partial x^1}{\partial X^1}\frac{\partial x^2}{\partial X^2} - \frac{\partial x^1}{\partial X^2}\frac{\partial x^2}{\partial X^1} = 1. \tag{3.2a}$$

$$\frac{\partial^2 x^1}{\partial t^2} = \alpha\left(\frac{\partial^2 x^1}{\partial (X^1)^2} + \frac{\partial^2 x^1}{\partial (X^2)^2}\right) - \frac{\partial p}{\partial X^1}\frac{\partial x^2}{\partial X^2} + \frac{\partial p}{\partial X^2}\frac{\partial x^2}{\partial X^1}, \tag{3.2b}$$

$$\frac{\partial^2 x^2}{\partial t^2} = \alpha\left(\frac{\partial^2 x^2}{\partial (X^1)^2} + \frac{\partial^2 x^2}{\partial (X^2)^2}\right) - \frac{\partial p}{\partial X^2}\frac{\partial x^1}{\partial X^1} + \frac{\partial p}{\partial X^1}\frac{\partial x^1}{\partial X^2}, \tag{3.2c}$$

where $\alpha = 2(a + b)\rho_0 > 0$ is a constant related to material parameters of the Mooney-Rivlin model and constant mass density ρ_0.

Equations (3.2) form a system of constraint equation (from the incompressibility condition $J = 1$) and two linear wave equations with nonlinear first order coupling in p. Of importance, the dynamics of the current model are solely neo-Hookean since the Mooney-Rivlin parameter b does not appear independently in the PDE system.

In addition, equations (3.2) admit a Cauchy-Kovalevskaya (CK) form (see Theorem 3.2.1). As such, all conservation laws of the current model can be obtained through the direct method, as will be discussed in Section 3.4.

Theorem 3.2.1. *Equations (3.2) admit a Cauchy-Kovalevskaya form with respect to X^1 for $x_2^2 \neq 0$, and with respect to X^2 for $x_1^1 \neq 0$.*
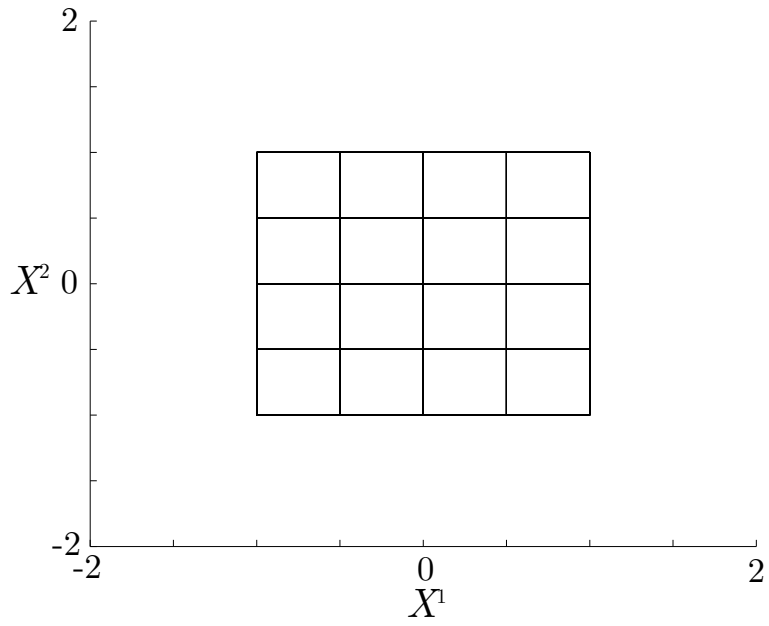
(a) Reference Mesh
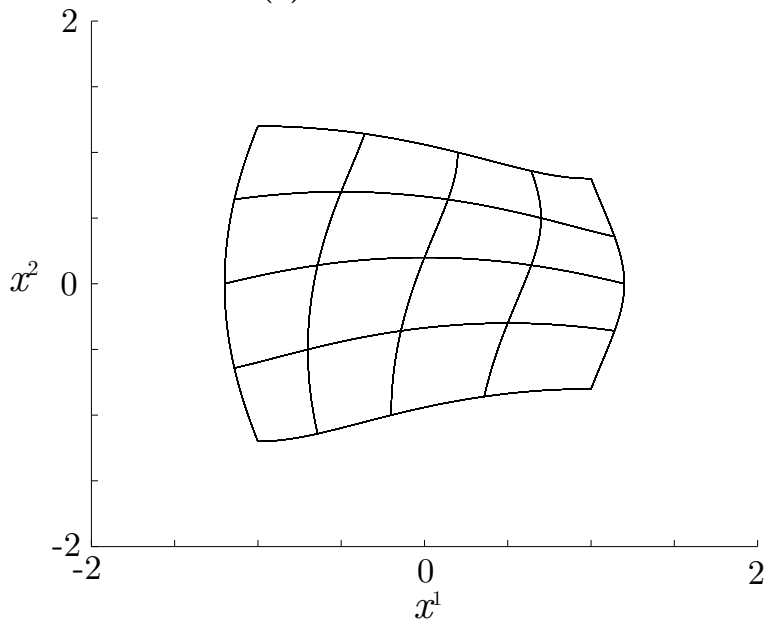
(b) Deformed Mesh

Figure 3.1: Deformation of sample mesh at time t under a planar deformation with coordinate dependence (3.1).

Proof. Let $x_2^2 \neq 0$. We wish to show equations (3.2) can be written in CK-form with respect to X^1.

First, we may write equation (3.2a) in the form

$$0 = \frac{\partial x^1}{\partial X^1} - \frac{1 + \frac{\partial x^1}{\partial X^2}\frac{\partial x^2}{\partial X^1}}{\frac{\partial x^2}{\partial X^2}}, \tag{3.3}$$

with leading derivative x_1^1.

Upon substituting (3.3) for x_1^1 and its derivatives into (3.2b), we can solve this equation for p_1.

$$
\begin{aligned}
0 = \frac{\partial p}{\partial X^1} - &\frac{1}{\left(\left(\frac{\partial x^1}{\partial(X^2)}\right)^2 + \left(\frac{\partial x^2}{\partial(X^2)}\right)^2\right)\left(\frac{\partial x^2}{\partial X^2}\right)^2}\left(\alpha\left(\frac{\partial^2 x^1}{\partial(X^2)^2}\left(\frac{\partial x^2}{\partial X^2}\right)^3\right.\right.\\
&-\frac{\partial x^1}{\partial X^2}\frac{\partial^2 x^2}{\partial(X^2)^2}\left(\frac{\partial x^2}{\partial X^2}\right)^2 - \frac{\partial x^1}{\partial X^2}\frac{\partial^2 x^2}{\partial(X^2)^2}\left(\frac{\partial x^2}{\partial X^1}\right)^2 + \frac{\partial x^2}{\partial X^2}\frac{\partial^2 x^1}{\partial(X^2)^2}\left(\frac{\partial x^2}{\partial X^1}\right)^2\\
&\left.-\frac{\partial x^2}{\partial X^1}\frac{\partial^2 x^2}{\partial(X^2)^2} - \frac{\partial x^2}{\partial X^2}\frac{\partial^2 x^2}{\partial X^1\partial X^2}\right) - \frac{\partial^2 x^1}{\partial t^2}\left(\frac{\partial x^2}{\partial X^2}\right)^3 + \frac{\partial x^1}{\partial X^2}\frac{\partial x^2}{\partial X^2}\frac{\partial p}{\partial X^2}\\
&\left.+\frac{\partial p}{\partial X^2}\frac{\partial x^2}{\partial X^1}\frac{\partial x^2}{\partial X^2}\left(\frac{\partial x^1}{\partial X^2}\right)^2 + \frac{\partial p}{\partial X^2}\frac{\partial x^2}{\partial X^1}\left(\frac{\partial x^2}{\partial X^2}\right)^3 + \frac{\partial x^1}{\partial X^2}\frac{\partial^2 x^2}{\partial t^2}\left(\frac{\partial x^2}{\partial X^2}\right)^2\right),
\end{aligned}
\tag{3.4}
$$

where $(x_2^1)^2 + (x_2^2)^2 \neq 0$ by the assumption $x_2^2 = 0$.

We substitute (3.3) for x_1^1 and its derivatives, as well as (3.4) for p_1 and its derivatives, into (3.2c). As such, we can solve this equation for the highest derivative of x^2 with respect to X^1:

$$
\begin{aligned}
0 = \frac{\partial^2 x^2}{\partial(X^1)^2} - &\frac{1}{\left(\left(\frac{\partial x^1}{\partial(X^2)}\right)^2 + \left(\frac{\partial x^2}{\partial(X^2)}\right)^2\right)\left(\frac{\partial x^2}{\partial X^2}\right)^2}\left(-\frac{\partial x^1}{\partial X^2}\frac{\partial^2 x^1}{\partial(X^2)^2}\left(\frac{\partial x^2}{\partial X^2}\right)^3\right.\\
&-\frac{\partial^2 x^2}{\partial(X^2)^2}\left(\frac{\partial x^2}{\partial X^2}\right)^4 + \frac{\partial x^1}{\partial X^2}\frac{\partial x^2}{\partial X^1}\frac{\partial^2 x^2}{\partial(X^2)^2} - \frac{\partial x^1}{\partial X^2}\frac{\partial x^2}{\partial X^2}\frac{\partial^2 x^1}{\partial(X^2)^2}\left(\frac{\partial x^2}{\partial X^1}\right)^2\\
&+\left(\frac{\partial x^2}{\partial X^1}\right)^2\frac{\partial^2 x^2}{\partial(X^2)^2}\left(\frac{\partial x^1}{\partial X^2}\right)^2 + \frac{\partial x^1}{\partial X^2}\frac{\partial^2 x^2}{\partial X^1\partial X^2}\frac{\partial x^2}{\partial X^2}\\
&\left.+\frac{1}{\alpha}\left(\frac{\partial^2 x^2}{\partial t^2}\left(\frac{\partial x^2}{\partial X^2}\right)^4 + \frac{\partial x^1}{\partial X^2}\frac{\partial^2 x^1}{\partial t^2}\left(\frac{\partial x^2}{\partial X^2}\right)^3 + \frac{\partial p}{\partial X^2}\left(\frac{\partial x^2}{\partial X^2}\right)^3\right)\right).
\end{aligned}
\tag{3.5}
$$

Thus, we have a system with leading derivatives p_1, x_1^1, and x_{11}^2, proving that was to be shown.

The proof that equations (3.2) can be written in CK-form with respect to X^2 for $x_1^1 \neq 0$ can be shown by a symmetric argument. $\square$

3.3 Symmetry Analysis

Using the symmetry analysis methods outlined in Chapter 1, we study equations (3.2). We first present the Lie point symmetries admitted by the system. Afterwards, we provide an example of the invariants for the multi-parameter Lie group of point transformations. Then, we perform reductions to the PDE system for several sets of interesting invariants.

3.3.1 Admitted Lie Point Symmetries

Theorem 3.3.1. *The equations of motion* (3.2) *are invariant under the local Lie groups of point transformations with infinitesimal generators*

$$\mathrm{R}^1 = \frac{\partial}{\partial t} \, , \ \mathrm{R}^2 = \frac{\partial}{\partial X^1} \, , \ \mathrm{R}^3 = \frac{\partial}{\partial X^2} \, , \ \mathrm{R}^4 = F^1(t) \frac{\partial}{\partial p} \, ,$$

$$\mathrm{R}^5 = X^2 \frac{\partial}{\partial X^1} - X^1 \frac{\partial}{\partial X^2} \, , \ \mathrm{R}^6 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} \, ,$$

$$\mathrm{R}^7 = F^2 \frac{\partial}{\partial x^1} - F_{t,t}^2 \, x^1 \frac{\partial}{\partial p} \, , \ \mathrm{R}^8 = F^3 \frac{\partial}{\partial x^2} - F_{t,t}^3 \, x^2 \frac{\partial}{\partial p} \, ,$$

$$\mathrm{R}^9 = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + X^1 \frac{\partial}{\partial X^1} + X^2 \frac{\partial}{\partial X^2} + t \frac{\partial}{\partial t} \, ,$$

where $F^1(t)$, $F^2(t)$, *and* $F^3(t)$ *are arbitrary functions of time.*

Proof. The above Lie groups of point transformations follow from applying the constructive algorithm outlined in Chapter 1. $\square$

Of importance, no special symmetry classification occurs for particular values of $\alpha > 0$.

We now derive the global group of transformations associated with the infinitesimal generators in 3.3.1. Below, ϵ^i is the parameter corresponding to R^i for $i = 1, \ldots, 9$.

Symmetries R^1, R^2, and R^3 correspond to translations in t, X^1, and X^2, respectively:

$$R^1: \quad t^* = t + \epsilon^1, \quad (X^1)^* = X^1, \quad (X^2)^* = X^2, \quad (x^1)^* = x^1, \quad (x^2)^* = x^2, \quad p^* = p.$$

$$R^2: \quad t^* = t, \quad (X^1)^* = X^1 + \epsilon^2, \quad (X^2)^* = X^2, \quad (x^1)^* = x^1, \quad (x^2)^* = x^2, \quad p^* = p.$$

$$R^3: \quad t^* = t, \quad (X^1)^* = X^1, \quad (X^2)^* = X^2 + \epsilon^3, \quad (x^1)^* = x^1, \quad (x^2)^* = x^2, \quad p^* = p.$$

The symmetry R^4 time dependent translations in p by a function $F^1(t)$:

$$R^4: \quad t^* = t, \quad (X^1)^* = X^1, \quad (X^2)^* = X^2, \quad (x^1)^* = x^1, \quad (x^2)^* = x^2, \quad p^* = p + \epsilon^4 F^1(t).$$

Symmetries R^5 and R^6 correspond to rotations in the independent and dependent spatial variables respectively.

$$R^5: \quad t^* = t, \quad (X^1)^* = \cos(\epsilon^5)X^1 - \sin(\epsilon^5)X^2, \quad (X^2)^* = \cos(\epsilon^5)X^2 + \sin(\epsilon^5)X^1,$$
$$(x^1)^* = x^1, \quad (x^2)^* = x^2, \quad p^* = p.$$

$$R^6: \quad t^* = t, \quad (X^1)^* = X^1, \quad (X^2)^* = X^2,$$
$$(x^1)^* = \cos(\epsilon^6)x^1 - \sin(\epsilon^6)X^2, \quad (x^2)^* = \cos(\epsilon^6)x^2 + \sin(\epsilon^6)x^1, \quad p^* = p.$$

Here, R^7 and R^8 are generalized boost symmetries in x^1 and x^2 respectively, and reduce to Galilean boost symmetries for $F^2 = t$ and $F^3 = t$.

$$R^7: \quad t^* = t, \quad (X^1)^* = X^1, \quad (X^2)^* = X^2,$$
$$(x^1)^* = x^1 + \epsilon^7 F^2(t), \quad (x^2)^* = x^2, \quad p^* = p - \epsilon^7 F^2_{tt} x^1 - \frac{1}{2}(\epsilon^7)^2 F^2 F^2_{tt}.$$

$$R^8: \quad t^* = t, \quad (X^1)^* = X^1, \quad (X^2)^* = X^2,$$
$$(x^1)^* = x^1, \quad (x^2)^* = x^2 + \epsilon^8 F^3(t), \quad p^* = p - \epsilon^8 F^3_{tt} x^2 - \frac{1}{2}(\epsilon^8)^2 F^3 F^3_{tt}.$$

The symmetry R^9 corresponds to scaling in all variables but p.

$$R^9: \quad t^* = e^{\epsilon^9} t, \quad (X^1)^* = e^{\epsilon^9} X^1, \quad (X^2)^* = e^{\epsilon^9} X^2,$$
$$(x^1)^* = e^{\epsilon^9} x^1, \quad (x^2)^* = e^{\epsilon^9} x^2, \quad p^* = p.$$

3.3.2 Example of General Invariants

Recall from Chapter 1, Lie Symmetries admit quantities (called invariants) that do not change under the action of the Lie group of point transformations. We find invariants for the linear combination of admitted Lie point symmetries in Theorem 3.3.1.

The general form of the infinitesimal generator for the multi-parameter Lie group of point transformations for $\{R\}_{i=1}^9$ is

$$R = \sum_{i=1}^9 c^i \, R^i, \tag{3.6}$$

where c^i are arbitrary parameters. Of note, we use the set of parameters $\{c^i\}_{i=1}^9$ to make clear where each symmetry generator has contributed. In application, one would reduce the number of parameters to a minimum by reparametrizing several parameters into one.

To find the invariants of (3.6), we must solve the characteristic system $RI = 0$, which is

$$\frac{dt}{c^1 + c^9 t} = \frac{dX^1}{c^2 + c^5 \, X^2 + c^9 \, X^1} = \frac{dX^2}{c^3 - c^5 \, X^1 + c^9 \, X^2}$$
$$= \frac{dx^1}{c^7 \, F^2\,(t) + c^6 \, x^2 + c^9 \, x^1} = \frac{dx^2}{c^8 \, F^3\,(t) - c^6 \, x^1 + c^9 \, x^2} \tag{3.7}$$
$$= \frac{dp}{c^4 \, F^1\,(t) - c^7 \left(\frac{d^2 F^2}{dt^2}\,(t)\right) x^1 - c^8 \left(\frac{d^2 F^3}{dt^2}\,(t)\right) x^2}.$$

Note that the differentials in the independent variables have coefficients which only depend on the independent variables. The dependent variables follow the same pattern, except for the arbitrary functions of t. Thus, the characteristic system may be solved for the invariants depending on the independent variables first, followed by those depending on the dependent variables

Recall that we also have a choice of which independent variable to solve the characteristic system with respect to. As a result, the invariants can take different forms based upon this choice. Note it may be the case that the characteristic system cannot be solved in terms of a particular independent variable, but can be with respect to another.

Invariants Depending on the Independent Variables

To provide an example of the general invariants, we solve the characteristic system (3.7) for the invariants with respect to time (assuming $c^9 \neq 0$). This yields invariants which depend

on the independent variables as

$$I^1 = \frac{1}{\omega^1 + t} \left[\left(X^1 + \frac{\omega^2 - \omega^3 \omega^5}{1 + (\omega^5)^2} \right) \cos\left(\omega^5 \ln\left(\omega^1 + t\right)\right) \right.$$
$$\left. - \left(X^2 + \frac{\omega^3 + \omega^2 \omega^5}{1 + (\omega^5)^2} \right) \sin\left(\omega^5 \ln\left(\omega^1 + t\right)\right) \right], \qquad (3.8a)$$

$$I^2 = \frac{1}{\omega^1 + t} \left[\left(X^1 + \frac{\omega^2 - \omega^3 \omega^5}{1 + (\omega^5)^2} \right) \sin\left(\omega^5 \ln\left(\omega^1 + t\right)\right) \right.$$
$$\left. + \left(X^2 + \frac{\omega^3 + \omega^2 \omega^5}{1 + (\omega^5)^2} \right) \cos\left(\omega^5 \ln\left(\omega^1 + t\right)\right) \right], \qquad (3.8b)$$

where we have reparametrized the constants such that $\omega^i = c^i/c^9$ to provide clarity.

Consider now the structure of the invariants. First, the invariants do not contain the parameters corresponding to Lie point symmetries R^4, R^6, R^7, and R^8; this is expected since the independent variables are invariant under these point transformations. Second, the symmetry parameters give an indication of the associated point transformation in the invariants. For example, ω^1 corresponds to time translations after scaling by c^9, and ω^5 to rotations after scaling by c^9. In addition, we may exclude the action of certain symmetries (except scaling) by setting the corresponding parameter to zero. For example, to exclude time translations, we set $\omega^1 = 0$. To exclude the scaling symmetry, one would need to resolve the characteristic system (3.7) from the beginning for $c^9 = 0$.

Invariants Depending on the Dependent Variables

We now consider the invariants which depend on the dependent variables. Solving the characteristic system (3.7) in the general case yields large, unwieldy integrals. As such, we comment upon the expected form of these invariants, and then find the invariants for a simpler example important to later study.

As with the invariants dependent on the independent variables, we solve the dependent variable portion of the characteristic system with respect to time. A consequence is that the invariants depend explicitly upon time and the dependent variables. As a side note, consider the arbitrary functions of time in R^4, R^7, and R^8, and suppose we solve the characteristic system (3.7) with respect to, e.g., X^1. In this case, t would now be a function of X^1. As such, we would need to solve the system of invariants of the independent variables for t, and

substitute this t into each $F^i(t)$. Upon this substitution, solving for the invariants of the dependent variables may become significantly more complicated.

We now find invariants depending on the dependent variables (3.7) excluding the boost symmetries R^7 and R^8. These invariants are found as

$$V^1 = \frac{1}{\omega^1 + t} \left(x^1 \cos \left(\omega^6 \ln \left(\omega^1 + t \right) \right) - x^2 \sin \left(\omega^6 \ln \left(\omega^1 + t \right) \right) \right), \tag{3.9a}$$

$$V^2 = \frac{1}{\omega^1 + t} \left(x^1 \sin \left(\omega^6 \ln \left(\omega^1 + t \right) \right) + x^2 \cos \left(\omega^6 \ln \left(\omega^1 + t \right) \right) \right), \tag{3.9b}$$

$$V^3 = p - \omega^4 \int \frac{F^1(t)}{\omega^1 + t} \mathrm{d}t, \tag{3.9c}$$

where we have again reparametrized the constants such that $\omega^i = c^i / c^9$ for clarity.

To make the invariants simpler, we may exclude the time translation symmetry by setting $c^1 = 0$, the hydrostatic pressure translation by setting $c^4 = 0$, and the rotational symmetry by setting $c^6 = 0$. To exclude the scaling symmetry, one would need to solve the characteristic system (3.7) from scratch with $c^9 = 0$.

3.3.3 Symmetry Reductions for Several Sets of Invariants

We now present the reduced equations of motion (3.2) for several sets of simpler invariants by use of the invariant form method presented in Chapter 1. As well, we comment on the reduced equations type and how the equations of motion have changed under the reduction. We do not present reductions under the more general invariants (3.8) and (3.9) since it would be uninformative as to reduction methodology.

In the reductions below, we state the invariants and how the variables X^1, X^2, t, x^1, and x^2, depend upon them.

Reduction under Rotational Invariance of Independent Variables

We consider here a reduction of order by the invariant form method for the rotation symmetry R^5 in the independent variables. Afterwards, we consider a particular solution type in cylindrical coordinates $(r(R, \Theta, Z), \theta(R, \Theta, Z), z(R, \Theta, Z))$, where lowercase symbols correspond to the Eulerian frame and uppercase symbols to the Lagrangian frame. The motion

of interest in the latter case is deformations in which the material is not rotated (i.e. $\theta = \Theta$), and the Eulerian radius r is independent of the Lagrangian angle Θ (i.e. $r = r(R, t)$).

Reduction under R^5 First, for the rotation symmetry R^5 in the independent variables, the invariants are

$$I^1 = \sqrt{(X^1)^2 + (X^2)^2}, \qquad I^2 = t \tag{3.10a}$$

$$V^1(I^1, I^2) = x^1, \qquad V^2(I^1, I^2) = x^2, \qquad V^3(I^1, I^2) = p. \tag{3.10b}$$

Note there is no dependence on the angle $\theta = \arctan(X^1/X^2)$ each V^i.

Upon direct substitution of the invariants into the PDE system (3.2a) as per the invariant form method, we obtain the expression

$$1 = 4X^1 X^2 V_1^1 V_1^1 - 4X^1 X^2 V_1^1 V_1^1 \equiv 0,$$

which is a contradiction. Thus, there does not exist a solution to the system (3.2) that is invariant under rotations R^5 alone. (Of note, this result is also arrived at in Lei and Blume [76]).

Special Rotation Deformation   We now analyze a particular case in which the material undergoes zero rotation, and the radius r in the Eulerian frame is independent of the Lagrangian angle Θ. The coordinate dependence corresponding to this type of motion in cylindrical coordinates is

$$x^1 = r(F, t) \cos \theta, \quad x^2 = r(R, t) \sin \theta, \quad p = p(R, t). \tag{3.11}$$

Upon changing variables in (3.2) and substituting (3.11), the equations of motion (3.2) become

$$\det \mathbf{F} = \frac{r}{R} \frac{\partial r}{\partial R} \equiv 1, \tag{3.12a}$$

$$\frac{\partial^2 r}{\partial t^2} = \alpha \left(\frac{\partial^2 r}{\partial R^2} + \frac{1}{R} \frac{\partial r}{\partial R} - \frac{r}{R^2} \right) - \frac{r}{R} \frac{\partial p}{\partial R}, \tag{3.12b}$$

where the incompressibility condition (3.2a) is transformed into (3.12a), and (3.2b) and (3.2c) are transformed into the same equation (3.12b).

We now solve (3.12a)-(3.12b). Solving (3.12a) for r yields the solution

$$r = \sqrt{R^2 + f(t)}, \tag{3.13}$$

where $f(t)$ is an arbitrary function of time.

Consider the point $R = 0$. As such, we have that $r(0) = \sqrt{f(t)}$, where $f(t) \geq 0$ for r to be real. For $f(t) > 0$, the point $R = 0$ will be deformed into a circle with radius $r = \sqrt{f(t)} > 0$. This deformation requires the formation of a void within the solid at the origin for time $t > 0$, resulting in a non-smooth solution in r. As such, the only smooth solution for r occurs if $f(t) \equiv 0$, which is the stationary solution $r = R$. In turn, equation (3.12b) reduces to $p_R = 0$ for all R, yielding the hydrostatic pressure p as $p = p(t)$ which is then fixed by appropriate initial and boundary values.

