Some Large-Scale Regularity Results for Linear Elliptic Equations with Random Coefficients and on the Well-Posedness of Singular Quasilinear SPDEs

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Die Annahme der Dissertation wurde empfohlen von:

1. Professor Dr. Felix Otto (MPI MIS Leipzig)
2. Professor Dr. Hendrik Weber (University of Bath, UK)

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Abstract

This thesis is split into two parts, the first one is concerned with some problems in stochastic homogenization and the second addresses a problem in singular SPDEs. In the part on stochastic homogenization we are interested in developing large-scale regularity theories for random linear elliptic operators by using estimates for the homogenization error to transfer regularity from the homogenized operator to the heterogeneous one at large scales. In the whole-space case this has been done by Gloria, Neukamm, and Otto through means of a homogenization-inspired Campanato iteration. Here we are specifically interested in boundary regularity and as a model setting we consider random linear elliptic operators on the half-space with either homogeneous Dirichlet or Neumann boundary data. In each case we obtain a large-scale $C^{1,\alpha}$-regularity theory and the main technical difficulty turns out to be the construction of a sublinear homogenization corrector that is adapted to the boundary data. The case of Dirichlet boundary data is taken from a joint work with Julian Fischer. In an attempt to head towards a percolation setting, we have also included a chapter concerned with the large-scale behaviour of harmonic functions on a domain with random holes assuming that these are "well-spaced".

In the second part of this thesis we would like to provide a pathwise solution theory for a singular quasilinear parabolic initial value problem with a periodic forcing. The difficulty here is that the roughness of the data limits the regularity the solution such that it is not possible to define the nonlinear terms in the equation. A well-posedness result, therefore, comes with two steps: 1) Giving meaning to the nonlinear terms and 2) Showing that with this meaning the equation has a solution operator with some continuity properties. The solution theory that we develop in this contribution is a perturbative result in the sense that we think of the solution of the initial value problem as a perturbation of the solution of an associated periodic problem, which has already been handled in a work by Otto and Weber. The analysis in this part relies entirely on estimates for the heat semigroup. The results in the second part of this thesis will be in an upcoming joint work with Felix Otto and Jonas Sauer.

Parts of this thesis are contained in the following works:

- J. Fischer and C. Raithel. *Liouville Principles and a Large-Scale Regularity Theory for Random Elliptic Operators on the Half-Space.* SIAM J. Math. Anal. 49 (1): 82-114, 2017.

- C. Raithel. *A Large-Scale Regularity Theory for Random Elliptic Operators on the Half-Space with Homogeneous Neumann Boundary Data.* arXiv preprint, 2017 arXiv:1703.04328

Dedication: To Michael E. Dishaw and all the people that still miss him.

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Contents

0.1 Stochastic Homogenization

The first focus of my PhD work was the stochastic homogenization of linear elliptic PDEs. At the core of this theory is the qualitative result that if the space of bounded coefficient fields with a fixed ellipticity ratio is endowed with a stationary and ergodic (see below) probability measure, then it is possible to find a constant coefficient field a_{hom} such that for almost every heterogeneous coefficient field a the constant coefficient operator $-\nabla \cdot a_{hom} \nabla$ approximates $-\nabla \cdot a \nabla$ at large scales. The requirements of stationarity and ergodicity on the ensemble can be motivated on the intuitive level using that homogenization is really the averaging of the coefficient field that happens when we "zoom-out". To encourage this averaging we should avoid coefficient fields that exhibit atypical large-scale geometries (e. g. radial symmetry) or that are sampled differently in different regions. This can be formalized as requiring that the probability measure (with expectation $\langle \cdot \rangle$) satisfies:

- *Ergodicity*: Any shift-invariant random variable must be $\langle \cdot \rangle$-almost surely constant, which corresponds to a qualitative assumption of decorrelation on large scales.
- *Stationarity*: The expectation $\langle \cdot \rangle$ is shift-invariant.

The qualitative theory of stochastic homogenization was developed in the late 70s and early 80s (see [48] by Papanicolaou and Varadhan and [39] by Kozlov) and holds under qualitative ergodicity as defined above. However, more recently, quantitative results, e. g. convergence rates (in the scale) for the heterogeneous operator to the homogeneous one, have come into vogue and always require a quantification of the ergodicity assumption. Choosing an appropriate notion of "quantified ergodicity" is in itself nontrivial and there are various notions available. One intuitive way to quantify ergodicity is to impose a specific decay rate on the correlation between random variables depending on $a|_U$ and $a|_V$, where $U, V \subseteq \mathbb{R}^d$, as the distance $d(U, V)$ increases. This is the version of quantified ergodicity used in the initial work on error estimates in '86 [52]. In this paper the author, Yurinskii, was able to obtain a suboptimal, but non-trivial, convergence rate. Ten years later came the work of Naddaf and Spencer [45] and then Naddaf and Conlon [18], in which quantitative mixing conditions on the ensemble were encoded in terms of a *spectral gap condition*. In particular, they only consider ensembles that satisfy a Poincaré inequality with zero-average over the probability space, where the role of the gradient in the classical Poincaré inequality is played by a "vertical derivative" that measures the dependence of a random variable on the coefficient field. Using the framework of a spectral gap, the authors were able to, for small ellipticity ratio, obtain optimal bounds for the random fluctuations of the energy density of the gradient of an object called the "corrector."

Quantitative results in stochastic homogenization rely on the random fluctuations and deterministic growth of a random field called the "corrector." For a fixed coefficient field a and a direction $\xi \in \mathbb{R}^d$ the *whole-space corrector in the direction* ξ, denoted ϕ_ξ, corrects the linear function $\xi \cdot x$, which is a_{hom}-harmonic, to be a-harmonic. By a function being a-harmonic we mean that it is in the kernel of the operator $-\nabla \cdot a\nabla$. So, in particular, the corrected function $\phi_\xi + \xi \cdot x$ is a distributional solution of

$$-\nabla \cdot a\nabla(\phi_\xi + \xi \cdot x) = 0 \qquad \text{on} \quad \mathbb{R}^d. \tag{1}$$

Notice that, since the map $\xi \mapsto \phi_\xi$ may be taken to be linear, to specify the corrector for any $\xi \in \mathbb{R}^d$ it suffices to construct $(\phi_{b_1}, ..., \phi_{b_d})$, where $\{b_1, ..., b_d\}$ is a basis of $\mathbb{R}^d$. Of course, as stated, the equation (1) has the trivial solution $-\xi \cdot x$. This, however, is not a reasonable choice for the corrector ϕ_ξ as, by its very nature, the corrector should be sublinear. The question of the uniqueness (up to addition of a random constant) of the sublinear corrector ϕ was only recently settled in the affirmative and follows, e.g., from the analysis provided in [31].

In quantitative homogenization, it is convenient to introduce a dual quantity to the corrector ϕ_ξ for $\xi \in \mathbb{R}^d$; namely, a vector potential $\sigma_{\xi jk}$ for the flux correction $a(\xi + \nabla\phi_\xi) - a_{hom}\xi$. Notice that this is the difference between the flux in the direction ξ in the microscopic picture and in the homogenized picture. In particular, $\sigma_{\xi jk}$ is skew-symmetric in the last two indices ($\sigma_{\xi jk} = -\sigma_{\xi kj}$) and distributionally satisfies

$$\nabla_k \cdot \sigma_{\xi jk} = e_j \cdot (a(\xi + \nabla\phi_\xi) - a_{hom}\xi), \tag{2}$$

where we use the notation $\nabla_k \cdot \sigma_{\xi jk} = \sum_{k=1}^d \partial_k \sigma_{\xi jk}$. Cleary the equation (2) does not uniquely define σ as it has a gauge invariance in the sense that one may perturb a solution by any solenoidal random field without penalty. The object σ is often called the "flux corrector", which is notation taken from periodic homogenization, where it is classical. In the stochastic setting σ was first constructed by Gloria, Neukamm, and Otto in [31], where the choice of gauge ensured that σ is sublinear. As in the case of the corrector, for each $j, k \in \mathbb{R}^d$ the flux corrector is determined by $(\sigma_{b_i jk}, ..., \sigma_{b_d jk})$, where $\{b_1, ..., b_d\}$ is a basis of $\mathbb{R}^d$.

To see how the corrector is a natural object in the theory of homogenization, we, for a fixed coefficient field a, study the operator $-\nabla a(\frac{\cdot}{\epsilon})\nabla$ as

$\epsilon \to 0$. Here, it is natural to do an asymptotic expansion at two scales, the microscopic scale at which the coefficients oscillate and the macroscopic scale on which the right-hand side oscillates, of the solution of the equation with rescaled coefficients $a(\frac{\cdot}{\epsilon})$. This yields the *2-scale expansion*:

$$u_\epsilon \approx u_{hom} + \epsilon \sum_{i=1}^{d} \frac{\partial u_{hom}}{\partial x_i} \phi_i \left(\frac{x}{\epsilon}\right) + \text{higher-order (in } \epsilon \text{) terms,} \tag{3}$$

where d denotes the dimension, u_ϵ denotes the solution to the equation with rescaled coefficients, u_{hom} denotes the solution to the homogenized equation, and ϕ_i is the corrector introduced above. Notice that, neglecting higher-order terms, the 2-scale expansion implies that the question of the existence of a corrector with sublinear growth is equivalent to that of whether homogenization occurs.

With this coarse overview of stochastic homogenization in-hand, we now state that the main focus of Part 1 of this thesis is the construction of boundary correctors; i. e. functions solving the corrector equation on a domain, but also satisfying certain boundary data. As a motivation for the usefulness of these objects, let us return to periodic homogenization and consider the Dirichlet problem with periodic coefficients. Here, the expected error estimates for the convergence of u_ϵ to u_{hom} are not obtained by naively using the 2-scale expansion with the standard periodic cell-correctors (solutions of the corrector equations posed on the torus) since the use of the cell-correctors in this ansatz introduces oscillations at the boundary. Of course, a natural reaction to this is to introduce an object that not only solves the corrector equation, but also satisfies homogeneous Dirichlet boundary data. This is a boundary corrector, which may then be used to obtain more refined error estimates. In periodic homogenization the use of boundary correctors goes back as far as the seminal work of Avellaneda and Lin [11], in which various uniform in ϵ (up to the boundary) regularity theories are developed for the solutions u_ϵ. In their work they use a local version of a Dirichlet corrector, which solves the corrector equation on the domain and *locally* at the boundary satisfies homogeneous Dirichlet boundary data. More recently, the strategy of Avellaneda and Lin in [11] was generalized to the Neumann setting by Kenig, Lin, and Shen [38]. The Neumann case is more difficult because the boundary condition of both u_ϵ and the boundary corrector are dependent on the scale.

Even in periodic homogenization the construction of Dirichlet correctors already motivates interesting questions. In particular, since this boundary corrector may constructed by correcting the cell-corrector with a function that is a-harmonic on the domain and "cancels-out" the cell-corrector at the boundary of the domain, the study of the boundary corrector and the study of the oscillating Dirichlet problem go hand-in-hand. The treatment of this homogenization problem is much more difficult than the standard Dirichlet problem as one must also be concerned with the homogenization of the boundary data, which is also oscillating. Even in periodic homogenization this is an exciting problem that has recently received much attention. The literature on this goes back to Allaire and Amar in the 90s [1] and it has been, more recently, treated by Gérard-Varet and Masmoudi [26, 27]. Interestingly, it turns out that homogenization in this framework depends on the resonance of the directions of periodicity with the tangent planes of the domain. Assuming that the period-cell is the unit-cube, the favorable situation is that within some (small enough) neighborhood of every boundary point there exists a tangent plane with Diophantine slope. There is, in particular, an important recent paper of Armstrong, Kuusi, Mourrat, and Prange [5] on the homogenization of the oscillating Dirichlet problem in uniformly convex domains. Here, their method yields the optimal (up to the loss of ϵ^{δ} for arbitrarily small $\delta > 0$) convergence rate for $u_\epsilon \to u_{hom}$ in L^q for $q \in [2, \infty)$ in $d \geq 4$ and improves previous regularity results for the homogenized boundary data. Following subsequent work of Shen and Zhuge [51], the method of [5] yields the optimal rates in $d = 2, 3$. The result of [51], which adapts the method of [5] to the Neumann setting, is helpful in lower dimensions because the authors obtain better regularity for the homogenized boundary data.

While boundary correctors are a quite classical theme in periodic homogenization, in stochastic homogenization their use has only recently begun to be phased-in. To make this point, in Chapter 1 of Part 1 of this thesis, which was a joint work with Julian Fischer [22], we gave the, to my knowledge, first construction of a (half-space) Dirichlet corrector in the stochastic setting [1]. In a subsequent work [50], given in Chapter 2 of Part 1, I then extended the methods of [22] to also construct a (half-space) Neumann corrector. In these contributions, the primary motivation for the construction of these boundary correctors is actually qualitative: We wanted to obtain a large-scale almost sure $C^{1,\alpha}$- regularity theory for the heterogeneous coefficient operator $-\nabla\cdot a\nabla$ on the (upper) half-space with homogeneous Dirichlet or Neumann boundary data.

To motivate our meaning of "large-scale $C^{1,\alpha}$-regularity", we recall that Hölder regularity is equivalent to a certain approximability of a function by

[1] Previously, a $C^{0,1}$-regularity theory up to the boundary was proved in both the Dirichlet and Neumann cases in the *almost periodic* setting by Armstrong and Shen [8]. Their method also relies on boundary correctors and it is likely that their method extends to the stochastic setting.

polynomials. To capture this concept, in the classical setting one introduces the notion of *tilt-excess*, which compares a solution of the elliptic equation $-\nabla \cdot a\nabla u = 0$ to the space of affine polynomials $x \mapsto \xi \cdot x + c$ in the squared energy norm. So, for a function u, the tilt-excess on the ball $B_r := \{|x| < r\}$ is defined as

$$\mathrm{Exc}^C(r) := \inf_{\xi \in \mathbb{R}^d} \fint_{B_r} |\nabla u - \xi|^2 \, dx, \tag{4}$$

where the superscript "C" stands for "classical". Differentiability properties of the function u are then encoded in decay properties of the tilt-excess in the radius r. For solutions to the Laplace equation $-\Delta u = 0$ on $\mathbb{R}^d$, the tilt-excess displays decay in the radius r according to

$$\mathrm{Exc}^C(r) \leq \left(\frac{r}{R}\right)^2 \mathrm{Exc}^C(R) \tag{5}$$

for any pair of radii $0 < r \leq R$. When valid for balls $B_r(x_0) := \{|x - x_0| < r\}$ around any center $x_0 \in \mathbb{R}^d$, this excess decay property entails $C^{1,1}$-regularity of solutions.

In the first chapter of this thesis we are inspired by a previous work of Gloria, Neukamm, and Otto [31], in which they introduce a version of the excess that is modified to take into account the homogenization framework. In particular, for a given heterogeneous coefficient field a one replaces the linear functions $\xi \cdot x$ in (4) with the a-linear functions $\phi_\xi + \xi \cdot x$, where ϕ_ξ is the homogenization corrector in (1). For this modified version of the excess, they then find that, assuming the corrector and corresponding flux corrector are sublinear, for any Hölder exponent $\alpha \in (0,1)$ there exists a minimal radius $r_\alpha^*(a) > 0$ above which a $C^{1,\alpha}$-excess decay holds for a-harmonic functions. Using that $\alpha > 0$, this large-scale $C^{1,\alpha}$-excess decay is then upgraded to a large-scale mean-value property and a $C^{1,\alpha}$-Liouville principle. Here, $C^{1,\alpha}$-Liouville principle means that the space of a-harmonic functions u satisfying the growth condition $|u(x) - u(0)| \lesssim |x|^{1+\alpha}$ has the same dimension as in the constant coefficient setting; in fact, they find that this space consists the a-linear functions .

We emphasize that the Liouville principle proved by Gloria, Neukamm, and Otto is *not* proved for all uniformly elliptic, bounded coefficient fields, but only for those for which one has access to a sublinear corrector and flux corrector. From [31] we know that this sublinear pair (ϕ, σ) exists almost surely for stationary and ergodic ensembles, where these properties ensure that almost surely one avoids the classical counterexamples to Hölder regularity. To see this graphically, we take a look at a plot of the coefficient field constructed by Meyers in his counterexample in the scalar case (see Example 3 in [49]):

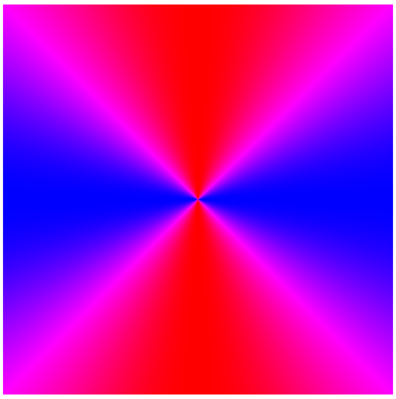

Figure 1: Here, of course, we consider the situation $d = 2$. *Picture borrowed from Julian Fischer.*

Notice that this coefficient field is radially symmetric. In the systems case there is the classical counterexample of De Giorgi (see Section 9.1.1 in [30]), which shows that a-harmonic functions may not even be locally bounded. However, the coefficient field in this counterexample is also non-generic under our assumptions.

In Part 1 of this thesis we develop three different large-scale regularity theories *à la* Gloria, Neukamm, and Otto. While we use scalar notation throughout, all of our results also hold for systems. In each chapter we must introduce a different version of the excess that is modified to take into account both the homogenization framework (which was already done in [31]) and the presence of the additional boundary data. Since the Liouville principle is obtained as a corollary of the excess decay, the excess should in each case be comparing the a-harmonic functions to the space that we expect in the Liouville principle. In each of the three chapters we use this principle as the guide by which we define our excess and also the boundary correctors. For clarity, we consider the case of homogeneous Dirichlet boundary data on the half-space as an example: Here, motivated by [31] and the Liouville principle in the constant coefficient case, we expect that a-harmonic functions u on $\mathbb{R}^d_+$ such that $u = 0$ on $\partial\mathbb{R}^d_+$ and $|u(x) - u(0)| \lesssim |x|^{1+\alpha}$ will be of the form $x_d + \phi_d^D$, where ϕ_d^D solves

$$\begin{aligned} -\nabla \cdot a\nabla(\phi_d^D + x_d) &= 0 && \text{in} \quad \mathbb{R}^d_+, \\ \phi_d^D &= 0 && \text{on} \quad \partial\mathbb{R}^d_+. \end{aligned}$$

This means that we should define the excess of u on B_r^+ , the half-ball of radius $r > 0$, as

$$\mathrm{Exc}^D(r) := \inf_{b\in\mathbb{R}} \fint_{B_r^+} |\nabla(u - b(x_d + \phi_d^D))|^2 \, dx,$$

where the superscript "D" stands for "Dirichlet".

The general approach for the proof of the excess decay result in [31] is to consider the solution of the heterogeneous-coefficient problem as a perturbation of the solution of the homogenized problem via the 2-scale expansion. Looking at the 2-scale expansion given in (3), notice that the left-hand side is the object that one is interested in developing a large-scale regularity theory for and the right-hand side is accessible through means of constant coefficient regularity theory applied to u_{hom} and homogenization results detailing the growth of the corrector. This perturbative method for obtaining regularity is not new and was pioneered in the periodic setting by Avellaneda and Lin in [11], which we have already mentioned. Due to the loss of compactness going from the periodic to the stochastic framework, the methods of Avellaneda and Lin, however, fail in the stochastic setting and new methods are required to compensate. Due to this difficultly there was a long lull between [11] and the corresponding theories now available in the stochastic setting.

The first group that was able to overcome the difficulties introduced by the stochastic framework was Armstrong and Smart. In particular, in [10] they are able to obtain an almost sure large-scale $C^{0,1}$-regularity theory in the scalar case under the assumptions of symmetric coefficients and a stationary ensemble with a finite range of dependence (if $\mathrm{dist}(U, V) \geq c$ for some $c \in \mathbb{R}_{>0}$ then $a|_U$ and $a|_V$ are stochastically independent). The arguments developed by Armstrong and Smart rely on a convergence rate for the homogenization error, which is obtained by studying certain subadditive energies associated with the heterogeneous coefficient equation. This initial work was followed by a contribution of Armstrong and Mourrat [7], in which the method of [10] was extended to the case of systems and stationary ensembles satisfying an *α-mixing condition*, which is a prescribed power-law decay of the correlations. For a complete overview of the techniques developed by Armstrong, Smart, Mourrat, and collaborators one can reference the recent book [3].

While the work of Armstrong and Smart was the first to implement the general philosophy of Avellaneda and Lin in the stochastic setting, as already mentioned, the contribution that directly motivated my joint work with Julian Fischer is [31] by Gloria, Neukamm, and Otto (GNO). The method presented in [31] is an alternative approach to that of Armstrong *et al.*; in particular, instead of studying energy quantities associated with the heterogeneous coefficient equation, one actually studies the solutions themselves. While both approaches rely on an "excess decay," the excess of GNO is *intrinsic* in the sense that it is expressed in terms of the harmonic coordinates and that of Armstrong and Smart is expressed in terms of affine functions. Time-wise, [31] came between [10] and [7]. The theory in [31] is split into a qualitative statement, almost surely above some scale there is a $C^{1,\alpha}$-excess decay, and some quantitative results such as the quantification of the scale above which the excess decay occurs. Their qualitative result holds under the assumption of qualitative ergodicity, while the quantitative aspects of their work require the additional assumption that the ensemble satisfies a *coarsened logarithmic Sobolev inequality (cLSI).* To finish this review, it should also be mentioned that the two main directions (Armstrong *et al.* and GNO) were predated by a work of Marahrens and Otto [43] in which they prove a large-scale $C^{0,\alpha}$-regularity theory for $\alpha \in (0,1)$ using bounds on derivatives of the heterogeneous Green's function, which are derived *à la* Naddaf and Spencer under an LSI assumption on the ensemble.

Both of the directions detailed above have been generalized in many ways over the last couple of years. A sizeable extension of both settings was obtaining an almost sure large-scale $C^{k,\alpha}$-regularity theory (for $k \geq 1$). This was first done by Fischer and Otto in [21] and slightly later by Armstrong, Kuusi, and Mourrat in [4]. Both of the theories have also been extended to handle parabolic equations: This is work by Bella, Chiarini, and Fehrman [13] in the GNO framework and by Armstrong, Bordas, and Mourrat [2]. In the GNO framework the case of degenerate coefficient fields (satisfying a moment bound motivated by [17]) has been treated by Bella, Fehrman, and Otto in the continuum case [14] and was considered by Deuschel, Nguyen, and Slowik in the discrete setting [20]. The method of Armstrong and Smart was adapted to handle fully nonlinear equations under the assumption of strict, but not uniform, ellipticity [9]. There is also a work in which Armstrong and Lin obtain optimal bounds on the growth of the corrector for non-divergence form linear equations [6].

To give a bit of context for Chapter 3 we mention that another (very related) area of much recent interest has been the large-scale behaviour of harmonic functions on percolation clusters. Here, in order to obtain an interesting result it is not even necessary to consider the case of random coefficients as the randomness is already encoded into the percolation environment. For supercritical percolation clusters on $\mathbb{Z}^d$ under only the assumption of qualitative ergodicity there is a zeroth-order Liouville principle available, which was proved by Benjamini, Duminil-Copin, Kozma, and Yadin in [15]. For the random conductance model on supercritical clusters generated by Bernoulli bond percolation this was recently generalized by Armstrong and Dario to k-th order Liouville principles [19].

While in this thesis we only obtain qualitative results, the motivating work [31] actually contains many quantitative results. In particular, they also show that for any $\alpha \in (0,1)$, under a cLSI assumption, the minimal radius r_α^* has stretched exponential moments. Also, they show a variety of estimates, which then culminate in a H^1-error estimate for the 2-scale expansion with a certain stochastic integrability. While the scaling of the error estimate depends on the particular cLSI assumed and the dimension d and is optimal, this estimate is not optimal with respect to the stochastic integrability. It should be noted that, on the other hand, the methods of Armstrong, Smart, Mourrat, and collaborators yield optimal results in terms of stochastic integrability, but with non-optimal scaling.

Returning to the contents of this thesis, we notice that the main difficulty in adapting the argument of GNO to the three settings is, in each case, the construction of an appropriate sublinear boundary corrector. Traditionally (see [37]), and also in [31], the corrector on the whole-space is obtained by using the stationarity and qualitative ergodicity of the ensemble to rephrase the corrector equation over the probability space in terms of "horizontal derivatives" and solving it there using a Lax-Milgram argument, which is possible due to the finite mass of the space. In particular, we use that $e_i \in L^2(\Omega)$, where Ω is the probability space. In Chapter 3, which is different in character from the other two chapters, our assumption that the holes of the domain are well-spaced allows us to modify the classical argument to obtain the desired sublinear corrector. However, in Chapters 1 and 2 there are two immediately apparent issues that prevent us from modifying the classical approach: 1) Since a spatial shift may shift a point in $\mathbb{R}^d_+$ out of $\mathbb{R}^d_+$, stationarity is lost in the e_d-direction and 2) In the classical probabilistic construction of the corrector one first constructs the gradient $\nabla\phi$ as a random variable, takes the stationary extension to obtain a random field, and then uses the Poincaré lemma to find ϕ. This means that if we want a boundary corrector with Dirichlet boundary data then we must somehow encode the boundary condition into the probabilistic formulation, which is on the level of the gradient. It is unclear how to do this. So, we instead opt for an entirely deterministic construction in which we correct a whole-space corrector that we assume to exist and to satisfy a quantified sublinearity condition.

The construction of the boundary correctors in Chapters 1 and 2 is really the core of Part 1. As previously stated, the construction of these boundary correctors proceeds by correcting a whole-space corrector, for which we assume some quantified sublinearity, in such a way as to enforce the boundary data. The correction of the whole-space corrector is actually performed on iteratively higher scales in a fashion similar to the previous construction of higher-order correctors in [21]. In particular, we enforce the boundary data on successively large dyadic annuli: For annuli below a certain scale their contribution to the full correction (of the whole-space corrector) may be sufficiently controlled with standard energy estimates, we call these the *near-field contributions*, and to obtain appropriate estimates for the contributions of the larger annuli, which we call *far-field contributions*, we inductively make use of a large-scale regularity theory that we have initially up to some scale thanks to the near-field contributions. The quantified sublinearity condition on the whole-space corrector is enforced to ensure that the sum of all of the corrections converges and is sublinear.

As the construction of the boundary correctors in Chapters 1 and 2 is heavily motivated by the previous construction of Fischer and Otto for higher-order correctors [21], it comes as no surprise that the quantified sublinearity required of the whole-space corrector (ϕ, σ) in our construction is very similar to the notion required in [21]. In particular, introducing the notation

$$\delta_R := \frac{1}{R}\left(\fint_{B_R} |(\phi,\sigma)|^2\,dx\right)^{\frac{1}{2}}, \tag{7}$$

for the construction of the higher-order correctors in [21] one requires

$$\sum_{m=0}^{\infty} \delta_{2^m} < \infty \tag{8}$$

and in the first two chapters of Part 1 we require the slightly stronger

$$\sum_{m=0}^{\infty} m\delta_{2^m}^{\frac{1}{3}} < \infty. \tag{9}$$

Notice that, in fact, (9) is satisfied whenever an estimate of the form

$$\delta_r \lesssim \frac{1}{|\log r|^{6+\epsilon}} \qquad \text{for large } r$$

for arbitrarily small $\epsilon > 0$ holds.

To name a few examples, our results in Chapters 1 and 2 are applicable to the following cases of ensembles of random coefficient fields:

- Ensembles for which $a(x)$ is either equal to a positive definite matrix a_1 or equal to another positive definite matrix a_2, depending on whether x

is contained in a random set of balls of a given fixed radius, the centers of the balls being chosen according to a Poisson point process. A realization of this ensemble is shown in the left picture in Figure 2.

- Stationary ensembles with finite range of dependence (i. e. ensembles for which $a|_U$ and $a|_V$ are stochastically independent for any two sets $U, V \subset \mathbb{R}^d$ with $\operatorname{dist}(U, V) \geq c$) subject to uniform ellipticity and boundedness conditions. Note that the previous case is a particular case of this. That the condition (9) is satisfied almost surely for such ensembles follows, e. g. from the estimates in [32].
- Coefficient fields of the form $\xi(\tilde{a}(x))$, where $\tilde{a}$ denotes a matrix-valued stationary Gaussian random field subject to the decorrelation estimate

$$|\operatorname{Cov}(\tilde{a}(x), \tilde{a}(y))| \leq \frac{C}{|x-y|^\beta}$$

for some $\beta \in (0, d)$ and where $\xi : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ is a Lipschitz map taking values in a bounded uniformly elliptic subset of the matrices of dimension $d \times d$. That the condition (9) is satisfied is shown in [23]. A realization of this ensemble is shown in the right picture in Figure 2.

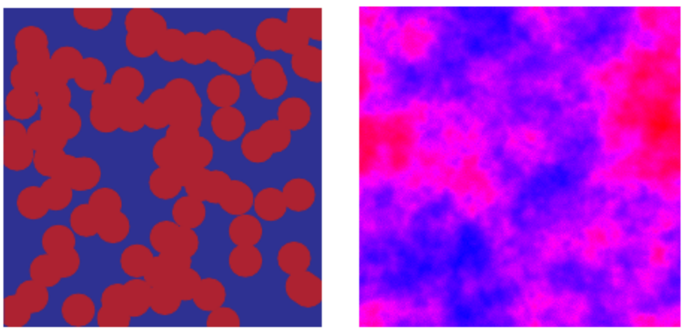

Figure 2: Two realizations of random coefficient fields satisfying (9). *Picture taken from [22].*

For cohesiveness we take the opportunity to say a couple more words about Chapter 3. As already mentioned, the original motivation for considering this situation was to move towards the percolation setting. However, in our formulation we only consider the situation of "well-spaced" holes, by which we mean that not only are the holes disjoint, but they are separated from each other by an additional "buffer" area. This means that we actually avoid the main problem, which is that in the percolation setting the holes may cut-out islands in the domain. If we were able to drop the disjointness assumption on the holes then the resulting work could be seen as a GNO-inspired continuum version of the first-order qualitative result obtained by Armstrong and Dario in [19], but with constant coefficients and for a hopefully broader class of percolation cluster (they only consider Bernoulli percolation). However, as previously noted, Armstrong and Dario not only provide Liouville principles up to arbitrary order, but also obtain many quantitative results.

Before continuing on to the second part of this thesis, we mention a future project that may be within reach given the tools developed in Chapters 1 and 2. Recently there has been much interest in homogenization when the underlying periodic or random structure is perturbed by either a local defect or an interface. In the periodic setting the situation of two periodic structure meeting at an interface has been treated by Blanc, Le Bris, and Lions in [16]. It would interesting to treat the corresponding situation in the stochastic setting: So, the situation in which the random coefficients of the elliptic operator on the whole-space are chosen according to different stationary and quantified ergodic probability measures on the upper- and lower half-spaces. Due to the distinction of the half-spaces this clearly generates an ensemble that is inherently non-stationary, which makes it inaccessible to the standard stochastic homogenization theory. If one tries to construct a homogenization corrector the logical thing is to concatenate the two whole-space correctors corresponding to the top and bottom ensembles and then to correct for the mismatch of the fluxes at the boundary and at the same time correct for the jump. If one tries to do this naively one obtains an overdetermined problem with two equations for the corrections and four boundary conditions. However, if instead of concatenating restricted whole-space correctors, we use Dirichlet correctors that already match up at the boundary of the half-spaces, then we would only have to correct the fluxes, which is probably possible using the methods given in Chapter 2. This could be a future work.

0.2 Pathwise Treatments of Singular SPDEs

The purpose of the second part of this thesis is to construct a stable solution operator for a quasilinear parabolic initial value problem in $1+1$ dimensions that is driven by a rough right-hand side. In particular, we are interested in obtaining a solution $W \in C^{\alpha}(\mathbb{R}^2_+)$ of

$$\partial_2 W - a(W)\partial_1^2 W + W = f \qquad \text{in} \quad \mathbb{R}^2_+ \tag{10a}$$

$$W = W_{int} \qquad \text{on} \quad \partial\mathbb{R}^2_+, \tag{10b}$$

where $a : \mathbb{R} \to [\lambda, 1]$ for $\lambda > 0$ is regular and, for some $\alpha \in (\frac{2}{3}, 1)$, we have that $W_{int} \in C^{\alpha}(\mathbb{R})$ and $f \in C^{\alpha-2}(\mathbb{R}^2)$. By "$C^{\alpha}(\mathbb{R}^2)$" when $\alpha > 0$ we are always referring to the parabolic Hölder space, which is defined in terms of the intrinsic parabolic metric. We give the definition of the negative Hölder space $C^{\alpha-2}(\mathbb{R}^2)$ in Definition 2.3. To simplify matters we assume that f and W_{int} are 1-periodic in space and f is also 1-periodic in time. For us "periodic" always means "1-periodic" and we, therefore, drop the "1". While in this thesis our arguments are worded in terms of one spatial dimension, our analysis carries through to the general case of $1+d$ dimensions.

As already mentioned above, in this thesis we do not finish the analysis of (10), which as we will see below requires a fixed point argument, but instead conclude after having proven all of the necessary ingredients for the argument. The actual fixed point argument and the extension of our results to the quasilinear case will be in a future work and will require a "small data" assumption. The determination of what should be proved to prepare for the fixed point argument comes from [47], in which they treat the space-time periodic version of the problem we are interested in (without a massive term); in particular, the fixed point argument that we plan to perform with the ingredients proved in thesis has a precursor in their Theorem 2. In general, the contents of this thesis should be viewed as the ingredients towards a generalization of the recent work by Otto and Weber [47] in which the authors treat the space-time periodic version of (10) (without the massive term). In [47], since they assume space-time periodicity, the authors choose to use elliptic notation in the sense that they call the time direction the "x_2-direction". While we are now interested in the initial value problem, in keeping with their convention we also choose to denote the time direction as the x_2-direction.

The issue that makes the analysis of (10) nonclassical is the rough driver on the right-hand side and also a lack of regularity for the initial condition. In particular, standard theory suggests that these inputs limit the regularity of the solution W to the extent that the nonlinear term has no classical meaning. To illustrate this, notice that the best regularity that we can expect for a solution of (10) is $W \in C^{\alpha}(\mathbb{R}^2_+)$, which translates into $\partial_1^2 W \in C^{\alpha-2}(\mathbb{R}^2_+)$ and $a(W) \in C^{\alpha}(\mathbb{R}^2)$. Since the Hölder regularities of these terms add up to something negative, i. e. $(\alpha-2)+\alpha<0$, there is no classical interpretation of the product $a(W)\partial_1^2 W$. Due to this issue, giving a solution theory for (10) first consists of constructing the singular product $a(W)\partial_1^2 W$, thereby making sense of the equation (10), and then making sure that the induced solution operator, which must still be constructed, satisfies some continuity properties. As we will discuss, it turns out that the stability of the solution operator can be guaranteed by certain analytic conditions on the singular product.

In the current contribution we would like to capitalize on the results of Otto and Weber by, loosely speaking, viewing the solution W of (10) as a perturbation of the solution of the space-time periodic problem considered by Otto and Weber. In order to see that this is a natural strategy we recall the classical method for obtaining solutions to quasilinear equations of the form (10). Classically, when the forcing and initial condition have a higher regularity, and one is not concerned with the definition of the nonlinear term, the solution W is obtained via a fixed point argument that relies on a solution theory for the linear problem associated to (10); i. e.

$$(\partial_2 - a\partial_1^2 + 1)W = f \qquad \text{in} \quad \mathbb{R}^2_+ \tag{11a}$$

$$W = W_{int} \qquad \text{on} \quad \partial\mathbb{R}^2_+, \tag{11b}$$

where $a \in C^\alpha(\mathbb{R}^2_+)$ and should be thought of as $a = a(W)$. In particular, to obtain a solution for (10) one looks for a fixed point of the map:

$$\overline{U} \mapsto a(\overline{U}) \mapsto W, \tag{12}$$

where the function $\overline{U}$, sampled from the expected solution space of the quasilinear problem, is mapped to $a = a(\overline{U})$, where the "a" on the left-hand side refers to the coefficients in (11) and the "a" on the right-hand side refers to the nonlinearity in (10), and this then gets mapped to the solution of (11) with coefficients $a = a(\overline{U})$. The desired fixed point of (12) is shown to exist by using the standard Schauder estimates and a smallness assumption on the data to show that (12) is a contraction.

Returning to our low-regularity setting we would like to emulate the argument from the previous paragraph, which means that our analysis of (10) hinges on our treatment of the associated linear problem (11). The point is that on the level of the linear problem it is very natural to take a perturbative ansatz for W in the sense that the solution W of (11) will be constructed as $W = U + u$, where $u \in C^\alpha(\mathbb{R}^2)$ solves

$$(\partial_2 - a^{ext}\partial_1^2 + 1)u = f \qquad \text{in} \quad \mathbb{R}^2 \tag{13}$$

with a^{ext} denoting some extension of a to $\mathbb{R}^2$, and $U \in C^\alpha(\mathbb{R}^2_+)$ solves

$$(\partial_2 - a\partial_1^2 + 1)U = 0 \qquad \text{in} \quad \mathbb{R}^2_+ \tag{14a}$$

$$U = W_{int} - u \qquad \text{on} \quad \partial\mathbb{R}^2_+. \tag{14b}$$

The solution theory for (11) is then obtained as the combination of our treatments of (13) and (14).

Even with the above splitting of W in-hand, however, W is *not* really a perturbation of the periodic problem treated by Otto and Weber. We see this already on the level of the linear problem (11) in which we would eventually like to make the identification $a := a(W^{ext})$, where W^{ext} is some extension of W, and, therefore, lose periodicity of the coefficients $a \in C^\alpha(\mathbb{R}^2)$ in x_2-direction. While this loss of periodicity in no way affects the construction of the singular product used in (13), which we can borrow from the treatment of the associated problem in [47], the construction of the solution operator found in Proposition 1 of [47] has to be slightly modified. In particular, in Proposition 1 of this contribution we essentially replace their periodicity in the x_2-direction by estimates for the heat semigroup that hold thanks to the presence of a massive term in (13). While the modification of the analytic arguments of Otto and Weber for the treatment of (13) constitutes much of the effort invested here, really the main new contribution in this work is the treatment of the the initial value problem (14). The point here is that, while in order to define the singular product $a^{ext}\partial_1^2 u$ in (13) it is necessary to take a pathwise approach that relies on stochastic ingredients, thanks to the rate of the blow-up of the heat-kernel at initial time the product $a\partial_1^2 U$ in (14) can actually be defined using classical methods. Using bounds for the heat semigroup it is then possible to treat the problem (14) using classical versions of techniques used for (13).

We return now to the issue of the construction of the singular products in (10), (11), (13), and (14). Once again, we try to piggy-back off of the strategy of Otto and Weber, which relies heavily on a concept that they introduce called "modelledness". To motivate this concept, notice that in the classical setting, i. e. $f \in C^\alpha(\mathbb{R}^2)$ and $W_{int} \in C^{\alpha+2}(\mathbb{R})$ for $\alpha > 0$, due to the local dependence of the solution W of (11) on the coefficients, we expect that W will "locally look like" the solution of (11) with frozen coefficients. In particular, letting $\overline{V}(\cdot, a_0) \in C^\alpha(\mathbb{R}^2_+)$ solve

$$(\partial_2 - a_0\partial_1^2 + 1)\overline{V}(\cdot, a_0) = f \qquad \text{in} \quad \mathbb{R}^2_+ \tag{15a}$$

$$\overline{V}(\cdot, a_0) = W_{int} \qquad \text{on} \quad \partial\mathbb{R}^2_+, \tag{15b}$$

for $a_0 \in [\lambda, 1]$, we think that locally, say around the point $x_0 \in \mathbb{R}^2$, the fluctuations of W solving (11) will look like those of $\overline{V}(\cdot, a(x_0))$. In order to turn this into a global condition one can use the concept of "modelledness" as defined in [47]. Roughly speaking, we say that W is *modelled after* the family $\{\overline{V}(\cdot, a_0)\}$ parameterized by $a_0 \in [\lambda, 1]$ if at any point it behaves locally like some member of this family in a $C^{2\alpha}$-way. Using this intuition suggests that the correct space in which to perform the fixed point argument for our rough analogue of (12) is up to some subtleties the function space $\{\overline{U} \in C^\alpha(\mathbb{R}^2_+) \,|\, \overline{U} \text{ is "modelled after" } \overline{V}(\cdot, a_0)\}$.

To solidify our notation and also because it is interesting to comment on the definition of “modelledness" within the framework of the previous literature, we right away give a formal definition, which is the same as that used in [47]. It is:

Definition 2.1 (Modelledness)**.** *Let* $\alpha \in (\frac{1}{2}, 1)$ *and* $\Omega \subseteq \mathbb{R}^2$. *Assume that for some* $I \in \mathbb{N}$ *we have functions* $(\overline{V}_1(\cdot, a_0), ..., \overline{V}_I(\cdot, a_0))$ *such that* $\overline{V}_i : \Omega \times \mathbb{R} \to \mathbb{R}$. *A function* $W : \Omega \to \mathbb{R}$ *is said to be modelled after* $(\overline{V}_1(\cdot, a_0), ..., \overline{V}_I(\cdot, a_0))$ *on* Ω *according to functions* $(a_1, ..., a_I)$ *and* $(\sigma_1, ..., \sigma_I)$ *with* $\sigma_i, a_i \in C^\alpha(\Omega)$ *if there exists a function* ν *such that*

$$M_\Omega := \sup_{x \neq y; x, y \in \Omega} \frac{1}{d^{2\alpha}(x, y)} \qquad (16)$$
$$|W(y) - W(x) - \sigma_i(x)(\overline{V}_i(y, a_i(x)) - \overline{V}_i(x, a_i(x))) - \nu(x)(y - x)_1|$$

is finite. We emphasise that here we use the Einstein summation convention that repeated indices are summed over.

We say that a function W *is trivially modelled after* $(\overline{V}_1(\cdot, a_0), ..., \overline{V}_I(\cdot, a_0))$ *and functions* $(a_1, ..., a_I)$ *if each* $\sigma_i = 0$. *Since* $\alpha \in (\frac{1}{2}, 1)$ *this is equivalent to the condition that* $W \in C^{2\alpha}(\Omega)$, *but additionally specifies a choice of model.*

If one compares our Definition 2.1 with Definition 1 of [47] it becomes apparent that we have added the extra dependence of the modelling on the domain Ω. This is because in our arguments, due to the perturbative ansatz that we take, we work with functions/ distributions defined on either all of $\mathbb{R}^2$ or only on $\mathbb{R}^2_+$. Since the domain is always clear we end up always dropping the subscript Ω on M_Ω.

We now interpret Definition 2.1. A quick inspection connects Definition 2.1 with our previous informal description of modelledness. In particular, the evaluation $a_0 = a(x)$ in Definition 2.1 can be motivated by recalling that in a neighborhood of $x_0 \in \mathbb{R}^2$ we want to think of W as “locally looking like" $\overline{V}(\cdot, a(x_0))$. Furthermore, the modulating functions σ_i in (16) come about since we are interested in Hölder regularity; in particular, for a fixed $x \in \mathbb{R}^2$ we would like to heuristically think of σ_i as being the derivative of W with respect to $V_i(\cdot, a_0)$. The ideological difference between M_Ω and the standard $C^{2\alpha}$-seminorm is that, instead of measuring closeness of increments to a basis of degree-1 polynomials (as one does in the classical setting), we measure it to the families of functions $\left(\left\{\overline{V}_1(\cdot, a_0)\right\}, ..., \left\{\overline{V}_I(\cdot, a_0)\right\}\right)$ with the feature that the actual basis of functions that we compare to, i. e. the specific members of the families used for $x \in \Omega$ in (16), varies according to the a_i. Due to the restriction that $\alpha > \frac{1}{2}$ we include the term $\nu(x)(x - y)_1$ to make sure that every $C^{2\alpha}$ function is trivially modelled. A second consequence of the restriction $\alpha > \frac{1}{2}$ is that if there exists a function ν making M_Ω finite, then this ν is unique.

After this discussion on modelling and our explanation for why the solution W of (11) should be modelled after the $\overline{V}(\cdot, a_0)$ solving (15), we will simply state that we expect the solution u of (13) to be modelled after $v(\cdot, a_0)$, the unique C^α- solution of

$$(\partial_2 - a_0\partial_1^2 + 1)v(\cdot, a_0) = f \qquad \text{in} \quad \mathbb{R}^2, \tag{17}$$

and that U solving the initial value problem (14) will be modelled after $V(\cdot, a_0) \in C^\alpha(\mathbb{R}^2_+)$, the solution of

$$(\partial_2 - a_0\partial_1^2 + 1)V(\cdot, a_0) = 0 \qquad \text{in} \quad \mathbb{R}^2_+ \tag{18a}$$

$$V(\cdot, a_0) = V_{int} \qquad \text{on} \quad \partial\mathbb{R}^2_+, \tag{18b}$$

with the identification $V_{int} = W_{int} - u$. This postulated modelling of u and U is not quite consistent with our previous intuition that $W = u + U$ should be modelled after $\overline{V}(\cdot, a_0)$. In particular, to show that W has the correct modelling it is necessary to have $V_{int} = W_{int} - v(\cdot, a_0)$ in (18), which essentially means that we must swap out u for $v(\cdot, a_0)$. Using bounds for the heat semigroup and the modelling of u after $v(\cdot, a_0)$, we perform this swapping in Theorem 1. This is one of the more significant new arguments in this contribution.

While it plays no significant role in the analysis, the reader familiar with [47] will notice that, opposed to the equation (62) in [47] solved by $v^{OW}(\cdot, a_0)$, our (17) has a massive term and we do not project the forcing f onto the space of mean-free functions. Since f is assumed to be space-time periodic, the solution $v(\cdot, a_0)$ of (17), obtained via Fourier methods, is also periodic and, thanks to the massive term, is the unique C^α-solution of (17) on $\mathbb{R}^2$. We notice also that in some sense the two changes to (62) of [47] that we make here compensate each other. In particular, in [47] they must include the projection since testing their equation with the constant function 1 yields

$$0 = \int_{\mathbb{T}^2} 1 \cdot (\partial_2 - a_0\partial_1^2)\, v^{OW}(\cdot, a_0)\, \mathrm{d}x = \int_{\mathbb{T}^2} Pf\, \mathrm{d}x$$

after two integration by parts. In contrast, in our setting thanks to the massive term the same calculation instead yields

$$\int_{\mathbb{T}^2} v(\cdot, a_0)\, \mathrm{d}x = \int_{\mathbb{T}^2} f\, \mathrm{d}x. \tag{19}$$

Returning to our reason for introducing modelledness, with this formalism in place the construction of the singular products can be broken down into a two step process. Using the notation $\tilde{\cdot}$ in order to denote even-reflection across the axis $x_2 = 0$ and $\overline{V}^{ext}(\cdot, a_0) = (v + \tilde{V}_{W_{int}-v(\cdot,a_0)})(\cdot, a_0)$, where the subscript on $\tilde{V}$ specifies the initial condition, we summarize these steps as:

***Step 1a**:* One uses stochastic ingredients and bounds for the heat semigroup to almost surely (for certain assumptions on the random distribution f) define families of *offline* reference products $\left\{\overline{V}^{ext}(\cdot, a_0) \diamond \partial_1^2 \overline{V}^{ext}(\cdot, a_0')\right\}$ and $\left\{\overline{V}^{ext}(\cdot, a_0) \diamond \partial_1^2 v(\cdot, a_0')\right\}$ indexed by a_0 and a_0' that satisfy some analytic estimates.

or

***Step 1b**:* One uses only the bounds for the heat semigroup to, for any $a \in C^{\alpha}(\mathbb{R}^2)$, show that $a\partial_1^2 \tilde{V}(\cdot, a_0)$ is classically defined as a distribution and also satisfies the desired analytic estimates.

***Step 2**:* With the reference products from Steps 1a or 1b in-hand one uses the analytic conditions they satisfy to transfer the definition of the singular product onto the functions we are actually interested in by using the appropriate modelling. This is done in two "reconstruction lemmas", which are called reconstruction lemmas because in the theory of Otto and Weber they play the role of Hairer's Reconstruction Theorem (Theorem 3.10 in [35]) in [35].

We remark that only Step 1a depends on probabilistic arguments, while Steps 1b and 2 may be proved in entirely deterministic ways. Notice that we have started to use " $\diamond$ " to emphasize the singular nature of products.

To give an example of the above discussed strategy we specifically consider the product $a^{ext} \diamond \partial_1^2 u$ in (13). We treat the problem (13) in Proposition 1, in which we assume that we have access to a well-behaved family of reference products $\left\{a^{ext} \diamond \partial_1^2 v(\cdot, a_0)\right\}$. These products are well-behaved in the sense that one has $C^{2\alpha-2}$ control over commutators involving regularization and multiplication. In particular, letting ψ_T be a smooth convolution kernel at scale T (discussed in detail in Section 2.1) one assumes that the L^{∞}-norm of the commutator (and two parameter derivatives) scales as

$$\sup_{a_0 \in [\lambda, 1]} \sup_{T \leq 1} (T^{\frac{1}{4}})^{2-2\alpha} \|[a^{ext}, (\cdot)_T] \diamond \partial_1^2 v(\cdot, a_0)\|_2 \lesssim 1, \tag{20}$$

where the subscript on the L^{∞}-norm indicates the parameter derivatives. Taking these reference products as input, we then use the *second* reconstruction lemma (Lemma 4) to swap $v(\cdot, a_0)$ for u, which is possible thanks to the modelling of u after $v(\cdot, a_0)$ and yields the desired singular product $a^{ext} \diamond \partial_1^2 u$. Since here we have assumed the reference products as input into Proposition 1, we have not needed to apply either Steps 1a or 1b.

To do as much work as possible towards treating the quasilinear problem in an upcoming work here, we also construct the reference products $\left\{\overline{V}^{ext}(\cdot, a_0)) \diamond \partial_1^2 v(\cdot, a_0')\right\}$ and post-process these reference products via both reconstruction lemmas; this is done in Sections 6 and 7. In particular, from our analysis of the map (12) and our choice to treat (11) with the perturbative ansatz, we know that in order to treat the quasilinear problem we should think of $a = a(W^{ext})$ with W^{ext} being some extension of the solution W of (11) with some admissible coefficients. Since the solution W is constructed as $W = u + U$, the extension that we take will be $W^{ext} = u + \tilde{U}$, which is modelled after $\overline{V}^{ext}(\cdot, a_0)$. In order to apply Proposition 1 with the coefficients $a^{ext} = a(W^{ext})$ we know from the previous paragraph that it is first necessary to post-process the reference products $\left\{\overline{V}^{ext}(\cdot, a_0)) \diamond \partial_1^2 v(\cdot, a_0')\right\}$ to give a family of well-behaved products $\left\{a(W^{ext}) \diamond \partial_1^2 v(\cdot, a_0)\right\}$ that satisfy (20). Since W^{ext} is modelled after $\overline{V}^{ext}(\cdot, a_0))$ and the regularity assumptions for a from (10) are tailored so that $a(W^{ext})$ is then also modelled after $(v + \tilde{V})(\cdot, a_0))$, the first reconstruction lemma allows us to swap $\overline{V}^{ext}(\cdot, a_0)$ for $a(W^{ext})$ in the reference products. Notice that heuristically we have a map

$$f(\omega) \overset{\psi}{\mapsto} \left\{\overline{V}^{ext}(\cdot, a_0)) \diamond \partial_1^2 v(\cdot, a_0)\right\} \overset{S}{\mapsto} a(W^{ext}) \diamond \partial_1^2 u \textit{ in } (13), \tag{21}$$

which is defined only for almost every realization of the noise (in a *pathwise* sense), where the first mapping requires probabilistic ingredients and the second is entirely deterministic in the sense that it represents the application of the reconstruction lemmas. We apply Theorem 1 in the case that $a^{ext} = a(W^{ext})$ in Theorem 2, which is proven in Section 7; this is really the main ingredient for the fixed point argument needed to treat the quasilinear problem.

The general framework that we use to construct the singular products and the reconstruction lemmas are taken essentially directly from [47]. We, furthermore, borrow their construction of the offline reference products $v^{OW}(\cdot, a_0) \diamond \partial_1^2 v^{OW}(\cdot, a_0')$ in order to obtain the singular products $v(\cdot, a_0) \diamond \partial_1^2 v(\cdot, a_0')$; here, the point is simply that the massive term in (17) and the lack of the projection applied to the forcing does not interfere with their arguments. We are thus able to avoid any new probabilistic arguments in the current contribution. It is then, however, still necessary to perturb these reference products in order to obtain all those needed to treat the initial value problem (10). Notice, in particular, that this perturbation of the reference products is consistent with the perturbative ansatz for W in the sense that we also think of $\overline{V}(\cdot, a_0)$ as a perturbation of $v(\cdot, a_0)$ by $V_{W_{int}-v(\cdot,a_0)}(\cdot, a_0)$. As we have already alluded to above, it is possible to perturb the offline products by using estimates for the heat semigroup.

Before moving on to review some previous literature, we would like to draw attention to the fact that, while here we choose to pursue a solution theory for (10) by using the perturbative method described in detail above, it would most likely also be possible to use an approach that more directly mimics the work of Otto and Weber. In particular, the arguments of Otto and Weber for the well-posedness of the periodic problem (10) rely on a regularization of f via convolution with ψ_T in which they then pass to the limit $T \to 0$. One of the main difficulties in the treatment of the initial value problem is the fact that many of the results in [47] are phrased in terms of convolution with ψ_T, which presupposes that one starts with a distribution defined on all of $\mathbb{R}^2$. In this contribution we bypass this issue by extending functions to negative times in various ways. However, since the derivatives in the nonlinearity in (10) are only in the x_1-direction, it seems that it might be possible to simply convolve in one direction, which would make the method more compatible with initial value problems. This, however, would call for a modification of the arguments for the construction of the offline products and the reconstruction lemmas in [47] and also complicate the treatment of the regularized problems in Proposition 1.

While in this thesis we work within the framework introduced by Otto and Weber, there are various other theories available for similar problems (e. g. Hairer's theory of regularity structures and the theory of paracontrolled distributions by Gubenielli, Imkeller, and Perkowski [42]). All of the methods have certain underlying themes that run throughout and that can be identified in the above discussion. One of the most prominent themes is the sharp separation between probabilistic and deterministic methods in the arguments. To see this separation represented in the literature, one can actually go back to the theory of rough paths developed by Lyons to treat singular SDEs (see the original work of Lyons [41] or the overview by Friz and Hairer [24], whose exposition we follow here).

In developing his theory of rough paths Lyons was interested in providing a constructive solution theory for SDEs of the form

$$dY_t = f_0(Y_t) + f(Y_t)dX_t \tag{22}$$

for two functions/ 1-forms f_0 and f in the case that the path X is irregular; an important example is the case that the equation is driven by a Brownian path $X = B$. Before the theory of rough paths it was known that (22) is not well-posed in a pathwise sense as there is no metric on the space of Brownian paths making the Itô map associated to (22) continuous [2]. To remedy the situation, Lyons notice that instead of considering X simply as a path, he could instead view it in conjunction with its iterated integrals $\mathbb{X}$ (defined below) and, thereby, factor the Itô map into two pieces as

$$X(\omega) \overset{\psi}{\mapsto} (X, \mathbb{X})(\omega) \overset{S}{\mapsto} Y(\omega). \tag{24}$$

Here, the first map ψ is measurable and the second map, called the *Itô-Lyons map*, is continuous in both the realization of the noise and the initial condition if we endow the space of paths enriched with their iterated integrals, called *rough paths*, with a certain "p-variation" rough path metric.

To tie this back to the work of Otto and Weber, notice that we should be comparing (21) and (24). In both (21) and (24) the definition of the map ψ requires probabilistic tools, while the construction of S is completely deterministic. Also, in both cases the maps ψ are "universal" in the sense that they do not depend on the form of the equation (13) or (22). While in (21) the map ψ almost surely yields offline products, in (24) a realization of the noise is almost surely mapped to itself coupled with its iterated integrals $\mathbb{X}$. As the name may indicate, for $n \in \mathbb{N}$ and $i_1, ..., i_n \in \{1, .., d\}$, the iterated integrals of order n are given by

$$X_{s,t}^{n, i_1 \dots i_n} = \int_s^t \int_s^{r_n} \dots \int_s^{r_2} \mathrm{d}B^{i_1}(r) \dots \mathrm{d}B^{i_n}(r). \tag{25}$$

In the case of Brownian motion ($X = B$) and $n = 2$ there is a one-parameter family of choices, two which are the Itô and Stratonvich integrals, for the integrals (25) that are obtained via the convergence of corresponding Riemann sums in probability.

In contrast to the probabilistic nature of ψ, in both (21) and (24) the map S is defined in a completely deterministic way. Since we have already

[2]In particular, it had been shown that there does not exist a separable Banach space $\mathcal{B}$ such that any Brownian path almost surely is contained in $\mathcal{B}$ and the map

$$(f, g) \mapsto \int_0^{\cdot} f(t)\dot{g}(t)\,\mathrm{d}t, \tag{23}$$

which maps smooth paths f and g to a continuous path, extends to a continuous mapping from $\mathcal{B} \times \mathcal{B} \to C([0,1])$. Since for $Y \in \mathbb{R}^2$ the map (23) for $f = B^1$ and $g = B^2$ solves the equation $\dot{Y}^1 = \dot{B}^1$ and $\dot{Y}^2 = Y^1 \dot{B}^2$, this shows that the Itô map, which maps the realization of the noise to the solution of (22), cannot in general be continuous.

discussed the map S in (21) and the construction of the solution maps for (11) and (10) in detail, we now only comment on S in (24). The construction of the solution map S in (24) requires a definition for the stochastic integral

$$\int_0^t f(Y_s)\,\mathrm{d}X_s, \tag{26}$$

which for regular f is only classically possible for arbitrary paths X and Y if on the Hölder scale the regularity of X and Y (call it $\alpha > 0$) satisfies $2\alpha > 1$. Lyons, however, observed that intuitively since Y solves (22), its fluctuations should at small scales look like those of X. As he has previously determined that the map

$$(X, \mathbb{X}) \mapsto \int_0^t f(X_s)\,\mathrm{d}X_s \tag{27}$$

from the space of rough paths with iterated integrals up to certain order with some (p-variation) metrics is continuous, he is then able to define (26) for a large enough class of paths Y that "locally look like" X such that he can perform a Picard iteration to obtain the solution of (22). Depending on the regularity of X it is necessary to have information on the iterated integrals up to different orders; in particular, the rougher X the more information one needs. In the case of Brownian motion it is only necessary to have access to the second order iterated integrals.

The theory of Lyons was enriched by the subsequent contribution of Gubinelli, who introduced the concept of a *controlled rough path* [33]. A key difference between the work of Gubinelli and Lyons is that Gubinelli fixes the path which Y locally looks like. In particular, for $\alpha \in (\frac{1}{3}, \frac{1}{2})$, one says that the path Y is controlled by the path Z if the increments of Y are approximated by those of Z in a $C^{2\alpha}$ way; i. e. there exists $\sigma \in C^\alpha$, sometimes called the *Gubinelli derivative*, such that

$$|Y_t - Y_s - \sigma(s)(Z_t - Z_s)| \lesssim |t-s|^{2\alpha} \tag{28}$$

holds. Notice that "modelledness" *à la* Otto and Weber above is essentially a higher dimensional version of Gubinelli's "controlledness" with the additional caveat that in "modelledness" one controls with a family of functions parameterized by $a_0 \in \mathbb{R}$, which is then modulated by the function $a \in C^\alpha$. Ignoring the 1-form f, Gubinelli then shows that if the integral in (26) is defined for $Y = Z$ such that a "commutator estimate" of the form

$$\left| \int_s^t Z_r\,\mathrm{d}X_r - Z_s(X_t - X_s) \right| \lesssim |t-s|^{2\alpha} \tag{29}$$

holds, then this suffices to define the integral in (26) for Y controlled by Z such that

$$\left|\int_s^t Y_r \,\mathrm{d}X_r - Y_s(X_t - X_s) - \sigma(s)\int_s^t (Z_r - Z_s)\,\mathrm{d}X_r\right| \lesssim |t-s|^{3\alpha}. \tag{30}$$

Gubinelli's approach is to encode the desired singular integral as the solution of an algebraic problem with some analytic constraints. Notice that the condition (29) corresponds to our estimates of the form (20), where we have replaced integration over time by convolution with a smooth kernel. In fact, (30) also has corresponding statements in the reconstruction lemmas that we borrow from Otto and Weber.

After the work of Lyons and the results of Gubenelli [33] for SDEs, the theory was extended to treat singular SPDEs, where the roughness in the noise is contained also in spatial variables. As explained in the introduction of [42], previously there had been various contributions in which singular SPDEs had been considered, but in these the irregularity in the noise has always been time-like, which essentially allowed for the reduction to the standard rough path framework. The issues that arise when considering singular SPDEs with noise also containing some spatial roughness are different than those for singular SDEs. In particular, as we have already seen in our discussion of the current work, the problem becomes that one cannot expect enough spatial regularity for the solution to classically make sense of the nonlinear terms appearing in the equation. Taking inspiration from the the work of Lyons and Gubinelli for SDEs, the issue of well-posedness for singular SPDEs was developed within three frameworks:

1) The theory of regularity structures by Hairer, which was first used to treat the KPZ equation [34] and then extended to a much more general framework in [35].

2) The theory of paracontrolled distributions developed by Gubinelli, Imkeller, and Perkowski in [42], which combines the results of Gubinelli in [33] with the paraproduct and corresponding paradifferential calculus introduced by Bony. Notice that the tools developed by Bony, relying on Littlewood-Paley theory, are Fourier analytic in nature.

3) The "parametric approach" of Otto and Weber, which we are interested in extending here.

We will not discuss [47] again, but focus instead (in much less detail) on the first of the two approaches listed above. The main idea of the theory of regularity structures is to, for a given singular SPDE, adapt one's notion of regularity by replacing the standard Taylor expansion by instead expanding in terms of nonlinear combinations of the noise. The reasoning for this resonates with much of the intuition already discussed in this introduction; in particular, the mindset is that there is no reason to believe that a solution will behave at small scales like a polynomial, but instead that it will behave like the solution of the associated linear problem, which is represented by convolving the noise with the appropriate Green's function. For a given singular SPDE, this then leads to the replacement of the standard Hölder spaces by analogous spaces defined in terms of a new basis of distributions, which now may have non-integer or even negative homogeneity. These new spaces are called spaces of *modelled distributions* (see Definition 3.1 in [35]). The singular SPDE one is trying to solve is then encoded in terms of a fixed point problem within the space of *modelled distributions* in which the necessary nonlinear terms have also been defined via a reconstruction lemma (see Theorem 3.10 of [35]) [3]. Hairer then also shows that the solutions obtained via this fixed point argument correspond to those obtained by passing to the limit in a sequence of classical solutions to regularized problems under suitable renormalization (subtraction of possibly infinite counter terms). As we have said above, Hairer first treated the KPZ equation within this framework and then developed it as a general strategy in [35]. There is a gentle introduction to the basics of the theory in the latter part of the notes [24] by Friz and Hairer.

The equations treated within the theories of Hairer and Gubinelli, Imkeller, and Perkowski satisfy an important subcriticality condition. To set-up for the explanation of this assumption, we write their equation as

$$\mathcal{L}Y = F(Y, f) \qquad \text{on} \quad \mathbb{R}^d, \tag{31}$$

where $\mathcal{L}$ is a linear operator and the right-hand side $F(Y, f)$ contains the nonlinearity and depends on the noise f. In this notation the Green's function corresponding to the associated linear operator $G_{\mathcal{L}}$, which we have mentioned heuristically in the previous paragraph, is a solution to $\mathcal{L}G_{\mathcal{L}} = \delta_0$. As explained by Hairer in [35], he must assume that the nonlinearity in F is subcritical in the sense that if he rescales the equation (31) in such a way that $\mathcal{L}Y_\epsilon$ and the noise f_ϵ are invariant then formally the nonlinear term will disappear at small scales. As an instructional example, he considers the KPZ equation posed on $S^1 \times \mathbb{R}_+$ with additive space-time white noise with

[3] One nice thing about the theory of Otto and Weber is that their reconstruction lemmas avoid the use of wavelets, which are heavily used in the proof of Hairer's result.

the identifications $\mathcal{L} = \partial_t - \partial_x^2$ and $F(Y, f) = (\partial_1 Y)^2 + f$. In this example setting $Y_\epsilon(x,t) = Y(\epsilon x, \epsilon^2 t)$ and $f_\epsilon(x,t) = \epsilon^{\frac{3}{2}} f(\epsilon x, \epsilon^2 t)$ yields that Y_ϵ solves

$$(\partial_t - \partial_x^2) Y_\epsilon = \epsilon^{\frac{1}{2}} (\partial_x Y_\epsilon)^2 + f_\epsilon.$$

Due to the prefactor $\epsilon^{\frac{1}{2}}$ the nonlinear term formally scales out as $\epsilon \to 0$. As pointed out in [46], in which the theory of Otto and Weber is extended to the case $f \in C^{\alpha-2}(\mathbb{R}^2)$ for $\alpha > 0$, this restriction of subcriticality corresponds to the restriction $\alpha > 0$. As is explained in detail in the introduction of [46], decreasing α forces one to work with higher-order/ nested versions of the concept of modellness.

While both the strategy of regularity structures and also the framework of paracontrolled distributions were originally developed for semilinear equations, following the work of Otto and Weber both methods were extended to the quasilinear case. In particular, this extension within the framework of regularity structures can be found in [29] by Gerencsér and Hairer and for similar results in the setting of paracontrolled distributions the reader can reference [25] by Furlan and Gubinelli. There is also the contribution [12] by Bailleul, Debussche, and Hofmanová, in which they essentially treat the same problem as Otto and Weber in [47], but as an initial value problem. Their results are included within those obtained by Furlan and Gubinelli, but avoid the extension of the theory contained in [25] by transforming the specific equation they are working with into a semilinear equation, which can then be treated with the already available results in [42]. As observed in the introduction of [25], due to their use of a transformation the method is quite rigid in the sense that it cannot, e. g. be adapted the case of $1 + d$ dimensions with $4 \geq d \geq 2$ and a matrix-valued diffusion coefficient. For a more detailed discussion of the methods used and how these works relate to one another the reader can consult the introductions of [25] and [28].

To draw this overview back to the current work, we should emphasize that in the frameworks of Hairer and Gubinelli, Imkeller, and Perkowski the issue of initial data was incorporated from the get-go. In particular, this is discussed in Subsection 7.2 of Hairer's original article on regularity structures [35] and is also present in the original article on paracontrolled distributions by Gubinelli, Imkeller, and Perkowski [42]. In each of these works and also in [12], the authors obtain local in time solutions, where the time cut-off depends on the realization of the noise. In contrast, thanks to the massive term in our parabolic operator, in the current contribution we obtain global in time solutions. Along the lines of enforcing boundary data, we would also like to mention that there is the recent work of Gerencsér and Hairer in which they consider the initial-boundary value problem. This work, in particular, inspired the perturbative ansatz used in the current contribution.

Just as in the first part of the introduction, we would like to mention some future directions. Since the problem that we consider in the second part of the thesis is subject to many assumptions, there are many possible generalizations. As already mentioned above, one interesting project might be to consider the same problem as in this thesis, but convolve (regularize) in just one direction. It would, of course, also be possible to consider the initial-boundary value problem. Another interesting possible extension would be to drop the periodicity assumption on the noise.

0.3 Connections and Underlying Themes

While the two parts of this thesis are disconnected enough to deserve their own introductions, they are very similar in terms of themes and techniques used. On a certain level it is obvious that this is the case since the equations considered in both sections are of the same class (elliptic or parabolic), which means that in both parts we have access to the same estimates and techniques. The connection, however, is deeper in the sense that in both parts of the thesis the problem is one of regularity for solutions to heterogenous coefficient operators, which is overcome by transferring results from corresponding constant coefficient operators. While in the first part of the thesis the problem is explicitly formulated in terms of regularity, in the second part of the thesis the regularity problem is hiding in the definition of the singular product "$a(u) \diamond \partial_1^2 u$". In both parts the issue of low-regularity for heterogeneous solutions is overcome by "transferring regularity" from a constant coefficient operator; in the first part of thesis this is done on large scales through the homogenized operator using an estimate for the homogenization error and in the second part of the thesis this is done by transferring the singular product via the reconstruction lemmas.

While both parts of this thesis rely on methods from classic elliptic regularity theory, the two parts actually rely on different classical methods. To flesh this out, recall that, as we have talked about above, Part I of this thesis is based on iterative De Giorgi type arguments in the form of a Campanato iteration that is adapted to the homogenization setting. Of course, an argument of this type implicitly uses the equivalence of Hölder and Campanato spaces, which allows us to work in Sobolev spaces and rely on energy methods. The second part of this thesis differentiates itself from the first in that we work exclusively in Hölder spaces. In this setting we capitalize on the

Krylov-Safanov approach to Schauder theory, which distinguishes itself in that it does not rely on the Green's function representation of the solution. This is important because in our argument we apply it not to solutions, but "approximate solutions".

To conclude this introduction, there is one last common thread to mention, which is that, while both results are probabilistic in nature, the arguments in both parts split naturally into separate deterministic and probabilistic steps. In both cases the application of the regularity theory as described in the previous paragraph is part of the deterministic portion, which takes as input the existence of a sublinear corrector in Part I or a well-behaved family of offline/ reference products in Part II. These inputs are shown to exist almost surely using probabilistic arguments for certain assumptions on either a random ensemble of coefficient fields or for a random Gaussian forcing. The arguments in this thesis are mainly deterministic in nature, resting on probabilistic results of others; e. g. the construction of the whole-space homogenization corrector in [31] or the construction of offline products in [47].

0.4 Notation

Throughout this thesis we will consistently use the following notational conventions:

We denote the number of spatial dimensions by d. The upper half-space $\{x \in \mathbb{R}^d : x_d > 0\}$ is indicated as $\mathbb{R}^d_+$ and the lower half-space is $\mathbb{R}^d_- = -\mathbb{R}^d_+$. By B_r, we denote the ball of radius r centered at the origin. The upper half-ball of radius r centered at the origin, i. e. $B_r \cap \mathbb{R}^d_+$, is denoted by B_r^+ and, correspondingly, the notation B_r^- is used for the set $-B_r^+$. By $B_r(x)$ we denote the ball of radius r with center $x \in \mathbb{R}^d$. For two sets M and N, the set $\{m \in M : m \notin N\}$ is denoted by $M \setminus N$.

When it is not important to keep track of constants, we use the notation "$\lesssim$" to mean "$\leq$ up to a universal constant". The dependence of the universal constant should be clear from the context, when it is not we use the notation "$\leq C(d, \lambda, \alpha)$", where "$C(d, \lambda, \alpha)$" denotes a generic constant depending on the quantities in the brackets. By "$a \ll b$" we mean that $a \leq C(d, \lambda, \alpha) b$ for some large enough constant $C(d, \lambda, \alpha)$.

For a measurable set $A \subset \mathbb{R}^d$, we denote its d-dimensional Lebesgue measure by $|A|$. By $\int_A f\,dx$ we denote the Lebesgue integral of the function f over the set A. By $\fint_A f\,dx$ we denote the average integral, i. e. $\frac{1}{|A|}\int_A f\,dx$.

We always use $e_i \in \mathbb{R}^d$ to denote the standard coordinate basis of $\mathbb{R}^d$. In particular, we have that $e_i = \nabla x_i$.The (possibly weak) partial derivative with respect to the ith coordinate will be denoted by ∂_i.

While in the first part of the thesis we only ever work with the standard Hölder spaces, in the second part we work with parabolic Hölder spaces.

Note on Labels: The numbering of the equations within the two parts is disjoint and self-contained. So, if in Chapter 1 of Part 1 we reference "(1)" we mean (1) in Section 0.1 of Part I titled "Notation Specific to Part I".