Reduction under Traveling Wave Coordinates

Using the linear combination of time and spatial translation symmetries $\mathrm{R} = c^1 \mathrm{R}^1 + c^2 \mathrm{R}^2 + c^3 \mathrm{R}^3$, we obtain the travelling wave invariants

$$I^1 = X^1 - \frac{c^2}{c^1}t, \qquad I^2 = X^2 - \frac{c^3}{c^1}t. \tag{3.14}$$

x^1, x^2, and p, are invariant under translations in the independent variables, leading to invariants V^i such that $x^1 = V^1(I^1, I^2)$, $x^2 = V^2(I^1, I^2)$, and $p = V^3(I^1, I^2)$. Thus, equations (3.2) are transformed to

$$V_1^1 V_2^2 - V_2^1 V_1^2 = 1, \tag{3.15a}$$

$$0 = \left(\alpha - \left(\frac{c^2}{c^1}\right)^2\right) V_{1,1}^1 + \left(\alpha - \left(\frac{c^3}{c^1}\right)^2\right) V_{2,2}^1$$

$$- 2\frac{c^2 c^3}{(c^1)^2} V_{1,2}^1 + V_2^3 V_1^2 - V_1^3 V_2^2, \tag{3.15b}$$

$$0 = \left(\alpha - \left(\frac{c^2}{c^1}\right)^2\right) V_{1,1}^2 + \left(\alpha - \left(\frac{c^3}{c^1}\right)^2\right) V_{2,2}^2$$

$$- 2\frac{c^2 c^3}{(c^1)^2} V_{1,2}^2 + V_1^3 V_2^1 - V_2^3 V_1^1. \tag{3.15c}$$

Clearly, the incompressibility condition has the same form under the coordinate transformation. The two coupled equations (3.2b) and (3.2c) are transformed into coupled PDEs of

a similar form, with the addition of terms containing mixed partial derivatives $V^i_{1,2}$. As well, the nonlinearity and coupling are due to the terms containing V^3 (related to the hydrostatic pressure).

Reduction under Time and Spatial Translation Symmetries plus Scaling Symmetry

For the linear combination of time and spatial translation symmetries with the scaling symmetry $R = c^1 R^1 + c^2 R^2 + c^3 R^3 + c^9 R^9$, the corresponding invariants are

$$I^1 = \frac{c^2 + c^9\, X^1}{c^9\, (c^1 + c^9\, t)}, \qquad I^2 = \frac{c^3 + c^9\, X^2}{c^9,\, (c^1 + c^9\, t)}. \tag{3.16a}$$

$$V^1(I^1,\, I^2) = \frac{1}{(c^1 + c^9 t)} x^1, \quad V^2(I^1,\, I^2) = \frac{1}{(c^1 + c^9 t)} x^2, \quad V^3(I^1,\, I^2) = p(I^1,\, I^2). \tag{3.16b}$$

Fixing the invariants I^1 and I^2 as constant and solving for X^1 and X^2 yields equations for straight lines in the X^1-t plane and X^2-t plane respectively.

By the invariant form method, equations (3.2) reduce for the invariants (3.16) to

$$V^1_1 V^2_2 - V^1_2 V^2_1 = 1, \tag{3.17a}$$

$$0 = \alpha \left(V^1_{1,1} + V^1_{2,2} \right) + V^3_2 V^2_1 - V^3_1 V^2_2$$
$$- (c^9)^2 \left((I^1)^2 V^1_{1,1} + 2 I^1 I^2 V^1_{1,2} + (I^2)^2 V^1_{2,2} \right), \tag{3.17b}$$

$$0 = \alpha \left(V^2_{1,1} + V^2_{2,2} \right) + V^3_1 V^1_2 - V^3_2 V^1_1$$
$$- (c^9)^2 \left((I^1)^2 V^2_{1,1} + 2 I^1 I^2 V^2_{1,2} + (I^2)^2 V^2_{2,2} \right). \tag{3.17c}$$

As in the previous reduction, the incompressibility condition does not change under the coordinate transformation. The equations (3.2b) and (3.2c) are autonomous, and have been transformed into non-autonomous equations (3.17b) and (3.17c).

Reduction under Time Translation and Spatial Rotation Invariance

For the time translation symmetry added to the rotation symmetry in the independent variables $R = c^1 R^1 + c^5 R^5$, we obtain the invariants I^1 and I^2 as

$$I^1 = X^1 \cos\left(\frac{c^5\, t}{c^1} \right) - X^2 \sin\left(\frac{c^5\, t}{c^1} \right), \tag{3.18a}$$

$$I^2 = X^1 \sin\left(\frac{c^5\, t}{c^1} \right) + X^2 \cos\left(\frac{c^5\, t}{c^1} \right). \tag{3.18b}$$

To develop intuition on what each invariant I^1 and I^2 look like in (X^1, X^2, t)-space, we consider the isocurves defined by setting I^1 and I^2 constant.

The isocurve in (X^1, X^2, t)-space defined by setting I^1 to a constant and solving (3.18a) for X^1 and X^2 is given as

$$X^1 = I^1 \cos\left(\frac{c^5 t}{c^1}\right), \qquad X^2 = -I^1 \sin\left(\frac{c^5 t}{c^1}\right),$$

which is the equation of a helix centered about the line $(X^1, X^2, t) = (0, 0, t)$. Similarly, the isocurve from setting I^2 to a constant and solving (3.18b) for X^1 and X^2 is

$$X^1 = I^2 \sin\left(\frac{c^5 t}{c^1}\right), \qquad X^2 = I^2 \cos\left(\frac{c^5 t}{c^1}\right),$$

which is the equation of a helix in (X^1, X^2, t)-space. As such, the isocurves defined by each invariant are transverse families of helix' in (X^1, X^2, t)-space that intersect for $I^1 = I^2$.

The invariants V^1 and V^2 are related to the dependent variables by $x^1 = V^1(I^1, I^2)$, $x^2 = V^2(I^1, I^2)$, and $p = V^3(I^1, I^2)$. Thus, equations (3.2) reduce to

$$V_1^1 V_2^2 - V_2^1 V_1^2 = 1, \tag{3.19a}$$

$$0 = \alpha \left(V_{1,1}^1 + V_{2,2}^1 \right) - V_1^3 V_2^2 + V_2^3 V_1^2$$
$$-\frac{(c^5)^2}{(c^1)^2} \left(V_{1,1}^1 \left(I^2\right)^2 - 2 V_{1,2}^1 I^1 I^2 + V_{2,2}^1 \left(I^1\right)^2 - V_1^1 I^1 - V_2^1 I^2 \right), \tag{3.19b}$$

$$0 = \alpha \left(V_{1,1}^2 + V_{2,2}^2 \right) + V_1^3 V_2^1 - V_2^3 V_1^1$$
$$-\frac{(c^5)^2}{(c^1)^2} \left(V_{1,1}^2 \left(I^2\right)^2 - 2 V_{1,2}^2 I^1 I^2 + V_{2,2}^2 \left(I^1\right)^2 - V_1^2 I^1 - V_2^2 I^2 \right). \tag{3.19c}$$

The incompressiblity condition is again unchanged under the transformation. As in the previous reduction with invariants (3.16), equations (3.2b) and (3.2c) are transformed into non-autonomous equations (3.19b) and (3.19c).

Reduction under Translation and Rotation Invariance of Independent Variables

For the sum of the translation symmetries and rotation symmetry in the independent variables $\mathrm{R} = c^1 \mathrm{R}^1 + c^2 \mathrm{R}^2 + c^3 \mathrm{R}^3 + c^5 \mathrm{R}^5$, we obtain the invariants

$$I^1 = \frac{1}{c^5} \left(\left(c^2 + c^5 X^2\right) \cos\left(\frac{c^5 t}{c^1}\right) + \left(-c^3 + c^5 X^1\right) \sin\left(\frac{c^5 t}{c^1}\right) \right), \tag{3.20a}$$

$$I^2 = \frac{1}{c^5} \left(\left(c^3 - c^5 X^1\right) \cos\left(\frac{c^5 t}{c^1}\right) + \left(c^2 + c^5 X^2\right) \sin\left(\frac{c^5 t}{c^1}\right) \right). \tag{3.20b}$$

We again consider the isocurves in (X^1, X^2, t)-space defined by setting each invariant in (3.20) constant.

For I^1 constant, solving equation (3.20a) for X^1 and X^2 to yields the following isocurve in (X^1, X^2, t)-space:

$$X^1 = I^1 \cos\left(\frac{c^5 t}{c^1}\right) - \frac{c^2}{c^5}, \qquad X^2 = I^1 \sin\left(\frac{c^5 t}{c^1}\right) + \frac{c^3}{c^5},$$

which is the equation of a helix with centered about the line $(X^1, X^2, t) = (-c^2/c^5, c^3/c^5, t)$. Similarly, setting I^2 to a constant and solving (3.20b) for X^1 and X^2 yields

$$X^1 = -I^2 \cos\left(\frac{c^5 t}{c^1}\right) + \frac{c^3}{c^5}, \qquad X^2 = I^2 \sin\left(\frac{c^5 t}{c^1}\right) - \frac{c^2}{c^5},$$

which is also the equation of a helix in (X^1, X^2, t)-space centered about the line $(X^1, X^2, t) = (c^3/c^5, -c^2/c^5, t)$.

The invariants V^1, V^2, and V^3 are related to the dependent variables through $x^1 = V^1(I^1, I^2)$, $x^2 = V^2(I^1, I^2)$, and $p = V^3(I^1, I^2)$. Under this transformation, equations (3.2) are reduced to (3.19a)-(3.19c), and so is equivalent to no spatial translations taken into account.

3.4   Conservation Laws

We now study the conservation laws of the system (3.2). We begin by deriving the admitted conservation law multipliers, and the resulting conservation law fluxes.

3.4.1   Conservation Law Multipliers

Using the algorithm outlined in Chapter 1, we seek zeroth and first order conservation law multipliers of equations (3.2).

Note that all conservation laws of a given PDE system arise from multipliers of the equivalent Cauchy-Kovalevskaya form of the system (see Section 1.4.3). Equations (3.2) admit a CK-form (see Theorem 3.2.1), but the resulting multipliers and conservation laws are difficult to work with. As such, we present the multipliers and conservation laws for equations (3.2) as written, noting that we have obtained all conservation laws (up to first

order); this was verified by a direct comparison with the results of studying the system in CK-form given in Theorem 3.2.1.

In the subsequent sections, it is understood that Λ^1 multiplies equation (3.2a), Λ^2 multiplies (3.2b), and Λ^3 multipliers (3.2c).

Theorem 3.4.1. *Equations* (3.2) *admit the following zeroth order conservation law multipliers:*

$$\Lambda^1 = F^1_{tt}x^1 + F^2_{tt}x^2 + F^3(t), \tag{3.21a}$$

$$\Lambda^2 = C^1 x^2 + F^1(t), \tag{3.21b}$$

$$\Lambda^3 = -C^1 x^1 + F^2(t), \tag{3.21c}$$

where $F^1(t)$, $F^2(t)$, *and* $F^3(t)$ *are arbitrary functions of* t; *and,* C^1 *is an arbitrary constant.*

Proof. The multipliers (3.21) follow from application of the constructive algorithm (outlined in Chapter 1) with multipliers of the form

$$\Lambda^i = \Lambda^i(X^1, X^2, t, x^1, x^2, p), \quad i = 1, 2, 3.$$

$\square$

Theorem 3.4.2. *Equations* (3.2) *admit the following first order conservation law multipliers (in addition to the zeroth order multipliers presented in Theorem 3.4.1)*

$$\Lambda^1 = C^2 p_t + C^3 p_1 + C^4 p_2 + C^5 (X^1 p_2 - X^2 p_1), \tag{3.22a}$$

$$\Lambda^2 = -C^2 x^1_t - C^3 x^1_1 - C^4 x^1_2 + C^5 (X^2 x^1_1 - X^1 x^1_2), \tag{3.22b}$$

$$\Lambda^3 = -C^2 x^2_t - C^3 x^2_1 - C^4 x^2_2 + C^5 (X^2 x^2_1 - X^1 x^2_2), \tag{3.22c}$$

where $\{C^i\}^5_{i=1}$ *are arbitrary constants.*

Proof. Using multipliers of the form

$$\Lambda^i = \Lambda^i(X^1, X^2, t, x^1, x^2, p, \partial x^1, \partial x^2, \partial p), \quad i = 1, 2, 3,$$

the multipliers (3.22) follow from the constructive method outlined in Chapter 1. $\square$

Note that singular first order conservation law multipliers arise in the proof of Theorem 3.4.2 due to the multiplier dependence upon x_1^1, x_2^1, x_1^2, and x_2^2. In particular, the singular first order multipliers are of the form

$$\Lambda^1(X^1, X^2, t, x^1, x^2, p, \partial x^1, \partial x^2, \partial p) = \frac{f^i(X^1, X^2, t, x^1, x^2, p)}{x_1^1 x_2^2 - x_2^1 x_1^2 - 1}, \tag{3.23}$$

which become singular on solutions to (3.2a). We disregard multipliers of this form since they can potentially lead to divergence expressions that are not conservation laws of the system (3.2) [22].

3.4.2 Divergence Conservation Law Forms

For each set of conservation law multipliers in Theorem 3.4.1 and 3.4.2, we apply the direct method of flux analysis (as in [22]) to construct the divergence conservation law forms of equations (3.2).

Recall that we assumed mass density ρ_0 was constant and was cancelled from the PDE system (3.2). As such, we assume the divergence conservation law equations in the following are implicitly multiplied by a factor of mass density ρ_0 to obtain the correct physical interpretation.

We utilize the notation D_i for the total derivative operator with respect to the ith independent variable, as presented in Chapter 1.

Conservation of Mass in the Eulerian Frame

We first make a note on the conservation of mass in the Eulerian frame. In Chapter 2, it was presented that conservation of mass can be represented by the relation $\rho_0 = J\rho$, where ρ_0 is the mass density in the Lagrangian frame, ρ is mass density in the Eulerian frame, and $J = \det \mathbf{F}$ is the determinant of the deformation gradient $\mathbf{F}$. As such, mass density in the Eulerian frame is constant due to the Lagrangian mass density assumed constant, and $J = 1$ from the incompressibility condition.

Conservation of Momentum in the Eulerian Frame

Recall that velocity $\vec{V}$ and spatial velocity $\vec{v}$ are equivalent. For the velocity $\vec{G}_t = \vec{V} = \vec{v}$, the model admits conservation of generalized momentum (in x^1, and x^2) and conservation of angular momentum (in x^3), in the Eulerian frame (multiplier set of $F^1(t)$, $F^2(t)$, and C^1, respectively). The divergence conservation law forms are as follows.

For arbitrary $F^1(t)$, we obtain the conservation of generalized momentum in x^1:

$$
\begin{aligned}
& D_t \left(F^1 x_t^1 - F_t^1 x^1 \right) \\
& -D_{X^1} \left(\alpha F^1 x_1^1 - p F^1 x_2^2 - \frac{1}{2} F_{tt}^1 \left(x^1 \right)^2 x_2^2 \right) \\
& -D_{X^2} \left(\alpha F^1 x_2^1 + p F^1 x_1^2 + \frac{1}{2} F_{tt}^1 \left(x^1 \right)^2 x_1^2 \right) = 0.
\end{aligned}
\tag{3.24}
$$

For arbitrary $F^2(t)$, we obtain the conservation of generalized momentum in x^2:

$$
\begin{aligned}
& D_t \left(F^2 x_t^2 - F_t^2 x^2 \right) \\
& -D_{X^1} \left(\alpha F^2 x_1^2 + p F^2 x_2^1 + F_{tt}^2 \left(x^2 \right)^2 x_2^1 \right) \\
& -D_{X^2} \left(\alpha F^2 x_2^2 - p F^2 x_1^1 - \frac{1}{2} F_{tt}^2 \left(x^2 \right)^2 x_1^1 \right) = 0.
\end{aligned}
\tag{3.25}
$$

Note that we have the conservation of linear momentum in x^1 for $F^1(t)$ a constant, and in x^2 for $F^2(t)$ a constant.

From the multiplier set with coefficient C^1, we obtain the conservation of angular momentum in x^3:

$$
\begin{aligned}
& D_t \left(x^1 x_t^2 - x_t^1 x^2 \right) \\
& +D_{X^1} \left(\alpha \left(x_1^1 x^2 - x^1 x_1^2 \right) - p x^1 x_2^1 - p x^2 x_2^2 \right) \\
& +D_{X^2} \left(\alpha \left(x_2^1 x^2 - x^1 x_2^2 \right) + p x^1 x_1^1 + p x^2 x_1^2 \right) = 0.
\end{aligned}
\tag{3.26}
$$

Except for the incompressibility condition, the equations of motion were constructed from the balance of linear momentum, and so it is expected that linear momentum is conserved. In addition, conservation of angular momentum is also an expected result since it is equivalent to the equation $\mathbf{F}\mathbf{P}^T = \mathbf{P}\mathbf{F}^T$, which holds identically for the current model due to the assumed isotropy (see Chapter 2, Section 2.4.2).

Conservation of Energy

Applying the multiplier set with coefficient C^2, we obtain the conservation of energy:

$$D_t \left(\frac{\alpha}{2} \left(\left(x_1^1\right)^2 + \left(x_2^1\right)^2 + \left(x_1^2\right)^2 + \left(x_2^2\right)^2 \right) + \frac{1}{2} \left(x_t^1\right)^2 + \frac{1}{2} \left(x_t^2\right)^2 \right)$$

$$-D_{X^1} \left(\alpha \left(x_t^1 x_1^1 + x_t^2 x_1^2 \right) - p x_t^1 x_2^2 + p x_2^1 x_t^2 \right) \tag{3.27}$$

$$-D_{X^2} \left(\alpha \left(x_t^1 x_2^1 + x_t^2 x_2^2 \right) + p x_t^1 x_1^2 - p x_1^1 x_t^2 \right) = 0.$$

Here, kinetic energy K is given as

$$K = \frac{\rho_0}{2} \left(x_t^1\right)^2 + \frac{\rho_0}{2} \left(x_t^2\right)^2,$$

and potential energy P as

$$P = \rho_0 \left(\frac{\alpha}{2} \left(\left(x_1^1\right)^2 + \left(x_2^1\right)^2 + \left(x_1^2\right)^2 + \left(x_2^2\right)^2 \right) \right).$$

Generalized Incompressibility Condition

The multiplier set with function $F^3(t)$ corresponds to a generalized version of the incompressibility condition, which has divergence conservation law form

$$D_t \left(\int F^3 dt \right) - D_{X^1} \left(F^3 x_1^1 x_2^2 \right) + D_{X^2} \left(F^3 x_1^1 x_1^2 \right) = 0. \tag{3.28}$$

If one expands the above expression and simplifies (assuming $F^3(t) \neq 0$), then the incompressiblity condition (3.2a) is obtained.

Conservation of Momentum in the Lagrangian Frame

There are three additional conservation laws for two components of linear momentum, and one component of angular momentum, in the Lagrangian (or material) frame (corresponding to the multiplier sets with coefficient C^3, C^4, and C^5, respectively). Note these momenta are also referred to as material momenta [83].

The vectors for the linear momentum ($\mathfrak{p}$) and angular momentum ($\vec{X} \times \mathfrak{p}$) in the Lagrangian frame are defined, respectively, as [83]:

$$\mathfrak{p} = \rho_0 \mathbf{F}^T \vec{x}_t = \rho_0 (x_t^1 x_1^1 + x_t^2 x_1^2) \hat{i} + \rho_0 (x_t^1 x_2^1 + x_t^2 x_2^2) \hat{j}, \tag{3.29}$$

$$\vec{X} \times (\mathfrak{p}) = \rho_0 (X^1 (x_t^1 x_2^1 + x_t^2 x_2^2) - X^2 (x_t^1 x_1^1 + x_t^2 x_1^2)) \hat{k}, \tag{3.30}$$

where ρ_0 is the Lagrangian mass density; and, $\hat{i}$, $\hat{j}$, and $\hat{k}$ are the standard Cartesian basis vectors in the Lagrangian frame. Here, the coefficients of $\hat{i}$ and $\hat{j}$ in the (3.29) are the linear momenta in the Lagrangian frame, and the coefficient of $\hat{k}$ in (3.29) is the angular momentum in the Lagrangian frame [83].

For C^3, we obtain the conservation of linear momentum in X^1:

$$
\mathrm{D}_t \left(x_t^1 x_1^1 + x_t^2 x_1^2 \right)
$$
$$
+\mathrm{D}_{X^1} \left(\alpha \left(\left(x_2^1\right)^2 + \left(x_2^2\right)^2 - \left(x_1^1\right)^2 - \left(x_1^2\right)^2 \right) + p - \frac{1}{2}\left(x_t^1\right)^2 - \frac{1}{2}\left(x_t^2\right)^2 \right) \qquad (3.31)
$$
$$
-\mathrm{D}_{X^2}\left(\alpha\left(x_1^1 x_2^1 + x_1^2 x_2^2 \right)\right) = 0.
$$

For C^4, we obtain the conservation of linear momentum in X^2:

$$
\mathrm{D}_t \left(x_t^1 x_2^1 + x_t^2 x_2^2 \right) - \mathrm{D}_{X^1}\left(\alpha\left(x_1^1 x_2^1 + x_1^2 x_2^2 \right) \right)
$$
$$
+\mathrm{D}_{X^2} \left(\alpha \left(\left(x_1^1\right)^2 + \left(x_1^2\right)^2 - \left(x_2^1\right)^2 - \left(x_2^2\right)^2 \right) + p - \frac{1}{2}\left(x_t^1\right)^2 - \frac{1}{2}\left(x_t^2\right)^2 \right) = 0. \qquad (3.32)
$$

For C^5, we obtain the conservation of angular momentum in X^3 (i.e. the $\hat{k}$ component of (3.30)):

$$
\mathrm{D}_t \left(X^2(x_t^1 x_1^1 + x_t^2 x_1^2) - X^1(x_t^1 x_2^1 + x_t^2 x_2^2) \right)
$$
$$
+\mathrm{D}_{X^1} \left(\frac{\alpha}{2} X^2 \left(-\left(x_1^1\right)^2 - \left(x_1^2\right)^2 + \left(x_2^1\right)^2 + \left(x_2^2\right)^2 \right) \right.
$$
$$
\left. +\alpha X^1\left(x_1^1 x_2^1 + x_1^2 x_2^2 \right) + X^2 p - \frac{1}{2}X^2\left(x_t^1\right)^2 - \frac{1}{2}X^2\left(x_t^2\right)^2 \right) \qquad (3.33)
$$
$$
+\mathrm{D}_{X^2} \left(\frac{\alpha}{2} X^1 \left(-\left(x_1^1\right)^2 - \left(x_1^2\right)^2 + \left(x_2^1\right)^2 + \left(x_2^2\right)^2 \right) \right.
$$
$$
\left. -\alpha X^2\left(x_1^1 x_2^1 + x_1^2 x_2^2 \right) - X^1 p + \frac{1}{2}X^1\left(x_t^1\right)^2 + \frac{1}{2}X^1\left(x_t^2\right)^2 \right) = 0.
$$

We would like to remark about the form of material momenta here and in the literature. The linear material momentum (3.29) and angular material momentum (3.30) are written in the form used by Marsden and Hughes [83]. Other researchers have defined these quantities in varying forms. For example, linear material momenta is written as $\mathfrak{p} = -\rho_0 \mathbf{F}^t \vec{x}_t$ in [28] and [85], in which the latter author calls this 'pseudomomentum'. It should be noted that Maugin [85] normally discusses pseudomomentum with respect to 'forces' acting on, e.g., defects and cracks in material solids. Similarly, material angular momentum takes on different forms in the literature (see, e.g., [45, 85, 86]).

3.4.3 Connection Between Conservation Laws and Symmetries

Recall from Chapter 1 that Lie symmetries of a variational system of partial differential equations are connected to the conservation laws of the system through Noether's theorem. For a PDE system to be variational, it is required that the linearizing operator of the system is self-adjoint. If so, under Boyer's formulation of Noether's theorem, the evolutionary form of the Lie symmetries admitted by the associated Lagrangian to the PDE system are equivalent to the conservation law multipliers of the equations of motion.

In addition, the Lie symmetries admitted by the Lagrangian are also Lie symmetries admitted by the associated Euler-Lagrange equations. As such, it is possible to obtain conservation laws of a variational PDE system from Lie symmetries admitted by the same system without knowing the Lagrangian. However, it is important to note the converse is not necessarily true: a Lie symmetry admitted by the Euler-Lagrange equations is not necessarily a Lie symmetry admitted by the associated Lagrangian. A symmetry of this type is called non-variational symmetry. An example would of a non-variational symmetry admitted by a variational PDE system is the scaling symmetry [14] [96].

We show the system (3.2) is non-variational in the proof of the following theorem.

Theorem 3.4.3. *Equations* (3.2) *do not form a variational system of PDEs.*

Proof. To show the system is non-variational, it is sufficient to show inequality of one entry in the matrices defining the linearizing operator L and its adjoint operator L* (see Section 1.5.2). We do so by considering the matrix entry of each operator corresponding to the incompressibility condition (3.2a).

By the formula in Chapter 1, the first row of the linearizing operator corresponds to the first equation of the system being linearized (in our case, (3.2a)), where the ith column corresponds to the ith dependent variable. Entries in the linearizing operator are found through equation (1.122) in Section 1.5.2. As such, the first row of the linearizing operator L is

$$L^1[\vec{U}] = \left[U_2^2 \frac{\partial}{\partial X^1} - U_1^2 \frac{\partial}{\partial X^2}, \quad -U_2^1 \frac{\partial}{\partial X^1} + U_2^2 \frac{\partial}{\partial X^2}, \quad 0 \right].$$

We find the first row of the adjoint operator as

$$\text{L}^{*1}[\vec{U}] = \left[-U_2^2 \frac{\partial}{\partial X^1} + U_1^2 \frac{\partial}{\partial X^2}, \quad U_2^1 \frac{\partial}{\partial X^1} - U_1^1 \frac{\partial}{\partial X^2}, \quad 0 \right],$$

where the entries are found through formula (1.123) presented in Section 1.5.2.

Clearly, L cannot be equal to L* since the first rows are not equivalent. As such, the system of PDEs is non-variational. □

In light of Theorem 3.4.3, there does not exist a direct correspondence between the admitted Lie symmetries and conservation laws of the PDE system (3.2). However, one can still compare the symmetries and conservation laws of the system of PDEs (for example, see [24]), which is instructive as to how one would proceed if the PDE system were variational. To make the comparison, the Lie point symmetries in Section 3.3.1 are expressed in evolutionary form, which are presented in the left column of Table 3.1.

In the current system (3.2), there are only eight conservation law multipliers (of zeroth and first order) and nine Lie point symmetries. The scaling symmetry R^9 is not similar to any zeroth or first order conservation law multiplier obtained in Section 3.4.1. As such, we provide the evolutionary form of R^9 separately.

$$\hat{\text{R}}^9 = \left(x^1 - X^1 x_1^1 - X^2 x_2^1 - t x_t^1 \right) \frac{\partial}{\partial x^1} + \left(x^2 - X^1 x_1^2 - X^2 x_2^2 - t x_t^2 \right) \frac{\partial}{\partial x^2}$$
$$+ \left(-X^1 p_1 - X^2 p_2 - t p_t \right) \frac{\partial}{\partial p}.$$

To compare the Lie point symmetries and conservation law multipliers, we must determine which infinitesimal $\hat{\eta}^i$ corresponds to which multiplier Λ^j. For a variational system, the correspondence is given based on the Euler operator used to derive the particular equation (i.e. $\Lambda^j = \hat{\eta}^i$ for an equation derived by applying E_{u^i} to the Lagrangian, where u^i is the ith dependent variable). In the non-variational case, we must compare symmetry infinitesimals and conservation law multipliers to establish a correspondence (if one exists).

From a preliminary comparison, each $\hat{\eta}^i$ in Lie symmetry generator $\hat{\text{R}}^6$ corresponds to the set of conservation law multipliers with constant C^1. This yields the correspondence between certain Lie point symmetry infinitesimals and conservation law multipliers. Comparing the remaining Lie point symmetries and first order conservation laws based on this correspondence, we obtain similarity between certain symmetry infinitesimals and multipliers

with equality in particular cases. These similarities and equalities can be seen in Table 3.1, where we also note the physical meaning of each symmetry and conserved quantity associated with each set of multipliers. The consistent difference between Lie symmetry infinitesimals $\hat{\eta}^i$ and the conservation law multiplier Λ^i is a negative sign on one multiplier relative to the others (under the current sign convention). For example, this can be seen in the first row of Table 3.1, in which all the infinitesimals have the same sign whereas the multipliers undergo a change in sign from one to the next. These differences can be attributed to the system not being variational.

3.5 Discussion

In this Chapter, a detailed symmetry and conservation law analysis was performed for a two dimensional model of an incompressible Mooney-Rivlin solid. It was shown the Mooney-Rivlin strain energy density yields the same equations of motion as derived for the neo-Hookean strain energy density, and that these equations admit a Cauchy-Kovalevskaya form. In the symmetry analysis of Section 3.3, Lie point symmetries of the governing PDE system were classified, invariants for various linear combinations of the symmetry generators were obtained, and the system of equations was reduced for several sets of these invariants. Afterwards, in Section 3.4, the zeroth and first order conservation law multipliers were classified, as well as the resulting divergence conservation law forms. The physical interpretation of the conserved densities was then discussed. The system (3.2) is then shown to be non-variational (i.e. not a system of Euler-Lagrange equations). In the spirit of [24], symmetries and conservation law multipliers were compared, with differences and similarities noted where they arise.

We now discuss work in the literature related to the incompressible two-dimensional Mooney-Rivlin model.

Lie point symmetries and invariant solutions of the incompressible planar neo-Hookean model are studied by Lei and Blume in [76]. The Lie symmetry classification and reductions of the PDE system discussed in this paper can be compared to that done in the current chapter since the model studied in this Chapter is equivalent to the planar neo-Hookean

Table 3.1: Comparison of Lie point symmetries in evolutionary form to zeroth and first order conservation law multipliers of the system (3.2).

Symmetry Generator	Similar Conservation Law Multiplier
$\hat{R}^1 = -x_t^1 \frac{\partial}{\partial x^1} - x_t^2 \frac{\partial}{\partial x^2} - p_t \frac{\partial}{\partial p}$ Time Translation	$\Lambda^1 = p_t, \quad \Lambda^2 = -x_t^1, \Lambda^3 = -x_t^2$ Conservation of Energy
$\hat{R}^2 = -x_1^1 \frac{\partial}{\partial x^1} - x_1^2 \frac{\partial}{\partial x^2} - p_1 \frac{\partial}{\partial p}$ Translation in X^1	$\Lambda^1 = p_1, \quad \Lambda^2 = -x_1^1, \Lambda^3 = -x_1^2$ Conservation of Momentum in X^1
$\hat{R}^3 = -x_2^1 \frac{\partial}{\partial x^1} - x_2^2 \frac{\partial}{\partial x^2} - p_2 \frac{\partial}{\partial p}$ Translation in X^2	$\Lambda^1 = p_2, \quad \Lambda^2 = -x_2^1, \Lambda^3 = -x_2^2$ Conservation of Momentum in X^2
$\hat{R}^4 = F^1(t) \frac{\partial}{\partial p}$ Time dependent Translation in p	$\Lambda^1 = F^3(t), \Lambda^2 = 0, \quad \Lambda^3 = 0$ Generalized Incompressiblity Condition
$\hat{R}^5 = \left(-X^2 x_1^1 + X^1 x_2^1\right) \frac{\partial}{\partial x^1} +$ $\left(-X^2 x_1^2 + X^1 x_2^2\right) \frac{\partial}{\partial x^2} +$ $\left(-X^2 p_1 + X^1 p_2\right) \frac{\partial}{\partial p}$ X^1-X^2 Rotations	$\Lambda^1 = (X^1 p_2 - X^2 p_1), \Lambda^2 =$ $(X^2 x_1^1 - X^1 x_2^1), \Lambda^3 = (X^2 x_1^2 - X^1 x_2^2)$ Angular Momentum in Lagrangian Frame
$\hat{R}^6 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$ x^1-x^2 Rotations	$\Lambda^1 = 0, \quad \Lambda^2 = x^2, \Lambda^3 = -x^1$ Angular Momentum in Eulerian Frame
$\hat{R}^7 = F^2 \frac{\partial}{\partial x^1} - F_{t,t}^2 x^1 \frac{\partial}{\partial p}$ Generalized Boost in x^1	$\Lambda^1 = F_{tt}^1 x^1, \Lambda^2 = F^1(t), \quad \Lambda^3 = 0$ Generalized Momentum in x^1
$\hat{R}^8 = F^3 \frac{\partial}{\partial x^2} - F_{t,t}^3 x^2 \frac{\partial}{\partial p}$ Generalized Boost in x^2	$\Lambda^1 = F_{tt}^2 x^2, \quad \Lambda^2 = 0, \Lambda^3 = F^2(t)$ Generalized Momentum in x^2

model.

Cheviakov and Ganghoffer [35] analyze conditions such that a constitutive law for a hyperelastic material admits a natural (stress free) state under zero deformation. In addition, they study the equivalence transformations and Lie symmetries of the planar compressible Mooney-Rivlin model (also called the Ciarlet-Mooney-Rivlin model).

A discussion of materials best modelled by Mooney-Rivlin elasticity, as well as applications of other elasticity models, can be found in Bower [26].

Dai [38] studies nonlinear dispersive wave solutions for a Mooney-Rivlin elastic rod.

Applications of numerical methods to the incompressible Mooney-Rivlin model can be found in [94].

Chapter 4

Fiber Reinforced Hyperelastic Materials

4.1   Introduction

In this chapter, analytical properties are analyzed for three types of motion in an incompressible fiber reinforced Mooney-Rivlin solid with constant mass density ρ_0. In particular, we are interested in the Lie symmetry and conservation laws of admitted by the PDE system governing each motion. In Section 4.2, the general equations of motion for a Mooney-Rivlin solid reinforced with two fiber families is recalled from Chapter 2.

Section 4.3 is dedicated to the study of a one dimensional incompressible fiber reinforced Mooney-Rivlin model. After the equations of motion are presented, a numerical simulation is studied to develop intuition of solution behaviour. Then, admitted Lie symmetries of the PDE system are classified, and the equations of motion are reduced for particular symmetries. Afterwards, conservation law multipliers are classified for the system, and the conservation law form of the equations are derived and discussed. Finally, the Hodograph transformation is outlined and applied to the equations of motion to obtain a linear non-constant coefficient PDE system, after which we determine if it can be mapped into one of constant coefficients using a point transformation.

In Section 4.4, a two dimensional fiber reinforced model for which the displacement of the solid is transverse to the $X^1 X^2$-plane is studied. First, the PDE system governing the motion is derived, and one dimensional solutions are discussed. After, Lie symmetries of the system are classified, and conservation laws are obtained through the direct method (i.e. by multipliers) and discussed.

Another model in three dimensions with two transverse displacements orthogonal to the X^3-axis is analyzed in Section 4.5. As in the first two dimensional model, the one dimensional

solutions to the model equations are discussed. After, Lie symmetries of the PDE system are classified, and conservation laws are analyzed.

Lastly, a two dimensional two fiber family model is presented in Section 4.6, which may be studied in future work.

4.2 General Equations of Motion for Fiber Reinforced Mooney-Rivlin Elasticity

Recall the equations governing the motion of a fiber reinforced, homogeneous, incompressible hyperelastic material with zero external body forces are (c.f. (2.39))

$$\det \mathbf{F} = 1, \tag{4.1a}$$

$$\rho_0 \frac{\partial^2 x^i}{\partial t^2} = \sum_j^3 \frac{\partial P^{ij}}{\partial X^j}, \quad i = 1, 2, 3, \tag{4.1b}$$

$$P^{ij} = \rho_0 \frac{\partial W}{\partial F^{ij}} - p(F^{-1})^{ji}, \quad i, j = 1, 2, 3, \tag{4.1c}$$

$$\mathbf{PF}^T = \mathbf{FP}^T, \tag{4.1d}$$

where $x^i = x^i(\vec{X}, t) \in \Omega \subset \mathbb{R}^3$ are Eulerian points, $X^i \in \Omega_0 \subset \mathbb{R}^3$ are Lagrangian points, P^{ij} are the components of the First Piola-Kirchhoff stress $\mathbf{P}$, $p = p(\vec{X}, t)$ is the hydrostatic pressure, and $W = W(I^1, I^2, I^3, I^4, I^5, I^6, I^7)$ is the strain energy density defined in terms of the matrix invariants (c.f. (2.35) and (2.47))

$$I^1(\mathbf{C}) = \text{Tr}\mathbf{C}, \tag{4.2a}$$

$$I^2(\mathbf{C}) = \frac{1}{2} \left((\text{Tr}\mathbf{C})^2 - \text{Tr}(\mathbf{C}^2) \right), \tag{4.2b}$$

$$I^3(\mathbf{C}) = \det \mathbf{C}, \tag{4.2c}$$

$$I^4(A^1, \mathbf{C}) = \left(A^1 \right)^T \mathbf{C} \left(A^1 \right), \tag{4.2d}$$

$$I^5(A^1, \mathbf{C}) = \left(A^1 \right)^T \mathbf{C}^2 \left(A^1 \right), \tag{4.2e}$$

$$I^6(A^2, \mathbf{C}) = \left(A^2 \right)^T \mathbf{C} \left(A^2 \right), \tag{4.2f}$$

$$I^7(A^2, \mathbf{C}) = \left(A^2 \right)^T \mathbf{C}^2 \left(A^2 \right), \tag{4.2g}$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, $\mathbf{C}^2 = \mathbf{CC}$, and A^1 and A^2 correspond to unit vectors corresponding to the direction of the two fiber families.

The strain energy density W can be written as the sum of the isotropic component $W_{\text{iso}} = W_{\text{iso}}(I^1, I^2, I^3)$ and anisotropic component $W_{\text{aniso}} = W_{\text{aniso}}(I^4, I^5, I^6, I^7)$, which account for the underlying material and fiber reinforcement respectively. In the current Chapter, we consider a Mooney-Rivlin isotropic strain energy density (2.42), and the anisotropic strain energy density (2.51) with $n = 2$ originally studied by Basciano et al [10]; that is,

$$W = aI^1 + bI^2 + q\left(I^4 - 1\right)^2 + s\left(I^6 - 1\right)^2, \tag{4.3}$$

where a, b, q, $s > 0$ are material parameters, I^1 and I^2 are the Mooney-Rivlin invariants, and I^4 and I^6 are invariants specific to two fiber families.

Of importance, the form of the strain energy density (4.3) allows for a simplification of system (4.1).

Theorem 4.2.1. *For the strain energy density* (4.3), *equation* (4.1d) *is an identity.*

Proof. The proof follows by direct calculation. $\qquad\square$

Remark 4.2.2. For $b = 0$ and $(I^4 - 1)$ sufficiently small, the strain energy density (4.3) approximates (up to addition of a constant) the Holzapfel, Gasser, and Ogden model [59],

$$W = a(I^1 - 1) + \frac{k^1}{k^2}\left(e^{k^2(I^4-1)^2} - 1\right) + \frac{k^1}{k^2}\left(e^{k^2(I^6-1)^2} - 1\right) \tag{4.4}$$

for $k^1 = q = s$ and k^2 material parameters. Here, it was assumed the fiber families were nearly perpendicular. Parameters for (4.4) have been determined for the adventia layer of an artery as $a = 0.15$ kPa, $k^1 = 0.5620$ kPa, and $k^2 = 0.7112$ [59].

4.3 One Dimensional Transverse Motion

We now turn our attention to one dimensional transverse motion of a incompressible Mooney-Rivlin material reinforced with a single fiber family.

4.3.1 Equations of Motion

We first state the fiber angle convention and the coordinate dependency for the motion under consideration. After, we present the specific equations of motion to be studied, and as well as particular properties of these equations.

The single fiber family is oriented at an angle γ clockwise from the X^1-axis in the $X^1 X^3$-plane with unit vector

$$A^1 = \begin{bmatrix} \cos(\gamma) \\ 0 \\ \sin(\gamma) \end{bmatrix}. \tag{4.5}$$

Note that we do not consider $\gamma < 0$ or $\gamma > \pi/2$ since the same dynamics can be obtained by flipping the system about the X^3-axis. For example, if $\gamma = -\pi/3$, the material can be flipped about the X^3-axis to yield the equivalent model with $\gamma = \pi/3$

Displacement $G(X^1, t)$ of the solid from equilibrium along the X^3-axis is described by the coordinate dependence

$$\vec{X} = \begin{bmatrix} X^1 \\ X^2 \\ G\left(X^1, t\right) + X^3 \end{bmatrix}. \tag{4.6}$$

As such, the incompressibility condition is identically satisfied, and the hydrostatic pressure is given by $p = p(X^1, t)$.

For clarity, we make the change of notation $X^1 = x$ for the remainder of the current Section. As such, the equations of motion for an incompressible Mooney Rivlin solid with fiber family oriented at an angle γ from the x-axis are derived from (4.3)-(4.6) as
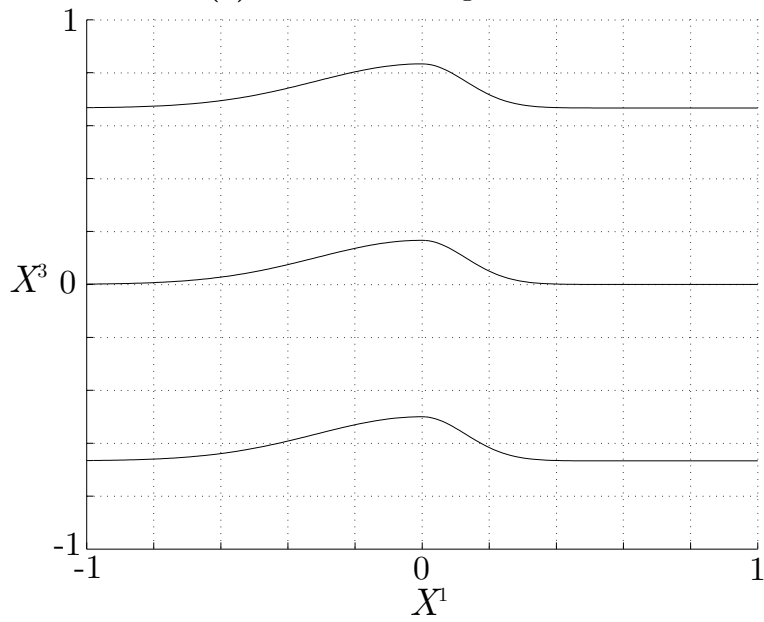
$$\frac{\partial^2 G}{\partial t^2} = \left(\alpha + \beta \cos^2(\gamma) \left(3\cos^2(\gamma) \left(\frac{\partial G}{\partial x}\right)^2 + 6\cos(\gamma)\sin(\gamma)\frac{\partial G}{\partial x} + 2\sin^2\gamma\right)\right) \frac{\partial^2 G}{\partial x^2}, \tag{4.7a}$$

$$0 = \frac{\partial p}{\partial x} - 2\beta\rho_0 \cos^3\gamma \left(\cos\gamma \frac{\partial G}{\partial x} + \sin\gamma\right) \frac{\partial^2 G}{\partial x^2}, \tag{4.7b}$$

where $G = G(x, t)$, $p = p(x, t)$, $\alpha = 2(a+b) > 0$ and $\beta = 4q > 0$ are constants related to the material parameters, and ρ_0 is mass density.
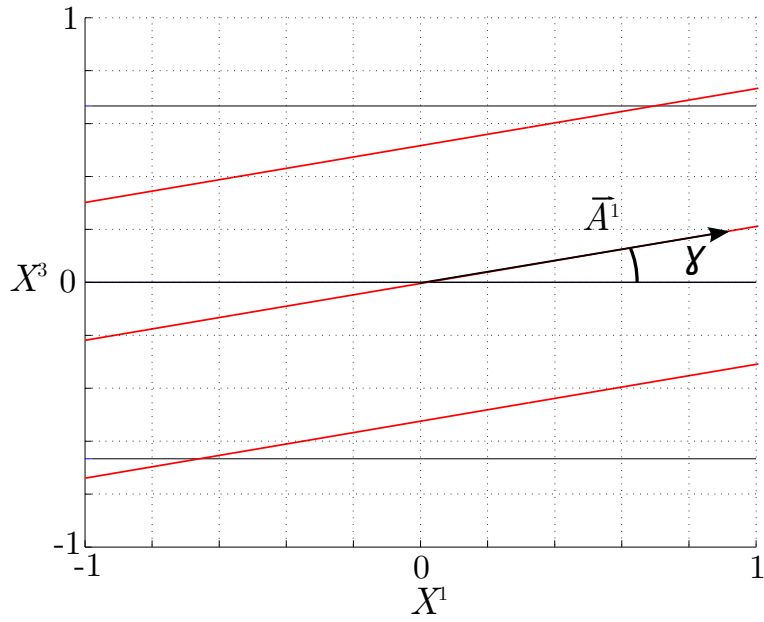
(a) Reference Configuration

(b) Deformed Configuration

Figure 4.1: Sample deformation of solid along the X^3-axis with fiber family (red lines) oriented along $\vec{A}^1$ at an angle γ counter-clockwise from the X^1-axis in the $X^1 X^3$-plane.

We now make several remarks about equations (4.7).

The above system of equations (4.7) is neo-Hookean due to the Mooney-Rivlin parameter b not appearing independently of the neo-Hookean parameter a. As such, we may assume the hyperelastic matrix is neo-Hookean.

Note that equation (4.7b) admits the conservation law form

$$D_x \left(p - \beta \rho_0 \cos^3 \gamma \left[\cos \gamma (G_x)^2 + 2 \sin \gamma G_x \right] \right) = 0, \tag{4.8}$$

which is solved for p as

$$p = \beta \rho_0 \cos^3 \gamma \left(\cos \gamma \left(\frac{\partial G}{\partial x} \right)^2 + 2 \sin \gamma \frac{\partial G}{\partial x} \right) + f(t), \tag{4.9}$$

for an arbitrary $f(t)$. Thus, p can be determined for a known solution G to equation (4.7a). As such, we may study equation (4.7a) alone and obtain p through (4.9).

Of importance, equation (4.7a) is in Cauchy-Kovalevskaya form with respect to t. As such, all symmetries may be obtained through the algorithm presented in Chapter 1, and all conservation laws of (4.7a) can be found by the direct method [22]. Additionally, given analytic initial data, there exist unique analytic solutions to equation (4.7a) within some neighbourhood of the origin by the Cauchy-Kovalevskaya Theorem [46].

Two important cases of the current model are the restriction of the fiber family parallel to the x-axis (i.e. $\gamma = 0$) and perpendicular to the x-axis (i.e. $\gamma = \pi/2$). The former results in equation (4.7a) reducing to

$$\frac{\partial^2 G}{\partial t^2} = \left(\alpha + 3\beta \left(\frac{\partial G}{\partial x} \right)^2 \right) \frac{\partial^2 G}{\partial x^2}. \tag{4.10}$$

The coefficient of G_{xx} in the current equation is of a simpler form than in (4.7a), suggesting the dynamics are different for $\gamma = 0$ compared to $\gamma \neq 0$. Indeed, equation (4.10) is invariant under reflection about the X^3-axis, given by the discrete transformation

$$t^* = t, \quad x^* = -x, \quad G^* = G, \quad (G_x)^* = -G_x,$$

whereas equation (4.7a) is not invariant under this transformation for $\gamma \in (0, \pi/2)$.

For $\gamma = \pi/2$, the fibers will be parallel to the displacement, which implies the fiber family should have no effect on the motion. Indeed, equation (4.7a) reduces identically to the linear

wave equation

$$\frac{\partial^2 G}{\partial t^2} = \alpha \frac{\partial^2 G}{\partial x^2}, \tag{4.11}$$

and, from (4.9), we have $p = f(t)$. The general solution of the linear wave equation is well known (see, e.g., [56]).

4.3.2 Numerical Solutions

In the current section, we present a numerical simulation of equation (4.10) and (4.9) using the Maple 18 pdsolve/numeric command. The details relevant to the numerical simulation are first presented, as well as a plot of G as a function of x at various times t. Afterwards, the hydrostatic pressure p is plotted from equation (4.9) based on the numerical solution for G. Lastly, we discuss the results of the numerical simulation.

The pdsolve/numeric command can numerically solve a given scalar PDE in time and space, given the problem has sufficient initial and boundary values. The default numerical scheme used by pdsolve/numeric is an implicit, centered finite difference scheme of second order in time and space [82]. If the scalar PDE contains second or higher order derivatives in time, then pdsolve/numeric first restates the problem as a PDE system with additional variables such that the highest derivative in time is first order.

For the current numerical simulation, the material parameters are fixed as $\alpha = 1$ and $\beta = 1/3$ out of convenience. Note that a different choice of $\alpha > 0$ will change the overall wave speed, whereas a different $\beta > 0$ will alter the effect of the fiber family on the wave speed. We also specify a Gaussian initial condition $G(x, 0) = \exp(-x^2)$ with zero initial velocity $G_t(x, 0) = 0$ on the space interval $[-8, 8]$. To remove boundary effects from the simulation, the x-interval is taken as [-20,20]. For simplicity, Dirichlet boundary conditions $G(-20, t) = G(20, t) = \exp(-400)$ are chosen. The space step is fixed as $\Delta x = 0.02$ such that the simulation uses 1000 spatial points. The time step is chosen as $\Delta t = \Delta x/2 = 0.01$ due to the relatively small difference in numerical solutions at $t = 4$ for $\Delta t = 0.01$ and $\Delta t = 0.02$ (see Figure 4.3).