Part I

Large-Scale Boundary Regularity for Random Linear Elliptic Operators

0.1 Notation Specific to Part I

We denote the space of uniformly elliptic (with ellipticity ratio $\lambda > 0$) and bounded coefficient fields $a(x) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ as

$$\Omega = \left\{a(x) \,|\, |a(x)\xi| \le |\xi| \text{ and } \lambda|\xi|^2 \le |\xi \cdot a(x)\xi| \text{ for } \xi \in \mathbb{R}^d \text{ for a.e. } x \in \mathbb{R}^d\right\}. \tag{1}$$

Recall that the equation satisfied by the whole-space corrector in the direction $\xi \in \mathbb{R}^d$, denoted ϕ_ξ, is

$$-\nabla \cdot a\nabla(\phi_\xi + \xi \cdot x) = 0 \qquad \text{on} \quad \mathbb{R}^d \tag{2}$$

and we use the notation $\phi_{e_i} = \phi_i$.

Recall that the whole-space flux corrector $\sigma_{\xi jk}$ is skew-symmetric in the last two indices (it satisfies $\sigma_{\xi jk} = -\sigma_{\xi kj}$) and its defining equation reads

$$\nabla_k \cdot \sigma_{\xi jk} = e_j \cdot (a(e_i + \nabla\phi_i) - a_{hom}e_i), \tag{3}$$

where we use the notation $\nabla_k \cdot \sigma_{\xi jk} = \sum_{k=1}^d \partial_k \sigma_{\xi jk}$. Just as for the corrector, we use the convention $\sigma_{ijk} = \sigma_{e_i jk}$.

We measure the sublinearity of the generalized corrector (ϕ, σ) in terms of the quantity

$$\delta_R := \frac{1}{R}\left(\fint_{B_R} |(\phi,\sigma)|^2\,dx\right)^{\frac{1}{2}}. \tag{4}$$

We define the functions space:

$$H^1_{bdd}(\mathbb{R}^d_+) = \left\{u \in H^1(\mathbb{R}^d_+) : \text{supp}(u) \subseteq B_r \text{ for some } r > 0\right\}.$$

In our arguments in Chapters 1 and 2 it is occasionally important to track certain constants:

- $C_P(d)$: The maximum of the Poincaré constant of B_1 with homogeneous Dirichlet boundary conditions, the Poincaré constant of B_1^+ with homogeneous Dirichlet boundary conditions on $\partial\mathbb{R}^d_+ \cap B_1$, the Poincaré constant of B_1^+ with zero average, and 1.

- $C_I(d)$: The maximum of the universal constant in the constant coefficient regularity estimate (1.37) below and 1.

Also in Chapters 1 and 2 we use the convention that subscripts on a vector or tensor coming before a comma refer to components and subscripts after a comma refer to a scale (not to taking a partial derivative). Taking an example from Chapter 1: the expression $\sigma^D_{djk,M}$ refers to the component djk of a modified flux corrector, which has been adapted on scales $\leq 2^M r_0$ for some dyadic base scale $r_0 > 0$.

Chapter 1

A Large-Scale Regularity Theory for Random Linear Elliptic Operators on $\mathbb{R}^d_+$ with Homogeneous Dirichlet Boundary Data

1.1 Set-up

In this section we are interested in the large-scale regularity of solutions $u \in H^1_{loc}(\mathbb{R}^d_+)$ to the problem

$$-\nabla \cdot a\nabla u = 0 \qquad \text{in} \quad \mathbb{R}^d_+, \tag{1.1a}$$

$$u = 0 \qquad \text{on} \quad \partial\mathbb{R}^d_+, \tag{1.1b}$$

where a is the restriction of a random coefficient field on $\mathbb{R}^d$ to $\mathbb{R}^d_+$. As already mentioned in the introduction of the thesis, the main result of this section is the construction of a sublinear Dirichlet boundary corrector for the haf-space, which for $\alpha \in (0,1)$ allows us to prove a large-scale $C^{1,\alpha}$- regularity theory for solutions of (1.1) and an associated Liouville-type theorem.

Following the strategy that we have set out in the introduction, in order to obtain a large-scale $C^{1,\alpha}$- excess decay, we must introduce a modified version of the excess used in [31] that also takes into consideration the homogeneous Dirichlet boundary data enforced in (1.1b). As the $C^{1,\alpha}$-Liouville principle that we want to show is a corollary of the excess decay, the space that one compares u to in the definition of the excess should be the space that one expects to recover in Liouville principle; i.e. it should be the space of a-harmonic functions on $\mathbb{R}^d_+$ satisfying homogeneous Dirichlet boundary data and the growth condition $|u(x) - u(0)| \lesssim |x|^{1+\alpha}$. In the constant coefficient case when $a = a_{hom}$ this space consists of the linear functions cx_d for $c \in \mathbb{R}$. Extrapolating from this, we expect that in the case of heterogeneous coefficients a the functions satisfying (1.1) and the above listed growth property will be of the form $c(x_d + \phi^D_d)$ for $c \in \mathbb{R}$, where ϕ^D_d should satisfy

$$-\nabla \cdot a\nabla(x_d + \phi^D_d) = 0 \qquad \text{in} \quad \mathbb{R}^d_+, \tag{1.2a}$$

$$\phi^D_d = 0 \qquad \text{on} \quad \partial\mathbb{R}^d_+. \tag{1.2b}$$

Notice that the boundary condition (1.2b) is imposed due to (1.1b). Assuming that it is possible to construct a sublinear ϕ_d^D, we then define the Dirichlet half-space excess of u satisfying (1.1) on the half-ball of radius $r > 0$ as

$$\mathrm{Exc}^D(r) := \inf_{b\in\mathbb{R}} \fint_{B_r^+} |\nabla(u - b(x_d + \phi_d^D))|^2 \, dx. \tag{1.3}$$

With these conventions we summarize the main results of this chapter as:

"Theorem". *Let $\alpha \in (0,1)$ and let a be a random coefficient field sampled according to a stationary and ergodic ensemble for which almost surely there exists a whole-space generalized corrector (ϕ, σ) such that (9) is satisfied. Then $\langle\cdot\rangle$- almost surely, whenever the pair (ϕ,σ) satisfying (9) exists, the following hold:*

i) There exists a Dirichlet half-space corrector in the direction e_d, denoted ϕ_d^D, which satisfies (1.2) and also the sublinear growth condition.

$$\lim_{r\to\infty} \frac{1}{r}\left(\fint_{B_r^+} |\phi_d^D|^2 \, dx\right)^{\frac{1}{2}} = 0.$$

Additionally letting $u \in H^1_{loc}(\mathbb{R}^d_+)$ be a weak solution of (1.1) we have that:

ii) There exists a finite $r^*_\alpha(a) > 0$ such that u satisfies the $C^{1,\alpha}$-excess decay estimate

$$Exc^D(r) \lesssim \left(\frac{r}{R}\right)^{2\alpha} Exc^D(R)$$

for any pair of radii $R \geq r \geq r^*_\alpha(a)$.

iii) If u satisfies a growth condition of the form

$$\lim_{r\to\infty} \frac{1}{r^{1+\alpha}}\left(\fint_{B_r^+} |u|^2 \, dx\right)^{\frac{1}{2}} = 0$$

for some $\alpha \in (0,1)$, then it is a multiple of the "perturbed coordinate function"

$$x \mapsto x_d + \phi_d^D(x).$$

1.1.1 Notation Specific to Chapter 1

By $\dot{H}^1_0(\mathbb{R}^d_+)$, we denote the space of locally integrable functions u with square-integrable gradient and vanishing trace on $\partial\mathbb{R}^d_+$, equipped with the norm $||v||_{\dot{H}^1_0(\mathbb{R}^d_+)} := (\int_{\mathbb{R}^d_+} |\nabla v|^2 \, dx)^{\frac{1}{2}}$.

1.2 Main Results

We now give the full statements of the two theorems contained in this chapter. The first theorem ensures the almost sure existence of a Dirichlet half-space corrector with sublinear growth behavior. As already emphasized above, the key assumption of the theorem is the existence of a whole-space generalized corrector (ϕ, σ) satisfying (9). The theorem that we obtain is:

Theorem 1. *Let $a \in \Omega$, where Ω is defined in (1). Assume that corresponding to a there exists a whole-space generalized corrector (ϕ, σ) satisfying the growth condition*

$$\sum_{m=0}^{\infty} m\delta_{2^m}^{\frac{1}{3}} < \infty. \tag{1.4}$$

Then there exists a generalized Dirichlet half-space corrector (ϕ^D, σ^D) with the following properties:

i) For $i \neq d$, the correctors ϕ_i^D and flux correctors σ_i^D coincide with the restriction of the whole-space correctors to the half-space; i.e. we have that $\phi_i^D = \phi_i|_{\mathbb{R}^d_+}$ and $\sigma_i^D = \sigma_i|_{\mathbb{R}^d_+}$.

ii) The corrector ϕ_d^D is adapted to the boundary data in the sense that it is distributional solution of

$$-\nabla \cdot a\nabla(\phi_d^D + x_d) = 0 \qquad in \quad \mathbb{R}^d_+, \tag{1.5a}$$

$$\phi_d^D = 0 \qquad on \quad \partial\mathbb{R}^d_+. \tag{1.5b}$$

iii) The adapted flux corrector σ_d^D is the vector potential for the flux correction corresponding to $x_d + \phi_d^{\mathbb{H}}$ in the sense that it is skew-symmetric in the third index (k) and is a distributional solution of

$$\nabla_k \cdot \sigma_{djk}^D = e_j \cdot (a(e_d + \nabla\phi_d^D) - a_{hom}e_d) \qquad in \quad \mathbb{R}^d_+. \tag{1.6}$$

iv) The generalized half-space corrector grows sublinearly in the sense that

$$\delta_r^D := \frac{1}{r}\left(\fint_{B_r^+} |(\phi^D, \sigma^D)|^2 \, dx + \fint_{B_r^-} \sum_{i=1}^{d-1} |\phi_i|^2 \, dx \right)^{\frac{1}{2}} \tag{1.7}$$

satisfies

$$\lim_{r\to\infty} \delta_r^D = 0.$$

In particular, for any $\alpha \in (0,1)$ there exists a finite radius $r_\alpha^*(a) > 0$ for which the condition (1.9) below is satisfied.

In fact, our proof shows that a quantitative estimate on the sublinear growth of the whole-space corrector in the form

$$\delta_r \leq \frac{C}{r^\gamma}$$

for some $\gamma \in (0,1]$ may be turned into an estimate for the generalized half-space corrector of the form

$$\delta_r^D \leq \frac{\tilde{C}}{r^{\gamma/3}}. \tag{1.8}$$

This bound is a consequence of more precise estimates on the right-hand sides of the inequalities (1.45), (1.47), and (1.49) in the proof below. However, one should not expect the estimate (1.8) to be optimal, which is why we did not emphasize this quantitative bound in our theorem.

In our second theorem we take advantage of the existence of a sublinear generalized Dirichlet half-space corrector (ϕ^D, σ^D) when it exists by showing a large-scale $C^{1,\alpha}$- decay of the excess defined in (1.3).

Theorem 2. *Let $a \in \Omega$, where Ω is defined in (1). For any Hölder exponent $\alpha \in (0,1)$ there exists a constant $C_\alpha(d,\lambda)$ such that the following statements hold:*

Suppose that for $R > 0$ there exists a generalized Dirichlet half-space corrector (ϕ^D, σ^D) such that ϕ_d^D satisfies (1.5) on B_R^+, ϕ_i^D for $i \neq d$ is the restriction of a corrector ϕ_i on B_R to B_R^+, and σ^D is skew-symmetric and solves (1.6) on B_R^+. Suppose, furthermore, that the pair (ϕ^D, σ^D) is sublinear on large scales in the sense that

$$\delta_r^D \leq \frac{1}{C_\alpha(d,\lambda)} \text{ for all } r \geq r_\alpha^* \tag{1.9}$$

for some radius $0 < r_\alpha^* < R$. Here δ_r^D is defined in (1.7).

Let $u \in H^1(B_R^+)$ solve (1.1) on B_R^+, i. e. let u be a solution to the problem

$$\begin{aligned} -\nabla \cdot a\nabla u &= 0 && \text{in} \quad B_R^+, \\ u &= 0 && \text{on} \quad B_R \cap \partial\mathbb{R}_+^d. \end{aligned}$$

Then, we find that:

i) For any $r \in [r_\alpha^*, R]$ we have that

$$Exc^D(r) \lesssim \left(\frac{r}{R}\right)^{2\alpha} Exc^D(R), \tag{1.10}$$

where the adapted excess is defined in (1.3) *and the universal constant depends on* α, d, *and* λ.

ii) For $r \in [r_\alpha^*, R]$ *the function*

$$b \mapsto \fint_{B_r^+} |\nabla u - b(e_d + \nabla \phi_d^D)|^2 \, dx$$

is coercive in the sense that

$$\fint_{B_r^+} |\nabla u - b(e_d + \nabla \phi_d^D)|^2 \, dx \geq c(\alpha, d, \lambda)|b - b_{min}|^2 \tag{1.11}$$

for some $b_{min} \in \mathbb{R}$.

iii) For $r \in [r_{\frac{1}{2}}^*, R]$ *the mean-value property*

$$\fint_{B_r^+} |\nabla u|^2 \, dx \leq C_{Mean}(d, \lambda) \fint_{B_R^+} |\nabla u|^2 \, dx \tag{1.12}$$

holds for some constant $C_{Mean}(d, \lambda)$.

Combining Theorem 1 with Theorem 2 and post-processing the excess decay via the Caccioppoli inequality, yields the following $C^{1,\alpha}$- Liouville principle:

Corollary 1. *Let* $a \in \Omega$, *where* Ω *is defined in* (1). *Suppose that for a there exists a generalized corrector* (ϕ, σ) *satisfying* (1) *and* (3) *and also the growth condition* (1.4). *Then if* $u \in H^1_{loc}(\mathbb{R}^d_+)$ *is a-harmonic with homogeneous Dirichlet boundary conditions (i. e. if u solves* (1.1)*) and satisfies the growth condition*

$$\lim_{r \to \infty} \frac{1}{r^{1+\alpha}} \left(\fint_{B_r^+} |u|^2 \, dx \right)^{\frac{1}{2}} = 0 \tag{1.13}$$

for some $0 < \alpha < 1$, *then u must be of the form*

$$u = b \cdot (x_d + \phi_d^D)$$

for some $b \in \mathbb{R}$. *Here,* ϕ_d^D *is the Dirichlet half-space corrector constructed in Theorem 1.*

1.3 Construction of the Generalized Dirichlet Half-Space Corrector

We first give an outline of our strategy and then prove all of our claims.

1.3.1 Outline of Strategy

We now give an exposition of our strategy for the construction of the generalized Dirichlet half-space corrector (Theorem 1).

Step 1– (Construction of a sublinear ϕ_d^D up to a certain scale)

Our approach for the construction of ϕ_d^D is to adapt the whole-space corrector ϕ_d to the Dirichlet boundary conditions on $\partial\mathbb{R}^d_+$. We would like to achieve this by subtracting from ϕ_d a sublinearly growing function $\tilde{\varphi}$ that is a-harmonic on $\mathbb{R}^d_+$ and equals ϕ_d on the boundary, i. e. by setting $\phi_d^D := \phi_d - \tilde{\varphi}$ with $\tilde{\varphi}$ being a sublinearly growing solution to the problem

$$-\nabla \cdot a\nabla\tilde{\varphi} = 0 \qquad \text{in } \mathbb{R}^d_+, \tag{1.14a}$$

$$\tilde{\varphi} = \phi_d \qquad \text{on } \partial\mathbb{R}^d_+. \tag{1.14b}$$

As (1.14) is a linear equation, we can decompose the right-hand side of (1.14b) into contributions from dyadic annuli, solve the corresponding problems, and then add the solutions to obtain $\tilde{\varphi}$. We will show that this sum converges and sums to a sublinearly growing function.

Pursuing this strategy, let $r_0 = 2^{m_0}$ for $m_0 \in \mathbb{N}$ be a generic dyadic radius. Let $\{\eta_m | -1 \leq m\}$ be a radial partition of unity with $\operatorname{supp}(\eta_{-1}) \subset \{x \in \mathbb{R}^d : |x| \leq r_0\}$ and $\operatorname{supp}(\eta_m) \subset \{x \in \mathbb{R}^d : r_0 2^{m-1} \leq |x| \leq r_0 2^{m+1}\}$ for $m \geq 0$; suppose that η_m satisfies an estimate of the form $|\nabla\eta_m| \leq \frac{4}{r_0 2^m}$. Also, for $L_m \in (0, r_0 2^{m+1}]$ consider one-dimensional cutoff functions $S_m(x) = S_m(x_d)$ satisfying $S_m(x) = 1$ for $|x_d| \leq L_m$ and $S_m(x) = 0$ for $|x_d| \geq 2L_m$; suppose that $|\nabla S_m| \leq \frac{2}{L_m}$. Note that we shall later choose $L_m \ll r_0 2^{m+1}$.

Introducing the cutoffs $\chi_m(x) := \eta_m(x) S_m(x)$, we then consider the Lax-Milgram solutions $\varphi_m \in \dot{H}^1_0(\mathbb{R}^d_+)$ to the problem

$$-\nabla \cdot a\nabla\varphi_m = \nabla \cdot a\nabla(\chi_m\phi_d) \qquad \text{in} \quad \mathbb{R}^d_+, \tag{1.15a}$$

$$\varphi_m = 0 \qquad \text{on} \quad \partial\mathbb{R}^d_+. \tag{1.15b}$$

Defining $\varphi_M^\Sigma := \sum_{m=-1}^M \varphi_m$ and $\tilde{\varphi}_M^\Sigma := \varphi_M^\Sigma + \sum_{m=-1}^M \chi_m\phi_d$, we see that

$$\phi_{d,M}^D := \phi_d - \tilde{\varphi}_M^\Sigma \tag{1.16}$$

solves the corrector equation (1.5a) in $\mathbb{R}^d_+$ with homogeneous Dirichlet boundary conditions on $\partial\mathbb{R}^d_+ \cap B_{r_0 2^M}$.

Backpedaling a bit, notice that in order to obtain the Lax-Milgram solution it is important to write down the correct weak formulation; in particular, since we only ever test (1.15) with functions $v \in H^1_{bdd}(\mathbb{R}^d_+)$, we can neglect boundary terms and write the weak formulation of (1.15) as

$$\int_{\mathbb{R}^d_+} \nabla v \cdot a \nabla \varphi_m = - \int_{\mathbb{R}^d_+} \nabla v \cdot a \nabla (\chi_m \phi_d). \tag{1.17}$$

For this weak formulation it is then easy to run a Lax-Milgram argument in $\dot{H}^1_0(\mathbb{R}^d_+)$ to obtain φ_m. Here we use the homogeneous Sobolev space with Dirichlet boundary data (which is a Hilbert space thanks to the enforced boundary data) in order to avoid having to use the Poincaré inequality on $\mathbb{R}^d_+$, which is not on a finite domain. This treatment is sufficient since $H^1_{bdd}(\mathbb{R}^d_+) \subset \dot{H}^1(\mathbb{R}^d_+)$.

To estimate the size of the modification $\tilde{\varphi}^{\Sigma}_M$ on a half-ball B^+_r, we split the contributions φ_m into two groups: the "near-field contributions", which are the φ_m such that the inclusion $\mathrm{supp}(\chi_m) \subseteq B_{16r}$ holds, and the "far-field contributions", which are the φ_m for which $\mathrm{supp}(\chi_m) \cap B^+_{4r} = \emptyset$ holds.

Having distinguished between the two types of contributions, we then first estimate the near-field contributions. As we shall see, this can be done with the standard energy estimate for the equation (1.15) and an appropriate estimate for $\chi_m \phi_d$. The energy norm of the term $\chi_m \phi_d$ in turn may be made small by an appropriate choice of L_m.

Lemma 1. *Let the assumptions of Theorem 1 be satisfied and $m \geq -1$. Then there exists $L_m \ll r_0 2^{m+1}$ and a constant $C_1(d, \lambda)$ such that the following is true: For any $r > 0$ the estimates*

$$\left(\fint_{B^+_r} |\nabla (\chi_m \phi_d)|^2 \, dx \right)^{\frac{1}{2}} \leq C_1(d, \lambda) \left(\frac{r_0 2^{m+1}}{r} \right)^{\frac{d}{2}} \delta^{\frac{1}{3}}_{r_0 2^{m+1}} \tag{1.18}$$

and

$$\left(\fint_{B^+_r} |\nabla \varphi_m|^2 \, dx \right)^{\frac{1}{2}} \leq C_1(d, \lambda) \left(\frac{r_0 2^{m+1}}{r} \right)^{\frac{d}{2}} \delta^{\frac{1}{3}}_{r_0 2^{m+1}} \tag{1.19}$$

hold. In particular, for any $r \geq \frac{1}{16} r_0 2^{m+1}$ the function φ_m satisfies the bound

$$\left(\fint_{B^+_r} |\nabla \varphi_m|^2 \, dx \right)^{\frac{1}{2}} \leq C_2(d, \lambda) \min \left\{ 1, \left(\frac{r_0 2^{m+1}}{r} \right)^{\frac{d}{2}} \right\} \delta^{\frac{1}{3}}_{r_0 2^{m+1}} \tag{1.20}$$

with $C_2 := C_{Mean} C_1 8^d$.

For the construction of ϕ^D_d we will need the estimate (1.20) on B^+_r also for the far-field contributions, which we obtain in Step 3 below.

Step 2– (Construction of a sublinearly growing σ_d^D up to a certain scale)

Having constructed an intermediate corrector $\phi_{d,M}^D$ (see (1.16)) for which the desired homogeneous Dirichlet boundary data have been partially enforced in the sense that $\phi_{d,M}^D = 0$ on $\partial\mathbb{R}_+^d \cap B_{2^M r_0}$, we need to construct a corresponding flux corrector $\sigma_{d,M}^D$ according to (1.6). Again, our approach is to adapt the whole-space flux corrector σ_d to take into account the modification $\phi_{d,M}^D - \phi_d$ of the corrector by adding a correction $\psi_{jk,M}$. In particular, we construct sublinearly growing functions $\psi_{jk,M}$ that satisfy

$$-\nabla_k \cdot \psi_{jk,M} = e_j \cdot \big(a(e_d + \nabla\phi_{d,M}^D) - a(e_d + \nabla\phi_d)\big) \quad \text{in } \mathbb{R}_+^d \tag{1.21}$$

and define

$$\sigma_{djk,M}^D := \sigma_{djk} - \psi_{jk,M}.$$

In order to ensure the skew-symmetry of $\sigma_{d,M}^D$, we need to construct the $\psi_{,M}$ to be skew-symmetric.

It turns out that a suitable ansatz is

$$\psi_{jk,M} := \partial_k v_{j,M} - \partial_j v_{k,M} \tag{1.22}$$

with the components of $v_{,M} : \mathbb{R}_+^d \to \mathbb{R}^d$ solving the equations

$$-\Delta v_{j,M} = e_j \cdot (a(e_d + \nabla\phi_{d,M}^D) - a(e_d + \nabla\phi_d)) \quad \text{in } \mathbb{R}_+^d, \tag{1.23a}$$
$$v_{j,M} = 0 \quad \text{for } j \neq d \text{ on } \partial\mathbb{R}_+^d, \tag{1.23b}$$
$$\partial_d v_{d,M} = 0 \quad \text{on } \partial\mathbb{R}_+^d. \tag{1.23c}$$

To see that (1.22) is a good ansatz we first notice that the skew-symmetry condition is already built-in. Furthermore, differentiating the equation (1.23), we infer

$$-\Delta(\nabla_k \cdot v_{k,M}) = 0 \quad \text{in } \mathbb{R}_+^d, \tag{1.24a}$$
$$\nabla_k \cdot v_{k,M} = 0 \quad \text{on } \partial\mathbb{R}_+^d. \tag{1.24b}$$

By the zeroth-order Liouville principle for harmonic functions with homogeneous Dirichlet boundary conditions on $\mathbb{R}_+^d$, sublinear growth of $\nabla_k \cdot v_{k,M}$ entails that $\nabla_k \cdot v_{k,M} = 0$ on all of $\mathbb{R}_+^d$. This leads, as desired, to the conclusion

$$\begin{aligned}-\nabla_k \cdot \psi_{jk,M} &= \sum_{k=1}^d (\partial_k\partial_j v_{k,M} - \partial_k^2 v_{j,M})\\ &= \partial_j(\nabla_k \cdot v_{k,M}) - \Delta v_{j,M}\\ &= e_j \cdot (a(e_d + \nabla\phi_{d,M}^D) - a(e_d + \nabla\phi_d)).\end{aligned}$$

So, the moral of the story is that in order to obtain a solution to (1.21) it suffices to construct solutions $v_{j,M}$ to (1.23) for which $\nabla_k \cdot v_{k,M}$ is a sublinearly growing function.

To construct such a solution $v_{j,M}$, notice that, as $\phi^D_{d,M} - \phi_d$ is a-harmonic on $\mathbb{R}^d_+$, we may rewrite the right-hand side in (1.23a) as

$$\begin{aligned} e_j \cdot (a(\nabla\phi^D_{d,M} - \nabla\phi_d)) &= e_j \cdot a(\nabla\phi^D_{d,M} - \nabla\phi_d) + x_j \nabla \cdot (a(\nabla\phi^D_{d,M} - \nabla\phi_d)) \\ &= \nabla \cdot (x_j a(\nabla\phi^D_{d,M} - \nabla\phi_d)). \end{aligned}$$

Our strategy, just like in Step 1, is now to work with a decomposition into contributions from dyadic annuli: Reusing the partition of unity η_n from Step 1, we consider the Lax-Milgram solutions $v^n_{j,M}$ of the problems

$$-\Delta v^n_{j,M} = \nabla \cdot (\eta_n x_j a(\nabla\phi^D_{d,M} - \nabla\phi_d)) \quad \text{in } \mathbb{R}^d_+, \tag{1.25a}$$

$$v^n_{j,M} = 0 \quad \text{for } j \neq d \text{ on } \partial\mathbb{R}^d_+, \tag{1.25b}$$

$$\partial_d v^n_{d,M}(x) = 0 \quad \text{on } \partial\mathbb{R}^d_+. \tag{1.25c}$$

Again, just like in Step 1, in order to apply a Lax-Milgram argument it is important to write down the correct weak formulation, which in this case is

$$\int_{\mathbb{R}^d_+} \nabla u \cdot \nabla v^n_{j,M} \mathrm{d}x = -\int_{\mathbb{R}^d_+} \eta_n x_j \nabla u \cdot a(\nabla\phi^D_{d,M} - \nabla\phi_d) \mathrm{d}x \tag{1.26}$$

for test functions $u \in H^1_{bdd}(\mathbb{R}^d_+)$. For the case $j \neq d$ we again apply Lax-Milgram to the space $\dot{H}^1_0(\mathbb{R}^d_+)$ just as we have done in the previous step to the solution $v^n_{j,M}$.

To find the solution $v^n_{d,M}$ satisfying (1.25a) on $\mathbb{R}^d_+$ and (1.25c) on $\partial\mathbb{R}^d_+$, we must instead apply Lax-Milgram to the space $L^{\frac{2d}{d-2}}(\mathbb{R}^d_+) \cap \dot{H}^1(\mathbb{R}^d_+)$ when $d > 2$ and $BMO(\mathbb{R}^d_+) \cap \dot{H}^1(\mathbb{R}^d_+)$ when $d = 2$, in each case endowed with the inner-product inherited from the homogeneous Sobolev space. We consider the case $d > 2$ and remark that the case $d = 2$ is exactly the same. Notice first that $H^1_{bbd}(\mathbb{R}^d_+) \subseteq L^{2d/(d-2)}(\mathbb{R}^d_+) \cap \dot{H}^1(\mathbb{R}^d_+)$ thanks to the Sobolev embedding and that (1.26) is the weak formulation of (1.25) for this class of test functions when $j = d$ since then $x_d = 0$ on $\partial\mathbb{R}^d_+$. For the actual Lax-Milgram argument we notice that:

i) The space $L^{2d/(d-2)}(\mathbb{R}^d_+) \cap \dot{H}^1(\mathbb{R}^d_+)$ endowed with $\langle \cdot, \cdot \rangle_{\dot{H}^1}$ is complete and, therefore, a Hilbert space thanks to the Sobolev embedding.

ii) The integral on the right-hand side of (1.26) is well-defined due to the compact support of η_n.

iii) The bilinear form on the left-hand side of (1.26) is clearly coercive.

iv) The right-hand side of (1.26) defines a bounded operator on $L^{\frac{2d}{d-2}}(\mathbb{R}^d_+) \cap \dot{H}^1(\mathbb{R}^d_+)$. In particular, we notice that

$$\begin{aligned}&\int_{\mathbb{R}^d_+} |\eta_n x_j \nabla u \cdot a(\nabla \phi^D_{d,M} - \nabla \phi_d)| dx \\ \leq & C(d,\lambda,n) \left(\int_{\mathbb{R}^d_+} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{B_{r_0 2^{n+1}}} |\nabla \phi^D_{d,M} - \nabla \phi_d|^2 \, dx \right)^{\frac{1}{2}},\end{aligned}$$

where we have used the compact support of η_n and the boundedness of a. As $\nabla \phi^D_{d,M} - \nabla \phi_d \in H^1_{loc}(\mathbb{R}^d)$ it follows that the right-hand side of (1.26) is a bounded operator.

Having checked all of the criterion, we find that we may apply Lax-Milgram to obtain a solution $v^n_{d,M} \in L^{2d/(d-2)}(\mathbb{R}^d_+) \cap \dot{H}^1(\mathbb{R}^d_+)$ to (1.26). Notice that the reason we do not use the space $\dot{H}^1(\mathbb{R}^d_+)$ with the additional restriction that $\fint_{B_1^+} u \, dx = 0$ for our Lax-Milgram argument is that this space does not contain $H^1_{bdd}(\mathbb{R}^d)$.

In order to obtain $v_{j,M}$, we intend to sum all of the contributions. However, to ensure that on a half-ball B_r^+ the "far-field contributions" – i. e. the $v^n_{j,M}$ with $2^{n+1} r_0 \geq 16r$ – do not destroy the smallness of the sum $\sum_{n=-1}^{\infty} \nabla v^n_{j,M}$, we subtract-off the initial linear growth of $v^n_{j,M}$. For this purpose we introduce the notation

$$b^n_{j,M} =: \begin{cases} 0 & \text{if } n = -1 \\ \nabla v^n_{j,M}(0) & \text{if } n \neq -1. \end{cases} \tag{1.27}$$

Notice that $b^n_{jk,M} = 0$ unless $n \neq -1$ and either $j = d$ and $k \neq d$ or $j \neq d$ and $k = d$. We then obtain the following estimate:

Lemma 2. *Let the assumptions of Theorem 1 be satisfied. Let $M \geq -1$ and $n \geq -1$. Then for $r \geq r_0$ and $j,k \in \{1,...,d\}$ we have the estimate*

$$\begin{aligned}&\frac{1}{r} \left(\fint_{B_r^+} |\partial_k (v^n_{j,M} - b^n_{j,M} \cdot x)|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq C_3(d,\lambda) \min \left\{ 1, \frac{r_0 2^{n+1}}{r} \right\} \left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla \phi^D_{d,M} - \nabla \phi_d|^2 \, dx \right)^{\frac{1}{2}}\end{aligned}$$

with $C_3(d,\lambda) := 4 C_4 C_I$.

This estimate immediately enables us to pass to the limit $N \to \infty$ in the sum $\sum_{n=-1}^{N} (v^n_{j,M} - b^n_{j,M} \cdot x)$, which we do in the next lemma.

Lemma 3. *Let the assumptions of Theorem 1 be satisfied and let the* L_m *be chosen as in Lemma 1. Then for* $r \geq r_0$ *and* $j \in \{1, ..., d\}$ *the series* $\sum_{n=-1}^{\infty}(v^n_{j,M} - b^n_{j,M} \cdot x)$ *converges absolutely in* $H^1(B_r^+)$ *to a limit* $v_{j,M}$*. For this limit, the function* $\psi_{jk,M} = \partial_k v_{j,M} - \partial_j v_{k,M}$ *satisfies the equation*

$$-\nabla_k \cdot \psi_{jk,M} = e_j \cdot a(\nabla\phi^D_{d,M} - \nabla\phi_d) \qquad \text{in } \mathbb{R}^d_+ \tag{1.28}$$

and for $r \geq r_0$ *and* $j, k \in \{1, ..., d\}$ *we have the estimate*

$$\begin{aligned}&\frac{1}{r}\left(\fint_{B_r^+} |\psi_{jk,M}|^2 \, dx\right)^{\frac{1}{2}} \\ &\leq 2C_3(d,\lambda)\sum_{n=-1}^{\infty}\min\left\{1, \frac{r_0 2^{n+1}}{r}\right\}\left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla\phi^D_{d,M} - \nabla\phi_d|^2 \, dx\right)^{\frac{1}{2}}.\end{aligned} \tag{1.29}$$

Step 3– (Inductively building a sublinear corrector on larger scales)

In the previous two steps the base radius r_0 according to which we construct the η_m was arbitrary. We now choose r_0 independently of m in such a way that the estimate (1.20) does not only hold under the condition that $r \geq \frac{1}{16}r_0 2^{m+1}$, but more generally for any $r \geq r_0$.

To extend the inequality (1.20) to also to the far-field contributions, we shall crucially rely on the mean-value property (1.12) for a-harmonic functions. To this aim, we proceed by induction in m; to show (1.20) for φ_{m+1} for all $r \geq r_0$, we use the intermediate generalized Dirichlet half-space corrector $(\phi^D_{d,m}, \sigma^D_{d,m})$ and establish that it satisfies the estimate (1.9) for $\alpha = \frac{1}{2}$ for $r \geq r_0$, which by Theorem 2 entails the mean-value property (1.12) for a-harmonic functions on scales $r \in [r_0, r_0 2^m]$ with $R = r_0 2^m$.

The point is that we have to choose r_0 large enough so that for every $m \geq -1$ the intermediate generalized corrector $(\phi^D_{d,m}, \sigma^D_{d,m})$ satisfies $r_0 \geq r^*_{\frac{1}{2},m}$, where have used the notation that the minimal radius for $\alpha = \frac{1}{2}$ of the mth intermediate corrector (in the sense of (1.9)) is $r^*_{\frac{1}{2},m}$. We show that this is possible in the next lemma:

Lemma 4. *Let the assumptions of Theorem 1 be satisfied – in particular, suppose that there exists a whole-space generalized corrector* (ϕ, σ) *corresponding to* a *that satisfies* (1.4) *– and let the* L_m *be chosen as in Lemma 1. Then there exists* $r_0 > 0$ *independent of* $M \in \{-1, 0, 1, 2, \ldots\}$ *with the* *following property: If the φ_m satisfy the estimate*

$$\left(\fint_{B_r^+} |\nabla\varphi_m|^2\,dx\right)^{\frac{1}{2}} \leq C_2 \min\left\{1, \left(\frac{r_0 2^{m+1}}{r}\right)^{\frac{d}{2}}\right\}\delta_{r_0 2^{m+1}}^{\frac{1}{3}} \tag{1.30}$$

for all $r \geq r_0$ and all $m \in \{-1,\ldots,M\}$ (recall the definition $C_2 := C_{Mean}C_1 8^d$), then $(\phi_{d,M}^D, \sigma_{d,M}^D)$ satisfies the smallness condition (1.9) for $\alpha = \frac{1}{2}$ and all $r \geq r_0$, i. e. we have

$$\delta_r^D \leq \frac{1}{C_{\frac{1}{2}}(d,\lambda)}.$$

As a consequence, in this case φ_{M+1} also satisfies the estimate (1.30) for all $r \geq r_0$.

Note that the start of the induction, i. e. the estimate (1.30) for $m = -1$, is provided by Lemma 1.

Step 4– (Passage to the limit $M \to \infty$)

In the last step, we pass to the limit $M \to \infty$ to obtain ϕ_d^D and σ_d^D as the limits of the sequences $\{\phi_{d,M}^D\}_M$ and $\{\sigma_{d,M}^D\}_M$, thereby establishing Theorem 1.

1.3.2 Proofs

Proofs for Step 1 – (Estimates for the modification of the corrector ϕ_d in the near-field case)

As we have already mentioned above, Lemma 1 is a consequence of appropriate energy estimates for the defining equation of φ_m and a suitable bound for $\chi_m\phi_d$. Here comes the argument:

Proof of Lemma 1. Let us abbreviate $R := r_0 2^{m+1}$. Testing the weak formulation (1.17) with φ_m and estimating using the uniform ellipticity and boundedness of a yields

$$\left(\int_{\mathbb{R}_+^d} |\nabla\varphi_m|^2\,dx\right)^{\frac{1}{2}} \lesssim \left(\int_{B_R^+} |\phi_d\nabla\chi_m|^2\,dx\right)^{\frac{1}{2}} + \left(\int_{B_R^+} |\chi_m\nabla\phi_d|^2\,dx\right)^{\frac{1}{2}}. \tag{1.31}$$

We treat the two terms on the right-hand side separately. For the first, using our definition of χ_m and $L_m \leq R$, we find that

$$\left(\int_{B_R^+} |\phi_d \nabla\chi_m|^2\,dx\right)^{\frac{1}{2}} \lesssim \quad \frac{R^{\frac{d}{2}}}{L_m}\left(\fint_{B_R^+} |\phi_d|^2\,dx\right)^{\frac{1}{2}} \leq \frac{R^{\frac{d+2}{2}}}{L_m}\delta_R. \tag{1.32}$$

For our treatment of the second term on the right-hand side of (1.31), we even-reflect χ_m across the axis $\{x_d = 0\}$ such that it is defined on $\mathbb{R}^d$. We may then test the whole-space corrector equation (1) with $\chi_m^2(\phi_d + x_d)$. After using Young's inequality and the uniform ellipticity of a, this yields

$$\int_{\mathbb{R}^d} \chi_m^2|\nabla\phi_d + e_d|^2\,dx \lesssim \int_{\mathbb{R}^d} |\nabla\chi_m|^2|\phi_d + x_d|^2\,dx. \tag{1.33}$$

Now notice that we have $\operatorname{supp}(\chi_m) \subseteq [-R,R]^{d-1}\times[-2L_m, 2L_m]$; in particular, on $\operatorname{supp}(\chi_m)$ we have $|x_d| \leq 2L_m$. The triangle inequality in $L^2(B_R)$, the estimate (1.33), and the bound $|\nabla\chi_m| \leq \frac{C}{L_m}$ then yield

$$\begin{aligned}
&\int_{B_R} |\chi_m\nabla\phi_d|^2\,dx \lesssim \int_{B_R}\chi_m^2\,dx + \int_{B_R}\chi_m^2|\nabla\phi_d + e_d|^2\,dx\\
&\lesssim |\operatorname{supp}(\chi_m)| + \frac{1}{L_m^2}\int_{\operatorname{supp}(\chi_m)} |\phi_d|^2 + |x_d|^2\,dx\\
&\lesssim |\operatorname{supp}(\chi_m)| + \frac{R^d}{L_m^2}\fint_{B_R}|\phi_d|^2\,dx\\
&\lesssim R^{d-1}L_m + \frac{R^{d+2}}{L_m^2}\delta_R^2.
\end{aligned}$$

The second term on the right-hand side of (1.31) is, therefore, estimated by

$$\left(\int_{B_R^+} |\chi_m\nabla\phi_d|^2\,dx\right)^{\frac{1}{2}} \lesssim R^{\frac{d-1}{2}}L_m^{\frac{1}{2}} + \frac{R^{\frac{d+2}{2}}}{L_m}\delta_R. \tag{1.34}$$

Together, (1.34), (1.32), and (1.31) give that

$$\begin{aligned}
&\left(\fint_{B_r^+}|\nabla\varphi_m|^2\,dx\right)^{\frac{1}{2}} + \left(\fint_{B_r^+}|\nabla(\chi_m\phi_d)|^2\,dx\right)^{\frac{1}{2}}\\
&\lesssim \left(\frac{R}{r}\right)^{\frac{d}{2}}\frac{R}{L_m}\delta_R + \left(\frac{R}{r}\right)^{\frac{d}{2}}\left(\frac{L_m}{R}\right)^{\frac{1}{2}}.
\end{aligned}$$

Choosing $L_m := \epsilon R = \epsilon r_0 2^{m+1}$, we can optimize this expression in ϵ. Plugging in the optimal $\epsilon = \delta_R^{\frac{2}{3}}$ yields

$$\left(\fint_{B_r^+}|\nabla\varphi_m|^2\,dx\right)^{\frac{1}{2}} + \left(\fint_{B_r^+}|\nabla(\chi_m\phi_d)|^2\,dx\right)^{\frac{1}{2}} \leq C_1\left(\frac{r_0 2^{m+1}}{r}\right)^{\frac{d}{2}}\delta_{r_0 2^{m+1}}^{\frac{1}{3}}.$$

This directly gives (1.18) and (1.19). By the definition of C_2, for $r \geq \frac{1}{16} r_0 2^{m+1}$ this also entails the estimate (1.20).

$\square$

Step 2 – (Estimates for the modification of the flux corrector σ)

The following bound forms the basis for the estimates on the size of the modification ψ_{jk} of the flux correction σ_d. It is obtained by an energy estimate for $v_{j,M}^n$ and a mean-value property of harmonic functions.

Lemma 5. *Using the notation from Section 1.3, let $M \geq -1$, $n \geq -1$, and abbreviate $R := r_0 2^{n+1}$. Then there exists a constant $C_4 = C_4(d)$ such that for any $r \geq \frac{1}{16} R$ the estimate*

$$\left(\fint_{B_r^+} |\nabla v_{j,M}^n - b_{j,M}^n|^2 \, dx \right)^{\frac{1}{2}} \leq C_4 R \left(\fint_{B_R^+} |\nabla \phi_{d,M}^D - \nabla \phi_d|^2 \, dx \right)^{\frac{1}{2}}$$

holds.

Proof. Testing the weak formulation (1.26) with the solution $v_{j,M}^n$ and using the property $\mathrm{supp}(\eta_n) \subset \{|x| \leq R\}$ as well as the boundedness of a, we obtain the energy estimate

$$\left(\int_{\mathbb{R}_+^d} |\nabla v_{j,M}^n|^2 \, dx \right)^{\frac{1}{2}} \leq R \left(\int_{B_R^+} |\nabla \phi_{d,M}^D - \nabla \phi_d|^2 \, dx \right)^{\frac{1}{2}}. \tag{1.35}$$

We then notice that when $n \neq -1$ the functions $\partial_k v_{j,M}^n$ are harmonic in $\{|x| < \frac{R}{4}\}$ with homogeneous Neumann boundary conditions on $\partial \mathbb{R}_+^d \cap \{|x| < \frac{R}{4}\}$ whenever $b_{j,M}^n \neq 0$. In this case we have access to a classic mean-value property. By (1.27) we then deduce that

$$\begin{aligned} |b_{j,M}^n| = |\nabla v_{j,M}^n(0)| &\leq C(d) \left(\fint_{B_{\frac{r}{4}}} |\nabla v_{j,M}^n|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq C(d) R \left(\fint_{B_R^+} |\nabla \phi_{d,M}^D - \nabla \phi_d|^2 \, dx \right)^{\frac{1}{2}}. \end{aligned} \tag{1.36}$$

The lemma is now an easy consequence of (1.35) and (1.36). $\square$

Our next goal is to prove Lemma 2. For this we recall the following basic fact for harmonic functions: For any harmonic function w on B_R^+ with either homogeneous Dirichlet or homogeneous Neumann boundary conditions on $\partial\mathbb{R}^d_+ \cap B_R$, for any $r \in (0, \frac{r}{4}]$ we have

$$\left(\fint_{B_r^+} |w - w(0)|^2 \, dx \right)^{\frac{1}{2}} \leq C_I(d) \frac{r}{R} \left(\fint_{B_R^+} |w|^2 \, dx \right)^{\frac{1}{2}}. \tag{1.37}$$

This inequality follows from the regularity estimate (1.54) below and the Caccioppoli estimate for harmonic function on B_R^+ with homogeneous Neumann or Dirichlet boundary conditions on $\partial\mathbb{R}^d_+ \cap B_R$. These Caccioppoli estimates are completely elementary, but are included in this thesis as Lemma 6 of Chapters 1 and Lemma 5 of Chapter 2.

Proof of Lemma 2. For a given radius r, we separately consider the case of a near-field contribution, defined as contributions for which n satisfies $r_0 2^{n+1} \leq 16r$, and the case of a far-field contribution, when $r_0 2^{n+1} > 16r$. Notice that, since $r \geq r_0$, $n = -1$ always corresponds to a near-field contribution.

For the near-field contributions, by Lemma 5 we have the estimate

$$\begin{aligned}
&\frac{1}{r} \left(\fint_{B_r^+} |\partial_k (v^n_{j,M} - b^n_{j,M} \cdot x)|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq C_4 \frac{r_0 2^{n+1}}{r} \left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla \phi^D_{d,M} - \nabla \phi_d|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq 16 C_4 \min\left\{ 1, \frac{r_0 2^{n+1}}{r} \right\} \left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla \phi^D_{d,M} - \nabla \phi_d|^2 \, dx \right)^{\frac{1}{2}}. \qquad (1.38)
\end{aligned}$$

Next, we address the far-field contributions. Notice again that $\partial_k v^n_{j,M} - b^n_{jk,M}$ is harmonic in $B^+_{r_0 2^{n-1}}$ and satisfies either homogeneous Dirichlet or homogeneous Neumann boundary conditions on $\partial\mathbb{R}^d_+ \cap B_{r_0 2^{n-1}}$ (depending on j and k). Furthermore, we have $\partial_k v^n_{j,M}(0) - b^n_{jk,M} = 0$ and $r \leq r_0 2^{n-3}$. Therefore, an application of (1.37) to $w := \partial_k v^n_{j,M} - b^n_{jk,M}$ followed by Lemma 5, the latter applied with $r := r_0 2^{n-1}$ and $R := r_0 2^{n+1}$, yields the desired bound

$$\begin{aligned}
&\frac{1}{r} \left(\fint_{B_r^+} |\partial_k (v^n_{j,M} - b^n_{j,M} \cdot x)|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq C_I \frac{1}{r_0 2^{n-1}} \left(\fint_{B^+_{r_0 2^{n-1}}} |\partial_k v^n_{j,M} - b^n_{jk,M}|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq 4 C_4 C_I \left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla \phi^D_{d,M} - \nabla \phi_d|^2 \, dx \right)^{\frac{1}{2}}.
\end{aligned}$$

□

To complete this step we give the proof of Lemma 3:

Proof of Lemma 3. By Lemma 2, for any $r > 0$ absolute convergence in $H^1(B_r^+)$ of the series

$$\sum_{n=-1}^{\infty} (v_{j,M}^n - b_{j,M}^n \cdot x)$$

towards a limit $v_{j,M}$ follows once we have established an estimate of the form

$$\sum_{n=-1}^{\infty} \left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla \phi_{d,M}^D - \nabla \phi_d|^2 \, dx \right)^{\frac{1}{2}} < \infty. \tag{1.39}$$

Note that since $v_{j,M}^n$ satisfies the weak formulation (1.26), the difference $v_{j,M}^n - b_{j,M}^n \cdot x$ also does for test functions $u \in H^1_{bdd}(\mathbb{R}^d)$. Using Lemma 2 and assuming that we have shown the estimate (1.39) we can the sum over the contributions n and apply Fubini's theorem to obtain

$$\int_{\mathbb{R}^d_+} \nabla u \cdot \nabla v_{j,M} \mathrm{d}x = - \int_{\mathbb{R}^d_+} x_j \nabla u \cdot a(\nabla \phi_{d,M}^D - \nabla \phi_d) \mathrm{d}x.$$

Lemma 2 also gives the bound

$$\begin{aligned} &\frac{1}{r} \left(\fint_{B_r^+} |\partial_k v_{j,M}|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq C_3(d, \lambda) \sum_{n=-1}^{\infty} \min \left\{ 1, \frac{r_0 2^{n+1}}{r} \right\} \left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla \phi_{d,M}^D - \nabla \phi_d|^2 \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

Thus, the estimate (1.29) is a direct consequence of Lemma 2. Again, once we have established (1.39), this bound entails sublinear growth of the function $\nabla_k \cdot v_{k,M}$ in the sense

$$\lim_{r \to \infty} \frac{1}{r} \left(\fint_{B_r^+} |\nabla_k \cdot v_{k,M}|^2 \, dx \right)^{\frac{1}{2}} = 0;$$

this follows from the dominated convergence theorem. Recalling the derivation of (1.21), we then deduce that $\psi_{jk,M}$ indeed satisfies (1.28).

Now it only remains to show (1.39). Here we use that for any $m \in \{-1, \dots, M\}$ the bounds (1.18) and (1.19), applied with $r := r_0 2^{n+1}$, entail that

$$\left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla \varphi_m|^2 \, dx \right)^{\frac{1}{2}} + \left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla (\chi_m \phi_d)|^2 \, dx \right)^{\frac{1}{2}} \lesssim 2^{\frac{d(m-n)}{2}} \delta^{\frac{1}{3}}_{r_0 2^{m+1}}.$$

Taking the sum with respect to m and recalling that

$$\phi^D_{d,M} - \phi_d = - \sum_{m=-1}^{M} (\varphi_m + \chi_m \phi_d),$$

we obtain the bound

$$\left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla \phi^D_{d,M} - \nabla \phi_d|^2 \, dx \right)^{\frac{1}{2}} \lesssim 2^{-\frac{dn}{2}} \sum_{m=-1}^{M} 2^{\frac{dm}{2}} \delta^{\frac{1}{3}}_{r_0 2^{m+1}}.$$

This directly implies (1.39).

□

Step 3 – (Estimates for the modification of the corrector ϕ_d in the far-field case)

Proof of Lemma 4. For the moment, let $r_0 = 2^{m_0} > 0$ be an arbitrary dyadic radius for which the φ_m with $m \in \{-1, \dots, M\}$ satisfy (1.30) for all $r \geq r_0$. By the triangle inequality in $L^2(B^+_r)$ and the Poincaré inequality on B^+_r with homogeneous Dirichlet boundary conditions on $\partial \mathbb{R}^d_+ \cap B_r$, writing $\phi_d - \tilde{\varphi}^\Sigma_M = (1 - \sum_{m=-1}^{M} \chi_m) \phi_d - \varphi^\Sigma_M$ we get

$$\begin{aligned} &\frac{1}{r} \left(\fint_{B^+_r} \left|\phi_d - \tilde{\varphi}^\Sigma_M\right|^2 + \left|\sigma_d - \psi_{,M}\right|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq \frac{1}{r} \left(\fint_{B^+_r} |(\phi_d, \sigma_d)|^2 \, dx \right)^{\frac{1}{2}} + \frac{1}{r} \left(\fint_{B^+_r} |\psi_{,M}|^2 \, dx \right)^{\frac{1}{2}} \\ &\quad + C_P \left(\fint_{B^+_r} |\nabla \varphi^\Sigma_M|^2 \, dx \right)^{\frac{1}{2}}. \end{aligned} \tag{1.40}$$

Notice that for $r \geq r_0$ Lemma 3 yields

$$\frac{1}{r} \left(\fint_{B^+_r} |\psi_{,M}|^2 \right)^{\frac{1}{2}}$$

$$\leq 2d^2 C_3 \sum_{n=-1}^{\infty} \min\left\{1, \frac{r_0 2^{n+1}}{r}\right\} \left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla \tilde{\varphi}^{\Sigma}_M|^2 \, dx \right)^{\frac{1}{2}}. \tag{1.41}$$

Using our assumption that the φ_m with $m \in \{-1, \ldots, M\}$ satisfy (1.30) for any $r \geq r_0$ – and, therefore in particular for $r := r_0 2^{n+1}$ – gives that

$$\begin{aligned}
&\sum_{n=-1}^{\infty} \min\left\{1, \frac{r_0 2^{n+1}}{r}\right\} \left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla \varphi^{\Sigma}_M|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq C_2 \sum_{m=-1}^{M} \sum_{n=-1}^{\infty} \min\left\{1, 2^{\frac{d(m-n)}{2}}\right\} \delta^{\frac{1}{3}}_{r_0 2^{m+1}} \\
&\leq C_2 \sum_{m=-1}^{M} \Big(m + 1 + \frac{1}{1 - 2^{-\frac{d}{2}}} \Big) \delta^{\frac{1}{3}}_{r_0 2^{m+1}}.
\end{aligned} \tag{1.42}$$

Furthermore, we may use that χ_m is supported in $B^+_{r_0 2^{m+1}} \backslash B^+_{r_0 2^{m-1}}$ for $m \neq -1$ and (1.18) (applied with $r := r_0 2^{n+1}$) to get that

$$\begin{aligned}
&\sum_{m=-1}^{M} \sum_{n=-1}^{\infty} \min\left\{1, \frac{r_0 2^{n+1}}{r}\right\} \left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla (\chi_m \phi_d)|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq \sum_{m=-1}^{M} \sum_{n=m-1}^{\infty} \left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla (\chi_m \phi_d)|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq C_1 \sum_{m=-1}^{M} \sum_{n=m-1}^{\infty} 2^{\frac{d(m-n)}{2}} \delta^{\frac{1}{3}}_{r_0 2^{m+1}} \\
&\leq C_1 \sum_{m=-1}^{M} \frac{2^{\frac{d}{2}}}{1 - 2^{-\frac{d}{2}}} \delta^{\frac{1}{3}}_{r_0 2^{m+1}}.
\end{aligned} \tag{1.43}$$

Then, continuing (1.41) with (1.42) and (1.43) yields

$$\begin{aligned}
&\frac{1}{r} \left(\fint_{B^+_r} |\psi_{,M}|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq 2d^2 C_3 (C_1 + C_2) \sum_{m=-1}^{M} \Big(m + 1 + \frac{2^{\frac{d}{2}}}{1 - 2^{-\frac{d}{2}}} \Big) \delta^{\frac{1}{3}}_{r_0 2^{m+1}} \\
&\leq 2d^2 C_3 (C_1 + C_2) \sum_{k=m_0}^{M+m_0+1} \Big(k + \frac{2^{\frac{d}{2}}}{1 - 2^{-\frac{d}{2}}} \Big) \delta^{\frac{1}{3}}_{2^k}.
\end{aligned}$$

To treat the other term of (1.40) we again use (1.30), which gives

$$\left(\fint_{B_r^+} |\nabla \varphi_M^\Sigma|^2 \, dx\right)^{\frac{1}{2}} \le C_2 \sum_{m=-1}^{M} \min\left\{1, \left(\frac{r_0 2^{m+1}}{r}\right)^{\frac{d}{2}}\right\} \delta_{r_0 2^{m+1}}^{\frac{1}{3}}$$
$$\le C_2 \sum_{k=m_0}^{M+m_0+1} \delta_{2^k}^{\frac{1}{3}}.$$

So, for $r \ge r_0$ we arrive at

$$\frac{1}{r}\left(\fint_{B_r^+} \left|\phi_d - \tilde{\varphi}_M^\Sigma\right|^2 + |\sigma_d - \psi_{,M}|^2 \ dx\right)^{\frac{1}{2}}$$
$$\le \delta_r + 2d^2 C_3 (C_1 + C_2) \sum_{k=m_0}^{\infty} \left(k + \frac{2^{\frac{d}{2}}}{1 - 2^{-\frac{d}{2}}}\right) \delta_{2^k}^{\frac{1}{3}}$$
$$+ C_P C_2 \sum_{k=m_0}^{\infty} \delta_{2^k}^{\frac{1}{3}}. \tag{1.44}$$

As a consequence of the estimate (1.44), our assumption (1.4) allows us to choose $r_0 = 2^{m_0}$ large enough – independently of M – such that for $(\phi_{d,M}^D, \sigma_{d,M}^D)$ the estimate (1.9) is satisfied for $\alpha = \frac{1}{2}$ and $r \ge r_0$.

Thus, we infer the estimate (1.30) for φ_{M+1}: The case $\frac{r_0 2^{(M+1)+1}}{r} \le 16$ has already been treated in Lemma 1; it just remains to extend the estimate to the case $\frac{r_0 2^{(M+1)+1}}{r} > 16$. As $(\phi_{d,M}^D, \sigma_{d,M}^D)$ is a Dirichlet half-space corrector on B_R^+ with $R := r_0 2^M$ which satisfies (1.9) for $\alpha = \frac{1}{2}$ and $r \ge r_0$, Theorem 2 is applicable and yields the mean-value property (1.12) for a-harmonic functions on $B_{r_0 2^M}^+$ with homogeneous Dirichlet boundary conditions on $\partial\mathbb{R}_+^d \cap B_{r_0 2^M}$. Since φ_{M+1} is indeed a-harmonic in $B_{r_0 2^M}^+$ with homogeneous Dirichlet boundary conditions on $\partial\mathbb{R}_+^d \cap B_{r_0 2^M}$, we deduce for $r \in [r_0, r_0 2^M]$ using in the second step the estimate (1.19) for $r := r_0 2^M$

$$\left(\fint_{B_r^+} |\nabla \varphi_{M+1}|^2 \, dx\right)^{\frac{1}{2}} \le C_{Mean} \left(\fint_{B_{r_0 2^M}^+} |\nabla \varphi_{M+1}|^2 \, dx\right)^{\frac{1}{2}}$$
$$\le C_{Mean} C_1 2^d \delta_{r_0 2^{(M+1)+1}}^{\frac{1}{3}}.$$

This shows (1.30) for φ_{M+1} and $r \in [r_0, r_0 2^M]$.

□

Step 4 – (Passage to the limit $M \to \infty$)

Proof of Theorem 1. Let the L_m be chosen as in Lemma 1. Let $r_0 = 2^{m_0}$ be chosen as in Lemma 4. By Lemma 4, the estimate (1.30) then holds for all $m \geq -1$ (the start of the induction, i. e. (1.30) for $m = -1$, is provided by Lemma 1).

For $i \neq d$ we then choose $\phi_i^D := \phi_i|_{\mathbb{R}^d_+}$ and $\sigma_{ijk}^D := \sigma_{ijk}|_{\mathbb{R}^d_+}$. By our assumption (1.4), we therefore have to verify the assertion on sublinear growth iv) in our theorem only for ϕ_d^D and σ_d^D.

Part 1: The corrector ϕ_d^D.

We first show that the series $\sum_{m=-1}^{\infty} \varphi_m$ converges absolutely in $H^1(B_r^+)$ for all $r \geq r_0$. By the Poincaré inequality for functions in $H^1(B_r^+)$ with homogeneous Dirichlet boundary conditions on $\partial\mathbb{R}^d_+ \cap B_r$, it suffices to calculate (using (1.30))

$$\sum_{m=-1}^{\infty} \left(\fint_{B_r^+} |\nabla \varphi_m|^2 \, dx \right)^{\frac{1}{2}} \leq C_2 \sum_{m=-1}^{\infty} \delta_{r_0 2^{m+1}}^{\frac{1}{3}} \leq C_2 \sum_{k=m_0}^{\infty} \delta_{2^k}^{\frac{1}{3}}$$

and to use the summability of the $\{\delta_{2^k}^{\frac{1}{3}}\}_k$ (see (1.4)).

Again, combining (1.30) with the Poincaré inequality yields for the sum $\varphi := \sum_{m=-1}^{\infty} \varphi_m = \lim_{M\to\infty} \varphi_M^{\Sigma}$

$$\begin{aligned} \frac{1}{r} \left(\fint_{B_r^+} |\varphi|^2 \, dx \right)^{\frac{1}{2}} &\leq \sup_M \frac{1}{r} \left(\fint_{B_r^+} |\varphi_M^{\Sigma}|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq C_P C_2 \sum_{k=m_0}^{\infty} \min\left\{1, \left(\frac{2^k}{r}\right)^{\frac{d}{2}}\right\} \delta_{2^k}^{\frac{1}{3}} \end{aligned} \tag{1.45}$$

for all $r \geq r_0$.

Next, we show that $\{\sum_{m=-1}^{M} \chi_m \phi_d\}_M$ forms a Cauchy sequence in $H^1(B_r^+)$ for all $r \geq r_0$. Using the fact that $\chi_m \phi_d$ vanishes outside of $B_{r_0 2^{m+1}} \setminus B_{r_0 2^{m-1}}$ (except for $m = -1$, for which $\chi_{-1}\phi_d$ vanishes outside of B_{r_0}), the Poincaré inequality for functions in $H^1(B_{r_0 2^{m+1}})$ that vanish on $\partial B_{r_0 2^{m+1}} \cap \mathbb{R}^d_+$ yields that for any $r > 0$

$$\left(\int_{B_r} |\chi_m \phi_d|^2 \, dx \right)^{\frac{1}{2}} \lesssim r \left(\int_{B_r} |\nabla(\chi_m \phi_d)|^2 \, dx \right)^{\frac{1}{2}}. \tag{1.46}$$

Using (1.18) and again $\operatorname{supp}(\chi_m) \subset B_{r_0 2^{m+1}} \setminus B_{r_0 2^{m-1}}$, we see that

$$\sum_{m=-1}^{\infty} \left(\fint_{B_r} |\nabla(\chi_m \phi_d)|^2 \, dx \right)^{\frac{1}{2}} \leq 2^d C_1 \sum_{m=-1}^{\infty} \min\left\{1, \left(\frac{r_0 2^{m+1}}{r}\right)^{\frac{d}{2}}\right\} \delta_{r_0 2^{m+1}}^{\frac{1}{3}}$$

$$\leq 2^d C_1 \sum_{k=m_0}^{\infty} \min\left\{1, \left(\frac{2^k}{r}\right)^{\frac{d}{2}}\right\} \delta_{2^k}^{\frac{1}{3}}. \tag{1.47}$$

So, $\left\{\sum_{m=-1}^{M} \chi_m \phi_d\right\}_M$ forms a Cauchy sequence in $H^1(B_r^+)$.

The function $\tilde{\varphi} := \varphi + \sum_{m=-1}^{\infty} \chi_m \phi_d = \lim_{M\to\infty} \tilde{\varphi}_M^{\Sigma}$ is a weak solution of the problem (1.14): (1.14) is satisfied on B_r by all $\tilde{\varphi}_M^{\Sigma}$ for which $r_0 2^M \geq r$ holds. Thus, (1.14) carries over to the limit $M \to \infty$ for arbitrarily big radii r. Therefore (1.14) holds globally for the limit $\tilde{\varphi}$, which entails that $\phi_d^D = \phi_d - \tilde{\varphi}$ solves (1.5).

By (1.45), (1.46), and (1.47), via the dominated convergence theorem our assumption (1.4) implies that $\tilde{\varphi}$ and, therefore, $\phi_d^D = \phi_d - \tilde{\varphi}$ are sublinear in the sense

$$\lim_{r\to\infty} \frac{1}{r} \left(\fint_{B_r} |\phi_d^D|^2 \, dx \right)^{\frac{1}{2}} = 0.$$

Part 2: The vector potential σ_d^D.

We now show that $\{\psi_{jk,M}\}_M$ forms a Cauchy sequence in $L^2(B_r^+)$ for all $r \geq r_0$; furthermore, we show that the limit ψ_{jk} has sublinear growth. To this aim, observe that the differences $v_{j,M+1}^n - v_{j,M}^n$ are weak solutions to the problem

$$-\Delta(v_{j,M+1}^n - v_{j,M}^n) = -\nabla \cdot (\eta_n x_j a \nabla(\varphi_{M+1} + \chi_{M+1}\phi_d)) \quad \text{in } \mathbb{R}_+^d, \tag{1.48a}$$

$$v_{j,M+1}^n - v_{j,M}^n = 0 \quad \text{if } j \neq d \text{ on } \partial\mathbb{R}_+^d, \tag{1.48b}$$

$$\partial_d(v_{d,M+1}^n - v_{d,M}^n) = 0 \quad \text{on } \partial\mathbb{R}_+^d. \tag{1.48c}$$

To shorten the subsequent computations, let us use the convention $v_{j,-2}^n \equiv 0$ and $b_{j,-2}^n = 0$; then (1.48) holds also for $M = -2$.

Estimating analogously to the proof of Lemma 2– note that the only difference between the equation satisfied by $v_{j,M}^n$ and the equation satisfied by $v_{j,M+1}^n - v_{j,M}^n$ is the right-hand side, we deduce that for any $r \geq r_0$

$$\frac{1}{r} \left(\fint_{B_r^+} \left|\partial_k \left(v_{j,M+1}^n - v_{j,M}^n - (b_{j,M+1}^n - b_{j,M}^n) \cdot x\right)\right|^2 dx \right)^{\frac{1}{2}}$$

$$\leq C_3 \min\left\{1, \frac{r_0 2^{n+1}}{r}\right\} \left(\fint_{B_{r_0 2^{n+1}}^+} |\nabla(\varphi_{M+1} + \chi_{M+1}\phi_d)|^2 \, dx \right)^{\frac{1}{2}}.$$

Taking the sum with respect to n and using Fubini's theorem, we deduce that the limits $v_{j,M}$ of the series $\sum_{n=-1}^{\infty}(v_{j,M}^n - b_{j,M}^n \cdot x)$ satisfy

$$\frac{1}{r}\left(\fint_{B_r^+} |\partial_k(v_{j,M+1} - v_{j,M})|^2\,dx\right)^{\frac{1}{2}}$$
$$\le C_3 \sum_{n=-1}^{\infty} \min\left\{1, \frac{r_0 2^{n+1}}{r}\right\}\left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla(\varphi_{M+1} + \chi_{M+1}\phi_d)|^2\,dx\right)^{\frac{1}{2}}.$$

Taking the sum with respect to M and estimating the right-hand side by the inequality (1.30) and the estimate (1.18) – both inequalities applied with r replaced by $r_0 2^{n+1}$ and m replaced by $M+1$ – (note again that $\chi_{M+1}\phi_d$ vanishes on $B^+_{r_0 2^{n+1}}$ in case $r_0 2^{M+1-1} \ge r_0 2^{n+1}$ and $M+1 \neq -1$), we infer

$$\frac{1}{r}\sum_{M=-2}^{\infty}\left(\fint_{B_r^+} |\partial_k(v_{j,M+1} - v_{j,M})|^2\,dx\right)^{\frac{1}{2}}$$
$$\le C_3(C_2 + 2^d C_1)\sum_{n=-1}^{\infty}\min\left\{1, \frac{r_0 2^{n+1}}{r}\right\}\sum_{M=-2}^{\infty}\min\left\{1, 2^{d(M+1-n)/2}\right\}\delta^{\frac{1}{3}}_{r_0 2^{M+1+1}}. \tag{1.49}$$

Now, by this estimate and the dominated convergence theorem, it is sufficient to show

$$\sum_{n=-1}^{\infty}\sum_{M=-2}^{\infty}\min\left\{1, 2^{d(M+1-n)/2}\right\}\delta^{\frac{1}{3}}_{r_0 2^{M+1+1}} < \infty \tag{1.50}$$

in order to obtain both the Cauchy sequence property of $\nabla v_{j,M}$ in $L^2(B_r)$ and the sublinearity property

$$\lim_{r\to\infty}\frac{1}{r}\left(\fint_{B_r^+}|\partial_k v_j|^2\,dx\right)^{\frac{1}{2}} \le \lim_{r\to\infty}\sup_{M\ge -1}\frac{1}{r}\left(\fint_{B_r^+}|\partial_k v_{j,M}|^2\,dx\right)^{\frac{1}{2}} = 0.$$

Note that by $\psi_{jk} := \partial_k v_j - \partial_j v_k$ and $\sigma^D_{djk} = \sigma_{djk} - \psi_{jk}$, this estimate then directly implies the desired result

$$\lim_{r\to\infty}\frac{1}{r}\left(\fint_{B_r^+}|\sigma^D_d|^2\,dx\right)^{\frac{1}{2}} = 0.$$

Furthermore, the $\psi_{jk,M}$ are solutions to the equation (1.21). Since we can pass to the limit $M\to\infty$ in the weak formulation of (1.21) for any smooth compactly supported test function, this shows that the limit $\sigma^D_{djk} := \lim_{M\to\infty}(\sigma_{djk} - \psi_{jk,M})$ solves the equation (1.6).

To see that (1.50) holds, we just need to estimate

$$\sum_{n=-1}^{\infty}\sum_{M=-2}^{\infty}\min\left\{1,2^{d(M+1-n)/2}\right\}\delta_{r_0 2^{M+1+1}}^{\frac{1}{3}}$$
$$\leq \sum_{M=-2}^{\infty}\left(M+2+\frac{1}{1-2^{-\frac{d}{2}}}\right)\delta_{r_0 2^{M+1+1}}^{\frac{1}{3}} = \sum_{k=m_0}^{\infty}\left(k-m_0+\frac{1}{1-2^{-\frac{d}{2}}}\right)\delta_{2^k}^{\frac{1}{3}}$$

and use the summability property (1.4). This finishes the proof of our theorem. □

1.4 Proofs of the Large-Scale Regularity Results

1.4.1 Constant Coefficient Regularity and Caccioppoli Estimate

Before proving Theorem 2, we first prove the following Caccioppoli inequality:

Lemma 6. *Let a be a coefficient field satisfying the ellipticity and boundedness assumptions as in* (1)*. For any a-harmonic function u on B_R^+ subject to homogeneous Dirichlet boundary conditions on $\partial\mathbb{R}^d_+ \cap \partial B_R^+$, the estimate*

$$\left(\fint_{B^+_{\frac{r}{2}}} |\nabla u|^2\,dx\right)^{\frac{1}{2}} \lesssim \frac{1}{R}\left(\fint_{B_R^+} |u|^2\,dx\right)^{\frac{1}{2}}. \tag{1.51}$$

holds.

Proof. Testing the equation

$$-\nabla\cdot(a\nabla u) = 0 \quad \text{in } B_R^+$$

with $\eta^2 u$, where η is a radial cut-off with $\eta\equiv 1$ in $B_{\frac{r}{2}}$, $\eta\equiv 0$ outside of B_R, $0\leq\eta\leq 1$ everywhere, and $|\nabla\eta|\leq\frac{4}{R}$, we get

$$\int_{B_R^+} \eta^2\nabla u\cdot a\nabla u + 2\eta u\nabla\eta\cdot a\nabla u\,dx = 0.$$

Note that the boundary terms vanish as $\eta^2 u$ is zero on ∂B_R^+. Using the uniform ellipticity of a and Young's inequality allows us to write

$$\lambda\int_{B_R^+}\eta^2|\nabla u|^2\,dx \leq 2\int_{B_R^+}|\eta u\nabla\eta\cdot a\nabla u|\,dx \leq \int_{B_R^+}\frac{\lambda}{2}\eta^2|\nabla u|^2+\frac{2}{\lambda}|\nabla\eta|^2u^2\,dx.$$

The properties of η finish the argument. □

The following classical regularity properties of constant coefficient elliptic equations will play a crucial rule in the derivation of the excess decay estimate.

Lemma 7. *Let v be a weak solution to the constant coefficient equation $-\nabla \cdot (a_{hom} \nabla v) = 0$ in $B^+_{R'}$ with homogeneous Dirichlet boundary conditions on $B^+_{R'} \cap \partial \mathbb{R}^d_+$, where a_{hom} is a positive definite matrix. Then there exists some $\beta = \beta(d, \lambda) > 0$ such that for any positive $\rho \leq \frac{1}{2} R'$ and any positive $r \leq \frac{1}{2} R'$ the following estimates hold:*

$$r^2 \sup_{B^+_r} |\nabla^2 v|^2 \lesssim \left(\frac{r}{R'}\right)^2 \fint_{B^+_{R'}} |\nabla v|^2 \, dx, \tag{1.52a}$$

$$\int_{B^+_{R'} \setminus B^+_{R'-2\rho}} |\nabla v|^2 \, dx \lesssim R' \left(\frac{\rho}{R'}\right)^\beta \int_{\partial B^+_{R'}} |\nabla^{tan} v|^2 \, dS, \tag{1.52b}$$

$$\sup_{B^+_{R'-\rho}} (|\nabla^2 v|^2 + \frac{1}{\rho^2} |\nabla v|^2) \lesssim \frac{1}{\rho^2} \left(\frac{R'}{\rho}\right)^d \fint_{B^+_{R'}} |\nabla v|^2 \, dx. \tag{1.52c}$$

Proof. For the third estimate, notice that if $x' \in S$ where $S = B^+_{R'-\rho} \cap \{x_d \geq \frac{\rho}{2}\}$ then v is a_{hom}-harmonic on $B_{\frac{\rho}{2}}(x')$. Therefore, for these x' we have the inner regularity estimate

$$\sup_{y \in B_{\rho/4}(x')} \rho^2 |\nabla^2 v(y)|^2 + \sup_{y \in B_{\rho/4}(x')} |\nabla v(y)|^2 \lesssim \frac{1}{\rho^d} \int_{B_{\frac{\rho}{2}}(x')} |\nabla v|^2 \, dx, \tag{1.53}$$

which follows by an iterated use of the Caccioppoli inequality on balls to derive an H^k estimate for k large enough and a subsequent use of the Sobolev embedding.