To obtain a plot for the hydrostatic pressure p, $\alpha = 1$, $\beta = 1/3$, $\gamma = 0$, and $\rho_0 = 1$, are
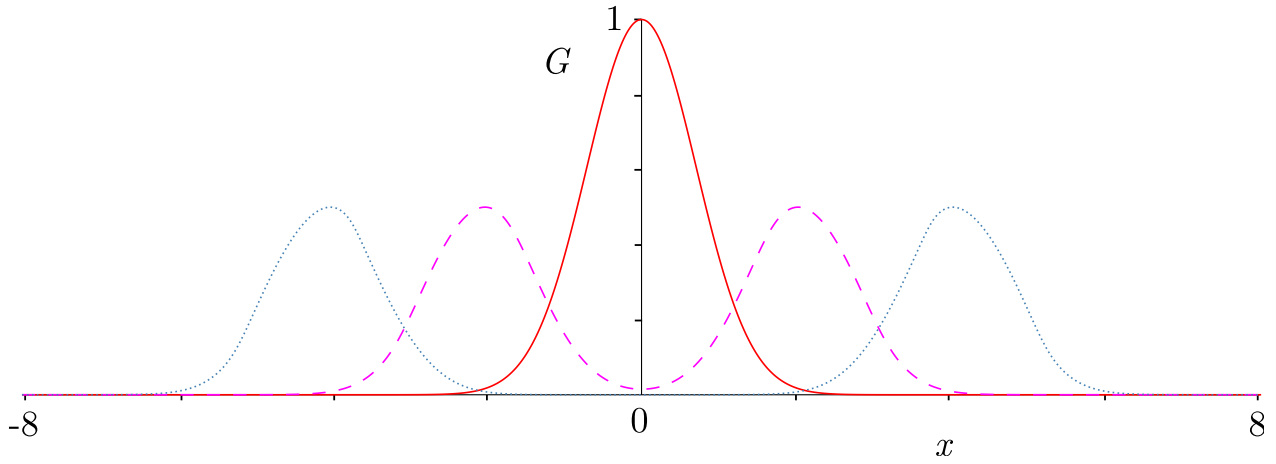
Figure 4.2: Plot of numerical solution of (4.10) for $\alpha = 1$, $\beta = 1/3$, space step = 0.02, and time step 0.01. The colors red (solid line), magenta (dashed line), and blue (dotted line) correspond to the solution at times $t = 0, 2, 4$, respectively.
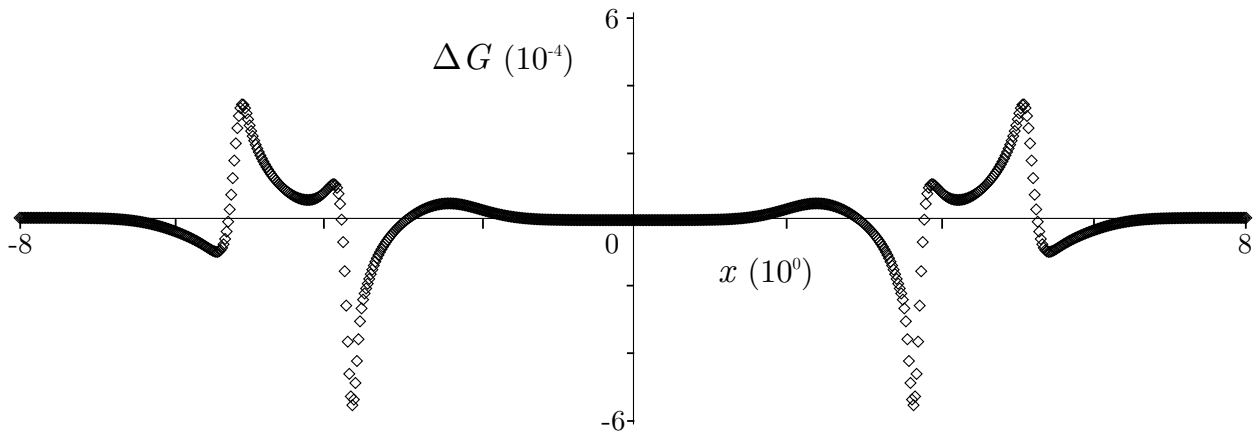

Figure 4.3: Plot of difference ΔG between numerical solutions G of (4.10) for time steps $\delta t_1 = 0.01$ and $\delta t_2 = 0.02$ at time $t = 4$.
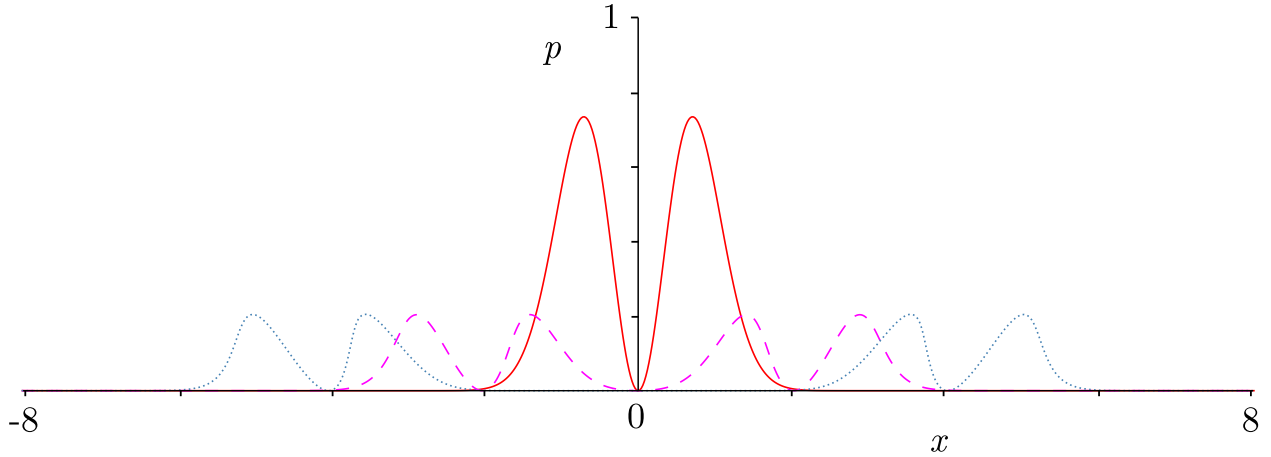
Figure 4.4: Plot of numerical solution of (4.9) based on solution to (4.10) at times $t = 0, 2, 4$ in colors red (solid line), magenta (dashed line), and blue (dotted line), respectively.

substituted into (4.9) to yield

$$p = \frac{1}{3}\left(\frac{\partial G}{\partial x}\right) + f(t). \tag{4.12}$$

The arbitrary function $f(t)$ is taken as $f(t) = 0$ to determine the behaviour of p due only to the solution G of (4.10).

In order to plot p from (4.12) and the numerical solution G in Figure 4.2, an approximation of G_x at each point x and t is required. For the second order accurate centered difference approximation,

$$\frac{\partial G}{\partial x}(x_j, t_i) \approx \frac{G(x_{j+1}, t_i) - G(x_{j-1}, t_i)}{2\Delta x},$$

the plot of p is given in Figure 4.2.

We comment now on the numerical solution in Figures 4.2 and 4.4. First, the solution G to (4.10) in Figure 4.2 start as a single Gaussian that separates into two profiles travelling apart in time. These profiles exhibit a change in shape as time increases. In particular, the velocity of the profile is smaller in magnitude near $G \approx 0$ and the peak. Indeed, this can be seen in (4.10) by observing the 'wave speed' $\alpha + \beta(G_x)^2$ is larger away from $G \approx 0$ and the peak for $\alpha > 0$ and $\beta > 0$. Figure 4.4 shows that p also separates into two travelling profiles that approach sawtooth-like shapes as time increases, indicating that a a pressure shock may form for larger t.

The particular Maple files used for this analysis can be found within Appendix A.

4.3.3   Lie Point Symmetries

We now classify the Lie point symmetries of equations (4.7a) for the parameters $\alpha > 0$, $\beta > 0$, and $\gamma \in [0, \pi/2)$. After, we present the corresponding global groups of point transformations, and discuss the physical interpretation of the symmetries. For $\gamma = \pi/2$, equation (4.7a) becomes the well studied linear wave equation, for which the symmetries can be found in [66].

Table 4.1: Lie point symmetry classification of (4.7a).

Parameters	Symmetries
arbitrary	$\mathrm{X}^1 = \frac{\partial}{\partial t}$, $\mathrm{X}^2 = \frac{\partial}{\partial x}$, $\mathrm{X}^3 = \frac{\partial}{\partial G}$, $\mathrm{X}^4 = t\frac{\partial}{\partial G}$, $\mathrm{X}^5 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + G\frac{\partial}{\partial G}$
$4\alpha \leq \beta, \quad \gamma = \cos^{-1}\left(\sqrt{\frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4\alpha}{\beta}}\right)}\right)$	X^1, X^2, X^3, X^4, X^5, $\mathrm{X}^6 = \cos(\gamma)x\frac{\partial}{\partial x} + 2\cos(\gamma)t\frac{\partial}{\partial t} - \sin(\gamma)x\frac{\partial}{\partial G}$

Theorem 4.3.1. *The Lie point symmetry classification of the system* (4.7a) *for $\gamma \neq \pi/2$ is given in Table* (4.1).

Proof. Proof of Theorem 4.3.1 follows from the methodology presented in Chapter 1. □

We now derive the global groups of point transformations equivalent to the Lie point symmetries in Theorem 4.3.1. As in Chapter 3, ϵ^i corresponds to X^i for $i = 1, \ldots, 6$.

$$\mathrm{X}^1: \quad t^* = t + \epsilon^1, \quad x^* = x, \qquad G^* = G;$$
$$\mathrm{X}^2: \quad t^* = t, \qquad x^* = x + \epsilon^2, \quad G^* = G;$$
$$\mathrm{X}^3: \quad t^* = t, \qquad x^* = x, \qquad G^* = G + \epsilon^3;$$
$$\mathrm{X}^4: \quad t^* = t, \qquad x^* = x, \qquad G^* = G + \epsilon^4 t;$$
$$\mathrm{X}^5: \quad t^* = e^{\epsilon^5}t, \qquad x^* = e^{\epsilon^5}x, \qquad G^* = e^{\epsilon^5}G.$$

Here, we can clearly see that $\{X^i\}_{i=1}^3$ correspond to invariance under translations of the independent variables, X^4 to time-dependent translation of G, and X^5 to a uniform scaling of all variables. Invariance under X^1 and X^2 is expected due to equation (4.7a) being autonomous.

For the restriction of material parameters such that $4\alpha \leq \beta$ and γ is fixed by

$$\cos^2(\gamma) = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4\alpha}{\beta}}\right), \qquad (4.13)$$

then equation (4.7a) is invariant under the additional Lie symmetry X^6 with global group of point transformations

$$t^* = e^{2(\cos\gamma)\epsilon^6}\, t, \quad x^* = e^{(\cos\gamma)\epsilon^6}\, x, \quad G^* = G + (\tan\gamma)\left(1 - e^{(\cos\gamma)\epsilon^6}\right)x.$$

Thus, X^6 corresponds to equation (4.7a) invariant under a scaling of the independent variables and x-dependent translation in the variable G.

Note there is one solution $\gamma = \pi/4$ for $4\alpha = \beta$ to (4.13), and two solutions about $\pi/4$ for $4\alpha < \beta$. As such, we conclude the symmetry X^6 is plausible for the fiber bundle having a strong effect on the motion relative to the underlying Mooney-Rivlin matrix.

Remark 4.3.2. For $b = 0$ and $(I^4 - 1)$ sufficiently small, we may adapt the parameters for the (4.4) strain energy density discussed in Section 4.2 for our model as $\alpha = 0.3$ and $\beta = 2.2480$ (omitting units kPa) [59]. Clearly, $4\alpha = 1.2 \leq 0.2480 = \beta$, so X^6 may arise as a symmetry for this model.

4.3.4 Reductions and Solutions

We now study invariant solutions of equation (4.7a) for particular admitted Lie symmetries.

Travelling Wave Solution

Upon substituting the *ansatz* $G(x,t) = V(z)$ for $z = x - ct$ with wave speed c into (4.7a), the resulting ODE is

$$0 = \left(\alpha + 3\beta\cos^4\gamma\left(\frac{dV}{dI}\right)^2 + 6\beta\cos^3\gamma\sin\gamma\frac{dV}{dI} + 2\beta\cos^2\gamma\sin^2\gamma - c^2\right)\frac{d^2V}{dI^2}, \qquad (4.14)$$

This equation admits only the trivial solution,

$$G(x, t) = V(z) = C^1 z + C^2, \tag{4.15}$$

where C^1 and C^2 are constants of integration.

This solution is the equation of a line in (x, G)-space that travels to the left or right for negative or positive wave speed c respectively. This solution is physical only for the stationary solution $G(x, t) = C^2$.

Special Symmetry Reduction

Suppose that $4\alpha < \beta$ and $\gamma = \cos^{-1}\left(\sqrt{\frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4\alpha}{\beta}}\right)}\right)$. Then equation (4.7a) admits the symmetry $X^6 = \cos(\gamma)x\frac{\partial}{\partial x} + 2\cos(\gamma)t\frac{\partial}{\partial t} - \sin(\gamma)x\frac{\partial}{\partial G}$, which has the following invariants:

$$I = \frac{t}{x^2},$$

$$V = G + \tan(\gamma)x.$$

Substituting $V = V(I)$ and $\gamma = \cos^{-1}\left(\sqrt{\frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4\alpha}{\beta}}\right)}\right)$ into (4.7a) and simplifying yields the ODE

$$0 = \left(2\omega I^4(V')^2 - 1\right)V'' + 3\omega I^3(V')^3, \tag{4.16}$$

where $\omega = 12(\beta - 2\alpha \pm \beta\sqrt{\beta - 4(\alpha/\beta)})$ a constant. The sign in each $\pm$ matches the sign taken for the angle γ.

For any solution V to (4.16), the function $G = V - \tan(\gamma)x$ will be unbounded due to the $\tan(\gamma)x$ term. As such, reduction of (4.7a) using the special symmetry X^6 yields non physical solutions.

4.3.5 Conservation Laws

We now study the conservation laws to the system (4.7a) through the direct method. We classify the admitted conservation law multipliers based on the parameter γ, and present the resulting conserved densities.

Conservation Law Multipliers

We first present the zeroth order conservation law multipliers of equation (4.7a) through the methodology presented in Chapter 1. Afterwards, we seek specific first order multipliers of (4.7a) associated with important physical quantities.

Theorem 4.3.3. *The zeroth order conservation law multipliers of* (4.7a) *are*

$$\Lambda = C^1 t + C^2.$$

Proof. The above zeroth order conservation law multipliers are found through the constructive method outlined in Chapter 1 with multiplier of the form

$$\Lambda = \Lambda(x, t, G).$$

$\square$

Of importance, no special cases arise in the classification of conservation law multipliers of equation (4.7a).

We now present specific first order conservation law multipliers of system (4.7a). The conservation law multipliers below are found by intuition based on the form of (4.7a) and the work done in Chapter 3.

Theorem 4.3.4. *Equation* (4.7a) *admits the first order conservation law multipliers*

$$\Lambda = C^3 G_t + C^4 G_x.$$

Proof. The proof follows from multiplying (4.7a) by each of G_t and G_x, and verifying the Euler operator annihilates the resulting equations. $\square$

Remark 4.3.5. We do not present the classification of first order conservation law multipliers of (4.7a) in general. This is due to difficulty in solving the determining equations for the conservation law multipliers, which in each case of the classification contain the determining equation

$$\frac{\partial^2}{\partial G_t{}^2}\Lambda^1 = \frac{\frac{\partial^2}{\partial G_x{}^2}\Lambda^1}{\alpha + 12\,G_x{}^2\cos^4\gamma\frac{\beta}{4} + 24\,G_x\,\cos^3\gamma\frac{\beta}{4}\sin\gamma - 8\,\cos^4\gamma\frac{\beta}{4} + 8\,\cos^2\gamma\frac{\beta}{4}}, \qquad (4.17)$$

where $\Lambda = \Lambda(x, t, G, G_t, G_x)$. The general solution to the quasilinear equation (4.17) is not known (to the authors current awareness), and so we cannot proceed in determining the first order conservation law multipliers by the direct method.

Divergence Conservation Law Form

We now present the divergence conservation law form of the system (4.7a) for the multipliers in Theorems 4.3.3 and 4.3.4.

Conservation of Energy

For the first order multiplier corresponding to C^3, we have the conservation of energy.

$$
\begin{aligned}
&D_t \left(\frac{1}{2}(G_t)^2 + \frac{1}{2}(G_x)^2 + \beta \cos^2 \gamma (G_x)^2 \left[\frac{1}{4} \cos^2 \gamma (G_x)^2 + \cos \gamma \sin \gamma G_x + \sin^2 \gamma \right] \right) \\
&\quad - D_x \left(G_t G_x \left(\alpha + \beta \cos^2 \gamma \left[\cos^2 \gamma (G_x)^2 + 3 \cos \gamma \sin \gamma G_x + 2 \sin^2 \gamma \right] \right) \right) = 0.
\end{aligned}
$$

Conservation of Linear Momentum in the Eulerian Frame

For multiplier corresponding to C^2, we have the conservation of linear momentum in the Eulerian frame of reference.

$$
D_t (G_t) - D_x \left(G_x \left(\alpha + \beta \cos^2 \gamma \left[\cos^2 \gamma (G_x)^2 + 3 \cos \gamma \sin \gamma G_x + 2 \sin^2 \gamma \right] \right) \right) = 0.
$$

Center of Mass Theorem

In the current notation for Eulerian and Lagrangian quantities, consider the quantity

$$
\rho_0(tx_t^i - x), \quad i = 1, 2, 3. \tag{4.18}
$$

The *center of mass theorem* in the continuum is equivalent to conservation of each quantity in (4.18). This observation is due to Caviglia and Morro [30, 31], from comparison with a discrete version derived by Hill [58].

For the multiplier corresponding to C^1, we obtain the conservation law in x^3 of the center of mass theorem.

$$
D_t (tG_t - G) - D_x \left(tG_x \left(\alpha + \beta \cos^2 \gamma \left[\cos^2 \gamma (G_x)^2 + 3 \cos \gamma \sin \gamma G_x + 2 \sin^2 \gamma \right] \right) \right) = 0.
$$

Note that the center of mass theorem holds identically in the x^1 and x^2 directions for equation (4.7a) since $x^1 = X^1$ and $x^2 = X^2$, which implies

$$\mathrm{D}_t \left(\rho_0 (t x_t^i - x^i) \right) \equiv 0, \quad i = 1, 2.$$

Conservation of Linear Momentum in the Lagrangian Frame

For multiplier set corresponding to C^4, we have the conservation of material linear momentum in the Lagrangian frame of reference, for which the divergence conservation law form is

$$\mathrm{D}_t \left(G_x G_t \right)$$
$$-\mathrm{D}_x \left(\frac{1}{2}(G_t)^2 + \frac{\alpha}{2}(G_x)^2 + \beta \cos^2 \gamma (G_x)^2 \left[\frac{3}{4} \cos^2 \gamma (G_x)^2 + 2 \cos \gamma \sin \gamma G_x + \sin^2 \gamma \right] \right) = 0.$$

4.3.6 Hodograph Transformation and Invertible Mappings

We now present the Hodograph transformation for a quasi-linear second order PDE based on [114]. Afterwards, we discuss whether the resulting equations can be mapped into a constant coefficient wave equation [20].

Consider a quasi-linear second order PDE of the form

$$\frac{\partial^2 G}{\partial t^2} = c \left(\frac{\partial G}{\partial x} \right) \frac{\partial^2 G}{\partial x^2}, \tag{4.19}$$

where c is a function of G_x. We introduce two dependent variables $u = G_x$ and $v = G_t$. As such, equation (4.19) becomes the system

$$\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial t} - c(u) \frac{\partial u}{\partial x} = 0 \tag{4.20}$$

Assuming the Jacobian $J = \frac{\partial(u,v)}{\partial(x,t)} \neq 0$, we apply the Hodograph transformation as outlined in [87] [114] to swap the independent and dependent variables of (4.20). The transformation has the form

$$u_x = J^{-1} t_v, \quad u_t = -J^{-1} x_v, \quad v_x = -J^{-1} t_u, \quad v_t = J^{-1} x_u, \tag{4.21}$$

where $x = x(u, v)$ and $t = t(u, v)$ is assumed in the right hand side of each relation in (4.21).

After substituting (4.21) into (4.20) and cancelling the common factor of J^{-1}, we obtain the system

$$\frac{\partial x}{\partial v} - \frac{\partial t}{\partial u} = 0, \qquad \frac{\partial x}{\partial u} - c(u) \frac{\partial t}{\partial v} = 0.$$

Eliminating x from this system of equations yields the linear wave equation with non-constant coefficients,

$$\frac{\partial^2 t}{\partial u^2} - c(u)\frac{\partial^2 t}{\partial v^2} = 0. \tag{4.22}$$

Invertible Mapping Theorem

Of interest is if a linear wave equation of the form (4.22) can be mapped (through an invertible mapping) to one with constant coefficients. We first consider the following theorem by Bluman [20].

Theorem 4.3.6. *A linear wave equation* (4.22) *with non-constant coefficients can be mapped into one with constant coefficients by an invertible mapping if and only if the 'wave speed' $K(u) = \pm\sqrt{c(u)}$ is a solution to the differential equation*

$$\frac{d}{du}\left(\frac{K^2 K'''}{2KK'' - (K')^2}\right) = 0, \tag{4.23}$$

where primes denote differentiation.

For our particular equation (4.7a), the 'wave speed' $K(u)$ is

$$K(u) = \frac{\pm 1}{\sqrt{\alpha + \beta u^2}}, \tag{4.24}$$

where $\alpha > 0$ and $\beta > 0$ are constants.

This leads us to the following theorem regarding (4.22) with wave speed (4.24).

Theorem 4.3.7. *Equation* (4.22) *with $K(u)$ given by* (4.24) *cannot be mapped by an invertible mapping into a linear PDE with constant coefficients.*

Proof. Consider the linear wave equation (4.22) with wave speed (4.24). Suppose that this equation can be mapped into a linear wave equation with constants coefficients. By Theorem 4.3.6, the wave speed must be a solution to the differential equation (4.23). By substituting the wave speed (4.24) into (4.23) and simplifying, we obtain the equation

$$\pm\frac{3\beta\left(20\beta^3 u^6 - 58\alpha\beta^2 u^4 + 21\alpha^2\beta u^2 - 6\alpha^3\right)}{\left(3\beta^3 u^6 + 4\alpha\beta^2 u^4 - \alpha^2\beta u^2 - 2\alpha^3\right)\left(3\beta u^2 - 2\alpha\right)\sqrt{\alpha + \beta u^2}} = 0, \tag{4.25}$$

where $\alpha > 0$ and $\beta > 0$. Since u is arbitrary, the left hand side of the above equation is zero if and only if $\beta = 0$, which is a contradiction. $\qquad\square$

4.4 Motion Transverse to a Plane

In this section, we study the motion of a displacement G transverse to a plane of a solid embedded with a fiber family.

4.4.1 Equations of Motion

The coordinate dependence for a displacement $G(X^1, X^2, t)$ transverse to the $X^1 X^2$ plane can be written as (without a loss of generality)

$$\vec{X} = \begin{bmatrix} X^1 \\ X^2 \\ G\left(X^1, X^2, t\right) + X^3 \end{bmatrix}. \tag{4.26}$$

See Figure 4.5 for a sample deformation of this type.

By observation of (4.26), the incompressibility condition is identically satisfied. The hydrostatic pressure then takes the form $p(X^1, X^2, t)$.