For $x' \in S_\partial$, where $S_\partial = \partial \mathbb{R}^d_+ \cap B^+_{R'-\rho}$, we get an analogous estimate for half-balls: In this case, the result can also be shown by proving H^k regularity estimates for k large enough followed by the Sobolev embedding. The derivation of H^k-type regularity estimates is again standard: One may proceed by repeatedly using the Caccioppoli estimate for v and its tangential (higher) derivatives $\partial_{i_1} \ldots \partial_{i_{k-1}} v$ with $i_1, \ldots, i_{k-1} \neq d$. To obtain estimates on higher derivatives which involve multiple derivatives in the normal direction e_d – only estimates for derivatives containing a single normal derivative are provided by the aforementioned applications of the Caccioppoli inequality – one directly uses the equation satisfied by v. Thus, for $x' \in S_\partial$ we have

$$\sup_{y \in B^+_{\frac{\rho}{2}}(x')} \rho^2 |\nabla^2 v(y)|^2 + \sup_{y \in B^+_{\frac{\rho}{2}}(x')} |\nabla v(y)|^2 \lesssim \frac{1}{\rho^d} \int_{B^+_\rho(x')} |\nabla v|^2 \, dx. \tag{1.54}$$

The estimate (1.52a) is an immediate consequence of (1.54) with $\rho := R'$ and $x' = 0$. To obtain (1.52c) let

$$s = \sup_{x' \in S} \sup_{y \in B_{\rho/4}(x')} (|\nabla^2 v(y)|^2 + \frac{1}{\rho^2} |\nabla v(y)|^2),$$

$$s_\partial = \sup_{x'\in S_\partial} \sup_{y\in B^+_{\frac{\rho}{2}}(x')} (|\nabla^2 v(y)|^2 + \frac{1}{\rho^2}|\nabla v(y)|^2).$$

Using (1.53) and (1.54), we may then write

$$\begin{aligned}
&\sup_{x\in B^+_{R'-\rho}} (|\nabla^2 v|^2 + \frac{1}{\rho^2}|\nabla v|^2) \le \max\{s, s_\partial\}\\
&\lesssim \sup_{x'\in S\cup S_\partial} \frac{1}{\rho^{d+2}} \int_{B_\rho(x')\cap\mathbb{R}^d_+} |\nabla v|^2\, dx \lesssim \frac{1}{\rho^2}\left(\frac{R'}{\rho}\right)^d \fint_{B^+_{R'}} |\nabla v|^2\, dx,
\end{aligned}$$

finishing the proof of (1.52c).

Finally, for the inequality (1.52b) we first extend v to $B_{R'}$ by odd-reflection. The extended v satisfies the elliptic equation

$$-\nabla\cdot(\tilde{a}_{hom}\nabla v) = 0 \qquad \text{in } B_{R'}$$

with

$$(\tilde{a}_{hom})_{ij} = \begin{cases} (a_{hom})_{ij} & \text{for } x_d > 0,\\ (a_{hom})_{ij} & \text{for } x_d < 0 \text{ and } i \ne d, j\ne d,\\ -(a_{hom})_{ij} & \text{for } x_d < 0 \text{ and } i = d, j\ne d,\\ -(a_{hom})_{ij} & \text{for } x_d < 0 \text{ and } i \ne d, j = d,\\ (a_{hom})_{ij} & \text{for } x_d < 0 \text{ and } i = j = d. \end{cases}$$

If we then let $\bar{v}$ be the harmonic extension of $v|_{\partial B_{R'}}$ to $B_{R'}$, we have the estimate $||\nabla\bar{v}||_{L^{\frac{2}{1-\beta}}(B_{R'})} \lesssim R'^{\frac{1-d\beta}{2}}||\nabla^{tan} v||_{L^2(\partial B_{R'})}$, provided that $\beta > 0$ is not too large. This is a consequence of interpolation between

$$\|\nabla\bar{v}\|_{L^{\frac{2d}{d-1}}(B_{R'})} \lesssim \|\nabla^{tan} v\|_{L^2(\partial B_{R'})}, \tag{1.55}$$

which results from decomposing v in terms of the spherical harmonics on $\partial B_{R'}$ and using Bochner's identity, and

$$\|\nabla\bar{v}\|_{L^2(B_{R'})} \lesssim R'\|\nabla^{tan} v\|_{L^2(\partial B_{R'})}, \tag{1.56}$$

which follows from (1.55) via Hölder's inequality. We then use Meyers' estimate [44], which states that for any $\beta > 0$ small enough (depending on d and λ), the solution $v-\bar{v}$ to the equation

$$\begin{aligned}
-\nabla\cdot(\tilde{a}_{hom}\nabla(v-\bar{v})) &= \nabla\cdot(\tilde{a}_{hom}\nabla\bar{v}) && \text{in } B_{R'},\\
v-\bar{v} &= 0 && \text{on } \partial B_{R'}
\end{aligned}$$

satisfies the bound $||\nabla(v-\bar{v})||_{L^{\frac{2}{1-\beta}}(B_{R'})} \lesssim ||\tilde{a}_{hom}\nabla\bar{v}||_{L^{\frac{2}{1-\beta}}(B_{R'})}$. Combining this estimate with the bound on $\bar{v}$ obtained above yields that

$$||\nabla v||_{L^{\frac{2}{1-\beta}}(B^+_{R'})} \lesssim R'^{\frac{1-d\beta}{2}}||\nabla^{tan}v||_{L^2(\partial B^+_{R'})}.$$

It then follows by Hölder's inequality that

$$\left(\int_{B^+_{R'}\setminus B^+_{R'-2\rho}} |\nabla v|^2\,dx\right)^{\frac{1}{2}} \leq |B^+_{R'}\setminus B^+_{R'-2\rho}|^{\frac{\beta}{2}}\left(\int_{B^+_{R'}} |\nabla v|^{\frac{2}{1-\beta}}\,dx\right)^{\frac{1-\beta}{2}}$$
$$\lesssim (R')^{\frac{1-\beta}{2}}\rho^{\frac{\beta}{2}}\left(\int_{\partial B^+_{R'}} |\nabla^{tan}v|^2\,dS\right)^{\frac{1}{2}},$$

concluding the proof of (1.52b). □

1.4.2 Proof of Theorem 2: A Large-Scale $C^{1,\alpha}$- Excess Decay and Mean-Value Property

We now turn to the proof of the excess decay estimate.

Proof of Theorem 2. For convenience throughout this proof we make use of the Einstein summation convention, i. e. whenever an index appears twice in an expression, summation with respect to the index is implied.

Step 1 – (Comparison of u to solution of homogenized problem/ Main step of the argument)

In the first step of the proof, we show that for each $r < R$ there exists $b \in \mathbb{R}$ such that the estimate

$$\begin{aligned} &\fint_{B^+_r} |\nabla u - b(e_d + \nabla\phi^D_d)|^2\,dx \\ &\lesssim \left(\left(\frac{r}{R}\right)^2(1+\delta^2) + \left(\frac{R}{r}\right)^d \delta^{2\beta/(d+2+\beta)}\right)\fint_{B^+_R} |\nabla u|^2\,dx \end{aligned} \tag{1.57}$$

is valid, with the abbreviation

$$\delta := \max\left\{\delta^D_{2r}, \delta^D_R\right\}.$$

Note that for $r \in \left[\frac{R}{4}, R\right]$ the estimate trivially holds for $b = 0$. It is, therefore, sufficient to show (1.57) for $r \leq \frac{r}{4}$. To do this, we first choose a radius $R' \in (\frac{R}{2}, R)$ such that

$$\int_{\partial B^+_{R'}} |\nabla^{tan} u|^2 \, dx \lesssim \frac{1}{R} \int_{B^+_R \setminus B^+_{\frac{r}{2}}} |\nabla u|^2 \, dx \lesssim \frac{1}{R} \int_{B^+_R} |\nabla u|^2 \, dx. \tag{1.58}$$

We know that such a radius exists by writing the middle integral in polar coordinates and using that $\nabla^{tan} u = 0$ on $\partial \mathbb{R}^d_+ \cap B_R$.

Let v be the a_{hom}-harmonic function that coincides with u on $\partial B^+_{R'}$. To show the estimate (1.57) we compare ∇u to ∇v corrected as suggested by the two-scale expansion as given in (3) of Section 0.1 of the Introduction. Notice that, due to the boundary conditions of v, we know that $\nabla v(0)$ only has a normal component. This observation allows us to write

$$\begin{aligned} &\int_{B^+_r} |\nabla u - \partial_d v(0)(e_d + \nabla \phi^D_d)|^2 \, dx \\ \lesssim &\int_{B^+_r} |(\nabla v - \nabla v(0))(\mathrm{id} + \nabla \phi^D)|^2 \, dx \\ &+ \int_{B^+_r} |\nabla u - \partial_i v(e_i + \nabla \phi^D_i)|^2 \, dx. \end{aligned} \tag{1.59}$$

Notice that the second term on the right-hand side corresponds to one piece of the gradient of the "homogenization error" coming from the ansatz for v given by the two-scale expansion. To estimate this term, we first derive an estimate for the gradient of

$$w := u - (v + \eta \phi^D_i \partial_i v),$$

where η is a cut-off with $0 \le \eta \le 1$, $\eta \equiv 1$ in $B^+_{R'-2\rho}$, $\eta \equiv 0$ outside of $B^+_{R'-\rho}$, and $|\nabla \eta| \le \frac{4}{\rho}$. We will later optimize the width of the boundary-layer introduced by ρ, but for the moment we only assume that $0 < \rho \le \frac{1}{4} R'$.

The function w distributionally satisfies the equation

$$-\nabla \cdot (a \nabla w) = \nabla \cdot ((1-\eta)(a - a_{hom}) \nabla v + (\phi^D_i a - \sigma^D_i) \nabla(\eta \partial_i v)) \quad \text{in } B^+_{R'}. \tag{1.60}$$

To see this, one uses that u is a-harmonic, that ϕ^D_i solves the corrector equation (1.5) on $B^+_{R'}$ when $i = d$ and the whole-space corrector equation (1) when $i \neq d$, and the defining property (1.6) of σ^D, which gives

$$\begin{aligned} &- \nabla \cdot (a \nabla w) \\ &= \nabla \cdot \big(a \nabla v + \eta \partial_i v a \nabla \phi^D_i\big) + \nabla \cdot (\phi^D_i a \nabla(\eta \partial_i v)) \\ &= \nabla \cdot \big((1-\eta) a \nabla v + \eta \partial_i v a(e_i + \nabla \phi^D_i)\big) + \nabla \cdot (\phi^D_i a \nabla(\eta \partial_i v)) \\ &= \nabla \cdot ((1-\eta) a \nabla v) + \nabla(\eta \partial_i v) \cdot a(e_i + \nabla \phi^D_i) + \nabla \cdot (\phi^D_i a \nabla(\eta \partial_i v)) \end{aligned}$$

$$= \nabla \cdot ((1-\eta)(a-a_{hom})\nabla v) + \nabla(\eta\partial_i v)\cdot(a(e_i+\nabla\phi_i^D) - a_{hom}e_i)$$
$$+ \nabla\cdot(\phi_i^D a\nabla(\eta\partial_i v))$$
$$= \nabla \cdot ((1-\eta)(a-a_{hom})\nabla v) + \nabla(\eta\partial_i v)\cdot(\nabla\cdot\sigma_i^D) + \nabla\cdot(\phi_i^D a\nabla(\eta\partial_i v))$$

distributionally on $B^+_{R'}$. To complete the calculation, we use the skew-symmetry of the vector potential σ^D_{ijk} in the form $\nabla(\eta\partial_i v)\cdot(\nabla\cdot\sigma_i^D) = -\nabla\cdot(\sigma_i^D\nabla(\eta\partial_i v))$.

Notice that, due to the cut-off η, the boundary conditions of ϕ_d^D, and the boundary conditions of v, w satisfies homogeneous Dirichlet boundary conditions on $\partial B^+_{R'}$. Therefore, the standard energy estimate for the equation (1.60) reads

$$\left(\int_{B^+_{R'}} |\nabla w|^2\,dx\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{\lambda}\left(\int_{B^+_{R'}} |(1-\eta)(a-a_{hom})\nabla v + (\phi_i^D a - \sigma_i^D)\nabla(\eta\partial_i v)|^2\,dx\right)^{\frac{1}{2}}.$$

The boundedness of a and a_{hom} and the properties of η then imply

$$\int_{B^+_{R'-2\rho}} |\nabla u - \partial_i v(e_i+\nabla\phi_i^D)|^2\,dx$$
$$\lesssim \int_{B^+_{R'}\setminus B^+_{R'-2\rho}} |\nabla v|^2\,dx + \int_{B^+_{R'-\rho}} |(\phi^D,\sigma^D)|^2(|\nabla^2 v|^2 + \frac{1}{\rho^2}|\nabla v|^2)\,dx. \quad (1.61)$$

Due to the conditions that we have placed on r, ρ, and R' we have $r \leq R'-2\rho$. Therefore the second term on the right-hand side of (1.59) can be estimated by the formula (1.61). This yields

$$\int_{B_r^+} |\nabla u - \partial_d v(0)(e_d+\nabla\phi_d^D)|^2\,dx$$
$$\lesssim \int_{B_r^+} |\nabla v - \nabla v(0)|^2|\mathrm{id}+\nabla\phi^D|^2\,dx$$
$$+ \int_{B^+_{R'}-B^+_{R'-2\rho}} |\nabla v|^2\,dx + \int_{B^+_{R'-\rho}} |(\phi^D,\sigma^D)|^2(|\nabla^2 v|^2 + \frac{1}{\rho^2}|\nabla v|^2)\,dx$$
$$\leq r^2 \sup_{B_r^+}|\nabla^2 v|^2 \int_{B_r^+} |\mathrm{id}+\nabla\phi^D|^2\,dx$$
$$+ \int_{B^+_{R'}\setminus B^+_{R'-2\rho}} |\nabla v|^2\,dx + \sup_{B^+_{R'-\rho}}(|\nabla^2 v|^2 + \frac{1}{\rho^2}|\nabla v|^2)\int_{B_R^+} |(\phi^D,\sigma^D)|^2\,dx.$$
$$(1.62)$$

To further process this estimate, we exploit that v solves the constant coefficient equation $-\nabla \cdot (a_{hom}\nabla v) = 0$ in $B^+_{R'}$ with homogeneous Dirichlet boundary conditions on $\partial\mathbb{R}^d_+ \cap B_{R'}$; thus the estimates (1.52) are available. Furthermore, notice that the difference $v-u$ solves

$$\begin{aligned} -\nabla \cdot (a_{hom}\nabla(v-u)) &= \nabla \cdot (a_{hom}\nabla u) && \text{in} \quad B^+_{R'}, \\ v-u &= 0 && \text{on} \quad \partial B^+_{R'}. \end{aligned}$$

Testing this equation with $v-u$ and using Young's inequality yields

$$\int_{B^+_{R'}} |\nabla v|^2\,dx \leq 2\int_{B^+_{R'}} |\nabla u|^2\,dx + 2\int_{B^+_{R'}} |\nabla (v-u)|^2\,dx \lesssim \int_{B^+_{R'}} |\nabla u|^2\,dx. \tag{1.63}$$

Applying (1.52) and (1.63) to the equation (1.62), and using that $R' \in (\frac{R}{2}, R)$ as well as (1.58) and the equality $\nabla^{tan}u = \nabla^{tan}v$ on $\partial B^+_{R'}$, gives that

$$\begin{aligned} &\fint_{B^+_r} |\nabla u - \partial_d v(0)(e_d + \nabla\phi^D_d)|^2\,dx \\ &\lesssim \Bigg(\left(\frac{r}{R}\right)^2 \fint_{B^+_r} |\mathrm{id} + \nabla\phi^D|^2\,dx \\ &\qquad + \left(\frac{R}{r}\right)^d \left(\left(\frac{\rho}{R}\right)^\beta + \left(\frac{R}{\rho}\right)^{d+2} (\delta^D_R)^2 \right) \Bigg) \fint_{B^+_R} |\nabla u|^2\,dx. \end{aligned} \tag{1.64}$$

Now, we choose a specific ρ. Recall that we required $0 < \rho \leq \frac{1}{4}R'$. By varying ρ subject to this condition, we can obtain $\frac{\rho}{R} = s$ for any $s \in (0, \frac{1}{8}]$. We select ρ to satisfy $\frac{\rho}{R} = \min\{(\delta^D_R)^{2/(d+2+\beta)}, \frac{1}{8}\}$. Plugging this into (1.64) and using $\delta^D_R \leq 1$ (which we may assume by choosing $C_\alpha(d,\lambda)$ large enough) results in

$$\begin{aligned} &\fint_{B^+_r} |\nabla u - \partial_d v(0)(e_d + \nabla\phi^D_d)|^2\,dx \\ &\lesssim \left(\left(\frac{r}{R}\right)^2 \fint_{B^+_r} |\mathrm{id} + \nabla\phi^D|^2\,dx + \left(\frac{R}{r}\right)^d (\delta^D_R)^{2\beta/(d+2+\beta)} \right) \fint_{B^+_R} |\nabla u|^2\,dx. \end{aligned}$$

For the first integral on the right-hand side, notice that $x_d + \phi^D_d$ is a-harmonic in B^+_{2r} and vanishes on $\partial\mathbb{R}^d_+$. So, to estimate $\fint_{B^+_r} |e_d + \nabla\phi^D_d|^2\,dx$ we may use (1.51). To handle the terms of the form $e_i + \nabla\phi^D_i$ for $i \neq d$, we use the whole-space Caccioppoli estimate. We find that

$$\begin{aligned}\fint_{B_r^+} |\mathrm{id} + \nabla\phi^D|^2\,dx &\lesssim \fint_{B_r^+} |e_d + \nabla\phi_d^D|^2\,dx + \sum_{i=1}^{d-1} \fint_{B_r} |e_i + \nabla\phi_i^D|^2\,dx \\ &\lesssim \frac{1}{r^2}\left(\fint_{B_{2r}^+} |x_d + \phi_d^D|^2\,dx + \sum_{i=1}^{d-1} \fint_{B_{2r}} |x_i + \phi_i^D|^2\,dx\right).\end{aligned} \tag{1.65}$$

Young's inequality yields

$$\frac{1}{r^2}\left(\fint_{B_{2r}^+} |x_d + \phi_d^D|^2\,dx + \sum_{i=1}^{d-1} \fint_{B_{2r}} |x_i + \phi_i^D|^2\,dx\right) \lesssim \left(1 + (\delta_{2r}^D)^2\right). \tag{1.66}$$

We can then conclude that

$$\begin{aligned}&\fint_{B_r^+} |\nabla u - \partial_d v(0)(e_d + \nabla\phi_d^D)|^2\,dx \\ &\lesssim \left(\left(\frac{r}{R}\right)^2 (1 + \delta^2) + \left(\frac{R}{r}\right)^d \delta^{2\beta/(d+2+\beta)}\right) \fint_{B_R^+} |\nabla u|^2\,dx,\end{aligned} \tag{1.67}$$

where $\delta := \max\{\delta_{2r}^D, \delta_R^D\}$.

Step 2– (Proof of the Dirichlet half-space excess decay.)

For any two radii $\tilde{r}$ and $\tilde{R}$ with $r^* \le \tilde{r} \le \tilde{R} \le R$, we can rephrase (1.67) in terms of Exc^D: Notice that for any $b \in \mathbb{R}$ the function $u - b(x_d + \phi_d^D)$ is a-harmonic on $B_{\tilde{R}}^+$ with homogeneous Dirichlet boundary conditions on $\partial\mathbb{R}_+^d \cap B_R$. Applying (1.67) to $u - b(x_d + \phi_d^D)$ and taking the infimum with respect to b yields

$$\mathrm{Exc}^D(\tilde{r}) \le C(d,\lambda)\left(\left(\frac{\tilde{r}}{\tilde{R}}\right)^2 (1 + \delta^2) + \left(\frac{\tilde{R}}{\tilde{r}}\right)^d \delta^{2\beta/(d+2+\beta)}\right) \mathrm{Exc}^D(\tilde{R}).$$

Letting $\theta = \tilde{r}/\tilde{R}$ and using $\delta \le 1$ gives that

$$\mathrm{Exc}^D(\tilde{r}) \le C(d,\lambda)\left(2\theta^2 + \delta^{2\beta/(d+2+\beta)}\theta^{-d}\right) \mathrm{Exc}^D(\tilde{R}), \tag{1.68}$$

where the fixed constant $C(d,\lambda)$ comes from (1.67) and where we have used $\delta \le \frac{1}{C_\alpha(d,\lambda)} \le 1$ (the latter inequality holding w. l. o. g.).

We now choose θ and the constant $C_\alpha(d,\lambda)$ in the smallness condition (1.9) in such a way that

$$C(d,\lambda)(2\theta^2 + \delta^{2\beta/(d+2+\beta)}\theta^{-d}) \le \theta^{2\alpha} \tag{1.69}$$

is satisfied. To do this we first select $\theta \in (0,1)$ such that $2C(d,\lambda)\theta^2 \le \frac{1}{2}\theta^{2\alpha}$ holds. We then select the constant $C_\alpha(d,\lambda)$ in (1.9) to be large enough to ensure $C(d,\lambda)\delta^{2\beta/(d+2+\beta)}\theta^{-d} \le \frac{1}{2}\theta^{2\alpha}$. This entails the estimate

$$\mathrm{Exc}^D(\theta\tilde{R}) \le \theta^{2\alpha}\mathrm{Exc}^D(\tilde{R}) \tag{1.70}$$

for all $\tilde{R} \in [\frac{1}{\theta}r_\alpha^*, R]$.

The half-space excess decay estimate for arbitrary r, R with $r_\alpha^* \le r \le R$ follows by iterating the estimate (1.70). As this procedure is both straightforward and a standard argument, we omit it.

Step 3 – (Proof of the coercivity of the excess expression.)

As the left-hand side of (1.11) is a second-order polynomial in b, to establish the desired result it is sufficient to show an estimate of the form

$$\fint_{B_r^+} |b(e_d + \nabla\phi_d^D)|^2\,dx \ge \frac{1}{2^{d+2}}|b|^2. \tag{1.71}$$

We take η to be a cutoff with $\eta \equiv 1$ in $B^+_{\frac{r}{2}}$, $\eta \equiv 0$ outside B_r^+, $0 \le \eta \le 1$ everywhere, and $|\nabla\eta| \le \frac{2}{r}$. We then have

$$\begin{aligned}\fint_{B_r^+} |b(e_d + \nabla\phi_d^D)|^2\,dx &\ge |b|^2 \fint_{B_r^+} \eta|e_d + \nabla\phi_d^D|^2\,dx \\ &\ge |b|^2 \fint_{B_r^+} \eta\,dx \left| e_d + \frac{1}{\fint_{B_r^+}\eta\,dx}\fint_{B_r^+} \eta\nabla\phi_d^D\,dx\right|^2 \\ &\ge |b|^2 \fint_{B_r^+} \eta\,dx \left| e_d - \frac{1}{\fint_{B_r^+}\eta\,dx}\fint_{B_r^+} \phi_d^D\nabla\eta\,dx\right|^2.\end{aligned} \tag{1.72}$$

Notice that the second of the above inequalities follows from an application of Jensen's inequality. Also, in the third inequality the boundary term has vanished due to the Dirichlet boundary conditions satisfied by ϕ_d^D.

Another use of Hölder's inequality yields that

$$\frac{1}{\fint_{B_r^+}\eta\,dx}\left|\fint_{B_r^+} \phi_d^D\nabla\eta\,dx\right| \le 2^{d+1}\delta_r^D.$$

We may assume that $C_\alpha(d,\lambda)$ in (2.10) is chosen large enough to ensure that $2^{d+1}\delta_r^D \le \frac{1}{2}$. Estimating $\fint_{B_r^+}\eta\,dx \ge (\frac{1}{2})^d$, we see that (1.71) now follows from (1.72).

Step 4– (Proof of the mean-value property.)

Let $r^*_{\frac{1}{2}} \leq r \leq R$; denote by b_ρ the value of b for which the infimum in the definition of the tilt-excess $\mathrm{Exc}^D(\rho)$ is attained. We then have

$$\begin{aligned}
\fint_{B_r^+} |\nabla u|^2\,dx &\lesssim \mathrm{Exc}^D(r) + |b_r|^2 \\
&\lesssim \mathrm{Exc}^D(R) + |b_r|^2 \\
&\lesssim \fint_{B_R^+} |\nabla u|^2\,dx + |b_R|^2 + |b_r - b_R|^2. \qquad (1.73)
\end{aligned}$$

Here, we have used (1.65), (1.66), and $\delta^D_{2r} \leq 1$ for the first inequality, the excess decay for $\alpha = \frac{1}{2}$ for the second, and the definition of the adapted excess and Young's inequality for the third.

To complete our argument it remains to estimate $|b_R|^2$ and $|b_r - b_R|^2$. First, by (1.71) and the triangle inequality, we easily infer

$$|b_R|^2 \lesssim \fint_{B_R^+} |b_R(e_d + \nabla\phi^D_d)|^2\,dx \lesssim \mathrm{Exc}^D(R) + \fint_{B_R^+} |\nabla u|^2\,dx \lesssim \fint_{B_R^+} |\nabla u|^2\,dx.$$

To estimate $|b_r - b_R|$, let $\rho \in [\max\{r^*_{\frac{1}{2}}, \frac{r}{2}\}, R]$. Then the coercivity property (1.71) and the triangle inequality entail

$$\begin{aligned}
|b_\rho - b_R|^2 &\lesssim \fint_{B_\rho^+} |(b_\rho - b_R)e_d + (b_\rho - b_R)\nabla\phi^D_d|^2\,dx \\
&\lesssim \mathrm{Exc}^D(\rho) + \mathrm{Exc}^D(R) \\
&\lesssim \fint_{B_R^+} |\nabla u|^2\,dx.
\end{aligned}$$

Choose $N \in \mathbb{N}_0$ such that $\frac{R}{2^{N+1}} \leq r \leq \frac{R}{2^N}$. The triangle inequality, the coercivity (1.71), and the excess decay for $\alpha = \frac{1}{2}$ then allows us to write

$$\begin{aligned}
|b_r - b_R|^2 &\leq \left(|b_r - b_{R2^{-N}}| + \sum_{n=1}^{N} |b_{R2^{-n}} - b_{R2^{-(n-1)}}| \right)^2 \\
&\lesssim \left(\sum_{n=0}^{N} \left(\mathrm{Exc}^D(R2^{-n})\right)^{\frac{1}{2}} \right)^2 \lesssim \left(\sum_{n=0}^{N} 2^{-n/2}\mathrm{Exc}^D(R)^{\frac{1}{2}} \right)^2 \\
&\lesssim \mathrm{Exc}^D(R).
\end{aligned}$$

In total, (1.73) therefore entails the desired mean-value property.

□

1.4.3 A $C^{1,\alpha}$ Liouville Principle

Using the excess decay proved above we then obtain a $C^{1,\alpha}$- Liouville principle. Here is the argument:

Proof of Corollary 1. The Caccioppoli estimate from Lemma 6 shows that the growth condition (1.13) implies that

$$\lim_{R\to\infty} \frac{1}{R^{2\alpha}} \fint_{B_R^+} |\nabla u|^2 \, dx = 0.$$

This, in turn, gives that

$$\lim_{R\to\infty} \frac{1}{R^{2\alpha}} \mathrm{Exc}^D(R) = 0.$$

By Theorem 1 and Theorem 2 there exists a radius $r_\alpha^* > 0$ such that the excess decay (1.10) holds for $R \geq r \geq r_\alpha^*$. In particular, keeping r fixed and passing to the limit $R \to \infty$, we deduce $\mathrm{Exc}^D(r) = 0$ for any $r \geq r_\alpha^*$. Since the coercivity property (1.11) implies that the infimum in the definition of the excess is attained and since we have $u = 0$ on $\partial\mathbb{R}^d_+$, we find that for all $r \geq r_\alpha^*$ there exists $b \in \mathbb{R}$ such that $u(x) = b(x_d + \phi_d^D)$ in B_r^+. $\square$

Chapter 2

A Large-Scale Regularity Theory for Random Linear Elliptic Operators on $\mathbb{R}^d_+$ with Homogeneous Neumann Boundary Data

2.1 Set-Up

In this chapter we are interested in the large-scale regularity of solutions to linear elliptic equations with random coefficients and homogeneous Neumann boundary data. This is, of course, a logical continuation of the previous chapter in which we have considered the situation for homogeneous Dirichlet boundary data. In the current chapter we work with the following model case: Let $u \in H^1_{loc}(\overline{\mathbb{R}}^d_+)$, where this is the space of functions such that for any $\Omega \Subset \mathbb{R}^d$ we have that $u \in H^1(\Omega \cap \mathbb{H}^d_+)$, be a distributional solution of

$$-\nabla \cdot (a\nabla u) = 0 \qquad \text{in} \quad \mathbb{R}^d_+, \tag{2.1a}$$

$$e_d \cdot a\nabla u = 0 \qquad \text{on} \quad \partial\mathbb{R}^d_+, \tag{2.1b}$$

where a is the restriction to the half-space of a coefficient field $a(x) : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ that is bounded and uniformly elliptic on $\mathbb{R}^d$.

As in the previous chapter, the current chapter consists mainly of two theorems. In the first theorem, assuming that for a given realization of the random coefficients a there exists a whole-space generalized corrector satisfying (1.4) from the previous chapter, we construct a sublinear Neumann boundary corrector for the half-space, which we denote (ϕ^N, σ^N). Our construction of the Neumann half-space corrector in this theorem is heavily motivated by the construction of the Dirichlet half-space corrector in the previous chapter. The main difference between the construction we use in the previous chapter and that here is the energy estimate that we use to treat the near-field contributions of the correction to the corrector (see Lemma 1 of the current chapter). As in the Dirichlet setting, in the second theorem we use the sublinearity of (ϕ^N, σ^N) to prove a large-scale $C^{1,\alpha}$- excess decay for solutions of (2.1) and then, in a corollary, we obtain the desired $C^{1,\alpha}$-Liouville principle.

Just like in the previous chapter, we must modify the excess used in [31] to take into account the boundary condition (2.1b). To see how to so this we should formulate the Liouville principle that we expect to show. Again, we first consider the constant coefficient setting: In the constant coefficient setting, when $a = a_{hom}$, if a function u solves (2.1) and conforms to the growth condition $|u(x) - u(0)| \lesssim |x|^{1+\alpha}$, then u is expected to be of the form $b \cdot x + c$ for $b \in \mathbb{R}^d$ such that $e_d \cdot a_{hom} b = 0$ and $c \in \mathbb{R}$. Here the condition $e_d \cdot a_{hom} b = 0$ comes from the boundary condition (2.1b). Defining the set

$$B := \left\{ b \in \mathbb{R}^d \mid e_d \cdot a_{hom} b = 0 \right\}, \tag{2.2}$$

in analogue to when $a = a_{hom}$, if (2.1) has heterogeneous coefficients we expect that functions u satisfying the same conditions will be of the form $u = b \cdot x + \phi_b^N + c$, where $b \in B$ and $c \in \mathbb{R}^d$. So, for $b \in B$ we should construct a sublinear Neumann half-space corrector ϕ_b^N solving

$$-\nabla \cdot a \nabla(\phi_b^N + b \cdot x) = 0 \qquad \text{in} \quad \mathbb{R}^d_+, \tag{2.3a}$$

$$e_d \cdot a \nabla(\phi_b^N + b \cdot x) = 0 \qquad \text{on} \quad \partial\mathbb{R}^d_+ \tag{2.3b}$$

The Neumann half-space excess of a function u satisfying (2.1) on the half-ball of radius $r > 0$ is then given by

$$\mathrm{Exc}^N(r) := \inf_{\tilde{b} \in B} \fint_{B_r^+} |\nabla u - (\tilde{b} + \nabla \phi_{\tilde{b}}^N)|^2 \, dx. \tag{2.4}$$

With these notions in-hand we can now summarize the main results of this chapter. To avoid excessive repetition we summarize the current results by referencing those of the previous chapter:

“Theorem". *With the same assumptions as in the summary of results in Section 1 of the previous chapter, we $\langle\cdot\rangle$- almost surely obtain analogues of i) – iv). It is not necessary to restate results i)- iii) as the those from the previous chapter hold here with the replacement of the superscripts “D” by “N”. The result iv) is altered in the sense that if u solves* (2.1) *and satisfies the subquadratic growth assumption* (1.13)*, then it will be of the form $u = b \cdot x + \phi_b^N + c$ with $b \in B$ and $c \in \mathbb{R}^d$. Here B is defined in* (2.2)*.*

2.1.1 Notation Specific to Chapter 2

Throughout this chapter will often use that since both of the maps $\xi \in \mathbb{R}^d \mapsto \phi_\xi$ and $\xi \in \mathbb{R}^d \mapsto \sigma_\xi$ can be chosen to be linear, using the definition of $\delta_r(\phi, \sigma)$ given in (4) and Jensen's inequality, we have that for any orthonormal basis

$\{b_1, ..., b_d\}$ of $\mathbb{R}^d$

$$
\begin{aligned}
&\frac{1}{r}\left(\fint_{B_r}\sum_{i=1}^{d}\left(|\phi_{b_i}|^2+\sum_{j,k=1}^{d}|\sigma_{b_ijk}|^2\right)\,dx\right)^{\frac{1}{2}}\\
\leq&\frac{1}{r}\left(\frac{d+1}{2}\fint_{B_r}\sum_{i=1}^{d}\left(\sum_{w=1}^{d}|b_i\cdot e_w|^2|\phi_{e_w}|^2+\sum_{j,k,w=1}^{d}|b_i\cdot e_w|^2|\sigma_{e_wjk}|^2\right)\,dx\right)^{\frac{1}{2}}\\
\leq&\left(\frac{d(d+1)}{2}\right)^{\frac{1}{2}}\delta_r(\phi,\sigma).
\end{aligned}
\tag{2.5}
$$

We use $H^1_{loc}(\overline{\mathbb{R}}^d_+)$ to denote the space of functions u such that for any compact set $U \Subset \mathbb{R}^d$ we have that $u \in H^1(U \cap \mathbb{R}^d_+)$.

We use the notational conventions that $\partial\widehat{B}^+_R = \partial B^+_R \setminus \partial\mathbb{R}^d_+$ and $\partial\underline{B}^+_R = \partial B^+_R \cap \partial\mathbb{R}^d_+$.

2.2 Main Results

We now give the full statement of the two theorems and the Liouville principle that arises as a corollary. In this first theorem we construct a sublinear Neumann generalized corrector:

Theorem 1. *Let $a \in \Omega$ and let $\{b_i\}$ be an orthonormal basis of $\mathbb{R}^d$ such that $b_i \in B$ for $i \neq d$. Here, B is given by* (2.2) *and Ω is defined in* (1)*. Assume that there exists a whole-space generalized corrector (ϕ, σ) satisfying* (2) *and* (3) *and the quantified sublinear growth condition* (1.4) *(see Theorem 1 of the previous chapter). Then there exists a generalized Neumann half-space corrector (ϕ^N, σ^N) satisfying the following properties:*

i) The Neumann half-space corrector $\phi^N_{b_d}$ and flux corrector $\sigma^N_{b_d}$ are the restriction of ϕ_{b_d} and σ_{b_d} respectively to the half-space, i.e. $\phi^N_{b_d} = \phi_{b_d}|_{\mathbb{R}^d_+}$ and $\sigma^N_{b_d} = \sigma_{b_d}|_{\mathbb{R}^d_+}$.

ii) For $i \neq d$ the Neumann half-space corrector $\phi^N_{b_i}$ is a weak solution of

$$-\nabla \cdot a\nabla(\phi_{b_i}^N + b_i \cdot x) = 0 \qquad in \quad \mathbb{R}_+^d, \tag{2.6a}$$
$$e_d \cdot a\nabla(\phi_{b_i}^N + b_i \cdot x) = 0 \qquad on \quad \partial\mathbb{R}_+^d, \tag{2.6b}$$

where the class of test functions is given by $H^1_{bdd}(\mathbb{R}^d_+)$.

iii) For $i \neq d$ *and* $j \in \{1, ..., d\}$ *the Neumann half-space flux corrector* $\sigma^N_{b_ij}$ *is a distributional solution of*

$$\nabla_k \cdot \sigma^N_{b_ijk} = e_j \cdot \big(a(\nabla\phi^N_{b_i} + b_i) - a_{hom} b_i\big) \qquad in \quad \mathbb{R}^d_+. \tag{2.7}$$

Furthermore, $\sigma^N_{b_ijk}$ *is skew-symmetric in* j *and* k.

iv) The generalized Neumann half-space corrector (ϕ^N, σ^N) *is sublinear in the sense that*

$$\delta_r^N(\phi^N, \sigma^N) := \frac{1}{r}\left(\sum_{i=1}^{d-1} \fint_{B_r^+} |(\phi^N_{b_i} - \fint_{B_r^+} \phi^N_{b_i}\, dx, \sigma^N_{b_i})|^2\, dx \right.$$
$$\left. + \fint_{B_r} |(\phi^N_{b_d}, \sigma^N_{b,d})|^2\, dx \right)^{\frac{1}{2}} \tag{2.8}$$

satisfies

$$\lim_{r\to\infty} \delta_r^N(\phi^N, \sigma^N) = 0. \tag{2.9}$$

Just like in Chapter 1 the estimates used to prove Theorem 1 guarantee a growth rate for the generalized Neumann half-space corrector. In particular, just like in the previous chapter: If the whole-space generalized corrector is sublinear in the sense that

$$\delta_r \lesssim \frac{1}{r^\gamma}$$

for $\gamma > 0$, then the generalized Neumann half-space corrector that we construct satisfies

$$\delta_r^N \lesssim \frac{1}{r^{\gamma/3}}.$$

The sublinear pair (ϕ^N, σ^N) constructed in Theorem 1 is then used to prove Theorem 2.

Theorem 2. *Let* $a \in \Omega$. *Then for all Hölder exponents* $\alpha \in (0, 1)$ *there exists a constant* $C_\alpha(d, \lambda)$ *such that if for a radius* $R > 0$ *there exists a* *generalized Neumann half-space corrector satisfying i) − iii) from Theorem 1 on* B_R^+ *and there exists a minimal radius* $r_\alpha^* > 0$ *for which*

$$\delta_r^N(\phi^N, \sigma^N) \leq \frac{1}{C_\alpha(d,\lambda)} \quad \text{if} \quad r > r_\alpha^*, \tag{2.10}$$

the following properties hold:

Let $u \in H^1(B_R^+)$ *be a-harmonic with no-flux boundary conditions on* $\partial \underline{B}_R^+$*, i. e. let* u *be a weak solution of*

$$\begin{aligned} -\nabla \cdot (a\nabla u) &= 0 \qquad && \text{in} \quad B_R^+, \\ e_d \cdot a\nabla u &= 0 && \text{on} \quad \partial \underline{B}_R^+, \end{aligned}$$

where the class of test functions is given by

$$\left\{ u \in H^1(\mathbb{R}_+^d) \,:\, supp(u) \subset B_r \text{ for some } R > r > 0 \right\}.$$

Then using the definition (2.4) *the following hold:*

i) For $r \in [r_\alpha^*, R]$ *the excess decay estimate given by*

$$Exc^N(r) \lesssim \left(\frac{r}{R}\right)^{2\alpha} Exc^N(R) \tag{2.11}$$

holds.

ii) For $r \in [r_\alpha^*, R]$ *the tilt-excess functional*

$$\tilde{b} \in \mathbb{R}^d \mapsto \fint_{B_r^+} |\nabla u - (\tilde{b} + \nabla \phi_{\tilde{b}})|^2 \, dx$$

is coercive.

iii) There exists $C_{Mean}(d,\lambda) \geq 1$ *such that for* $r \in [r_{\frac{1}{2}}^*, R]$ *the mean-value property*

$$\fint_{B_r^+} |\nabla u|^2 \, dx \leq C_{Mean} \fint_{B_R^+} |\nabla u|^2 \, dx \tag{2.12}$$

holds.

Post-processing the excess decay, just as we have done in the previous chapter, we obtain the following $C^{1,\alpha}$-Liouville principle as a corollary:

Corollary 1. *For* $a \in \Omega$ *assume that there exists a whole-space generalized corrector* (ϕ, σ) *satisfying the growth condition* (1.4) *and let* B *be given by* (2.2)*. Under these conditions we have that if* $u \in H^1_{loc}(\overline{\mathbb{R}}_+^d)$ *solves* (2.1) *and satisfies the subquadratic growth condition* (1.13) *for some* $\alpha \in (0,1)$*, then* $u = \tilde{b} \cdot x + \phi_{\tilde{b}}^N + c$ *for some* $\tilde{b} \in B$ *and* $c \in \mathbb{R}$*.*

2.3 Construction of the Generalized Neumann Half-Space Corrector

Recall that for $i \in \{1, ..., d-1\}$ we would like to construct the half-space-adapted corrector $\phi^N_{b_i}$ satisfying (2.6) and the corresponding skew-symmetric flux correctors. We proceed in a manner similar to the previous chapter and try to correct the available whole-space generalized corrector that is assumed to exist. We follow the same basic strategy as laid out in detail in Section 1.2 of Chapter 1 and even reuse the same cut-off functions. In particular, for an arbitrary initial radius $r_0 \geq 1$ we let:

i) $\{\eta_n\}_{n\geq -1}$ be a smooth radial partition of unity subordinate to the covering of $\mathbb{R}^d$ by $\{B_{r_0 2^{n+1}} \setminus B_{r_0 2^{n-1}}\}_{n\geq 0} \cup B_{r_0}$ such that $|\nabla \eta_n| \leq \frac{4}{r_0 2^n}$.

ii) For each set in the cover we define a smooth one-dimensional cut-off function $L_n(x) = L_n(x_d)$ satisfying $|L_n(x_d)| = 1$ for $|x_d| \leq l_n$ and $L_n(x_d) = 0$ for $|x_d| \geq 2l_n$ such that $|\nabla L_n| \leq \frac{2}{l_n}$.

Following the previous chapter, specific values for the heights l_n are chosen in the proof of Lemma 1. For any radius $r > 0$, to measure the size of the corrections to ϕ_{b_i} and σ_{b_i} on B_r^+, we again split the dyadic annuli into two groups: near-field contributions, when $16r > r_0 2^{n+1}$, and far-field contributions, when $16r \leq r_0 2^{n+1}$.

Step 1– (Estimate for the near-field contributions.)

The correction to ϕ_{b_i}, which we will call φ_{b_i}, that will enforce the desired boundary condition is a weak solution of

$$-\nabla \cdot (a\nabla \varphi_{b_i}) = 0 \qquad \text{in} \quad \mathbb{R}^d_+, \tag{2.13a}$$

$$e_d \cdot a\nabla \varphi_{b_i} = -e_d \cdot a\nabla(\phi_{b_i} + b_i \cdot x) \qquad \text{on} \quad \partial\mathbb{R}^d_+ \tag{2.13b}$$

where the class of test functions is given by $H^1_{bdd}(\mathbb{R}^d_+)$. For the boundary condition (2.13b) recall from the definition of the whole-space corrector that $a\nabla(\phi_{b_i} + b_i \cdot x)$ is a solenoidal field, which means that $e_d \cdot a\nabla(\phi_{b_i} + b_i \cdot x)$ has a trace in $H^{-\frac{1}{2}}(\partial\mathbb{R}^d_+)$. To solve (2.13) we again split $\mathbb{R}^d_+$ into dyadic annuli (indexed by n) and for each $n \in \{-1, 0, 1, ..\}$ seek a solution $\varphi^n_{b_i}$ to

$$-\nabla \cdot (a\nabla \varphi^n_{b_i}) = 0 \qquad \text{in} \quad \mathbb{R}^d_+, \tag{2.14a}$$

$$e_d \cdot a\nabla \varphi^n_{b_i} = -\eta_n e_d \cdot a\nabla(\phi_{b_i} + b_i \cdot x) \qquad \text{on} \quad \partial\mathbb{R}^d_+. \tag{2.14b}$$

The ansatz for the correction is then $\varphi_{b_i} = \sum_{n=-1}^{\infty} \varphi^n_{b_i}$, which makes the ansatz for the Neumann half-space corrector $\phi^N_{b_i} = \phi_{b_i} + \sum_{n=-1}^{\infty} \varphi^n_{b_i}$. Of course, just as in the Dirichlet case, we must check that summing over the corrections is possible and that the sum is sublinear.

For fixed $n \geq -1$ we find solutions to (2.14) using Lax-Milgram arguments similar to those from the previous chapter. In particular, running a Lax-Milgram argument in $L^{2d/(d-2)}(\mathbb{R}^d_+) \cap \dot{H}^1(\mathbb{R}^d_+)$ when $d > 2$ and in $BMO(\mathbb{R}^d_+) \cap \dot{H}^1(\mathbb{R}^d_+)$ when $d = 2$ (in both cases with the inner-product inherited from the homogeneous Sobolev space), we obtain solutions $\tilde{\varphi}^n_{b_i}$ of the following weak formulation

$$\int_{\mathbb{R}^d_+} \nabla u \cdot a\nabla\tilde{\varphi}^n_{b_i}\, dx = -\int_{\partial\mathbb{R}^d_+} u\eta_n e_d \cdot a\nabla(\phi_{b_i} + b_i \cdot x)\, dS. \tag{2.15}$$

Right away we remark that the desired space of test function for (2.14), $H^1_{bdd}(\mathbb{R}^d_+)$, is contained in the Lax-Milgram spaces and for these functions (2.15) is the weak formulation of (2.14). With the Lax-Milgram solution $\tilde{\varphi}^n_{b_i}$ in-hand we let $\varphi^n_{b_i} = \tilde{\varphi}^n_{b_i} - \fint_{B_1^+} \tilde{\varphi}^n_{b_i}\, dx$.

The actual Lax-Milgram argument is essentially the same as that in Step 2 of Section 1.3 of the previous chapter. The only missing ingredient is checking that the right-hand side of (2.15) defines a bounded operator on the Lax-Milgram space. For this we assume that $d > 2$ and remark that the case $d = 2$ is the same; we then write

$$\begin{aligned}
&\left|\int_{\partial\mathbb{R}^d_+} u\eta_n e_d \cdot a\nabla(\phi_{b_i} + b_i \cdot x)\, dS\right| \\
\leq &\int_{\mathbb{R}^d_+} |\nabla(u\eta_n) \cdot a\nabla(\phi_{b_i} + b_i \cdot x)|\, dx \\
\leq &C(d,\lambda,n)\left(\int_{\mathbb{R}^d_+} |\nabla u|^2\, dx\right)^{\frac{1}{2}} \left(\int_{B_{r_0 2^{n+1}}} |\nabla(\phi_{b_i} + b_i \cdot x)|^2\, dx\right)^{\frac{1}{2}},
\end{aligned}$$

where we have used the compact support of η_n, the boundedness of a, the critical Sobolev embedding, and that $\phi_{b_i} + b_i \cdot x$ is a-harmonic. As $\phi_{b_i} + b_i \cdot x \in H^1_{loc}(\mathbb{R}^d)$ it follows that the right-hand side of (2.15) is a bounded operator.

It then remains to show that the ansatz $\sum_{n=-1}^{\infty} \varphi^n_{b_i}$ converges and is sublinear. For this purpose we introduce the notation $\phi^N_{b_i,N} = \phi_{b_i} + \sum_{n=-1}^{N} \varphi^n_{b_i}$ and notice that the solution $\varphi^n_{b_i}$ of (2.14) also solves

$$-\nabla \cdot (a(\nabla\varphi^n_{b_i} + \eta_n L_n \nabla(\phi_{b_i} + b_i \cdot x)) = -\nabla \cdot (\eta_n L_n a\nabla(\phi_{b_i} + b_i \cdot x)) \quad \text{in} \mathbb{R}^d_+. \tag{2.16a}$$

$$e_d \cdot a(\nabla\varphi^n_{b_i} + \eta_n L_n \nabla(\phi_{b_i} + b_i \cdot x)) = 0 \quad \text{on} \partial\mathbb{R}^d_+, \tag{2.16b}$$

where we slipped the vertical cut-off L_n into the forcing on the right-hand side.

In Lemma 1 we then show that we may choose the heights $l_n > 0$ so that the standard energy estimate for (2.16) provides a sufficient bound for the size of the near-field contributions. This lemma should be seen as the replacement of Lemma 1 from the previous chapter.

Lemma 1. *Assume that the conditions of Theorem 1 are satisfied. Then there exists a constant $C_1(d, \lambda)$ so that for every $n \geq -1$ there is a height $l_n > 0$ such that for any $r > 0$ and $i \in \{1, .., d-1\}$ the bound*

$$\left(\fint_{B_r^+} |\nabla \varphi_{b_i}^n|^2 \, dx \right)^{\frac{1}{2}} \leq C_1(d, \lambda) \left(\frac{r_0 2^{n+1}}{r} \right)^{\frac{d}{2}} \delta_{r_0 2^{n+1}}^{\frac{1}{3}} \tag{2.17}$$

holds. In particular, when $16r > r_0 2^{n+1}$ we have that

$$\left(\fint_{B_r^+} |\nabla \varphi_{b_i}^n|^2 \, dx \right)^{\frac{1}{2}} \leq C_2(d, \lambda) \min \left\{ 1, \left(\frac{r_0 2^{n+1}}{r} \right)^{\frac{d}{2}} \right\} \delta_{r_0 2^{n+1}}^{\frac{1}{3}} \tag{2.18}$$

with $C_2 := C_{Mean} C_1 8^d$.

Proof. We set $R = r_0 2^{n+1}$ and notice that we may assume $\fint_{B_R^+} \varphi_{b_i}^n \, dx = 0$. Testing (2.16) with $\varphi_{b_i}^n$ and only integrating by parts on the left-hand side yields

$$\begin{aligned} \int_{\mathbb{R}^d_+} \nabla \varphi_{b_i}^n \cdot a \nabla \varphi_{b_i}^n \, dx = & - \int_{\mathbb{R}^d_+} \eta_n L_n \nabla \varphi_{b_i}^n \cdot a \nabla (\phi_{b_i} + b_i \cdot x) \, dx \\ & - \int_{\mathbb{R}^d_+} \varphi_{b_i}^n L_n \nabla \eta_n \cdot a \nabla (\phi_{b_i} + b_i \cdot x) \, dx \\ & - \int_{\mathbb{R}^d_+} \varphi_{b_i}^n \eta_n \nabla L_n \cdot a \nabla (\phi_{b_i} + b_i \cdot x) \, dx. \end{aligned} \tag{2.19}$$

To begin we process the third term. As $i \in \{1, ..., d-1\}$ by the definition of B we know that $e_d \cdot a_{hom} b_i = 0$, which by the equation for the whole-space flux corrector implies that

$$\nabla_k \cdot \sigma_{b_i dk} = e_d \cdot a \nabla (\phi_{b_i} + b_i \cdot x) \qquad \text{in} \quad \mathbb{R}^d \tag{2.20}$$

is satisfied in the distributional sense. We use this in the following computation, in which k indexes the entries of the vector $\sigma_{b_i d}$ and we use Einstein's summation convention. Making use of the identity $\nabla L_n = \partial_d L_n e_d$, which holds since we have assumed that $L_n(x) = L_n(x_d)$, we write:

$$
\begin{aligned}
\int_{\mathbb{R}^d_+} \varphi^n_{b_i}\eta_n \nabla L_n \cdot a\nabla(\phi_{b_i} + b_i \cdot x)\, dx &= \int_{\mathbb{R}^d_+} \varphi^n_{b_i}\eta_n \partial_d L_n e_d \cdot a\nabla(\phi_{b_i} + b_i \cdot x)\, dx \\
&= -\int_{\mathbb{R}^d_+} \partial_d L_n \partial_k(\varphi^n_{b_i}\eta_n)\sigma_{b_i dk}\, dx \\
&\quad - \int_{\mathbb{R}^d_+} \varphi^n_{b_i}\eta_n \partial^2_d L_n \sigma_{b_i dd}\, dx \\
&= -\int_{\mathbb{R}^d_+} \partial_d L_n \partial_k(\varphi^n_{b_i}\eta_n)\sigma_{b_i dk}\, dx.
\end{aligned}
\tag{2.21}
$$

Notice that the boundary terms in the first integration by parts vanish by the definitions of L_n and η_n and the last equality follows from the skew-symmetry of σ_{b_i}. Making use of (2.21), the uniform ellipticity and boundedness of a, and the Poincaré inequality with zero-average on B^+_R, we find that (2.19) implies

$$
\begin{aligned}
&\int_{\mathbb{R}^d_+} |\nabla \varphi^n_{b_i}|^2\, dx \\
&\lesssim \int_{\operatorname{supp}(\eta_n L_n)} |\nabla(\phi_{b_i} + b_i \cdot x)|^2\, dx + \int_{B^+_R} |\partial_d L_n \eta_n \sigma_{b_i d}|^2\, dx \\
&+ R^2 \left(\int_{\operatorname{supp}(\eta_n L_n)} |\nabla \eta_n|^2 |\nabla(\phi_{b_i} + b_i \cdot x)|^2\, dx + \int_{B^+_R} |\partial_d L_n \partial_k \eta_n \sigma_{b_i dk}|^2\, dx \right).
\end{aligned}
\tag{2.22}
$$

We can simplify this expression by recalling that $|\nabla \eta_n| \leq \frac{8}{R}$ and $|\partial_d L_n| \leq \frac{2}{l_n}$, which allows us to write

$$
\int_{\mathbb{R}^d_+} |\nabla \varphi^n_{b_i}|^2\, dx \lesssim \int_{\operatorname{supp}(\eta_n L_n)} |\nabla(\phi_{b_i} + b_i \cdot x)|^2\, dx + \frac{1}{l_n^2} \int_{B^+_R} |\sigma_{b_i d}|^2\, dx. \tag{2.23}
$$

To finish our argument we treat the first term on the right-hand side of (2.23). The key ingredient here is the standard Caccioppoli estimate, which we apply in a box-wise sense in $\operatorname{supp}(\eta_n L_n)$ by covering the domain with cubes of side length $4l_n$. If we denote the (d-dimensional) cube with center $z \in \mathbb{R}^d$ and side length $l \in \mathbb{R}$ by $C_l(z)$, we may find a set of points

$$
\begin{aligned}
S = \Big\{ z \in \mathbb{R}^d \;\Big|\; &|\operatorname{supp}(\eta_n L_n) \setminus \cup_{z \in S} C_{4l_n}(z)| = 0 \\
&\sum_{z \in S} \chi_{C_{6l_n}(z)}(x) \leq 2^d \text{ for all } x \in \mathbb{R}^d \Big\}.
\end{aligned}
\tag{2.24}
$$

Then, for each box $C_{4l_n}(z)$ we let $\tilde{C}_{4l_n,6l_n,z}$ denote the smooth cut-off of $C_{4l_n}(z)$ in the box of side length $6l_n$ centered around it. In particular, we require that

$$\tilde{C}_{4l_n,6l_n,z}(x) = \begin{cases} 1 & \text{if } x \in C_{4l_n}(z) \\ 0 & \text{if } x \notin C_{6l_n}(z) \end{cases} \tag{2.25}$$

and that $|\nabla \tilde{C}_{4l_n,6l_n,z}| \leq \frac{2}{l_n}$. For each $z \in S$ we test the whole-space corrector equation (2) with $(\tilde{C}_{4l_n,6l_n,z})^2 \eta_n^2(\phi_{b_i} + b_i \cdot (x - z))$. After using the uniform ellipticity and boundedness of a and Young's inequality we obtain that

$$\int_{C_{4l_n}(z)\cap B_R} |\nabla(\phi_{b_i} + b_i \cdot x)|^2 \, dx \lesssim \frac{1}{l_n^2} \int_{C_{6l_n}(z)\cap B_R} |\phi_{b_i} + b_i \cdot (x - z)|^2 \, dx, \tag{2.26}$$

where we have also used that we may choose l_n to satisfy $l_n \ll R$. Breaking up $\operatorname{supp}(\eta_n L_n)$ into the cubes $C_{4l_n}(z)$ with centers $z \in S$ and applying (2.26) on each cube, we find that

$$\begin{aligned}
\int_{\operatorname{supp}(\eta_n L_n)} |\nabla(\phi_{b_i} + b_i \cdot x)|^2 \, dx &\leq \sum_{z\in S} \int_{C_{4l_n}(z)\cap B_R} |\nabla(\phi_{b_i} + b_i \cdot x)|^2 \, dx \\
&\lesssim \frac{1}{l_n^2} \sum_{z\in S} \int_{C_{6l_n}(z)\cap B_R} |\phi_{b_i} + b_i \cdot (x - z)|^2 \, dx \\
&\lesssim \frac{1}{l_n^2} \left(\int_{B_R} |\phi_{b_i}|^2 \, dx + \sum_{z\in S} \int_{C_{6l_n}(z)\cap B_R} |x - z|^2 \, dx \right) \\
&\lesssim R^d \left(\frac{R}{l_n}\right)^2 \delta_R^2 + R^{d-1} l_n.
\end{aligned} \tag{2.27}$$

Here we have used that the longest diagonal in a d-dimensional box of side length $6l_n$ has length $6l_n d^{\frac{1}{2}}$ and also (2.5).

Combining (2.27) with (2.23) we find that for all $r > 0$ it holds that

$$\fint_{B_r^+} |\nabla \varphi_{b_i}^n|^2 \, dx \lesssim \left(\frac{R}{r}\right)^d \left(\left(\frac{R}{l_n}\right)^2 \delta_R^2 + \frac{l_n}{R} \right). \tag{2.28}$$

Letting $l_n = \alpha R$ and plugging in the optimal $\alpha = \delta_R^{\frac{2}{3}}$ yields (2.17). Lastly, we note that (2.18) is a trivial consequence of (2.17) and that $C_{Mean} \geq 1$.

$\square$

Step 2– (Estimates for $\sigma^N_{b_i}$)

In this step we construct intermediate Neumann half-space flux correctors $\sigma^N_{b_i,N}$ that correspond to the $\phi^N_{b_i,N}$ from the last step. The reader may notice that the superscript "N" is being used to indicate "Neumann" and the subscript "N" indicates the scale to which the corrector has been adapted. We construct $\sigma^N_{b_ij,N}$ as a distributional solution of

$$\nabla_k \cdot \sigma^N_{b_ijk,N} = e_j \cdot (a(b_i + \nabla \phi^N_{b_i,N}) - a_{hom} b_i) \qquad \text{in} \quad \mathbb{R}^d_+. \tag{2.29}$$

Just as in the construction in the Dirichlet case (see Section 1.3), our strategy here is to correct σ_{b_ijk} with a modification $\psi_{b_ijk,N}$ that satisfies

$$\nabla_k \cdot \psi_{b_ijk,N} = e_j \cdot (a(b_i + \nabla \phi^N_{b_i,N}) - a(b_i + \nabla \phi_{b_i})) \qquad \text{in} \quad \mathbb{R}^d_+. \tag{2.30}$$

Taking the ansatz $\sigma^N_{b_ijk,N} = \sigma_{b_ijk} + \psi_{b_ijk,N}$, we must then ensure that $\psi_{b_ijk,N}$ is sublinear and skew-symmetric in j and k.

As the reader will notice, the construction of the correction $\psi_{b_ijk,N}$ is essentially the same as the construction of the analogous correction in the previous chapter. In particular, decomposing $\mathbb{R}^d$ into the dyadic annuli from the last step, we find Lax-Milgram solutions $v^n_{b_ij,N} : \mathbb{R}^d_+ \to \mathbb{R}$ of

$$-\Delta v^n_{b_ij,N} = \nabla \cdot (\eta_n x_j a(\nabla \phi^N_{b_i,N} - \nabla \phi_{b_i})) \qquad \text{in } \mathbb{R}^d_+, \tag{2.31a}$$

$$v^n_{b_ij,N} = 0 \qquad \text{for } j \neq d \text{ on } \partial\mathbb{R}^d_+, \tag{2.31b}$$

$$\partial_d v^n_{b_id,N} = 0 \qquad \text{on } \partial\mathbb{R}^d_+. \tag{2.31c}$$

When $j \neq d$ we find a solution $v^n_{b_ij,N} \in \dot{H}^1(\mathbb{R}^d_+)$ and when $j = d$ we run our Lax-Milgram argument in $L^{\frac{2d}{d-2}}(\mathbb{R}^d_+) \cap \dot{H}^1(\mathbb{R}^d_+)$ if $d > 2$ or in $BMO(\mathbb{R}^d_+) \cap \dot{H}^1(\mathbb{R}^d_+)$ if $d = 2$. For details on the Lax-Milgram arguments see Section 1.3. We then take the ansatz

$$\psi_{b_ijk,N} := \partial_k v_{b_ij,N} - \partial_j v_{b_ik,N}, \tag{2.32}$$

where we have summed over the dyadic annuli indexed by n in the sense that $v_{b_ij,N}$ " $=$ " $\sum_n v^n_{b_ij,N}$; the " $=$ " is included since the sum on the right-hand side does not converge unless we subtract-off the linear factor. In particular, we set

$$c^n_{b_ij,N} =: \begin{cases} 0 & \text{if } n = -1 \\ \nabla v^n_{b_ij,N}(0) & \text{if } n \neq -1 \end{cases} \tag{2.33}$$

and

$$d^n_{b_ij,N} =: \begin{cases} 0 & \text{if } j \neq d \\ \fint_{B_1^+} (v^n_{b_id,N} - c^n_{b_id,N} \cdot x) \,\mathrm{d}x & \text{if } j = d \end{cases} \tag{2.34}$$

and define

$$v_{b_ij,N} := \sum_{n=-1}^{\infty} (v^n_{b_ij,N} - c^n_{b_ij,N} \cdot x - d^n_{b_ij,N}). \tag{2.35}$$

In Lemma 2 below we estimate the growth of each contribution $v^n_{b_ij,N} - c^n_{b_ij,N} \cdot x - d^n_{b_ij,N}$ in the half-ball B^+_r and then post-process this result in Lemma 3 to establish that the sum on the right-hand side of (2.35) converges. Since seeing that (2.32) is an appropriate ansatz is exactly the same argument as in the Dirichlet case we do not repeat the argument here, but instead ask that the reader consults Section 1.3. To summarize the result: The ansatz (2.32) for ψ_{b_i} holds if $\nabla_k \cdot v_{b_ik,N}$ is sublinear.

As indicated above, we start with Lemma 2:

Lemma 2. *Assume that the conditions of Theorem 1 hold and the heights l_n are chosen according to Lemma 1. Then for $N \geq -1$, $n \geq -1$, $j,k \in \{1, ..., d\}$, and $i \in \{1, ..., d-1\}$ it holds that*

$$\begin{aligned} &\frac{1}{r}\left(\fint_{B^+_r} |\partial_k(v^n_{b_ij,N} - c^n_{b_ij,N} \cdot x)|^2 \, dx\right)^{\frac{1}{2}} \\ &\leq C_3(d) \min\left\{1, \frac{r_0 2^{n+1}}{r}\right\}\left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla \phi^N_{b_i,N} - \nabla \phi_{b_i}|^2 \, dx\right)^{\frac{1}{2}} \end{aligned} \tag{2.36}$$

for all $r > 0$ and $C_3(d) := 16 C_4(d) C_I(d)$, where the constant $C_4(d)$ is specified in the proof.

Proof. Throughout this proof we use the notation $R := r_0 2^{n+1}$. The argument given here is a combination of Lemma 5 and Lemma 2 of the previous chapter. First recall that the weak formulation of (2.31) for any $j \in \{1, ..., d\}$ that we use in the Lax-Milgram argument to obtain $v^n_{b_ij,N}$ does not include boundary terms (since the class of test functions we are interested in is $H^1_{bdd}(\mathbb{R}^d_+)$). Testing this weak formulation with $v^n_{b_ij,N}$ and combining the resulting energy estimate with the observation

$$|c^n_{b_ij,N}| \leq C(d)\left(\fint_{B^+_{\frac{r}{4}}} |\nabla v^n_{b_ij,N}|^2 \, \mathrm{d}x\right)^{\frac{1}{2}}, \tag{2.37}$$

gives that for any radius $r \geq \frac{1}{16}R$ the relation

$$\left(\fint_{B_r^+} |\nabla v_{b_ij,N}^n - c_{b_ij,N}^n|^2 \, dx \right)^{\frac{1}{2}} \leq C_4(d) R \left(\fint_{B_R^+} |\nabla \phi_{b_i,N}^N - \nabla \phi_{b_i}|^2 \, dx \right)^{\frac{1}{2}} \tag{2.38}$$

holds for some $C_4(d) \geq 1$. Notice that (2.37) relies on the classical mean-value property for harmonic functions with homogeneous Dirichlet or Neumann boundary data.

To obtain (2.36) for a radius $r > 0$ we then split the contributions (indexed by n) into near-field and far-field contributions. The desired estimate (2.36) follows immediately from (2.38) for the near-field contributions, i.e. when $r \geq \frac{1}{16}R$. To treat the far-field contributions we notice that $\partial_k v_{b_ij,N}^n - e_k \cdot c_{b_ij,N}^n$ is harmonic on $B_{r_0 2^{n-1}}^+$ with homogeneous Dirichlet or homogeneous Neumann boundary conditions on $\partial \underline{B}_{r_0 2^{n-1}}^+$ (depending on j and k). As $r < \frac{1}{16}R$ we may first apply (1.37) and then (2.38) in the following way:

$$\begin{aligned} &\frac{1}{r} \left(\fint_{B_r^+} |\partial_k (v_{b_ij,N}^n - c_{b_ij,N}^n \cdot x)|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq C_I \frac{1}{r_0 2^{n-1}} \left(\fint_{B_{r_0 2^{n-1}}^+} |\partial_k (v_{b_ij,N}^n - c_{b_ij,N}^n \cdot x)|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq 4 C_4 C_I \left(\fint_{B_{r_0 2^{n+1}}^+} |\nabla \phi_{b_i,N}^N - \nabla \phi_{b_i}|^2 \, dx \right)^{\frac{1}{2}}. \end{aligned} \tag{2.39}$$

□

We then formally check the ansatz (2.32). Here is the argument:

Lemma 3. *Assume that the conditions in Theorem 1 hold and the heights l_n are chosen according to Lemma 2. Let $i \in \{1, ..., d-1\}$, $j \in \{1, ..., d\}$, and $N \geq -1$. Then we have that the sum on the right-hand side of* (2.35)

$$\sum_{n=-1}^{\infty} (v_{b_ij,N}^n - c_{b_ij,N}^n \cdot x - d_{b_ij,N}^n) \tag{2.40}$$

converges absolutely in $H^1(B_r^+)$ for any $r > 0$. The expression $\nabla_k \cdot v_{b_ik,N}$ is sublinear in the sense that

$$\lim_{r \to \infty} \frac{1}{r} \left(\fint_{B_r^+} |\nabla_k \cdot v_{b_ik,N}|^2 \, dx \right)^{\frac{1}{2}} = 0.$$

And, lastly, for $r > 0$ and $k \in \{1, ..., d\}$ the ansatz $\psi_{b_ijk,N}$ satisfies the estimate

$$
\begin{aligned}
&\frac{1}{r}\left(\fint_{B_r^+} |\psi_{b_ijk,N}|^2\,dx\right)^{\frac{1}{2}} \\
&\leq 2C_3(d,\lambda)\sum_{n=-1}^{\infty}\min\left\{1, \frac{r_0 2^{n+1}}{r}\right\}\left(\fint_{B^+_{r_0 2^{n+1}}} |\nabla\phi^N_{b_i,N} - \nabla\phi_{b_i}|^2\,dx\right)^{\frac{1}{2}}.
\end{aligned}
\tag{2.41}
$$

Proof. This lemma is the counterpart of Lemma 4 in the previous chapter; due to the similarities we skip some details. By Lemma 2 it is clear that if

$$
\sum_{n=-1}^{\infty}\left(\fint_{B_{r_0 2^{n+1}}} |\nabla\phi^N_{b_i,N} - \nabla\phi^N_{b_i}|^2\,dx\right)^{\frac{1}{2}} < \infty \tag{2.42}
$$

then $\sum_{n=-1}^{\infty}\nabla(v^n_{b_ij,N} - c^n_{b_ij,N}\cdot x)$ converges absolutely in $L^2(B_r^+)$ for any $j \in \{1, ..., d\}$ and any $r > 0$. Conveniently, (2.42) follows easily from the identity $\phi^N_{b_i,N} - \phi_{b_i} = \sum_{n=-1}^{N}\varphi^n_{b_i}$ and (2.17) from Lemma 1.

When $j \neq d$ the homogeneous Dirichlet boundary data of $v^n_{b_ij,N} - c^n_{b_ij,N}\cdot x - d^n_{b_ij,N}$ on $\partial\mathbb{R}^d_+$ allows us to upgrade this to the absolute convergence of $\sum_{n=-1}^{\infty}(v^n_{b_ij,N} - c^n_{b_ij,N}\cdot x - d^n_{b_ij,N})$ in $H^1(B_r^+)$. To treat the case $j = d$ we use that

$$
\fint_{B_1^+} v^n_{b_id,N} - c^n_{b_id,N}\cdot x - d^n_{b_id,N}\,dx = 0 \tag{2.43}
$$

for all $n \geq -1$. For any $u \in H^1(B_r^+)$ such that $\fint_{B_1^+} u\,\mathrm{d}x = 0$ notice that a combination of Young's inequality, Jensen's inequality, and the Poincaré inequality with zero average on B_r^+ yields

$$
\begin{aligned}
\int_{B_r^+}|u|^2\,dx &= \int_{B_r^+}\left|u - \fint_{B_1^+} u\,dx\right|^2 dx \\
&\lesssim \int_{B_r^+}\left|u - \fint_{B_r^+} u\,dx\right|^2 dx + r^d\int_{B_1^+}\left|u - \fint_{B_r^+} u\,dx\right|^2 dx \\
&\lesssim r^{d+2}\int_{B_r^+}|\nabla u|^2\,dx.
\end{aligned}
\tag{2.44}
$$

This, in particular, implies that we also obtain absolute convergence in $H^1(B_r^+)$ in the case $j = d$.

To finish, notice that the sublinearity of $\nabla_k\cdot v_{b_ik,N}$ follows from Lemma 2 and the bound (2.42) using the dominated convergence theorem. The estimate (2.41) for $\psi_{b_ij,N}$ also follows from Lemma 2.

□

Step 3– (Construction of the generalized Neumann half-space corrector)

Just as in Step 3 of Section 1.3 in the previous chapter, we now choose a specific r_0, which is large enough so that for all $N \geq -1$ the generalized intermediate Neumann half-space corrector $(\phi^N_{,N}, \sigma^N_{,N})$ satisfies condition (2.10) from Theorem 2 for $\alpha = \frac{1}{2}$ and $r \geq r_0$. Furthermore, for this choice of r_0 and for $i \neq d$ we find that $\varphi^n_{b_i}$ satisfies (2.18) for any $r \geq r_0$ and $n \geq -1$.

Lemma 4. *Assume that the conditions of Theorem 1 hold and the heights l_n are chosen according to Lemma 2. Then there exists a dyadic radius $r_0 = 2^{n_0}$ that does not depend on N such that the following statements hold:*

If for all $n \in \{-1, ..., N\}$ and $i \in \{1, ..., d-1\}$ it holds that

$$\left(\fint_{B_r^+} |\nabla \varphi^n_{b_i}|^2 \, dx \right)^{\frac{1}{2}} \leq C_2(d, \lambda) \min \left\{ 1, \left(\frac{r_0 2^{n+1}}{r} \right)^{\frac{d}{2}} \right\} \delta^{\frac{1}{3}}_{r_0 2^{n+1}} \tag{2.45}$$

for $r \geq r_0$ then $(\phi^N_{,N}, \sigma^N_{,N})$– where we let $\phi^N_{b_d,N} = \phi_{b_d}|_{\mathbb{R}^d_+}$ and $\sigma^N_{b_d,N} = \sigma_{b_d}|_{\mathbb{R}^d_+}$ for all $N \geq -1$– satisfies condition (2.10) from Theorem 2 for $r \geq r_0$ and $\alpha = \frac{1}{2}$, i.e.

$$\delta^N_r(\phi^N_N, \sigma^N_N) \leq \frac{1}{C_{\frac{1}{2}}(d, \lambda)} \quad \textit{for} \quad r \geq r_0. \tag{2.46}$$

Furthermore, then (2.45) holds for $\varphi^{N+1}_{b_i}$ for all $i \in \{1, ..., d-1\}$ and $r \geq r_0$.