We now discuss the expected dynamics of the fiber reinforced system based on the orientation of a fiber family with respect to the $X^1 X^2$-plane. If a fiber family is oriented perpendicular to the plane, then the system becomes isotropic about the X^3-direction and there should be no effect on the motion. As well, the dynamics should be the same for any orientation of a single fiber in the $X^1 X^2$-plane. With this in mind, we consider the fiber family to have orientation vector

$$A^1 = \begin{bmatrix} \cos(\gamma) \\ 0 \\ \sin(\gamma) \end{bmatrix}, \tag{4.27}$$

where γ is a constant angle $\gamma \in [0, \pi/2]$. Here, A^1 is taken in the $X^1 X^3$-plane to observe the effect of a fiber family outside of the plane. The choice to measure the angle γ with respect to the $X^1 X^2$ plane is taken such that our convention is consistent with the one dimensional case studied in Section 4.3. Additionally, this allows us to more easily compare the one dimensional results to the two dimensional results to follow.
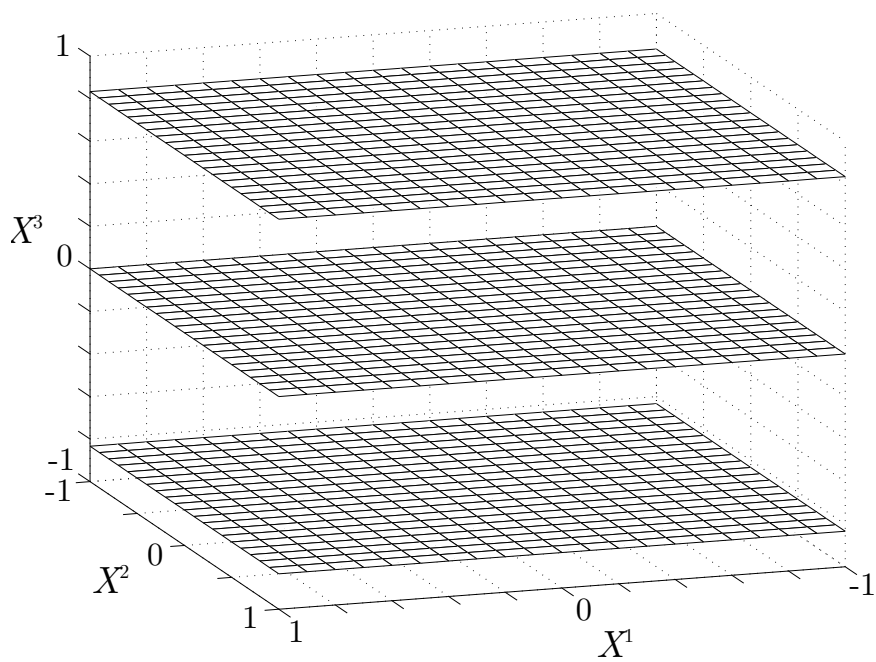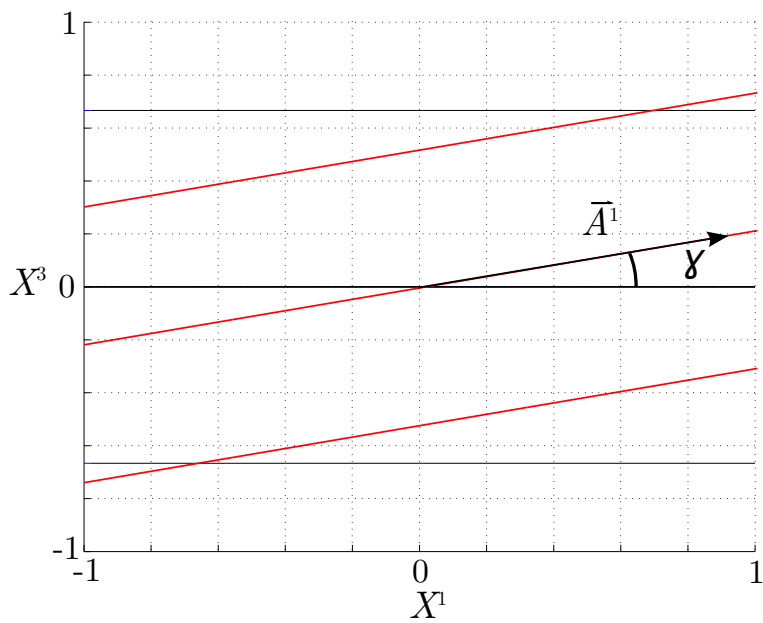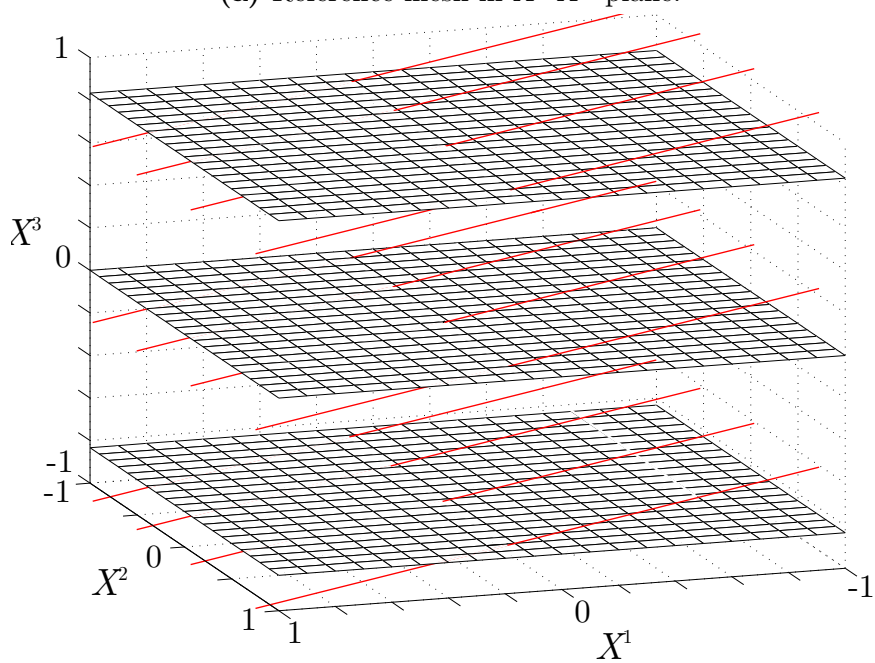
(a) Reference Mesh
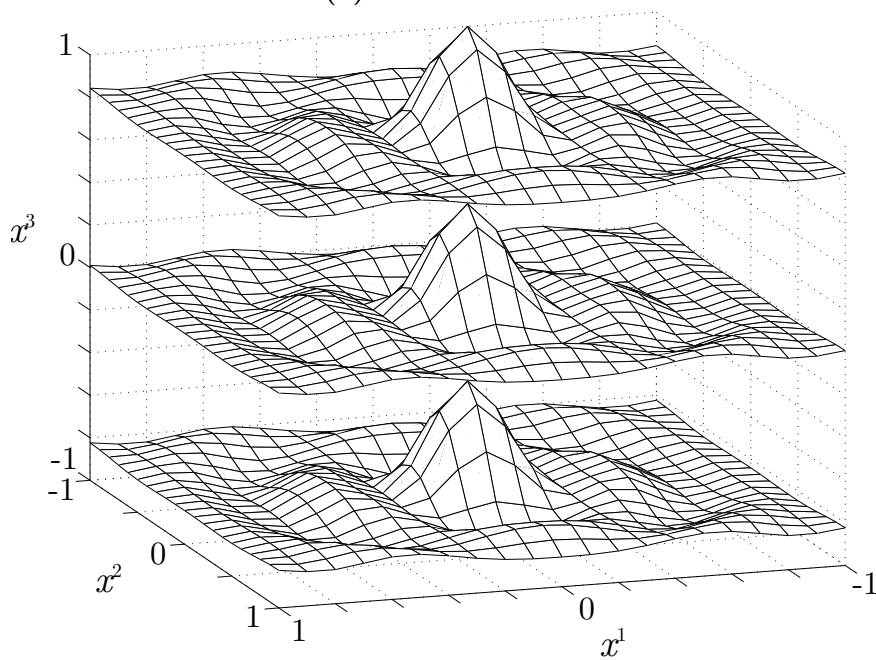
(b) Deformed Mesh

Figure 4.5: Deformation of sample meshes under transverse to plane motion.

(a) Reference mesh in $X^1 X^3$-plane.

(b) Reference mesh at side angle.

Figure 4.6: Fiber bundle in reference configuration.

By use of the angle convention in (4.27), the equations of motion for an incompressible Mooney Rivlin solid with fiber family oriented at an angle γ from the X^1-axis in the $X^1 X^3$-plane are derived from (4.1)-(4.3), (4.26), and (4.27) as

$$\frac{\partial^2 G}{\partial t^2} = 2\,(a+b)\left(\frac{\partial^2 G}{\partial (X^1)^2} + \frac{\partial^2 G}{\partial (X^2)^2}\right)$$

$$+4q\cos^2(\gamma)\left(3\cos^2(\gamma)\left(\frac{\partial G}{\partial X^1}\right)^2 + 6\cos(\gamma)\sin(\gamma)\frac{\partial G}{\partial X^1} + 2\sin^2\gamma\right)\frac{\partial^2 G}{\partial (X^1)^2}, \quad (4.28a)$$

$$0 = \frac{\partial p}{\partial X^1} + 2b\rho_0\left(\frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial(X^2)^2} - \frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial(X^1)\partial(X^2)}\right)$$

$$-8q\cos^3\gamma\left(\cos\gamma\frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial(X^1)^2} + \sin\gamma\frac{\partial^2 G}{\partial(X^1)^2}\right), \quad (4.28b)$$

$$0 = \frac{\partial p}{\partial X^2} + 2b\rho_0\left(\frac{\partial G}{\partial(X^2)}\frac{\partial^2 G}{\partial(X^1)^2} - \frac{\partial G}{\partial(X^1)}\frac{\partial^2 G}{\partial(X^1)\partial(X^2)}\right), \quad (4.28c)$$

where $a, b > 0$ are the Mooney-Rivlin material parameters, $q > 0$ is the fiber family parameter, and ρ_0 is the mass density.

We now discuss several properties of the system (4.28). First, note that these equations reduce to a Mooney-Rivlin system for $q = 0$, as well as a neo-Hookean model under the further restriction $b = 0$. Second, we see that for the fiber bundle parallel to X^3 (i.e. $\gamma = \pi/2$), the system again reduces to the Mooney-Rivlin model.

In addition, equations (4.28b) and (4.28c) contain no explicit time derivatives. As such, these equations can be viewed to define p in terms of G. Furthermore, one may first study the motion in G alone, and then solve for p as in the 1D case. However, we may not ignore equations (4.28b) and (4.28c) without first considering if there is some differential consequence in G upon elimination of p from this subsystem . To remove p, we compute the compatibility conditions on the derivatives of p, which yields the equation (written in a divergence form for clarity)

$$0 = D_{X^1}\left(b\frac{\partial G}{\partial X^2}\left(\frac{\partial^2 G}{\partial (X^1)^2} + \frac{\partial^2 G}{\partial (X^2)^2}\right)\right)$$

$$+D_{X^2}\left(4q\cos^3\gamma\frac{\partial^2 G}{\partial(X^1)^2}\left(\cos\gamma\frac{\partial G}{\partial X^1} + \sin\gamma\right) - b\frac{\partial G}{\partial X^1}\left(\frac{\partial^2 G}{\partial(X^1)^2} + \frac{\partial^2 G}{\partial(X^2)^2}\right)\right). \quad (4.29)$$

As such, we cannot ignore equations (4.28b) and (4.28c) in our analysis of the motion of G alone as we had done for the one dimensional case. However, by having this differential

consequence explicitly, we have the option to analyze (4.28a) to determine G with (4.29) as a restriction, and then can solve for p using (4.28b) and (4.28c).

4.4.2   Remarks on One Dimensional Solutions

We now discuss connections between solutions of the current two dimensional system (4.28) and those of the one dimensional system (4.7). We first show how solutions to the one dimensional system are also solutions to the two dimensional system. Afterwards, we briefly consider if the superposition of the discussed one dimensional solutions is also a solution.

Consider the one dimensional restriction $G = A(X^1, t)$ in the system (4.28). As such, the model equations become

$$\frac{\partial^2 A}{\partial t^2} = \left(2(a + b) + 4q\cos^2(\gamma) \left(\cos^2(\gamma) \left(3 \left(\frac{\partial A}{\partial X^1} \right)^2 - 2 \right) \right. \right.$$
$$\left. \left. + 6\cos(\gamma)\sin(\gamma)\frac{\partial A}{\partial X^1} + 2 \right) \right) \frac{\partial^2 A}{\partial X^{1^2}}, \tag{4.30a}$$

$$0 = \frac{\partial p}{\partial X^1} - 8q\rho_0 \cos^3 \gamma \left(\cos\gamma \frac{\partial A}{\partial X^1}\frac{\partial^2 A}{\partial (X^1)^2} + \sin\gamma \frac{\partial^2 A}{\partial (X^1)^2} \right), \tag{4.30b}$$

$$\frac{\partial p}{\partial X^2} = 0. \tag{4.30c}$$

Thus, the hydrostatic pressure takes the form $p = p(X^1, t)$, and equations (4.28) reduce exactly to the system (4.7).

Consider the one dimensional case such that $G = B(X^2, t)$. As such, the system (4.28) reduces to

$$\frac{\partial^2 B}{\partial t^2} = 2(a + b)\frac{\partial^2 B}{\partial (X^2)^2}, \tag{4.31a}$$

$$\frac{\partial p}{\partial X^1} = 0, \tag{4.31b}$$

$$\frac{\partial p}{\partial X^2} = 0. \tag{4.31c}$$

Clearly, the system (4.28) has been reduced to the linear wave equation with the hydrostatic pressure $p = p(t)$ as some function of time. For an initial boundary value problem (IBVP) with sufficient data, the general solution of the linear wave equation (4.31a) can be obtained, for example, as a Fourier series through separation of variables [56].

A natural question to ask is if $A(X^1, t)$ and $B(X^2, t)$ solve equations (4.30a) and (4.31a) respectively, then is the superposition $G(X^1, X^2, t) = A(X^1, t) + B(X^2, t)$ a physical solution to the system (4.28) We explore this question in the following.

Recall that we may solve the system (4.28) by first solving (4.28a) and (4.29) for G, and then substituting this solution into (4.28b) and (4.28c) to obtain p. As such, we first consider substituting $G(X^1, X^2, t) = A(X^1, t) + B(X^2, t)$ into the differential consequence (4.29), for which we obtain the separable PDE

$$0 = 2b \left(\frac{\partial A}{\partial X^1} \frac{\partial^3 B}{\partial (X^2)^3} - \frac{\partial^3 A}{\partial (X^1)^3} \frac{\partial B}{\partial X^2} \right). \tag{4.32}$$

Before proceeding, we note that to solve the linear wave equation (4.31a) for B, we may use the separation of variables solution $B = \tau(t)\chi(X^2)$ to obtain the system of ODEs

$$\tau'' = -\lambda \tau, \tag{4.33a}$$

$$\chi'' = \frac{-\lambda}{2(a+b)} \chi, \tag{4.33b}$$

where λ is a constant from the separation of variables, and prime denotes differentiation with respect to the appropriate variable. This system can be solved for physical initial and boundary values (as in, e.g., a vibrating string), yielding τ and χ as linear combinations of sines and cosines in t and X^2 respectively [56]. Here, the restriction $\lambda \geq 0$ is determined from the boundary conditions. Without finding the general solution χ to (4.33b), we proceed to solve the separated equation (4.32) with $B = \chi(X^2)\tau(t)$.

Upon separating (4.32) with respect to X^1 and X^2 and substituting $B = \chi(X^2)\tau(t)$, we obtain

$$\frac{\left(\frac{\partial^3 A}{\partial (X^1)^3} \right)}{\frac{\partial A}{\partial X^1}} = \frac{\chi'''}{\chi'}. \tag{4.34}$$

Clearly, each side of the above equation must be constant as each of X^1, X^2, and t vary. So, upon setting each side of (4.34) to the constant $-k$, we obtain the following system of equations in A and χ:

$$A_{111} = -kA_1, \tag{4.35a}$$

$$\chi''' = -k\chi', \tag{4.35b}$$

where k is a constant. Furthermore, we may simplify (4.35b) by substituting (4.33b) to obtain

$$\left(\frac{-\lambda}{2(a+b)} - k\right)\chi' = 0. \tag{4.36}$$

Equation (4.36) admits two solutions: χ is constant, or $2(a+b)k = \lambda$. The case when χ is constant corresponds to the trivial solution $B(x,t) = C^1 t + C^2$ of (4.31a) for constants $\{C^i\}_{i=1}^2$. This implies equation (4.29) is identically zero, and the system (4.28) reduces to the one dimensional system (4.7) discussed in Section 4.3.

In the case when $2(a+b)k = \lambda$, we proceed to solve (4.35a) for A. Note there are two distinct cases for k to consider: $k = 0$ when $\lambda = 0$; or, $k > 0$ when $\lambda \geq 0$.

For $k = 0$, B is stationary (i.e. constant), and so equation (4.32) is identically zero. Thus, $A(X^1, t)$ is any solution to the one dimensional equation (4.7a) in Section 4.3.

In the case when $k > 0$, (4.35a) has a solution of the form

$$A(X^1, t) = f^1(t) + f^2(t)\sin(\sqrt{k}X^1) + f^3(t)\cos(\sqrt{k}X^1). \tag{4.37}$$

Upon substituting (4.37) this into (4.30a) and solving for each $\{f^i(t)\}_{i=1}^3$, one obtains the solution A as

$$A(X^1, t) = \kappa^2 t + \kappa^1, \tag{4.38}$$

where $\{\kappa^i\}_{i=1}^2$ are constants. This is the trivial solution to (4.30a) and (4.32).

In summary, we have shown that the only solutions of the form $G(X^1, X^2, t) = A(X^1, t) + B(X^2, t)$ to equations (4.28a) and (4.29) are: A is a solution to the one dimensional equation (4.30a) and B is constant; or, B is a solution to the linear wave equation (4.31a) and A is the trivial solution to (4.30a).

4.4.3   Lie Point Symmetries

We now classify the Lie point symmetries of (4.28a) and (4.29) for material parameters $a > 0$, $b > 0$, $q > 0$, and $\gamma \in [0, \pi/2]$. After, we derive the equivalent global group of point transformations for each symmetry, and discuss the physical interpretation of each symmetry.

We do not consider equations (4.28b) and (4.28c) in our analysis since we are only interested in the symmetries of G. However, we must include the differential consequence (4.29) in our study. This is because equation (4.29) limits the Lie symmetry determining equations for G in the same way that equations (4.28b) and (4.28c) would have.

Theorem 4.4.1. *The Lie point symmetry classification of the system* (4.28a) *and* (4.29) *for material parameters $a > 0$, $b > 0$, $q > 0$, and $\gamma \in [0, \pi/2]$, is given in Table* (4.4.1).

Proof. Proof of Theorem 4.4.1 follows from the methodology presented in Chapter 1. $\qquad\square$

Table 4.2: Lie point symmetry classification of (4.28a) and (4.29).

Parameters	Symmetries
arbitrary	$Y^1 = \frac{\partial}{\partial t}$, $Y^2 = \frac{\partial}{\partial X^1}$, $Y^3 = \frac{\partial}{\partial X^2}$, $Y^4 = \frac{\partial}{\partial G}$, $Y^5 = t\frac{\partial}{\partial G}$, $Y^6 = X^1\frac{\partial}{\partial X^1} + X^2\frac{\partial}{\partial X^2} + t\frac{\partial}{\partial t} + G\frac{\partial}{\partial G}$
$\gamma = \pi/2$ or $q = 0$	Y^1, Y^2, Y^3, Y^4, Y^5, Y^6, $Y^7 = -X^2\frac{\partial}{\partial X^1} + X^1\frac{\partial}{\partial X^2}$, $Y^8 = G\frac{\partial}{\partial G}$

The Lie point symmetries for $\gamma = \pi/2$ or $q = 0$ correspond to the fiber bundle perpendicular to the $X^1 X^2$-plane, which is equivalent to the isotropic Mooney-Rivlin model with no fibers present. It is clear from the symmetry classification that the admitted Lie point symmetries in the anisotropic model (γ arbitrary) are fewer in number compared to those of the isotropic model ($\gamma = \pi/2$ or $q = 0$).

We now derive the global groups of point transformations equivalent to the Lie point symmetries in Theorem 4.4.1. As in the preceding sections, the parameter ϵ^i corresponds to Y^i for $i = 1, \ldots, 8$.

For the material parameters arbitrary, the global one-parameter groups of point transfor-

mation equivalent to Lie point symmetries $\{Y^i\}_{i=1}^6$ and Y^8 are:

$$Y^1: \quad t^* = t + \epsilon^1, \quad (X^1)^* = X^1, \qquad (X^2)^* = X^2, \qquad (G)^* = G;$$

$$Y^2: \quad t^* = t, \qquad (X^1)^* = X^1 + \epsilon^2, \quad (X^2)^* = X^2, \qquad (G)^* = G;$$

$$Y^3: \quad t^* = t, \qquad (X^1)^* = X^1, \qquad (X^2)^* = X^2 + \epsilon^3, \quad (G)^* = G;$$

$$Y^4: \quad t^* = t, \qquad (X^1)^* = X^1, \qquad (X^2)^* = X^2, \qquad (G)^* = G + \epsilon^4;$$

$$Y^5: \quad t^* = t, \qquad (X^1)^* = X^1, \qquad (X^2)^* = X^2, \qquad (G)^* = G + \epsilon^5 t;$$

$$Y^6: \quad t^* = e^{\epsilon^6} t, \quad (X^1)^* = e^{\epsilon^6} X^1, \quad (X^2)^* = e^{\epsilon^6} X^2, \quad (G)^* = e^{\epsilon^6} G.$$

Here, we can clearly see that $\{Y^i\}_{i=1}^4$ correspond to invariance under translations, Y^5 to time-dependent translations of G, and Y^6 to scaling of all variables. Invariance of (4.28a) and (4.29) under Y^1, S^2, and Y^3 was expected since (4.28a) and (4.29) are autonomous.

For $\gamma = \pi/2$ or $q = 0$, equations (4.28a) and (4.29) admit two additional Lie point symmetries Y^7 and Y^8, which have the equivalent global groups of transformations

$$Y^7: \quad t^* = t, \quad (X^1)^* = \cos(\epsilon^7)X^1 - \sin(\epsilon^7)X^2,$$

$$(X^2)^* = \sin(\epsilon^7)X^1 + \cos(\epsilon^7)X^2, \quad (G)^* = G,$$

$$Y^8: \quad t^* = t, \qquad (X^1)^* = X^1, \qquad (X^2)^* = X^2, \qquad (G)^* = e^{\epsilon^8} G.$$

Clearly, Y^7 corresponds to invariance under rotations of the independent spatial variables, and Y^8 to a scaling of the dependent variables G.

4.4.4 Conservation Laws

We classify the conservation laws of equation (4.28a) for $a > 0$, $b > 0$, $q > 0$, and $\gamma \in 0, \pi/2$). The equations is written in Cauchy-Kovalevskaya form with respect to t, and so all conservation laws may be found as non-trivial conservation law multipliers with a one-to-one correspondence to the conserved quantities. It is clear that conservation laws admitted by (4.28a) also hold for any solution of the system (4.28).

Conservation Law Multipliers

We now proceed to find the zeroth order conservation laws of the equations of motion.

Theorem 4.4.2. *Any zeroth order conservation law multiplier $\Lambda(t, X^2)$ of the wave equation*

$$\frac{\partial^2 G}{\partial t^2} = 2(a+b)\frac{\partial^2 G}{\partial X^{2^2}} \tag{4.39}$$

is also a conservation law multiplier of equation (4.28a).

Proof. Equation (4.28a) can be written in the form

$$\frac{\partial^2 G}{\partial t^2} = 2(a+b)\frac{\partial^2 G}{\partial (X^2)^2}$$

$$+\mathrm{D}_1\left(4q\cos^4(\gamma)\left(\frac{\partial G}{\partial X^1}\right)^3 + 12q\cos^3(\gamma)\sin\gamma\left(\frac{\partial G}{\partial X^1}\right)^2 + 2(a+b+\sin^2\gamma)\frac{\partial G}{\partial X^1}\right), \tag{4.40}$$

which is equation (4.39) plus a divergence in X^1.

For any function $f(t, X^2)$, we have that $f(t, X^2)\mathrm{D}_1(\Phi^1) = \mathrm{D}_1(f(t, X^2)\Phi^1)$. As such, any conservation law multiplier $\Lambda(t, X^2)$ of equation (4.39) is also a multiplier of (4.28a), as desired. $\square$

Of importance, the conservation law multipliers Λ of equation (4.7a) in Theorem 4.4.2 are the only zeroth order.

Corollary 4.4.3. *The zeroth order conservation law multipliers of the linear wave equation (4.39)*

$$\Lambda^1 = F^1\left(X^2 - \frac{t}{(2(a+b))^{\frac{1}{2}}}\right) + F^2\left(X^2 + \frac{t}{(2(a+b))^{\frac{1}{2}}}\right) \tag{4.41}$$

are the only zeroth order conservation law multipliers of (4.28a).

Proof. The above zeroth order conservation law multipliers are found through the constructive method outlined in Chapter 1 with multipliers of the form

$$\Lambda = \Lambda(X^1, X^2, t, G).$$

The completeness of this result follows from equation (4.28a) being in Cauchy-Kovalevskaya form. $\square$

Of importance, no special conservation laws appear for particular values of a, b, q, or γ.

We now present the first order multiplier of equation (4.28a) for the conservation of energy.

Theorem 4.4.4. *Equation* (4.28a) *admits the first order conservation law multiplier*

$$\Lambda = G_t. \tag{4.42}$$

Proof. The proof follows from multiplying (4.7a) by G_t and verifying the Euler operator annihilates the resulting equation. $\qquad\square$

Divergence Conservation Law Form

The divergence conservation law forms for the multipliers in Theorems 4.4.3 and 4.4.4 are now presented.

Conservation of Generalized Quantities

The model admits conservation of generalized quantities for multipliers F^1 and F^2 in Theorem 4.4.3. The divergence conservation law forms are as follows.

From the multiplier $F^1 + F^2$, we obtain

$$\mathrm{D}_t \left((F^1 + F^2)G_t - ((F^1)' + (F^2)')G \right)$$

$$-\mathrm{D}_1 \left((F^1 + F^2) \left(2(a+b)G_1 + 4e \cos^2 \gamma \left[\cos^2 \gamma (G_1)^3 + 3 \cos \gamma \sin \gamma (G_1)^2 + 2 \sin^2 \gamma G_1 \right] \right) \right)$$

$$\tag{4.43}$$

$$-\mathrm{D}_2 \left(2(a+b)(F^1 + F^2)G_2 + \sqrt{2(a+b)}((F^1)' + (F^2)')G \right) = 0,$$

where prime denotes differentiation with respect to the functions argument. Note we may set $F^2 = 0$ to obtain the divergence conservation law for the multiplier F^1 alone, and vice versa for F^2.