Proof. Let $r > 0$. Young's inequality, the Poincaré inequality with zero average on B_r^+, and the calculation (2.5) give that

$$\frac{1}{r} \left(\fint_{B_r^+} \left| \phi_{b_i} + \sum_{n=-1}^{N} \varphi^n_{b_i} - \fint_{B_r^+} (\phi_{b_i} + \sum_{n=-1}^{N} \varphi^n_{b_i}) \, dx \right|^2 \mathrm{d}x \right. \tag{2.47}$$

$$\left. + \fint_{B_r^+} |\sigma_{b_i} + \psi_{b_i,N}|^2 \, \mathrm{d}x + \fint_{B_r} |\phi_{b_d}|^2 + |\sigma_{b_d}|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}$$

$$\leq \delta_r + \sum_{i=1}^{d-1} \left(\frac{2}{r} \left(\fint_{B_r^+} |\psi_{b_i,N}|^2 \, dx \right)^{\frac{1}{2}} + C_P \sum_{n=-1}^{N} \left(\fint_{B_r^+} |\nabla \varphi^n_{b_i}|^2 \, dx \right)^{\frac{1}{2}} \right). \tag{2.48}$$

We estimate the second and third terms on the right-hand side of (2.48) separately. We start with the second term, for any $i \in \{1, ..., d-1\}$ an application of Lemma 3 and assumption (2.45) yield that

$$
\begin{aligned}
\frac{1}{r}\left(\fint_{B_r^+} |\psi_{b_i,N}|^2\,dx\right)^{\frac{1}{2}} &\le 2d^2 C_3 \sum_{m=-1}^{\infty}\sum_{n=-1}^{N} \min\left\{1, \frac{r_0 2^{m+1}}{r}\right\} \\
&\qquad\qquad \times \left(\fint_{B^+_{r_0 2^{m+1}}} |\nabla \varphi^n_{b_i}|^2\,dx\right)^{\frac{1}{2}} \\
&\le 2d^2 C_3 C_2 \sum_{n=-1}^{N}\sum_{m=-1}^{\infty} \min\left\{1, 2^{d(n-m)/2}\right\} \delta^{\frac{1}{3}}_{r_0 2^{n+1}} \\
&\le 2d^2 C_3 C_2 \sum_{n=n_0}^{N+n_0+1} \left(n - n_0 + \frac{1}{1-2^{-\frac{d}{2}}}\right) \delta^{\frac{1}{3}}_{2^n}.
\end{aligned}
\tag{2.49}
$$

To treat the third term, for any $i \in \{1, ..., d-1\}$ assumption (2.45) gives that

$$
\begin{aligned}
\sum_{n=-1}^{N} \left(\fint_{B_r^+} |\nabla \varphi^n_{b_i}|^2\,dx\right)^{\frac{1}{2}} &\le C_2 \sum_{n=-1}^{N} \min\left\{1, \left(\frac{r_0 2^{n+1}}{r}\right)^{\frac{d}{2}}\right\} \delta^{\frac{1}{3}}_{r_0 2^{n+1}} \\
&\le C_2 \sum_{n=n_0}^{N+n_0+1} \delta^{\frac{1}{3}}_{2^n}.
\end{aligned}
\tag{2.50}
$$

Combining these three estimates with (2.48) gives that

$$
\begin{aligned}
\delta^N_r(\phi^N_{,N}, \sigma^N_{,N}) &\le 4\left(\frac{d(d+1)}{2}\right)^{\frac{1}{2}} \delta_r \\
&\qquad + 4d^2 C_3 C_2 C_P \sum_{n=n_0}^{\infty} \left(n - n_0 + 1 + \frac{1}{1-2^{-\frac{d}{2}}}\right) \delta^{\frac{1}{3}}_{2^n}.
\end{aligned}
\tag{2.51}
$$

By our assumption (1.4) on the whole-space generalized corrector we find that we can choose the initial radius $r_0 = 2^{n_0}$ large enough, in a manner independent of N, such that (2.46) holds.

We then show that $\varphi^{N+1}_{b_i}$ satisfies (2.45) for all $r \ge r_0$. Notice that (2.45) for $r \ge r_0$ such that $\varphi^{N+1}_{b_i}$ is a near-field contribution, i.e. when $\frac{r_0 2^{N+2}}{r} \le 16$, has already been shown in Lemma 1. We, therefore, restrict ourselves to the case when $r \le r_0 2^{N-2}$. By the argument above, $(\phi^N_{,N}, \sigma^N_{,N})$ satisfies the conditions of Theorem 2 with $\alpha = \frac{1}{2}$, $R = r_0 2^N$, and $r^*_{\frac{1}{2}} \le r_0$. Therefore, we may apply the mean-value property (2.12) to $\varphi^{N+1}_{b_i}$, which is a-harmonic on

$B^+_{r_0 2^N}$ with no-flux boundary data on $\partial\mathbb{R}^d_+ \cap B_{r_0 2^N}$. Following our application of (2.12) by a use of (2.17) from Lemma 1 allows us to write

$$\begin{aligned} \left(\fint_{B_r^+} |\nabla \varphi_{b_i}^{N+1}|^2 \, dx \right)^{\frac{1}{2}} &\leq C_{Mean} \left(\fint_{B^+_{r_0 2^N}} |\nabla \varphi_{b_i}^{N+1}|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq C_{Mean} C_1 2^d \delta^{\frac{1}{3}}_{r_0 2^{N+2}}. \end{aligned} \tag{2.52}$$

□

Step 4–(Passing to the limit $N \to \infty$)

Proof of Theorem 1. We split this proof into two parts.

Part 1: The corrector $\phi^N_{b_i}$.

Since the first couple of contributions, i.e. $\varphi^n_{b_i}$ for $n \in \{-1, 0, 1, 2, 3\}$, are near-field contributions for any $r > r_0$, by an induction argument Lemma 4 implies that for any $r \geq r_0$ the inequality

$$\sum_{n=-1}^{\infty} \left(\fint_{B_r^+} |\nabla \varphi^n_{b_i}|^2 \, dx \right)^{\frac{1}{2}} \leq C_2 \sum_{n=n_0}^{\infty} \delta^{\frac{1}{3}}_{2^n} \tag{2.53}$$

holds. Furthermore, the Poincaré inequality with zero average applied in the form (2.44) with $u = \varphi^n_{b_i}$ gives that

$$\sum_{n=-1}^{\infty} \left(\fint_{B_r^+} |\varphi^n_{b_i}|^2 \, dx \right)^{\frac{1}{2}} \lesssim r^{\frac{d+2}{2}} \sum_{n=-1}^{\infty} \left(\fint_{B_r^+} |\nabla \varphi^n_{b_i}|^2 \, dx \right)^{\frac{1}{2}}. \tag{2.54}$$

Therefore, thanks to the assumption (1.4), the sum $\sum_{n=-1}^{\infty} \varphi^n_{b_i}$ converges absolutely in $H^1(B_r^+)$. We must still show that this limit φ_{b_i} is sublinear. For this we use Lemma 4 combined with the Poincaré inequality with zero average to obtain

$$\begin{aligned} \frac{1}{r} \left(\fint_{B_r^+} \left| \varphi_{b_i} - \fint_{B_r^+} \varphi_{b_i} \, dx \right|^2 dx \right)^{\frac{1}{2}} &\leq C_P \left(\fint_{B_r^+} |\nabla \varphi_{b_i}|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq C_P \sup_N \sum_{n=-1}^{N} \left(\fint_{B_r^+} |\nabla \varphi^n_{b_i}|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq C_P C_2 \sum_{n=n_0}^{\infty} \min\left\{ 1, \left(\frac{2^n}{r} \right)^{\frac{d}{2}} \right\} \delta^{\frac{1}{3}}_{2^n}, \end{aligned} \tag{2.55}$$

which is sufficient due to (1.4) and the dominated convergence theorem for sums.

We then take $\phi_{b_i}^N = \phi_{b_i} + \varphi_{b_i}$ for $i \neq d$ and $\phi_{b_d}^{\mathbb{H}} = \phi_{b_d}|_{\mathbb{H}_+^d}$. The relation (2.55) and (1.4) yield the desired sublinearity property

$$\lim_{r\to\infty} \frac{1}{r} \left(\sum_{i=1}^{d-1} \fint_{B_r^+} \left| \phi_{b_i}^N - \fint_{B_r^+} \phi_{b_i}^N \, dx \right|^2 dx + \fint_{B_r} |\phi_{b_d}^N|^2 \, dx \right)^{\frac{1}{2}} = 0.$$

Part 2: The flux corrector $\sigma_{b_i}^N$.

We now pass to the limit $N \to \infty$ in the sequence $\{\psi_{b_ijk,N}\}_N$ by showing that it is a Cauchy sequence in $L^2(B_r^+)$ for all $r > r_0$. First, we notice that $v_{b_ij,N+1}^n - v_{b_ij,N}^n$ satisfies the equation

$$\begin{aligned} -\Delta(v_{b_ij,N+1}^n - v_{b_ij,N}^n) &= \nabla \cdot (\eta_n x_j a \nabla \varphi_{b_i}^{N+1}) && \text{in } \mathbb{R}_+^d, && (2.56a)\\ v_{b_ij,N+1}^n - v_{b_ij,N}^n &= 0 && \text{for } j \neq d \text{ on } \partial\mathbb{R}_+^d, && (2.56b)\\ \partial_d(v_{b_id,N+1}^n - v_{b_id,N}^n) &= 0 && \text{on } \partial\mathbb{R}_+^d. && (2.56c) \end{aligned}$$

We consider this equation for $N \geq -2$ and adopt the notation $v_{b_ij,-2}^n = 0$ and $c_{b_ij,-2}^n = 0$. Repeating the argument from the proof of Lemma 2, gives that

$$\begin{aligned} &\frac{1}{r} \left(\fint_{B_r^+} |\partial_k(v_{b_ij,N+1}^n - v_{b_ij,N}^n) - e_k \cdot (c_{b_ij,N+1}^n - c_{b_ij,N}^n)|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq C_3 \min\left\{1, \frac{r_0 2^{n+1}}{r}\right\} \left(\fint_{B_{r_0 2^{n+1}}^+} |\nabla(\phi_{b_i,N+1}^N - \phi_{b_i,N}^N)|^2 \, dx \right)^{\frac{1}{2}} \qquad (2.57)\\ &= C_3 \min\left\{1, \frac{r_0 2^{n+1}}{r}\right\} \left(\fint_{B_{r_0 2^{n+1}}^+} |\nabla \varphi_{b_i}^{N+1}|^2 \, dx \right)^{\frac{1}{2}}. \end{aligned}$$

Summing in n, we find that the $v_{b_ij,N}$ satisfy

$$\begin{aligned} &\frac{1}{r} \left(\fint_{B_r^+} |\partial_k(v_{b_ij,N+1} - v_{b_ij,N})|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq C_3 \sum_{n=-1}^{\infty} \min\left\{1, \frac{r_0 2^{n+1}}{r}\right\} \left(\fint_{B_{r_0 2^{n+1}}^+} |\nabla \varphi_{b_i}^{N+1}|^2 \, dx \right)^{\frac{1}{2}}. \end{aligned} \qquad (2.58)$$

We then sum (2.58) over N, which as we assume that $r \geq r_0$, by (2.45) yields

$$\begin{aligned} &\frac{1}{r} \sum_{N=-2}^{\infty} \left(\fint_{B_r^+} |\partial_k (v_{b_i j, N+1} - v_{b_i j, N})|^2 \, dx \right)^{\frac{1}{2}} \\ &\leq C_3 C_2 \sum_{N=-2}^{\infty} \sum_{n=-1}^{\infty} \min \left\{ 1, \frac{r_0 2^{n+1}}{r} \right\} \min \left\{ 1, 2^{d(N+1-n)/2} \right\} \delta_{r_0 2^{N+2}}^{\frac{1}{3}}. \end{aligned} \tag{2.59}$$

To complete our argument we notice that

$$\begin{aligned} &\sum_{N=-2}^{\infty} \sum_{n=-1}^{\infty} \min\{1, 2^{d(N+1-n)/2}\} \delta_{r_0 2^{N+2}}^{\frac{1}{3}} \\ &\leq \sum_{N=n_0}^{\infty} (N + \frac{1}{1 - 2^{-\frac{d}{2}}}) \delta_{2^N}^{\frac{1}{3}} \\ &< \infty, \end{aligned} \tag{2.60}$$

by the assumption (1.4). By (2.59), (2.60), and the definition $\psi_{b_i jk, N} = \partial_k v_{b_i j, N} - \partial_j v_{b_i k, N}$ we find that $\{\psi_{b_i jk, N}\}_N$ is a Cauchy sequence in $L^2(B_r^+)$ for all $r > 0$. We may, therefore, on every half-ball B_r^+ pass to the limit, which we denote as $\psi_{b_i jk}$. Also following from (2.59) and (2.60), this time using the dominated convergence theorem for sums, is the sublinearity property:

$$\lim_{r \to \infty} \frac{1}{r} \left(\fint_{B_r^+} |\psi_{b_i jk}|^2 \, dx \right)^{\frac{1}{2}} = 0. \tag{2.61}$$

We then take $\sigma_{b_i jk}^N = \sigma_{b_i jk} + \psi_{b_i jk}$ for $i \neq d$ and $\sigma_{b_d jk}^N = \sigma_{b_d jk}|_{\mathbb{R}^d_+}$. The relation (2.61) and (1.4) give the desired sublinearity property

$$\lim_{r \to \infty} \frac{1}{r} \left(\sum_{i=1}^{d-1} \fint_{B_r^+} \left|\sigma_{b_i}^N\right|^2 \, dx + \fint_{B_r} |\sigma_{b_d}^N|^2 \, dx \right)^{\frac{1}{2}} = 0.$$

□

2.4 Proofs of the Large-Scale Regularity Results

2.4.1 Constant Coefficient Regularity and Caccioppoli Estimate

We first recall two basic lemmas, which correspond to Lemmas 6 and 7 in the previous chapter. The first is a Caccioppoli estimate for a-harmonic functions with no-flux boundary data on the half-space; it is completely elementary and only included for completeness:

Lemma 5. *Let $a \in \Omega$, where Ω is defined by (1), and $r > 0$. Then for any function u that is a-harmonic on B_{2r}^+ and has no-flux boundary conditions on $\partial \underline{B}_{2r}^+$ the estimate*

$$\int_{B_r^+} |\nabla u|^2 \, dx \lesssim \frac{1}{r^2} \int_{B_{2r}^+} |u|^2 \, dx \tag{2.62}$$

holds.

Proof. Let η denote a radial cut-off such that $\eta(x) \equiv 1$ when $|x| \leq r$, $\eta(x) \equiv 0$ when $|x| \geq 2r$, and $|\nabla \eta(x)| \leq \frac{2}{r}$. We test the equation

$$-\nabla \cdot (a \nabla u) = 0 \qquad \text{in} \quad B_{2r}^+, \tag{2.63a}$$

$$e_d \cdot a(\nabla u) = 0 \qquad \text{on} \quad \partial \underline{B}_{2r}^+ \tag{2.63b}$$

with $\eta^2 u$. The boundary term vanishes on $\partial \underline{B}_{2r}^+$ due to the no-flux boundary condition (2.63b) and also on $\partial \widehat{B}_{2r}^+$ due to the cut-off η. Using the uniform ellipticity and boundedness of a and Young's inequality gives

$$\lambda \int_{B_{2r}^+} \eta^2 |\nabla u|^2 \, dx \leq \int_{B_{2r}^+} \frac{\lambda}{2} \eta^2 |\nabla u|^2 + \frac{2}{\lambda} |\nabla \eta|^2 u^2 \, dx. \tag{2.64}$$

To finish the argument one absorbs the first term on the right-hand side of (2.64) into the left-hand side and uses the properties of η. □

We will also use the following facts from constant coefficient regularity theory:

Lemma 6. *Let $a_{hom} \in \Omega$ be constant, where Ω is defined by (1), and fix $R > 0$. Let v be a_{hom}-harmonic on B_R^+ with no-flux boundary conditions on $\partial \underline{B}_R^+$; i.e., v solves*

$$-\nabla \cdot (a_{hom} \nabla v) = 0 \qquad \textit{in} \quad B_R^+, \tag{2.65a}$$

$$v = u \qquad \textit{on} \quad \partial \widehat{B}_R^+, \tag{2.65b}$$

$$e_d \cdot a_{hom} \nabla v = 0 \qquad \textit{on} \quad \partial \underline{B}_R^+ \tag{2.65c}$$

for some function $u \in H^{\frac{1}{2}}(\partial B_r^+)$. Then for any positive $\rho \leq \frac{R}{2}$ and $r \leq \frac{R}{2}$ there exists a $\beta(d, \lambda) > 0$ such that the following estimates hold:

$$\sup_{x \in B_r^+} |\nabla^n v(x)|^2 \lesssim \left(\frac{1}{R}\right)^{2(n-1)} \fint_{B_R^+} |\nabla v|^2 \, dx \qquad \textit{for any } n \geq 1, \tag{2.66a}$$

$$\int_{A'} |\nabla v|^2 \, dx \lesssim (R)^{1-\beta} \rho^\beta \int_{\partial \widehat{B}_R^+} |\nabla^{tan} u|^2 \, dS, \tag{2.66b}$$

$$\textit{and} \sup_{x\in A''} |\nabla^n v(x)|^2 \lesssim \left(\frac{1}{\rho}\right)^{2(n-1)} \left(\frac{R}{\rho}\right)^d \fint_{B_R^+} |\nabla v|^2\, dx \quad \textit{for any } n \geq 1, \tag{2.66c}$$

where we have used the notation

$$\begin{aligned} A' &= (B_R^+ \setminus B_{R-2\rho}^+) \cup (B_R^+ \cap \{\, x \,|\, x_d \leq 2\rho\}) \ \textit{and} \\ A'' &= B_{R-\rho}^+ \setminus \{\, x \,|\, x_d \leq \rho\}. \end{aligned} \tag{2.67}$$

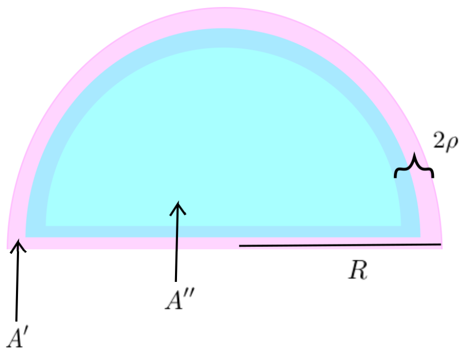

Figure 2.1: In this figure the domain A'' is colored blue and the domain A' is colored violet.

Proof. The third estimate (2.66c) follows from the observation that for all $x \in A''$ we have the inner regularity estimate

$$\sup_{y\in B_{\frac{\rho}{2}}(x)} |\nabla^n v(y)|^2 \lesssim \frac{1}{\rho^{d+2(n-1)}} \int_{B_\rho(x)} |\nabla v|^2\, dx. \tag{2.68}$$

Just as already observed in Lemma 7 of the previous chapter, this follows from an application of the Sobolev embedding and noting that all of the components of $\nabla^n v$ are a_{hom}-harmonic in $B_\rho(x)$, which allows for an iterative application of the Caccioppoli estimate. We obtain (2.66c) by writing:

$$\begin{aligned} \sup_{x\in A''} |\nabla^n v(x)|^2 &\leq \sup_{x\in A''} \sup_{y\in B_{\frac{\rho}{2}}(x)} |\nabla^n v(y)|^2 \\ &\lesssim \sup_{x\in A''} \frac{1}{\rho^{d+2(n-1)}} \int_{B_\rho(x)} |\nabla v|^2\, dx \\ &\lesssim \frac{1}{\rho^{2(n-1)}} \left(\frac{R}{\rho}\right)^d \fint_{B_R^+} |\nabla v|^2\, dx. \end{aligned} \tag{2.69}$$

The first estimate (2.66a) is shown in a similar manner. In particular, we again use the Sobolev embedding and iterate the Caccioppoli inequality (2.62) by differentiating (2.65). However, this procedure only yields the inequality (2.62) for higher derivatives involving at most one derivative in the e_d direction (as $\partial_d v$ does not satisfy (2.65c)). Using a standard argument, one obtains the required estimates for higher derivatives involving multiple normal derivatives. In particular, one expresses $\partial_d^n v$ in terms of $\partial^\beta v$ where $|\beta| = n$ and $\beta_d = n-1$ by using the equation (2.65) and proceeds inductively.

The second estimate is shown in exactly the same way as the second estimate of Lemma 7 in the previous chapter. To avoid too much repetition, we do not repeat the argument here.

□

2.4.2 Proof of Theorem 2: A Large-Scale $C^{1,\alpha}$- Excess Decay and Mean-Value Property

We are now ready to prove Theorem 2. Here comes the argument:

Proof of Theorem 2. We use the Einstein summation convention. Thanks to the linearity of the map $\xi \mapsto \phi_\xi^N$, we may re-write the expression in (2.4) as

$$\mathrm{Exc}^N(r) = \inf_{\xi \in \mathbb{R}^d} \fint_{B_r^+} \left| \nabla u - \sum_{i=1}^{d-1} \langle b_i, \xi \rangle (b_i + \nabla \phi_{b_i}^N) \right|^2 dx. \tag{2.70}$$

We proceed in our proof by following the same steps as in Theorem 2 of Chapter 1.

Step 1 – (Comparison of u to solution of homogenized problem/ Main step of the argument)

In this step we show that there exists $\beta > 0$ and a radius $r' > 0$ such that for any radius r such that $r' \le r \le R$ there exists a $\xi \in \mathbb{R}^d$ such that

$$\begin{aligned} &\fint_{B_r^+} \left| \nabla u - \sum_{i=1}^{d-1} \langle b_i, \xi \rangle (b_i + \nabla \phi_{b_i}^N) \right|^2 dx \\ &\lesssim \left(\left(\frac{R}{r} \right)^{2(d+1)} \delta^{\beta/(d+3)} + \left(\frac{r}{R} \right)^2 \right) \fint_{B_R^+} |\nabla u|^2 \, dx, \end{aligned} \tag{2.71}$$

where $\delta = \max(\delta_{2r}^N, \delta_R^N)$. This should be see as the analogue of (1.57) from the previous chapter.

To set-up our argument, we first notice that (2.71) is clear for $r \in \left[\frac{r}{4}, R\right]$ with the choice $\xi = 0$ and we, therefore, assume that $r < \frac{r}{4}$. Also, we let

$R' \in \left[\frac{r}{2}, R\right]$ be a radius such that

$$\int_{\partial \widehat{B}^+_{R'}} |\nabla^{tan} u|^2 \, dS \le \frac{1}{R} \int_{B^+_R \setminus B^+_{\frac{r}{2}}} |\nabla^{tan} u|^2 \, dx, \tag{2.72}$$

which can be seen to exist by writing the second integral in polar coordinates. In this argument we use two smooth cut-offs: First, a one-dimensional cut-off $L(x) = L(x_d)$ that satisfies $L(x_d) = 1$ if $|x_d| \le \rho$ and $L(x_d) = 0$ if $|x_d| \ge 2\rho$. Second, a function η that satisfies $\eta(x) = 1$ if $|x| \le R' - 2\rho$ and $\eta(x) = 0$ if $|x| \ge R' - \rho$. We assume that $0 < \rho \le \frac{r}{2}$ and both $|\nabla L_d| \le \frac{2}{\rho}$ and $|\nabla \eta| \le \frac{2}{\rho}$.

The core of the argument is to consider u as a perturbation of v satisfying (2.65) from Lemma 6 with the coefficients $a_{hom} \in \mathbb{R}^{d\times d}$ being the homogenized coefficients. In particular, v is taken to satisfy

$$-\nabla \cdot (a_{hom} \nabla v) = 0 \qquad \text{in} \quad B^+_{R'}, \tag{2.73a}$$

$$v = u \qquad \text{on} \quad \partial \widehat{B}^+_{R'}, \tag{2.73b}$$

$$e_d \cdot a_{hom} \nabla v = 0 \qquad \text{on} \quad \partial \underline{B}^+_{R'}. \tag{2.73c}$$

Interpreting the boundary condition (2.73c) in the distributional sense, one may find a solution $v \in H^1(B^+_{R'})$ to this equation using a Lax-Milgram argument. Thanks to Sobolev embedding we may actually interpret (2.73c) in a pointwise sense. Decomposing $\nabla v = \langle b_i, \nabla v\rangle b_i$ and using that $b_i \in B$ when $i \ne d$ gives that

$$0 = e_d \cdot a_{hom} \nabla v(0) = e_d \cdot a_{hom} b_d \langle b_d, \nabla v(0)\rangle. \tag{2.74}$$

As $e_d \cdot a_{hom} b_d \ne 0$ this implies that $\langle b_d, \nabla v(0)\rangle = 0$.Having made this observation, we use Young's inequality to write

$$\begin{aligned} &\int_{B^+_r} \left| \nabla u - \sum_{i=1}^{d-1} \langle b_i, \nabla v(0)\rangle (b_i + \nabla \phi^N_{b_i}) \right|^2 dx \\ \lesssim &\int_{B^+_r} |\nabla (u - v) - (1 - L)\langle b_i, \nabla v\rangle \nabla \phi^N_{b_i})|^2 \, dx \\ &+ \int_{B^+_r} |\langle b_i, \nabla v - \nabla v(0)\rangle (b_i + \nabla \phi^N_{b_i})|^2 \, dx \\ &+ \int_{B^+_r} |L \langle b_i, \nabla v\rangle \nabla \phi^N_{b_i}|^2 \, dx \end{aligned} \tag{2.75}$$

and then treat the three terms on the right-hand side separately.

We begin with the first term. Let $w = u - v - \eta(1 - L)\langle b_i, \nabla v\rangle \phi^N_{b_i}$ denote the ansatz for the homogenization error given by two-scale expansion. Since

$r \le R' - 2\rho$ we have that

$$\begin{aligned}&\int_{B_r^+} |\nabla(u-v) - (1-L)\langle b_i, \nabla v\rangle \nabla\phi_{b_i}^N)|^2\,dx \\ &\lesssim \int_{B_r^+} |\nabla w|^2\,dx + \int_{B_r^+} |\nabla((1-L)\langle b_i, \nabla v\rangle)\phi_{b_i}^N|^2\,dx.\end{aligned} \tag{2.76}$$

Using the equations for the Neumann half-space corrector and flux corrector, and the properties of u and v we derive an equation for the homogenization error analogous to (1.60) from the Theorem 2 of the previous chapter. In particular, we find that w is a weak solution to

$$\begin{aligned}-\nabla\cdot(a\nabla w) = \nabla\cdot\big(&(1-\eta(1-L))(a-a_{hom})\nabla v - \sigma_{b_i}^N\nabla(\eta(1-L)\langle b_i,\nabla v\rangle) \\ &+a\nabla(\eta(1-L)\langle b_i,\nabla v\rangle)\phi_{b_i}^N\big) \quad \text{in } B_{R'}^+.\end{aligned} \tag{2.77}$$

in the sense that we can can test the equation with functions from $H^1_{bdd}(\mathbb{R}^d_+)$. Testing (2.77) with w and using Hölder's inequality and the uniform ellipticity and boundedness of a and a_{hom}, we obtain

$$\begin{aligned}\int_{B_{R'}^+} |\nabla w|^2\,dx \lesssim \int_{A'} |\nabla v|^2\,dx + \sup_{x\in A''}\left(|\nabla^2 v|^2 + \frac{1}{\rho^2}|\nabla v|^2\right) \\ \times \sum_{i=1}^{d-1}\int_{A''}\left(\left|(\phi_{b_i}^N, \sigma_{b_i}^N)\right|^2 + \left|\phi_{b_d}^N, \sigma_{b_d}^N\right|^2\right)dx.\end{aligned} \tag{2.78}$$

Conveniently, we may also bound the second term of (2.76) in terms of the second term of (2.78). Applying (2.66b) and (2.66c) and using that we have chosen R' according to (2.72), then gives that

$$\begin{aligned}&\int_{B_r^+} |\nabla(u-v) - (1-L)\langle b_i, \nabla v\rangle\nabla\phi_{b_i}^N|^2\,dx \\ &\lesssim \left(\frac{\rho}{R}\right)^{\beta}\int_{B_R^+}|\nabla u|^2\,dx + \left(\frac{R}{\rho}\right)^{d+2}(\delta_R^N)^2\int_{B_{R'}^+}|\nabla v|^2\,dx.\end{aligned} \tag{2.79}$$

We continue and bound the second term on the right-hand side of (2.75). Here, an application of (2.66a) for $n=2$ yields

$$\begin{aligned}&\int_{B_r^+} |\langle b_i, \nabla v - \nabla v(0)\rangle(b_i + \nabla\phi_{b_i}^N)|^2\,dx \\ &\lesssim r^2 \sup_{x\in B_r^+}|\nabla^2 v|^2 \int_{B_r^+}|b_i + \nabla\phi_{b_i}^N|^2\,dx \\ &\lesssim \left(\frac{r}{R}\right)^2 ⨍_{B_{R'}^+}|\nabla v|^2\,dx \int_{B_r^+}|b_i+\nabla\phi_{b_i}^N|^2\,dx.\end{aligned} \tag{2.80}$$

Notice that for $i = d$ the whole-space Caccioppoli estimate and for $i \neq d$ the estimate (2.62) together imply that

$$\fint_{B_r^+} |b_i + \nabla \phi_{b_i}^N|^2 \, dx \lesssim 1 + (\delta_{2r}^N)^2. \tag{2.81}$$

The combination of (2.80) and (2.81) then gives

$$\begin{aligned} &\int_{B_r^+} |\langle b_i, \nabla v - \nabla v(0) \rangle (b_i + \nabla \phi_{b_i}^N)|^2 \, dx \\ &\lesssim r^d (1 + (\delta_{2r}^N)^2) \left(\frac{r}{R}\right)^2 \fint_{B_{R'}^+} |\nabla v|^2 \, dx. \end{aligned} \tag{2.82}$$

Lastly, we treat the third term on the right-hand side of (2.75). An application of (2.66a) for $n = 1$ gives

$$\int_{B_r^+} |L\langle b_i, \nabla v \rangle \nabla \phi_{b_i}^N|^2 \, dx \lesssim \fint_{B_{R'}^+} |\nabla v|^2 \, dx \int_{B_r^+ \cap \{x_d \leq 2\rho\}} |\nabla \phi_{b_i}^N|^2 \, dx. \tag{2.83}$$

To treat the right-hand side of (2.83) we modify the box-wise Caccioppoli argument used in Lemma 1. Using the same notation (the d-dimensional box with center $z \in \mathbb{R}^d$ and length $l \in \mathbb{R}$ is denoted as $C_l(z)$), we cover $B_r^+ \cap \{x_d \leq 2\rho\}$ with boxes of width 4ρ with centers taken from a set

$$S = \left\{ z \in \mathbb{R}^d \;\middle|\; \begin{aligned} &|B_r^+ \cap \{x_d \leq 2\rho\} \setminus \cup_{z \in S} C_{4\rho}(z)| = 0, \quad \cup_{z \in S} C_{6\rho}(z) \subseteq B_{2r}, \\ &\text{and } \sum_{z \in S} \chi_{C_{6\rho}(z)}(x) \leq 2^d \text{ for all } x \in \mathbb{R}^d \end{aligned} \right\} \tag{2.84}$$

Then we let $\tilde{C}_{4\rho,6\rho,z}$ be the cut-off of $C_{4\rho}(z)$ in the box of side length 6ρ centered around it (see (2.25) for the definition). When $i \neq d$, for each $z \in S$, we test the Neumann half-space corrector equation (2.6) with $(\tilde{C}_{4\rho,6\rho,z})^2(\phi_{b_i}^N + b_i \cdot (x - z) - \fint_{B_{2r}^+} \phi_{b_i}^N \, dx)$. This gives

$$\begin{aligned} &\int_{C_{4\rho}(z) \cap \mathbb{R}_+^d} |\nabla \phi_{b_i}^N + b_i|^2 \, dx \\ &\lesssim \frac{1}{\rho^2} \int_{C_{6\rho}(z) \cap \mathbb{R}_+^d} |\phi_{b_i}^N + b_i \cdot (x - z) - \fint_{B_{2r}^+} \phi_{b_i}^N \, dx|^2 \, dx, \end{aligned} \tag{2.85}$$

where the boundary term has vanished due to the boundary condition (2.6b). When $i = d$ testing the whole-space corrector equation (2) with $(\tilde{C}_{4\rho,6\rho,z})^2(\phi_{b_d} +$

$b_d \cdot (x-z))$ gives that

$$\int_{C_{4\rho}(z)\cap\mathbb{R}^d_+} |\nabla\phi^N_{b_d} + b_d|^2\,dx \lesssim \frac{1}{\rho^2}\int_{C_{6\rho}(z)} |\phi_{b_d} + b_d\cdot(x-z)|^2\,dx. \tag{2.86}$$

Summing over the $z \in S$ as in (2.27) gives that

$$\int_{B^+_r\cap\{x_d\le 2\rho\}} |\nabla\phi^N_{b_i}|^2\,dx \lesssim r^{d-1}\rho + \left(\frac{r}{\rho}\right)^2 r^d(\delta^N_{2r})^2. \tag{2.87}$$

Combining (2.87) with (2.83) then allows us to conclude:

$$\int_{B^+_r} |L\langle b_i,\nabla v\rangle\nabla\phi^N_{b_i}|^2\,dx \lesssim \left(r^{d-1}\rho + \left(\frac{r}{\rho}\right)^2 r^d(\delta^N_{2r})^2\right) ⨍_{B^+_{R'}} |\nabla v|^2\,dx. \tag{2.88}$$

Having treated all three terms on the right-hand side of (2.75), with the estimates (2.79), (2.82), and (2.88), we may now write

$$\begin{aligned}
&⨍_{B^+_r}\left|\nabla u - \sum_{i=1}^{d-1}\langle b_i,\nabla v(0)\rangle(b_i+\nabla\phi^N_{b_i})\right|^2 dx\\
&\lesssim \left(\left(\frac{R}{r}\right)^d\left(\frac{R}{\rho}\right)^{d+2}(\delta^N_R)^2 + \left(\frac{r}{R}\right)^2(1+(\delta^N_{2r})^2) + \frac{\rho}{r} + \left(\frac{r}{\rho}\right)^2(\delta^N_{3r})^2\right)\\
&\quad\times ⨍_{B^+_{R'}}|\nabla v|^2\,dx + \left(\frac{R}{r}\right)^d\left(\frac{\rho}{R}\right)^\beta ⨍_{B^+_R}|\nabla u|^2\,dx\\
&\lesssim \left(\left(\frac{R}{r}\right)^{2(d+1)}\left(\frac{r}{\rho}\right)^{d+2}\delta^2 + \left(\frac{r}{R}\right)^2(1+\delta^2) + \frac{\rho}{r}\right) ⨍_{B^+_{R'}}|\nabla v|^2\,dx\\
&\quad + \left(\frac{R}{r}\right)^d\left(\frac{\rho}{r}\right)^\beta ⨍_{B^+_R}|\nabla u|^2\,dx.
\end{aligned} \tag{2.89}$$

Here, we have used the notation $\delta = \max\left\{\delta^N_R, \delta^N_{2r}\right\}$ and that $r \le \frac{r}{4}$ and $\rho \le \frac{r}{2}$. To post-process this estimate we do two things: derive an apriori estimate for $\|\nabla v\|_{L^2(B^+_{R'})}$ and choose a specific width ρ for the boundary layer introduced by the cut-offs η and L. The apriori estimate for ∇v follows from the equation satisfied by the difference $v-u$:

$$-\nabla\cdot(a_{hom}\nabla(v-u)) = \nabla\cdot a_{hom}\nabla u \quad \text{in} \quad B^+_{R'}, \tag{2.90a}$$

$$v-u = 0 \quad \text{on} \quad \partial\widehat{B}^+_{R'}, \tag{2.90b}$$

$$e_d\cdot a_{hom}\nabla(v-u) = -e_d\cdot a_{hom}\nabla u \quad \text{on} \quad \partial\underline{B}^+_{R'}. \tag{2.90c}$$

Testing (2.90) with $v-u$ and using Hölder's inequality then yields that

$$\int_{B^+_{R'}} |\nabla(v-u)|^2\,dx \lesssim \int_{B^+_{R'}} |\nabla u|^2, \tag{2.91}$$

which by Young's inequality gives

$$\int_{B^+_{R'}} |\nabla v|^2\,dx \lesssim \int_{B^+_{R'}} |\nabla u|^2. \tag{2.92}$$

We then turn to choosing the width ρ. Recall that the only assumption on ρ was that $\rho \in (0, \frac{r}{2}]$. By varying ρ within this interval we may obtain $\frac{\rho}{r} = s$ for any $s \in (0, \frac{1}{4}]$. We set ρ to satisfy $\frac{\rho}{r} = \min\left\{\frac{1}{4}, \delta^{2/(d+3)}\right\}$.

These observations allow us to, for sufficiently large r and R, re-write (2.89) as

$$\begin{aligned}&\fint_{B^+_r} \left|\nabla u - \sum_{i=1}^{d-1} \langle b_i, \nabla v(0)\rangle (b_i + \nabla \phi^N_{b_i})\right|^2\,dx \\ &\lesssim \left(\left(\frac{R}{r}\right)^{2(d+1)} \delta^{2/(d+3)} + \left(\frac{r}{R}\right)^2 + \left(\frac{R}{r}\right)^d \delta^{2\beta/(d+3)}\right) \fint_{B^+_R} |\nabla u|^2\,dx.\end{aligned} \tag{2.93}$$

Notice that here "sufficiently large r and R" means $R \geq r \geq r'$ for the minimal radius $r' > 0$ guaranteeing that $\delta \leq 1$. Using that $0 < \beta < 1$ and $\frac{R}{r} \geq 1$ then yields (2.71).

Step 2– (Proof of the Neumann half-space excess decay)

We apply the result of the first step to any two radii $\tilde{r}$ and $\tilde{R}$ such that $r' \leq \tilde{r} \leq \tilde{R} \leq R$. Notice that thanks to (2.6) the function $\tilde{u}_c = u - \sum_{i=1}^{d-1} \langle b_i, c\rangle (b_i + \phi^N_{b_i})$ is a-harmonic with no-flux boundary conditions on $\partial \underline{B}^+_R$ for any $c \in \mathbb{R}^d$. Applying (2.71) to these functions and taking the infimum over $c \in \mathbb{R}^d$ allows us to write:

$$\mathrm{Exc}^N(\tilde{r}) \lesssim \left(\theta^{-2(d+1)} \delta^{\beta/(d+3)} + \theta^2\right) \mathrm{Exc}^N(\tilde{R}), \tag{2.94}$$

where we have used the notation $\theta = \frac{\tilde{r}}{\tilde{R}}$. Thanks to condition (2.10) and $\alpha < 1$ we may choose θ and $C_\alpha(d,\lambda)$ such that

$$\theta^{-2(d+1)} \delta^{\beta/(d+3)} + \theta^2 \leq \theta^{2\alpha}. \tag{2.95}$$

is satisfied above some minimal radius $r'' \geq r' > 0$. Making these choices, we obtain that

$$\mathrm{Exc}^N(\theta \tilde{R}) \lesssim \theta^{2\alpha} \mathrm{Exc}^N(\tilde{R}), \tag{2.96}$$

whenever $r'' \leq \theta\tilde{R}$. Iterating this estimate finishes the argument for the excess decay.

Step 3– (Proof of the coercivity of the excess expression)

This argument is exactly the same as in the previous chapter. In particular, we must show that

$$\fint_{B_r^+} |\nabla u - (\tilde{b} + \nabla\phi_{\tilde{b}})|^2 \, dx \to \infty \quad \text{as} \quad |\tilde{b}| \to \infty \tag{2.97}$$

and, by the triangle inequality in $L^2(B_r^+)$, it suffices to prove that

$$\fint_{B_r^+} |\tilde{b} + \nabla\phi_{\tilde{b}}^N|^2 \, dx \geq \left(\frac{1}{16}\right)^{d+1} |\tilde{b}|^2. \tag{2.98}$$

To show this we insert a smooth cut-off function η into the left-hand side of (2.98), where $\eta = 1$ on $B_{\frac{r}{2}}^+ \cap \{x_d \geq \frac{r}{4}\}$, $\eta = 0$ outside of B_r^+, $0 \leq \eta \leq 1$, and $|\nabla\eta| \leq \frac{12}{r}$. It is clear that

$$\fint_{B_r^+} |\tilde{b} + \nabla\phi_{\tilde{b}}^N|^2 \, dx \geq \fint_{B_r^+} \eta^2 |\tilde{b} + \nabla\phi_{\tilde{b}}^N|^2 \, dx. \tag{2.99}$$

Jensen's inequality and an integration by parts, in which the boundary term cancels due to the cut-off η, then yield

$$\begin{aligned}
&\fint_{B_r^+} \eta^2 |\tilde{b} + \nabla\phi_{\tilde{b}}^N|^2 \, dx \\
&\geq \left(\fint_{B_r^+} \eta \, dx\right)^2 \left|\fint_{B_r^+} \frac{\eta}{\fint_{B_r^+} \eta \, dx} (\tilde{b} + \nabla\phi_{\tilde{b}}^N) \, dx\right|^2 \\
&\geq \left(\fint_{B_r^+} \eta \, dx\right)^2 \left|\tilde{b} + \frac{1}{\fint_{B_r^+} \eta \, dx} \fint_{B_r^+} \eta \nabla\phi_{\tilde{b}}^N \, dx\right|^2 \\
&= \left(\fint_{B_r^+} \eta \, dx\right)^2 \left|\tilde{b} - \frac{1}{\fint_{B_r^+} \eta \, dx} \fint_{B_r^+} \nabla\eta \left(\phi_{\tilde{b}}^N - \fint_{B_r^+} \phi_{\tilde{b}}^N \, dx\right) \, dx\right|^2.
\end{aligned} \tag{2.100}$$

Notice that

$$\left(\frac{1}{4}\right)^d \leq \fint_{B_r^+} \eta \, dx, \tag{2.101}$$

which, along with an application of Hölder's inequality, implies

$$\frac{1}{\fint_{B_r^+} \eta \, dx} \left|\fint_{B_r^+} \nabla\eta \left(\phi_{\tilde{b}}^N - \fint_{B_r^+} \phi_{\tilde{b}}^N \, dx\right) \, dx\right| \leq 4^{d+2} |\tilde{b}| \delta_r^N. \tag{2.102}$$

By (2.10) we can choose $C_\alpha(d,\lambda)$ large enough such that $4^{d+1}|\tilde{b}|\delta_r^N \le |\tilde{b}|/2$ for all $r \ge r'''$. Combining this with (2.102) and (2.100), we conclude (2.98).

Remark: The minimal radius $r_\alpha^* > 0$ from the statement of the theorem is then chosen to be $r_\alpha^* = \max(r', r'', r''')$.

Step 4– (Proof of the mean-value property)

For any radius $r \in [r^*_{\frac{1}{2}}, R]$ we let $\tilde{b}_r \in B$ satisfy

$$\mathrm{Exc}^N(r) = \fint_{B_r^+} |\nabla u - (\tilde{b}_r + \nabla \phi^N_{\tilde{b}_r})|^2 \, dx. \tag{2.103}$$

It then holds that

$$\begin{aligned} \fint_{B_r^+} |\nabla u|^2 \, dx \lesssim& \mathrm{Exc}^N(r) + |\tilde{b}_r|^2 \\ \lesssim& \mathrm{Exc}^N(R) + |\tilde{b}_r|^2 \\ \lesssim& \fint_{B_R^+} |\nabla u|^2 \, dx + |\tilde{b}_R|^2 + |\tilde{b}_r - \tilde{b}_R|^2. \end{aligned} \tag{2.104}$$

Here, the first inequality follows from Young's inequality and (2.81), the second uses the excess decay from Step 2, and the third is obtained like the first.

We must bound $|\tilde{b}_R|^2$ and $|\tilde{b}_r - \tilde{b}_R|^2$. The first bound is a simple consequence of (2.98), the definition of the Neumann half-space excess, and Young's inequality:

$$|\tilde{b}_R|^2 \lesssim \fint_{B_R^+} |\tilde{b}_R + \nabla \phi^N_{\tilde{b}_R}|^2 \, dx \lesssim \mathrm{Exc}^N(R) + \fint_{B_R^+} |\nabla u|^2 \, dx \lesssim \fint_{B_R^+} |\nabla u|^2 \, dx. \tag{2.105}$$

To obtain an estimate for the difference $|\tilde{b}_r - \tilde{b}_R|^2$ we first notice if $R - r \le \frac{r}{2}$ then the coercivity property (2.98), the excess decay, and Young's inequality give

$$\begin{aligned} |\tilde{b}_r - \tilde{b}_R|^2 \lesssim& \fint_{B_r^+} |\tilde{b}_r - \tilde{b}_R + (\nabla \phi^N_{\tilde{b}_r} - \nabla \phi^N_{\tilde{b}_R})|^2 \, dx \\ \lesssim& \mathrm{Exc}^N(r) + \mathrm{Exc}^N(R) \\ \lesssim& \mathrm{Exc}^N(R). \end{aligned} \tag{2.106}$$

Notice that the condition that $r \in [\frac{r}{2}, R]$ is used for the second inequality. To finish, we iterate (2.106). Dropping the assumption that $r \in [\frac{r}{2}, R]$, let

$n = \lfloor \log_{\frac{1}{2}}(\frac{R}{r}) \rfloor$. The excess decay for $\alpha = \frac{1}{2}$ then gives that

$$
\begin{aligned}
|\tilde{b}_r - \tilde{b}_R|^2 &\leq \left(|\tilde{b}_r - \tilde{b}_{R2^{-n}}| + \sum_{m=1}^{n} |\tilde{b}_{R2^{-m}} - \tilde{b}_{R2^{-(m-1)}}| \right)^2 \\
&\lesssim \left(\sum_{m=0}^{n} (\mathrm{Exc}^N(R2^{-m}))^{\frac{1}{2}} \right)^2 \\
&\lesssim \left(\sum_{m=0}^{n} 2^{-\frac{m}{2}} (\mathrm{Exc}^N(R))^{\frac{1}{2}} \right)^2 \\
&\lesssim \mathrm{Exc}^N(R) \lesssim \fint_{B_R^+} |\nabla u|^2 \, dx.
\end{aligned}
\tag{2.107}
$$

The mean-value property then follows from (2.104), (2.105), and (2.107).

□

2.4.3 A $C^{1,\alpha}$ - Liouville Principle

With the excess decay in-hand it is now easy to prove the $C^{1,\alpha}$- Liouville principle; in particular, the proof is exactly the same as in the previous chapter. We include it for completeness:

Proof of Corollary 1. With Lemma 5 the assumption (1.13) of subquadratic growth can be processed to yield

$$\lim_{r \to \infty} \frac{1}{r^{2\alpha}} \fint_{B_r^+} |\nabla u|^2 \, dx = 0. \tag{2.108}$$

By the definition of "Exc^N" this implies that

$$\lim_{r \to \infty} \frac{1}{r^{2\alpha}} \mathrm{Exc}^N(r) = 0. \tag{2.109}$$

Our condition on the whole-space generalized corrector guarantees that the excess decay (2.11) holds above the minimal radius $r_\alpha^* > 0$. So, for all $\tilde{r} > r_\alpha^* > 0$ we have that

$$\mathrm{Exc}^N(\tilde{r}) \leq \left(\frac{\tilde{r}}{r} \right)^{2\alpha} \mathrm{Exc}^N(r) \tag{2.110}$$

for any $r > \tilde{r}$. Due to (2.109) this implies that $\mathrm{Exc}^N(\tilde{r}) = 0$ for all $\tilde{r} \geq r_\alpha^*$. Since the infimum in the definition of the half-space-adapted tilt-excess is attained, this implies that

$$u = \tilde{b}_{\tilde{r}} \cdot x + \phi^N_{\tilde{b}_{\tilde{r}}} + c_{\tilde{r}} \text{ on } B^+_{\tilde{r}} \tag{2.111}$$

for some constants $\tilde{b}_{\tilde{r}} \in \mathbb{R}^d$ and $c_{\tilde{r}} \in \mathbb{R}$. □

Chapter 3

A Large-Scale Regularity Theory for Harmonic Functions with No-Flux Boundary-Data On Randomly Perforated Domains with “Well-Separated" Holes

3.1 Set-up

As a continuation of the previous two chapters, in this chapter we are interested in the large-scale regularity properties of harmonic functions on perforated domains, where we assume that the perforations do not intersect and are separated by a certain buffer. The motivation for considering such a domain is to start heading towards the percolation setting, but, due to our simplifying assumptions, the current chapter is a far cry from this intended goal. This chapter is included in this thesis because the construction of the sublinear corrector is closer to the classical technique given in [31] and, therefore, uses different tools than the previous two chapters.

The exact situation that we consider is as follows: let $X \subset \mathbb{R}^d$ be a set of discrete points that are indexed by an index set $H \subseteq \mathbb{N}$ such that for distinct $i, j \in H$ we have that $B_2(x_i) \cap B_2(x_j) = \emptyset$; we call such a set of points *well-separated.* In this chapter we consider domains of the form

$$P = \mathbb{R}^d \setminus \cup_{i \in H} B_1(x_i), \tag{3.1}$$

where H is the index set of some well-separated set of discrete points X. We are interested in the large-scale regularity of $u \in H^1_{loc}(\overline{P})$, where this space consists of functions such that $u \in H^1(P \cap V)$ for any $V \Subset \mathbb{R}^d$, solving

$$-\Delta u = 0 \qquad \text{in} \quad P, \tag{3.2a}$$

$$\nu \cdot \nabla u = 0 \qquad \text{on} \quad \partial P. \tag{3.2b}$$

Unlike in the previous two chapters, we assume that the dimension $d \geq 3$ so that the set P is always simply connected. We again use scalar notation, but the arguments we use extend to the case of systems.

In contrast to Chapters 1 and 2, the randomness in this chapter is encoded into the choice of domain. In particular, if for any subset $P \subset \mathbb{R}^d$ we let χ_P

denote the indicator function of the set P, in all that follows we will consider μ to be a stationary and ergodic probability measure on the space

$$\Omega = \{I_P \,|\, I_P = \mathrm{Id}\chi_P \text{ with } P \text{ as in (3.1)}\} \tag{3.3}$$

that is supported on the set

$$\tilde{\Omega} = \{I_P \,|\, I_P = \mathrm{Id}\chi_P \text{ with } P \text{ as in (3.1) with well-separated holes}\}\,. \tag{3.4}$$

We construct an example of such a stationary and ergodic measure μ on Ω: Let Ω_0 be the set of all sets of discrete points in $\mathbb{R}^d$, on which the Poisson point-process induces a stationary and ergodic measure. Fixing a set $X \in \Omega_0$ we define a self-map of Ω_0 such that any point $x \in X$ such that there exists another point $y \in X$ satisfying $B_2(x) \cap B_2(y) \neq \emptyset$ is removed from the set. Clearly, the set of points in the image of this map, say $\tilde{X}$, induce coefficient fields $I_{P(\tilde{X})}$ that have well-separated holes. Calling this map f we then consider the pushforward measure $f_*\mu$ on Ω. We then check that $f_*\mu$ is still stationary and ergodic:

- *(Stationarity)* If $I \in \Omega$ then for any $x \in \mathbb{R}^d$ we have that $f^{-1}(I) + x = f^{-1}(I(\cdot + x))$. By the stationarity of μ we find that $f_*\mu(I) = f_*\mu(I(\cdot + x))$.
- *(Ergodicity)* Let $g \in L^2(\Omega)$ satisfy $g(I(\cdot+x)) = g(I(\cdot))$ for $\mathcal{L}$-a.e. $x \in \mathbb{R}^d$ and $f_*\mu$-a.e. $I \in \Omega$. This then implies that $g \circ f \in L^2(\Omega_0)$ is also shift invariant. By the ergodicity of μ the function $g \circ f$ is μ-a.s. equal to $\int_\Omega g \circ f \,\mathrm{d}\mu(I)$ on Ω_0. Using the definition of the pushforward measure, we find that for $f_*\mu$- a.e. $I \in \Omega$ we can write $g(I) = \int_\Omega g \circ f \,\mathrm{d}\mu(I_P) = \int_{\tilde{\Omega}} g \,\mathrm{d}f_*\mu(I_P)$.

As we have done in the previous two chapters, in order to obtain our results we adapt the notion of excess used in [31] to the perforated domain. As the final goal of the analysis given this chapter would be a $C^{1,\alpha}$- Liouville principle for functions satisfying (3.2), we can motivate the form of the excess with the expected form of this Liouville principle. Towards this we recall that in the non-random situation, which is in this case simply the Laplace equation on $\mathbb{R}^d$, the subquadratic harmonic functions are simply linear functions. Going from this, one would expect that the subquadratic functions satisfying (3.2) are of the form $\xi \cdot x + \phi_\xi + c$ for $\xi \in \mathbb{R}^d$ and $c \in \mathbb{R}$, where the adapted corrector $\phi_\xi \in H^1_{loc}(\bar{P})$ is a weak solution of

$$-\nabla \cdot I_P \nabla(\phi_\xi + \xi \cdot x) = 0 \qquad \text{in} \quad P, \tag{3.5a}$$

$$\nu \cdot \nabla(\phi_\xi + \xi \cdot x) = 0 \qquad \text{on} \quad \partial P. \tag{3.5b}$$

We arrive at the following definition: The perforated domain excess of a solution $u \in H^1_{loc}(\overline{P})$ of (3.2) on $B_r \cap P$ is given by

$$\mathrm{Exc}^H(r) = \inf_{\xi \in \mathbb{R}^d} \fint_{B_r \cap P} |\nabla u - \nabla(\xi \cdot x + \phi_\xi(I_P, x))|^2 \, \mathrm{d}x, \tag{3.6}$$

where the superscript "H" stands for "*holes*".

The first theorem that we prove in this chapter μ-almost surely provides us with a sublinear pair (ϕ, σ) that is adapted to the perforated domain. Our construction of the adapted corrector in this chapter is not like in the previous chapters; in particular, the adapted corrector we use in this chapter arises naturally as the limit of whole-space correctors obtained via the argument presented in Gloria, Neukamm, and Otto in [31]. The adapted corrector is taken as input into the second theorem in which we prove a large-scale $C^{1,\alpha}$-excess decay, but, unlike our previous chapters, we do not go as far as deriving a mean-value property or obtaining a Liouville principle. These results would be immediate given our Theorems 1 and 2 and the arguments already given in the previous chapters, but this would further increase the length of this thesis without providing any new ideas.

Before going into detail, we mention that the main technical difference between the situation we consider in this chapter and previously is that here our homogenized equation,

$$-\nabla \cdot I_{hom} \cdot \nabla u_{hom} = 0 \qquad \text{in} \quad \mathbb{R}^d \tag{3.7}$$

with solution $u_{hom} \in H^1_{loc}(\mathbb{R}^d)$, is posed on a different domain than the original equation (3.2). Notice that here the homogenized coefficients I_{hom} are defined analogously to the standard case; in particular, for $\xi \in \mathbb{R}^d$ the homogenized coefficients are determined by the relation

$$I_{hom}\xi = \int_\Omega I_P(0)(\xi + \nabla\phi_\xi(I_P, 0)) \, \mathrm{d}\mu(I_P).$$

The fact that u_{hom} and u are defined on different domains makes it not so clear how to define the homogenization error "w", which we have used as a technical tool in the proofs of Theorem 2 in the previous chapters.

3.2 Main Results

As indicated above, in our first theorem we assume that we have a stationary and ergodic measure μ on Ω and show the almost sure existence of a sublinear generalized adapted corrector.

Theorem 1. *Let μ be a stationary and ergodic measure on Ω defined in* (3.3) *that is supported on $\tilde{\Omega}$ defined in* (3.4). *There exist two random fields $\phi(I_P, x)$ and $\sigma(I_P, x)$ that fulfil the following conditions:*

i) For every $i \in \{1, ..., d\}$ the adapted corrector in the direction e_i, $\phi_i(I_P, \cdot)$, is a weak solution of (3.5) *μ-almost surely.*

ii) For every $i, j, k \in \{1, ..., d\}$ the adapted flux corrector $\sigma_{ijk}(I_P, \cdot)$ is a distributional solution of

$$\nabla_k \cdot \sigma_{ijk}(I_P, \cdot) = e_j \cdot q_i(I_P, \cdot) \qquad in \quad \mathbb{R}^d \tag{3.8}$$

μ-almost surely, where the current correction on the right-hand side is

$$q_i(I_P, x) = \begin{cases} \nabla\phi_i(I_P, x) + e_i - \int_\Omega I_P(0)(\nabla\phi_i(I_P, 0) + e_i)\, d\mu(I_P) & \text{if} \quad x \in P \\ -\int_\Omega I_P(0)(\nabla\phi_i(I_P, 0) + e_i)\, d\mu(I_P) & \text{if} \quad x \notin P. \end{cases} \tag{3.9}$$

The σ_{ijk}, furthermore, satisfy the skew-symmetry condition $\sigma_{ijk} = -\sigma_{ikj}$.

iii) The random fields ϕ and σ are μ-almost surely sublinear in the sense that

$$\lim_{r\to\infty} \delta_r(I_P) = 0 \tag{3.10}$$

where we use the notation

$$\delta_r(I_P) = \frac{1}{r^2} \fint_{B_r\cap P} \sum_{i=1}^{d} \left(\left|\phi_i(I_P, x) - \fint_{B_r\cap P} \phi_i(I_P, x)\, dx\right|^2 + \sum_{j,k=1}^{d} \left|\sigma_{ijk}(I_P, x) - \fint_{B_r\cap P} \sigma_{ijk}(I_P, x)\, dx\right|^2 \right) dx.$$

We use the generalized adapted corrector constructed in Theorem 1 to prove the following excess decay:

Theorem 2. *Let μ be a stationary and ergodic measure on Ω defined in* (3.3) *that is supported on $\tilde{\Omega}$ defined in* (3.4). *For every Hölder exponent $\alpha \in (0, 1)$ there exists a constant $C_\alpha(d) > 0$ such that if*

$$\delta_r(I_P) \le \frac{1}{C_\alpha(d)} \qquad for \quad r \ge r^*_\alpha(I_P) \tag{3.11}$$

for a minimal radius $r^*_\alpha(I_P) > 0$, then for $u \in H^1_{loc}(\overline{P})$ a weak solution of (3.2) *the excess decay*

$$Exc^H(r) \lesssim \left(\frac{r}{R}\right)^{2\alpha} Exc^H(R) \tag{3.12}$$

is satisfied for $R \geq r \geq r^*_\alpha(I_P)$. *We find that the minimal radius* μ*-almost surely satisfies* $r^*_\alpha(I_P) < \infty$.

All of the proofs in this chapter hinge on the following elementary lemma concerning the harmonic extension of a function $u \in H^1_{loc}(\overline{P})$ into the holes of P. This lemma is, in particular, the reason we assume that the holes of P are well-separated. As this lemma is the foundation of this chapter, we state and immediately prove it:

Lemma 1. *Let* $P = \mathbb{R}^d \setminus \cup_{i\in H} B_1(x_i)$ *as in* (3.1) *such that the holes are well-separated and let* $u \in H^1_{loc}(\overline{P})$. *Define the harmonic extension of* u *into the holes, which we denote* u_{ext}, *as*

$$u_{ext}(x) = \begin{cases} u(x) & \text{if} \quad x \in P \\ u_i^E(x) & \text{if} \quad x \in B_1(x_i) \text{ for some } i \in H \end{cases} \tag{3.13}$$

where for every $i \in H$ *we have that* $u_i^E : B_1(x_i) \to \mathbb{R}^d$ *solves*

$$\begin{aligned} -\Delta u_i^E &= 0 \qquad && \text{in} \quad B_1(x_i), \\ u_i^E &= u && \text{on} \quad \partial B_1(x_i). \end{aligned} \tag{3.14}$$

Then we obtain the following two results:

i) The bound

$$\fint_{B_r} |\nabla u_{ext}|^2 \, dx \lesssim \fint_{B_{2r}\cap P} |\nabla u|^2 \, dx \tag{3.15}$$

holds for $r \geq 3$.

ii) Using the notation

$$\Omega_{good} = \{I_P \in \Omega \,|\, 0 \in P\}, \tag{3.16}$$

if we have the additional information that $\nabla u_{ext}(I_P, x)$ *is a stationary random field in* $L^2(\Omega)$ *then* (3.15) *implies that*

$$\int_\Omega |\nabla u_{ext}(I_P, 0)|^2 d\mu(I_P) \lesssim \int_{\Omega_{good}} |\nabla u(I_P, 0)|^2 d\mu(I_P). \tag{3.17}$$

Proof of Lemma 1. We start by proving i). Fix $r \geq 3$ and index the holes of P that intersect B_r nontrivially by $H' \subseteq H$. For any $i \in H$ denote $A_i = B_2(x_i) \setminus B_1(x_i)$ and notice that thanks to the well-separatedness of the holes of P, these annuli are disjoint. Also, as the inequality (3.15) only involves gradients we may assume that for a fixed $i \in H$ we have that $\fint_{A_i} u \, \mathrm{d}x = 0$.

The trace theorem applied to $u \in H^1(A_i)$ followed by the Poincaré inequality gives that

$$\|u\|_{H^{\frac{1}{2}}(\partial A_i)} \lesssim \|\nabla u\|_{L^2(A_i)} + \|u\|_{L^2(A_i)} \lesssim \|\nabla u\|_{L^2(A_i)}. \tag{3.18}$$

Furthermore, since the trace operator $\gamma : H^1(B_1(x_i)) \to H^{\frac{1}{2}}(\partial B_1(x_i))$ has a continuous right inverse there exists v defined on $B_1(x_i)$ such that $v = u$ on $\partial B_1(x_i)$ and the bound

$$\|\nabla v\|_{L^2(B_1(x_i))} \lesssim \|u\|_{H^{\frac{1}{2}}(\partial A_i)} \tag{3.19}$$

holds. Of course, the harmonic extension u^E satisfies

$$\|\nabla u_i^E\|_{L^2(B_1(x_i))} \leq \|\nabla v\|_{L^2(B_1(x_i))}, \tag{3.20}$$

which, when hooked-up with (3.18), implies that

$$\|\nabla u_i^E\|_{L^2(B_1(x_i))} \lesssim \|\nabla u\|_{L^2(A_i)}. \tag{3.21}$$

To finish the argument for (3.15) notice that

$$\begin{aligned} \int_{B_r} |\nabla u_{ext}|^2 \, \mathrm{d}x &= \int_{P \cap B_r} |\nabla u|^2 \, \mathrm{d}x + \sum_{i \in H'} \int_{B_1(x_i)} |\nabla u_i^E|^2 \, \mathrm{d}x \\ &\lesssim 2 \int_{P \cap B_{2r}} |\nabla u|^2 \, \mathrm{d}x, \end{aligned}$$

where we have use dthat $u_{ext} = u$ on P, the bound (3.21), and that the A_i are disjoint.

For the pat $ii)$ of this lemma we first re-write (3.15) as

$$\fint_{B_r} |\nabla u_{ext}|^2 \, \mathrm{d}x \lesssim \fint_{B_{2r}} |I_P(x) \nabla u|^2 \, \mathrm{d}x$$

and then take the expectation of both sides. To finish we apply Fubini's theorem and the stationarity of the ensemble to obtain

$$\begin{aligned} \fint_{B_r} \int_\Omega |\nabla u_{ext}(I_P, x)|^2 \, \mathrm{d}\mu(I_P) \, \mathrm{d}x &\lesssim \fint_{B_{2r}} \int_\Omega |I_P(x) \nabla u(I_P, x)|^2 \, \mathrm{d}\mu(I_P) \, \mathrm{d}x \\ &= \fint_{B_{2r}} \int_\Omega |I_P(0) \nabla u(I_P, 0)|^2 \, \mathrm{d}\mu(I_P) \, \mathrm{d}x \end{aligned}$$

which after also using the stationarity of μ on the left-hand side gives (3.17) as desired. □

3.3 Construction of the Generalized Adapted Corrector

With Lemma 1 in-hand we proceed to prove Theorem 1. Here comes the argument:

Proof of Theorem 1. The proof is broken into two parts: In the first part we prove *i)* and *ii)* by constructing (ϕ, σ) and in the second part we prove the sublinearity property *iii)*. Throughout this proof we will make recurring use of the convention

$$\Omega_{good} = \{I_P \in \Omega \,|\, 0 \in P\},$$

which we have already introduced in Lemma 1, and the corresponding notion

$$\Omega_{bad} = \Omega \setminus \Omega_{good} = \{I_P \in \Omega \,|\, 0 \notin P\}.$$

Part 1: Construction of (ϕ, σ)

We construct (ϕ, σ) in a manner very similar to Lemma 1 in [31].

Step 1– (Construction of ϕ*)*

For a fixed $i \in \{1, ..., d\}$ we construct the random field $\phi_i(I_P, x)$, the adapted corrector in the direction e_i. For any $I_P \in \Omega$ and $0 < \epsilon$ we define

$$I_P^\epsilon = I_P + \epsilon \mathrm{Id} \chi_{\mathbb{R}^d \setminus P}$$

and let

$$\mathcal{X} = \left\{ g \in L^2(\Omega, \mathbb{R}^d) \;\middle|\; D_j g_k = D_k g_j \text{ and } \int_\Omega g \, \mathrm{d}\mu(I_P) = 0 \right\}, \qquad (3.22)$$

where D_j denotes the “horizontal derivative" given by

$$(D_j g)(I_P) = \lim_{h \to 0} \frac{g(I_P(\cdot + he_j)) - g(I_P(\cdot))}{h} \qquad (3.23)$$

and the curl-free condition is meant in the distributional sense.

We first notice that $\mathcal{X}$ is a closed subspace of $L^2(\Omega, \mathbb{R}^d)$. To see this we take a $L^2(\Omega, \mathbb{R}^d)$- convergent sequence $\{g^l \in \mathcal{X}\}_l$ with limit $g^\infty \in \mathcal{X}$. For $\varphi \in H^1(\Omega)$, where this space is defined in terms of the horizontal derivatives (3.23), we use Hölder's inequality to write

$$\left| \int_\Omega D_j \varphi g_k^\infty - D_k \varphi g_j^\infty \, \mathrm{d}\mu(I_P) \right|$$

$$\lesssim \int_\Omega |(D_j\varphi g_k^\infty - D_k\varphi g_j^\infty) - (D_j\varphi g_k^l - D_k\varphi g_j^l)| \,\mathrm{d}\mu(I_P)$$
$$\lesssim \|D_j\varphi\|_{L^2(\Omega)}\|g_k^\infty - g_k^l\|_{L^2(\Omega)} + \|D_k\varphi\|_{L^2(\Omega)}\|g_j^\infty - g_j^l\|_{L^2(\Omega)},$$

which yields the curl-free condition for g^∞ after passing to the limit $l \uparrow \infty$. Showing the vanishing average condition is essentially the same; in particular, notice that for every l we may write

$$\left|\int_\Omega g^\infty \mathrm{d}\mu(I_P)\right| \lesssim \|1\|_{L^2(\Omega)}\|g^\infty - g^l\|_{L^2(\Omega)}, \tag{3.24}$$

which gives the vanishing average condition for g^∞ after take the limit $l \uparrow \infty$.

Aside from the closedness of $\mathcal{X}$ we also notice that the coefficient field I_P^ϵ is, for fixed $\epsilon > 0$, uniformly elliptic. Using these two observations we notice that an application of Lax-Milgram provides us with a $\tilde{g}^\epsilon \in \mathcal{X}$ such that for every $g \in \mathcal{X}$ it holds that

$$\int_\Omega g \cdot I_P^\epsilon(0)(\tilde{g}^\epsilon + e_i)\,\mathrm{d}\mu(I_P) = 0. \tag{3.25}$$

Notice that in this application of Lax-Milgram we have crucially used the finite mass of the probability space. In order to fix notation for the second part of this proof, we remark that it is shown in [31] that the stationary extension of $\tilde{g}^\epsilon$, which we will denote $\tilde{g}^\epsilon(I_P(\cdot + x)) = \bar{\tilde{g}}^\epsilon(I_P, x)$, satisfies $\bar{\tilde{g}}^\epsilon(I_P, x) = \nabla\phi^\epsilon(I_P, x)$, where $\phi^\epsilon(I_P, x)$ is a whole-space corrector for the coefficient field I_P^ϵ and is μ-almost surely sublinear.

Testing (3.25) with $\tilde{g}^\epsilon$ yields

$$\left(\int_\Omega |(I_P^\epsilon(0))^{\frac{1}{2}}\tilde{g}^\epsilon|^2\,\mathrm{d}\mu(I_P)\right)^{\frac{1}{2}} \leq 1. \tag{3.26}$$

When $I_P \in \Omega_{good}$, as $0 \in P$, we know that $|\tilde{g}^\epsilon \cdot I_P^\epsilon(0)\tilde{g}^\epsilon|^2 = |\tilde{g}^\epsilon|^2$ and (3.26) gives that

$$\left(\int_{\Omega_{good}} |\tilde{g}^\epsilon|^2\,\mathrm{d}\mu(I_P)\right)^{\frac{1}{2}} \leq 1, \tag{3.27}$$

which implies that there exists a $\tilde{g}^0 \in L^2(\Omega_{good}, \mathbb{R}^d)$ such that $\tilde{g}^\epsilon \xrightarrow{\epsilon\downarrow 0} \tilde{g}^0$. We extend the definition of $\tilde{g}^0$ to all of Ω in the following way:

$$\tilde{g}^0(I_P) = \begin{cases} \tilde{g}^0(I_P) & \text{for} \quad I_P \in \Omega_{good} \\ 0 & \text{if} \quad I_P \in \Omega_{bad}. \end{cases} \tag{3.28}$$

Notice that since for $\varphi \in H^1(\Omega)$ it holds that $D\varphi \in \mathcal{X}$, the relation (3.25) implies that

$$\int_\Omega D\varphi \cdot I_P^\epsilon(0)(\tilde{g}^\epsilon + e_i)\,\mathrm{d}\mu(I_P) = 0. \tag{3.29}$$

We would like to pass to the limit $\epsilon \downarrow 0$ in this equation. To do this we write

$$\begin{aligned}&\int_\Omega D\varphi \cdot (I_P^\epsilon(0)(\tilde{g}^\epsilon + e_i) - I_P(0)(\tilde{g}^0 + e_i))\,\mathrm{d}\mu(I_P)\\ =&\int_\Omega D\varphi \cdot \big((I_P^\epsilon(0) - I_P(0))e_i + (I_P^\epsilon(0) - I_P(0))\,\tilde{g}^\epsilon - I_P(0)(\tilde{g}^0 - \tilde{g}^\epsilon)\big)\,\mathrm{d}\mu(I_P).\end{aligned} \tag{3.30}$$

The first term is immediately bounded as

$$\int_\Omega |D\varphi \cdot (I_P^\epsilon(0) - I_P(0))e_i|\,\mathrm{d}\mu(I_P) \le \epsilon \left(\int_\Omega |D\varphi|^2\,\mathrm{d}\mu(I_P)\right)^{\frac{1}{2}}. \tag{3.31}$$

For the second term we use that (3.26) implies

$$\left(\int_\Omega |\tilde{g}^\epsilon|^2 \mathrm{d}\mu(I_P)\right)^{\frac{1}{2}} \le \epsilon^{-\frac{1}{2}},$$

which allows us to use Hölder's inequality to write

$$\int_\Omega |D\varphi \cdot (I_P^\epsilon(0) - I_P(0))\tilde{g}^\epsilon|\,\mathrm{d}\mu(I_P) \le \epsilon^{\frac{1}{2}} \left(\int_\Omega |D\varphi|^2\,\mathrm{d}\mu(I_P)\right)^{\frac{1}{2}}. \tag{3.32}$$

Combining (3.30), (3.31), and (3.32) we find that

$$\begin{aligned}&\left|\int_\Omega D\varphi \cdot (I_P^\epsilon(0)(\tilde{g}^\epsilon + e_i) - I_P(0)(\tilde{g}^0 + e_i))\,\mathrm{d}\mu(I_P)\right|\\ \lesssim&\,\epsilon^{\frac{1}{2}}\|D\varphi\|_{L^2(\Omega)} + \left|\int_\Omega D\varphi \cdot I_P(0)(\tilde{g}^0 - \tilde{g}^\epsilon)\,\mathrm{d}\mu(I_P)\right|,\end{aligned}$$

which when combined with (3.29), after passing to the limit $\epsilon \downarrow 0$ and using the weak convergence of $\tilde{g}^\epsilon$ to $\tilde{g}^0$ in $L^2(\Omega_{good}, \mathbb{R}^d)$, yields that

$$\int_\Omega D\varphi \cdot I_P(0)(\tilde{g}^0 + e_i)\,\mathrm{d}\mu(I_P) = 0. \tag{3.33}$$

We now use the notation $\varphi(I_P(\cdot + x)) = \bar{\varphi}(I_P, x)$ and $\tilde{g}^0(I_P(\cdot + x)) = \bar{\tilde{g}}^0(I_P, x)$. Using the relation $D_j\varphi(I_P(\cdot + x)) = \partial_j\bar{\varphi}(I_P, x)$, where "$\partial_j$" refers to the standard partial derivative in the spatial variable, we use the stationarity and ergodicity of μ to convert (3.33) to

$$\int_{\mathbb{R}^d} \nabla(\bar{\varphi}(I_P, x)) \cdot I_P(x)(\bar{\tilde{g}}^0(I_P, x) + e_i)\,\mathrm{d}x = 0, \tag{3.34}$$

which holds μ-almost surely for any $\varphi \in H^1(\Omega)$ such that $\varphi(I_P, \cdot)$ has uniformly bounded support for all $I_P \in \Omega$. In particular, notice that if $\bar{f}(I_P, \cdot)$ is a stationary random field such that $\operatorname{supp}(\bar{f}(I_P, \cdot)) \subset B_r$ for all $I_P \in \Omega$ and the corresponding random variable $\bar{f}(I_P, 0) = f \in L^1(\Omega)$ then

$$\begin{aligned}
\int_{\mathbb{R}^d} \bar{f}(I_P, x)\,\mathrm{d}x &\overset{\mu-\text{a.s.}}{=} \int_\Omega \int_{\mathbb{R}^d} \bar{f}(I_P, x)\,\mathrm{d}x\,\mathrm{d}\mu(I_P) \\
&= \int_\Omega \int_{B_r} \bar{f}(I_P, x)\,\mathrm{d}x\,\mathrm{d}\mu(I_P) \\
&= \int_{B_r} \int_\Omega \bar{f}(I_P, x)\,\mathrm{d}\mu(I_P)\,\mathrm{d}x \\
&= |B_r| \int_\Omega \bar{f}(I_P, 0)\,\mathrm{d}\mu(I_P).
\end{aligned} \tag{3.35}$$

Here, the first equality holds μ-almost surely thanks to the erogodicity of the ensemble since the left-hand side is a shift-invariant random variable and we have used the Fubini-Tonelli theorem to swap the integrals in the third step. In order to obtain (3.34) we must take f to be the integrand in (3.33).

We, furthermore, notice that for $\varphi \in H^1(\Omega)$ such that $\operatorname{supp}(\varphi) \subseteq \Omega_{good}$ by the definition (3.28) the distributional curl-free condition passes to the limit; i. e. we have that

$$\int_\Omega (D_j\varphi)\tilde{g}_k^0 - (D_k\varphi)\tilde{g}_j^0\,\mathrm{d}\mu(I_P) = 0. \tag{3.36}$$

Once again assuming that $\bar{\varphi}(I_P, \cdot)$ has uniformly bounded support for all $I_P \in \Omega$ and now additionally $\operatorname{supp}(\varphi) \subseteq \Omega_{good}$, using (3.35) we obtain that

$$\int_{\mathbb{R}^d} \partial_j\varphi(I_P, x)\bar{\tilde{g}}_k^0(I_P, x) - \partial_k\varphi(I_P, x)\bar{\tilde{g}}_j^0(I_P, x)\,\mathrm{d}x = 0 \tag{3.37}$$

μ-almost surely. So, μ-almost surely $\bar{\tilde{g}}^0(I_P, \cdot)$ is distributionally curl-free on P, which is simply connected since $d \geq 3$. Using the Poincaré lemma this implies that μ-almost surely there exists a scalar potential $\phi_i(I_P, \cdot) \in H^1_{loc}(P)$ such that $\bar{\tilde{g}}^0(I_P, x) = \nabla\phi_i(I_P, x)$ on P. Of course, we may, furthermore, impose that $⨍_{B_1} \phi_i(I_P, x)\mathrm{d}x = 0$ for each realization $I_P \in \Omega$ for which this potential exists.

To finish our argument we, for a fixed realization $I_{P^*} \in \Omega$ such that (3.34) and (3.37) hold, make the substitution $\bar{\tilde{g}}^0(I_{P^*}, x) = \nabla\phi_i(I_{P^*}, x)$ in (3.34). Then for any test function $\overline{\varphi} \in H^1_{bbd}(\mathbb{R}^d)$ we define the random variable

$$\varphi(I_P) = \begin{cases} \overline{\varphi}(x) & \text{for} \quad I_P = I_{P^*}(\cdot + x) \\ 0 & \qquad\quad \text{otherwise,} \end{cases} \tag{3.38}$$

the stationary extension of which satisfies $\varphi(I_{P^*}, x) = \overline{\varphi}(x)$ and also $\varphi(I_P, x)$ has uniformly bounded support for all $I_P \in \Omega$. These identifications allow us to conclude that $\phi_i(I_{P^*}, x)$ satisfies the weak formulation of the corrector equation (3.5) on P^*.

Step 2– (Construction of σ)

For a fixed $i \in \{1, ..., d\}$ we now construct the random field $\sigma_i(I_P, x)$, which is matrix valued since we have dropped two indices, using the strategy given in [31]. In particular, we let

$$\mathcal{Y} = \left\{ \tilde{b} \in L^2(\Omega, \mathbb{R}^{d\times d}_{sym}) \;\middle|\; D_k \tilde{b}_{lm} = D_m \tilde{b}_{lk} \text{ and } \int_\Omega \tilde{b}_{kl} \,\mathrm{d}\mu(I_P) = 0 \right\} \tag{3.39}$$

and introduce the three-tensor $b_{jkl} \in \mathcal{Y}$, which is the orthogonal projection of $q_j I \in L^2(\Omega, \mathbb{R}^{d\times d}_{sym})$ onto $\mathcal{Y}$ with $q = I_P(0)(\tilde{g}^0 + e_i) - \int_\Omega I_P(0)(\tilde{g}^0 + e_i) \,\mathrm{d}\mu(I_P)$. We then obtain two relations for b_j:

i) Notice that if $\varphi \in H^2(\Omega)$ then $D^2\varphi \in \mathcal{Y}$. So, we may, in particular, write

$$0 = \int_\Omega D^2\varphi : (b_j - q_j I) \,\mathrm{d}\mu(I_P) = \int_\Omega (\text{trace } D^2\varphi)(\text{trace } b_j - q_j) \,\mathrm{d}\mu(IP), \tag{3.40}$$

where we have used orthogonality and the curl-free condition in $\mathcal{Y}$. In particular, we use that the relation

$$\begin{aligned}
\int_\Omega D_l D_l \varphi b_{jkk} \,\mathrm{d}\mu(I_P) &= -\int_\Omega D_l \varphi D_l b_{jkk} \,\mathrm{d}\mu(I_P) \\
&= -\int_\Omega D_l \varphi D_k b_{jkl} \,\mathrm{d}\mu(I_P) \\
&= \int_\Omega D_l D_k \varphi b_{jkl} \,\mathrm{d}\mu(I_P)
\end{aligned}$$

holds for all $l, k \in \{1, ..., d\}$. Due to ergodicity the space $\{\text{trace } D^2\varphi \mid \varphi \in H^2(\Omega)\}$ is dense in $\{\varphi \in L^2(\Omega) \mid \int_\Omega \varphi \,\mathrm{d}\mu(I_P) = 0\}$. To see this choose some ψ in the orthogonal complement of the L^2-closure of the first space; we show that ψ is μ-almost surely equal to some constant. This is almost immediate since for all $\tilde{\varphi} \in H^2(\Omega)$ we have that

$$\int_\Omega \psi \,\text{trace}(D^2\tilde{\varphi}) \,\mathrm{d}\mu(I_P) = -\int_\Omega D\psi \cdot D\tilde{\varphi} \,\mathrm{d}\mu(I_P) = 0,$$

which implies that $D\psi = 0$ μ-almost surely and by ergodicity this implies that ψ is μ-almost surely a constant. Combining this density observation with $\int_\Omega q_j \,\mathrm{d}\mu(I_P) = \int_\Omega \operatorname{trace} b_j \,\mathrm{d}\mu(I_P) = 0$, the equality (3.40) implies that

$$\operatorname{trace} b_j = b_{jkk} = q_j \tag{3.41}$$

μ-almost surely.

ii) We find that

$$D_l D_l b_{kkj} = D_j D_k b_{kll} \tag{3.42}$$

distributionally, where we have used the curl-free assumption in $\mathcal{Y}$. Therefore, if $\varphi \in H^2(\Omega)$ such that $D\varphi(I_P, \cdot)$ has uniformly bounded support for all $I_P \in \Omega$ we may use (3.33) to write

$$\begin{aligned}\int_\Omega \operatorname{trace}(D^2\varphi) b_{kkj} \,\mathrm{d}\mu(I_P) &= \int_\Omega D_l D_l \varphi b_{kkj} \,\mathrm{d}\mu(I_P) \\ &= \int_\Omega D_k D_j \varphi b_{kll} \,\mathrm{d}\mu(I_P) \\ &\overset{(3.41)}{=} \int_\Omega D_k D_j \varphi q_k \,\mathrm{d}\mu(I_P) \\ &= 0.\end{aligned} \tag{3.43}$$

By density this implies that

$$\sum_{k=1}^{d} b_{kkj} = 0 \tag{3.44}$$

μ-almost surely.

To complete the argument we pass to the stationary extensions of the b_{jkl}, which are denoted $\bar{b}_{jkl}(I_P, x)$. The curl-free assumption in $\mathcal{Y}$ implies that $D_l b_{jkm} = D_m b_{jkl}$ distributionally in the probability space, which then translates into the condition

$$\partial_l \bar{b}_{jkm}(I_P, x) = \partial_m \bar{b}_{jkl}(I_P, x) \tag{3.45}$$

which μ-almost surely holds in a distributional sense in $\mathbb{R}^d$. Combining this curl-free condition with $d \geq 3$ allows one to apply the Poincaré lemma, which then for every $I_P \in \Omega$ such that (3.45) holds gives $\sigma_{jk}(I_P, \cdot)$ such that

$$b_{jkl}(I_P, x) - b_{kjl}(I_P, x) = \partial_l \sigma_{jk}(I_P, x) \tag{3.46}$$

holds distributionally on $\mathbb{R}^d$. The vanishing average assumption in $\mathcal{Y}$ implies that $\int_\Omega \nabla \sigma_{jk} \,\mathrm{d}\mu(I_P) = 0$ and we may, furthermore, impose that for every

$I_P \in \Omega$ for which σ is defined it holds that $\fint_{B_1} \sigma_{jk}(I_P, x)\,\mathrm{d}x = 0$, which implies that $\sigma_{jk}(I_P, x)$ retains the skew-symmetry of $b_{jkl}(I_P, x) - b_{kjl}(I_P, x)$ in j and k. After using stationarity and ergodicity to move to physical space (just like in our argument for the construction of ϕ), the equalities (3.41) and (3.44) yield that μ-almost surely

$$\partial_l \sigma_{jl}(I_P, x) = b_{jll}(I_P, x) - b_{jjl}(I_P, x) = q_j(I_P, x) \tag{3.47}$$

distributionally in $\mathbb{R}^2$.

Part 2: Almost sure sublinearity of (ϕ, σ)

Since σ is constructed so that $\int_\Omega \nabla \sigma_{ijk}\,\mathrm{d}\mu(I_P) = 0$ and $\nabla \sigma_{ijk}$ is stationary for all $i, j, k \in \{1, ..., d\}$, the standard sublinearity argument from [31, Proof of Corollary 1: Step 1] applies and shows that

$$\lim_{r \to \infty} \frac{1}{r} \fint_{B_r} \left| \sigma_{ijk}(I_P, \cdot) - \fint_{B_r} \sigma_{ijk}(I_P, \cdot)\,\mathrm{d}x \right|^2 \mathrm{d}x = 0 \tag{3.48}$$

μ-almost surely. Due to the disjointness assumption on the holes of P we have that $r^d \lesssim |B_r \cap P|$ uniformly in r and, therefore, we may write

$$\begin{aligned} &\frac{1}{r} \fint_{B_r \cap P} \left| \sigma_{ijk}(I_P, \cdot) - \fint_{B_r \cap P} \sigma_{ijk}(I_P, \cdot)\,\mathrm{d}x \right|^2 \mathrm{d}x \\ \lesssim &\frac{1}{r} \fint_{B_r} \left| \sigma_{ijk}(I_P, \cdot) - \fint_{B_r} \sigma_{ijk}(I_P, \cdot)\,\mathrm{d}x \right|^2 \mathrm{d}x, \end{aligned}$$

which by (3.48) then implies the desired sublinearity property for σ.

It then only remains to show the sublinearity of ϕ, which we do in the steps below. To avoid the notation becoming too bulky we drop the index i from ϕ_i and instead always write ϕ. Also, in this argument we make use of the ϕ^ϵ, which we have introduced in our construction of ϕ.

Step 1– (The harmonic extension of the corrector into the holes)

We consider $\phi_{ext}(I_P, x)$, the harmonic extension as defined in Lemma 1 of the corrector $\phi(I_P, x)$ into the holes of P. We first observe that $\nabla \phi_{ext}$ is stationary on $\mathbb{R}^d$. To see this we recall from the previous step that $\nabla \phi^\epsilon \rightharpoonup \nabla \phi$ in $L^2(\Omega_{good})$. Using the ergodicity of the ensemble we find that this implies μ-almost surely that $\nabla \phi^\epsilon \rightharpoonup \nabla \phi$ in $L^2_{loc}(P)$. As the $\nabla \phi^\epsilon$ are stationary, by the uniqueness of weak limits we have that for fixed $z \in \mathbb{R}^d$ it holds that $\nabla \phi(I_P(\cdot + z), x) = \nabla \phi(I_P, x + z)$ on $P - z$. This, of course, means that on

$P-z$ we have that $\phi(I_P(\cdot+z),x)=\phi(I_P,x+z)+C$. The uniqueness of the harmonic extension then implies that $\phi_{ext}(I_P,x+z)=\phi_{ext}(I_P(\cdot+z),x)+C$ for $x\in\mathbb{R}^d$, which implies that $\nabla\phi_{ext}$ is stationary.

We would then like to show that, in fact, $\nabla\phi^\epsilon(I_P,\cdot)\rightharpoonup\nabla\phi_{ext}(I_P,\cdot)$ in $L^2_{loc}(\mathbb{R}^d)$ μ-almost surely. We already know that μ-almost surely $\nabla\phi^\epsilon(I_P,\cdot)\rightharpoonup\nabla\phi_{ext}(I_P,\cdot)$ in $L^2_{loc}(P)$, which means that, fixing a realization I_P for which this convergence holds, we only need to concentrate on the holes, of which we fix $B_1(x_i)$. Since our claim is on the level of the gradients, we may assume that the ϕ^ϵ and ϕ have average 0 on an annulus of radius 1 around $B_1(x_i)$, which we denote A. Using the Poincaré inequality on this annulus we then find that $\phi^\epsilon\rightharpoonup\phi_{ext}$ in $H^1_{loc}(A)$, which implies that $\phi^\epsilon\rightharpoonup\phi_{ext}$ in $H^{\frac{1}{2}}(\partial B_1(x_i))$ because the trace operator is continuous. To finish our argument we notice that $\phi^\epsilon-\phi_{ext}$ is the unique harmonic function in $H^1(B_1(x_i))$ with the boundary data $\phi^\epsilon-\phi$. This implies our claim since the solution operator of the Laplacian with Dirichlet boundary data is again linear and bounded, which implies that it preserves weak convergence.

We then process the above observation to obtain that

$$\lim_{\epsilon\to 0}\int_\Omega\nabla\phi^\epsilon-\nabla\phi_{ext}\,\mathrm{d}\mu(I_P)=0,\tag{3.49}$$

which follows via Birkhoff's ergodic theorem after recalling the stationarity of $\nabla\phi^\epsilon-\nabla\phi_{ext}\in L^2(\Omega)$. Since for every $\epsilon>0$ we have that $\int_\Omega\nabla\phi^\epsilon\,\mathrm{d}\mu(I_P)=0$ this implies that $\int_\Omega\nabla\phi_{ext}\,\mathrm{d}\mu(I_P)=0$.

Step 2– (Non-degeneracy of the homogenized coefficients)

Let $\xi\in\mathbb{R}^d$. Since $\int_\Omega\nabla\phi^\epsilon_\xi\,\mathrm{d}\mu(I_P)=0$ and $\nabla\phi^\epsilon_\xi$ is, furthermore, curl-free we have that $\nabla\phi^\epsilon_\xi\in\mathcal{X}$. This, in particular, implies that the relation

$$\int_\Omega\nabla\phi^\epsilon_\xi\cdot I_P(0)(\nabla\phi_\xi+\xi)\mathrm{d}\mu(I_P)=0\tag{3.50}$$

holds, which allows us to write:

$$\begin{aligned}\xi\cdot I_{hom}\xi&=\int_\Omega\xi\cdot I_P(0)(\xi+\nabla\phi_\xi)\,\mathrm{d}\mu(I_P)\\&=\int_\Omega(\xi+\nabla\phi^\epsilon_\xi)\cdot I_P(0)(\xi+\nabla\phi_\xi)\,\mathrm{d}\mu(I_P)\\&=\int_\Omega(\xi+\nabla\phi^\epsilon_\xi)\cdot I_P(0)(\xi+\nabla\phi^\epsilon_\xi)\,\mathrm{d}\mu(I_P)\\&\quad+\int_\Omega(\xi+\nabla\phi^\epsilon_\xi)\cdot I_P(0)(\nabla\phi_\xi-\nabla\phi^\epsilon_\xi)\,\mathrm{d}\mu(I_P)\end{aligned}\tag{3.51}$$

$$
\begin{aligned}
&\geq \int_{\Omega_{good}} |\xi + \nabla\phi_\xi^\epsilon|^2 \,\mathrm{d}\mu(I_P) + \int_\Omega \xi \cdot I_P(0)(\nabla\phi_\xi - \nabla\phi_\xi^\epsilon)\,\mathrm{d}\mu(I_P) \\
&\quad + \int_\Omega \nabla\phi_\xi^\epsilon \cdot I_P(0)(\nabla\phi_\xi - \nabla\phi_\xi^\epsilon)\,\mathrm{d}\mu(I_P).
\end{aligned}
$$

To treat the third term on the right-hand side we use both (3.50) and (3.25) to notice that

$$
\begin{aligned}
&\int_\Omega \nabla\phi_\xi^\epsilon \cdot I_P(0)(\nabla\phi_\xi - \nabla\phi_\xi^\epsilon)\,\mathrm{d}\mu(I_P) \\
=&\int_\Omega \nabla\phi_\xi^\epsilon \cdot I_P(0)(\nabla\phi_\xi + \xi - (\nabla\phi_\xi^\epsilon + \xi))\,\mathrm{d}\mu(I_P) \\
=&-\int_\Omega \nabla\phi_\xi^\epsilon \cdot I_P(0)(\nabla\phi_\xi^\epsilon + \xi)\,\mathrm{d}\mu(I_P) \\
=&\int_{\Omega_{bad}} \nabla\phi_\xi^\epsilon I_P^\epsilon(0)(\nabla\phi_\xi^\epsilon + \xi)\,\mathrm{d}\mu(I_P) \\
=&\epsilon \int_{\Omega_{bad}} |\nabla\phi_\xi^\epsilon|^2 \,\mathrm{d}\mu(I_P) + \int_{\Omega_{bad}} \nabla\phi_\xi^\epsilon \cdot I_P^\epsilon(0)\xi\,\mathrm{d}\mu(I_P).
\end{aligned} \tag{3.52}
$$

Combining (3.51) and (3.52) yields

$$
\begin{aligned}
\xi \cdot I_{hom}\xi \geq &\int_{\Omega_{good}} |\xi + \nabla\phi_\xi^\epsilon|^2 \,\mathrm{d}\mu(I_P) \\
&+ \int_\Omega \xi \cdot I_P(0)(\nabla\phi_\xi - \nabla\phi_\xi^\epsilon)\,\mathrm{d}\mu(I_P) + \int_{\Omega_{bad}} \nabla\phi_\xi^\epsilon \cdot I_P^\epsilon(0)\xi\,\mathrm{d}\mu(I_P),
\end{aligned} \tag{3.53}
$$

where we, furthermore, have that

$$
\lim_{\epsilon\to 0} \left| \int_\Omega \xi \cdot I_P(0)(\nabla\phi_\xi - \nabla\phi_\xi^\epsilon)\,\mathrm{d}\mu(I_P) + \int_{\Omega_{bad}} \nabla\phi_\xi^\epsilon \cdot I_P^\epsilon(0)\xi\,\mathrm{d}\mu(I_P) \right| = 0. \tag{3.54}
$$

To see (3.54) one uses the weak convergence of $\nabla\phi_\xi^\epsilon \rightharpoonup \nabla\phi_\xi$ on Ω_{good} and, furthermore, processes (3.26) to give $\|\epsilon^{\frac{1}{2}}\nabla\phi_\xi^\epsilon\|_{L^2(\Omega_{bad})} \leq |\xi|$, which after an application of Hölder's inequality gives

$$
\left| \int_{\Omega_{bad}} \nabla\phi_\xi^\epsilon \cdot I_P^\epsilon(0)\xi\,\mathrm{d}\mu(I_P) \right| \leq \|\epsilon^{\frac{1}{2}}\nabla\phi_\xi^\epsilon\|_{L^2(\Omega_{bad})}\epsilon^{\frac{1}{2}}|\xi| \leq \epsilon^{\frac{1}{2}}|\xi|^2. \tag{3.55}
$$

To finish our argument we notice that by the definition of the coefficient field I_P^ϵ we have that $\phi_\xi^\epsilon(I_P,\cdot)+\xi\cdot x = ((\phi_\xi^\epsilon(I_P,\cdot)+\xi\cdot x)|_P)_{ext}$; Recall, furthermore, that $\nabla\phi_\xi^\epsilon(I_P,x)$ is a stationary random field. Therefore, we may apply

(3.17) along with Jensen's inequality and the condition $\int_\Omega \nabla\phi_\xi^\epsilon \,\mathrm{d}\mu(I_P) = 0$ to obtain that

$$\begin{aligned}\int_{\Omega_{good}} |\xi + \nabla\phi_\xi^\epsilon|^2 \,\mathrm{d}\mu(I_P) \gtrsim& \int_\Omega |\xi + \nabla\phi_\xi^\epsilon|^2 \,\mathrm{d}\mu(I_P) \\ \gtrsim& \left|\int_\Omega \xi + \nabla\phi_\xi^\epsilon \,\mathrm{d}\mu(I_P)\right|^2 \\ =& |\xi|^2.\end{aligned} \tag{3.56}$$

This is then combined with (3.53) and (3.54) to give

$$\xi \cdot I_{hom}\xi \gtrsim |\xi|^2. \tag{3.57}$$

Step 3– (The μ-almost sure sublinearity of ϕ)

For this argument we let $\{\eta_r\}_{r>0}$ be a family of Gaussian convolution kernels, where the subscript "r" denotes the rescaling of the kernel η_1 on the scale r. We use a superscript "r" to denote convolution with η_r; e.g. we let $\phi_{ext}(I_P,\cdot) * \eta_r(x) = \phi_{ext}^r(I_P,x)$ and $\phi^\epsilon(I_P,\cdot) * \eta_r(x) = \phi^{\epsilon,r}(I_P,x)$. Also, sometimes we will use the notation $(\cdot)_r = \cdot * \eta_r$. The stationarity of the random fields $\nabla\phi_{ext}^r$ and $\nabla\phi^{\epsilon,r}$ follows immediately from the stationarity of $\nabla\phi_{ext}$ and $\nabla\phi^\epsilon$.

We begin our sublinearity argument with an application of the triangle inequality in $L^2(\mathbb{R}^d)$ and by passing to the harmonic extension

$$\begin{aligned}&\frac{1}{R}\inf_{c\in\mathbb{R}}\left(\fint_{B_R\cap P} |\phi - c|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ \lesssim&\frac{1}{R}\inf_{c\in\mathbb{R}}\left(\fint_{B_R} |\phi_{ext} - c|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ \lesssim&\frac{1}{R}\left(\inf_{c\in\mathbb{R}}\left(\fint_{B_R} |\phi_{ext}^r - c|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(\fint_{B_R} |\phi_{ext}^r - \phi_{ext}|^2 \mathrm{d}x\right)^{\frac{1}{2}}\right) \\ \lesssim&\left(\fint_{B_R} |\nabla\phi_{ext}^r|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \frac{r}{R}\left(\fint_{B_R} |\nabla\phi_{ext}|^2 \mathrm{d}x\right)^{\frac{1}{2}}.\end{aligned} \tag{3.58}$$

Notice that the first inequality of (3.58) holds due to the observation that $r^d \lesssim |B_r \cap P|$ uniformly in r. Also, the last inequality is due to the Poincaré inequality with zero average and a standard convolution estimate.

To treat the first term on the right-hand side of (3.58), we use that $\nabla\phi_{ext}^r$ is stationary and apply the maximal ergodic theorem [40, Chapter 1: Corollary

2.2]. This gives that

$$\lim_{R\to\infty}\left(\fint_{B_R}|\nabla\phi^r_{ext}|^2\mathrm{d}x\right)^{\frac{1}{2}}\le\left(\int_\Omega|\nabla\phi^r_{ext}|^2\,\mathrm{d}\mu(I_P)\right)^{\frac{1}{2}}\tag{3.59}$$

μ-almost surely. We then seek to control $\left(\int_\Omega|\nabla\phi^r_{ext}|^2\,\mathrm{d}\mu(I_P)\right)^{\frac{1}{2}}$ for which we use that I_{hom} is nondegenerate: For every e_i there is $\xi_i\in\mathbb{R}^d$ such that $e_i=I_{hom}\xi_i$. Therefore, we may write

$$\begin{aligned}\int_\Omega\partial_i\phi^r_{ext}\,\mathrm{d}\mu(I_P)&=\int_\Omega\nabla\phi^r_{ext}\cdot I_{hom}\xi_i\,\mathrm{d}\mu(I_P)\\&=-\int_\Omega\nabla\phi^r_{ext}\cdot(I_P(0)(\nabla\phi_{\xi_i}+\xi_i)-I_{hom}\xi_i)\,\mathrm{d}\mu(I_P)\\&\quad+\int_\Omega\nabla\phi^r_{ext}\cdot I_P(0)(\nabla\phi_{\xi_i}+\xi_i)\,\mathrm{d}\mu(I_P)\\&=-\int_\Omega\nabla\phi^r_{ext}\cdot(I_P(0)(\nabla\phi_{\xi_i}+\xi_i)-I_{hom}\xi_i)\,\mathrm{d}\mu(I_P),\end{aligned}\tag{3.60}$$

where for the last equality we use that $\nabla\phi^r_{ext}$ is distributionally curl-free and that $\int_\Omega\nabla\phi^r_{ext}\,\mathrm{d}\mu(I_P)=0$ by Step 1. To continue (3.60) we write

$$\begin{aligned}\left|\int_\Omega\partial_i\phi^r_{ext}\,\mathrm{d}\mu(I_P)\right|&\le\left(\int_\Omega|\nabla\phi_{ext}|^2\,\mathrm{d}\mu(I_P)\right)^{\frac{1}{2}}\\&\quad\times\left(\int_\Omega|(I_P(0)(\nabla\phi_{\xi_i}+\xi_i)-I_{hom}\xi_i)_r|^2\,\mathrm{d}\mu(I_P)\right)^{\frac{1}{2}}\\&\lesssim\left(\int_\Omega|(I_P(0)\nabla\phi_{\xi_i}+\xi_i)_r-I_{hom}\xi_i|^2\,\mathrm{d}\mu(I_P)\right)^{\frac{1}{2}}.\end{aligned}\tag{3.61}$$

By Von Neumann's ergodic theorem, since $I_P(0)(\nabla\phi_{\xi_i}+\xi_i)$ is stationary, $\int_\Omega I_P(0)(\nabla\phi_{\xi_i}+\xi_i)\,\mathrm{d}\mu(I_P)=I_{hom}\xi_i$, and $I_P(0)(\nabla\phi_{\xi_i}+\xi_i)\in L^2(\Omega)$, we have that

$$\lim_{r\to\infty}\left(\int_\Omega|(I_P(0)(\nabla\phi_{\xi_i}+\xi_i))_r-I_{hom}\xi_i|^2\,\mathrm{d}\mu(I_P)\right)^{\frac{1}{2}}=0.\tag{3.62}$$

Since (3.62) holds for all $i\in\{1,...,d\}$, (3.61) gives that

$$\lim_{r\to\infty}\left|\int_\Omega\nabla\phi^r_{ext}\,\mathrm{d}\mu(I_P)\right|=0\tag{3.63}$$

and for any $\epsilon>0$ we can choose $r_0>0$ such that

$$\left|\int_\Omega\nabla\phi^{r_0}_{ext}\,\mathrm{d}\mu(I_P)\right|^2\le\frac{\epsilon}{2}.\tag{3.64}$$

Another application of Von Neumann's ergodic theorem implies that we may choose $\rho_0 > 0$ such that for $\rho \geq \rho_0$ we have that

$$\int_\Omega \left|(\nabla \phi_{ext}^{r_0})_\rho - \int_\Omega \nabla \phi_{ext}^{r_0} \, \mathrm{d}\mu(I_P)\right|^2 \, \mathrm{d}\mu(I_P) \leq \frac{\epsilon}{2}. \tag{3.65}$$

We then find that

$$\begin{aligned} &\int_\Omega \left|(\nabla \phi_{ext}^{r_0})_{\rho_0}\right|^2 \, \mathrm{d}\mu(I_P) \\ \leq &\int_\Omega \left|(\nabla \phi_{ext}^{r_0})_\rho - \int_\Omega \nabla \phi_{ext}^{r_0} \, \mathrm{d}\mu(I_P)\right|^2 \, \mathrm{d}\mu(I_P) + \left|\int_\Omega \nabla \phi_{ext}^{r_0} \, \mathrm{d}\mu(I_P)\right|^2 \\ \leq &\epsilon. \end{aligned} \tag{3.66}$$

By the semigroup property of the Gaussian convolution kernel, we obtain that for $r \geq \sqrt{r_0^2 + \rho_0^2}$ it holds that

$$\left(\int_\Omega |\nabla \phi_{ext}^r|^2 \, \mathrm{d}\mu(I_P)\right)^{\frac{1}{2}} \leq \epsilon^{\frac{1}{2}}. \tag{3.67}$$

We then fix any $r \geq \sqrt{r_0^2 + \rho_0^2}$ and combine (3.58), (3.59), (3.17), and (3.67) to write

$$\begin{aligned} \lim_{R\to\infty} \frac{1}{R} \inf_{c\in\mathbb{R}} \left(\fint_{B_R\cap P} |\phi - c|^2 \mathrm{d}x\right)^{\frac{1}{2}} &\leq \epsilon^{\frac{1}{2}} + \lim_{R\to\infty} \frac{r}{R} \left(\fint_{B_R} |\nabla \phi_{ext}|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq \epsilon^{\frac{1}{2}}, \end{aligned} \tag{3.68}$$

which, in particular, proves that

$$\lim_{R\to\infty} \frac{1}{R} \fint_{B_R\cap P} |\phi - \fint_{B_R\cap P} \phi \, \mathrm{d}x|^2 \, \mathrm{d}x = 0 \tag{3.69}$$

as desired.

□

3.4 Proof of the Large-Scale $C^{1,\alpha}$- Excess Decay

3.4.1 Constant Coefficient Regularity and Caccioppoli Estimate

Just as in the previous two chapters, for our proof of Theorem 2 we require access to a particular Caccioppoli estimate and to some constant coefficient regularity estimates. We start with the Caccioppoli estimate:

Lemma 2. *Let $r > 0$ and $u \in H^1(B_r \cap P)$ solve (3.2) on $B_r \cap P$; i.e. u solves*

$$\begin{aligned} -\Delta u &= 0 \qquad && in \quad B_r \cap P, \\ \nu \cdot \nabla u &= 0 \qquad && on \quad B_r \cap \partial P. \end{aligned} \tag{3.70}$$

Then for any $0 < \rho < \frac{r}{2}$ we find that

$$\int_{B_\rho \cap P} |\nabla u|^2 \, dx \lesssim \frac{1}{\rho^2} \int_{B_{2\rho} \cap P} |u|^2 \, dx. \tag{3.71}$$

Proof of Lemma 2. Letting η be a smooth radial cut-off for B_ρ in $B_{2\rho}$ such that $|\nabla \eta| \leq \frac{2}{\rho}$, we test the equation (3.70) with $\eta^2 u$ and obtain

$$\int_{B_r \cap P} 2u\eta \nabla \eta \cdot \nabla u + \eta^2 |\nabla u|^2 \, \mathrm{d}x = 0. \tag{3.72}$$

Of course, the boundary terms have vanished due to the no-flux boundary data of u and the cut-off η. The claim then follows from Young's inequality and the properties of the cut-off η. □

Here are the constant coefficient regularity estimates that we use:

Lemma 3. *Let $v \in H^1(B_r)$ be a solution to*

$$\begin{aligned} -\nabla \cdot I_{hom} \nabla v &= 0 \qquad && in \quad B_r, \\ v &= u_{ext} \qquad && on \quad \partial B_r \end{aligned} \tag{3.73}$$

for some $u_{ext} \in H^{\frac{1}{2}}(\partial B_r)$ and some constant uniformly elliptic coefficient $I_{hom} \in M_{d \times d}(\mathbb{R})$. Then for any $x \in B_r$ and $\rho > 0$ such that $B_{2\rho}(x) \subseteq B_r$ we have that

$$\sup_{y \in B_\rho(x)} |\nabla^n v| \lesssim \left(\frac{1}{\rho}\right)^{n-1} \left(\fint_{B_{2\rho}(x)} |\nabla v|^2 \, dx \right)^{\frac{1}{2}} \quad \textit{for all } n \in \mathbb{N} \tag{3.74}$$

and for $0 < \delta < \frac{r}{2}$ it holds that

$$\int_{B_r \setminus B_{r-\delta}} |\nabla v|^2 \, dx \lesssim r^{\frac{d-1}{d}} \delta^{\frac{1}{d}} \int_{\partial B_r} |\nabla^{tan} u_{ext}|^2 \, dS. \tag{3.75}$$

Proof. While the proof of (3.74) is exactly the same as in Lemma 6 of Chapter 1 and Lemma 7 of Chapter 2, the proof of (3.75) actually sees some simplification because now v solves a *constant coefficient* equation on B_r. In particular, we have access to the Calderón-Zygmund estimate for (3.73), which gives that

$$\left(\int_{B_r} |\nabla v|^p \, \mathrm{d}x\right)^{1/p} \leq C(p,d) \left(\int_{B_r} |\nabla \bar{v}|^p \, \mathrm{d}x\right)^{1/p}$$

for all $1 < p < \infty$, where $\bar{v}$ is the harmonic extension of $v|_{\partial B_r}$. We upgrade this to

$$\left(\int_{B_r} |\nabla v|^p \, \mathrm{d}x\right)^{1/p} \lesssim \left(\int_{\partial B_r} |\nabla^{tan} u_{ext}|^2 \, \mathrm{d}x\right)^{1/2}, \tag{3.76}$$

where $p = \frac{d-2}{2d}$ using (1.55). The desired relation then follows from an application of Hölder's inequality as

$$\int_{B_r \setminus B_{r-\delta}} |\nabla v|^2 \, \mathrm{d}x \leq |B_r \setminus B_{r-\delta}|^{\frac{1}{d}} \left(\int_{B_r} |\nabla v|^p \, \mathrm{d}x\right)^{2/p}. \tag{3.77}$$

□

3.4.2 Proof of Theorem 2: A Large-Scale $C^{1,\alpha}$- Excess Decay

With these results in-hand we now prove Theorem 2. A substantial difference between the case treated here and the cases of Dirichlet or Neumann boundary data treated earlier is that the homogenized and heterogeneous problems are posed on different domains. As already mentioned in the introduction, we overcome this by insisting that the holes of P are well-separated and using our Lemma 1. Here comes the argument:

Proof of Theorem 2.

The main idea of this argument is the same way as the corresponding results in the previous chapters. In particular, the excess decay follows from the following observation: There exists a minimal radius $r' > 0$ such that for any two radii $R \geq r \geq r'$ there is a $\xi \in \mathbb{R}^d$ such that

$$\begin{aligned} &\fint_{B_r \cap P} |\nabla u - \xi \cdot (\nabla \phi(I_P, x) + x)|^2 \, \mathrm{d}x \\ &\lesssim \left(\left(\frac{r}{R}\right)^2 + \left(\frac{R}{r}\right)^d (\delta_R^{\frac{1}{(d+4)(d+1)}} + \delta_R^{\frac{d}{d+1}})\right) \fint_{B_R} |\nabla u_{ext}|^2 \, \mathrm{d}x. \end{aligned} \tag{3.78}$$

We now prove this claim. Just as in the previous chapters, we may assume that $r \in (0, R/8]$ since the relation (3.78) holds trivially for $r \in (R/8, R]$

with the choice $\xi = 0$. As was shown in Lemma 1 we may extend u (defined on P) to u_{ext} (defined on $\mathbb{R}^d$) such that for $r \geq 3$ the estimate (3.15) holds. With u_{ext} in-hand, we know that there exists a radius $R' \in \left[\frac{R}{2}, R\right]$ such that the estimate

$$\int_{\partial B_{R'}} \left|\nabla^{tan} u_{ext}\right|^2 \, \mathrm{d}S \lesssim \frac{1}{R} \int_{B_R} |\nabla u_{ext}|^2 \, \mathrm{d}x \tag{3.79}$$

holds. For our argument, we introduce a smooth radial cut-off function η that satisfies $\eta = 1$ on $B_{R'-3\rho}$, $\eta = 0$ on $\mathbb{R}^d \setminus B_{R'-2\rho}$, and $|\nabla \eta| \lesssim \frac{4}{\rho}$. We choose ρ subject to the constraint $\rho \in [1, \frac{R}{8}]$.

The main idea of the proof of (3.78) is to compare u to v solving

$$\begin{aligned} -\nabla \cdot I_{hom} \nabla v = 0 \qquad & \text{in} \quad B_{R'}, \\ v = u_{ext} \qquad & \text{on} \quad \partial B_{R'}. \end{aligned} \tag{3.80}$$

Making this comparison in the squared energy norm we (as in the previous two chapters) write

$$\begin{aligned} &\int_{B_r \cap P} |\nabla u - \partial_i v(0) \cdot (e_i + \nabla \phi_i(I_P, x))|^2 \, \mathrm{d}x \\ \lesssim &\int_{B_r \cap P} |(\mathrm{Id} + \nabla \phi(I_P, x))(\nabla v - \nabla v(0))|^2 \, \mathrm{d}x \\ &+ \int_{B_r \cap P} |\nabla u - \partial_i v(e_i + \nabla \phi_i(I_P, x))|^2 \, \mathrm{d}x. \end{aligned} \tag{3.81}$$

We then introduce the homogenization error w, defined on $P \cap B_{R'}$, which is given by

$$w = u - v - \eta \partial_i v \phi_i(I_P, x). \tag{3.82}$$

Notice that thanks to the cut-off η the homogenization error w vanishes on $\partial B_{R'} \cap P$. Using that $r \leq R' - 3\rho$ we then re-write (3.81) as

$$\begin{aligned} &\int_{B_r \cap P} |\nabla u - \partial_i v(0) \cdot (e_i + \nabla \phi_i(I_P, x))|^2 \, \mathrm{d}x \\ \lesssim &\int_{B_r \cap P} |(\nabla v - \nabla v(0))(\mathrm{Id} + \nabla \phi(I_P, x))|^2 \, \mathrm{d}x \\ &+ \int_{B_{R'} \cap P} |\nabla w|^2 \, \mathrm{d}x + \int_{B_{R'} \cap P} |\nabla(\eta \partial_i v) \phi_i(I_P, x))|^2 \, \mathrm{d}x \end{aligned} \tag{3.83}$$

Again, to continue the estimate (3.83) we would like to obtain an energy estimate for w. To do this we notice that w is a distributional solution of

$$-\Delta w = \nabla \cdot ((1 - \eta)(\mathrm{Id} - I_{hom}) \nabla v + (\phi_i - \sigma_i) \nabla(\eta \partial_i v)) \qquad \text{on } P \cap B_{R'} \tag{3.84}$$

where test functions are taken from $C_0^\infty(B_{R'} \cap P)$. Since $w \notin C_0^\infty(B_{R'} \cap P)$, when we test (3.84) with w we pick-up boundary terms; in particular, after an application of Hölder's inequality, we can write

$$\begin{aligned}
&\int_{B_{R'}\cap P} |\nabla w|^2 \,\mathrm{d}x \\
&\lesssim \left(\int_{B_{R'}\cap P} |(1-\eta)(\mathrm{Id} - I_{hom})\nabla v + (\phi_i - \sigma_i)\nabla(\eta\partial_i v)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_{B_{R'}\cap P} |\nabla w|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&\quad + \left| \int_{B_{R'}\cap \partial P} w\nu \cdot (\nabla w + (1-\eta)(\mathrm{Id} - I_{hom})\nabla v + (\phi_i - \sigma_i)\nabla(\eta\partial_i v)) \,\mathrm{d}S \right|.
\end{aligned} \tag{3.85}$$

The non-boundary terms are simple to treat using the constant coefficient regularity results from Lemma 3, that $v = u_{ext}$ on $\partial B_{R'}$, and that R' was chosen such that (3.79) holds. In particular, in the same manner as in the previous two chapters we may apply Lemma 3 as

$$\begin{aligned}
&\left(\int_{B_{R'}\cap P} |(1-\eta)(\mathrm{Id} - I_{hom})\nabla v + (\phi_i - \sigma_i)\nabla(\eta\partial_i v)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{B_{R'}\setminus B_{R'-3\rho}} |\nabla v|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} + R^{\frac{d+2}{2}} \sup_{y\in B_{R'-2\rho}} |\nabla(\eta\partial_i v)(y)|\delta_R \\
&\lesssim (R^{\frac{d-1}{d}}\rho^{\frac{1}{d}})^{\frac{1}{2}} \left(\int_{\partial B_{R'}} |\nabla^{tan} v|^2 \,\mathrm{d}S \right)^{\frac{1}{2}} + R^{\frac{d}{2}} \left(\frac{R}{\rho} \right)^{\frac{d+2}{2}} \delta_R \left(\fint_{B_{R'}} |\nabla v|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&\overset{(3.79)}{\lesssim} \left(\frac{\rho}{R} \right)^{\frac{1}{2d}} R^{\frac{d}{2}} \left(\fint_{B_{R'}} |\nabla u_{ext}|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} + R^{\frac{d}{2}} \left(\frac{R}{\rho} \right)^{\frac{d+2}{2}} \delta_R \left(\fint_{B_{R'}} |\nabla v|^2 \,\mathrm{d}x \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.87}$$

We then treat the boundary term in (3.85). Using the no-flux boundary data of u and $x_i + \phi_i(I_P, x)$ we rewrite the boundary term as

$$\int_{B_{R'}\cap \partial P} w\nu \cdot (\nabla w + (1-\eta)(\mathrm{Id} - I_{hom})\nabla v + (\phi_i - \sigma_i)\nabla(\eta\partial_i v)) \,\mathrm{d}S \tag{3.88}$$

$$\begin{aligned}
= -\int_{B_{R'}\cap \partial P} \Big(&(w\eta\partial_i v)\nu \cdot (e_i + \nabla\phi_i(I_P, x)) \\
&+ w(1-\eta)\nu \cdot I_{hom}\nabla v + w\nu \cdot \sigma_i \nabla(\eta\partial_i v) \Big) \,\mathrm{d}S
\end{aligned}$$

$$= -\int_{B_{R'}\cap\partial P} w(1-\eta)\nu\cdot I_{hom}\nabla v\,\mathrm{d}S - \int_{B_{R'}\cap\partial P} w\nu\cdot\sigma_i\nabla(\eta\partial_i v)\,\mathrm{d}S.$$

For the first term on the right-hand side we find that the relation

$$\begin{aligned}
&\left|\int_{B_{R'}\cap\partial P} w(1-\eta)\nu\cdot I_{hom}\nabla v\,\mathrm{d}S\right| \\
=&\left|\int_{B_{R'}\cap P} \nabla\cdot(w(1-\eta)I_{hom}\nabla v)\,\mathrm{d}x\right| \\
\lesssim&\int_{B_{R'}\cap P} |(1-\eta)\nabla w\cdot I_{hom}\nabla v| + |w\nabla\eta\cdot I_{hom}\nabla v|\,\mathrm{d}x \\
\lesssim&\|\nabla w\|_{L^2(B_{R'}\cap P)}\|\nabla v\|_{L^2(B_{R'}\setminus B_{R'-3\rho})} \\
&+\frac{1}{\rho}\|w\|_{L^2((B_{R'-2\rho}\setminus B_{R'-3\rho})\cap P)}\|\nabla v\|_{L^2(B_{R'}\setminus B_{R'-3\rho})}
\end{aligned} \tag{3.89}$$

holds. Notice that in the first step we have used the homogeneous Dirichlet boundary data of w on $\partial B_{R'}\cap P$ and in the second step that v solves (3.80). Also, we have used that $|I_{hom}| \le 1$, which follows immediately from the definition. We would then like to apply the inequality

$$\frac{1}{\rho}\|w\|_{L^2((B_{R'-2\rho}\setminus B_{R'-3\rho})\cap P)} \lesssim \|\nabla w\|_{L^2(B_{R'}\cap P)}, \tag{3.90}$$

which, when combined with (3.89), gives

$$\begin{aligned}
&\left|\int_{B_{R'}\cap\partial P} w(1-\eta)\nu\cdot I_{hom}\nabla v\,\mathrm{d}S\right| \\
\lesssim&\|\nabla w\|_{L^2(B_{R'}\cap P)}\|\nabla v\|_{L^2(B_{R'}\setminus B_{R'-3\rho})} \\
\lesssim&\left(\frac{\rho}{R}\right)^{\frac{1}{2d}} R^{\frac{d}{2}}\left(\fint_{B_{R'}}|\nabla u_{ext}|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{B_{R'}\cap P}|\nabla w|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.91}$$

Here we have treated $\|\nabla v\|_{L^2(B_{R'}\setminus B_{R'-3\rho})}$ just like in (3.87).

In order to use (3.90) we notice that the trivial extension w^0 of w onto P, defined as

$$w^0(x) = \begin{cases} w(x) & \text{if } x\in B_{R'}\cap P \\ 0 & \text{if } x\in P\setminus B_{R'}, \end{cases}$$

is in $H^1(P)$ and the harmonic extension $w^0_{ext}\in H^1(\mathbb{R}^d)$ satisfies the standard Poincaré inequality

$$\frac{1}{\rho}\|w^0_{ext}\|_{L^2(B_{R'}\setminus B_{R'-3\rho})} \lesssim \|\nabla w^0_{ext}\|_{L^2(B_{R'})}. \tag{3.92}$$

To obtain (3.90) we index the holes of P that nontrivially intersect $B_{R'}$ by $J \subset \mathbb{N}$; i.e. the holes are $B_1(x_j)$ for $j \in J$. The right-hand side of (3.92) can then be treated as

$$\|\nabla w^0_{ext}\|^2_{L^2(B_{R'})} \leq \|\nabla w\|^2_{L^2(B_{R'}\cap P)} + \sum_{j\in J} \|\nabla w^0_{ext}\|^2_{L^2(B_1(x_j))} \tag{3.93}$$

By Lemma 1 and the definition of the extension w^0_{ext} we know that for each hole $B_1(x_j)$ the bound

$$\int_{B_1(x_j)} |\nabla w^0_{ext}|^2 \,\mathrm{d}x \lesssim \int_{B_2(x_j)\setminus B_1(x_j)} |\nabla w^0_{ext}|^2 \,\mathrm{d}x = \int_{(B_2(x_j)\setminus B_1(x_j))\cap P} |\nabla w|^2 \,\mathrm{d}x$$

holds, which thanks to the disjointness assumption on the holes of P allows us to post-process (3.92) and (3.93) to give the desired (3.90).

We then treat the second boundary term on the right-hand side of (3.88). For this we index the holes of P that intersect $B_{R'-2\rho}$ nontrivially by $J \subset \mathbb{N}$. Again employing the extension w^0_{ext} from the previous paragraph, we then use the divergence theorem and the definition of η to write:

$$\begin{aligned}
&\left|\int_{\partial P\cap B_{R'}} w\nu\cdot\sigma_i\nabla(\eta\partial_i v)\,\mathrm{d}S\right| \\
\leq &\sum_{j\in J}\left|\int_{\partial B_1(x_j)\cap B_{R'}} w^0_{ext}\nu\cdot\sigma_i\nabla(\eta\partial_i v)\,\mathrm{d}S\right| \\
= &\sum_{j\in J}\left|\int_{B_1(x_j)} \nabla\cdot(w^0_{ext}\sigma_i\nabla(\eta\partial_i v))\,\mathrm{d}S\right| \\
\leq &\sup_{y\in B_{R'-\rho}} |\nabla(\eta\partial_i v)(y)| \left(\int_{B_{R'}} |\nabla w^0_{ext}|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B_{R'}\cap P} |\sigma|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&+ \sup_{y\in B_{R'-\rho}} |\nabla\cdot\nabla(\eta\partial_i v)(y)| \left(\int_{B_{R'}} |w^0_{ext}|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B_{R'}\cap P} |\sigma|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&+ \sum_{j\in J}\left|\int_{B_1(x_j)} w^0_{ext}(\nabla\cdot\sigma_i)\cdot\nabla(\eta\partial_i v)\,\mathrm{d}x\right|.
\end{aligned} \tag{3.94}$$

The constant coefficient regularity from Lemma 3 and the Poincaré inequality for functions with homogeneous Dirichlet boundary data applied to w^0_{ext}

allows us to treat the first and second terms on the right-hand as

$$
\begin{aligned}
&\sup_{y\in B_{R'-\rho}} |\nabla(\eta\partial_i v)(y)| \left(\int_{B_{R'}} |\nabla w^0_{ext}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B_{R'}\cap P} |\sigma|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\
&+ \sup_{y\in B_{R'-\rho}} |\nabla^2(\eta\partial_i v)(y)| \left(\int_{B_{R'}} |w^0_{ext}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B_{R'}\cap P} |\sigma|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\
\lesssim & R^{\frac{d}{2}} \left(\frac{R}{\rho} + 1 + \frac{R^2}{\rho^2}\right) \left(\frac{R}{\rho}\right)^{\frac{d}{2}} \left(\fint_{B_{R'}} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B_{R'}} |\nabla w^0_{ext}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \delta_R.
\end{aligned}
\tag{3.95}
$$

We can complete this estimate by noticing again that, in fact, by Lemma 1 and the definition of w^0_{ext} we have that

$$
\left(\int_{B_{R'}} |\nabla w^0_{ext}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \lesssim \left(\int_{B_{R'}} |\nabla w_{ext}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{3.96}
$$

To treat the third term on the right-hand side of (3.94) we work a bit harder and use the equation (3.8). In particular, we can rewrite the term as

$$
\begin{aligned}
&\sum_{j\in J} \left| \int_{B_1(x_j)} w^0_{ext}(\nabla\cdot\sigma_i)\cdot\nabla(\eta\partial_i v) \,\mathrm{d}x \right| \\
=&\sum_{j\in J} \left| \int_{B_1(x_j)} w^0_{ext}(I_{hom}e_i)\cdot(\partial_i v\nabla\eta + \eta\nabla\partial_i v) \,\mathrm{d}x \right| \\
=&\sum_{j\in J} \left| \int_{B_1(x_j)} w^0_{ext}(I_{hom}e_i)\cdot\partial_i v\nabla\eta \,\mathrm{d}x \right|,
\end{aligned}
$$

where in the last line we have used the equation for v. We then use Hölder's inequality, the Poincaré inequality for functions with homogeneous Dirichlet boundary data applied to w^0_{ext}, (3.75) along with (3.79) applied to v, and Lemma 1 to bound

$$
\begin{aligned}
&\sum_{j\in J} \left| \int_{B_1(x_j)} w^0_{ext}(I_{hom}e_i)\cdot\partial_i v\nabla\eta \,\mathrm{d}x \right| \\
\leq&\frac{1}{\rho} \left(\int_{B_{R'}\setminus B_{R'-3\rho}} |w^0_{ext}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B_{R'}\setminus B_{R'-3\rho}} |\nabla v|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\
\lesssim&\left(\frac{\rho}{R}\right)^{\frac{1}{2d}} R^{\frac{d}{2}} \left(\int_{B_{R'}} |\nabla w^0_{ext}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\fint_{B_R} |\nabla u_{ext}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\lesssim \left(\frac{\rho}{R}\right)^{\frac{1}{2d}} R^{\frac{d}{2}} \left(\int_{B_{R'}\cap P} |\nabla w|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \left(\fint_{B_R} |\nabla u_{ext}|^2 \, \mathrm{d}x\right)^{\frac{1}{2}}$$

Combining these calculations with (3.94) and (3.95) and the apriori estimate $\|\nabla v\|_{L^2(B_{R'})} \lesssim \|\nabla u_{ext}\|_{L^2(B_{R'})}$, which we obtain as a consequence of the energy estimate for the equation

$$\begin{aligned} -\nabla \cdot I_{hom}\nabla(v - u_{ext}) &= \nabla \cdot I_{hom}\nabla u_{ext} && \text{in} \quad B_{R'}, \\ v - u_{ext} &= 0 && \text{on} \quad \partial B_{R'}, \end{aligned}$$

we find that

$$\begin{aligned} &\left| -\int_{\partial P\cap B_{R'}} w\nu \cdot \sigma_i \nabla(\eta \partial_i v) \, \mathrm{d}S \right| \\ \lesssim & R^{\frac{d}{2}} \left(\left(\frac{R}{\rho}\right)^{\frac{d}{2}+2} \delta_R + \left(\frac{\rho}{R}\right)^{\frac{1}{2d}} \right) \left(\fint_{B_{R'}} |\nabla u_{ext}|^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{B_{R'}\cap P} |\nabla w|^2 \, \mathrm{d}x\right)^{\frac{1}{2}}. \end{aligned} \tag{3.97}$$

This finishes our treatment of the boundary terms in the energy estimate of w. We now combine (3.85), (3.91), and (3.97) to obtain

$$\fint_{B_{R'}\cap P} |\nabla w|^2 \, \mathrm{d}x \lesssim \left(\frac{R}{r}\right)^{d} \left(\left(\frac{\rho}{R}\right)^{\frac{1}{d}} + \left(\frac{R}{\rho}\right)^{d+4} \delta_R^2 \right) \fint_{B_{R'}} |\nabla u_{ext}|^2 \, \mathrm{d}x. \tag{3.98}$$

Returning to the inequality (3.83) and further using the above apriori estimate, we notice that the sum of the first and third terms can also be bounded using Lemma 3 as

$$\begin{aligned} &\fint_{B_r} |(\nabla v - \nabla v(0))(\mathrm{Id} + \nabla\phi(I_P, x))|^2 \, \mathrm{d}x + \fint_{B_{R'}} |\nabla(\eta\partial_i v)\phi_i(I_P, x))|^2 \, \mathrm{d}x \\ \lesssim & \left(\left(\frac{r}{R}\right)^2 (\delta_R^2 + 1) + \left(\frac{R}{\rho}\right)^{d+2} \delta_R^2 \right) \fint_{B_{R'}} |\nabla u_{ext}|^2 \, \mathrm{d}x. \end{aligned} \tag{3.99}$$

This observation in combination with (3.83), (3.98), $\delta_R \leq 1$, and the relations of all the various radii then yields that

$$\begin{aligned} &\fint_{B_r} |\nabla u - \nabla v(0) \cdot (e_i + \nabla\phi_i(I_P, x))|^2 \, \mathrm{d}x \\ \lesssim & \left(\left(\frac{r}{R}\right)^2 + \left(\frac{R}{r}\right)^{d} \left(\frac{\rho}{R}\right)^{\frac{1}{d}} + \left(\frac{R}{r}\right)^{d} \left(\frac{R}{\rho}\right)^{d+4} \delta_R \right) \fint_{B_R} |\nabla u_{ext}|^2 \, \mathrm{d}x. \end{aligned} \tag{3.100}$$

We post-process this estimate by choosing the width of the boundary layer ρ such that

$$\frac{\rho}{R} = \min\left(\frac{1}{8}, \delta_R^{\frac{d}{(d+4)(d+1)}}\right).$$

Plugging in this choice of ρ yields

$$\begin{aligned}
&\fint_{B_r} |\nabla u - \nabla v(0) \cdot (e_i + \nabla \phi_i(I_P, x))|^2 \, \mathrm{d}x \\
&\lesssim \left(\left(\frac{r}{R}\right)^2 + \left(\frac{R}{r}\right)^d (\delta_R^{\frac{1}{(d+4)(d+1)}} + \delta_R^{\frac{d}{d+1}})\right) \fint_{B_R} |\nabla u_{ext}|^2 \, \mathrm{d}x.
\end{aligned}$$

With (3.78) in-hand, one rephrases it in terms of the excess and finds a ratio of radii $\theta = \frac{R}{r}$ and minimal radius $r^* > 0$ such that if $R \geq r \geq r^*$ the excess decay holds. To obtain the excess decay for any $R \geq r \geq r^*_\alpha(I_P)$ one then iterates this estimate. As there are no fundamental differences between the argument used here and that in the previous two chapters, we omit this. $\square$

Part II

A Pathwise Approach to a Quasilinear Initial Value Problem

Chapter 1

A Pathwise Approach to a Quasilinear Initial Value Problem

1.1 Set-up and Overview of our Strategy

In this part of the thesis we are interested in developing a pathwise solution theory for a quasilinear initial value problem, which is singular in the sense that the data limits the regularity of the solution such that the nonlinear terms of the equation have no classical meaning. In particular, assuming that $\alpha \in (\frac{2}{3}, 1)$ we are interested in constructing a continuous solution operator for

$$\partial_2 W - a(W)\partial_1^2 W + W = f \qquad \text{in} \quad \mathbb{R}^2_+ \tag{1.1a}$$

$$W = W_{int} \qquad \text{on} \quad \partial\mathbb{R}^2_+, \tag{1.1b}$$

where $a : \mathbb{R} \to [\lambda, 1]$ for $\lambda > 0$ is regular, $W_{int} \in C^\alpha(\mathbb{R})$, and $f \in C^{\alpha-2}(\mathbb{R}^2)$. Throughout this part all functions and distributions are assumed to be 1-periodic in the x_1-direction and f is, furthermore, 1-periodic in the x_2-direction. For an overview of the difficulties and the strategy we use in this contribution the reader is asked to reference the introduction. In the current section our main goal is to solidify our notation and formally state our results.

1.1.1 Definitions and Tools

We have already introduced the concept of modelledness in the introduction, but, for the convenience of the reader, we repeat the definition here:

Definition 1 (Modelledness). *Let $\alpha \in (\frac{1}{2}, 1)$ and $\Omega \subseteq \mathbb{R}^2$. Assume that for some $I \in \mathbb{N}$ we have functions $(\overline{V}_1(\cdot, a_0), ..., \overline{V}_I(\cdot, a_0))$ such that $\overline{V}_i : \Omega \times \mathbb{R} \to \mathbb{R}$. A function $W : \Omega \to \mathbb{R}$ is said to be modelled after $(\overline{V}_1(\cdot, a_0), ..., \overline{V}_I(\cdot, a_0))$ on Ω according to functions $(a_1, ..., a_I)$ and $(\sigma_1, ..., \sigma_I)$ with $\sigma_i, a_i \in C^\alpha(\Omega)$*

if there exists a function ν *such that*

$$M_\Omega := \sup_{x \neq y; x,y \in \Omega} \frac{1}{d^{2\alpha}(x,y)} \quad (1.2)$$
$$|W(y) - W(x) - \sigma_i(x)(\overline{V}_i(y, a_i(x)) - \overline{V}_i(x, a_i(x))) - \nu(x)(y-x)_1|$$

is finite. We emphasise that here we use the Einstein summation convention that repeated indices are summed over.

We say that a function W *is trivially modelled after* $(\overline{V}_1(\cdot, a_0), ..., \overline{V}_I(\cdot, a_0))$ *according to* $(a_1, ..., a_I)$ *if each* $\sigma_i = 0$*. Since* $\alpha \in (\frac{1}{2}, 1)$ *this is equivalent to the condition that* $W \in C^{2\alpha}(\Omega)$*, but additionally specifies a choice for the functions* $(a_1, ..., a_I)$*.*

For a discussion of this definition see the paragraphs following Definition 2.1 in the introduction.

We have explained in the introduction that we expect the solution W of (1.1) to be modelled after $\overline{V}(\cdot, a_0)$ solving

$$(\partial_2 - a_0\partial_1^2 + 1)\overline{V}(\cdot, a_0) = f \qquad \text{in} \quad \mathbb{R}^2_+, \qquad (1.3a)$$
$$\overline{V}(\cdot, a_0) = W_{int} \qquad \text{on} \quad \partial\mathbb{R}^2_+, \qquad (1.3b)$$

where this function decomposes as $\overline{V}(\cdot, a_0) = v(\cdot, a_0) + V(\cdot, a_0)$ for $v(\cdot, a_0)$ solving

$$(\partial_2 - a_0\partial_1^2 + 1)v(\cdot, a_0) = f \qquad \text{in} \quad \mathbb{R}^2 \qquad (1.4)$$

and $V(\cdot, a_0)$ solving

$$(\partial_2 - a_0\partial_1^2 + 1)V(\cdot, a_0) = 0 \qquad \text{in} \quad \mathbb{R}^2_+, \qquad (1.5a)$$
$$V(\cdot, a_0) = V_{int} \qquad \text{on} \quad \partial\mathbb{R}^2_+ \qquad (1.5b)$$

with the initial condition (1.5b) chosen as $V_{int} = W_{int} - v(\cdot, a_0)$. Also explained in the introduction is that the solution W of (1.1) is obtained via a fixed point argument that takes as input a solution theory for the linear problem

$$(\partial_2 - a\partial_1^2 + 1)W = f \qquad \text{in} \quad \mathbb{R}^2_+, \qquad (1.6a)$$
$$W = W_{int} \qquad \text{on} \quad \partial\mathbb{R}^2_+. \qquad (1.6b)$$

The problem (1.6) is, in turn, treated via a perturbative ansatz in which we first handle the forcing *à la* Otto and Weber and then show that the initial condition can be enforced classically. In this thesis we treat the linear problem (1.6) and develop all of the ingredients we need for the fixed point argument, but leave the actual argument to an upcoming contribution. Really, given the results in this thesis, the fixed point argument can now be directly adapted from the corresponding argument in [47].

When we treat (1.6) with the perturbative ansatz explained in the introduction, in order to enforce the initial condition, we must handle the initial value problem

$$(\partial_2 - a^{ext}\partial_1^2 + 1)U = 0 \qquad \text{in} \quad \mathbb{R}^2_+, \tag{1.7a}$$
$$U = W_{int} - u \qquad \text{on} \quad \partial\mathbb{R}^2_+, \tag{1.7b}$$

where $u \in C^\alpha(\mathbb{R}^2)$ is the solution of

$$(\partial_2 - a^{ext}\partial_1^2 + 1)u = f \qquad \text{in} \quad \mathbb{R}^2 \tag{1.8}$$

and a^{ext} is an extension of a to $\mathbb{R}^2$. Even though it is counterintuitive, the initial value problems we treat in this contribution are sensitive to the definition of the coefficients also for negative times. This is a result of the whole-space nature of the singular product, which is defined in terms of convolutions on the whole-space.

Our treatment of (1.7) is again perturbative. In particular, we would like to construct U by correcting the ansatz

$$q := V(\cdot, \bar{a}(\cdot)), \tag{1.9}$$

where $V(\cdot, a_0)$ solves (1.5) with initial condition $V_{int} = W_{int} - u$ and $\bar{a}$ solves

$$(\partial_2 - \partial_1^2)\bar{a} = 0 \qquad \text{in} \quad \mathbb{R}^2_+, \tag{1.10a}$$
$$\bar{a} = a \qquad \text{on} \quad \partial\mathbb{R}^2_+. \tag{1.10b}$$

Notice that the choice of q as an ansatz is quite intuitive: The most straightforward choice of ansatz would be $V(\cdot, a(\cdot))$, but we would like to vary the parameter a_0 in a smooth way for which purpose we introduce $\bar{a}$. We choose to define $\bar{a}$ without a massive term in order to ensure that $\bar{a} \geq \lambda$.

As we have mentioned in the introduction, whenever we write $C^\alpha(\mathbb{R}^2)$ for $\alpha > 0$ we are referring to the *parabolic* Hölder space that is defined in terms of the distance function

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|^{\frac{1}{2}} \tag{1.11}$$

for any two $x, y \in \mathbb{R}^2$. We must still define what we mean by "C^α" when $\alpha \in (1, 2)$ and "$C^{\alpha-2}$" when $\alpha \in (0, 1) \cup (1, 2)$; here are the definitions:

Definition 2 (Negative Hölder seminorm). *Let $\alpha \in (0,1)\cup(1,2)$. We define the $C^{\alpha-2}$-seminorm of a function u as*

$$[u]_{\alpha-2} := \inf_{(u^1,u^2)} \left([u^1]_\alpha + [u^2]_\alpha\right), \tag{1.12}$$

where the infimum is taken over pairs of functions (u^1, u^2) *such that* $u = \partial_1^2 u^1 + \partial_2 u^2$*. We may always assume that a near optimal pair* (u^1, u^2) *in the above sense satisfies* $u^1(0) = u^2(0) = 0$*. When* $\alpha \in (1,2)$ *we define*

$$[u]_\alpha := [\partial_1 u]_{\alpha-1}. \tag{1.13}$$

We notice below that for our purposes in this contribution it is (most likely) possible to use a weaker version of (1.12), but for convenience in this thesis we choose to use the definition given above.

As a side effect of losing periodicity in the x_2-direction, we sometimes must work with a local version of the standard Hölder seminorm.

Definition 3 (Local Hölder seminorm). *Let* $\alpha \in (0,1) \cup (1,2)$*. We define local versions of the* C^α *and* $C^{\alpha-2}$*-seminorms. In particular, for* $\alpha \in (0,1)$ *we let*

$$[u]_\alpha^{loc} := \sup_{x,y \in \mathbb{R}^2 \ s.t.\ d(x,y) \leq 1} \frac{|u(x) - u(y)|}{d^\alpha(x,y)} \tag{1.14}$$

and if $\alpha \in (1,2)$ *then* $[u]_\alpha^{loc}$ *is defined like* (1.13)*, but with the local version of the seminorm on the right-hand side. The definition* (1.12) *is adapted in the same way.*

Whenever we are only interested in the local properties of the C^α-seminorm in the sense of wanting to bound the modulus of continuity, then having an estimate for the *local* C^α-seminorm is sufficient. To avoid confusion, notice that if we let

$$C^\alpha_{loc}(\mathbb{R}^2) = \left\{ u \,|\, u \in C^\alpha(\Omega) \text{ for all } \Omega \Subset \mathbb{R}^2 \right\}$$

be the classical local Hölder space and

$$C^\alpha_{equicts} = \left\{ u \,|\, [u]_\alpha^{loc} < \infty \right\},$$

then clearly the inclusion $C^\alpha_{equicts}(\mathbb{R}^2) \subset C^\alpha_{loc}(\mathbb{R}^2)$ holds.

Before moving on, we introduce some notation that we use throughout this contribution. First, for a function u we use the convention

$$\|u\| := \|u\|_{L^\infty},$$

where the domain should always be clear from the context. Furthermore, we will often consider families of functions that are parameterized by either $a_0 \in [\lambda, 1]$ or $a_0, a_0' \in [\lambda, 1]$; we use the notation

$$\|u(\cdot, a_0, a_0')\|_{j,k} := \|\partial_{a_0}^j \partial_{a_0'}^k u(\cdot, a_0, a_0')\|$$
$$\text{and} \quad \|u(\cdot, a_0)\|_j := \|\partial_{a_0}^j u(\cdot, a_0)\|.$$

We use the same convention for the Hölder norms and seminorms; i.e. we write

$$\begin{aligned} \|u(\cdot, a_0, a_0')\|_{\alpha,j,k} &:= \|\partial_{a_0}^j \partial_{a_0'}^k u(\cdot, a_0, a_0')\|_{\alpha}, \\ \|u(\cdot, a_0)\|_{\alpha,j} &:= \|\partial_{a_0}^j u(\cdot, a_0)\|_{\alpha}, \\ [u(\cdot, a_0, a_0')]_{\alpha,j,k} &:= [\partial_{a_0}^j \partial_{a_0'}^k u(\cdot, a_0, a_0')]_{\alpha}, \\ \text{and} \quad [u(\cdot, a_0)]_{\alpha,j} &:= [\partial_{a_0}^j u(\cdot, a_0)]_{\alpha}. \end{aligned}$$

Of course, the analogous definitions can be written down for the local version of the Hölder seminorm.

While Definition 2 above is the standard definition that we use for the negative Hölder seminorms appearing here, we will often also work with an equivalent formulation, which is developed in Lemma 1 and relies on convolution with a specific kernel (ψ_1 already mentioned in the introduction) at scales $T \leq 1$. The convolution kernel that we choose to use is the same as that used by Otto and Weber in [47] and is most easily defined in terms of its Fourier transform in the sense that

$$\hat{\psi}_T(k) = \exp(-T(k_1^4 + k_2^2)). \tag{1.15}$$

This definition immediately implies that ψ_T is a Schwarz function. The reason for choosing the kernel ψ_T is that it is the semigroup associated with the elliptic operator $\partial_1^4 - \partial_2^2$, which is positive and has the same relative scaling between x_1 and x_2 as the parabolic operator $\partial_2 - \partial_1^2\,(+1)$. Usually, throughout this exposition, we use the convention

$$(\cdot)_T = \cdot * \psi_T;$$

occasionally, we even drop the parentheses and simply use the subscript T.

We now go down a laundry list of useful properties for ψ_T. To prove some of these properties we rely on the change of coordinates

$$\hat{x} = (\hat{x}_1, \hat{x}_2) = \left(\frac{x_1}{T^{\frac{1}{4}}}, \frac{x_2}{T^{\frac{1}{2}}} \right), \tag{1.16}$$

which we use many times throughout this part of the thesis. Here is the list of properties:

- Using (1.15) and the notation (1.16) we find that

  $$\psi_T(x_1, x_2) = (T^{\frac{1}{4}})^{-3} \psi_1 (\hat{x}_1, \hat{x}_2) \tag{1.17}$$

  for any $T > 0$. This means that $\|\psi_T\|_{L^1} = \|\psi_1\|_{L^1}$.

- *(Bound on the moments of ψ_T)* For any $i, j \geq 0$, $\alpha > 0$, and $y \in \mathbb{R}^2$ we have that

$$\int_{\mathbb{R}^2} |d^\alpha(x,y)||\partial_1^i \partial_2^j \psi_T(x-y)| \, \mathrm{d}x \lesssim (T^{\frac{1}{4}})^{\alpha-i-2j}. \tag{1.18}$$

  To see this we may assume that $y = 0$. We then rescale using (1.16) and write

$$\int_{\mathbb{R}^2} |d^\alpha(x,0)||\partial_1^i \partial_2^j \psi_T(x)| \, \mathrm{d}x = \int_{\mathbb{R}^2} (T^{\frac{1}{4}})^{\alpha-i-2j} |d^\alpha(\hat{x},0)||\partial_1^i \partial_2^j \psi_1(\hat{x})| \, \mathrm{d}\hat{x},$$

  which yields (1.18) after using that ψ_1 is a Schwarz function.