Of interest is the above divergence conservation law form encompasses an infinite number of conservation laws. Three particular conservation laws encompassed in the above general divergence expression are the conservation of linear momentum in x^3 (for $F^1 + F^2 = 1$).

Conservation of Energy

The model admits conservation of energy for the multiplier G_t. The divergence conservation

law form is

$$D_t \left(\frac{1}{2}(G_t)^2 + (a+b) \left((G_1)^2 + (G_1)^2 \right) \right.$$

$$\left. +q\cos^2\gamma \left(\cos^2\gamma(G_1)^4 + 4\cos\gamma\sin\gamma(G_1)^3 + 4\sin^2\gamma(G_1)^2 \right) \right) \qquad (4.44)$$

$$-D_{X^1} \left(2(a+b)G_2G_t + q\cos^2\gamma \left(8\sin^2\gamma G_1G_t + 12\cos\gamma\sin\gamma(G_1)^2G_t + 4\cos^2\gamma(G_1)^3G_t \right) \right)$$

$$-D_{X^2} \left(2(a+b)G_2G_t \right) = 0.$$

Here, we have the kinetic energy density K as

$$K = \frac{\rho_0}{2}(G_t)^2,$$

and potential energy density is P as

$$P = \rho_0 \left((a+b) \left((G_1)^2 + (G_1)^2 \right) \right.$$

$$\left. +q\cos^2\gamma \left(\cos^2\gamma(G_1)^4 + 4\cos\gamma\sin\gamma(G_1)^3 + 4\sin^2\gamma(G_1)^2 \right) \right).$$

Comparing P to the strain energy density

$$W = (a+b)(3 + (G_1)^2 + (G_2)^2) + q\cos^2\gamma \left(\cos^2\gamma(G_1)^4 + 4\cos\gamma\sin\gamma(G_1)^3 + 4\sin^2\gamma(G_1)^2 \right),$$

we observe that $P = (W - 3(a+b))$. As such, the potential energy is equal to the strain energy density up to addition of a constant.

4.5 Motion Transverse to an Axis

We now study another type of two-dimensional motion in which the solid is displaced transverse to an axis.

4.5.1 Equations of Motion

Motion accounting for the displacement $G^1(X^3, t)$ in the X^1-direction and $G^2(X^3, t)$ in the X^2-direction (see Figure 4.7) can be written for the coordinate dependence (without a loss

of generality) as

$$\vec{X} = \begin{bmatrix} G^1 \left(X^3, t \right) + X^1 \\ G^2 \left(X^3, t \right) + X^2 \\ X^3 \end{bmatrix}. \tag{4.45}$$

As a result, the incompressibility condition is identically satisfied. The hydrostatic pressure p is of the form $p = p(X^3, t)$.

Similar to the other two cases of motion studied, we consider the following unit vector to describe the fiber family

$$A = \begin{bmatrix} \sin(\delta) \\ 0 \\ \cos(\delta) \end{bmatrix}, \tag{4.46}$$

where $\delta \in [0, \pi/2]$ is related to the previous angle convention (in Section 4.3 by $\delta = \pi/2 - \gamma$. Note that while this vector measures the orientation of the fiber family with respect to a different reference axis than the motions in Sections 4.3 and 4.4, it follows the same convention in that the orientation of the fiber family is measured at an angle with respect to an independent spatial variable.
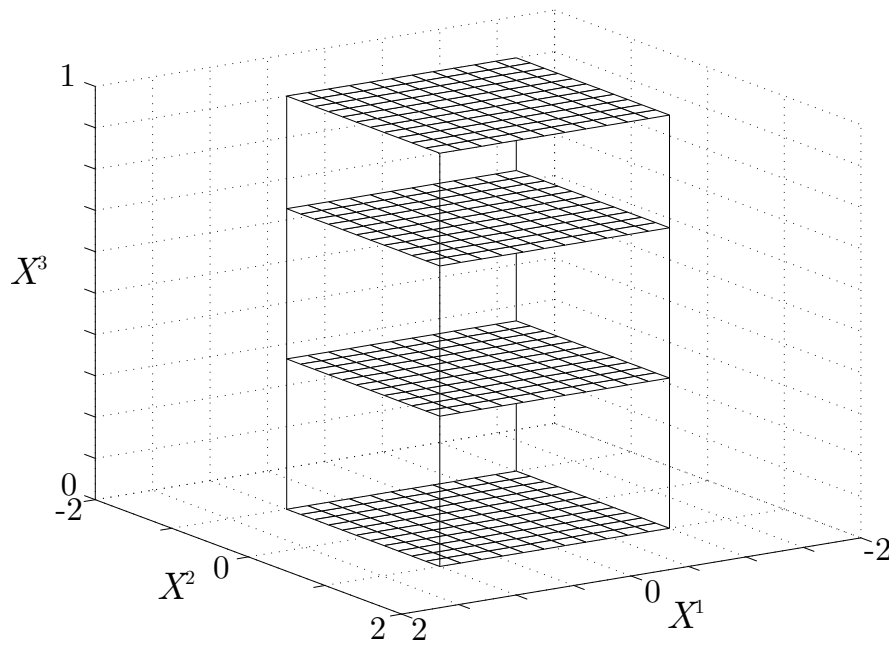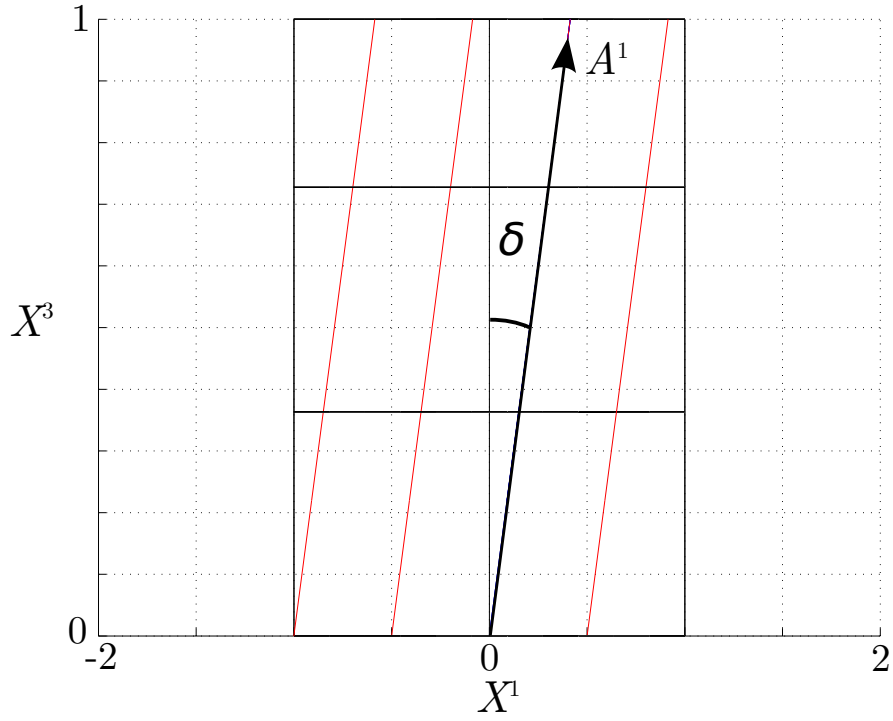
(a) Reference Mesh
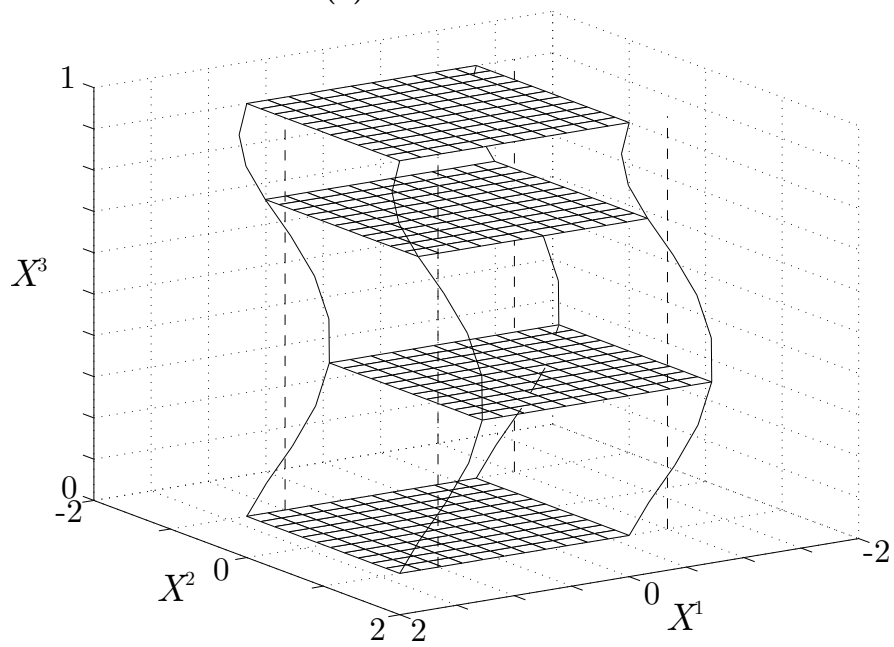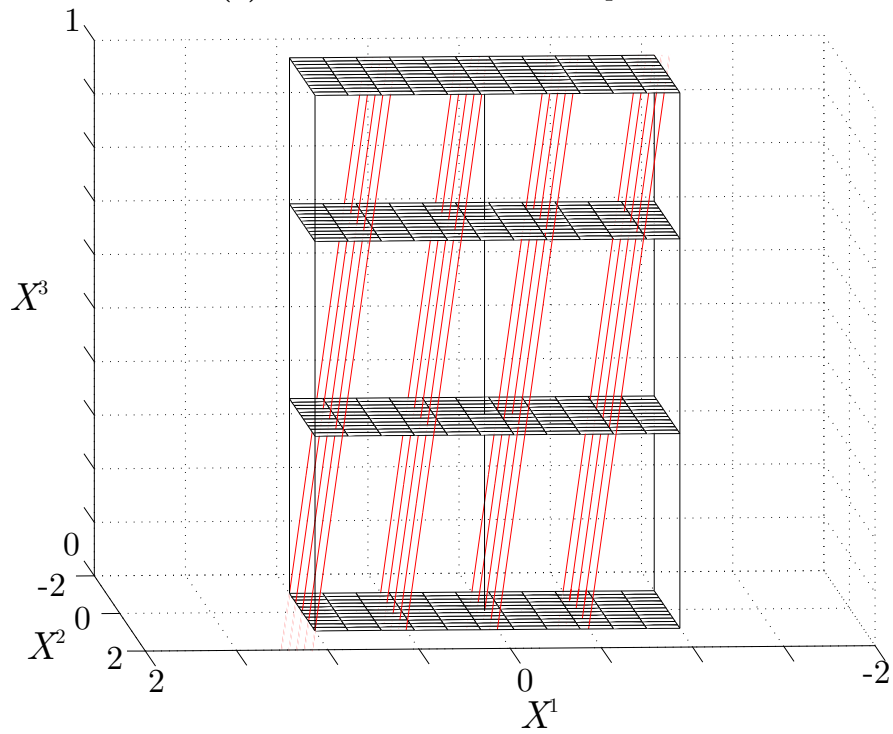

(b) Deformed Mesh

Figure 4.7: Deformation of sample meshes under transverse to axis motion.

(a) Reference mesh in $X^1 X^3$-plane.

(b) Reference mesh at side angle.

Figure 4.8: Fiber bundle in reference configuration.

The equations of motion for arbitrary angle δ are derived from (4.3)-(4.1) and (4.46) as

$$\frac{\partial^2 G^1}{\partial t^2} = \alpha \frac{\partial^2 G^1}{\partial (X^3)^2}$$

$$+\beta \cos^2(\delta) \left[\cos^2 \delta \left(2\frac{\partial G^1}{\partial X^3}\frac{\partial G^2}{\partial X^3}\frac{\partial^2 G^2}{\partial(X^3)^2} + \frac{\partial^2 G^1}{\partial(X^3)^2}\left(\frac{\partial G^2}{\partial X^3}\right)^2 \right. \right.$$

$$\left. +3\left(\frac{\partial G^1}{\partial X^3}\right)^2 \frac{\partial^2 G^1}{\partial(X^3)^2} - 2\frac{\partial^2 G^1}{\partial(X^3)^2} \right) \qquad (4.47a)$$

$$\left. +2\cos\delta\sin\delta\left(3\frac{\partial G^1}{\partial X^3}\frac{\partial^2 G^1}{\partial(X^3)^2} + \frac{\partial G^2}{\partial X^3}\frac{\partial^2 G^2}{\partial(X^3)^2}\right) + 2\frac{\partial^2 G^1}{\partial(X^3)^2} \right],$$

$$\frac{\partial^2 G^2}{\partial t^2} = \alpha \frac{\partial^2 G^2}{\partial (X^3)^2}$$

$$+\beta \cos^2(\delta)\left[\cos^2\delta\left(2\frac{\partial G^1}{\partial X^3}\frac{\partial G^2}{\partial X^3}\frac{\partial^2 G^1}{\partial(X^3)^2} + \left(\frac{\partial G^1}{\partial X^3}\right)^2\frac{\partial^2 G^2}{\partial(X^3)^2} + 3\left(\frac{\partial G^2}{\partial X^3}\right)^2\frac{\partial^2 G^2}{\partial(X^3)^2}\right)\right. \qquad (4.47b)$$

$$\left. +2\cos\delta\sin\delta\left(\frac{\partial^2 G^1}{\partial(X^3)^2}\frac{\partial G^2}{\partial X^3} + \frac{\partial G^1}{\partial X^3}\frac{\partial^2 G^2}{\partial(X^3)^2}\right)\right],$$

$$\frac{\partial p}{\partial X^3} = 2\beta\rho_0\cos^3\delta\left(\cos\delta\left(\frac{\partial G^1}{\partial X^3}\frac{\partial^2 G^1}{\partial(X^3)^2} + \frac{\partial G^2}{\partial X^3}\frac{\partial^2 G^2}{\partial(X^3)^2}\right) + \sin\delta\frac{\partial^2 G^1}{\partial(X^3)^2}\right). \qquad (4.47c)$$

where $\alpha = 2(a+b) > 0$, $\beta = 4q > 0$, and ρ_0 is mass density. The parametrization in terms of α and β is done to provide clarity in the following study, and to keep with the notation used in the one dimensional motion (Section 4.3).

We now comment on the form of the equations (4.47). First, the neo-Hookean and Mooney-Rivlin parameters a and b do not appear independently in (4.47), and so the dynamics are fundamentally neo-Hookean. Additionally, the dynamics due to the fiber family only disappear if the fiber family assumed not present (i.e. $q = 0$), which is contrary to the two dimensional motion in Section 4.4. In the case when $q = 0$ or $\delta = \pi/2$, (4.47) reduce to a system of uncoupled, one dimensional, linear wave equations in G^1 and G^2, and an equation defining $p = p(t)$. The linear wave equation is well-studied, with solutions found in, for example, [56].

Of significance, the equation (4.47c) defines p in terms of G^1 and G^2 (as in the one dimensional case, Section 4.3), and can be put into the divergence conservation law form

$$D_3\left(4q\cos^3\delta(\cos\delta((G_3^1)^2 + (G_3^2)^2) + 2\sin\delta G_3^1) - p\right) = 0. \qquad (4.48)$$

Equation (4.48) can be solved to obtain the family of solutions

$$p = 2\beta\rho_0\cos^3\delta\left(\frac{\cos\delta}{2}\left(\left(\frac{\partial G^1}{\partial y^3}\right)^2 + \left(\frac{\partial G^2}{\partial y^3}\right)^2\right) + \sin\delta\frac{\partial G^1}{\partial y^3}\right) + f(t), \qquad (4.49)$$

where $f(t)$ is a function of time to be fixed by the initial and boundary conditions. As such, through the remainder of the current Chapter, we study equations (4.47a)-(4.47b), noting that any solution G^1 and G^2 can be substituted into (4.49) to obtain p. With that said, we focus our study on equations (4.47), which are highly coupled and nonlinear.

Additionally, the system (4.47) is currently written in Cauchy-Kovalevskaya form with respect to t, and the right hand side of each equation is analytic in its arguments. Thus, solutions G^1 and G^2 to these equations exist and are unique in some neighbourhood about the origin, by the Cauchy-Kovalevskaya theorem [46]. Additionally, there exists a one-to-one correspondence between conservation law multipliers and equivalence classes of conservation laws.

4.5.2 Remarks on One Dimensional Solutions

We now consider the motion being restricted to one dimension. In the case where $G^1 = G^1(X^3, t)$ and $G^2 \equiv 0$, equation (4.47c) becomes identically zero and (4.47a) reduces to the quasilinear wave equation,

$$\frac{\partial^2 G^1}{\partial t^2} = \left(\alpha + \beta\cos^2(\delta)\left(\cos^2(\delta)\left(3\left(\frac{\partial G^1}{\partial X^3}\right)^2 - 2\right) + 6\cos(\delta)\sin(\delta)\frac{\partial G^1}{\partial X^3} + 2\right)\right)\frac{\partial^2 G^1}{\partial(X^3)^2}, \qquad (4.50)$$

which is equivalent to (4.7a) in Section 4.3 up to a relabelling of parameters and variables. As such, any solution G^1 to (4.50) with $G^2 \equiv 0$ is also a solution to the system (4.47).

Now, for the case $G^2 = G^2(X^3, t)$ and $G^1 \equiv 0$, (4.47a) becomes identically zero and (4.47b) reduces to the quasilinear wave equation

$$\frac{\partial^2 G^2}{\partial t^2} = \left(\alpha + \beta\left(\frac{\partial G^2}{\partial X^3}\right)^2\right)\frac{\partial^2 G^2}{\partial(X^3)^2}, \qquad (4.51)$$

which is equivalent to (4.10) up to a relabelling. As such, any solution G^2 to equation (4.51) with $G^1 \equiv 0$ is a solution to the system (4.47).

137

4.5.3 Lie Point Symmetries

We now classify the Lie point symmetries of equations (4.47a) and (4.47b) for $\alpha > 0$, $\beta > 0$, and $\delta \in [0, \pi/2)$. Symmetries for equations (4.47a) and (4.47b) in the case when $\delta = \pi/2$ are equivalent to those of the linear wave equation, which are studied in [66].

Theorem 4.5.1. *The Lie point symmetry classification of the system* (4.47a) *and*(4.47b) *for material parameters* $\alpha > 0$, $\beta > 0$, *and* $\delta \in [0, \pi/2)$ *is given in Table* (4.3).

Proof. Proof of Theorem 4.5.1 follows from the methodology presented in Chapter 1. $\qquad\square$

Table 4.3: Lie point symmetry classification of (4.47a) and (4.47b).

Parameters	Symmetries
arbitrary	$Z^1 = \frac{\partial}{\partial t}$, $Z^2 = \frac{\partial}{\partial X^3}$, $Z^3 = \frac{\partial}{\partial G^1}$, $Z^4 = \frac{\partial}{\partial G^2}$, $Z^5 = t\frac{\partial}{\partial G^1}$, $Z^6 = t\frac{\partial}{\partial G^2}$, $Z^7 =$ $-\cos(\delta)G^2\frac{\partial}{\partial G^1} + (\sin(\delta)X^3 + \cos(\delta)G^1)\frac{\partial}{\partial G^2}$, $Z^8 = X^3\frac{\partial}{\partial X^3} + t\frac{\partial}{\partial t} + G^1\frac{\partial}{\partial G^1} + G^2\frac{\partial}{\partial G^2}$
$4\alpha \leq \beta$ and $\cos^2(\delta) = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4\alpha}{\beta}}\right)$	Z^1 , Z^2 , Z^3 , Z^4 , Z^5 , Z^6 , Z^7 , Z^8 , $Z^9 = \cos(\delta)X^3\frac{\partial}{\partial X^3} + 2\cos(\delta)t\frac{\partial}{\partial t} - \sin(\delta)X^3\frac{\partial}{\partial G^1}$

We now derive the global groups of point transformations equivalent to the Lie point symmetries in Theorem 4.5.1. As before, the parameter ϵ^i corresponds to Z^i for $i = 1, \ldots, 9$.

For the material parameters arbitrary, the global one-parameter groups of point transfor-

mation equivalent to Lie point symmetries $\{Z^i\}_{i=1}^{6}$ and Z^8 are:

$$
\begin{array}{lllll}
Z^1: & t^* = t + \epsilon^1, & (X^3)^* = X^3, & (G^1)^* = G^1, & (G^2)^* = G^2; \\
Z^2: & t^* = t, & (X^3)^* = X^3 + \epsilon^2, & (G^1)^* = G^1, & (G^2)^* = G^2; \\
Z^3: & t^* = t, & (X^3)^* = X^3, & (G^1)^* = G^1 + \epsilon^3, & (G^2)^* = G^2; \\
Z^4: & t^* = t, & (X^3)^* = X^3, & (G^1)^* = G^1, & (G^2)^* = G^2 + \epsilon^4; \\
Z^5: & t^* = t, & (X^3)^* = X^3, & (G^1)^* = G^1 + \epsilon^5 t, & (G^2)^* = G^2; \\
Z^6: & t^* = t, & (X^3)^* = X^3, & (G^1)^* = G^1, & (G^2)^* = G^2 + \epsilon^6 t; \\
Z^8: & t^* = e^{\epsilon^8} t, & (X^3)^* = e^{\epsilon^8} X^3, & (G^1)^* = e^{\epsilon^8} G^1, & (G^2)^* = e^{\epsilon^8} G^2.
\end{array}
$$

From the transformations, we can see that $\{Z^i\}_{i=1}^{4}$ correspond to invariance under translations, Z^5 and Z^6 to time-dependent translations of the dependent variables, and Z^8 to scaling of all variables. Invariance of (4.47a) and (4.47b) under the translations Z^1 and Z^2 are expected since equations (4.47a) and (4.47b) are autonomous.

For $\delta \neq \pi/2$, the global group of point transformations equivalent to Z^7 has the form

$$
t^* = t, \quad (X^3)^* = X^3,
$$

$$
(G^1)^* = \cos(\cos(\delta)\epsilon^8)G^1 - \sin(\cos(\delta)\epsilon^8)G^2 + \tan(\delta)(\cos(\cos(\delta)\epsilon^8) - 1)X^3,
$$

$$
(G^2)^* = \cos(\cos(\delta)\epsilon^8)G^2 + \sin(\cos(\delta)\epsilon^8)G^1 + \tan(\delta)\sin(\cos(\delta)\epsilon^8)X^3.
$$

Here, we can see that Z^7 corresponds to a rotation in the variables G^1 and G^2, with an X^3-dependent translation of G^1 and G^2. For $\delta = 0$, we obtain the standard rotation symmetry in G^1 and G^2. Setting $\delta = \pi/2$, we obtain the global group of point transformations for Z^7 as

$$
t^* = t, \quad (X^3)^* = X^3, \quad (G^1)^* = G^1 \quad (G^2)^* = G^2 + \epsilon^7 X^3,
$$

which is an X^3 dependent translation of G^2.

For the restriction of material parameters such that $4\alpha \leq \beta$ and δ is fixed to a solution of

$$
\cos^2(\delta) = \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{4\alpha}{\beta}}\right), \tag{4.52}
$$

then, we have the additional Lie symmetry Z^9 with equivalent transformation

$$t^* = e^{2\cos(\delta)\epsilon^9}\, t, \quad (X^3)^* = e^{\cos(\delta)\epsilon^9}\, X^3,$$

$$(G^1)^* = G^1 + \tan(\delta)\left(1 - e^{\cos(\delta)\epsilon^9}\right) X^3, \quad (G^2)^* = G^2.$$

Thus, Z^9 corresponds to a special scaling in the independent variables and a special X^3 translation in the dependent variable G.

Note that there are two solutions to (4.52) about $\pi/4$ for $4\alpha < \beta$, and $\delta = \pi/4$ for $4\alpha = \beta$. Thus, the resulting symmetry Z^9 is plausible since multiple angles exist for which (4.52) is true.

Remark 4.5.2. For $b = 0$ and $(I^4 - 1)$ sufficiently small, we may adapt the parameters for the (4.4) strain energy density discussed in Section 4.2 for our model as $\alpha = 0.3$ and $\beta = 2.2480$ (omitting units kPa) [59]. Clearly, $4\alpha = 1.2 \leq 0.2480 = \beta$, so Z^9 may arise as a symmetry for this model.

4.5.4 Conservation Laws

We now classify the conservation law multiplers of equations (4.47a) and (4.47b). As written, equations (4.47a) and (4.47b) are in Cauchy-Kovalevskaya form with respect to t. Thus, all conservation laws can be found from non-trivial multipliers, and there is a one-to-one correspondence between these multipliers and divergence conservation law form (up to an equivalence class of conservation laws). It is clear that conservation laws of (4.47a) and (4.47b) also hold on solutions to the system (4.47) and (4.47b)

As in Chapter 3, it is understood that Λ^1 multiplies (4.47a), and Λ^2 multiplies (4.47b).

Conservation Law Multipliers

The zeroth and first order conservation law multipliers of equations (4.47a) and (4.47b) are as classified as follows.