- *(Semigroup property of ψ_T)* For any distribution u and any two scales $t, T > 0$ we have that $(u * \psi_t) * \psi_T = u * (\psi_t * \psi_T)$ and by (1.15), furthermore, $\psi_t * \psi_T = \psi_{t+T}$. This yields that

$$(u_t)_T = u_{t+T}. \tag{1.19}$$

- For any $i, j \geq 0$ such that $i + j \geq 1$ and $u \in C^\alpha(\mathbb{R}^2)$ we have that

$$\begin{aligned} \int_{\mathbb{R}^2} \partial_1^i \partial_2^j u(y) \psi_T(x-y) \, \mathrm{d}y &= \int_{\mathbb{R}^2} (u(y) - u(x)) \partial_1^i \partial_2^j \psi_T(x-y) \, \mathrm{d}y \\ &\leq [u]_\alpha \int_{\mathbb{R}^2} d^\alpha(x,y) \partial_1^i \partial_2^j \psi_T(x-y) \, \mathrm{d}y \\ &\lesssim [u]_\alpha (T^{\frac{1}{4}})^{\alpha-i-2j}, \end{aligned} \tag{1.20}$$

  where for the first equality we have used an integration by parts in which the boundary terms vanish because ψ_T is a Schwarz function. The second inequality follows from (1.18).

- *(Monotonicity of the L^∞-norm in terms of the convolution scale)* For any distribution u we find that for $T \geq t > 0$ it holds that

$$\|u * \psi_T\| \lesssim \|u * \psi_t\| \|\psi_{T-t}\|_{L^1} = \|u * \psi_t\| \|\psi_1\|_{L^1} \lesssim \|u * \psi_t\|, \tag{1.21}$$

  where we have used (1.19) and Young's inequality in the first inequality and then (1.17).

We can now formulate an alternative version of the $C^{\alpha-2}$-seminorm for $\alpha \in (0,1)$. An analogue of the following lemma is a necessary component for linking the probabilistic and deterministic components of the solution theory in [47] and, therefore, also here.

Lemma 1. *Let $\alpha \in (0,1)$ and $\Omega \subseteq \mathbb{R}^2$ be convex. Then a distribution f on $\mathbb{R}^2$ that is periodic in the x_1-direction satisfies*

$$[f]_{\alpha-2}^{loc} \lesssim \sup_{T \leq 1} (T^{\frac{1}{4}})^{2-\alpha} \|f_T\|_\Omega \tag{1.22}$$

and

$$[f]_{\alpha-2;\Omega} \gtrsim \sup_{T\le 1}(T^{\frac{1}{4}})^{2-\alpha}\|f_T\|_\Omega. \tag{1.23}$$

If, in addition to the convexity of Ω*, we have that* $\|f*\psi_\epsilon\|_\Omega<\infty$ *for all* $\epsilon>0$*, then*

$$[f]_{\alpha;\Omega} \lesssim \sup_{T\le 1}(T^{\frac{1}{4}})^{-\alpha}\|f_T\|_\Omega. \tag{1.24}$$

As already alluded to above, this lemma is an analogue of Lemma 9 in [47]. As we will see in our proof of this result in Section 5, things are complicated by the loss of periodicity in the x_2-direction. In particular, this makes it impossible to use the method of proof in [47] and we instead adapt an argument from a work by Ignat and Otto [36] in which they analyze a singular version of a nonlinear elliptic equation, which they derive as a model for the magnetization ripple. In fact, compared to the work of Ignat and Otto we are in some sense in a slightly favorable situation because their convolution kernel is not a Schwarz function.

To reiterate, this alternative formulation of the $C^{\alpha-2}$-seminorm is useful when working with the singular products. In particular, the family of offline reference products $\{v(\cdot,a_0)\diamond\partial_1^2 v(\cdot,a_0')\}$ indexed by $a_0,a_0'\in[\lambda,1]$ that we borrow from [47] comes with a commutator estimate of the form

$$\sup_{a_0,a_0'\in[\lambda,1]}\sup_{T\le 1}(T^{\frac{1}{4}})^{2-2\alpha}\left\|\partial_{a_0}^i\partial_{a_0'}^j[v(\cdot,a_0),(\cdot)_T]\diamond\partial_1^2 v(\cdot,a_0')\right\|\lesssim 1. \tag{1.25}$$

As we will see below, this family of offline products only exist almost surely for a random forcing f satisfying certain criterion (see Section 1.2). By the above lemma we intuitively think of (1.25) as $C^{2\alpha-2}$-control for the commutator. Inputting these reference products into the reconstruction lemmas, in order to define a new singular product we then pass to the limit in a sequence of distributions that are uniformly controlled in the sense of the right-hand side of (1.22). Being able to pass to the limit then relies on the equicontinuity that is obtained from the left-hand side of (1.22)[1].

In the current contribution we treat (1.6) with the goal of using a fixed point argument to treat (1.1). As mentioned in the introduction this means

[1]Here it becomes clear that there is some ambiguity in our definition of the $C^{\alpha-2}$-seminorm. In particular, the expression that we use should do two things: 1) It should satisfy an analogue of Lemma 1 and 2) For a sequence of distributions uniform control of the $C^{\alpha-2}$-seminorm should allow for the application of the Arzelà-Ascoli theorem to pass to a limit. Another choice that would work and which would simplify the proof of Lemma 1 would be the weaker seminorm

$$[f]_{\alpha-2}^{loc}=\inf_{(u^1,u^2,u^3)}\left([u^1]_\alpha^{loc}+[u^2]_\alpha^{loc}+[u^3]_\alpha^{loc}+\|u^3\|\right),$$

where the infimum is taken over triplets (u^1,u^2,u^3) such that $u=\partial_1^2u^1+\partial_2u^2+u^3$.

that we would like to think of $a = a(\overline{W})$, where the a on the left-hand side is the coefficient field in (1.6), the a on the right-hand side is that in (1.1), and $\overline{W}$ is sampled from the solution space of (1.6). For the purpose of defining the requisite singular products we should choose the assumptions on the nonlinearity a in (1.1) such that if W is modelled after some family of functions $\{\overline{V}(\cdot, a_0)\}$ then $a(W)$ is as well. To motivate the correct assumptions on a we paraphrase the corresponding lemma from [47]:

"Lemma 1 of [47]." *i) Supposed that W is modelled after $\overline{V}(\cdot, a_0)$ according to a and σ both of class $C^\alpha(\mathbb{R}^2)$ with modelling constant M and, furthermore, that the function b is twice differentiable. Then $b(W)$ is also modelled after $\overline{V}(\cdot, a_0)$ on $\mathbb{R}^2$, but according to a and $\mu := b'(W)\sigma$ with modelling constant $\tilde{M}$ bounded as*

$$\tilde{M} \lesssim \|b'\|M + \|b''\|[W]_\alpha^2. \tag{1.26}$$

ii) Suppose for $i = 0, 1$ that the function W_i is modelled after $\overline{V}_i(\cdot, a_0)$ according to a_i and σ_i with modelling constant M_i. Furthermore, assume that $W_1 - W_0$ is modelled after $(\overline{V}_1(\cdot, a_0), \overline{V}_0(\cdot, a_0))$ according to (a_1, a_0) and $(\sigma_1, -\sigma_0)$ with modelling constant δM and that b is three times differentiable. Under these assumptions we find that $b(W_1) - b(W_0)$ is modelled after $(\overline{V}_1(\cdot, a_0), \overline{V}_0(\cdot, a_0))$ according to (a_1, a_0) and $(\mu_1 := b'(W_1)\sigma_1, -\mu_0 := -b'(W_0)\sigma_0)$ with modelling constant $\delta\tilde{M}$ and $\|b(W_1) - b(W_0)\|_\alpha$ bounded as

$$\begin{aligned}\delta\tilde{M} &\lesssim \|b'\|\delta M + \|b''\| \max_i [W_i]_\alpha [W_1 - W_0]_\alpha \\ &\quad + \frac{1}{2}\|b'''\|\|W_1 - W_0\| \max_i [W_i]^2 \\ &\quad + \|b''\|\|W_1 - W_0\| \max_i M_i\end{aligned}$$

$$\textit{and} \quad \|b(W_1) - b(W_0)\|_\alpha \lesssim \|b'\|\|W_1 - W_0\|_\alpha + \|b''\|\|W_1 - W_0\| \max_i [W_i]_\alpha$$

For the proof of this result we ask that the reader consults the paper of Otto and Weber. As we will see in Section 7, the second part of this lemma is used for the treatment of the quasilinear problem (1.1). For now we use this lemma to motivate the following assumptions on the nonlinearity a in (1.1):

$$a \in [\lambda, 1] \quad \text{and} \quad \|a'\|, \|a''\|, \|a'''\| \leq 1, \tag{1.27}$$

where the first assumption is the standard non-degeneracy condition.

As we have already mentioned, throughout this contribution we will extend various functions defined only for positive times to negative times. We will do this in two ways:

Definition 4 (Extensions to negative times)**.** *For a function f defined on $\mathbb{R}^2_+$ we use $\tilde{f}$ to denote the even-reflection across the axis $\{x \,|\, x_2 = 0\}$ and f^E to denote the trivial extension by* 0*. So, in particular, we have that*

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if} \quad x \in \mathbb{R}^2_+ \\ f(\tilde{x}) & \text{if} \quad x \in \mathbb{R}^2_-, \end{cases}$$

where we use the convention $\tilde{x} = (x_1, |x_2|)$ for $x = (x_1, x_2)$, and

$$f^E(x) := \begin{cases} f(x) & \text{if} \quad x \in \mathbb{R}^2_+ \\ 0 & \text{if} \quad x \in \mathbb{R}^2_-. \end{cases}$$

1.1.2 Offline Products Borrowed from Otto and Weber

We now essentially summarize Section 3 of [47] and, in doing so, find that their arguments can be used to define the offline products $v(\cdot, a_0) \diamond \partial_1^2 v(\cdot, a_0')$ for $v(\cdot, a_0)$ solving (1.4) almost surely assuming a random forcing f satisfying conditions to be specified below. As noted in the introduction, there are two differences between their setting (in which $v^{OW}(\cdot, a_0)$ solves

$$(\partial_2 - a_0\partial_1^2)v^{OW}(\cdot, a_0) = P(f) \qquad \text{on} \quad \mathbb{R}^2, \tag{1.28}$$

where P denotes the projection onto periodic mean-free functions) and ours: 1) The forcing in (1.4) is not assumed to have vanishing average and 2) We have a massive term. Since the arguments in [47] are lengthy and complicated, we do not repeat their entire exposition here and instead specifically indicate which steps in their proofs are affected by the differences in the current setting. For more details on their results the reader can consult Section 3 of [47] and for proofs Section 5.

Just as in [47], throughout this section we assume that f is a stationary centered space-time periodic Gaussian distribution that can be expressed in terms of its Fourier series as

$$f(x) = \sum_{k \in (2\pi\mathbb{Z})^2} \sqrt{\hat{C}(k)} e^{ik\cdot x} Z_k, \tag{1.29}$$

where the coefficients $\sqrt{\hat{C}}$ are real-valued, non-negative, and even in k and the Z_k are complex-valued centered Gaussians that, aside from the conditions that $Z_k = \overline{Z}_{-k}$ and $\langle Z_k Z_{-l} \rangle = \delta_{k,l}$, are independent. In their exposition, Otto and Weber encode the standard regularity assumption on f, i.e. that

$$\sup_{T \le 1} (T^{\frac{1}{4}})^{2-\alpha} \| f_T \| \lesssim 1, \tag{1.30}$$

in terms of the Fourier coefficients; in particular, they assume that there exist constants $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\alpha \in (0,1)$ such that

$$\lambda_1 + \lambda_2 = -1 + 2\alpha \qquad \text{and} \qquad \lambda_1, \frac{\lambda_2}{2} < 1 \tag{1.31}$$

and for which

$$\hat{C}(k) \leq \frac{1}{(1+|k_1|)^{\lambda_1}(\sqrt{1+|k_2|})^{\lambda_2}} \tag{1.32}$$

for any $k = (k_1, k_2) \in (2\pi\mathbb{Z})^2$. The difference in our setting introduced due to the lack of a projection on the right-hand side of (1.4), is summarized as the lack of the condition that $\hat{C}(0) = 0$. As discussed in the introduction, the massive term in (1.4) instead provides a bound on $\hat{C}(0)$ in terms of the universal constant in (1.30), which is sufficient for all of Otto and Weber's arguments in this section to still work.

Of course, the conditions listed above for the $C(k)$ are not always equivalent to (1.30) and the assumptions on the random distribution f are already used to show the relation between the two conditions. In particular, we summarize Lemma 6 of [47] as:

"Lemma 6 of [47]." *Let f be a stationary centered space-time periodic Gaussian distribution represented as* (1.29) *with coefficients satisfying* (1.32) *for some constants $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\alpha \in (0,1)$. We use the notation $f_T = f * \psi'_T$, where ψ' is any Schwarz function satisfying $\int_{\mathbb{R}^2} \psi' \, dx = 1$. Then for any $p < \infty$ and $\alpha' < \alpha$ the bound*

$$\left\langle \left(\sup_{T \leq 1} (T^{\frac{1}{4}})^{2-\alpha'} \|f_T\| \right)^p \right\rangle^{\frac{1}{p}} \lesssim 1 \tag{1.33}$$

holds. Furthermore, for $\kappa \in [0,4]$ there is the bound

$$\left\langle \left(\sup_{\epsilon \in (0,1]} \sup_{T \leq 1} (T^{\frac{1}{4}})^{2-\alpha'+\kappa} (\epsilon^{\frac{1}{4}})^{-\kappa} \|(f_\epsilon)_T - f_T\| \right)^p \right\rangle^{\frac{1}{p}} \lesssim 1. \tag{1.34}$$

Notice that in both (1.33) *and* (1.34) *the universal constants depends on α' and p.*

The combination of (1.33) and (1.34) shows that in fact the bound (1.33) holds for f replaced by $f_\epsilon = f * \psi'_\epsilon$ uniformly in $\epsilon > 0$. Notice also that the convolution kernel ψ' used here must not necessarily be the ψ_1 used in the deterministic arguments. In particular, the specific choice of ψ_1 becomes important when we start needing the semigroup property (e. g. in the reconstruction lemmas) or in the proof of Lemma 1.

Throughout the construction of the offline products in [47], one writes $v^{OW}(\cdot, a_0)$ in terms of the space-time periodic zero-average Green's function corresponding to the operator $\partial_2 - a_0\partial_1^2$. Of course, in our setting this is replaced by expressing $v(\cdot, a_0)$ in terms of the space-time periodic Green's function of $\partial_2 - a_0\partial_1^2 + 1$. This can easily be determined in terms of the coefficients of its Fourier series, which are given by

$$\hat{G}(k, a_0) = \frac{1}{a_0k_1^2 - ik_2 + 1} = \frac{a_0k_1^2 + ik_2 + 1}{a_0^2k_1^4 + 2a_0k_1^2 + k_2^2 + 1}. \tag{1.35}$$

This is slightly different than (132) in [47] due to the massive term. We can then write

$$\begin{aligned} \hat{v}(k, a_0) &= \frac{1}{a_0k_1^2 - ik_2 + 1}\hat{f}(k) \\ \text{and} \quad \widehat{\partial_1^2 v}(k, a_0) &= \frac{k_1^2}{a_0k_1^2 - ik_2 + 1}\hat{f}(k). \end{aligned} \tag{1.36}$$

Just as in [47] we can observe that defining the offline products $\partial_{a_0}^i\partial_{a_0'}^j v(\cdot, a_0) \diamond \partial_1^2 v(\cdot, a_0')$ for $i, j > 0$ is essentially the same as for $i = j = 0$ since for any number of parameter derivatives the symbol of the Fourier multipliers are bounded (see (133) of [47]).

The general procedure of Otto and Weber for defining their offline products is to regularize the terms via convolution with ψ_ϵ' (already introduced in Lemma 6 of [47]), multiply the regularized terms, and then to show that, under the assumptions on f listed above, the regularized products can almost surely be renormalized so that they converge as $\epsilon \to 0$. In particular, using the notation

$$\mathfrak{c}^{(2)}(\epsilon, a_0, a_0') := \langle v_\epsilon(\cdot, a_0)\partial_1^2 v_\epsilon(\cdot, a_0')\rangle, \tag{1.37}$$

where $\langle\cdot\rangle$ denotes the expectation, they show that for f satisfying the assumptions almost surely the renormalized products

$$v_\epsilon(\cdot, a_0) \diamond \partial_1^2 v_\epsilon(\cdot, a_0') := v_\epsilon(\cdot, a_0) \diamond \partial_1^2 v_\epsilon(\cdot, a_0') - \mathfrak{c}^{(2)}(\epsilon, a_0, a_0') \tag{1.38}$$

and the corresponding commutators

$$[v_\epsilon(\cdot, a_0), (\cdot)_T] \diamond \partial_1^2 v_\epsilon(\cdot, a_0') = v_\epsilon(\cdot, a_0)(\partial_1^2 v_\epsilon(\cdot, a_0'))_T - (v_\epsilon(\cdot, a_0)\partial_1^2 v_\epsilon(\cdot, a_0'))_T \tag{1.39}$$

converge and satisfy the appropriate estimates. Of course, then they define

$$v(\cdot, a_0) \diamond \partial_1^2 v(\cdot, a_0') := \lim_{\epsilon\to 0} v_\epsilon(\cdot, a_0) \diamond \partial_1^2 v_\epsilon(\cdot, a_0'). \tag{1.40}$$

In [47] the authors also give a specific expression for $\mathfrak{c}_\epsilon^2$ based on (1.35) in the sense of (1.36) and discuss the convergence of these constants as $\epsilon \to 0$. Since the Green's function has changed due to the presence of a massive term, this result will change as well. In particular, their Lemma 7 is replaced by:

"***Almost* Lemma 7 of [47]."** *For any $\epsilon > 0$ we have that*

$$\begin{aligned} &\mathfrak{c}^{(2)}(\epsilon, a_0, a_0') := \langle v_\epsilon(\cdot, a_0)\partial_1^2 v_\epsilon(\cdot, a_0')\rangle \\ &= \sum_{k\in(2\pi\mathbb{Z})^2\setminus\{0\}} \frac{(-a_0a_0'k_1^4 + k_2^2 - (a_0 + a_0')k_1^2)k_1^2\,\hat{C}(k)|(\psi_\epsilon')^2(k)|}{(a_0^2k_1^4 + 2a_0k_1^2 + k_2^2 + 1)((a_0')^2k_2^4 + 2a_0'k_1^2 + k_2^2 + 1)}. \end{aligned} \tag{1.41}$$

The proof of (1.41) follows from the proof of Lemma 7 in [47] by simply replacing their (132) by our (1.35).

The main result of Otto and Weber in which they pass to the limit in (1.38) and (1.39) is the following lemma, in which we have replaced $v^{OW}(\cdot, a_0)$ by $v(\cdot, a_0)$.

"***Almost* Proposition 2 / Part *i)* of Theorem 1 of [47] "** *Let f be a centered, one-periodic, stationary Gaussian random distribution for which there exist $\lambda_1, \lambda_2 > 0$ and $\alpha \in (0,1)$ such that* (1.32) *holds. Under these assumptions we find that for $i, j \geq 0$ the renormalized products $\partial_{a_0}^i\partial_{a_0'}^j v_\epsilon(\cdot, a_0) \diamond \partial_1^2 v_\epsilon(\cdot, a_0')$ defined in terms of* (1.38) *converge almost surely as $\epsilon \to 0$ uniformly in the parameters a_0 and a_0' in every $C^{\alpha'-2}$ space for $\alpha' < \alpha$. For every $p > 0$ we then have that*

$$\left\langle \left(\sup_{a_0, a_0' \in [\lambda, 1]} \sup_{T\leq 1} (T^{\frac{1}{4}})^{2-2\alpha'} \left\| \partial_{a_0}^i \partial_{a_0'}^j [v(\cdot, a_0), (\cdot)_T] \diamond \partial_1^2 v(\cdot, a_0') \right\| \right)^p \right\rangle^{\frac{1}{p}} \lesssim 1, \tag{1.42}$$

where the universal constant depends on i, j, α', and p.

The modification of the argument found in [47] is so minor that there is no new content here as opposed to Proposition 2 of [47] (which is then used in part *i)* of Theorem 1 in [47]). We only even mention the modification for completeness since this is a thesis.

The main tool that Otto and Weber use to obtain the convergence mentioned above and (1.42) is their Lemma 8, which they then use in the form of their Corollary 4. Replacing their $v^{OW}(\cdot, a_0)$ by $v(\cdot, a_0)$ we essentially quote their result as:

"*Almost* Lemma 8/ Corollary 4 of [47]" *Assume that we have f satisfying the same assumptions as in the previous statement and let $\epsilon > 0$ and $a_0, a_0' \in [\lambda, 1]$. Using the notation* (1.38) *and* (1.39) *we then find that for $i, j \geq 0$ the bound*

$$\left\langle \left([\partial_{a_0}^i v_\epsilon(\cdot, a_0), (\cdot)_T] \diamond \partial_{a_0'}^j \partial_1^2 v_\epsilon(\cdot, a_0') \right)^2 \right\rangle^{\frac{1}{2}} \lesssim (T^{\frac{1}{4}})^{2\alpha-2} \tag{1.43}$$

holds and additionally for $0 \leq \kappa \ll 1$ (where " $\ll$ " may depend on λ_1 and λ_2) we have that

$$\left\langle \left(\partial_\epsilon \left(\left[\partial_{a_0}^i v_\epsilon(\cdot, a_0), (\cdot)_T \right] \diamond \partial_{a_0'}^j \partial_1^2 v_\epsilon(\cdot, a_0') \right) \right)^2 \right\rangle^{\frac{1}{2}} \lesssim \frac{(T^{\frac{1}{4}})^{2\alpha-2-\kappa}}{\epsilon^{1-\frac{\kappa}{4}}}. \tag{1.44}$$

The proof of this result relies on the boundedness of the symbol of the Fourier multiplier corresponding to convolution with the Green's function (possibly with parameter derivatives). Since we have already observed above that this boundedness still holds, their argument does not undergo any relevant changes when being adapted to our current setting. We remark that in the work of Otto and Weber the renormalization of the classical products is necessary in the proof of Lemma 8.

In the first two steps of their proof of Proposition 2 they use the assumed stationarity on f, the equivalence of moments of random variables in the second Wiener chaos (over the Gaussian field f), the Sobolev inequality in terms of parameter derivatives, and Fubini's theorem (to switch e. g. an integration over a_0 and the expectation) to upgrade (1.43) and (1.44) to

$$\left\langle \left(\sup_{a_0, a_0' \in [\lambda, 1]} \sup_{T \leq 1} (T^{\frac{1}{4}})^{2-2\alpha'} \left\| \partial_{a_0}^i \partial_{a_0'}^j [v_\epsilon(\cdot, a_0), (\cdot)_T] \diamond \partial_1^2 v_\epsilon(\cdot, a_0') \right\| \right)^p \right\rangle^{\frac{1}{p}} \lesssim 1 \tag{1.45}$$

and

$$\left\langle \sup_{a_0, a_0' \in [\lambda, 1]} \sup_{T \leq 1} (T^{\frac{1}{4}})^{2-2\alpha'} \left(\partial_\epsilon \left(\left[\partial_{a_0}^i v_\epsilon(\cdot, a_0), (\cdot)_T \right] \diamond \partial_{a_0'}^j \partial_1^2 v_\epsilon(\cdot, a_0') \right) \right)^p \right\rangle^{\frac{1}{p}} \lesssim \epsilon^{\frac{\kappa}{4}-1}. \tag{1.46}$$

The results (1.45) and (1.46) are then post-processed in Step 3 using Jensen's inequality and the triangle inequality in L^p to give

$$\left\langle \left(\sup_{\epsilon\in(0,1]} \sup_{a_0,a_0'\in[\lambda,1]} \sup_{T\leq 1} (T^{\frac{1}{4}})^{2-2\alpha'} \left\| \partial^i_{a_0}\partial^j_{a_0'}[v_\epsilon(\cdot,a_0),(\cdot)_T]\diamond \partial_1^2 v_\epsilon(\cdot,a_0') \right\| \right)^p \right\rangle^{\frac{1}{p}} \lesssim 1, \tag{1.47}$$

where we emphasize that now the supremum over $\epsilon \in (0,1]$ is included. From there, in Step 5 the authors show that for all $p<\infty$, $\alpha'<\alpha$, $\kappa \ll 1$, and with the notation

$$A(T,a_0,a_0',\epsilon) = \partial^i_{a_0}\partial^j_{a_0'}[v_\epsilon(\cdot,a_0),(\cdot)_T]\diamond \partial_1^2 v_\epsilon(\cdot,a_0'),$$

there exists a Hölder estimate of the form

$$\begin{aligned}\Bigg\langle \Bigg(&\sup_{\epsilon_1\neq\epsilon_2\in(0,1]} \sup_{a_0,a_0'\in[\lambda,1]} \sup_{T\leq 1} (T^{\frac{1}{4}})^{2-2\alpha'+\kappa}|\epsilon_2-\epsilon_1|^{-\frac{\kappa}{4}} \\ &\times \|A(T,a_0,a_0',\epsilon_1)-A(T,a_0,a_0',\epsilon_2)\| \Bigg)^p \Bigg\rangle^{\frac{1}{p}} \lesssim 1.\end{aligned} \tag{1.48}$$

To conclude, in their Step 6 they then write

$$(v_\epsilon \diamond \partial_1^2 v_\epsilon)_T = v_\epsilon(\partial_1^2 v_\epsilon)_T - [v_\epsilon,(\cdot)_T]\diamond\partial_1^2 v_\epsilon$$

and notice that using their analogue of Lemma 1 along with the bounds (1.33), (1.34), (1.47), and (1.48) they can pass to the limit $\epsilon\to 0$ in $C^{\alpha-2}(\mathbb{R}^2)$.

1.1.3 Summary of the Construction of the New Singular Products

To construct a solution operator for the quasilinear problem (1.1) we require two new families of reference products:

1) The reference products $\left\{F\diamond\partial_1^2\tilde{V}(\cdot,a_0)\right\}$ indexed by $a_0\in[\lambda,1]$, where $F\in C^\alpha(\mathbb{R}^2)$ and $V(\cdot,a_0)$ solves (1.5) with $V_{int}\in C^\alpha(\mathbb{R})$.
2) The reference products $\left\{\tilde{V}(\cdot,a_0)\diamond\partial_1^2 v(\cdot,a_0')\right\}$ indexed by $a_0,a_0'\in[\lambda,1]$.

From these building blocks we can define all of the reference products mentioned in the introduction. As we have already discussed, in order to treat the linear problem (1.6), it is only necessary to construct the first family of reference products. However, since our final goal is to treat the quasilinear problem, we also construct the second new family of reference products. In order to keep things streamlined, we move all of our work on the singular product used for the quasilinear problem to an appendix in Section 6. This appendix contains the construction of the second family of reference products, the reconstruction lemma that allows us to swap out the first factor of the reference products, and a corollary in which we put the first reconstruction lemma into the form that we plan to use in our treatment of the quasilinear problem. We apply this corollary in Theorem 2 (see Section 1.4) in order to provide the main step necessary for the upcoming fixed point argument.

As we having already tried to emphasize in the introduction, the main take-away message from our construction of the new reference products is that these products are actually classical. In particular, it seems that in order to address the well-posedness of (1.1) *à la* Otto and Weber, we do not require any new stochastic bounds. In particular, using the notation

$$\overline{V}^{ext}(\cdot, a_0) = (v + \tilde{V}_{W_{int}-v(\cdot,a_0)})(\cdot, a_0), \tag{1.49}$$

where the subscript "$W_{int} - v(\cdot, a_0)$" specifies the initial condition in (1.5), we should think of the reference products $\overline{V}^{ext}(\cdot, a_0) \diamond \partial_1^2 \overline{V}^{ext}(\cdot, a_0')$ indexed by $a_0, a_0' \in [\lambda, 1]$ (already mentioned in the introduction) as classical perturbations of the offline products that we borrow from Otto and Weber. In particular, we simply define the reference products in a linear way:

$$\begin{aligned}
&\overline{V}^{ext}(\cdot, a_0) \diamond \partial_1^2 \overline{V}^{ext}(\cdot, a_0') \\
&:= \overline{V}^{ext}(\cdot, a_0) \diamond \partial_1^2 \tilde{V}_{W_{int}-v(\cdot,a_0')}(\cdot, a_0') + \tilde{V}_{W_{int}-v(\cdot,a_0)}(\cdot, a_0) \diamond \partial_1^2 v(\cdot, a_0') \\
&+ v(\cdot, a_0) \diamond \partial_1^2 v(\cdot, a_0'),
\end{aligned}$$

where the first term on the right-hand side is defined via "1)" above, the second term is defined via "2)", and the third term is borrowed from Otto and Weber as detailed in the previous subsection.

We construct the first family of new reference products as an application of the following general lemma:

Lemma 2. *Let $\alpha \in (0,1)$. Assume that $F \in C^\alpha(\mathbb{R}^2)$ and for G, a function defined on $\mathbb{R}^2$, there exists a constant $C(G) \in \mathbb{R}$ satisfying*

$$|\partial_1^2 G(x)| \lesssim C(G)(|x_2|^{\frac{\alpha-2}{2}} + |x_2|^{\frac{2\alpha-2}{2}}) \tag{1.50}$$

for every $x \in \mathbb{R}^2$. Under these assumptions the product $F\partial_1^2 G$ is well-defined as a distribution on $\mathbb{R}^2$ and

$$\sup_{T \le 1} (T^{\frac{1}{4}})^{2-2\alpha} \| [F, (\cdot)_T] \diamond \partial_1^2 G \| \lesssim C(G)[F]_\alpha^{loc}. \tag{1.51}$$

In order to obtain the first family of new reference products we would like to make the choice $G = \tilde{V}(\cdot, a_0)$ for fixed $a_0 \in [\lambda, 1]$. As we will see below, this choice of G actually only requires the factor with growth $|x_2|^{\frac{\alpha-2}{2}}$ on the right-hand side. The factor of growth $|x_2|^{\frac{2\alpha-2}{2}}$ is included in (1.50) because we will in one step of Proposition 2 consider the case $G = \tilde{q}$ with q as in (1.9).

The admissibility of the choice $G = \tilde{V}(\cdot, a_0)$ depends on the first of the bounds derived in the next lemma. The next lemma, while only containing elementary estimates, can be viewed as the key ingredient that allows us to extend the analysis of Otto and Weber to the initial value problem setting.

Lemma 3 (Semigroup bounds). *Let $\alpha \in (0,1)$ and $V(\cdot, a_0)$ solve* (1.5) *with initial condition $V_{int} \in C^\alpha(\mathbb{R})$. Then the following observations hold:*

i) For $2 \geq k \geq 0$ and $j \geq 0$ such that $k + j \geq 1$ the $\partial_{a_0}^j \partial_1^k V(\cdot, a_0)$ are well-defined distributions. Furthermore, for $x \in \mathbb{R}^2_+$ they satisfy

$$\sup_{a_0 \in [\lambda,1]} |\partial_{a_0}^j \partial_1^k V(x, a_0)| \lesssim [V_{int}]_\alpha \, x_2^{\frac{\alpha-k}{2}}. \tag{1.52}$$

ii) For $j \geq 0$ and a fixed time $x_2 \in \mathbb{R}_+$ we have the L^∞-estimate

$$\sup_{a_0 \in [\lambda,1]} \|\partial_{a_0}^j V(\cdot, x_2, a_0)\| \lesssim e^{-x_2} \|V_{int}\|. \tag{1.53}$$

iii) For $0 \leq j \leq 3$ the relation

$$[\partial_{a_0}^j V(\cdot, a_0)]_\alpha \lesssim \|V_{int}\|_\alpha \tag{1.54}$$

holds.

iv) For $0 \leq j \leq 1$ and $x, y \in \mathbb{R}^2_+$ we have that

$$\begin{aligned} &\sup_{a_0 \in [\lambda,1]} |\partial_{a_0}^j V(x, a_0) - \partial_{a_0}^j V(y, a_0)| \\ &\lesssim \|V_{int}\|_\alpha (|x_2|^{-\frac{\alpha}{2}} + |y_2|^{-\frac{\alpha}{2}}) d^{2\alpha}(x, y). \end{aligned} \tag{1.55}$$

v) If $V(\cdot, a_0)$ solves (1.5) *without the massive term, then the estimates* (1.52) *and* (1.55) *still hold. The estimate* (1.53) *still holds in a modified form; in particular, there is no factor of e^{-x_2} on the right-hand side. Also,* (1.54) *holds in a modified form in the sense that the right-hand side is the seminorm $[V_{int}]_\alpha$.*

With Lemma 3 in-hand, we are then in the position to post-process Lemma 2 and construct the first of the families of new reference products:

Corollary 1 (New reference products for the linear problem). *Let $\alpha \in (0,1)$, $F \in C^\alpha(\mathbb{R}^2)$, and $V(\cdot, a_0)$ solve* (1.5) *with initial condition $V_{int} \in C^\alpha(\mathbb{R})$. We then find that the product $F\partial_1^2\tilde{V}(\cdot, a_0)$ is well-defined as a distribution on $\mathbb{R}^2$ for any $a_0 \in [\lambda, 1]$ and the bound*

$$\sup_{a_0\in[\lambda,1]} \sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\| [F, (\cdot)_T] \diamond \partial_1^2\tilde{V}(\cdot, a_0)\| \lesssim [V_{int}]_\alpha [F]_\alpha^{loc} \tag{1.56}$$

holds.

Right away, since it is an important tool throughout this entire contribution, we mention that Lemma 3 is proved mainly by expressing $V(\cdot, a_0)$ in terms of the heat-kernel; i. e., using the notation

$$G(a_0, x_1, x_2) = \frac{1}{(4\pi a_0 x_2)^{\frac{1}{2}}} e^{\frac{-x_1^2}{4x_2 a_0} - x_2}, \tag{1.57}$$

for any $x \in \mathbb{R}^2_+$ we write

$$V(x, a_0) = \int_{\mathbb{R}} V_{int}(y) G(a_0, x_1 - y, x_2)\mathrm{d}y. \tag{1.58}$$

Often, we will use the convenient change of variables $z = \frac{x_1 - y}{(4x_2 a_0)^{\frac{1}{2}}}$ and the relations

$$\frac{\partial z}{\partial y} = \frac{-1}{(4x_2 a_0)^{\frac{1}{2}}} \quad \text{and} \quad \frac{\partial z}{\partial a_0} = -\frac{1}{2} z a_0^{-1}. \tag{1.59}$$

With the first new family of reference products constructed, on the level of the linear problem (1.6) we can then move to the reconstruction lemmas. Recall that for the quasilinear problem (1.1) we will require two reconstruction lemmas, but for the linear problem (1.6) given our assumptions we only need the second reconstruction lemma. As already stated, we put the first reconstruction lemma and the construction of the second family of new reference products into the appendix. For now, we state the second reconstruction lemma:

Lemma 4 (Modified Lemma 4 of [47]). *Let $\alpha \in (\frac{2}{3}, 1)$ and $I \in \mathbb{N}$. We have $F \in C^\alpha(\mathbb{R}^2)$, I families of functions $\{w_1(\cdot, a_0), ..., w_I(\cdot, a_0)\}$, I families of distributions $\left\{F \diamond \partial_1^2 w_1(\cdot, a_0), ..., F \diamond \partial_1^2 w_I(\cdot, a_0)\right\}$, and I constants $N_i \in \mathbb{R}$ such that the bounds*

$$\sup_{a_0\in[\lambda,1]} [w_i(\cdot, a_0)]_{\alpha,1} \leq N_i \tag{1.60}$$

$$\text{and} \quad \sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha} \sup_{a_0\in[\lambda,1]} \left\|[F, (\cdot)_T] \diamond \partial_1^2 w_i(\cdot, a_0)\right\|_1 \leq NN_i \tag{1.61}$$

hold. Then for a function $u \in C^\alpha(\mathbb{R}^2)$ that is modelled after $(w_1, ..., w_I)$ according to $a \in [\lambda, 1]$ such that $[a]_\alpha \leq 1$ and $(\sigma_1, ..., \sigma_I)$ all of class C^α, there exists a unique distribution $F \diamond \partial_1^2 u$ such that

$$\lim_{T \to 0} \| [F, (\cdot)_T] \diamond \partial_1^2 u - \sigma_i E [F, (\cdot)_T] \diamond \partial_1^2 w_i \| = 0, \tag{1.62}$$

where E denotes the evaluation of a function of (x, a_0) at $(x, a(x))$. The distribution $F \diamond \partial_1^2 u$ has finite local $C^{\alpha-2}$- seminorm and satisfies the bound

$$\sup_{T \leq 1} (T^{\frac{1}{4}})^{2-2\alpha} \| [F, (\cdot)_T] \diamond \partial_1^2 u \| \lesssim [F]_\alpha M + \|\sigma_i\|_\alpha N N_i. \tag{1.63}$$

In this lemma we assume that all functions and distributions are periodic in the x_1-direction.

The proof of this lemma, while essentially the same as in [47], is included for completeness. There are small differences due to the loss periodicity in the x_2-direction, but they are negligible and do not fundamentally alter the argument. Therefore, occasionally, when the arguments of [47] remain unchanged and their omission does not detract from an understanding of the bigger picture, we refer to [47] for details.

1.1.4 Discussion and Statement of our Results

Carrying out our analysis of (1.6) via the perturbative ansatz that we have discussed in detail in the introduction and above, we obtain the following:

Theorem 1 (Analysis of the Linear Problem)**.** *Let $\alpha \in \left(\frac{2}{3}, 1\right)$.*

i) (Construction of Solution Operator) Assume that we are given:

- *A space-time periodic distribution f and a constant $N_0 \in \mathbb{R}$ such that*

$$\sup_{T \leq 1} (T^{\frac{1}{4}})^{2-\alpha} \|f_T\| \leq N_0. \tag{1.64}$$

- *A function $a^{ext} \in C^\alpha(\mathbb{R}^2)$ that is periodic in the x_1-direction and satisfies $a^{ext} \in [\lambda, 1]$ and $[a^{ext}]_\alpha \ll 1$. Furthermore, $a^{ext}|_{\mathbb{R}^2_+} = a$ on $\mathbb{R}^2_+$.*
- *A periodic function $W_{int} \in C^\alpha(\mathbb{R})$ and a constant $N_0^{int} \in \mathbb{R}$ such that*

$$\|W_{int}\|_\alpha \leq N_0^{int}. \tag{1.65}$$

- *A family of distributions* $\left\{a^{ext} \diamond \partial_1^2 v(\cdot, a_0)\right\}$ *indexed by* $a_0 \in [\lambda, 1]$ *and a constant* $N \in \mathbb{R}$ *such that* $[a^{ext}]_\alpha \leq N \leq 1$ *and*

$$\sup_{a_0 \in [\lambda,1]} \sup_{T \leq 1} (T^{\frac{1}{4}})^{2-2\alpha} \left\| \left[a^{ext}, (\cdot)_T\right] \diamond \partial_1^2 v(\cdot, a_0)\right\|_2 \lesssim N N_0; \tag{1.66}$$

furthermore, for every $a_0 \in [\lambda, 1]$ *it holds that*

$$\sup_{T \leq 1} (T^{\frac{1}{4}})^{2-\alpha} \left\|a^{ext} \diamond \partial_1^2 v(\cdot, a_0)\right\| < \infty. \tag{1.67}$$

Under these assumptions there exists a solution $W \in C^\alpha(\mathbb{R}^2_+)$ *of*

$$(\partial_2 - a \diamond \partial_1^2 + 1)W = f \qquad \text{in} \quad \mathbb{R}^2_+, \tag{1.68a}$$

$$W = W_{int} \qquad \text{on} \quad \partial\mathbb{R}^2_+ \tag{1.68b}$$

that may be decomposed as $W = u + U$, *where* $u \in C^\alpha(\mathbb{R}^2)$ *solves*

$$(\partial_2 - a^{ext} \diamond \partial_1^2 + 1)u = f \qquad \text{in} \quad \mathbb{R}^2 \tag{1.69}$$

and is modelled after $v(\cdot, a_0)$ *solving* (1.4) *according to* a^{ext} *on* $\mathbb{R}^2$ *and* $U \in C^\alpha(\mathbb{R}^2_+)$ *solves*

$$(\partial_2 - a \diamond \partial_1^2 + 1)U = 0 \qquad \text{in} \quad \mathbb{R}^2_+, \tag{1.70a}$$

$$U = W_{int} - u \qquad \text{on} \quad \partial\mathbb{R}^2_+ \tag{1.70b}$$

and is modelled after $V(\cdot, a_0)$ *solving* (1.5) *with* $V_{int} = W_{int} - v(\cdot, a_0)$ *according to* a *on* $\mathbb{R}^2_+$. *The solution* U *is, furthermore, decomposed as* $U = q + w$ *for* q *as in* (1.9) *and* $w \in C^{2\alpha}(\mathbb{R}^2_+)$ *such that* $w = 0$ *on* $\partial\mathbb{R}^2_+$. *The solution* W *is unique within the class of functions admitting such as splitting.*

We, furthermore, find that

$$\|W\|_\alpha \lesssim N_0(N + 1) + N_0^{int} \tag{1.71}$$

and $W^{ext} = u + \tilde{q} + w^E$ *is modelled after* $(v + \tilde{V})(\cdot, a_0)$ *according to* a^{ext} *on* $\mathbb{R}^2$ *such that*

$$M \lesssim (N + 1)(N_0 + N_0^{int}). \tag{1.72}$$

ii) (Stability) Let $i, j = 0, 1$. *Assume that we are given:*

- *Space-time periodic distributions* f_i *satisfying* (1.64) *and a constant* $\delta N_0 \in \mathbb{R}$ *such that*

$$\sup_{T \leq 1} (T^{\frac{1}{4}})^{2-\alpha} \|(f_1 - f_0)_T\| \leq \delta N_0. \tag{1.73}$$

- *Two functions* $a_i^{ext} \in C^\alpha(\mathbb{R}^2)$ *satisfying the assumptions of part i) and a constant* $\delta N \in \mathbb{R}$ *such that*

$$\|a_1^{ext} - a_0^{ext}\|_\alpha \le \delta N. \tag{1.74}$$

- *Families of products* $\{a_i^{ext} \diamond \partial_1^2 v_j(\cdot, a_0)\}$ *indexed by* $a_0 \in [\lambda, 1]$ *satisfying*

$$\sup_{a_0\in[\lambda,1]} \sup_{T\le 1} (T^{\frac{1}{4}})^{2-2\alpha} \left\| [a_i^{ext}, (\cdot)_T] \diamond \partial_1^2 v_0(\cdot, a_0) - [a_i^{ext}, (\cdot)_T] \diamond \partial_1^2 v_1(\cdot, a_0) \right\|_1 \\ \le N\delta N_0, \tag{1.75}$$

and

$$\sup_{a_0\in[\lambda,1]} \sup_{T\le 1} (T^{\frac{1}{4}})^{2-2\alpha} \left\| [a_0^{ext}, (\cdot)_T] \diamond \partial_1^2 v_i(\cdot, a_0) - [a_1^{ext}, (\cdot)_T] \diamond \partial_1^2 v_i(\cdot, a_0) \right\|_1 \\ \le \delta N N_0. \tag{1.76}$$

- *Periodic functions* $W_{int,i} \in C^\alpha(\mathbb{R})$ *satisfying* (1.65) *and* $\delta N_0^{int} \in \mathbb{R}$ *satisfying*

$$\|W_{int,1} - W_{int,0}\|_\alpha \le \delta N_0^{int}. \tag{1.77}$$

Under these assumptions, denoting W_i *as the solution of* (1.68) *provided by i) that corresponds to* f_i, $W_{int,i}$, *and* a_i^{ext}, *we find that*

$$\|W_1 - W_0\|_\alpha \lesssim (N_0 + N_0^{int})\delta N + \delta N_0(N+1) + \delta N_0^{int} \tag{1.78}$$

and $u_1 + \tilde{q}_1 + (w_1)^E - (u_0 + \tilde{q}_0 + (w_0)^E)$ *is modelled after* $((v_1 + \tilde{V}_1)(\cdot, a_0), (v_0 + \tilde{V}_0)(\cdot, a_0))$ *according to* (a_1^{ext}, a_0^{ext}) *and* $(1, -1)$ *on* $\mathbb{R}^2$ *such that the modelling constant* δM *satisfies*

$$\delta M \lesssim \delta N(N_0 + N_0^{int} + 1) + \delta N_0(N+1) + \delta N_0^{int}. \tag{1.79}$$

While it may already be clear, we again specify the singular products used in the initial value problem (1.68), the equation (1.69), and the initial value problem (1.70). We go backwards and start with (1.70). Here, we know that $\tilde{q} + w^E$ is modelled after $\tilde{V}(\cdot, a_0)$, which allows us to define $a^{ext} \diamond \partial_1^2(\tilde{q} + w^E)$ using Lemma 4 and the reference products from Corollary 1. The product $a \diamond \partial_1^2 U$ in (1.70) is then defined as

$$a \diamond \partial_1^2 U := a^{ext} \diamond \partial_1^2 \left(\tilde{q} + w^E\right)|_{\mathbb{R}^2_+}. \tag{1.80}$$

The singular product used in (1.69) is rather obvious; in particular, since u is modelled after $v(\cdot, a_0)$, we can obtain $a^{ext} \diamond \partial_1^2 u$ via Lemma 4 taking the reference products $a^{ext} \diamond \partial_1^2 v(\cdot, a_0)$ as input. Lastly, the product $a \diamond \partial_1^2 W$ is obtained as

$$a \diamond \partial_1^2 W := a^{ext} \diamond \partial_1^2 (u + \tilde{q} + w^E)|_{\mathbb{R}^2_+}, \tag{1.81}$$

where the product on the right-hand side is obtained via Lemma 4 with the reference products $a^{ext} \diamond \partial_1^2 (v + \tilde{V})(\cdot, a_0) := a^{ext} \diamond \partial_1^2 v(\cdot, a_0) + a^{ext} \diamond \partial_1^2 V(\cdot, a_0)$ as input.

As we have discussed previously, the proof of Theorem 1 mainly comes down to combining Proposition 1, in which we take care of the forcing, and Proposition 2, in which we enforce the initial condition. The proof of our Proposition 1 can be seen as a variation on the proof of Proposition 1 in [47] in the sense that we substitute for the periodicity of the coefficients in the x_2-direction with a massive term in (1.82). The full statement of Proposition 1 is:

Proposition 1 (Modified Proposition 1 of [47]). *Let $\alpha \in \left(\frac{2}{3}, 1\right)$. For both parts of this proposition we adopt the assumptions and notations from Theorem 1.*

i) (Construction of Solution Operator) There exists a unique $u \in C^\alpha(\mathbb{R}^2)$ that is modelled after $v(\cdot, a_0)$ according to a^{ext} such that

$$(\partial_2 - a^{ext} \diamond \partial_1^2 + 1)u = f \qquad \text{in} \quad \mathbb{R}^2. \tag{1.82}$$

We may, furthermore, bound the modelling constant M and C^α-norm of u as

$$M + \|u\|_\alpha \lesssim N_0(N + 1). \tag{1.83}$$

ii) (Stability) Let $i = 0, 1$. Denoting the solutions given by the part i) corresponding to a_i^{ext} and f_i as u_i, we find that $u_1 - u_0$ is modelled after $(v_1(\cdot, a_0), v_0(\cdot, a_0))$ according to (a_1^{ext}, a_0^{ext}) and $(1, -1)$. The modelling constant δM and the C^α-norm of $u_1 - u_0$ satisfy

$$\delta M + \|u_0 - u_1\|_\alpha \lesssim \delta N (N_0 + 1) + (N + 1)\delta N_0. \tag{1.84}$$

Below comes the complete statement of Proposition 2. Indeed, this proposition turns out to follow in an entirely classical manner from the bounds proved in Lemma 3. Here is the statement:

Proposition 2. *Let* $\alpha \in \left(\frac{2}{3}, 1\right)$*.*

i) (Construction of Solution Operator) Assume that we are given a periodic initial condition $V_{int} \in C^{\alpha}(\mathbb{R})$ *and* $a^{ext} \in C^{\alpha}(\mathbb{R}^2)$ *satisfying the criterion of part i) of Theorem 1.*

Under these assumptions there exists a unique function $U \in C^{\alpha}(\mathbb{R}^2_+)$ *such that*

$$(\partial_2 - a \diamond \partial_1^2 + 1)U = 0 \qquad in \quad \mathbb{R}^2_+, \tag{1.85a}$$

$$U = V_{int} \qquad on \quad \partial\mathbb{R}^2_+ \tag{1.85b}$$

and U *may be decomposed as* $U = q + w$ *for* q *defined in* (1.9) *and* $w \in C^{2\alpha}(\mathbb{R}^2_+)$ *such that* $w = 0$ *on* $\partial\mathbb{R}^2_+$*. We find that* $\tilde{q} + w^E$ *is modelled after* $\tilde{V}(\cdot, a_0)$ *according to* a^{ext} *on* $\mathbb{R}^2$ *and this modelling satisfies*

$$M \lesssim \|a^{ext}\|_{\alpha}\|V_{int}\|_{\alpha}. \tag{1.86}$$

Also, the C^{α}*-norm of the solution may be bounded as*

$$\|U\|_{\alpha} \lesssim \|V_{int}\|_{\alpha}. \tag{1.87}$$

ii) (Stability) Let $i = 0, 1$*. Assume that we are given two periodic initial conditions* $V_{int,i} \in C^{\alpha}(\mathbb{R})$ *and two functions* $a_i^{ext} \in C^{\alpha}(\mathbb{R}^2)$ *all satisfying the assumptions of part i).*

Denoting the solutions of (1.85) *given by part i) as* U_i*, we find that* $\tilde{q}_1 + (w_1)^E - (\tilde{q}_0 + (w_0)^E)$ *is modelled after* $(\tilde{V}_1(\cdot, a_0), \tilde{V}_0(\cdot, a_0))$ *according to* (a_1^{ext}, a_0^{ext}) *and* $(1, -1)$ *such that the modelling constant* δM *satisfies*

$$\delta M \lesssim \|V_{int,0} - V_{int,1}\|_{\alpha} + \|a_0^{ext} - a_1^{ext}\|_{\alpha} \max_i \|V_{int,i}\|_{\alpha}. \tag{1.88}$$

Also, the C^{α}*-norm of* $U_1 - U_0$ *satisfies*

$$\|U_1 - U_0\|_{\alpha} \lesssim \|V_{int,0} - V_{int,1}\|_{\alpha} + \|a_0^{ext} - a_1^{ext}\|_{\alpha} \max_i \|V_{int,i}\|_{\alpha}. \tag{1.89}$$

Aside from the bounds of Lemma 3, which we exploit heavily in the proof of Proposition 2, the workhorse for both Proposition 1 and Proposition 2 is the following PDE lemma that is adapted from Lemma 5 of [47].

Lemma 5 (Modified Lemma 5 of [47])**.** *Let* $\alpha \in \left(\frac{1}{2}, 1\right)$ *and* $I \in \mathbb{N}$*. Assume we have* I *families of distributions* $\{f_1(\cdot, a_0), ..., f_I(\cdot, a_0)\}$ *indexed by* $a_0 \in [\lambda, 1]$ *and* I *constants* $N_i \in \mathbb{R}$ *such that*

$$\sup_{T \leq 1} (T^{\frac{1}{4}})^{2-\alpha} \sup_{a_0 \in [\lambda, 1]} \|f_{iT}(\cdot, a_0)\|_1 \leq N_i \tag{1.90}$$

and also a function $a \in [\lambda, 1]$ *satisfying* $[a]_\alpha \ll 1$. *Let the function* u *on* $\mathbb{R}^2_+$ *be modelled after* $(v_1(\cdot, a_0), ..., v_I(\cdot, a_0))$ *according to* a *and* $(\sigma_1, ..., \sigma_I)$ *and satisfy*

$$\sup_{T \le 1} (T^{\frac{1}{4}})^{2-2\alpha} \|(\partial_2 - a\partial_1^2 + 1)u_T - \sigma_i E f_{iT}(\cdot, a_0)\| \le K \tag{1.91}$$

for some $K \in \mathbb{R}$, *where* E *denotes evaluation of a function of* (x, a_0) *at* $(x, a(x))$. *We, furthermore, assume that* u *has a finite local* C^α*-seminorm. Under these assumptions, the modelling constant of* u *and the* C^α*-norm are bounded as*

$$M + \|u\|_\alpha \lesssim K + \|\sigma_i\|_\alpha N_i \tag{1.92}$$

In the setting of Proposition 1, the purpose of this lemma is to quantify the regularity of an "approximate solution" u of $(\partial_2 - a^{ext}\partial_1^2 + 1)\cdot = f$ on $\mathbb{R}^2$ using that additionally it is modelled after $v(\cdot, a_0)$ according to a^{ext}. In our proof of Proposition 1 this is helpful because to solve (1.82) we regularize the right-hand side, which gives a family of regularized solutions that are "approximate solutions" in the sense of (1.91). The estimates obtained from (1.92) allow us to pass to the limit in the regularization. In Proposition 2 we use Lemma 5 in order to obtain the correction $w \in C^{2\alpha}$, but because of the higher regularity available we can set $\sigma_i = 0$ and our application of the lemma is much simplified compared to its use in Proposition 1. For part *ii*) of the propositions, Lemma 5 is also formulated for right-hand sides that are the linear combination of some $\sigma_i f_i$.

Much of the argument for Lemma 5 relies on the analysis of a parabolic equation with massive term and frozen coefficient $a(x_0)$ for $x_0 \in \mathbb{R}^2$ that is satisfied by $u_T - \sigma(x_0)v_T(\cdot, a(x_0))$. The analysis of this equation is reminiscent of the Krylov-Safanov approach to Schauder theory in the sense that $u_T - \sigma(x_0)v_T(\cdot, a(x_0))$ is decomposed into two components $u_T - \sigma(x_0)v_T(\cdot, a(x_0)) = w_< + w_>$, where one is a "near-field" contribution and one is a "far-field" contribution. The near field-contribution $w_<$ is chosen to solve the equation satisfied by $u_T - \sigma(x_0)v_T(\cdot, a(x_0))$ (without the massive term which is moved to the right-hand side and viewed as part of the forcing) with the forcing restricted to a ball $B_L(x_0)$ for some $L > 0$. Thanks to the closeness of points in this ball to x_0 the classical regularity estimates that we obtain for $w_<$ are sufficient for our lemma. The far-field contribution is then defined as $w_> = u_T - \sigma(x_0)v_T(\cdot, a(x_0)) - w_<$, which solves a constant coefficient parabolic equation (without a massive term) on the ball $B_L(x_0)$ with zero right-hand side. Thanks to the zero right-hand side, the classical regularity estimates for $w_>$ are also sufficient. The modelling assumption on u is mainly used to then move from statements about the convolved object $u_T - \sigma(x_0)v_T(\cdot, a(x_0))$ back to the actual object of interest $u - \sigma(x_0)v(\cdot, a(x_0))$.

Combining the result of Theorem 1 with the families of reference products constructed/ borrowed as described in the previous two subsections, the reference products constructed in Section 6, and the reconstruction lemmas, we obtain the below Theorem 2. This will be the main ingredient in the fixed point argument that is used to treat the quasilinear problem.

Theorem 2. *Let* $\alpha \in \left(\frac{2}{3}, 1\right)$.

i) Assume that f *satisfies* (1.64)*; let* a*, the nonlinearity in* (1.1)*, satisfy* (1.27)*; and* $W_{int} \in C^\alpha(\mathbb{R})$ *be periodic and satisfy* (1.65)*. Let* $\overline{W} \in C^\alpha(\mathbb{R}^2)$ *such that* $[\overline{W}]_\alpha \ll 1$ *be modelled after* $(v+\tilde{V})(\cdot, a_0)$*, where* $V(\cdot, a_0)$ *has initial condition* $V_{int} = W_{int} - v(\cdot, a_0)$*, according to* $\bar{a} \in C^\alpha(\mathbb{R}^2)$*. We, furthermore, assume that we have access to a family of* $C^{\alpha-2}$ *offline reference products* $\left\{v(\cdot, a_0) \diamond \partial_1^2 v(\cdot, a_0')\right\}$ *indexed by* $a_0, a_0' \in [\lambda, 1]$ *satisfying*

$$\sup_{a_0, a_0' \in [\lambda, 1]} \sup_{T \le 1} (T^{\frac{1}{4}})^{2-2\alpha} \left\| [v(\cdot, a_0), (\cdot)_T] \diamond \partial_1^2 v(\cdot, a_0') \right\|_{2,2} \lesssim N_0^2. \tag{1.93}$$

Using the notation

$$a_{ext} := a(\overline{W}) \quad \text{and} \quad a := a_{ext}|_{\mathbb{R}^2_+}, \tag{1.94}$$

we then find that there exists a unique solution $W \in C^\alpha(\mathbb{R}^2)$ *of*

$$(\partial_2 - a \diamond \partial_1^2 + 1)W = f \qquad \text{in} \quad \mathbb{R}^2_+, \tag{1.95a}$$

$$W = W_{int} \qquad \text{on} \quad \partial\mathbb{R}^2_+ \tag{1.95b}$$

that may be decomposed as $W = u + U = u + q + w$ *as indicated in Theorem 1, where* u *is modelled after* $v(\cdot, a_0)$ *according to* a_{ext} *and* $\tilde{q} + w^E$ *is modelled after* $\tilde{V}(\cdot, a_0)$ *with initial condition* $V_{int} = W_{int} - v(\cdot, a_0)$ *according to* a_{ext}*. This solution satisfies*

$$\|W\|_\alpha \lesssim N_0(\tilde{M} + 1) + N_0^{int} \tag{1.96}$$

with the notation

$$\tilde{M} := M_{a_{ext}} + N_0 + N_0^{int}, \tag{1.97}$$

where $M_{a_{ext}}$ *denotes the modelling of* a_{ext} *after* $(\tilde{V} + v)(\cdot, a_0)$ *(see* (1.26)*), and the extension* $u + \tilde{q} + w^E$ *is modelled after* $(v + \tilde{V})(\cdot, a_0)$ *according to* a_{ext} *such that*

$$M \lesssim (N_0 + N_0^{int})(\tilde{M} + 1). \tag{1.98}$$

ii) (Stability) Let $i, j = 0, 1$. Assume that each f_i satisfies (1.64) and together they satisfy (1.73); the initial conditions $W_{int,i} \in C^\alpha(\mathbb{R})$ are periodic, independently satisfy (1.65), and together (1.77). We consider $\overline{W}_i \in C^\alpha(\mathbb{R}^2)$ satisfying $[\overline{W}_i]_\alpha \ll 1$ that are modelled after $(v_i + \tilde{V}_i)(\cdot, a_0)$ according to $\bar{a}_i$, where $V_i(\cdot, a_0)$ has initial condition $V_{int,i} = W_{int,i} - v_i(\cdot, a_0)$. We, furthermore, assume that we have access to four families of $C^{\alpha-2}$ distributions $\{v_i(\cdot, a_0) \diamond \partial_1^2 v_j(\cdot, a_0')\}$ indexed by $a_0, a_0' \in [\lambda, 1]$ satisfying (1.93) and

$$\begin{aligned} \sup_{a_0, a_0' \in [\lambda,1]} \sup_{T \le 1} (T^{\frac{1}{4}})^{2-2\alpha} \big\| [v_i(\cdot, a_0), (\cdot)_T] \diamond \partial_1^2 v_0(\cdot, a_0') \\ - [v_i(\cdot, a_0), (\cdot)_T] \diamond \partial_1^2 v_1(\cdot, a_0') \big\|_{1,1} \\ \le N_0 \delta N_0, \end{aligned} \tag{1.99}$$

and

$$\begin{aligned} \sup_{a_0, a_0' \in [\lambda,1]} \sup_{T \le 1} (T^{\frac{1}{4}})^{2-2\alpha} \big\| [v_0(\cdot, a_0), (\cdot)_T] \diamond \partial_1^2 v_i(\cdot, a_0') \\ - [v_1(\cdot, a_0), (\cdot)_T] \diamond \partial_1^2 v_i(\cdot, a_0') \big\|_{1,1} \\ \le \delta N_0 N_0. \end{aligned} \tag{1.100}$$

Using the notation

$$a_{i,ext} := a(\overline{W}_i) \quad \text{and} \quad a_i := a_{i,ext}|_{\mathbb{R}^2_+} \tag{1.101}$$

and letting $W_i = u_i + q_i + w_i$ indicate the solution to (1.95) with coefficients a_i, forcing f_i, and initial condition $W_{int,i}$. We then find that

$$\|W_1 - W_0\|_\alpha \lesssim \delta\tilde{M}(N_0 + N_0^{int}) + \delta N_0(\tilde{M} + 1) + \delta N_0^{int} \tag{1.102}$$

and $u_1 + \tilde{q}_1 + (w_1)^E - (u_0 + \tilde{q}_0 + (w_0)^E)$ is modelled after $((v_1 + \tilde{V}_1)(\cdot, a_0), (v_0 + \tilde{V}_0)(\cdot, a_0))$ according to (a_1, a_0) and $(1, -1)$ on $\mathbb{R}^2$ such that the modelling constant δM satisfies

$$\delta M \lesssim \delta\tilde{M}(N_0 + N_0^{int} + 1) + \delta N_0(\tilde{M} + 1) + \delta N_0^{int}. \tag{1.103}$$

Here we have used the notation

$$\begin{aligned} &\delta\tilde{M} \\ &:= \delta M_{a_{ext}} + (N_0 + N_0^{int} + 1)(\|\bar{a}_1 - \bar{a}_0\|_\alpha + \|\overline{W}_1 - \overline{W}_0\|_\alpha) + \delta N_0 + \delta N_0^{int}, \end{aligned}$$

where $\delta M_{a_{ext}}$ corresponds to the modelling of $a_{ext,1} - a_{ext,0}$ after $((v_1 + \tilde{V}_1)(\cdot, a_0), (v_0 + \tilde{V}_0)(\cdot, a_0))$ according to $(\overline{a}_1, \overline{a}_0)$ and $(1, -1)$.

To see how we use Theorem 2, which we prove in Section 7, we sketch the fixed point argument that we plan to use for the quasilinear problem. In analogue to (12) and (21) of the introduction and using the notation from Theorem 2, we consider the map:

$$
\begin{gathered}
(\overline{W}, \overline{a}) \\
\big\downarrow \psi \\
\Big(a := a(\overline{W}), \big\{a \diamond \partial_1^2 v(\cdot, a_0)\big\}_{a_0}\Big) \\
\big\downarrow S \\
\big(W^{ext} = u + \tilde{q} + w^E, a(\overline{W})\big)\,.
\end{gathered} \tag{1.104}
$$

Here ψ consists of the construction/ borrowing of reference products and S indicates the application of Theorem 1. Of course, the main task is to show that this map is in fact a contraction on a certain space of modelled functions, which follows from the bounds obtained in Theorem 2. Notice that the condition $[\overline{W}]_\alpha \ll 1$ in Theorem 2 will require us to consider only small data; this comment will be fleshed out in the up-coming contribution.

Also, a last comment, notice that in Theorem 2 we have again postulated the existence of the offline products of Otto and Weber. In order to apply the results we have quoted in Section 1.2 of this chapter, we notice that they imply that:

"Theorem 1 of [47], ii)" *Let f be a stationary, centered, Gaussian random distribution satisfying* (1.32)*, then there is a positive random constant η such that η^{-1} is in every stochastic L^p space for $p < \infty$, ηf satisfies* (1.64)*, and the offline products $\eta^2[v(\cdot, a_0), (\cdot)_T] \diamond \partial_1^2 v(\cdot, a_0')$ satisfy the assumption* (1.93).

This will be made formal in the upcoming contribution in which we actually handle the quasilinear problem.

1.2 Main PDE Ingredient: Proof of the *Krylov-Safanov* Lemma

We prove this lemma in a series of steps that follow the proof of Lemma 5 in [47]. The difference between their setting and ours is that in place of space-time periodicity we instead have a massive term in our parabolic operator; in essence, we replace compactness by an L^∞-estimate. Here comes the argument:

Proof of Lemma 5.

Step 1– (u is Lipschitz on large scales and bound for $[u]_\alpha^{loc}$)

We first show that for $x, y \in \mathbb{R}^2$ such that $d(x,y) \geq 1$ we have that

$$|u(x) - u(y)| \leq 2[u]_\alpha^{loc} d(x,y). \tag{1.105}$$

This follows easily from the triangle inequality: Fix $x, y \in \mathbb{R}^2$ and define the functions $\left\{\eta_k^x : \mathbb{R}^2 \to \mathbb{R}^2\right\}_{k\in\mathbb{Z}_{\geq 0}}$ such that if $k = 0$ then $\eta_0^x \equiv x$, if $k < d(x,y)$ then $\eta_k^x(y)$ is the point of intersection between the line connecting x and y and $\partial B_k(x)$, if $k = \lceil d(x,y) \rceil$ then $\eta_k^x(y) = y$, and finally if $k > \lceil d(x,y) \rceil$ then $\eta_k^x(y) = 0$. This notation allows us to write

$$\begin{aligned} |u(x) - u(y)| \leq & \sum_{1\leq k\leq\lceil d(x,y)\rceil} |u(\eta_{k-1}^x(y)) - u(\eta_k^x(y)))| \\ \leq & [u]_\alpha^{loc} (\lfloor d(x,y) \rfloor + (d(x,y) - \lfloor d(x,y) \rfloor)^\alpha) \\ \leq & 2\,[u]_\alpha^{loc}\, d(x,y). \end{aligned} \tag{1.106}$$

We would also like to obtain a bound for $[u]_\alpha^{loc}$ in terms of M. For this, let $x, y \in \mathbb{R}^2$ such that $d(x,y) \leq 1$ and notice that

$$\begin{aligned} \frac{|u(x) - u(y)|}{d^\alpha(x,y)} \leq & M d^\alpha(x,y) + \|\sigma_i\| \frac{|v_i(x, a(y)) - v_i(y, a(y))|}{d^\alpha(x,y)} + \frac{|\nu(y)(x-y)_1|}{d^\alpha(x,y)} \\ \lesssim & M + \|\sigma_i\| N_i, \end{aligned} \tag{1.107}$$

where in addition to the modelling we have used (1.370) and (1.322).

Step 2 – (Equations satisfied by u_T)

We now show that for every $x_0 \in \mathbb{R}^2$ and $T \in (0,1]$ the function u_T solves

$$(\partial_2 - a(x_0)\partial_1^2 + 1)(u_T - \sigma_i(x_0) v_{iT}(\cdot, a(x_0))) = g_{x_0}^T \qquad \text{on} \quad \mathbb{R}^2, \tag{1.108}$$

where $g_{x_0}^T(x)$ satisfies the estimate

$$|g_{x_0}^T(x)| \lesssim \tilde{N}((T^{\frac{1}{4}})^{2\alpha-2} + d^\alpha(x, x_0)(T^{\frac{1}{4}})^{\alpha-2}) \tag{1.109}$$

with $\tilde{N} = K + [a]_\alpha [u]_\alpha^{loc} + \|\sigma_i\|_\alpha N_i$ for any $x \in \mathbb{R}^2$.

We also find that u_T solves

$$(\partial_2 - a\partial_1^2 + 1)u_T = h^T \qquad \text{in} \quad \mathbb{R}^2, \tag{1.110}$$

where

$$\|h^T\| \lesssim K(T^{\frac{1}{4}})^{2\alpha-2} + \|\sigma_i\| N_i(T^{\frac{1}{4}})^{\alpha-2}. \tag{1.111}$$

We begin by showing that (1.108) is satisfied for $g^T_{x_0}$ to be determined and proving the bound (1.109). We notice that simple manipulations show that u_T solves

$$(\partial_2 - a(x_0)\partial_1^2 + 1)u_T = \sigma_i(x_0) f_{iT}(\cdot, a(x_0)) + g^T_{x_0} \qquad \text{in} \quad \mathbb{R}^2, \tag{1.112}$$

where

$$\begin{aligned} g^T_{x_0} =&(\partial_2 - a\partial_1^2 + 1)u_T - \sigma_i E f_{iT}(\cdot, a(x_0)) \\ &+ (a - a(x_0))\partial_1^2 u_T + (\sigma_i - \sigma_i(x_0)) E f_{iT}(\cdot, a(x_0)) \\ &+ \sigma_i(x_0)(E f_{iT}(\cdot, a(x_0)) - f_{iT}(\cdot, a(x_0))). \end{aligned} \tag{1.113}$$

For a fixed $x \in \mathbb{R}^2$ we then bound

$$\begin{aligned} &|g^T_{x_0}(x)| \\ \lesssim&\|(\partial_2 - a\partial_1^2 + 1)u_T - \sigma_i E f_{iT}(\cdot, a(x_0))\| + |a(x) - a(x_0)||\partial_1^2 u_T(x)| \\ &+ |(\sigma_i(x) - \sigma_i(x_0)) E f_{iT}(\cdot, a(x_0))| \\ &+ \|\sigma_i(x_0)(E f_{iT}(\cdot, a(x_0)) - f_{iT}(\cdot, a(x_0)))\| \\ \leq& K(T^{\frac{1}{4}})^{2\alpha-2} + [a]_\alpha d^\alpha(x, x_0)|\partial_1^2 u_T(x)| \\ &+ [\sigma_i]_\alpha d^\alpha(x, x_0) \sup_{a_0 \in [\lambda, 1]} \|f_{iT}(\cdot, a_0)\| \\ &+ \|\sigma_i\| [a]_\alpha d^\alpha(x, x_0) \sup_{a_0 \in [\lambda, 1]} \|f_{iT}(\cdot, a_0)\|_1 \\ \leq& K(T^{\frac{1}{4}})^{2\alpha-2} + [a]_\alpha d^\alpha(x, x_0)|\partial_1^2 u_T(x)| + \|\sigma_i\|_\alpha d^\alpha(x, x_0) N_i(T^{\frac{1}{4}})^{\alpha-2}, \end{aligned} \tag{1.114}$$

where we have used assumptions (1.90), (1.91), and $[a]_\alpha \leq 1$.

In order to obtain (1.109) it remains to show the bound

$$|\partial_1^2 u_T(x)| \lesssim [u]^{loc}_\alpha (T^{\frac{1}{4}})^{\alpha-2} \tag{1.115}$$

for every $x \in \mathbb{R}^2$. We do this using our result of Step 1 and (1.18), which allow us to write

$$\begin{aligned} &|\partial_1^2 u_T(x)| \\ =& \left| \int_{\mathbb{R}^2} (u(y) - u(x)) \partial_1^2 \psi_T(x - y) \, \mathrm{d}y \right| \\ \lesssim& [u]^{loc}_\alpha \left(\int_{B_1(x)} |\partial_1^2 \psi_T(x - y)| d^\alpha(x, y) \, \mathrm{d}y \right. \\ &\qquad \left. + \int_{B_1^c(x)} |\partial_1^2 \psi_T(x - y)| d(x, y) \, \mathrm{d}y \right) \\ \lesssim& [u]^{loc}_\alpha \left((T^{\frac{1}{4}})^{\alpha-2} + (T^{\frac{1}{4}})^{-1} \right). \end{aligned} \tag{1.116}$$

This is sufficient since we only consider $T \leq 1$. Plugging (1.115) into (1.114) yields the desired (1.109).

Using different manipulations we find that u_T solves

$$(\partial_2 - a\partial_1^2 + 1)u_T = h^T \qquad \text{in} \quad \mathbb{R}^2, \tag{1.117}$$

where

$$h^T(x) = (\partial_2 - a\partial_1^2 + 1)u_T - \sigma_i f_{iT}(\cdot, a(x)) + \sigma_i f_{iT}(\cdot, a(x)).$$

Using the assumptions (1.90) and (1.91) we obtain (1.111).

Step 3– (L^∞-estimates)

We first notice that for $T \in (0,1]$ the estimate

$$\|u_T\| \lesssim K(T^{\frac{1}{4}})^{2\alpha-2} + \|\sigma_i\| N_i (T^{\frac{1}{4}})^{\alpha-2} \tag{1.118}$$

holds. This follows easily from the previous step by applying Theorem 8.1.7 of [39] to (1.117) and using (1.111). Here, we rely on the massive term.