Theorem 4.5.3. *The zeroth order conservation law multipliers of the system* (4.47a) *and*

(4.47b) *are*

$$\Lambda^1 = -C^1 \cos \delta G^2 + C^2 t + C^3, \tag{4.53a}$$

$$\Lambda^2 = C^1(\sin \delta X^3 + \cos \delta G^1) + C^4 t + C^5. \tag{4.53b}$$

Proof. The above zeroth order conservation law multipliers are found through the constructive method outlined in Chapter 1 with multipliers of the form

$$\Lambda^1 = \Lambda^1(X^3, t, G^1, G^2),$$

$$\Lambda^2 = \Lambda^2(X^3, t, G^1, G^2).$$

□

Theorem 4.5.4. *The first order conservation law multipliers of the system* (4.47a) *and* (4.47b) *are*

$$\Lambda^1 = C^6 G_t^1 + C^7 G_3^1, \tag{4.54a}$$

$$\Lambda^2 = C^6 G_t^2 + C^7 G_3^2. \tag{4.54b}$$

Proof. The above first order conservation law multipliers are found through the constructive method outlined in Chapter 1 with multipliers of the form

$$\Lambda^1 = \Lambda^1(X^3, t, G^1, G^2, G_3^1, G_t^1, G_3^2, G_t^2),$$

$$\Lambda^2 = \Lambda^2(X^3, t, G^1, G^2, G_3^1, G_t^1, G_3^2, G_t^2).$$

□

Of importance, no special conservation laws for particular values of the material parameters.

Divergence Conservation Law Form

We now present the divergence conservation law forms of (4.47a) and (4.47b) for the multipliers in Theorems 4.5.3 and 4.5.4.

Conservation of Energy

The model admits conservation of energy (multiplier set with coefficient C^6). The divergence conservation law form is

$$
\mathrm{D}_t \left(\frac{1}{2}(G_t^1)^2 + \frac{1}{2}(G_t^2)^2 + \frac{\alpha}{2}\left((G_3^1)^2 + (G_3^2)^2\right) + \beta \cos^2 \delta \left((G_3^1)^2 \right.\right.
$$
$$
\left.\left. + \cos \delta \sin \delta ((G_3^1)^3 + G_3^1(G_3^2)^2)) + \frac{\beta}{4}\cos^2 \delta \left((G_3^1)^4 + (G_3^2)^4 + 2(G_3^1)^2(G_3^2)^2 - 4(G_3^1)^2\right) \right)
$$
$$
- \mathrm{D}_{X^3}\left(\alpha(G_t^1 G_3^1 + G_t^2 G_3^2) + 2\beta \cos^2 \delta G_t^1 G_3^1 \right. \tag{4.55}
$$
$$
+ \beta \cos^3 \delta \sin \delta \left(3(G_3^1)^3 + G_t^1(G_3^2)^2 + 2G_3^1 G_t^1 G_3^2\right)
$$
$$
+ \beta \cos^3 \delta \left(G_3^2 G_t^2(G_3^1)^2 + G_3^1 G_t^1(G_3^2)^2 + G_t^1(G_3^1)^3 + G_t^2(G_3^2)^3 - 2G_t^1 G_3^1\right)\right) = 0.
$$

Here, kinetic energy K is of the form

$$
K = \frac{\rho_0}{2}(G_t^1)^2 + \frac{\rho_0}{2}(G_t^2)^2,
$$

and potential energy P is of the form

$$
P = \rho_0 \left(\frac{\alpha}{2}\left((G_3^1)^2 + (G_3^2)^2\right) + \beta \cos^2 \delta \left((G_3^1)^2 \right.\right.
$$
$$
\left.\left. + \cos \delta \sin \delta ((G_3^1)^3 + G_3^1(G_3^2)^2)) + \frac{\beta}{4}\cos^2 \delta \left((G_3^1)^4 + (G_3^2)^4 + 2(G_3^1)^2(G_3^2)^2 - 4(G_3^1)^2\right) \right).
$$

Conservation of Momentum in Eulerian Frame

The model admits conservation of linear momentum in x^1 and x^2 (multiplier set of C^3 and C^5, respectively). Additionally, the system admits a conservation law related angular momentum (multiplier set of C^1). The divergence conservation law forms are as follows.

From the multiplier set with coefficient C^3, we obtain the conservation of linear momentum in x^1.

$$
\mathrm{D}_t\left(G_t^1\right) - \mathrm{D}_{X^3}\left(\alpha G_3^1 + 8q \cos^2 \delta\, G_3^1 \right.
$$
$$
+ \beta \cos^3 \delta \sin \delta \left(3(G_3^1)^2 + (G_3^2)^2\right) + \beta \cos^4 \delta \left((G_3^1)^3 - 2G_3^1 + G_3^1(G_3^2)^2\right)\right) = 0. \tag{4.56}
$$

From the multiplier set with coefficient C^5, we obtain the conservation of linear momentum of x^2.

$$
\mathrm{D}_t\left(G_t^2\right) - \mathrm{D}_{X^3}\left(G_3^2\left(\alpha + \beta \cos^3 \delta \left(\cos \delta \left((G_3^1)^2 + (G_3^2)^2\right) + 2\sin \delta G_3^1\right)\right)\right) = 0. \tag{4.57}
$$

From the multiplier set with coefficient C^1, we obtain a conservation related to the conservation of angular momentum.

$$\mathrm{D}_t \left(\cos\delta \left(G^1 G_t^2 - G^2 G_t^1 \right) + \sin\delta \left(X^3 G_t^2 + \alpha t G_3^2 \right) \right)$$
$$+ \mathrm{D}_{X^3} \left(\alpha \cos\delta \left(G^2 G_3^1 - G^1 G_3^2 \right) - \alpha \sin\delta \left(t G_t^2 + X^3 G_3^2 \right) \right.$$
$$+ \beta \cos^3\delta \left(2 G^2 G_3^1 - X^3 2 G_3^1 G_3^2 \right. \tag{4.58}$$
$$+ \cos\delta \sin\delta \left[-X^3 G_3^2 (G_3^1)^2 + G^2 (G_3^2)^2 - 2 G^1 G_3^1 G_3^2 + 3 G^2 (G_3^1)^2 - X^3 (G_3^2)^3 \right]$$
$$+ \cos^2\delta \left[2 X^3 G_3^1 G_3^2 + G^2 (G_3^1)^3 - G^1 G_3^2 (G_3^1)^2 - G^1 (G_3^2)^3 + G^2 G_3^1 (G_3^2)^2 - 2 G^2 G_3^1 \right] \right) = 0.$$

For $\delta = 0$, this divergence expression is conservation of angular momentum.

Center of Mass Theorem

Recall the center of mass theorem in the continuum is equivalent to three conservation laws with conserved densities (4.18) [30, 31],

$$\rho_0 (t x_t^i - x^i), \quad i = 1, 2, 3.$$

As such, the model admits the continuum analog of the center of mass theorem in x^1 and x^2.

From the multiplier set with coefficient C^2, we obtain the first conservation law of the center of mass theorem.

$$\mathrm{D}_t \left(t G_t^1 - G^1 \right) - \mathrm{D}_{X^3} \left(\alpha t G_3^1 + 2\beta \cos^2\delta \, t G_3^1 \right.$$
$$+ \beta \cos^3\delta \sin\delta \, t \left(3 (G_3^1)^3 + (G_3^2)^2 \right) + \beta \cos^4\delta \, t \left((G_3^1)^3 - 2 G_3^1 + G_3^1 (G_3^2)^2 \right) \right) = 0. \tag{4.59}$$

From the multiplier set with coefficient C^4, we obtain the second conservation law of the center of mass theorem.

$$\mathrm{D}_t \left(t G_t^2 - G^2 \right)$$
$$- \mathrm{D}_{X^3} \left(t G_3^2 \left[\alpha + \beta \cos^3\delta \left(\cos\delta ((G_3^1)^2 + (G_3^2)^2) \right) + 2 \sin\delta G_3^1 \right] \right) = 0. \tag{4.60}$$

Note that the third conservation law in the center of mass theorem holds identically for the system (4.47a) and (4.47b) since $x^3 = X^3$, which implies

$$\mathrm{D}_t \left(\rho_0 (t x_t^3 - x^3) \right) \equiv 0.$$

Conservation of Linear Momentum in Lagrangian Frame

For the multiplier set with coefficient C^7, we obtain the conservation of material linear momentum in the X^3 as

$$
\begin{aligned}
&\mathrm{D}_t \left(G_3^1 G_t^1 + G_3^2 G_t^2\right) \\
&+\mathrm{D}_{X^3} \left(-\frac{1}{2}(G_t^1)^2 - \frac{1}{2}(G_t^2)^2 - \frac{\alpha}{2}((G_3^1)^2 + (G_3^2)^2)\right. \\
&\quad -\beta \cos^2 \delta \left((G_3^1)^2 + 2\cos\delta\sin\delta \left[G_3^1(G_3^2)^2 + (G_3^1)^3\right]\right) \\
&\quad \left. -\frac{\beta}{4}\cos^4\delta \left(4(G_3^1)^2 - 3(G_3^2)^4 - 6(G_3^1)^2(G_3^2)^2 - 3(G_3^1)^4\right)\right) = 0.
\end{aligned}
\tag{4.61}
$$

4.6 Two Fiber Family Model

We briefly present a fiber reinforced incompressible Mooney-Rivlin model with two planar fiber families. In particular, the solid has a displacement $G(X^1, X^2, t)$ along the X^3-direction, which has the same coordinate dependence as the one fiber model in Section 4.4. As such, we recall the essential relations from Section 4.4 with few details to avoid repetition. We also discuss the similarities and difference of this model with the one fiber case.

We consider the coordinate dependence for a displacement $G(X^1, X^2, t)$ transverse to the $X^1 X^2$ plane,

$$
\vec{X} = \begin{bmatrix} X^1 \\ X^2 \\ G\left(X^1, X^2, t\right) + X^3 \end{bmatrix}.
\tag{4.62}
$$

A sample deformation of this type can be found in Figure 4.5 of Section 4.4.

By observation of (4.26), the incompressibility condition is identically satisfied, and we obtain hydrostatic pressure $p(X^1, X^2, t)$ as a Lagrange multiplier.

We consider two fiber families oriented along the $X^1 X^2$ plane with orientation vectors

$$
A^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A^2 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{bmatrix},
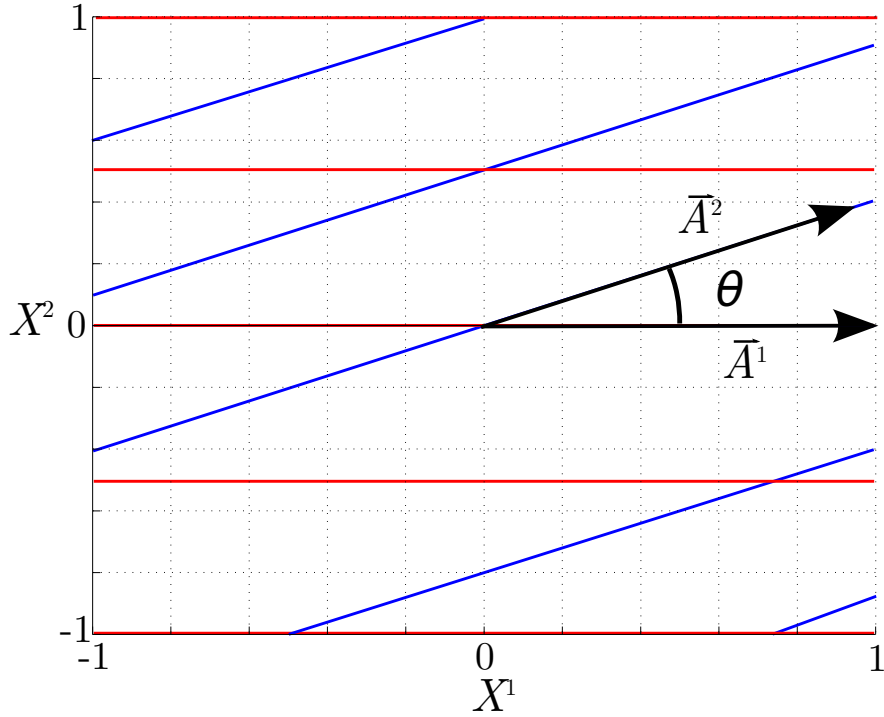\tag{4.63}
$$

Figure 4.9: Two planar fibers in Lagrangian configuration separated by an angle θ.

where $\theta \in [0, \pi/2]$ is a constant angle. Note we do not consider $\theta < 0$ or $\theta > \pi/2$ due to equivalence in dynamics of a two fiber system up to rotation about the X^3-axis. For example, if $\theta = 3\pi/4$, we could rotate the system by $3\pi/4$ to obtain an equivalent model as if we originally considered $\theta = \pi/4$.

For the orientation vectors in (4.63), the equations of motion for an incompressible Mooney Rivlin solid with two planar fiber families are derived from (4.3)-(4.1) and (4.62)-

(4.63) as

$$\frac{\partial^2 G}{\partial t^2} = 2(a+b)\left(\frac{\partial^2 G}{\partial (X^1)^2} + \frac{\partial^2 G}{\partial (X^2)^2}\right) + 12q\left(\frac{\partial G}{\partial X^1}\right)^2 \frac{\partial^2 G}{\partial (X^1)^2}$$

$$+12s\left[\left(\cos^4(\theta)\left(\frac{\partial^2 G}{\partial (X^1)^2} - \frac{\partial^2 G}{\partial (X^2)^2}\right) + 2\cos^3(\theta)\sin(\theta)\frac{\partial^2 G}{\partial X^1 \partial X^2}\right.\right.$$

$$\left. + \cos^2(\theta)\frac{\partial^2 G}{\partial (X^2)^2}\right)\left(\frac{\partial G}{\partial X^1}\right)^2$$

$$+ \left(-4\cos^4(\theta)\frac{\partial^2 G}{\partial X^1 \partial X^2} + 2\cos^3(\theta)\sin(\theta)\left(\frac{\partial^2 G}{\partial (X^1)^2} + \frac{\partial^2 G}{\partial (X^2)^2}\right)\right.$$

$$\left. + 4\cos^2(\theta)\frac{\partial^2 G}{\partial X^1 \partial X^2} + 2\cos(\theta)\sin(\theta)\frac{\partial^2 G}{\partial (X^2)^2}\right)\frac{\partial G}{\partial X^1}\frac{\partial G}{\partial X^2}$$

$$+ \left(\cos^4(\theta)\left(\frac{\partial^2 G}{\partial (X^2)^2} - \frac{\partial^2 G}{\partial (X^1)^2}\right) - 2\cos^3(\theta)\sin(\theta)\frac{\partial^2 G}{\partial X^1 \partial X^2}\right.$$

$$\left. + \cos^2(\theta)\left(\frac{\partial^2 G}{\partial (X^1)^2} - 2\frac{\partial^2 G}{\partial (X^2)^2}\right)\right.$$

$$\left.\left. + 2\cos(\theta)\sin(\theta)\frac{\partial^2 G}{\partial X^1 \partial X^2} + \frac{\partial^2 G}{\partial (X^2)^2}\right)\left(\frac{\partial G}{\partial X^2}\right)^2\right], \tag{4.64a}$$

$$0 = \frac{\partial p}{\partial X^1} + 2b\left(\frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial (X^2)^2} - \frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial X^1 \partial X^2}\right) - 8q\frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial (X^1)^2}$$

$$+8s\left[\cos^4\theta\left(-\frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial (X^1)^2} + 2\frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial X^1 \partial X^2} + \frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial (X^2)^2}\right)\right.$$

$$+ \cos^3\theta\sin\theta\left(-\frac{\partial^2 G}{\partial (X^1)^2}\frac{\partial G}{\partial X^2} + \frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial (X^2)^2} - 2\frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial X^1 \partial X^2}\right)$$

$$\left. - \cos^2\gamma\left(2\frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial X^1 \partial X^2} + \frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial (X^2)^2}\right) - \cos\gamma\sin\gamma\frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial (X^2)^2}\right], \tag{4.64b}$$

$$0 = \frac{\partial p}{\partial X^2} + 2b\left(\frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial (X^1)^2} - \frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial X^1 \partial X^2}\right)$$

$$+8s\left[\cos^4\theta\left(\frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial (X^1)^2} - \frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial (X^2)^2} + 2\frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial X^1 \partial X^2}\right)\right.$$

$$+ \cos^3\theta\sin\theta\left(-\frac{\partial^2 G}{\partial (X^1)^2}\frac{\partial G}{\partial X^1} + 2\frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial X^1 \partial X^2} + \frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial (X^2)^2}\right)$$

$$- \cos^2\gamma\left(-\frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial (X^1)^2} + 2\frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial (X^2)^2} - 2\frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial X^1 \partial X^2}\right)$$

$$\left. - \cos\gamma\sin\gamma\left(2\frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial X^1 \partial X^2} + \frac{\partial G}{\partial X^1}\frac{\partial^2 G}{\partial (X^2)^2}\right) - \frac{\partial G}{\partial X^2}\frac{\partial^2 G}{\partial (X^2)^2}\right]. \tag{4.64c}$$

The nonlinear terms arising from each fiber family in this PDE system are determined by observing the parameters q and s. Of importance, the two fiber families contribute separate terms, which is expected since fiber effects are included in the model through addition of individual anisotropic terms in the strain energy density W for each fiber family. In particular, the terms in the PDE system introduced by the fiber family oriented along the X^1-axis (parameter q) are quasilinear, whereas the second fiber family (parameter s) gives rise to many complicated nonlinear terms.

We now compare the current system with the one fiber model (4.28) for $\gamma = 0$, which we refer to as the one planar fiber family system.

First, the PDE system (4.64) is equivalent to the one planar fiber family system for $s = 0$, as well as for $\theta = 0$ up to parametrizing $q + s$ as a single constant. As such, the one planar fiber model may be viewed as a special case of the two planar fiber system.

Additionally, the admitted Lie symmetries of the two fiber family system are expected to be fewer in number than those for the one planar fiber system. This seems reasonable since adding more complicated nonlinear terms to a PDE results in additional symmetry determining equations, which in turn may restrict the solution space of the determining equations and thus the number of admitted Lie symmetries. Regarding conservation laws, the addition of a second fiber family to the one planar fiber family model will reduce the number of conservation laws since the two fiber model is nonlinear in the X^2-direction for $\gamma \neq 0$.

4.7 Discussion

In the current Chapter, analytical properties of three types of motion of an incompressible, isotropic, hyperelastic material interspersed with anisotropic fiber families were studied.

In Section 4.2, the necessary relationships are recalled from Chapter 2 to construct the equations of motion for a general fiber reinforced Mooney-Rivlin solid.

Section 4.3 is dedicated to the study of the one dimensional, one fiber reinforced Mooney-Rivlin model. A numerical simulation of the one dimensional equations of motion (4.10) and (4.9) is studied for the fiber oriented along the x-axis. The resulting numerical solution

showed a splitting of the Gaussian initial condition (taken with zero initial velocity) into two profiles traveling apart, suggesting the existence of some traveling wave solutions. The Lie symmetries of the PDE system are classified with respect to the material parameters. Of importance, for a special angle/s of the fiber with respect to the plane, an additional symmetry is admitted by the system that consists of a scaling of the independent variables and a translation of the displacement dependent on the Lagrangian variable x. Afterwards, invariant solutions of the system are derived for the travelling wave invariants, and for the special symmetry. Then, conservation laws of the governing PDE system are classified. Lastly, the Hodograph transformation was applied to the system, and it was determined the resulting linear non-constant coefficient PDE system cannot be mapped to one with constant coefficients by an invertible mapping.

In Section 4.4, the motion of a Mooney Rivlin material reinforced by a single fiber was studied for displacements transverse to a reference plane in the solid. It was determined that solutions to the one dimensional model in the previous Section are also solutions to the current system. Then, Lie symmetries of the equations of motion were classified, as well as the conservation laws.

Our attention was focused in Section 4.4 on another type of motion in Mooney-Rivlin fiber reinforced materials, for which there are two displacements orthogonal to each other and the X^3-axis. As in the previous model, it was determined that solutions to the one dimensional model in Section 4.3 are also solutions to the current PDE system. After, Lie symmetries of the equations of motion were classified, for which a special symmetry is admitted similar additional symmetry of the one dimensional model for the same angle between fiber of axis and direction of motion. Finally, conservation laws of the model were classified and discussed.

In the final Section, 4.6, a model of a Mooney-Rivlin material reinforced by two fiber families which as the same coordinate dependence as Section 4.4 was derived. We do not study this model in the current Thesis, but discuss it in the Concluding Remarks with respect to future study.

Concluding Remarks

We now provide a brief summary of the current Thesis and discuss future directions of research.

Summary

In Chapter 1, we outlined the fundamental notions to Lie symmetry and conservation law analysis in the context of studying differential equations. After providing a review of the fundamental definitions and theorems for Lie symmetry analysis, we presented the application of Lie symmetries to obtain the general solution of ordinary differential equations and invariant solutions of partial differential equations. We then discussed the direct method to obtain conservation laws for a given PDE system, particularly how to obtain conservation law multipliers by a constructive algorithm. Finally, we presented Noether's theorem and discussed how it connects Lie symmetries and the conservation laws for a variational system of PDEs. Within this discussion, we also outlined a method to determine if a given PDE system is variational. References to further work with Lie symmetry and conservation law analysis are provided in the chapter discussion.

The second chapter is dedicated to introducing the equations of motion governing hyperelasticity, which includes a brief review of the theory of elastodynamics. First, notions essential to describing the motion of a body through continuum mechanics are presented. After, the equations of motion for an elastic solid are given in terms of the Cauchy and First Piola-Kirchhoff stress' in the Eulerian and Lagrangian frames of reference, respectively. Afterwards, the assumptions of elasticity are outlined with respect to the First Piola-Kirchhoff stress. Lastly, the constitutive relations for various models of hyperelasticity are discussed, and the extension to fiber reinforced materials is presented.

A detailed example of the Lie symmetry and conservation law analysis methodology is

provided in Chapter 3. In particular, a two dimensional incompressible Mooney-Rivlin model is studied, for which the Lie symmetries and conservation laws are classified. Reductions of the PDE system governing the motion are performed for particular Lie symmetries. The model equations are determined to be a non-variational system. Admitted Lie symmetries and conservation laws of the model are compared, with similarities and differences noted therein.

Chapter 4 is dedicated to the study of analytical properties for one and two dimensional models of a fiber reinforced incompressible Mooney-Rivlin solid. In particular, we focused on three types of motion, including deformations transverse to a plane depending on one and two dimension(s), and deformations transverse to an axis. In the one dimensional transverse motion, we numerically simulated the system of PDEs to develop intuition as to the solution behaviour. Lie symmetries and conservation laws of the system are classified, and invariant solutions were studied for the traveling wave *ansatz* and a special symmetry. As well, we applied the Hodograph transformation to map the primary equation of motion to a linear non-constant coefficient PDE. We then verified this linear non-constant coefficient PDE could not be mapped to one with constant coefficients. For each of the two dimensional motions in the current model, we studied the one dimensional solutions admitted by the corresponding PDE system, and classified the Lie point symmetries and conservation laws of this system.

Future Directions

There are a number of potential directions for future research based on material within the current Thesis, of which we discuss a handful of possibilities here.

First, one may focus on a deeper Lie symmetry and conservation law analysis of the systems of partial differential equations studied in Chapters 3 and 4. This includes making further reductions to the PDE systems for admitted Lie symmetries, as well as solving the reduced systems of equations to obtain invariant solutions. One may also seek additional symmetries in each model, such as higher order, approximate, and nonlocal symmetries. In addition, further conservation laws for each PDE system can be sought, particularly higher order and nonlocal conservation laws.

One may also analyze other models of fiber reinforcement than those studied in Chapters 3 and 4 using the constitutive relations discussed in Chapter 2. Furthermore, one may study models for reinforcement by multiple fiber families, such as the model presented in Section 4.6. In particular, hyperelastic models with two fibers are of interest due to applications in modelling arterial tissue (see, for example, [10, 59, 60]).

Additionally, numerical simulations for each model presented in Chapters 3 and 4 can be performed. Of interest, these solutions could be compared to existing finite element simulations of fiber reinforced hyperelastic materials. Another avenue to explore would be the application of symmetry and conservation law based numerical schemes using the results obtained in the current Thesis.

References

[1] A.R Adem and C.M. Khalique. New exact solutions and conservation laws of a coupled Kadomtsev-Petviashvili system. *Computational Fluids*, 81:10–16, 2013.

[2] I.S. Akhatov, R.K. Gazizov, and N.K. Ibragimov. Nonlocal symmetries. Heauristic approach. *Journal of Soviet Mathematics*, 55(1):1401–1450, 1991.

[3] S.C. Anco and G. Bluman. Direct construction of conservation laws from field equations. *Physical Review Letters*, 78(15):2869, 1997.

[4] S.C. Anco and G. Bluman. Direct construction method for conservation laws of partial differential equations part I: Examples of conservation law classifications. *European Journal of Applied Mathematics*, 13(05):545–566, 2002.

[5] S.C. Anco and G. Bluman. Direct construction method for conservation laws of partial differential equations Part II: General treatment. *European Journal of Applied Mathematics*, 13(05):567–585, 2002.

[6] V. Baikov, R. Gazizov, and N. Ibragimov. Perturbations methods in group analysis. *Journal of Soviet Mathematics*, 55(1):1450–1490, 1989.