We also notice that for $T \in (0,1]$ the estimate

$$\begin{aligned} &\|u_T - \sigma_i(x_0) v_{iT}(\cdot, a(x_0))\|_{B_L(x_0)} \\ &\lesssim \tilde{N}((T^{\frac{1}{4}})^{2\alpha-2} + L^\alpha)(T^{\frac{1}{4}})^{\alpha-2} \end{aligned} \tag{1.119}$$

holds for $L > 0$. For this estimate we use the equation (1.110) and, letting $G(a(x_0), x_1, x_2)$ denote the heat-kernel as in (1.57), we write $u_T - \sigma_i(x_0) v_{iT}(\cdot, a(x_0))$ as

$$\begin{aligned} &u_T(x) - \sigma_i(x_0) v_{iT}(x, a(x_0)) \\ &= \int_0^\infty \int_{\mathbb{R}} g_{x_0}^T(x_1 - y, x_2 - s) G(a(x_0), y, s) \,\mathrm{d}y \,\mathrm{d}s. \end{aligned} \tag{1.120}$$

Combining (1.120) with the bound (1.109) and using the notation $x_0 = (x_{01}, x_{02})$, for $x \in B_L(x_0)$ we obtain

$$\begin{aligned} &|u_T(x) - \sigma_i(x_0) v_{iT}(x, a(x_0))| \\ \lesssim &\tilde{N} \int_0^\infty \int_{\mathbb{R}} ((T^{\frac{1}{4}})^{2\alpha-2} + (T^{\frac{1}{4}})^{\alpha-2}(|x_1 - y - x_{01}|^\alpha + |x_2 - s - x_{02}|^{\frac{\alpha}{2}})) \\ &\qquad \times |G(a(x_0), y, s)| \,\mathrm{d}y \,\mathrm{d}s. \\ \lesssim &\tilde{N} \Bigg((T^{\frac{1}{4}})^{2\alpha-2} + L^\alpha (T^{\frac{1}{4}})^{\alpha-2} \\ &\quad + \int_0^\infty \int_{\mathbb{R}} (T^{\frac{1}{4}})^{\alpha-2}(|y|^\alpha + s^{\frac{\alpha}{2}}) |G(a(x_0), y, s)| \,\mathrm{d}y \,\mathrm{d}s \Bigg). \end{aligned} \tag{1.121}$$

Here we have used that $B_L(x_0)$ refers to the "parabolic ball" and the exponential factor "e^{-x_2}" appearing in the Green's function due to the massive term. Using the rescaling (1.16) we then treat the integral on the bottom line of (1.121) as

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}} (T^{\frac{1}{4}})^{\alpha-2}(|y|^\alpha + s^{\frac{\alpha}{2}})|G(a(x_0), y, s)|\,\mathrm{d}y\,\mathrm{d}s \\
\lesssim&(T^{\frac{1}{4}})^{2\alpha-2}\int_0^\infty \int_{\mathbb{R}} (|\hat{y}|^\alpha + \hat{s}^{\frac{\alpha}{2}})|G(a(x_0), \hat{y}, \hat{s})|\,\mathrm{d}\hat{y}\,\mathrm{d}\hat{s} \\
\lesssim&(T^{\frac{1}{4}})^{2\alpha-2},
\end{aligned}$$

where we have again used the presence of the massive term. The previous estimate in combination with (1.121) gives the desired (1.119).

Step 4– (An excess decay)

In this step we show that for any two radii R and L such that $0 < R \ll L$, $T \in (0, 1]$, and $x_0 \in \mathbb{R}^2$ it holds that

$$\begin{aligned}
&\frac{1}{R^{2\alpha}} \inf_{l \in \mathrm{Span}\{1,x_1\}} \|u_T - \sigma_i(x_0)v_{iT}(\cdot, a(x_0)) - l\|_{B_R(x_0)} \\
\lesssim& \left(\frac{R}{L}\right)^{2(1-\alpha)} \frac{1}{L^{2\alpha}} \inf_{l \in \mathrm{Span}\{1,x_1\}} \|u_T - \sigma_i(x_0)v_{iT}(\cdot, a(x_0)) - l\|_{B_L(x_0)} \\
&+ \tilde{N}\left(\frac{L^2}{R^{2\alpha}(T^{\frac{1}{4}})^{2-2\alpha}} + \frac{L^{2+\alpha}}{R^{2\alpha}(T^{\frac{1}{4}})^{2-\alpha}}\right).
\end{aligned} \tag{1.122}$$

Alternatively, we have that

$$\begin{aligned}
&\inf_{l \in \mathrm{Span}\{1,x_1\}} \|u_T - \sigma_i(x_0)v_{iT}(\cdot, a(x_0)) - l\| \\
&\lesssim K(T^{\frac{1}{4}})^{2\alpha-2} + \|\sigma_i\| N_i (T^{\frac{1}{4}})^{\alpha-2}.
\end{aligned} \tag{1.123}$$

We first show (1.123). Here, we plug $l = 0$ into the left-hand side of (1.123) and use the triangle inequality along with (1.118), (1.21), and (1.322) to obtain

$$\begin{aligned}
&\inf_{l \in \mathrm{Span}\{1,x_1\}} \|u_T - \sigma_i(x_0)v_{iT}(\cdot, a(x_0)) - l\| \\
&\lesssim \|u_T\| + \|\sigma_i\| \sup_{a_0 \in [\lambda, 1]} \|v_{iT}(\cdot, a_0)\| \\
&\lesssim K(T^{\frac{1}{4}})^{2\alpha-2} + \|\sigma_i\| N_i (T^{\frac{1}{4}})^{\alpha-2}.
\end{aligned}$$

Showing (1.122) is the main technical step of this proof: In particular, on the ball $B_L(x_0)$ we decompose the function $u_T - \sigma_i(x_0)v_{iT}(\cdot, a(x_0))$ into a "near-field" and "far-field" contribution. Letting $w_<$ be the solution of

$$(\partial_2 - a(x_0)\partial_1^2)w_< = \chi_{B_L}(g_0^T - (u_T - \sigma_i(x_0)v_{iT}(\cdot, a(x_0)))) \qquad \text{in} \quad \mathbb{R}^2 \tag{1.124}$$

and defining $w_> = u_T - \sigma_i(x_0)v_{iT}(\cdot, a(x_0)) - w_<$, we find that $w_>$ satisfies

$$(\partial_2 - a(x_0)\partial_1^2)w_> = 0 \qquad \text{in} \quad B_L(x_0). \tag{1.125}$$

We may then use standard regularity theory to obtain the estimates

$$\|w_<\| \lesssim L^2(\|g_{x_0}^T\|_{B_L(x_0)} + \|u_T - \sigma_i(x_0)v_{iT}(\cdot, a(x_0))\|_{B_L(x_0)}) \tag{1.126}$$

and

$$\|\{\partial_1^2, \partial_2\}w_>\|_{B_{L/2}(x_0)} \lesssim L^{-2}\|w_> - l\|_{B_L(x_0)} \tag{1.127}$$

for any $l \in \text{Span}\{1, x_1\}$. The estimate (1.126) follows immediately from the heat-kernel representation of $w_<$ and the triangle inequality in L^∞ and (1.127) is proven via Bernstein's argument in Theorem 8.4.4 of [39] for $l = 0$. As is already mentioned in [47], one can easily reduce to the case that $l = 0$ since $w_> - l$ still solves (1.125) when $l \in \text{Span}\{1, x_1\}$.

We maneuver ourselves into a position to apply the estimates (1.126) and (1.127) by decomposing

$$u_T - \sigma_i(x_0)v_{iT}(\cdot, a(x_0)) = w_< + w_>$$

and using the triangle inequality to write

$$\|u_T - \sigma_i(x_0)v_{iT}(\cdot, a(x_0)) - l_R\|_{B_R(x_0)} \lesssim R^2\|\{\partial_1^2, \partial_2\}w_>\|_{B_R(x_0)} + \|w_<\|_{B_R(x_0)}$$

for $l_R = w_>(x_0) + \frac{\partial w_>}{\partial x_1}(x_0)x_1$. Using (1.126) and (1.127) along with (1.109) and (1.119) we may then continue this as

$$\begin{aligned} &R^2\|\{\partial_1^2, \partial_2\}w_>\|_{B_R(x_0)} + \|w_<\|_{B_R(x_0)} \\ &\lesssim \left(\frac{R}{L}\right)^2 \|w_> - l\|_{B_L(x_0)} + L^2\tilde{N}((T^{\frac{1}{4}})^{2\alpha-2} + L^\alpha(T^{\frac{1}{4}})^{\alpha-2}), \end{aligned}$$

which then yields the desired (1.122).

Step 4– (An equivalent definition of the modelling constant)

In this step we show that $M \sim M'$, where M' is defined as

$$M' := \sup_{x_0\in\mathbb{R}^2} \sup_{R>0} R^{-2\alpha} \inf_{l\in\text{Span}\{1,x_1\}} \|u - \sigma_i(x_0)v_i(\cdot, a(x_0)) - l\|_{B_R(x_0)}. \tag{1.128}$$

To show this we start with $M \lesssim M'$. For this we begin by observing that

$$\sup_{x_0\in\mathbb{R}^2} \inf_{l\in\operatorname{Span}\{1,x_1\}} \sup_{R>0} R^{-2\alpha}\|u-\sigma_i(x_0)v_i(\cdot,a(x_0))-l\|_{B_R(x_0)} \lesssim M' \tag{1.129}$$

for which we wlog assume that $x_0=0$. Let l_R denote the optimal l for a given radius R. We then notice that for any $R>0$ we have that $R^{-2\alpha}\|l_{2R}-l_R\|_{B_R} \lesssim M'$:

$$\begin{aligned}
&R^{-2\alpha}\|l_{2R}-l_R\|_{B_R}\\
\le& R^{-2\alpha}\|u-\sigma_i(0)v_i(\cdot,a(0))-l_R-u+\sigma_i(0)v_i(\cdot,a(0))+l_{2R}\|_{B_R}\\
\le& R^{-2\alpha}\left(\|u-\sigma_i(0)v_i(\cdot,a(0))-l_R\|_{B_R}+\|u-\sigma_i(0)v_i(\cdot,a(0))-l_{2R}\|_{B_{2R}}\right)\\
\lesssim& M'.
\end{aligned}$$

Writing $l_R=\nu_R x_1+c_R$ the above observation gives that $R^{1-2\alpha}|\nu_R-\nu_{2R}|+R^{-2\alpha}|c_R-c_{2R}| \lesssim M'$, which yields that $R^{1-2\alpha}|\nu_R-\nu_{R'}|+R^{-2\alpha}|c_R-c_{R'}| \lesssim M'$ for any $0<R'\le R$ because $\alpha\in\left(\frac{1}{2},1\right)$. We find that the sequences $\left\{\nu_{1/n}\right\}_n$ and $\left\{c_{1/n}\right\}_n$ are Cauchy and there exists $l=\nu x_1+c$ such that $R^{-2\alpha}\|l_R-l\|_{B_R(x_0)} \lesssim M'$ for all $R>0$. This observation yields (1.129). To complete the argument we notice that there is at most one l making the expression

$$\sup_{R>0} R^{-2\alpha}\|u-\sigma_i(0)v_i(\cdot,a(0))-l\|_{B_R}$$

finite since $2\alpha>1$. Therefore, thanks to the modelling of u, the optimal l on the left-hand side of (1.129) is given by $u(0)-\sigma_i(0)w_i(0,a(0))+\nu(0)x_1$. Repeating this argument for all $x_0\in\mathbb{R}^2$ then gives that $M\lesssim M'$, as desired.

To finish this step we show that $M'\lesssim M$. This direction follows from the observation that

$$M' \lesssim \sup_{x_0\in\mathbb{R}^2} \inf_{l\in\operatorname{Span}\{1,x_1\}} \sup_{R>0} R^{-2\alpha}\|u-\sigma_i(x_0)v_i(\cdot,a(x_0))-l\|_{B_R(x_0)}. \tag{1.130}$$

and the uniqueness observation that we have made in the previous paragraph.

Step 5– (Use of the modelling)

In this step we show that for $T\in(0,1]$, $L>0$, and $x_0\in\mathbb{R}^2$ the estimate

$$\begin{aligned}
&\frac{1}{(T^{\frac{1}{4}})^{2\alpha}}\|u_T-u-\sigma_i(x_0)(v_{iT}-v_i)(\cdot,a(x_0))\|_{B_L(x_0)}\\
&\lesssim M+\tilde{N}\left(\frac{L}{T^{\frac{1}{4}}}\right)^{\alpha},
\end{aligned} \tag{1.131}$$

holds. The argument here is taken essentially verbatim from Step 5 of Lemma 5 of [47]. For brevity we assume wlog that $x_0 = 0$ and use the notation $v_i(y, a(x)) = v_i(y, x)$. We fix $x \in B_L(0)$ and then use the triangle inequality to write

$$
\begin{aligned}
&|u_T(x) - u(x) - \sigma_i(0)(v_{iT} - v_i)(x, 0)| \\
\leq &\left| \int_{\mathbb{R}^2} (u(y) - u(x) - \sigma_i(x)(v_i(y, x) - v_i(x, x)) \right. \\
&\qquad\qquad\qquad \left. - \nu(x)(y_1 - x_1))\psi_T(x - y)\,\mathrm{d}y \right| \\
&+ \left| \int_{\mathbb{R}^2} (\sigma_i(x) - \sigma_i(0))(v_i(y, 0) - v_i(x, 0))\psi_T(x - y)\,\mathrm{d}y \right| \\
&+ \left| \int_{\mathbb{R}^2} \sigma_i(x)(v_i(y, x) - v_i(y, 0) - (v_i(x, x) - v_i(x, 0)))\psi_T(x - y)\,\mathrm{d}y \right|.
\end{aligned}
$$

Notice that we have used that $\psi_T(x)$ is even in x_1 to smuggle in the term $\nu(x)(y_1 - x_1)\psi_T(x - y)$ in the first line. We bound the first term on the right-hand side of the above expression using the modelledness of u and (1.18) by $M(T^{\frac{1}{4}})^{2\alpha}$. For the second term we use that $\sup_{a_0 \in [\lambda, 1]}[v_i(\cdot, a_0)]_\alpha \lesssim N_i$ thanks to (1.90) and (1.18) to obtain the bound $N_i[\sigma_i]_\alpha L^\alpha (T^{\frac{1}{4}})^\alpha$. The third term is treated as

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^2} \sigma_i(x)(v_i(y, x) - v_i(x, x) - (v_i(y, 0) - v_i(x, 0)))\psi_T(x - y)\,\mathrm{d}y \right| \\
\lesssim &[a]_\alpha L^\alpha \|\sigma_i\| \int_{\mathbb{R}^2} \int_0^1 |\partial_{a_0}(v_i(y, (1 - t)a(0) + ta(x)) \\
&\qquad\qquad\qquad - v_i(x, (1 - t)a(0) + ta(x)))|\,\mathrm{d}t\,\psi_T(x - y)\,\mathrm{d}y \\
\lesssim &[a]_\alpha L^\alpha \|\sigma_i\| N_i (T^{\frac{1}{4}})^\alpha,
\end{aligned}
$$

where we have used (1.322) to bound $[v_i(\cdot, a_0)]_\alpha$ and (1.18). Using our definition of $\tilde{N}$ from Step 2 we then obtain (1.131).

Step 6 – (Conclusion)

We now show that $M \lesssim \tilde{N}$. To begin, for $T \in (0, 1]$ and $x_0 \in \mathbb{R}^2$, we combine (1.123) and (1.131) to write

$$\begin{aligned}
&\frac{1}{R^{2\alpha}}\inf_{l\in\mathrm{Span}\{1,x_1\}}\|u-\sigma_i(x_0)v_i(\cdot,a(x_0))-l\|_{B_R(x_0)}\\
\lesssim&\frac{1}{R^{2\alpha}}\left((K+\|\sigma_i\|N_i)(T^{\frac{1}{4}})^{\alpha-2}+\|u_T-u-\sigma_i(x_0)(v_{iT}-v_i)(\cdot,a(x_0))\|_{B_L(x_0)}\right)\\
\lesssim&\frac{(T^{\frac{1}{4}})^{\alpha-2}}{R^{2\alpha}}(K+\|\sigma_i\|N_i)+\left(\frac{T^{\frac{1}{4}}}{R}\right)^{2\alpha}\left(M+\tilde{N}\left(\frac{L}{T^{\frac{1}{4}}}\right)^{\alpha}\right).
\end{aligned}\tag{1.132}$$

Alternatively, combining (1.122) with (1.131) we find that

$$\begin{aligned}
&\frac{1}{R^{2\alpha}}\inf_{l\in\mathrm{Span}\{1,x_1\}}\|u-\sigma_i(x_0)v_i(\cdot,a(x_0))-l\|_{B_R(x_0)}\\
\lesssim&\left(\frac{R}{L}\right)^{2(1-\alpha)}\frac{1}{L^{2\alpha}}\inf_{l\in\mathrm{Span}\{1,x_1\}}\|u-\sigma_i(x_0)v_i(\cdot,a(x_0))-l\|_{B_L(x_0)}\\
&+\tilde{N}\left(\frac{L^2}{R^{2\alpha}(T^{\frac{1}{4}})^{2-2\alpha}}+\frac{L^{2+\alpha}}{R^{2\alpha}(T^{\frac{1}{4}})^{2-\alpha}}\right)+\left(\frac{T^{\frac{1}{4}}}{R}\right)^{2\alpha}\left(M+\tilde{N}\left(\frac{L}{T^{\frac{1}{4}}}\right)^{\alpha}\right).
\end{aligned}\tag{1.133}$$

For the case that $R\leq 1$ we make use of (1.133) and let $L=\epsilon^{-1}R$ and $T^{\frac{1}{4}}=\epsilon R$ for some $\varepsilon\ll 1$; the restriction $R\leq 1$ guarantees that $T\leq 1$. Making these identifications we obtain

$$\begin{aligned}
&\sup_{R\leq 1}\frac{1}{R^{2\alpha}}\inf_{l\in\mathrm{Span}\{1,x_1\}}\|u-\sigma_i(x_0)v_i(\cdot,a(x_0))-l\|_{B_R(x_0)}\\
\lesssim&(\epsilon^{2-2\alpha}+\epsilon^{2\alpha})M+(\epsilon^{-(4-2\alpha)}+\epsilon^{-4}+1)\tilde{N},
\end{aligned}\tag{1.134}$$

where we have also used the equivalence $M\sim M'$. For $R\geq 1$ we alternatively use (1.132) and let $T^{\frac{1}{4}}=\epsilon$ and $L=\epsilon^{-1}R$, which gives

$$\begin{aligned}
&\sup_{R\geq 1}\frac{1}{R^{2\alpha}}\inf_{l\in\mathrm{Span}\{1,x_1\}}\|u-\sigma_i(x_0)v_i(\cdot,a(x_0))-l\|_{B_R(x_0)}\\
\lesssim&\epsilon^{\alpha-2}(K+\|\sigma_i\|N_i)+\epsilon^{2\alpha}M+\tilde{N}.
\end{aligned}\tag{1.135}$$

Combining (1.134) and (1.135) we find that

$$\begin{aligned}
&\sup_{R>0}\frac{1}{R^{2\alpha}}\inf_{l\in\mathrm{Span}\{1,x_1\}}\|u-\sigma_i(x_0)v_i(\cdot,a(x_0))-l\|_{B_R(x_0)}\\
\leq&(\epsilon^{2-2\alpha}+\epsilon^{2\alpha})M+(\epsilon^{\alpha-2}+\epsilon^{-(4-2\alpha)}+\epsilon^{-4}+1)\tilde{N}.
\end{aligned}\tag{1.136}$$

Using $M\sim M'$ and choosing ϵ small enough yields $M\lesssim\tilde{N}$.

After plugging in $\tilde{N}$ from Step 1 this gives

$$M \lesssim K + [a]_\alpha [u]_\alpha^{loc} + \|\sigma_i\|_\alpha N_i.$$

Using (1.107) and $[a]_\alpha \ll 1$ we then find that

$$M + [u]_\alpha^{loc} \lesssim K + \|\sigma_i\|_\alpha N_i.$$

Step 7 : (L^∞-bound on u)

To finish we show that

$$\|u\| \lesssim K + \|\sigma_i\| N_i + M. \tag{1.137}$$

To see this we first notice that by (1.118) with $T = 1$ we have that

$$\|u * \psi_1\| \lesssim K + \|\sigma_i\| N_i,$$

which we then use to bound

$$\begin{aligned}
|u(x)| &= \left| \int_{\mathbb{R}^2} u(x)\psi_1(y)\mathrm{d}y \right| \\
&\leq \left| \int_{\mathbb{R}^2} u(x-y)\psi_1(y)\mathrm{d}y \right| + \left| \int_{\mathbb{R}^2} (u(x-y) - u(x))\psi_1(y)\mathrm{d}y \right| \\
&\lesssim K + \|\sigma_i\| N_i + [u]_\alpha^{loc} \int_{\mathbb{R}^2} (|y| + |y|^\alpha)|\psi_1(y)|\mathrm{d}y \\
&\lesssim K + \|\sigma_i\| N_i + [u]_\alpha^{loc}
\end{aligned}$$

for $x \in \mathbb{R}^2$. Notice that in this calculation we have also used Step 1. □

1.3 Treatment of the Linear Problem

1.3.1 Proof of Proposition 1

As we have already mentioned in the introduction and also in Section 1.4 above, in the proof of this proposition we emulate the strategy of Otto and Weber in their proof of Proposition 1 in [47]. Aside from minor details, the main difference between the proof we present below and that in [47] is our use of the modified version of their Lemma 5 proved in the previous section.

Proof of Proposition 1. i) We start by showing existence and uniqueness for solutions of (1.82).

Step 1–(Regularization)

Assume that the conditions of Lemma 4 hold; throughout this step we adopt the notation of Lemma 4. Denote $w_{i\tau}(\cdot, a_0) = w_i(\cdot, a_0) * \psi_\tau$ for any $\tau > 0$ and *define*

$$F \diamond \partial_1^2 w_{i\tau}(\cdot, a_0) := (F \diamond \partial_1^2 w_i(\cdot, a_0))_\tau. \tag{1.138}$$

We would like to take the new reference products defined in (1.138) as input into Lemma 4 in order to obtain, for $u \in C^\alpha(\mathbb{R}^2)$ modelled after $w_{i\tau}(\cdot, a_0)$, a meaning for the singular product $F \diamond \partial_1^2 u$. To do this we must check that the family of distributions defined by (1.138) satisfies the conditions of Lemma 4. Once we have done this, we would like to further characterize the distribution $F \diamond \partial_1^2 u$. In particular, we show that if $\partial_1^2 u \in C^\alpha(\mathbb{R}^2)$, then we have that

$$F \diamond \partial_1^2 u = F \partial_1^2 u - \sigma_i E\left[F, (\cdot)_\tau\right] \diamond \partial_1^2 w_i. \tag{1.139}$$

First we check that the family of reference products given by (1.138) satisfies the assumptions of Lemma 4. We start with (1.60) for which we fix $x, z \in \mathbb{R}^2$ and $a_0 \in [\lambda, 1]$ and write

$$\begin{aligned}
&\sup_{a_0 \in [\lambda,1]} |w_{i\tau}(x, a_0) - w_{i\tau}(z, a_0)| \\
&\lesssim \sup_{a_0 \in [\lambda,1]} \left| \int_{\mathbb{R}^2} (w_i(x-y, a_0) - w_i(z-y, a_0)) \psi_\tau(y) \, \mathrm{d}y \right| \\
&\lesssim \sup_{a_0 \in [\lambda,1]} [w_i(\cdot, a_0)]_\alpha d^\alpha(x, z) \|\psi_\tau\|_{L^1(\mathbb{R}^2)}.
\end{aligned}$$

We obtain the same estimate for $\partial_{a_0} w_{i\tau}(\cdot, a_0)$ using exactly the same argument. Since we have assumed that the conditions of Lemma 4 hold, we find that

$$[w_{i\tau}(\cdot, a_0)]_{\alpha,1} \lesssim N_i.$$

It is also necessary to check that (1.61) holds. To do this, we first write

$$\begin{aligned}
&\sup_{a_0 \in [\lambda,1]} \sup_{T \le 1} (T^{\frac{1}{4}})^{2-2\alpha} \left\| [F, (\cdot)_T] \diamond \partial_1^2 w_{i\tau}(\cdot, a_0) \right\|_1 \\
&= \sup_{a_0 \in [\lambda,1]} \sup_{T \le 1} (T^{\frac{1}{4}})^{2-2\alpha} \left\| [F, (\cdot)_{T+\tau}] \diamond \partial_1^2 w_i(\cdot, a_0) \right\|_1 \\
&\lesssim \sup_{a_0 \in [\lambda,1]} \sup_{T \le 1} (T^{\frac{1}{4}})^{2-2\alpha} \Big(\left\| \left([F, (\cdot)_T] \diamond \partial_1^2 w_i(\cdot, a_0) \right)_\tau \right\|_1 \\
&\qquad + \left\| F \partial_1^2 w_{iT+\tau}(\cdot, a_0) - (F \partial_1^2 w_{iT})_\tau(\cdot, a_0) \right\|_1 \Big).
\end{aligned}$$

To bound the first term on the right-hand side by NN_i we can use (1.21) and the assumption (1.61). For the second term we may wlog assume that $\tau \leq T$ (the second line of the above calculation is symmetric in T and τ) in which case we can use (1.20) to write

$$\begin{aligned}
&\|F\partial_1^2 w_{iT+\tau}(\cdot,a_0) - (F\partial_1^2 w_{iT}(\cdot,a_0))_\tau\|_1 \\
&\lesssim [F]_\alpha \, \|\partial_1^2 w_{iT}(\cdot,a_0)\|_1 \left\| \int_{\mathbb{R}^2} |\psi_\tau(\cdot - y)| d^\alpha(\cdot,y)\, \mathrm{d}y \right\| \\
&\lesssim [F]_\alpha \, [w_i]_{\alpha,1} \, (T^{\frac{1}{4}})^{\alpha-2} (\tau^{\frac{1}{4}})^\alpha \\
&\lesssim [F]_\alpha \, N_i (T^{\frac{1}{4}})^{2\alpha-2}.
\end{aligned} \tag{1.140}$$

Combining these estimates we find that

$$\sup_{T\leq 1} (T^{\frac{1}{4}})^{2-2\alpha} \sup_{a_0\in[\lambda,1]} \left\| [F,(\cdot)_T] \diamond \partial_1^2 w_{i\tau} \right\|_1 \lesssim ([F]_\alpha + N) N_i. \tag{1.141}$$

Having verified the assumptions of Lemma 4, we then characterize the distribution $F \diamond \partial_1^2 u$ under the assumption that $\partial_1^2 u \in C^\alpha(\mathbb{R}^2)$. To obtain (1.139) we argue identically to [47] and notice that since

$$[F,(\cdot)_T] \diamond \partial_1^2 w_{i\tau}(\cdot,a_0) = [F,(\cdot)_{T+\tau}] \diamond \partial_1^2 w_i(\cdot,a_0),$$

which we have already used above, we have that $[F,(\cdot)_T] \diamond \partial_1^2 w_{i\tau}(\cdot,a_0) \to [F,(\cdot)_\tau] \diamond \partial_1^2 w_i(\cdot,a_0)$ uniformly for all $a_0 \in [\lambda,1]$ as $T \downarrow 0$. By (1.61) we find that this convergence is uniform in (x,a_0), which gives

$$\lim_{T\to 0} \|F\partial_1^2 u_T - (F\diamond \partial_1^2 u)_T - \sigma_i E\,[F,(\cdot)_\tau] \diamond \partial_1^2 w_i\| = 0$$

by using (1.62) for each $w_{i\tau}$. Since $\partial_1^2 u \in C^\alpha(\mathbb{R}^2)$ this yields

$$\lim_{T\to 0} \|F\partial_1^2 u - (F\diamond \partial_1^2 u)_T - \sigma_i E\,[F,(\cdot)_\tau] \diamond \partial_1^2 w_i\| = 0,$$

which by the uniqueness in Lemma 4 gives (1.139).

Step 2– (Analysis of the regularized problem)

In this step we show that for every $\tau \in (0,1)$ there exists a distributional solution $u^\tau \in C^{\alpha+2}(\mathbb{R}^2)$, modelled after $v_\tau(\cdot,a_0)$ according to a^{ext} and $\sigma = 1$, of the equation

$$(\partial_2 - a^{ext} \diamond \partial_1^2 + 1)u^\tau = f_\tau \qquad \text{in} \quad \mathbb{R}^2 \tag{1.142}$$

and that u^τ also classically solves

$$(\partial_2 - a^{ext}\partial_1^2 + 1)u^\tau = f_\tau - E\left[a^{ext},(\cdot)_\tau\right] \diamond \partial_1^2 v \qquad \text{in} \quad \mathbb{R}^2. \tag{1.143}$$

Notice that the statements (1.142) and (1.143) are, in fact, equivalent thanks to the previous step applied with the identifications $F = a^{ext}$, $I = 1$, $w_1(\cdot, a_0) = v(\cdot, a_0)$, and $\sigma_1 = 1$.

For now assume that $f_\tau - E\left[a^{ext}, (\cdot)_\tau\right] \diamond \partial_1^2 v \in C^\alpha(\mathbb{R}^2)$. In this case the existence of $u^\tau \in C^{\alpha+2}(\mathbb{R}^2)$ solving (1.143) is a simple consequence of standard Schauder theory. The argument for the existence of u^τ in this setting can be found in Theorem 8.7.3 of [39]. In particular, u^τ can be obtained using the standard heat-kernel formulation of solutions to the inhomogeneous heat equation with a massive term with an “initial" condition $u_\tau(x_1, -\infty) = 0$. We have actually already used this result once; in particular, see (1.120) in our proof of Lemma 5.

We must still show that $f_\tau + E[a^{ext}, (\cdot)_\tau] \diamond \partial_1^2 v \in C^\alpha(\mathbb{R}^2)$. For this we notice that $f_\tau \in C^\alpha(\mathbb{R}^2)$ since it is periodic and smooth. To see that $E\left[a^{ext}, (\cdot)_\tau\right] \diamond \partial_1^2 v \in C^\alpha(\mathbb{R}^2)$ we first remark that the L^∞-norm is bounded via the assumption (1.66). To finish we show that $[E\left[a^{ext}, (\cdot)_\tau\right] \diamond \partial_1^2 v]_\alpha^{loc} < \infty$. To begin we notice that by the definition of E we have

$$
\begin{aligned}
&\left[E\left[a^{ext}, (\cdot)_\tau\right] \diamond \partial_1^2 v\right]_\alpha^{loc} \\
&\lesssim [a^{ext}]_\alpha \left\|\left[a^{ext}, (\cdot)_\tau\right] \diamond \partial_1^2 v\right\|_1 + \sup_{a_0 \in [\lambda, 1]} \left[\left[a^{ext}, (\cdot)_\tau\right] \diamond \partial_1^2 v\right]_\alpha^{loc}.
\end{aligned} \tag{1.144}
$$

The first term on the right-hand side of (1.144) may be treated with our assumption (1.66). For the second term we write out

$$
\left[a^{ext}, (\cdot)_\tau\right] \diamond \partial_1^2 v(\cdot, a_0) = (a^{ext} \diamond \partial_1^2 v)_\tau(\cdot, a_0) - a^{ext} \partial_1^2 v_\tau(\cdot, a_0)
$$

and check that the terms on the right-hand side have a finite local C^α-seminorm. For the first term we use the assumption (1.67) and Lemma 1 to decompose $a^{ext} \diamond \partial_1^2 v = \partial_1^2 h^1 + \partial_2 h^2$ for some (h^1, h^2) that are near optimal in the sense of Definition 2. Fixing $x, z \in \mathbb{R}^2$ such that $d(x, z) \leq 1$ we integrate by parts, use that $[h^i]_\alpha^{loc} < \infty$, and apply (1.18) to write

$$
\begin{aligned}
&|(a^{ext} \diamond \partial_1^2 v)_\tau(x, a_0) - (a^{ext} \diamond \partial_1^2 v)_\tau(z, a_0)| \\
=&\left|\int_{\mathbb{R}^2} (h^1(x-y) - h^1(z-y))\partial_1^2 \psi_\tau(y)\, \mathrm{d}y \right. \\
&\qquad\qquad \left. - \int_{\mathbb{R}^2} (h^2(x-y) - h^2(z-y))\partial_2 \psi_\tau(y)\, \mathrm{d}y\right| \\
\lesssim&[a^{ext} \diamond \partial_1^2 v]_{\alpha-2}^{loc} (\tau^{\frac{1}{4}})^{-2} d^\alpha(x, z).
\end{aligned}
$$

The second term may be treated as:

$$
|a^{ext}(x)\partial_1^2 v_\tau(x, a_0) - a^{ext}(z)\partial_1^2 v_\tau(z, a_0)|
$$

$$\begin{aligned}
&\lesssim |a^{ext}(x) - a^{ext}(z)| \|\partial_1^2 v_\tau(\cdot, a_0)\| \\
&\quad + \|a^{ext}\| \int_{\mathbb{R}^2} |v(x-y, a_0) - v(z-y, a_0)| \, |\partial_1^2 \psi_\tau(y)| \, \mathrm{d}y \\
&\lesssim \|a^{ext}\|_\alpha [v(\cdot, a_0)]_\alpha (\tau^{\frac{1}{4}})^{-2} d^\alpha(x, z). \\
&\lesssim \|a^{ext}\|_\alpha N_0 (\tau^{\frac{1}{4}})^{-2} d^\alpha(x, z),
\end{aligned}$$

where we have used (1.20), (1.18), and (1.67).

Thanks to the high regularity of the $u^\tau \in C^{\alpha+2}(\mathbb{R}^2)$, we know that they are modelled after $v_\tau(\cdot, a_0)$ according to a and $\sigma = 1$.

Step 3– (Passing to the limit in the regularization)

In order to pass to the limit $\tau \to 0$ we apply Lemma 5 to the u^τ with $I = 1$, $f_1 = f_\tau$, $a = a^{ext}$, and $\sigma_1 = 1$. In our current setting N_1 from Lemma 5 is equal to N_0 from assumption (1.64). We must check that the u^τ are approximate solutions in the appropriate sense. To do this we start by convolving the equation (1.143) with ψ_T. This gives that

$$\begin{aligned}
&(\partial_2 - a^{ext}\partial_1^2 + 1) u_T^\tau - f_{\tau+T} \\
&= (a^{ext}\partial_1^2 u^\tau)_T - a^{ext}\partial_1^2 u_T^\tau - (E[a^{ext}, (\,\cdot\,)_\tau] \diamond \partial_1^2 v)_T \qquad \text{on} \quad \mathbb{R}^2,
\end{aligned}$$

where the right-hand side can be rewritten as

$$(a^{ext}\partial_1^2 u^\tau)_T - a^{ext}\partial_1^2 u_T^\tau - (E[a^{ext}, (\,\cdot\,)_\tau] \diamond \partial_1^2 v)_T = [a^{ext}, (\cdot)_T] \diamond \partial_1^2 u^\tau$$

by Step 1 and the regularity of the u^τ. By assumption (1.66), (1.141), (1.63) we then find that

$$\sup_{T \le 1} \|[a^{ext}, (\cdot)_T] \diamond \partial_1^2 u^\tau\| \lesssim [a^{ext}]_\alpha M_\tau + N_0([a^{ext}]_\alpha + N),$$

where M_τ refers to modelling of u^τ after $v_\tau(\cdot, a_0)$. So, for large enough $c \in \mathbb{R}$ we can set $K = c([a^{ext}]_\alpha M_\tau + N_0([a^{ext}]_\alpha + N)$.

After applying Lemma 5 to each u^τ and using that $[a^{ext}]_\alpha \ll 1$, we find that

$$M_\tau + \|u^\tau\|_\alpha \lesssim N_0(N+1). \tag{1.145}$$

This implies that the u^τ are uniformly bounded and equicontinuous, which means that up to a subsequence $u^\tau \to u$ uniformly as $\tau \to 0$. In order to see that the limiting $u \in C^\alpha(\mathbb{R}^2)$ solves (1.82) we must pass to the limit in (1.142). For this, we first notice that $f_\tau \rightharpoonup f$ and $\partial_2 u^\tau \rightharpoonup \partial_2 u$ distributionally. This only leaves us to check that $a^{ext} \diamond \partial_1^2 u^\tau \rightharpoonup a^{ext} \diamond \partial_1^2 u$, where $a^{ext} \diamond \partial_1^2 u$ is defined via Lemma 4 using the modelling of u after $v(\cdot, a_0)$ according to a^{ext} and $\sigma = 1$. This limiting modelling is the result of Definition 1 in tandem with the uniform in (x, a_0) convergence $v_\tau(\cdot, a_0) \to v(\cdot, a_0)$ and $u^\tau \to u$. Using (1.63) of Lemma 4 and (1.145) with (1.20), $\|a^{ext}\| \leq 1$, and (1.22), we find that the $a^{ext} \diamond \partial_1^2 u^\tau$ have uniformly bounded local $C^{\alpha-2}$ seminorm. This implies that $a^{ext} \diamond \partial_1^2 u^\tau \rightharpoonup h$ for some limiting distribution h.

Towards identifying the limit h, we remark that since

$$[a^{ext}, (\cdot)_T] \diamond \partial_1^2 v_\tau = [a^{ext}, (\cdot)_{T+\tau}] \diamond \partial_1^2 v,$$

and

$$a^{ext} \diamond \partial_1^2 v_\tau(\cdot, a_0) \rightharpoonup a^{ext} \diamond \partial_1^2 v(\cdot, a_0),$$

by the definition of $a^{ext} \diamond \partial_1^2 v_\tau(\cdot, a_0)$, we have that $[a^{ext}, (\cdot)_T] \diamond \partial_1^2 v_\tau \to [a^{ext}, (\cdot)_T] \diamond \partial_1^2 v$ as $\tau \to 0$ in a pointwise sense. After noticing that $\|[a^{ext}, (\cdot)_{T+\tau}] \diamond \partial_1^2 v\|_1$ is uniformly bounded this becomes $E[a^{ext}, (\cdot)_T] \diamond \partial_1^2 v_\tau \to E[a^{ext}, (\cdot)_T] \diamond \partial_1^2 v$. Combining our previous observations with the uniform bound

$$\|(a^{ext} \diamond \partial_1^2 u^\tau)_T - a^{ext} \partial_1^2 u_T - E[a^{ext}, (\cdot)_T] \diamond \partial_1^2 v_\tau\| \leq (T^{\frac{1}{4}})^{2\alpha-2} N_0(N+1),$$

which is a result of the triangle inequality along with (1.63), the uniform bounds on $\sup_{a_0 \in [\lambda,1]} \|v_\tau(\cdot, a_0)\|_{\alpha,1}$ and $\sup_{a_0 \in [\lambda,1]} \|[a^{ext}, (\cdot)_T] \partial_1^2 v_\tau(\cdot, a_0)\|_{\alpha,1}$, and (1.145), we see that for $T \in (0, 1]$ the convergence

$$\begin{aligned}(a^{ext} \diamond \partial_1^2 u^\tau)_T - a^{ext} \partial_1^2 u_T - E[a^{ext}, (\cdot)_T] \diamond \partial_1^2 v_\tau \\ \overset{*}{\rightharpoonup} (h)_T - a^{ext} \partial_1^2 u_T - E[a^{ext}, (\cdot)_T] \diamond \partial_1^2 v.\end{aligned}$$

holds weak-$*$ in $L^\infty(\mathbb{R}^2)$. By the lower-semicontinuity of the L^∞-norm with respect to weak-$*$ convergence, we then know that

$$\lim_{T \to 0} \limsup_{\tau \to 0} \|(a^{ext} \diamond \partial_1^2 u^\tau)_T - a^{ext} \partial_1^2 u_T - E[a^{ext}, (\cdot)_T] \diamond \partial_1^2 v_\tau\| = 0, \quad (1.146)$$

which by Step 10 of Proposition 1 in [47] gives that $a^{ext} \diamond \partial_1^2 u^\tau = h$.

Step 4– (Uniqueness)

We argue by contradiction and assume that there are two solutions u and u' satisfying (1.82) with the desired modelling. Subtracting the two we find that the difference $u - u'$ is now trivially modelled. We would then like to show that

$$a^{ext} \diamond \partial_1^2 u - a^{ext} \diamond \partial_1^2 u' = a^{ext} \diamond \partial_1^2 (u - u'), \quad (1.147)$$

where the products on the left-hand side are obtained via Lemma 4 using the modelling after $v(\cdot, a_0)$ according to a^{ext} and $\sigma = 1$ and the product on the right-hand side is obtained via the trivial modelling. Using the definitions of $a^{ext} \diamond \partial_1^2 u$ and $a^{ext} \diamond \partial_1^2 u'$ from Lemma 4 (i. e. that (1.62) holds) and that they have the same modelling, the triangle inequality gives that

$$\lim_{T \to 0} \|(a^{ext} \diamond \partial_1^2 u)_T - (a^{ext} \diamond \partial_1^2 u')_T - (a^{ext} \diamond \partial_1^2 (u - u'))_T\| = 0, \quad (1.148)$$

which then yields (1.147).

Having shown (1.147) we know that the difference $u - u'$ solves

$$(\partial_2 - a \diamond \partial_1^2 + 1)(u - u') = 0 \qquad \text{in} \quad \mathbb{R}^2 \quad (1.149)$$

to which we may then apply Lemma 5. In particular, we let $I = 1$, $f_1(\cdot, a_0) = 0$, $\sigma_1 = 0$, and $a = a^{ext}$. Convolving the equation (1.149) with ψ_T we find that $u - u'$ is an approximate solution in the sense of Lemma 5 with $K = [a^{ext}]_\alpha [u - u']_{2\alpha}$. Applying the lemma then gives that

$$[u - u']_{2\alpha} + \|u - u'\|_\alpha \lesssim [a^{ext}]_\alpha [u - u']_{2\alpha}, \quad (1.150)$$

which implies that $[u - u']_{2\alpha} + \|u - u'\|_\alpha = 0$ since $[a^{ext}]_\alpha \ll 1$.

ii) For this part we again follow the strategy of Otto and Weber in [47], but use our modified version of Lemma 5.

Step 5– (Interpolation of the data)

We linearly interpolate the coefficients and right-hand sides; in particular, we define a_s^{ext} and f_s as

$$a_s^{ext} := a_0^{ext}(1 - s) + a_1^{ext} s \quad \text{and} \quad f_s := f_0(1 - s) + f_1 s.$$

Correspondingly, we interpolate the solutions of (1.4) and set

$$v_s = v_0(1 - s) + v_1 s.$$

Notice that in order keep notation lean, in this section we occasionally suppress the dependence of v_s, v_1, and v_0 on the parameter a_0. Of course, $v_s(\cdot, a_0)$ solves (1.4) with right-hand side f_s. To make sure that Leibniz' rule is satisfied in s, i. e. that

$$\partial_s(a_s^{ext} \diamond \partial_1^2 v_s) = \partial_s a_s^{ext} \diamond \partial_1^2 v_s + a_s^{ext} \diamond \partial_1^2 \partial_s v_s, \quad (1.151)$$

we bi-linearly interpolate the offline products and define

$$a_s^{ext} \diamond \partial_1^2 v_s$$
$$:= (s-1)^2 a_0^{ext} \diamond \partial_1^2 v_0 + s(1-s)(a_0^{ext} \diamond \partial_1^2 v_1 + a_1^{ext} \diamond \partial_1^2 v_0) + s^2 a_1^{ext} \diamond \partial_1^2 v_1.$$

Using the triangle inequality and the uniqueness in Lemma 4 we find that

$$\begin{aligned} a_s^{ext} \diamond \partial_1^2 \partial_s v_s &= a_s^{ext} \diamond \partial_1^2 v_0 - a_s^{ext} \diamond \partial_1^2 v_1, \\ \partial_s a_s^{ext} \diamond \partial_1^2 v_s &= a_0^{ext} \diamond \partial_1^2 v_s - a_1^{ext} \diamond \partial_1^2 v_s, \\ \text{and} \quad a_s^{ext} \diamond \partial_1^2 \partial_{a_0} v_s &= \partial_{a_0} a_s^{ext} \diamond \partial_1^2 v_s, \end{aligned} \tag{1.152}$$

which along with the assumptions (1.66), (1.75), and (1.76) give that

$$\sup_{T\le 1} (T^{\frac{1}{4}})^{2-2\alpha} \|[a_s^{ext}, (\cdot)_T] \diamond \partial_1^2 \partial_s v_s\|_1 \lesssim N\delta N_0, \tag{1.153}$$

$$\sup_{T\le 1} (T^{\frac{1}{4}})^{2-2\alpha} \|[a_s^{ext}, (\cdot)_T] \diamond \partial_1^2 \partial_{a_0} v_s\|_1 \lesssim N N_0, \tag{1.154}$$

$$\text{and} \quad \sup_{T\le 1} (T^{\frac{1}{4}})^{2-2\alpha} \|[\partial_s a_s^{ext}, (\cdot)_T] \diamond \partial_1^2 v_s\|_1 \lesssim \delta N N_0. \tag{1.155}$$

As a last remark concerning new reference products, for $\tau > 0$ we can regularize all of the new reference products as in part *i)* and define, e. g. ,

$$a_s^{ext} \diamond \partial_1^2 v_{s\tau} := (a_s^{ext} \diamond \partial_1^2 v_s)_\tau. \tag{1.156}$$

Step 6– (A continuous curve of solutions u_s^τ and an equation for $\partial_s u_s^\tau$)

Using the same method as in part *i)* we find that for every $\tau > 0$ there exists a curve of $C^{\alpha+2}$ solutions u_s^τ for $s \in [0,1]$ of

$$(\partial_2 - a_s^{ext} \partial_1^2 + 1) u_s^\tau = f_{s\tau} - E_s[a_s^{ext}, (\cdot)_\tau] \diamond \partial_1^2 v_s \qquad \text{on} \quad \mathbb{R}^2, \tag{1.157}$$

where E_s denotes evaluation of a function of (x, a_0) at $(x, a_s^{ext}(x))$ and $f_{s\tau} = f_s * \psi_\tau$. Using the modelling of the u_s^τ after $v_{s\tau}$, where $v_{s\tau} = v_s * \psi_\tau$, according to a_s^{ext} and $\sigma = 1$ we find that by Step 1 of part *i)* the relation

$$a_s^{ext} \diamond \partial_1^2 u_s^\tau = a_s^{ext} \partial_1^2 u_s^\tau - E_s[a_s^{ext}, (\cdot)_\tau] \diamond \partial_1^2 v_s \tag{1.158}$$

holds. This allows us to rewrite (1.157) as

$$(\partial_2 - a_s^{ext} \diamond \partial_1^2 + 1) u_s^\tau = f_{s\tau} \qquad \text{on} \quad \mathbb{R}^2. \tag{1.159}$$

We would then like to differentiate (1.157) in terms of s in order to obtain an equation for $\partial_s u_s^\tau$. For this we notice that by (1.151) we have

$$\begin{aligned} &\partial_s E_s[a_s^{ext}, (\cdot)_\tau] \diamond \partial_1^2 v_s \\ =&E_s[\partial_s a_s^{ext}, (\cdot)_\tau] \diamond \partial_1^2 v_s + E_s[a_s^{ext}, (\cdot)_\tau] \diamond \partial_1^2 \partial_s v_s \\ &+ \partial_s a_s^{ext} E_s[a_s^{ext}, (\cdot)_\tau] \diamond \partial_1^2 \partial_{a_0} v_s, \end{aligned} \tag{1.160}$$

which means that $\partial_s u_s^\tau$ solves

$$(\partial_2 - a_s^{ext}\partial_1^2 + 1)\partial_s u_s^\tau - \Big(\partial_s f_{s\tau} + \partial_s a_s^{ext}\partial_1^2 u_s^\tau - E_s[\partial_s a_s^{ext}, (\cdot)_\tau] \diamond \partial_1^2 v_s$$
$$-E_s[a_s^{ext}, (\cdot)_\tau] \diamond \partial_1^2\partial_s v_s - \partial_s a_s^{ext} E_s[a_s^{ext}, (\cdot)_\tau] \diamond \partial_1^2 \partial_{a_0} v_s\Big) = 0 \qquad \text{on} \quad \mathbb{R}^2.$$

By standard Schauder theory this implies that $\partial_s u_s^\tau \in C^{\alpha+2}(\mathbb{R}^2)$ since the term in parentheses is in $C^\alpha(\mathbb{R}^2)$, which can be checked using the same tools as in Step 2. Due to the high regularity of $\partial_s u_s^\tau$ we know that it is modelled after $(\partial_s v_{s\tau}, \partial_{a_0} v_{s\tau})$ according to a_s^{ext} and $(1, \partial_s a_s^{ext})$. Using the identities

$$\partial_s a_s^{ext}\partial_1^2 u_s^\tau - E_s[\partial_s a_s^{ext}, (\cdot)_\tau] \diamond \partial_1^2 v_s = \partial_s a_s^{ext} \diamond \partial_1^2 u_s^\tau \tag{1.161}$$

and

$$\begin{aligned} &a_s^{ext}\partial_1^2\partial_s u_s^\tau - E_s[a_s^{ext}, (\cdot)_\tau] \diamond \partial_1^2\partial_s v_s - \partial_s a_s^{ext} E_s[a_s^{ext}, (\cdot)_\tau] \diamond \partial_1^2 \partial_{a_0} v_s \\ &= a_s^{ext} \diamond \partial_1^2 \partial_s u_s^\tau, \end{aligned} \tag{1.162}$$

which both follow from Step 1 given the high regularity of $\partial_s u_s^\tau$ and u_s^τ, we can rewrite the equation solved by $\partial_s u_s^\tau$ as

$$(\partial_2 - a_s^{ext} \diamond \partial_1^2 + 1)\partial_s u_s^\tau = \partial_s f_{s\tau} + \partial_s a_s^{ext} \diamond \partial_1^2 u_s^\tau \qquad \text{on} \quad \mathbb{R}^2. \tag{1.163}$$

Step 7– (Estimates for $\partial_s u_s^\tau$)

We now apply Lemma 5 to $\partial_s u_s^\tau$ with $I = 2$ and $f_1(\cdot, a_0) = \partial_s f_{s\tau}$, $\sigma_1 = 1$, $f_2(\cdot, a_0) = a_0 \partial_1^2 v_{s\tau}(\cdot, a_0)$, $\sigma_2 = \partial_s a_s^{ext}$, and $a = a_s^{ext}$. Notice that there is now a slight ambiguity in our notation since we use f_1 to denote the second forcing (f_1 in the assumptions of the proposition) and also for its identification in Lemma 5 (i.e. $\partial_s f_{s\tau}$). It is always clear which f_1 is meant.

To apply the lemma we first identify the constants N_1 and N_2 in (1.90). For N_1 we notice that since $\partial_s f_{s\tau} = f_{0\tau} - f_{1\tau}$ the relation

$$\sup_{T\le 1}(T^{\frac14})^{2-\alpha}\|(\partial_s f_{s\tau})_T\| \lesssim \sup_{T\le 1}(T^{\frac14})^{2-\alpha}\|(f_0 - f_1)_T\| \lesssim \delta N_0 \tag{1.164}$$

holds by (1.21). For N_2 we use (1.322), that $v_{s\tau}(\cdot, a_0)$ solves (1.4) with right-hand side $f_{s\tau}$, and (1.21) along with (1.20) to write

$$\sup_{a_0\in[\lambda,1]}\sup_{T\le 1}(T^{\frac14})^{2-\alpha}\|(a_0\partial_1^2 v_{s\tau}(\cdot, a_0))_T\| \lesssim [f_s]_{\alpha-2}\|\psi_\tau\|_{L^1(\mathbb{R}^2)} \lesssim N_0. \tag{1.165}$$

So, for our application of Lemma 5 for a large enough constant $c \in \mathbb{R}$ we can set $N_1 = c\delta N_0$ and $N_2 = cN_0$.

We then check that $\partial_s u_s^\tau$ is an approximate solution in the sense (1.91). For this we convolve (1.163) with ψ_T and notice that the calculation

$$\begin{aligned}
&\sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\|[a_s^{ext},(\cdot)_T]\diamond\partial_1^2\partial_s u_s^\tau + [\partial_s a_s^{ext},(\cdot)_T]\diamond\partial_1^2 u_s^\tau\| \\
&\leq \sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\|[a_s^{ext},(\cdot)_T]\diamond\partial_1^2\partial_s u_s^\tau\| \\
&\quad + \sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\|[\partial_s a_s^{ext},(\cdot)_T]\diamond\partial_1^2 u_s^\tau\| \\
&\lesssim [a_s^{ext}]_\alpha \delta M_s^\tau + \delta N N_0 + N\delta N_0 + [a_0^{ext} - a_1^{ext}]_\alpha M_s^\tau,
\end{aligned} \tag{1.166}$$

which follows from (1.63) of Lemma 4 and (1.153) and (1.154), is sufficient. Notice that for this we have also used that $N \leq 1$ and $\|a_1^{ext} - a_0^{ext}\|_\alpha \leq \delta N$. Here δM_s^τ belongs to the modelling of $\partial_s u_s^\tau$ after $(\partial_s v_{s\tau}, \partial_{a_0} v_{s\tau})$ according to a_s^{ext} and $(1, \partial_s a_s^{ext})$ and M_s^τ belongs to the modelling of u_s^τ after $v_{s\tau}$ according to a_s^{ext} and $\sigma = 1$. Since an application of the bounds obtained in part *i)* yields that $M_s^\tau \lesssim N_0(N+1)$, for large enough $c \in \mathbb{R}$ we can set

$$K = c([a_s^{ext}]_\alpha \delta M_s^\tau + \delta N(N_0+1) + N\delta N_0) \tag{1.167}$$

We can now apply Lemma 5 to the $\partial_s u_s^\tau$, which after using that $[a_s^{ext}]_\alpha \ll 1$, gives

$$\delta M_s^\tau + \|\partial_s u_s^\tau\|_\alpha \lesssim \delta N(N_0+1) + (N+1)\delta N_0. \tag{1.168}$$

Step 8–(Integration and passing to the limit)

Since we have (1.168) for all $s \in [0,1]$ we may integrate it up to obtain

$$\|u_1^\tau - u_0^\tau\|_\alpha \lesssim \left\|\int_0^1 \partial_s u_s^\tau \, \mathrm{d}s\right\|_\alpha \lesssim \delta N(N_0+1) + (N+1)\delta N_0. \tag{1.169}$$

To obtain a bound for δM^τ we notice that

$$\begin{aligned}
&\partial_s(u_s^\tau(y) - v_{s\tau}(y, a_s^{ext}(x))) \\
&= \partial_s u_s^\tau(y) - \partial_s v_{s\tau}(y, a_s^{ext}(x)) - \partial_s a_s^{ext}(x)\partial_{a_0} v_s(y, a_s^{ext}(x)),
\end{aligned} \tag{1.170}$$

which allows us to integrate up our bound on δM_s^τ to give that $u_1^\tau - u_0^\tau$ is modelled after $(v_{1\tau}, v_{0\tau})$ according to (a_1^{ext}, a_0^{ext}) and $(1,-1)$ with $\nu = \int_0^1 \nu_s \mathrm{d}s$. Here ν_s comes from the modelling of $\partial_s u_s^\tau$. We find that

$$\delta M^\tau \lesssim \delta N(N_0+1) + (N+1)\delta N_0. \tag{1.171}$$

We have already show in part *i)* that $u_i^\tau \to u_i$ uniformly, which allows us to pass to the limit in (1.169). In order to pass to the limit in the modelling we, furthermore, use that $v_{i\tau}(\cdot, a_i(\cdot)) \to v_i(\cdot, a_i(\cdot))$ uniformly. □

1.3.2 A Technical Lemma: Post-Processing of the Modelling

In order to obtain the correct modelling in Proposition 2 and in Theorem 1 we need the following technical lemma.

Lemma 6. *Let $\alpha \in (0,1)$.*

i) Assume the $a, a' \in C^\alpha(\mathbb{R}^2)$ so that $a = a'$ on the axis $\{x_2 = 0\}$. Then for $\tilde{V}(\cdot, a_0)$, the even-reflection of $V(\cdot, a_0)$ solving (1.3), and any distinct points $x, y \in \mathbb{R}^2$ we have that

$$\begin{aligned}&\left|\tilde{V}(x, a(y)) - \tilde{V}(y, a(y)) - (\tilde{V}(x, a'(y)) - \tilde{V}(y, a'(y)))\right| \\ &\lesssim \|V_{int}\|_\alpha([a]_\alpha + [a']_\alpha) d^{2\alpha}(x,y).\end{aligned} \tag{1.172}$$

ii) Assume that $a, a' \in C^\alpha(\mathbb{R}^2)$ with $\|a\|$ and $\|a'\| \leq 1$ and $V_{int} \in C^\alpha(\mathbb{R})$. Let $U \in C^\alpha(\mathbb{R}^2)$ be modelled after $\tilde{V}(\cdot, a_0)$ according to a with modelling constant M. Under these assumptions we find that if $a' = a$ on the axis $\{x_2 = 0\}$, then U is modelled after $\tilde{V}(\cdot, a_0)$ according to a' with modelling constant M' bounded as

$$M' \lesssim M + \|V_{int}\|_\alpha. \tag{1.173}$$

iii) For $i = 0, 1$ we have $a_i, a_i' \in C^\alpha(\mathbb{R}^2)$ with $\|a_i\|$ and $\|a_i'\| \leq 1$ and $V_{int,i} \in C^\alpha(\mathbb{R})$. We, furthermore, assume that $U \in C^\alpha(\mathbb{R}^2)$ is modelled after $(\tilde{V}_1(\cdot, a_0), \tilde{V}_0(\cdot, a_0))$ according to (a_1, a_0) and $(1, -1)$ with modelling constant δM. We then find that if $a_i' = a_i$ on the axis $\{x_2 = 0\}$, then U is modelled after $(\tilde{V}_1(\cdot, a_0), \tilde{V}_0(\cdot, a_0))$ according to (a_1', a_0') and $(1, -1)$ with modelling constant $\delta M'$ bounded as

$$\delta M' \lesssim \delta M + \|V_{int,1} - V_{int,0}\|_\alpha + \max_i \|V_{int,i}\|_\alpha(\|a_1 - a_0\|_\alpha + \|a_1' - a_0'\|_\alpha). \tag{1.174}$$

Proof. i) Notice that

$$\begin{aligned}&|\tilde{V}(x, a(y)) - \tilde{V}(y, a(y)) - (\tilde{V}(x, a'(y)) - \tilde{V}(y, a'(y)))| \\ &\lesssim \sup_{a_0 \in [\lambda, 1]} |\partial_{a_0}(\tilde{V}(x, a_0) - \tilde{V}(y, a_0))||a(y) - a'(y)|\end{aligned} \tag{1.175}$$

and bound the right-hand side in two ways as

$$\begin{aligned}
&\sup_{a_0\in[\lambda,1]} |\partial_{a_0}(\tilde{V}(x,a_0)-\tilde{V}(y,a_0))||a(y)-a'(y)| \\
&\lesssim \begin{cases} \|V_{int}\|_\alpha([a]_\alpha+[a']_\alpha)|y_2|^{\frac{\alpha}{2}}(|x_2|^{-\frac{\alpha}{2}}+|y_2|^{-\frac{\alpha}{2}})d^{2\alpha}(x,y) \\ \|V_{int}\|_\alpha([a]_\alpha+[a']_\alpha)|y_2|^{\frac{\alpha}{2}}d^{\alpha}(x,y). \end{cases}
\end{aligned} \tag{1.176}$$

Here, the first bound follows from (1.55) applied with $j=1$ to $V(\cdot,a_0)$ and the observation that $|a(y)-a'(y)|\lesssim([a]_\alpha+[a']_\alpha)|y_2|^{\frac{\alpha}{2}}$. The second bound in (1.176) follows from the same bound applied to $|a(y)-a'(y)|$, but (1.54) with $j=1$ applied to $V(\cdot,a_0)$. We now consider two cases: $|y_2|\leq 2|x_2|$ and $2|x_2|\leq|y_2|$.

Case 1– Assume that $|y_2|\leq 2|x_2|$. Then the first bound of (1.176) immediately gives

$$\begin{aligned}
&\sup_{a_0\in[\lambda,1]} |\partial_{a_0}(\tilde{V}(x,a_0)-\tilde{V}(y,a_0))||a(y)-a'(y)| \\
&\lesssim \|V_{int}\|_\alpha([a]_\alpha+[a']_\alpha)d^{2\alpha}(x,y).
\end{aligned}$$

Case 2– Assume that $|y_2|\geq 2|x_2|$. Then it is clear that $\frac{y_2}{2}\leq|x_2-y_2|$, which when combined with the second bound in (1.176) gives

$$\begin{aligned}
&\sup_{a_0\in[\lambda,1]} |\partial_{a_0}(\tilde{V}(x,a_0)-\tilde{V}(y,a_0))||a(y)-a'(y)| \\
&\lesssim \|V_{int}\|_\alpha([a]_\alpha+[a']_\alpha)d^{2\alpha}(x,y).
\end{aligned}$$

Together the two cases yield (1.172).

ii) This part is an easy corollary of part *i*). In particular, using (1.2) and the triangle inequality we write

$$\begin{aligned}
&|U(x)-U(y)-(\tilde{V}(x,a'(y))-\tilde{V}(y,a'(y)))| \\
&\lesssim Md^{2\alpha}+|\tilde{V}(x,a(y))-\tilde{V}(y,a(y))-(\tilde{V}(x,a'(y))-\tilde{V}(y,a'(y)))| \\
&\lesssim (M+\|V_{int}\|_\alpha)d^{2\alpha}(x,y).
\end{aligned}$$

iii) Now we use the Einstein summation convention. We apply the triangle inequality and for $x, y \in \mathbb{R}^2$ write

$$\begin{aligned}
&\Big|U(x) - U(y) - (-1)^{i+1}(\tilde{V}_i(x, a_i'(y)) - \tilde{V}(y, a_i'(y)))\Big| \\
&\lesssim \delta M d^{2\alpha}(x, y) \\
&+ |\tilde{V}_0(x, a_0(y)) - \tilde{V}_0(y, a_0(y)) - (\tilde{V}_0(x, a_0'(y)) - \tilde{V}_0(y, a_0'(y))) \\
&\quad - (\tilde{V}_1(x, a_1(y)) - \tilde{V}_1(y, a_1(y)) - (\tilde{V}_1(x, a_1'(y)) - \tilde{V}_1(y, a_1'(y))))|.
\end{aligned} \tag{1.177}$$

We now require some new notation. In particular, we let

$$a_i^t = t a_i + (1-t)a_i' \text{ for } i = 0, 1 \text{ and } a_s^t = s a_1^t + (1-s)a_0^t$$

and notice that

$$\begin{aligned}
\partial_s a_s^t &= a_1^t - a_0^t, \\
\partial_t a_s^t &= s(a_1 - a_1') + (1-s)(a_0 - a_0'), \\
\text{and} \quad \partial_t \partial_s a_s^t &= a_1 - a_0 - (a_1' - a_0').
\end{aligned} \tag{1.178}$$

We then calculate

$$\begin{aligned}
&\Big|\tilde{V}_1(x, a_1(y)) - \tilde{V}_1(x, a_1'(y)) - (\tilde{V}_0(x, a_0(y)) - \tilde{V}_0(x, a_0'(y))) \\
&\quad - \Big(\tilde{V}_1(y, a_1(y)) - \tilde{V}_1(y, a_1'(y)) - (\tilde{V}_0(y, a_0(y)) - \tilde{V}_0(y, a_0'(y)))\Big)\Big| \\
&= \left|\int_0^1 \int_0^1 \partial_s \partial_t (\tilde{V}_s(x, a_s^t(y)) - \tilde{V}_s(y, a_s^t(y)))\, \mathrm{d}s\, \mathrm{d}t\right| \\
&= \left|\int_0^1 \int_0^1 \partial_s ((\partial_{a_0} \tilde{V}_s(x, a_s^t(y)) - \partial_{a_0} \tilde{V}_s(y, a_s^t(y)))\partial_t a_s^t(y)) \mathrm{d}s\, \mathrm{d}t\right| \\
&\leq \int_0^1 \int_0^1 \Big(|\partial_{a_0}(\tilde{V}_1 - \tilde{V}_0)(x, a_s^t(y)) - \partial_{a_0}(\tilde{V}_1 - \tilde{V}_0)(y, a_s^t(y))|\, |\partial_t a_s^t(y)| \\
&\qquad + |\partial_{a_0}^2 \tilde{V}_s(x, a_s^t(y)) - \partial_{a_0}^2 \tilde{V}_s(y, a_s^t(y))|\, |\partial_t a_s^t(y)|\, |\partial_s a_s^t(y)| \\
&\qquad + |\partial_t \partial_s a_s^t(y)|\, |\partial_{a_0} \tilde{V}_s(x, a_s^t(y)) - \partial_{a_0} \tilde{V}_s(y, a_s^t(y))|\Big)\, \mathrm{d}s\, \mathrm{d}t.
\end{aligned} \tag{1.179}$$

To finish we bound the three terms on the right-hand side. Using the relations (1.178), the three terms are treated in a manner very similar to part i) above. In particular, for the first term using the exact same argument as in part i) yields that

$$\begin{aligned}
&|\partial_{a_0}(\tilde{V}_1 - \tilde{V}_0)(x, a_s^t(y)) - \partial_{a_0}(\tilde{V}_1 - \tilde{V}_0)(y, a_s^t(y))|\, |\partial_t a_s^t(y)| \\
&\lesssim \|V_{int,1} - V_{int,0}\|_\alpha d^{2\alpha}(x, y),
\end{aligned}$$

where we have used (1.54) and (1.55) with $j = 1$ applied to $(V_1 - V_0)(\cdot, a_0)$ and that $|a_s(y) - a'_s(y)| \lesssim |y_2|^{\frac{\alpha}{2}}$. For the second term of (1.179) we again use the same strategy as in part $i)$ with (1.54) and (1.55) with $j = 2$ applied to $V_s(\cdot, a_0)$ and the observation that $|\partial_t a_s^t(y)| \lesssim |y_2|^{\frac{\alpha}{2}}$. Additionally using that $\|\partial_s a_s^t(y)\| \lesssim \|a_1 - a_0\| + \|a'_1 - a'_0\|$, we obtain

$$\begin{aligned}&|\partial_{a_0}^2 \tilde{V}_s(x, a_s^t(y)) - \partial_{a_0}^2 \tilde{V}_s(y, a_s^t(y))|\, |\partial_t a_s^t(y)|\, |\partial_s a_s^t(y)| \\ &\lesssim \|V_{int,s}\|_\alpha (\|a_1 - a_0\| + \|a'_1 - a'_0\|) d^{2\alpha}(x, y).\end{aligned}$$

Getting to the last term of (1.179) we again use the same strategy, but this time to set-up we use

$$|(a_1 - a_0)(y) - (a'_1 - a'_0)(y)| \lesssim ([a_1 - a_0]_\alpha + [a'_1 - a'_0]_\alpha)|y_2|^{\frac{\alpha}{2}}$$

and either (1.55) with $j = 1$ and $k = 0$ applied to $V_s(\cdot, a_0)$ or (1.54) with $j = 1$. We then get that

$$\begin{aligned}&|\partial_t \partial_s a_s^t(y)|\, |\partial_{a_0} V_s(x, a_s^t(y)) - \partial_{a_0} V_s(y, a_s^t(y))| \\ &\lesssim \|V_{int,s}\|_\alpha ([a_1 - a_0]_\alpha + [a'_1 - a'_0]_\alpha) d^{2\alpha}(x, y),\end{aligned}$$

Combining these estimates finishes our argument.

□

1.3.3 Proof of Proposition 2

Proof of Proposition 2. The main idea of this proof is to postulate an ansatz for U, i.e. the function q as defined in (1.9), and then show that, by correcting q with some $w \in C^{2\alpha}(\mathbb{R}^2_+)$ such that $w = 0$ on $\partial\mathbb{R}^2_+$, we can set $U = q + w$.

i) As in Proposition 1 we start with existence and uniqueness, where now "uniqueness" means that the correction w is uniquely determined.

Step 1– (Modelling of q)

We first show that $\tilde{q}$ is modelled after $\tilde{V}(\cdot, a_0)$ according to a^{ext} on $\mathbb{R}^2$. For our argument we fix $x, y \in \mathbb{R}^2$ and use the notation $\tilde{x} = (x_1, |x_2|)$. The triangle inequality gives

$$\begin{aligned}&|V(\tilde{x}, \bar{a}(\tilde{x})) - V(\tilde{y}, \bar{a}(\tilde{y})) - (V(\tilde{x}, a^{ext}(y)) - V(\tilde{y}, a^{ext}(y)))| \\ \leq &|V(\tilde{x}, \bar{a}(\tilde{x})) - V(\tilde{x}, \bar{a}(\tilde{y}))| \\ &+ |V(\tilde{x}, \bar{a}(\tilde{y})) - V(\tilde{y}, \bar{a}(\tilde{y})) - (V(\tilde{x}, a^{ext}(y)) - V(\tilde{y}, a^{ext}(y)))|.\end{aligned} \tag{1.180}$$

To obtain the modelledness result we now use Lemma 3 and Lemma 6. The bounds from Lemma 3 in the versions mentioned in $v)$ are applicable to $\bar{a}$

with the identifications $a_0 = 1$ and $V_{int} = a$ thanks to (1.10). Also, part $i)$ of Lemma 6 is applicable with the identifications $a = \tilde{\bar{a}}$ and $a' = a^{ext}$. This immediately allows us to bound the second term of (1.180) as

$$\begin{aligned}&|V(\tilde{x}, \bar{a}(\tilde{y})) - V(\tilde{y}, \bar{a}(\tilde{y})) - (V(\tilde{x}, a^{ext}(y)) - V(\tilde{y}, a^{ext}(y)))|\\ &\lesssim \|V_{int}\|_\alpha([a^{ext}]_\alpha + [\bar{a}]_\alpha)d^{2\alpha}(x, y)\\ &\lesssim \|V_{int}\|_\alpha[a^{ext}]_\alpha d^{2\alpha}(x, y).\end{aligned} \tag{1.181}$$

For the first term on the right-hand side of (1.180) we write

$$\begin{aligned}&|V(\tilde{x}, \bar{a}(\tilde{x})) - V(\tilde{x}, \bar{a}(\tilde{y}))|\\ &\lesssim \sup_{a_0\in[\lambda,1]} |\partial_{a_0}V(\tilde{x}, a_0)|\,|\bar{a}(\tilde{y}) - \bar{a}(\tilde{x})|\\ &\lesssim \begin{cases}[V_{int}]_\alpha[a]_\alpha|x_2|^{\frac{\alpha}{2}}(|x_2|^{-\frac{\alpha}{2}} + |y_2|^{-\frac{\alpha}{2}})d^{2\alpha}(\tilde{x}, \tilde{y})\\ [V_{int}]_\alpha[a]_\alpha|x_2|^{\frac{\alpha}{2}}d^{\alpha}(\tilde{x}, \tilde{y}).\end{cases}\end{aligned} \tag{1.182}$$

Here, the first bound follows from (1.55) applied with $j = 0$ to $\bar{a}$ and (1.52) applied with $j = 1$ to $V(\cdot, a_0)$; the second comes from (1.54) applied to $\bar{a}$ and the same bound applied to $V(\cdot, a_0)$. Arguing exactly as in part $i)$ of Lemma 6 with the additional ingredient that $d(\tilde{x}, \tilde{y}) \lesssim d(x, y)$ we obtain that

$$|V(\tilde{x}, \bar{a}(\tilde{x})) - V(\tilde{x}, \bar{a}(\tilde{y}))| \lesssim \|V_{int}\|_\alpha[a]_\alpha d^{2\alpha}(x, y). \tag{1.183}$$

Combining (1.180), (1.181), and (1.183) we then find that $\tilde{q}$ is modelled after $\tilde{V}(\cdot, a_0)$ according to a^{ext} with modelling constant bounded as

$$M \lesssim \|V_{int}\|_\alpha[a^{ext}]_\alpha. \tag{1.184}$$

Step 2 – (Regularity for the forcing of the equation solved by w)

We now show that for every $x \in \mathbb{R}^2$ the bound

$$|(\partial_2 q - a\partial_1^2 q + q)^E(x)| \lesssim \|a\|_\alpha\|V_{int}\|_\alpha|x_2|^{\frac{2\alpha-2}{2}} \tag{1.185}$$

holds. Notice that on $\mathbb{R}^2_+$ the expression $\partial_2 q - a\partial_1^2 q + q$ is well-defined in a classical sense since q is smooth for positive times. To begin our argument for (1.185) we first apply Leibniz' rule to rewrite the expression $\partial_2 q - a\partial_1^2 q + q$ for $x \in \mathbb{R}^2_+$ as:

$$\begin{aligned}&(\partial_2 q - a\partial_1^2 q + q)(x)\\ =&\partial_2 V(x, \bar{a}(x)) + \partial_{a_0}V(x, \bar{a}(x))\partial_2\bar{a}(x) - a\partial_1^2 V(x, \bar{a}(x))\\ &- 2a\partial_1\partial_{a_0}V(x, \bar{a}(x))\partial_1\bar{a}(x) - a\partial_{a_0}V(x, \bar{a}(x))\partial_1^2\bar{a}(x)\\ &- a\partial_{a_0}^2 V(x, \bar{a}(x))(\partial_1\bar{a}(x))^2 + V(x, \bar{a}(x)).\end{aligned} \tag{1.186}$$

Thanks to (1.5) we have the relation $\partial_2 V(\cdot, a_0) = (a_0\partial_1^2 - 1)V(\cdot, a_0)$ on $\mathbb{R}^2_+$, which allows us to re-write the first term on the right-hand side of (1.186) as $\partial_2 V(x, \bar{a}(x)) = \bar{a}(x)\partial_1^2 V(x, \bar{a}(x)) - V(x, \bar{a}(x))$. In the same way, we can make the substitution $\partial_2\bar{a} = \partial_1^2\bar{a}$ using (1.10). We obtain the formal identity

$$\begin{aligned}(\partial_2 q - a\partial_1^2 q + q)(x) \\ =(\bar{a} - a)(x)\partial_1^2 V(x, \bar{a}(x)) + \partial_{a_0} V(x, \bar{a}(x))(1 - a(x))\partial_1^2\bar{a}(x) \\ - 2a(x)\partial_1\partial_{a_0} V(x, \bar{a}(x))\partial_1\bar{a}(x) - a(x)\partial_{a_0}^2 V(x, \bar{a}(x))(\partial_1\bar{a}(x))^2.\end{aligned} \tag{1.187}$$

We treat each term on the right-hand side of (1.187) separately by applying the bounds from Lemma 3.