[7] J.M. Ball. Some open problems in elasticity. In *Geometry, Mechanics, and Dynamics*, pages 3–59. Springer, 2002.

[8] D. Balzani, S. Brinkhues, and G.A. Holzapfel. Constitutive framework for the modeling of damage in collagenous soft tissues with application to arterial walls. *Computer Methods in Applied Mechanics and Engineering*, 213:139–151, 2012.

[9] Y. Basar and M. Itskov. Finite element formulation of the Ogden material model with application to rubber-like shells. *International Journal for Numerical Methods in Engineering*, 42(7):1279–1305, 1998.

[10] C.A. Basciano and C. Kleinstreuer. Invariants-based anisotropic constitutive models of healthy aneurysmal abdominal aortic wall. *Journal of Biomechanical Engineering*, 131:021009, 2009.

[11] S.M. Belkoff and R.C. Haut. A structural model used to evaluate the changing microstructure of maturing rat skin. *Journal of Biomechanics*, 24(8):711–720, 1991.

[12] E. Bessel-Hagen. Über die erhaltungssätze der elektrodynamik. *Mathematische Annalen*, 84(3-4):258–276, 1921.

[13] A. Bihlo, E. Dos Santos Cardoso-Bihlo, and R.O. Popovych. Complete group classification of a class of nonlinear wave equations. *Journal of Mathematical Physics*, 53:123515, 2012.

[14] S. Bludman and D.C. Kennedy. Invariant relationships deriving from classical scaling transformations. *Journal of Mathematical Physics*, 52(4):042902, 2011.

[15] G. Bluman. Use and construction of potential symmetries. *Mathematical and Computer Modelling*, 18(10):1–14, 1993.

[16] G. Bluman and A.F. Cheviakov. Nonlocally related systems, linearization and nonlocal symmetries for the nonlinear wave equation. *Journal of Mathematical Analysis and Applications*, 333:93–111, 2007.

[17] G. Bluman, A.F. Cheviakov, and N.M. Ivanova. Framework for nonlocally related partial differential equation systems and nonlocal symmetries: extension, simplification, and examples. *Journal of Mathematical Physics*, 47:113505, 2006.

[18] G. Bluman, A.F. Cheviakov, and N.M. Ivanova. Framework for nonlocally related partial differential equation systems and nonlocal symmetries: Extension, simplification, and examples. *Journal of Mathematical Physics*, 47(11):113505, 2006.

[19] G.W Bluman. *Construction of solutions to partial differential equations by the use of transformation groups.* PhD thesis, California Institute of Technology, 1968.

[20] G.W. Bluman. On mapping linear partial differential equations to constant coefficient equations. *SIAM Journal on Applied Mathematics*, 43(6):1259–1273, 1983.

[21] G.W. Bluman and S.C. Anco. *Symmetry and integration methods for differential equations*, volume 154. Springer, 2002.

[22] G.W. Bluman, A.F. Cheviakov, and S.C. Anco. *Applications of Symmetry Methods to Partial Differential Equations*, volume 168. Springer, 2010.

[23] G.W. Bluman and S. Kumei. *Symmetry and differential equations*, volume 81. Springer, 1989.

[24] G.W. Bluman and Temuerchaolu. Comparing symmetries and conservation laws of nonlinear telegraph equations. *Journal of Mathematical Physics*, 46(7):073513, 2005.

[25] G.W. Bluman, Temuerchaolu, and R. Sahadevan. Local and nonlocal symmetries for nonlinear telegraph equation. *Journal of Mathematical Physics*, 46:023505, 2005.

[26] A.F. Bower. *Applied mechanics of solids.* CRC Press, 2010.

[27] T.H. Boyer. Continuous symmetries and conserved currents. *Annals of Physics*, 42(3):445–466, 1967.

[28] M. Braun. The balance of material momentum at a shock wave propagating in a thermo-elastic material. *PAMM*, 7(1):1090109–1090110, 2007.

[29] G.I. Burde. Potential symmetries of the nonlinear wave equation $u_{tt} = (uu_x)_x$ and related exact and approximate solutions. *Journal of Physics A: Mathematical and General*, 34:5355–5371, 2001.

[30] G. Caviglia and A. Morro. Noether-type conservation laws for perfect fluid motions. *Journal of Mathematical Physics*, 28(5):1056–1060, 1987.

[31] G. Caviglia and A. Morro. Conservation laws for incompressible fluids. *International Journal of Mathematics and Mathematical Sciences*, 12(2):377–384, 1989.

[32] P. Chadwick. *Continuum mechanics: concise theory and problems.* Courier Dover Publications, 2012.

[33] C.S. Chang, H. Askes, and L.J. Sluys. Higher-order strain/higher-order stress gradient models derived from a discrete microstructure, with application to fracture. *Engineering Fracture Mechanics*, 69(17):1907–1924, 2002.

[34] A.F. Cheviakov. GeM software package for computation of symmetries and conservation laws of differential equations. *Computer Physics Communications*, 176(1):48–61, 2007.

[35] A.F. Cheviakov and J.-F Ganghoffer. Symmetry properties of two-dimensional Ciarlet-Mooney-Rivlin constitutive models in nonlinear elastodynamics. *Journal of Mathematical Analysis and Applications*, 396:625–639, 2012.

[36] P.G. Ciarlet. *Mathematical elasticity volume 1: three-dimensional elasticity*, volume 20. North-Holland Studies in Math and its Applications, 1988.

[37] P.A. Clarkson. Nonclassical symmetry reductions of the Boussinesq equation. *Chaos, Solitons & Fractals*, 5(12):2261–2301, 1995.

[38] H.-H. Dai. Nonlinear dispersive waves in a circular rod composed of a Mooney-Rivlin material. *London Mathematical Society Lecture Note Series*, pages 392–432, 2001.

[39] H. Demirkoparan, T.J. Pence, and A. Wineman. Torsion of a fiber reinforced hyperelastic cylinder for which the fibers can undergo dissolution and reassembly. 48(11):1179–1201, 2010.

[40] M. Destrade, B. Mac Donald, J.G. Murphy, and G. Saccomandi. At least three invariants are necessary to model the mechanical response of incompressible, transversely isotropic materials. *Computational Mechanics*, 52(4):959–969, 2013.

[41] M. Destrade and G. Saccomandi. Finite amplitude elastic waves propogating in compressible solids. *Physical Review E*, 72(6):016620, 2005.

[42] M. Destrade and G. Saccomandi. Transverse wave in nonlienarly elastic solids and the Milne-Pinney (or Ermakov) equation. *Mathematics and Mechanics of Solids*, 16(6):586–593, 2011.

[43] L. Dresner. *Similarity solutions of nonlinear partial differential equations.* Pitman Advanced Publishing Program, 1983.

[44] M.S. Dresselhaus, G. Dresselhaus, and A. Jorio. *Group theory: applications to the physics of condensed matter.* Springer, 2008.

[45] D.C. Fletcher. Conservation laws in linear elastodynamics. *Archive for Rational Mechanics and Analysis*, 60(4):329–353, 1976.

[46] G.B. Folland. *Introduction to partial differential equations.* Princeton University Press, 1995.

[47] Y.B. Fu and R.W. Ogden. *Nonlinear elasticity: theory and applications*, volume 281. Cambridge University Press, 2001.

[48] W.I. Fushchich and W.M. Shtelen. One approximate symmetry and approximate solutions of the non-linear wave equation with a small parameter. *Journal of Physics A: Mathematical and General*, 22:L887–L890, 1989.

[49] M.L. Gandarias and M.S. Bruzón. Reductions for some ordinary differential equations through nonlocal symmetries. *Journal of Nonlinear Mathematical Physics*, 18(1):123–133, 2013.

[50] M.L. Gandarias, M. Torrisi, and A. Valenti. Symmetry classification and optimal systems of a non-linear wave equation. *International Journal of Nonlinear Mechanics*, 39:389–398, 2004.

[51] C.S. Gardner. Kortweg de Vries equation and generalizations. IV. The Korteweg de Vries equation as a Hamiltonian system. *Journal of Mathematical Physics*, 12(8):1548–1551, 1971.

[52] C.S. Gardner, J.M. Greene, M.D. Kruskal, and R.M. Miura. Korteweg de Vries equation and generalizations. VI. Methods for exact solution. *Communications on Pure and Applied Mathematics*, 27(1):97–133, 1974.

[53] A.N. Gent. A new constitutive relation for rubber. *Rubber Chemistry and Technology*, 69(1):59–61, 1996.

[54] H. Goldstein, C. Poole, and J. Safko. *Classical Mechanics.* Addison Wesley, 2002.

[55] Z.Y. Guo, X.Q. Peng, and B. Moran. Mechanical response of neo-Hookean fiber reinforced incompressible nonlinearly elastic solids. *International journal of solids and structures*, 44(6):1949–1969, 2007.

[56] R. Haberman. *Applied partial differential equations: with Fourier series and boundary value problems*, volume 4. Pearson Prentice Hall Upper Saddle River, 2004.

[57] S. Hartmann. Parameter estimation of hyperelasticity relations of generalized polynomial-type with constraint conditions. *International Journal of Solids and Structures*, 38(44):7999–8018, 2001.

[58] E.L. Hill. Hamilton's principle and the conservation theorems of mathematical physics. *Reviews of Modern Physics*, 23(3):253, 1951.

[59] G.A. Holzapfel, T.C. Gasser, and R.W. Ogden. A new constitutive framework for arterial wall mechanics and a comparative study of material models. *Journal of Elasticity*, 61:1–48, 2000.

[60] G.A. Holzapfel and R.W. Ogden. Constitutive modelling of arteries. volume 466, pages 1551–1597, 2010.

[61] C.O. Horgan and J.G. Murphy. Limiting chain extensibility constitutive models of Valanis–Landel type. *Journal of Elasticity*, 86(2):101–111, 2007.

[62] C.O. Horgan and G. Saccomandi. Constitutive modelling of rubber-like and biological materials with limiting chain extensibility. *Mathematics and mechanics of solids*, 7(4):353–371, 2002.

[63] C.O. Horgan and G. Saccomandi. Constitutive models for compressible nonlinearly elastic materials with limiting chain extensibility. *Journal of Elasticity*, 77(2):123–138, 2004.

[64] C.O. Horgan and G. Saccomandi. A new constitutive theory for fiber-reinforced incompressible nonlinearly elastic solids. *Journal of the Mechanics and Physics of Solids*, 53(9):1985–2015, 2005.

[65] P.E. Hydon. *Symmetry methods for differential equations*. Cambridge University Press, 2000.

[66] N.H. Ibragimov. *CRC Handbook of Lie group analysis of differential equations*, volume 1. CRC Press, 1994.

[67] N.H. Ibragimov. *CRC Handbook of Lie group analysis of differential equations*, volume 2. CRC Press, 1994.

[68] N.H. Ibragimov. *Transformation groups applied to mathematical physics*, volume 3. Springer, 2001.

[69] M.I. Idiart and S. Lopez-Pamies. On the overall response of elastomeric solids with pressurized cavities. *Comptes Rendus Mecanique*, 340(4):359–368, 2012.

[70] N.M. Ivanova, C. Sophocleous, and Tracinà. Lie group analysis of two-dimensional variable-coefficient Burgers equation. *Zeitschrift für angewandte Mathematik und Physik*, 61:793–809, 2010.

[71] P. Kalita and R. Schaefer. Mechanical models of artery walls. *Archives of Computational Methods in Engineering*, 15(1):1–36, 2008.

[72] R. Kikuuwe, H. Tabuchi, and M. Yamamoto. An edge-based computationally efficient formulation of Saint Venant-Kirchhoff tetrahedral finite elements. *ACM Transactions on Graphics (TOG)*, 28(1):8, 2009.

[73] M.D. Kruskal, R.M. Miura, C.S. Gardner, and N.J. Zabusky. Korteweg de Vries equation and generalizations. V. Uniqueness and nonexistence of polynomial conservation laws. *Journal of Mathematical Physics*, 11(3):952–960, 1970.

[74] V. Kumar, R.K. Gupta, and R. Jiwari. Painlevé analysis, Lie symmetries, and exact solutions for variable coefficients Benjamin-Bona-Mahony-Burger (BBMB) equation. *Communications in Theoretical Physics*, 60:175–182, 2013.

[75] Y. Kwon. On Hadamard stability for compressible viscoelastic constitutive equations. *Journal of Non-Newtonian Fluid Mechanics*, 65(2):151–163, 1996.

[76] Hin-Chi Lei and Janet A Blume. Lie group and invariant solutions of the plane-strain equations of motion of a neo-Hookean solid. *International journal of non-linear mechanics*, 31(4):465–482, 1996.

[77] R.J. LeVeque. *Numerical methods for conservation laws*. Birkhauser, 1992.

[78] D. Levi and P. Winternitz. Lie groups and numerical solutions of differential equations: invariant discretization versus differential approximation. *Acta Polytechnica*, 53(5), 2013.

[79] S. Lie. *Theorie der transformationsgruppen, Vol. I.* B.G Teubner, Leipzig, 1880.

[80] S. Lie. *Theorie der transformationsgruppen, Vol. II.* B.G Teubner, Leipzig, 1890.

[81] S. Lie. *Theorie der transformationsgruppen, Vol. III.* B.G Teubner, Leipzig, 1893.

[82] Maplesoft. find numerical solution of time-based partial differential equations - Maple help:, 2014.

[83] J.E. Marsden and T.J.R. Hughes. *Mathematical Foundations of Elasticity*. Prentice-Hall, 1983.

[84] J.E. Marsden and T.S. Ratiu. *Introduction to mechanics and symmetry*. Springer, 1994.

[85] G.A. Maugin. *Material inhomogeneities in elasticity*, volume 3. CRC Press, 1993.

[86] G.A. Maugin and V.K. Kalpakides. A Hamiltonian formulation for elasticity and thermoelasticity. *Journal of Physics A: Mathematical and General*, 35(50):10775, 2002.

[87] S.V. Meleshko. *Methods for constructing exact solutions of partial differential equations: mathematical and analytical techniques with applications to engineering*. Springer, Dordrecht, 2006.

[88] J. Merodio and R.W. Ogden. Mechanical response of fiber-reinforced incompressible non-linearly elastic solids. *International Journal of Non-Linear Mechanics*, 40(2):213–227, 2005.

[89] R.M. Miura. Kortweg de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation. *Journal of Mathematical Physics*, 9(8):1202–1204, 1968.

[90] R.M. Miura, C.S. Gardner, and M.D. Kruskal. Kortweg de Vries equation and generalizations. II. Existence of conservation laws and constants of motion. *Journal of Mathematical physics*, 9(8):1204–1209, 1968.

[91] S. Murata. Nonclassical symmetry analysis for hyperbolic partial differential equation. *Communications in Nonlinear Science and Numerical Simulation*, 13:1472–1474, 2008.

[92] S. Murata. Nonclassical symmetry of nonlinear diffusion equations and factorization method. *Communications in Nonlinear Science and Numerical Simulation*, 15:3313–3315, 2010.

[93] R. Namani and P.V. Bayly. Shear wave propogation in anisotropic soft tissues and gels. pages 1117–1122. IEEE Engineering in Medicine and Biology Society, 2009.

[94] S. Nezamabadi, H. Zahrouni, and J. Yvonnet. Solving hyperelastic material problems by asymptotic numerical method. *Computational Mechanics*, 47(1):77–92, 2011.

[95] E. Noether. Invariante variationsprobleme. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, mathematisch-physikalische Klasse*, 1918:235–257, 1918.

[96] P.J. Olver. *Applications of Lie groups to differential equations*, volume 107. Springer Verlag, 2000.

[97] A. Ouhadan, E.H. El Kinani, and M. Hajar. Lie symmetries analysis for SIR model of epidemiology. *Applied Mathematical Sciences*, 7(92):4595–4604, 2013.

[98] L.V. Ovsiannikov. *Group analysis of differential equations*. Academic Press, 1982.

[99] R. Rebelo and F. Valiquette. Symmetry preserving numerical schemes for partial differential equations and their numerical tests. *Journal of Difference Equations and Applications*, 19(5):738–757, 2013.

[100] J.F Rodríguez, C. Ruiz, and M Doblaré. Mechanical stress in abdominal aortic aneurysms: influence of diameter, asymmetry, and material anisotropy. *Journal of Biomechanical Engineering*, 130:021023, 2008.

[101] J.F. Rodriquez, V. Alastrue, and M. Doblare. Finite element implementation of a stochastic three dimensional finite-strain damage model for fibrous soft tissue. *Computer Methods in Applied Mechanics and Engineering*, 197:946–958, 2008.

[102] P. Sáez, E. Peña, M.A. Martínez, and E. Kuhl. Computational modeling of hypertensive growth in the human carotid artery. *Computational Mechanics*, 53(6):1183–1196, 2014.

[103] M. Sharif and S. Waheed. Energy of the Bardeen model using an approxiamte symmetry method. *Physica Scripta*, 83:1–6, 2011.

[104] C.H. Shu and C.S. Gardner. Kortweg de Vries equation and generalizations. III. Derivation of the Kortweg de Vris equations and Burgers equation. *Journal of Mathematical Physics*, 10(3):536–539, 1969.

[105] I.N. Sneddon. Book review: Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity. *Bulletin of the American Mathematical Society*, 3:870–878, 1980.

[106] C.A. Taylor and J.D. Humphrey. Open problems in computational vascular biomechanics: hemodynamics and arterial wall mechanics. *Computer Methods in Applied Mechanics and Engineering*, 198(45):3514–3523, 2009.

[107] C. Truesdell. *Essays in the history of mechanics*. Springer, 1968.

[108] C. Truesdell and W. Noll. *The non-linear field theories of mechanics*. Springer, 2004.

[109] S. Umale, C. Deck, N. Bourdet, P. Dhumane, L. Soler, J. Marescaux, and R. Willinger. Experimental mechanical characterization of abdominal organs: liver, kidney and spleen. *Journal of the Mechanical Behaviour of Biomedical Materials*, 17:22–33, 2013.

[110] C. Wang and C. Truesdell. *Introduction to rational elasticity*, volume 1. Springer, 1973.

[111] Y. Wang, S.M. Son, S. Swartz, and N.C. Goulbourne. A mixed Von mises distribution for modeling soft biological tissues with two distributed fiber properties. *International Journal of Solids and Structures*, 49(21):2914–2923, 2012.

[112] J.A. Weiss, B.N. Maker, and S. Govindjee. Finite element implementation of incompressible, transversely isotropic hyperelasticity. *Computer Methods in Applied Mechanics and Engineering*, 135(1):107–128, 1996.

[113] P. Wriggers and J. Karelc. On enhanced strain methods for small and finite deformations of solids. *Computational Mechanics*, 18(6):413–428, 1996.

[114] E. Zauderer. *Partial differential equations of applied mathematics*, volume 71. John Wiley & Sons, 2011.

[115] Z. Zhang, B. Gao, and Y. Chen. Second-order approximate symmetry classification and optimal system of a class of perturbed nonlinear wave equations. *Communications in Nonlinear Science and Numerical Simulation*, 16:2709–2719, 2011.

Appendix A

Numerical Solution Code

The following is the Maple code for the numerical solution computed in Section 4.3.2.

```
>  restart;
>  with(DEtools):with(PDEtools):with(plots):with(LinearAlgebra):

>  # The problem is defined on the interval [-xend,xend].  NOTE: We use a
boundary larger than the one of interest (which is [-8,8]) to mitigate
boundary effects on the numerical method.
>  xend:=20:

>  # Specify space step of spatial coordinate for numerical method.
>  xspacing:=0.02:

>  # Obtain desired number of spatial points to consider based on space
stepsize (xspacing) and boundary (xend).
>  num_of_pts:=floor(2*xend/xspacing):

>  # Equation to be studied.
>  sys:= diff(G(t,x),t,t) = (alpha + 2*beta*(diff(G(t,x),x))^2)
*diff(G(t,x),x,x);
```

$$sys := \frac{\partial^2}{\partial t^2} G(t,x) = \left(\alpha + 2\beta \left(\frac{\partial}{\partial x} G(t,x) \right)^2 \right) \frac{\partial^2}{\partial x^2} G(t,x)$$

```
>  # Initial and boundary conditions to be used by pdsolve.
>  IBC:={G(0,x)=exp(-x^2),D[1](G)(0,x)=0,G(t,-xend)=exp(-(xend^2)),
G(t,xend)=exp(-(xend^2))};
```

$$IBC := \left\{ G(0,x) = e^{-x^2}, G(t,-20) = e^{-400}, G(t,20) = e^{-400}, D_1(G)(0,x) = 0 \right\}$$

```
>  # Specify the parameters in 'sys' for the numerical simulation.
>  alpha:=1:
>  beta:=1/3:

>  # Creating module from pdsolve/numeric for timestep equal to the
spacestep.
>  pdsxspace:=pdsolve(sys,IBC,numeric,G(t,x),time=t,range=-xend..xend,
spacestep=xspacing,timestep=xspacing):
```

```
>  # Creating module from pdsolve/numeric for timestep equal to half of
the spacestep.

>  pdshalfxspace:=pdsolve(sys,IBC,numeric,G(t,x),time=t,
range=-xend..xend,spacestep=xspacing,timestep=xspacing/2):



>  # Computing and plotting the solution for several times (t=0,2,4) on
interval [-8,8].

>  phalfxspace0:=pdshalfxspace:-plot(t=0,numpoints=num_of_pts,
color=red):
>  phalfxspace2:=pdshalfxspace:-plot(t=2,numpoints=num_of_pts,
color=magenta):
>  phalfxspace4:=pdshalfxspace:-plot(t=4,numpoints=num_of_pts,
color="SteelBlue"):
>  plots[display]({phalfxspace0,phalfxspace2,phalfxspace4},
view=[-8..8,0..1]);



>  # Determine solution G(t,x) at time t=4 (as a function of x).

>  Gfullspacing4:=pdsxspace:-value(t=4):
>  Ghalfspacing0:=pdshalfxspace:-value(t=0):
>  Ghalfspacing1:=pdshalfxspace:-value(t=1):
>  Ghalfspacing2:=pdshalfxspace:-value(t=2):
>  Ghalfspacing3:=pdshalfxspace:-value(t=3):
>  Ghalfspacing4:=pdshalfxspace:-value(t=4):



>  # Defining procedure to convert the functions above
(e.g.  Gfullspacing4) to vectors.

>  vector_output_of_fn:=proc(fnCN,xspacing,num_of_pts,xend) local H;
local m; local n;
>  m:=-xend;
>  n:=1;
>  H:=Vector(num_of_pts+1);
>  while m<=xend do
>  H[n]:= rhs(fnCN(m)[3]);
>  m:=m+xspacing;
>  n:=n+1;
>  end do;
>  H;
>  end proc:



>  # Creating vectors with entries having value of above functions
(e.g.  Gfullspacing4) at spatial points.
```

```
>   GValsfirst4:=vector_output_of_fn(Gfullspacing4,xspacing,num_of_pts,
xend):
>   GValshalf0:=vector_output_of_fn(Ghalfspacing0,xspacing,num_of_pts,
xend):
>   GValshalf1:=vector_output_of_fn(Ghalfspacing1,xspacing,num_of_pts,
xend):
>   GValshalf2:=vector_output_of_fn(Ghalfspacing2,xspacing,num_of_pts,
xend):
>   GValshalf3:=vector_output_of_fn(Ghalfspacing3,xspacing,num_of_pts,
xend):
>   GValshalf4:=vector_output_of_fn(Ghalfspacing4,xspacing,num_of_pts,
xend):


>   # Creating vector with spatial points.
>   f := proc (i) options operator, arrow; -xend+xspacing*(i-1) end proc :
>   vec_x:=Vector(num_of_pts+1,f):


>   # Calculating maximum value for display size of plot of difference
between timestep=spacestep and timestep=(1/2*spacestep.

>   difference_of_CN_fns:=GValshalf4-GValsfirst4:
>   h_disp:=1.1*max({max(difference_of_CN_fns),
-min(difference_of_CN_fns)}):


>   # Plotting difference between plots with timestep=spacestep and
timestep=(1/2*spacestep).

>   pointplot(vec_x,difference_of_CN_fns,view=[-8..8,-h_disp..h_disp]);


>   # Function to convert values of G to values of p (hydrostatic
pressure) based on the expression in equation (4.8) of Chapter 4.

>   g_vector_to_p:=proc(vector,xspacing) local H; local m; local n;
>   n:=2;
>   m:=Dimension(vector);
>   H:=Vector(m);
>   while n<=(m-1) do
>   H[n]:= ((vector(n+1) - vector(n-1))/(2*xspacing))^2;
>   n:=n+1;
>   end do;
>   H[1]:=H(2);
>   H[m]:=H(m-1);
>   H;
>   end proc:


>   # Plotting hydrostatic pressure p.
```

```
> pValshalf0:=g_vector_to_p(GValshalf0,xspacing):
> pValshalf2:=g_vector_to_p(GValshalf2,xspacing):
> pValshalf4:=g_vector_to_p(GValshalf4,xspacing):
> plot_of_p_half0:=plot(vec_x,pValshalf0,color=red):
> plot_of_p_half2:=plot(vec_x,pValshalf2,color=magenta):
> plot_of_p_half4:=plot(vec_x,pValshalf4,color="SteelBlue"):
> display(plot_of_p_half0,plot_of_p_half2,plot_of_p_half4,
view=[-8..8,0..1]);
```