Applying (1.53) and (1.54) both with $j = 0$ to $\bar{a}$, we find that

$$\|\bar{a}\| \lesssim \|a\| \leq 1 \quad \text{and} \quad [\bar{a}]_\alpha \lesssim [a]_\alpha \ll 1. \tag{1.188}$$

For the first term on the right-hand side of (1.187) we use that $|a(x)-\bar{a}(x)| \lesssim [a]_\alpha |x_2|^{\frac{\alpha}{2}}$, which we have already used in the previous step, combined with an application of (1.52) with $k = 2$ and $j = 0$ to $V(\cdot, a_0)$ to obtain

$$\left|(a - \bar{a})(x)\partial_1^2 V(x, \bar{a}(x))\right| \lesssim [a]_\alpha [V_{int}]_\alpha |x_2|^{\frac{2\alpha-2}{2}}. \tag{1.189}$$

For the second term we apply (1.52) with $j = 1$ and $k = 0$ to $V(\cdot, a_0)$, (1.52) with $k = 2$ and $j = 0$ to $\bar{a}$, and use (1.188) to find that

$$\left|\partial_{a_0} V(x, \bar{a}(x))(1 - a(x))\partial_1^2\bar{a}(x)\right| \lesssim [V_{int}]_\alpha [a]_\alpha |x_2|^{\frac{2\alpha-2}{2}}. \tag{1.190}$$

The third term is treated by (1.52) with $k = 1$ and $j = 1$ applied to $V(\cdot, a_0)$ and with $k = 1$ and $j = 0$ applied to $\bar{a}$, which gives

$$|a(x)\partial_{a_0}\partial_1 V(x, \bar{a}(x))\partial_1\bar{a}(x)| \lesssim [a]_\alpha [V_{int}]_\alpha |x_2|^{\frac{2\alpha-2}{2}}. \tag{1.191}$$

The last term on the right-hand side of (1.187) is handled by (1.52) with $k = 1$ and $j = 0$ applied to $\bar{a}$ and (1.53) with $j = 2$ applied to $V(\cdot, a_0)$:

$$\left|a(x)\partial_{a_0}^2 V(x, \bar{a}(x))(\partial_1\bar{a}(x))^2\right| \lesssim [a]_\alpha \|V_{int}\| |x_2|^{\frac{2\alpha-2}{2}}. \tag{1.192}$$

Together (1.189), (1.190), (1.191), and (1.192) give (1.185).

Step 3– (Construction of the correction w)

We now show that there exists $w \in C^{2\alpha}(\mathbb{R}^2_+)$ solving

$$(\partial_2 - a \diamond \partial_1^2 + 1)w = -(\partial_2 q - a\partial_1^2 q + q) \quad \text{in} \quad \mathbb{R}^2_+, \tag{1.193a}$$

$$w = 0 \quad \text{on} \quad \partial\mathbb{R}^2_+. \tag{1.193b}$$

In fact, we construct the solution w of (1.193) as a $C^{2\alpha}$-solution of

$$(\partial_2 - a^{ext} \diamond \partial_1^2 + 1)w = -(\partial_2 q - a\partial_1^2 q + q)^E \qquad \text{in} \quad \mathbb{R}^2 \tag{1.194}$$

and then show that $w|_{\mathbb{R}^2_-} = 0$. The construction of the correction w follows essentially the same procedure as our argument for the existence of a solution u of (1.82) in Proposition 1. In the current context, however, the argument from the previous proposition sees some simplification due to the higher regularity available here.

Step 3.1– (A specific form of the singular product) Let $u \in C^{2\alpha}(\mathbb{R}^2)$ and satisfy $\partial_1^2 u \in C^\alpha(\Omega)$ for $\Omega \subseteq \mathbb{R}^2$. Using the same argument as in Step 1 of Proposition 1, we find that the singular product $a \diamond \partial_1^2 u$ obtained using the trivial modelling of u via Lemma 4 coincides with the classical product on Ω.

Step 3.2– (Hölder bounds for g_τ) We now use the notation $g = -(\partial_2 q - a\partial_1^2 q + q)^E$ and $\mathbb{R}^2_L = \mathbb{R} \times (-\infty, -L]$ for $L > 0$. In the current step we let $\tau \in (0,1)$ and estimate $\|g_\tau\|_{\alpha;\mathbb{R}^2_L}$ for each $L \in (0,1)$ and $\|g_\tau\|_{\alpha;\mathbb{R}^2}$. In particular, for τ and $L \in (0,1)$ we show that

$$\|g_\tau\|_{\alpha;\mathbb{R}^2_L} \lesssim \|V_{int}\|_\alpha \|a\|_\alpha L^{-\frac{\alpha+2}{2}} (\tau^{\frac{1}{4}})^{2\alpha} \tag{1.195}$$

and

$$\|g_\tau\|_{\alpha;\mathbb{R}^2} \lesssim \|V_{int}\|_\alpha \|a\|_\alpha (\tau^{\frac{1}{4}})^{-2}.$$

We start by bounding the necessary seminorms and then the L^∞-norms. To estimate $[g_\tau]_{\alpha;\mathbb{R}^2_L}$ we use (1.310) of Corollary 2 and (1.185) to obtain

$$[g_\tau]_{\alpha;\mathbb{R}^2_L} \lesssim \|V_{int}\|_\alpha \|a\|_\alpha (\tau^{\frac{1}{4}})^{2\alpha} L^{-\frac{\alpha+2}{2}}.$$

To bound $[g_\tau]_{\alpha;\mathbb{R}^2}$ we again use (1.185), but now in combination with (1.309); we find that

$$[g_\tau]^{loc}_{\alpha;\mathbb{R}^2} \lesssim \|V_{int}\|_\alpha \|a\|_\alpha (\tau^{\frac{1}{4}})^{-2}.$$

This is, of course, only a bound on the local Hölder seminorm, which upgrades to a bound on the full seminorm with the addition of an L^∞-bound.

We bound the L^∞-norm $\|g_\tau\|_{\mathbb{R}^2_L}$ in the calculation (1.313) in Corollary 2. In particular, denoting the τ from (1.313) as τ' we apply (1.313) for $\tau' = T = \frac{\tau}{2}$, which gives

$$\|g_\tau\|_{\mathbb{R}^2_L} \lesssim \|V_{int}\|_\alpha \|a\|_\alpha L^{-\frac{\alpha+2}{2}} (\tau^{\frac{1}{4}})^{3\alpha}.$$

For the full L^∞-norm $\|g_\tau\|$ we can use (1.311), which for $T = \tau$ implies that

$$\|g_\tau\|_{\mathbb{R}^2} \lesssim \|V_{int}\|_\alpha \|a\|_\alpha (\tau^{\frac{1}{4}})^{2\alpha-2}.$$

Step 3.3– (Analysis of the regularized problem) In the last step we showed that $g_\tau \in C^\alpha(\mathbb{R}^2)$ for $\tau \in (0,1)$. This means that that we may find a solution $w^\tau \in C^{\alpha+2}(\mathbb{R}^2)$ of

$$(\partial_2 - a^{ext}\partial_1^2 + 1)w^\tau = g_\tau \qquad \text{in} \quad \mathbb{R}^2. \tag{1.196}$$

To obtain the desired correction w, we would like to pass to the limit $\tau \downarrow 0$ in the sequence of approximate solutions w^τ. As in Proposition 1, we do this with an application of Lemma 5 with $I = 1$, $f_1(\cdot, a_0) = 0$, $\sigma_1 = 0$, and $a = a^{ext}$ to the w^τ.

To apply Lemma 5 with the identifications made above, we check that the w^τ are approximate solutions of $(\partial_2 - a^{ext}\partial_1^2 + 1)\cdot = 0$ in the sense of (1.91) and determine an appropriate choice of K. Convolving (1.196) with ψ_T we obtain that w^τ solves

$$\begin{aligned}(\partial_2 - a^{ext}\partial_1^2 + 1)&(w^\tau)_T \\ &= (g_\tau)_T + (a^{ext}\partial_1^2 w^\tau)_T - a^{ext}\partial_1^2 (w^\tau)_T \qquad \text{in} \quad \mathbb{R}^2,\end{aligned} \tag{1.197}$$

where by (1.21), (1.185), and (1.311) we have that

$$\sup_{T\le 1}(T^{\frac{1}{4}})^{2-2\alpha}\|g_{\tau+T}\| \lesssim \|V_{int}\|_\alpha \|a\|_\alpha. \tag{1.198}$$

Furthermore, since $w^\tau \in C^{\alpha+2}(\mathbb{R}^2)$, Step 3.1 gives that

$$(a^{ext}\partial_1^2 w^\tau)_T - a^{ext}\partial_1^2 (w^\tau)_T = [a^{ext}, (\cdot)_T] \diamond \partial_1^2 w^\tau,$$

which, e.g. by (1.63) of Lemma 4, implies

$$\sup_{T\le 1}(T^{\frac{1}{4}})^{2-2\alpha}\|(a^{ext}\partial_1^2 w^\tau)_T - a^{ext}\partial_1^2 (w^\tau)_T\| \lesssim [w^\tau]_{2\alpha}[a^{ext}]_\alpha. \tag{1.199}$$

Combining (1.197), (1.198), and (1.199) we obtain that w^τ is indeed an approximate solution in the desired sense with $K = [a^{ext}]_\alpha [w^\tau]_{2\alpha} + \|V_{int}\|_\alpha \|a\|_\alpha$.

Applying Lemma 5 we find that

$$M = [w^\tau]_{2\alpha} \lesssim [a^{ext}]_\alpha [w^\tau]_{2\alpha} + \|a\|_\alpha \|V_{int}\|_\alpha,$$

which, after we use that $[a^{ext}]_\alpha \ll 1$, gives that

$$[w^\tau]_{2\alpha} \lesssim \|a\|_\alpha \|V_{int}\|_\alpha. \tag{1.200}$$

We also obtain the corresponding C^α-bound

$$\|w^\tau\|_\alpha \lesssim [a^{ext}]_\alpha [w^\tau]_{2\alpha} + \|a\|_\alpha \|V_{int}\|_\alpha \lesssim \|a\|_\alpha \|V_{int}\|_\alpha, \tag{1.201}$$

where we have used (1.200) and $[a^{ext}]_\alpha \le 1$.

Step 3.4– (Passing to the limit) We now pass to the limit $\tau \to 0$ in the sequence of approximate solutions w^τ. By (1.201) we can apply the Arzelà-Ascoli theorem, which implies that up to a subsequence $w^\tau \to w$ uniformly. In order to pass to the limit in (1.196), we first notice that $g_\tau \rightharpoonup g$ distributionally. We then show that the $a^{ext}\partial_1^2 w^\tau$ have uniformly bounded local $C^{\alpha-2}$-seminorms. This follows from the identity $a^{ext}\partial_1^2 w^\tau = a^{ext} \diamond \partial_1^2 w^\tau$, where the product on the right-hand side is obtained from Lemma 4 using the trivial modelling of w^τ. In particular, by (1.63), (1.201), and (1.20) we then have that

$$\begin{aligned}
&\sup_{T\le 1} (T^{\frac14})^{2-\alpha} \|(a^{ext}\partial_1^2 w^\tau)_T\| \\
&\lesssim \|a^{ext}\|_\alpha ([w^\tau]_{2\alpha} + \|V_{int}\|_\alpha) + \sup_{T\le 1} (T^{\frac14})^{2-\alpha} \|a^{ext}\partial_1^2 (w^\tau)_T\| \\
&\lesssim \|a^{ext}\|_\alpha \|V_{int}\|_\alpha,
\end{aligned}$$

which by (1.22) implies our claim. As previously observed in the proof of Proposition 1, the uniform bound on the local $C^{\alpha-2}$-seminorms implies that, up to a subsequence, $a^{ext}\partial_1^2 w^\tau \rightharpoonup h$ for some limiting distribution h as $\tau \to 0$. To see that $h = a^{ext} \diamond \partial_1^2 w$, where the product on the right-hand side is obtained via the trivial modelling from Lemma 4, we notice that

$$\lim_{T\to 0} \|(a^{ext}\partial_1^2 w^\tau)_T - a^{ext}\partial_1^2 (w^\tau)_T\| = 0 \tag{1.202}$$

for every $\tau > 0$ and that, furthermore, we have the convergence $(a^{ext}\partial_1^2 w^\tau)_T - a^{ext}\partial_1^2 (w^\tau)_T \to h_T - a^{ext}\partial_1^2 w_T$ in a pointwise sense. In order to see that this convergence holds weak-$*$ in $L^\infty(\mathbb{R}^2)$, we remark that by (1.63) and (1.200) the bound

$$\|(a^{ext}\partial_1^2 w^\tau)_T - a^{ext}\partial_1^2 (w^\tau)_T\| \lesssim \|a^{ext}\|_\alpha \|V_{int}\|_\alpha \tag{1.203}$$

holds uniformly in τ. Continuing as in Step 3 of Proposition 1 and using Step 10 of Proposition 1 of [47], we then find that $h = a \diamond \partial_1^2 w$ and, in particular, that w solves

$$(\partial_2 - a^{ext} \diamond \partial_1^2 + 1) w = g \qquad \text{in} \quad \mathbb{R}^2.$$

Since the bounds (1.200) and (1.201) are preserved under taking the limit $\tau \to 0$, we have that

$$\|w\|_\alpha + [w]_{2\alpha} \lesssim \|a\|_\alpha \|V_{int}\|_\alpha. \tag{1.204}$$

In order to see that w satisfies the initial condition (1.193b) we use (1.195). In particular, the classical Hölder estimate for (1.194) implies that

$$\|w_\tau\|_{\alpha+2;\mathbb{R}^2_L} \lesssim \|g_\tau\|_{\alpha;\mathbb{R}^2_L} \overset{(1.195)}{\lesssim} \|V_{int}\|_\alpha \|a\|_\alpha L^{-\frac{\alpha+2}{2}} (\tau^{\frac{1}{4}})^{2\alpha}$$

and passing to the limit $\tau \to 0$ yields that $w = 0$ on $\mathbb{R}^2_L$ for every $L > 0$. This means that the boundary condition (1.193b) is satisfied.

Step 3.5– (Uniqueness of the correction) In this step we show that the correction w solving (1.193) is unique. To see this we assume that we have two solutions w and w' of class $C^{2\alpha}$ on $\mathbb{R}^2_+$. We subtract them and use the same argument as in Step 4 of the proof of Proposition 1 to obtain that

$$\begin{aligned} (\partial_2 - a \diamond \partial_1^2 + 1)(w - w') &= 0 \qquad && \text{in} \quad \mathbb{R}^2_+ \\ w - w' &= 0 && \text{on} \quad \partial\mathbb{R}^2_+, \end{aligned}$$

where the singular product $a \diamond \partial_1^2(w - w')$ is obtained as the restriction of $a^{ext} \diamond \partial_1^2(w - w')^E$ to $\mathbb{R}^2_+$ and this is obtained via Lemma 4 using the trivial modelling. We also notice that by Step 3.1 we have that $a^{ext} \diamond \partial_1^2(w-w')^E = a^{ext}\partial_1^2(w - w')^E = 0$ on $\mathbb{R}^2_-$. In particular, we find that $w - w'$ solves

$$(\partial_2 - a^{ext} \diamond \partial_1^2 + 1)(w - w') = 0 \qquad \text{in} \quad \mathbb{R}^2,$$

which we can then take as input into Lemma 5. The proof of our claim then proceeds exactly as in Step 4 of Proposition 1 by showing that $\|w-w'\|_\alpha = 0$.

Step 4– (Conclusion)

To conclude we check the ansatz $U = q + w$. For this we first remark that because $q = V_{int}$ on $\partial\mathbb{R}^2_+$ and w satisfies (1.193b), the boundary condition (1.85b) holds. Furthermore, by (1.194) we have

$$\begin{aligned} &\partial_1^2(\tilde{q} + w) - a^{ext} \diamond \partial_1^2\tilde{q} - a^{ext} \diamond \partial_1^2 w + (\tilde{q} + w) \\ =&(\partial_1^2\tilde{q} - a^{ext} \diamond \partial_1^2\tilde{q} + \tilde{q}) - (\partial_1^2 q - a\partial_1^2 q + q)^E \qquad \text{in} \quad \mathbb{R}^2. \end{aligned} \tag{1.205}$$

To finish we show that

$$a^{ext} \diamond \partial_1^2\tilde{q} + a^{ext} \diamond \partial_1^2 w = a^{ext} \diamond \partial_1^2(\tilde{q} + w) \tag{1.206}$$

and

$$\big((\partial_1^2\tilde{q} - a^{ext} \diamond \partial_1^2\tilde{q} + \tilde{q}) - (\partial_1^2 q - a\partial_1^2 q + q)^E\big)\,|_{\mathbb{R}^2_+} = 0. \tag{1.207}$$

For (1.206) we first notice that since $w \in C^{2\alpha}(\mathbb{R}^2)$, by Step 1 we have that $\tilde{q} + w$ is modelled after $\tilde{V}(\cdot, a_0)$ according to a^{ext} and $\sigma = 1$ on $\mathbb{R}^2$. This allows us to define the product on the right-hand side of (1.206) via Lemma 4 with this modelling. The first product on the left-hand side is defined using the same modelling and the second product on the left-hand side is defined via the trivial modelling. Just like we have done previously in Step 4 of Proposition 1, we find that the triangle inequality and Lemma 4 may be combined to give

$$\lim_{T\to 0} \|(a^{ext} \diamond \partial_1^2(\tilde{q}+w))_T - (a^{ext} \diamond \partial_1^2 \tilde{q})_T - (a^{ext} \diamond \partial_1^2 w)_T\| = 0, \qquad (1.208)$$

which gives (1.206).

To finish checking the ansatz $U = q + w$, we notice that $a^{ext} \diamond \partial_1^2 \tilde{q}|_{\mathbb{R}^2_+}$ is the classical product, which shows (1.207). To see this we first show that $\tilde{q}$ satisfies (1.50), which implies that the product $a^{ext}\partial_1^2\tilde{q}$ is well-defined in a distributional sense. We then use (1.63) and (1.56) to write

$$\begin{aligned} &\lim_{T\to 0} \|(a^{ext} \diamond \partial_1^2 \tilde{q})_T - (a^{ext}\partial_1^2\tilde{q})_T\| \\ &\lesssim \lim_{T\to 0} \|[a^{ext}, (\cdot)_T] \diamond \partial_1^2 \tilde{q} - E[a^{ext}, (\cdot)_T]\partial_1^2 \tilde{V}(\cdot, a_0)\| \\ &\quad + \lim_{T\to 0} \|[a^{ext}, (\cdot)_T]\partial_1^2 \tilde{q} - E[a^{ext}, (\cdot)_T]\partial_1^2 \tilde{V}(\cdot, a_0)\|, \end{aligned} \qquad (1.209)$$

where we know by Lemma 4 that the first term goes to 0. We then treat the second term by performing a calculation that foreshadows our proof of Lemma 2. In particular, noting that here E denotes evaluation of a function of (x, a_0) at $(x, a^{ext}(x))$ we find that for $x \in \mathbb{R}^2$ the relation

$$\begin{aligned} &|[a^{ext}, (\cdot)_T]\partial_1^2 \tilde{q} - E[a^{ext}, (\cdot)_T]\partial_1^2 \tilde{V}(\cdot, a_0)| \\ &= \left| \int_{\mathbb{R}^2} (a^{ext}(x) - a^{ext}(x-y)) \right. \\ &\qquad \left. \times \partial_1^2(\tilde{V}(x-y, \tilde{\bar{a}}(x-y)) - \tilde{V}(x-y, a^{ext}(x)))\psi_T(y)\, \mathrm{d}y \right| \\ &\lesssim [a^{ext}]_\alpha \int_{\mathbb{R}^2} \sup_{a_0 \in [\lambda, 1]} |\partial_1^2 \partial_{a_0} \tilde{V}(x-y, a_0)| \\ &\qquad \times |y|^\alpha ([a^{ext}]_\alpha |y|^{\frac{\alpha}{2}} + |\tilde{\bar{a}}(x-y) - a^{ext}(x-y)|)|\psi_T(y)|\, \mathrm{d}y \\ &\lesssim [a^{ext}]_\alpha^2 [V_{int}]_\alpha \int_{\mathbb{R}^2} |x_2 - y_2|^{\frac{\alpha-2}{2}} |y|^\alpha (|y|^\alpha + |x_2 - y_2|^{\frac{\alpha}{2}})\psi_T(y)\, \mathrm{d}y \\ &\lesssim [a^{ext}]_\alpha^2 [V_{int}]_\alpha (T^{\frac{1}{4}})^{3\alpha-2} \end{aligned}$$

holds. Here we have used (1.52) and the bound $|\tilde{\bar{a}}(x) - a^{ext}(x)| \leq [a^{ext}]_\alpha |x_2|^{\frac{\alpha}{2}}$ for $x \in \mathbb{R}^2$. For the treatment of the integral in the last line refer to the proof of Lemma 2; the integral converges because ψ_1 is a Schwarz function. The above calculation shows that the second term on the right-hand side of (1.209) also vanishes, which implies that $a^{ext} \diamond \partial_1^2 \tilde{q} = a^{ext}\partial_1^2\tilde{q}$ as desired.

Quickly, before moving on, we prove (1.50) for $G = \tilde{q}$. Fix $x \in \mathbb{R}^2_+$ and recall from Step 2 that

$$\begin{aligned}\partial_1^2 q(x) =& \partial_1^2 V(x, \bar{a}(x)) - 2\partial_1 \partial_{a_0} V(x, \bar{a}(x)) \partial_1 \bar{a}(x) \\ & - \partial_{a_0} V(x, \bar{a}(x)) \partial_1^2 \bar{a}(x) - \partial_{a_0}^2 V(x, \bar{a}(x)) (\partial_1 \bar{a}(x))^2.\end{aligned} \tag{1.210}$$

As already discussed, the first term may be bounded as

$$|\partial_1^2 V(x, \bar{a}(x))| \lesssim [V_{int}]_\alpha |x_2|^{\frac{\alpha-2}{2}} \tag{1.211}$$

and the other terms as

$$\begin{aligned}&|2\partial_1 \partial_{a_0} V(x, \bar{a}(x)) \partial_1 \bar{a}(x)| + |\partial_{a_0} V(x, \bar{a}(x)) \partial_1^2 \bar{a}(x)| \\ &+ |\partial_{a_0}^2 V(x, \bar{a}(x)) (\partial_1 \bar{a}(x))^2| \\ \lesssim& \|V_{int}\|_\alpha [a]_\alpha |x_2|^{\frac{2\alpha-2}{2}}.\end{aligned} \tag{1.212}$$

To finish we prove the bounds (1.87) and (1.86). We start with (1.87) and use the triangle inequality, the estimate (1.204), (1.53) with $j = 0$ and $j = 1$ applied to $V(\cdot, a_0)$, (1.54) with $j = 0$, and (1.188) to write

$$[U]_\alpha \leq \sup_{a_0 \in [\lambda,1]} [V(\cdot, a_0)]_\alpha + \sup_{a_0 \in [\lambda,1]} \|\partial_{a_0} V(\cdot, a_0)\| [\bar{a}]_\alpha + [w]_\alpha \lesssim \|V_{int}\|_\alpha$$

and

$$\|U\|_\alpha \lesssim \|q\|_\alpha + \|w\|_\alpha \lesssim \|V_{int}\|_\alpha.$$

Likewise, for (1.86) we use (1.184) and (1.200) to write

$$M \lesssim [w]_{2\alpha} + [a^{ext}]_\alpha \|V_{int}\|_\alpha \lesssim \|a^{ext}\|_\alpha \|V_{int}\|_\alpha.$$

ii) In this part we consider two pairs $(a_i, V_{int,i})$ for $i = 0, 1$. Accordingly, we use the notation $q_i = V_i(\cdot, \bar{a}_i(\cdot))$, where $V_i(\cdot, a_0)$ solves (1.5) with initial condition $V_{int,i}$ and $\bar{a}_i$ solves (1.10) with initial condition a_i. In order to address the stability of the solution operator here, we use a strategy similar to the second part of Proposition 1 in combination with some classical estimates coming from Lemma 3.

Step 5– (Interpolation of the data)

We linearly interpolate the boundary data and coefficients and for $s \in [0, 1]$ define the objects a_s, a_s^{ext}, and $V_{int,s}$ as

$$a_s := a_1 s + a_0(1 - s), \tag{1.213}$$

$$a_s^{ext} := a_1^{ext} s + a_0^{ext}(1-s), \tag{1.214}$$

$$\text{and} \quad V_{int,s} := V_{int,1} s + V_{int,0}(1-s). \tag{1.215}$$

This induces the definition

$$q_s := V_s(\cdot, \bar{a}_s(\cdot)), \tag{1.216}$$

where $V_s(\cdot, a_0)$ solves (1.5) with initial condition $V_{int,s}$ and $\bar{a}_s$ solves (1.10) with initial condition a_s. That $\tilde{q}_s$ is modelled after $\tilde{V}_s(\cdot, a_0)$ according to $\tilde{a}_s^{ext}$ follows from the same argument as in Step 1 with the modelling constant $M_s \lesssim [a_s^{ext}]_\alpha \|V_{int,s}\|_\alpha$. Notice that unlike the situation in Proposition 1, here we not need to interpolate any reference products.

Step 6– (A continuous curve of solutions and an equation for $\partial_s w_s^\tau$)

Using the same methods as in part *i)* we find that, in analogue to (1.185), the relation

$$|(\partial_2 q_s - a_s \partial_1^2 q_s + q_s)^E(x)| \lesssim \|a_s\|_\alpha \|V_{int,s}\|_\alpha |x_2|^{\frac{2\alpha-2}{2}} \tag{1.217}$$

for $x \in \mathbb{R}^2$ holds. Feeding (1.217) into the machinery that we have developed in part *i*), we find that there exists a correction $w_s \in C^{2\alpha}(\mathbb{R}^2)$ solving (1.193) with right-hand side $\partial_2 q_s - a_s \partial_1^2 q_s + q_s$ and coefficients a_s and that this correction actually solves

$$(\partial_2 - a_s^{ext} \diamond \partial_1^2 + 1) w_s = g_s \qquad \text{in} \quad \mathbb{R}^2, \tag{1.218}$$

where $g_s = (\partial_2 q_s - a_s \partial_1^2 q_s + q_s)^E$. This solution w_s is obtained by taking the limit in $C^{2\alpha}(\mathbb{R}^2)$ of the sequence of regularized solutions w_s^τ of

$$(\partial_2 - a_s^{ext} \diamond \partial_1^2 + 1) w_s^\tau = (g_s)_\tau \qquad \text{in} \quad \mathbb{R}^2. \tag{1.219}$$

Using the same arguments as in Step 4 we find that $U_s = q_s + w_s$ solves (1.85) with coefficients a_s and initial condition $V_{int,s}$.

Noticing that by Step 3.1 when $\tau > 0$ the singular product in (1.219) is the classical product, we may differentiate the equation (1.219) with respect to s and find that $\partial_s w_s^\tau$ solves

$$(\partial_2 - a_s^{ext} \partial_1^2 + 1) \partial_s w_s^\tau = (\partial_s g_s)_\tau + \partial_s a_s^{ext} \partial_1^2 w_s^\tau \qquad \text{in} \quad \mathbb{R}^2. \tag{1.220}$$

Since the right-hand side of (1.220) is still of class C^α, we know by the standard Hölder estimate for (1.220) that $\partial_s w_s^\tau \in C^{\alpha+2}(\mathbb{R}^2)$. In particular, $\partial_s w_s^\tau$ is trivially modelled.

Step 7– (Estimates for $\partial_s w_s^\tau$)

We now apply Lemma 5 to the $\partial_s w_s^\tau$ with inputs $I = 2$, $f_1(\cdot, a_0) = \partial_s g_s$, $f_2(\cdot, a_0) = \partial_s a_s^{ext}\partial_1^2 w_s^\tau(\cdot, a_0)$, and $\sigma_1 = \sigma_2 = 0$. First, we must check that $\partial_s w_s^\tau$ is an approximate solution in the sense of (1.91). To begin we convolve (1.220) with ψ_T, which gives

$$\begin{aligned}&(\partial_2 - a_s^{ext}\partial_1^2 + 1)(\partial_s w_s^\tau)_T\\ &\quad= (\partial_s g_s)_{\tau+T} + (\partial_s a_s^{ext}\partial_1^2 w_s^\tau)_T - [a_s^{ext}, (\cdot)_T]\partial_1^2\partial_s w_s^\tau \qquad \text{in} \quad \mathbb{R}^2\end{aligned}$$

and we then show that

$$\begin{aligned}&\sup_{T\le 1}(T^{\frac{1}{4}})^{2-2\alpha}\|(\partial_s g_s)_{\tau+T} + (\partial_s a_s^{ext}\partial_1^2 w_s^\tau)_T - [a_s^{ext}, (\cdot)_T]\partial_1^2\partial_s w_s^\tau\|\\ \lesssim&[a_s^{ext}]_\alpha[\partial_s w_s^\tau]_{2\alpha} + \|a_0^{ext} - a_1^{ext}\|_\alpha \max_i \|V_{int,i}\|_\alpha + \|V_{int,0} - V_{int,1}\|_\alpha.\end{aligned} \tag{1.221}$$

Showing (1.221) turns out to be the bulk of the work in part *ii)*.

To maintain oversight in our argument for (1.221) we split it into three steps. The eventual application of Lemma 5 is included in Step 7.3.

Step 7.1– The main difficultly in showing (1.221) is obtaining the inequality

$$\sup_{T\le 1}(T^{\frac{1}{4}})^{2-2\alpha}\|(\partial_s g_s)_{\tau+T}\| \lesssim \|a_0 - a_1\|_\alpha \max_i \|V_{int,i}\|_\alpha + \|V_{int,0} - V_{int,1}\|_\alpha. \tag{1.222}$$

To obtain (1.222) we first notice that for $x \in \mathbb{R}^2$ we have that

$$\begin{aligned}\partial_s g_s(x) =& -\left((\partial_2 - a_s\partial_1^2 + 1)(V_1(x, \bar{a}_s(x)) - V_0(x, \bar{a}_s(x)))\right)^E\\ &- \left((\partial_2 - a_s\partial_1^2 + 1)(\overline{a_1 - a_0})\partial_{a_0} V_s(x, \bar{a}_s(x))\right)^E\\ &+ \left((a_1 - a_0)\partial_1^2 V_s(x, \bar{a}_s(x))\right)^E\end{aligned} \tag{1.223}$$

and then treat the terms on the right-hand side separately. The first term may be treated in the same way as in Step 2 using that $(V_1 - V_0)(\cdot, a_0)$ solves (1.5) with initial condition $V_{int,1} - V_{int,0}$ and $\bar{a}_s$ solves (1.10) with initial condition a_s. In particular, using the exact same argument and additionally (1.21), we obtain

$$\begin{aligned}&\sup_{T\le 1}(T^{\frac{1}{4}})^{2-2\alpha}\|((\partial_2 - a_s\partial_1^2 + 1)(V_1(\cdot, \bar{a}_s(\cdot)) - V_0(\cdot, \bar{a}_s(\cdot)))^E)_{\tau+T}\|\\ &\lesssim \|a_s\|_\alpha\|V_{int,1} - V_{int,0}\|_\alpha.\end{aligned} \tag{1.224}$$

We treat the second and third terms of (1.223) together. Using Leibniz' rule for $x \in \mathbb{R}^2_+$ we obtain (this calculation continues on the next page)

$$(\partial_2 - a_s\partial_1^2 + 1)(\overline{a_1 - a_0})\partial_{a_0} V_s(x, \bar{a}_s(x)) - (a_1 - a_0)\partial_1^2 V_s(x, \bar{a}_s(x))$$

$$
\begin{aligned}
=&\partial_{a_0} V_s(x,\bar{a}_s(x))\partial_2(\overline{a_1-a_0}) + (\overline{a_1-a_0})\partial_2\partial_{a_0}V_s(x,\bar{a}_s(x))\\
&+ (\overline{a_1-a_0})\partial_{a_0}^2 V_s(x,\bar{a}_s(x))\partial_2\bar{a}_s - a_s\partial_{a_0}V_s(x,\bar{a}_s(x))\partial_1^2(\overline{a_1-a_0})\\
&- 2a_s\partial_1(\overline{a_1-a_0})(\partial_1\partial_{a_0}V_s(x,\bar{a}_s(x)) + \partial_{a_0}^2 V_s(x,\bar{a}_s(x))\partial_1\bar{a}_s)\\
&- a_s(\overline{a_1-a_0})\Big(\partial_1^2\partial_{a_0}V_s(x,\bar{a}_s(x)) + 2\partial_1\partial_{a_0}^2 V_s(x,a_s(x))\partial_1\bar{a}_s(x)\\
&\qquad + \partial_{a_0}^3 V_s(x,a_s(x))(\partial_1\bar{a}_s)^2 + \partial_{a_0}^2 V_s(x,a_s(x))\partial_1^2\bar{a}_s\Big)\\
&+ (\overline{a_1-a_0})\partial_{a_0}V_s(x,\bar{a}_s(x))\\
&- (a_1-a_0)\Big(\partial_1^2 V_s(x,\bar{a}_s(x)) + 2\partial_1\partial_{a_0}V_s(x,\bar{a}_s(x))\partial_1\bar{a}_s\\
&\qquad +\partial_{a_0}^2 V_s((x,\bar{a}_s(x))(\partial_1\bar{a}_s)^2 + \partial_{a_0}V_s((x,\bar{a}_s(x))\partial_1^2\bar{a}_s\Big).
\end{aligned}
$$

Following the techniques used in Step 1 we make use of the identities

$$
\begin{aligned}
\partial_2(\overline{a_1-a_0}) &= \partial_1^2(\overline{a_1-a_0}) - (\overline{a_1-a_0}),\\
\partial_2\bar{a}_s &= \partial_1^2\bar{a}_s - \bar{a}_s,\\
\text{and}\quad \partial_2\partial_{a_0}V_s(x,\bar{a}_s(x)) &= \bar{a}_s(x)\partial_1^2\partial_{a_0}V_s(x,\bar{a}_s(x))\\
&\quad - \partial_{a_0}V_s(x,\bar{a}_s(x)) + \partial_1^2 V_s(x,\bar{a}_s(x)),
\end{aligned}
$$

where the last one comes from (1.253). In particular, plugging them in and rearranging terms we obtain

$$
\begin{aligned}
&(\partial_2 - a_s\partial_1^2 + 1)(\overline{a_1-a_0})\partial_{a_0}V_s(x,\bar{a}_s(x)) - (a_1-a_0)\partial_1^2 V_s(x,\bar{a}_s(x))\\
=&(1-a_s)\partial_{a_0}V_s(x,\bar{a}_s(x))\partial_1^2(\overline{a_1-a_0})\\
&+ (\overline{a_1-a_0})(\bar{a}_s(x) - a_s(x))\partial_1^2\partial_{a_0}V_s(x,\bar{a}_s(x))\\
&+ (\overline{a_1-a_0} - (a_1-a_0))\partial_1^2 V_s(x,\bar{a}_s(x)))\\
&+ (\overline{a_1-a_0})\partial_{a_0}^2 V_s(x,\bar{a}_s(x))(\partial_1^2\bar{a}_s - \bar{a}_s)\\
&- 2a_s\partial_1(\overline{a_1-a_0})(\partial_1\partial_{a_0}V_s(x,\bar{a}_s(x)) + \partial_{a_0}^2 V_s(x,\bar{a}_s(x))\partial_1\bar{a}_s)\\
&- a_s(\overline{a_1-a_0})\Big(2\partial_1\partial_{a_0}^2 V_s(x,a_s(x))\partial_1\bar{a}_s(x)\\
&\qquad + \partial_{a_0}^3 V_s(x,a_s(x))(\partial_1\bar{a}_s)^2 + \partial_{a_0}^2 V_s(x,a_s(x))\partial_1^2\bar{a}_s\Big)\\
&- (a_1-a_0)\Big(2\partial_1\partial_{a_0}V_s(x,\bar{a}_s(x))\partial_1\bar{a}_s + \partial_{a_0}^2 V_s((x,\bar{a}_s(x))(\partial_1\bar{a}_s)^2\\
&\qquad + \partial_{a_0}V_s((x,\bar{a}_s(x))\partial_1^2\bar{a}_s\Big)
\end{aligned}
$$

We treat each term on the right-hand side separately: Starting with the first term, we use that $\|a_s\| \leq 1$, an application of (1.52) with $k=2$ and $j=0$

to $\overline{a_1 - a_0}$, and with $k = 0$ and $j = 1$ applied to $V_s(\cdot, a_0)$ to obtain

$$|(1 - a_s(x))\partial_{a_0} V_s(x, \bar{a}_s(x))\partial_1^2(\overline{a_1 - a_0})(x)| \lesssim [a_1 - a_0]_\alpha [V_{int,s}]_\alpha x_2^{\frac{2\alpha-2}{2}}.$$

For the next term we use that $|(\bar{a}_s - a_s)(x)| \lesssim [a_s]_\alpha x_2^{\frac{\alpha}{2}}$, apply (1.52) with $k = 2$ and $j = 1$ to $V_s(\cdot, a_0)$, and (1.53) to $\overline{a_1 - a_0}$ in the form without the massive term to write

$$\begin{aligned}&|(\overline{a_1 - a_0})(x)(\bar{a}_s - a_s)(x)\partial_1^2\partial_{a_0} V_s(x, \bar{a}_s(x))|\\ &\lesssim \|a_1 - a_0\| [a_s]_\alpha [V_{int,s}]_\alpha x_2^{\frac{2\alpha-2}{2}}.\end{aligned}$$

The third term is treated in a similar fashion, using that

$$|(\overline{a_1 - a_0} - (a_1 - a_0))(x)| \lesssim [a_1 - a_0]_\alpha x_2^{\frac{\alpha}{2}}$$

and (1.52) applied with $k = 2$ and $j = 0$ to $V_s(\cdot, a_0)$. In particular, we obtain that

$$|(\overline{a_1 - a_0} - (a_1 - a_0))(x)\partial_1^2 V_s(x, \bar{a}_s(x))| \lesssim [a_1 - a_0]_\alpha [V_{int,s}]_\alpha x_2^{\frac{2\alpha-2}{2}}.$$

For the fourth term we use (1.53) in its version without the massive term applied to $\overline{a_1 - a_0}$, the relation $\|\bar{a}_s\| \leq 1$, (1.52) applied with $j = 2$ and $k = 0$ to $V_s(\cdot, a_0)$, and again with $k = 2$ and $j = 0$ to $\bar{a}_s$. We find that

$$\begin{aligned}&|\overline{a_1 - a_0}(x)\partial_{a_0}^2 V_s(x, \bar{a}_s(x))(\partial_1^2 \bar{a}_s - \bar{a}_s)(x)|\\ &\lesssim \|a_s\|_\alpha \|a_1 - a_0\| [V_{int,s}]_\alpha (x_2^{\frac{2\alpha-2}{2}} + x_2^{\frac{\alpha}{2}}).\end{aligned}$$

Continuing on, for the next term we apply (1.52) with $k = 1$ and $j = 0$ to $\overline{a_1 - a_0}$ and $\bar{a}_s$ and with $k = 1$ and $j = 1$ to $V_s(\cdot, a_0)$. Furthermore, we use (1.53) applied to $V_s(\cdot, a_0)$ with $j = 2$; combining these estimates and using that $\|a_s\| \leq 1$ we obtain

$$\begin{aligned}&|a_s(x)\partial_1\overline{a_1 - a_0}(x)(\partial_1\partial_{a_0} V_s(x, \bar{a}_s(x)) + \partial_{a_0}^2 V_s(x, \bar{a}_s(x))\partial_1\bar{a}_s(x))|\\ \lesssim&[a_1 - a_0]_\alpha \|V_{int,s}\|_\alpha (1 + [a_s]_\alpha) x_2^{\frac{2\alpha-2}{2}}.\end{aligned}$$

Again using the bound $\|a_s\| \leq 1$, (1.52) applied to $V_s(\cdot, a_0)$ with $k = 1$ or $k = 0$ and $j = 2$ and to $\bar{a}_s$ with $k = 1$ or $k = 2$ and $j = 0$, and (1.53) applied to $V_s(\cdot, a_0)$ with $j = 3$ and to $\overline{a_1 - a_0}$ with $j = 0$ we find that

$$\begin{aligned}&\Big|a_s(x)\overline{a_1 - a_0}(x)\left(2\partial_1\partial_{a_0}^2 V_s(x, a_s(x))\partial_1\bar{a}_s(x)\right.\\ &\qquad\left.+\partial_{a_0}^3 V_s(x, a_s(x))(\partial_1\bar{a}_s(x))^2 + \partial_{a_0}^2 V_s(x, a_s(x))\partial_1^2\bar{a}_s(x)\right)\Big|\\ \lesssim&\|a_1 - a_0\|([V_{int,s}]_\alpha [a_s]_\alpha + \|V_{int,s}\| [a_s]_\alpha^2)|x_2|^{\frac{2\alpha-2}{2}}\end{aligned}$$

Finally, we come to the last term for which we use (1.52) applied to $V_s(\cdot, a_0)$ with $k=1$ or $k=0$ and $j=1$ and to $\bar{a}_s$ with $k=1$ and $j=0$ and (1.53) applied to $V_s(\cdot, a_0)$ with $j=2$. We obtain

$$
\begin{aligned}
&\big|(a_1-a_0)(x)\,(2\partial_1\partial_{a_0}V_s(x,\bar{a}_s(x))\partial_1\bar{a}_s(x)\\
&\qquad\qquad +\partial_{a_0}^2V_s(x,\bar{a}_s(x))(\partial_1\bar{a}_s(x))^2+\partial_{a_0}V_s(x,\bar{a}_s(x))\partial_1^2\bar{a}_s(x)\big)\big|\\
\lesssim&\|a_1-a_0\|\left(([V_{int,s}]_\alpha[a_s]_\alpha+\|V_{int,s}\|_\alpha[a_s]_\alpha^2)x_2^{\frac{2\alpha-2}{2}}+\|V_{int,s}\|e^{-x_2}[a_s]_\alpha x_2^{\frac{\alpha-2}{2}}\right)\\
\lesssim&\|a_1-a_0\|[a_s]_\alpha\|V_{int,a}\|_\alpha x_2^{\frac{2\alpha-2}{2}}.
\end{aligned}
$$

Using that $\|a_s\|_\alpha \leq 1$ we then, in particular, find that for any $x\in\mathbb{R}^2$ it holds that

$$
\begin{aligned}
&\big|((\partial_2-a_s\partial_1^2+1)(\overline{a_1-a_0})\partial_{a_0}V_s(\cdot,\bar{a}_s(\cdot))-(a_1-a_0)\partial_1^2V_s(\cdot,\bar{a}_s(\cdot)))^E(x)\big|\\
\lesssim&\|a_1-a_0\|_\alpha\max_i\|V_{int,i}\|_\alpha|x_2|^{\frac{2\alpha-2}{2}},
\end{aligned}
$$

which using Corollary 2 then gives that

$$
\begin{aligned}
&\sup_{T\leq 1}(T^{\frac14})^{2-2\alpha}\Big\|\Big(\,((\partial_2-a_s\partial_1^2+1)(\overline{a_1-a_0})\partial_{a_0}V_s(\cdot,\bar{a}_s(\cdot))\\
&\qquad\qquad\qquad\qquad -(a_1-a_0)\partial_1^2V_s(\cdot,\bar{a}_s(\cdot)))^E\Big)_T\Big\|\\
\lesssim&\|a_1-a_0\|_\alpha\max_i\|V_{int,i}\|_\alpha.
\end{aligned}
$$

Combining this with (1.223) and (1.224) gives (1.222).

Step 7.2– To continue checking (1.221) we use the triangle inequality to write

$$
\begin{aligned}
&\sup_{T\leq 1}(T^{\frac14})^{2-2\alpha}\|(\partial_s a_s^{ext}\partial_1^2w_s^\tau)_T\|\\
\leq&\sup_{T\leq 1}(T^{\frac14})^{2-2\alpha}\|[\partial_s a_s^{ext},(\cdot)_T]\partial_1^2w_s^\tau\|+\sup_{T\leq 1}(T^{\frac14})^{2-2\alpha}\|\partial_s a_s^{ext}\partial_1^2(w_s^\tau)_T\|.
\end{aligned}
\tag{1.225}
$$

For the first term on the right-hand side we notice that by (1.63) of Lemma 4 and the analogue of (1.200) for w_s^τ, we have that

$$
\begin{aligned}
\sup_{T\leq 1}(T^{\frac14})^{2-2\alpha}\|[\partial_s a_s^{ext},(\cdot)_T]\partial_1^2w_s^\tau\|&\lesssim\|a_0^{ext}-a_1^{ext}\|_\alpha[w_s^\tau]_{2\alpha}\\
&\lesssim\|a_0^{ext}-a_1^{ext}\|_\alpha\|a_s\|_\alpha\|V_{int,s}\|_\alpha.
\end{aligned}
\tag{1.226}
$$

The second term on the right-hand side of (1.225) is again handled using (1.200). In particular, for $x\in\mathbb{R}^2$ we may then use (1.18) and that ψ_T is an even Schwarz function to write

$$
|\partial_s a_s^{ext}(\partial_1^2w_s^\tau)_T(x)|
$$

$$
\begin{aligned}
&\lesssim \|a_0^{ext} - a_1^{ext}\| \left| \int_{\mathbb{R}^2} (w_s^\tau(y) - w_s^\tau(x) - \partial_1 w_s^\tau(x)(y-x)_1) \partial_1^2 \psi_T(y-x)\, \mathrm{d}y \right| \\
&\lesssim \|a_0^{ext} - a_1^{ext}\| [w_s^\tau]_{2\alpha} (T^{\frac{1}{4}})^{2\alpha-2} \\
&\lesssim \|a_0^{ext} - a_1^{ext}\| \|a_s\|_\alpha \|V_{int,s}\|_\alpha (T^{\frac{1}{4}})^{2\alpha-2}.
\end{aligned}
$$

We then combine the last three relations and use $\|a_s\|_\alpha \leq 1$ to find that

$$
\sup_{T\leq 1} (T^{\frac{1}{4}})^{2-2\alpha} \|(\partial_s a_s^{ext} \partial_1^2 w_{s,\tau})_T\| \lesssim \|a_0^{ext} - a_1^{ext}\|_\alpha \|V_{int,s}\|_\alpha. \tag{1.227}
$$

Step 7.3– To finish checking (1.221), we again use (1.63) of Lemma 4 to find that

$$
\sup_{T\leq 1} (T^{\frac{1}{4}})^{2-2\alpha} \|[a_s^{ext}, (\cdot)_T] \partial_1^2 \partial_s w_s^\tau\| \lesssim [a_s^{ext}]_\alpha [\partial_s w_s^\tau]_{2\alpha}. \tag{1.228}
$$

We then combine (1.228) with the estimates already obtained in the previous (sub)steps and the triangle inequality to obtain (1.221).

Having shown (1.221) and using $[a_s^{ext}]_\alpha \ll 1$, we may apply Lemma 5 to obtain that

$$
\begin{aligned}
&\|\partial_s w_s^\tau\|_\alpha + [\partial_s w_s^\tau]_{2\alpha} \\
&\lesssim \|a_0^{ext} - a_1^{ext}\|_\alpha \max_i \|V_{int,i}\|_\alpha + \|V_{int,0} - V_{int,1}\|_\alpha.
\end{aligned} \tag{1.229}
$$

Step 8 – (Conclusion)

We would now like to finish by showing (1.88) and (1.89). Recall from part *i)* that the solutions of (1.85) are constructed as $U_i = q_i + w_i$ for $i = 0, 1$. Notice that, just like in Step 8 of Proposition 1, the bound (1.229) may be integrated over $s \in [0,1]$ to give

$$
\begin{aligned}
&\|w_0^\tau - w_1^\tau\|_\alpha + [w_0^\tau - w_1^\tau]_{2\alpha} \\
\lesssim &\|a_0^{ext} - a_1^{ext}\|_\alpha \max_i \|V_{int,i}\|_\alpha + \|V_{int,0} - V_{int,1}\|_\alpha.
\end{aligned} \tag{1.230}
$$

Passing to the limit $\tau \to 0$, we find that the bound (1.230) holds also for $w_0 - w_1$. It still remains to bound the C^α-norm of $q_1 - q_0$ and to quantify the modelling of $\tilde{q}_1 - \tilde{q}_0$ after $(\tilde{V}_1(\cdot, a_0), \tilde{V}_0(\cdot, a_0))$ according to $(1,-1)$ and (a_1^{ext}, a_0^{ext}).

We start by bounding the C^α-norm of $q_1 - q_0$. First, let $x \in \mathbb{R}^2_+$ and notice that

$$\begin{aligned}
&|q_1(x) - q_0(x)| \\
&\lesssim \sup_{a_0\in[\lambda,1]} \|(V_1 - V_0)(\cdot, a_0)\| + \sup_{a_0\in[\lambda,1]} \|\partial_{a_0} V_0(\cdot, a_0)\| \|\overline{a_1 - a_0}\| \\
&\lesssim \|V_{int,1} - V_{int,0}\| + \|V_{int,0}\|_\alpha \|a_1 - a_0\|,
\end{aligned} \tag{1.231}$$

where we have used (1.53). Moving on to the seminorm, we fix two distinct points $x, y \in \mathbb{R}^2_+$ and write

$$\begin{aligned}
&|(q_1 - q_0)(x) - (q_1 - q_0)(y)| \\
\leq& |V_1(x, \bar{a}_1(x)) - V_0(x, \bar{a}_0(x)) - (V_1(x, \bar{a}_1(y)) - V_0(x, \bar{a}_0(y)))| \\
&+ |V_1(x, \bar{a}_1(y)) - V_0(x, \bar{a}_0(y)) - (V_1(y, \bar{a}_1(y)) - V_0(y, \bar{a}_0(y)))|.
\end{aligned} \tag{1.232}$$

For the first term notice that

$$\begin{aligned}
&V_1(x, \bar{a}_1(x)) - V_0(x, \bar{a}_0(x)) - (V_1(x, \bar{a}_1(y)) - V_0(x, \bar{a}_0(y))) \\
=& \int_0^1 \partial_s (V_s(x, \bar{a}_s(x)) - V_s(x, \bar{a}_s(y)))\, \mathrm{d}s \\
=& \int_0^1 \Big(V_1(x, \bar{a}_s(x)) - V_0(x, \bar{a}_s(x)) - (V_1(x, \bar{a}_s(y)) - V_0(x, \bar{a}_s(y))) \\
&\quad + (\partial_{a_0} V_s(x, \bar{a}_s(x)) - \partial_{a_0} V_s(x, \bar{a}_s(y)))\overline{a_1 - a_0}(x) \\
&\quad + \partial_{a_0} V_s(x, \bar{a}_s(y))(\overline{a_1 - a_0}(x) - \overline{a_1 - a_0}(y)) \Big)\, \mathrm{d}s.
\end{aligned} \tag{1.233}$$

We then apply (1.53) with $j = 1$ to $(V_1 - V_0)(\cdot, a_0)$ and (1.54) to $\bar{a}_s$ as

$$\begin{aligned}
&|V_1(x, \bar{a}_s(x)) - V_0(x, \bar{a}_s(x)) - (V_1(x, \bar{a}_s(y)) - V_0(x, \bar{a}_s(y)))| \\
\lesssim& \sup_{a_0\in[\lambda,1]} \|\partial_{a_0}(V_1 - V_0)(\cdot, a_0)\| |\bar{a}_s(x) - \bar{a}_s(y)| \\
\lesssim& \|V_{int,1} - V_{int,0}\| [a_s]_\alpha d^\alpha(x, y).
\end{aligned} \tag{1.234}$$

For the second term of (1.233) we use (1.53) with $j = 2$ applied to $V_s(\cdot, a_0)$ and with $j = 0$ applied to $\overline{a_1 - a_0}$ to obtain

$$\begin{aligned}
&|(\partial_{a_0} V_s(x, \bar{a}_s(x)) - \partial_{a_0} V_s(x, \bar{a}_s(y)))\overline{a_1 - a_0}(x)| \\
\lesssim& \sup_{a_0\in[\lambda,1]} \|\partial^2_{a_0} V_s(\cdot, a_0)\| \, |\bar{a}_s(x) - \bar{a}_s(y)| \, \|\overline{a_1 - a_0}\| \\
\lesssim& \|V_{int,s}\|_\alpha [a_s]_\alpha \|a_1 - a_0\| d^\alpha(x, y).
\end{aligned}$$

The last term of (1.233) is bounded with (1.53) for $j = 1$ applied to $V_s(\cdot, a_0)$ and (1.54) for $j = 0$ applied to $\overline{a_1 - a_0}$:

$$|\partial_{a_0} V_s(x, \bar{a}_s(y))(\overline{a_1 - a_0}(x) - \overline{a_1 - a_0}(y))| \lesssim \|V_{int,s}\|_\alpha [a_1 - a_0]_\alpha d^\alpha(x, y).$$

Combining these estimates and using that $[a_s]_\alpha \leq 1$ gives

$$\begin{aligned} &|V_1(x, \bar{a}_1(x)) - V_0(x, \bar{a}_0(x)) - (V_1(x, \bar{a}_1(y)) - V_0(x, \bar{a}_0(y)))| \\ \lesssim &(\|V_{int,0} - V_{int,1}\|_\alpha + \|a_1 - a_0\|_\alpha \max_i \|V_{int,i}\|_\alpha) d^\alpha(x, y). \end{aligned} \tag{1.235}$$

A similar strategy can be used to bound the second term on the right-hand side of (1.232). In particular, we write

$$\begin{aligned} &|V_1(x, \bar{a}_1(y)) - V_0(x, \bar{a}_0(y)) - (V_1(y, \bar{a}_1(y)) - V_0(y, \bar{a}_0(y)))| \\ = &\left| \int_0^1 \partial_s (V_s(x, \bar{a}_s(y)) - V_s(y, \bar{a}_s(y))) \, \mathrm{d}s \right| \\ = &\int_0^1 \Big(|(V_1 - V_0)(x, \bar{a}_s(y)) - (V_1 - V_0)(y, \bar{a}_s(y))| \\ &\qquad + |\partial_{a_0} V_s(x, \bar{a}_s(y)) - \partial_{a_0} V_s(y, \bar{a}_s(y))| \, |\overline{a_1 - a_0}(y)| \Big) \, \mathrm{d}s \\ \lesssim &(\|V_{int,1} - V_{int,0}\|_\alpha + \|a_1 - a_0\| \max_i [V_{int,i}]_\alpha) d^\alpha(x, y), \end{aligned} \tag{1.236}$$

where we have again used (1.54) and (1.53). Together (1.232), (1.235), and (1.236) show that

$$[q_1 - q_0]_\alpha \lesssim \|a_1 - a_0\|_\alpha \max_i \|V_{int,i}\|_\alpha + \|V_{int,1} - V_{int,0}\|_\alpha. \tag{1.237}$$

We then also consider the modelling of $\tilde{q}_1 - \tilde{q}_0$ after $(\tilde{V}_1(\cdot, a_0), \tilde{V}_0(\cdot, a_0))$ according to (a_1^{ext}, a_0^{ext}) and $(1, -1)$ on $\mathbb{R}^2$. For distinct points $x, y \in \mathbb{R}^2_+$ we use the triangle inequality to write

$$\begin{aligned} &\Big| \tilde{V}_1(x, \tilde{\bar{a}}_1(x)) - \tilde{V}_0(x, \tilde{\bar{a}}_0(x)) - (\tilde{V}_1(y, \tilde{\bar{a}}_1(y)) - \tilde{V}_0(y, \tilde{\bar{a}}_0(y))) \\ &\quad - (\tilde{V}_1(x, a_1^{ext}(y)) - \tilde{V}_1(y, a_1^{ext}(y))) + (\tilde{V}_0(x, a_0^{ext}(y)) - \tilde{V}_0(y, a_0^{ext}(y))) \Big| \\ \lesssim &\Big| \tilde{V}_1(x, \tilde{\bar{a}}_1(x)) - \tilde{V}_0(x, \tilde{\bar{a}}_0(x)) - (\tilde{V}_1(x, \tilde{\bar{a}}_1(y)) - \tilde{V}_0(x, \tilde{\bar{a}}_0(y))) \Big| \\ &\quad + \Big| \tilde{V}_1(x, \tilde{\bar{a}}_1(y)) - \tilde{V}_1(x, a_1^{ext}(y)) - (\tilde{V}_0(x, \tilde{\bar{a}}_0(y)) - \tilde{V}_0(x, a_0^{ext}(y))) \\ &\quad - \Big(\tilde{V}_1(y, \tilde{\bar{a}}_1(y)) - \tilde{V}_1(y, a_1^{ext}(y)) - (\tilde{V}_0(y, \tilde{\bar{a}}_0(y)) - \tilde{V}_0(y, a_0^{ext}(y))) \Big) \Big|. \end{aligned} \tag{1.238}$$

Notice that the first term on the right-hand side is the same as in (1.232); however, we treat it slightly differently now. In particular, we treat this term in the same way as (1.182) in Step 1, which gives that

$$\begin{aligned}
&|(V_1(x,\bar{a}_s(x)) - V_0(x,\bar{a}_s(x))) - (V_1(x,\bar{a}_s(y)) - V_0(x,\bar{a}_s(y)))| \\
\lesssim & \sup_{a_0\in[\lambda,1]} |\partial_{a_0}(V_1 - V_0)(x,a_0))||\bar{a}_s(x) - \bar{a}_s(y)| \\
\lesssim & \|V_{int,1} - V_{int,0}\|_\alpha [a_s]_\alpha d^{2\alpha}(x,y).
\end{aligned}$$

Notice that here we have applied (1.53) with $j = 1$ to $(V_1 - V_0)(\cdot, a_0)$ and have either applied (1.54) to $\bar{a}_s$ or (1.55) both with $j = 0$. The second term of (1.238) is more involved, but has already been treated in part *iii*) of Lemma 6. In particular, making the identifications $a_i = \tilde{\bar{a}}_i$ and $a_i' = a_i^{ext}$, where the left-hand side is notation taken from Lemma 6, and using the result from the lemma, we find that the second term can be bounded by $\|V_{int,1} - V_{int,0}\|_\alpha + \max_i \|V_{int,i}\|_\alpha \|a_1^{ext} - a_0^{ext}\|_\alpha$. We conclude that $\tilde{q}_1 - \tilde{q}_0$ is modelled after $(\tilde{V}_1(\cdot, a_0), \tilde{V}_0(\cdot, a_0))$ according to (a_1^{ext}, a_0^{ext}) and $(1, -1)$ on $\mathbb{R}^2$ with modelling constant given by

$$M \lesssim \|V_{int,1} - V_{int,0}\|_\alpha + \max_i \|V_{int,i}\|_\alpha \|a_1^{ext} - a_0^{ext}\|_\alpha. \tag{1.239}$$

□

1.3.4 Proof of Theorem 1.

We proceed to our proof of Theorem 1, which now consists mainly of combining the two propositions and post-processing the modelling. Most of the work goes towards post-processing the modelling, which is done for part *i*) in Step 2 and for part *ii*) in Step 3. This post-processing relies on Lemma 6.

Proof of Theorem 1.

i) In this first part of our proof we combine the two propositions and give an appropriate solution of (1.68) with the correct modelling.

Step 1– (Checking the ansatz)

By Proposition 1 we know that there is a unique solution $u \in C^\alpha(\mathbb{R}^2)$ of (1.82) that is modelled after $v(\cdot, a_0)$ according to the extension a^{ext}. Proposition 2 gives a solution $U \in C^\alpha(\mathbb{R}^2_+)$ of (1.85) with initial condition $V_{int} = W_{int} - u|_{x_2=0}$ that decomposes as $U = q + w$ such that $\tilde{q} + w^E$ is modelled after $\tilde{V}(\cdot, a_0)$ according to a^{ext}. The ansatz for the solution W of (1.68) is then taken to be $W = u + U$.

To see that $W = u + U$ satisfies (1.68), we notice that the initial condition (1.68b) is clearly satisfied. In order to check that (1.68a) holds on $\mathbb{R}^2_+$ we show that the singular products from Proposition 1 and Proposition 2 are compatible in the sense that

$$a \diamond \partial_1^2 W = a \diamond \partial_1^2 u + a \diamond \partial_1^2 U. \tag{1.240}$$

Aside from $a \diamond \partial_1^2 u$, which is obtained via Lemma 4 from the reference products assumed to exist in (1.66), the other two singular products in (1.240) are defined in (1.80) and (1.81). The argument for (1.240) is essentially the same as for (1.147) in Step 4 of Proposition 1; in particular, by using Lemma 4 and the triangle inequality we find that

$$\lim_{T\to 0} \|(a^{ext} \diamond \partial_1^2 W)_T - (a^{ext} \diamond \partial_1^2 u)_T + (a^{ext} \diamond \partial_1^2 U)_T\| = 0,$$

which yields (1.240). The relation (1.71) is a consequence of (1.83) and (1.87).

Step 2– (Post-processing the modelling)

We would like to show that $\tilde{q} + w^E$ is modelled after $\tilde{V}(\cdot, a_0)$ with $V_{int} = W_{int} - v(\cdot, a_0)$ according to a^{ext}. Our first step towards showing this is introducing $a_{tr} : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$a_{tr}(x) = a(x_1, 0) \tag{1.241}$$

and using part *ii*) of Lemma 6 with the identifications $a = a^{ext}$ and $a' = a_{tr}$ to obtain that $\tilde{q} + w^E$ is modelled after $\tilde{V}(\cdot, a_0)$ with $V_{int} = W_{int} - u$ according to a_{tr}. In order to swap out the initial condition $V_{int} = W_{int} - u$ for the desired $V_{int} = W_{int} - v(\cdot, a_0)$ in this modelling, we let ν be associated to the modelling of u after $v(\cdot, a_0)$ and, for $V(\cdot, a_0)$ solving (1.5) with $V_{int} = u - v(\cdot, a_0)$, show that

$$\begin{aligned} &\left|\tilde{V}(x, a_{tr}(y)) - \tilde{V}(y, a_{tr}(y)) - \tilde{\nu}^{\int}(y)(x-y)_1\right| \\ &\lesssim N_0(N+1) d^{2\alpha}(x, y) \end{aligned} \tag{1.242}$$

for $x, y \in \mathbb{R}^2$ and

$$\nu^{\int}(y) = e^{-y_2} \int_{\mathbb{R}} \frac{1}{(4\pi a_{tr}(y) y_2)^{\frac{1}{2}}} \nu(s, 0) e^{\frac{-|y_1 - s|^2}{4 y_2 a_{tr}(y)}} \, \mathrm{d}s. \tag{1.243}$$

Once we have shown (1.242) we can use the triangle inequality to see that $\tilde{q} + w^E$ is modelled after $\tilde{V}(\cdot, a_0)$ with $V_{int} = W_{int} - v(\cdot, a_0)$ according to a_{tr}, now with modelling constant bounded by $(N_0^{int} + N_0)(N+1)$; by Lemma 6 this means that $\tilde{q}+w^E$ has the same modelling according to a^{ext}. By (1.173) and the results from Propositions 1 and 2 we find that (1.72) is satisfied.

The argument for (1.242) comes down to using the heat-kernel representation for $V(\cdot, a_0)$, that u is modelled after the $v(\cdot, a_0)$ according to a^{ext} on $\mathbb{R}^2$, and that $a^{ext}(x) = a_{tr}(x)$ when $x_2 = 0$. In particular, using the definition (1.243) we find that

$$\begin{aligned}
&\left|\tilde{V}(x, a_{tr}(y)) - \tilde{V}(y, a_{tr}(y)) - \tilde{\nu}^{\int}(y)(x-y)_1\right| \\
\lesssim & e^{-|y_2|}\left|\int_{\mathbb{R}}\Big(u(x_1 - z(4|x_2|a_{tr}(y))^{\frac{1}{2}}, 0) - u(y_1 - z(4|y_2|a_{tr}(y))^{\frac{1}{2}}, 0)\right. \\
&\quad - \Big(v((x_1 - z(4|x_2|a_{tr}(y))^{\frac{1}{2}}, 0), a(y_1 - z(4|y_2|a_{tr}(y))^{\frac{1}{2}}, 0)) \\
&\quad - v((y_1 - z(4|y_2|a_{tr}(y))^{\frac{1}{2}}, 0), a(y_1 - z(4|y_2|a_{tr}(y))^{\frac{1}{2}}, 0))\Big) \\
&\quad \left. - \nu(y_1 - z(4|y_2|a_{tr}(y))^{\frac{1}{2}}, 0)(x-y)_1\Big)\, e^{-z^2}\, \mathrm{d}z\right| \\
& + |e^{-|x_2|} - e^{-|y_2|}|(\|u\| + \sup_{a_0 \in [\lambda, 1]} \|v(\cdot, a_0)\|).
\end{aligned} \tag{1.244}$$

We first treat the second term on the right-hand side, which can be easily bounded by $|x_2 - y_2| N_0(N+1)$ due to the relation $|e^{-|x_2|} - e^{-|y_2|}| \lesssim ||x_2| - |y_2|| \leq |x_2 - y_2|$ and the bounds (1.83) and (1.322). This is sufficient when $d(x, y) \leq 1$; when $d(x, y) \geq 1$ we use the trivial bound $|e^{-|x_2|} - e^{-|y_2|}| \lesssim 1$ and $d^{2\alpha}(x, y) \geq 1$. For the first term we notice that in the case that $d(x, y) \geq 1$ the standard Hölder estimates (1.83) and (1.322) along with (1.370) and (1.83) yield the desired (1.242). Therefore, we must only still handle the case $d(x, y) \leq 1$, but this requires a bit more work. In particular We begin by using the modelling of u and the estimate (1.83) to bound this term as

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\Big(u(x_1 - z(4|x_2|a_{tr}(y))^{\frac{1}{2}}, 0) - u(y_1 - z(4|y_2|a_{tr}(y))^{\frac{1}{2}}, 0)\right. \\
&\quad - \Big(v((x_1 - z(4|x_2|a_{tr}(y))^{\frac{1}{2}}, 0), a(y_1 - z(4|y_2|a_{tr}(y))^{\frac{1}{2}}, 0)) \\
&\quad - v((y_1 - z(4|y_2|a_{tr}(y))^{\frac{1}{2}}, 0), a(y_1 - z(4|y_2|a_{tr}(y))^{\frac{1}{2}}, 0))\Big) \\
&\quad \left. - \nu(y_1 - z(4|y_2|a_{tr}(y))^{\frac{1}{2}}, 0)(x-y)_1\Big)\, e^{-z^2}\, \mathrm{d}z\right| \\
\lesssim & N_0(N+1) d^{2\alpha}(x, y) + \\
& \left|\int_{\mathbb{R}} \nu(y_1 - z(4|y_2|a_{tr}(y))^{\frac{1}{2}}, 0) z((4|x_2|a_{tr}(y))^{\frac{1}{2}} - (4|y_2|a_{tr}(y))^{\frac{1}{2}}) e^{-z^2}\, \mathrm{d}z\right|.
\end{aligned} \tag{1.245}$$

Continuing, we now split our treatment of the remaining term on the right-hand side into four cases:

Case 1– We assume that $|y_2|^{\frac{1}{2}} \geq d(x,y)$, $|x_2|^{\frac{1}{2}} \geq d(x,y)$, and $|y_2| \leq |x_2|$. Notice that the square-root function is Lipschitz on $\mathbb{R} \times [|y_2|, \infty)$ with Lipschitz constant $\frac{1}{2}|y_2|^{-\frac{1}{2}}$. In particular, we have that

$$|(4|x_2|a_{tr}(y))^{\frac{1}{2}} - (4|y_2|a_{tr}(y))^{\frac{1}{2}}| \lesssim |x_2 - y_2||y_2|^{-\frac{1}{2}}, \tag{1.246}$$

which, after using (1.365) in conjunction with (1.83), yields

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \nu(y_1 - z(4y_2a_{tr}(y))^{\frac{1}{2}}, 0) z((4x_2a_{tr}(y))^{\frac{1}{2}} - (4y_2a_{tr}(y))^{\frac{1}{2}}) e^{-z^2}\,\mathrm{d}z\right| \\
=&\left|\int_{\mathbb{R}} (\nu(y_1 - z(4y_2a_{tr}(y))^{\frac{1}{2}}, 0) - \nu(y_1, 0))\right. \\
&\qquad\qquad \left.\times z((4x_2a_{tr}(y))^{\frac{1}{2}} - (4y_2a_{tr}(y))^{\frac{1}{2}}) e^{-z^2}\,\mathrm{d}z\right| \\
\lesssim& N_0(N+1) \int_{\mathbb{R}^2} |z|^{2\alpha} |y_2|^{\frac{2\alpha-2}{2}} |x_2 - y_2| e^{-z^2}\,\mathrm{d}z \\
\lesssim& N_0(N+1) d^{2\alpha}(x,y).
\end{aligned}$$

Case 2– We assume that $|y_2|^{\frac{1}{2}} \geq d(x,y)$, $|x_2|^{\frac{1}{2}} \geq d(x,y)$, and $x_2 \leq y_2$. The only difference now is that we must use a different Lipschitz constant for the square-root function; namely, now we use $\frac{1}{2}|x_2|^{-\frac{1}{2}}$ as the Lipschitz constant on $\mathbb{R} \times [x_2, \infty)$. Following the same recipe as in the previous case and adding in a couple of uses of the triangle inequality we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \nu(y_1 - z(4|y_2|a_{tr}(y))^{\frac{1}{2}}, 0) z((4|x_2|a_{tr}(y))^{\frac{1}{2}} - (4|y_2|a_{tr}(y))^{\frac{1}{2}}) e^{-z^2}\,\mathrm{d}z\right| \\
\lesssim& N_0(N+1) \int_{\mathbb{R}} |z|^{2\alpha} |y_2|^{\frac{2\alpha-1}{2}} |x_2|^{-\frac{1}{2}} |x_2 - y_2| e^{-z^2}\,\mathrm{d}z \\
\lesssim& N_0(N+1) \int_{\mathbb{R}} |z|^{2\alpha} (|y_2 - x_2|^{\frac{2\alpha-1}{2}} + |x_2|^{\frac{2\alpha-1}{2}}) |x_2|^{-\frac{1}{2}} |x_2 - y_2| e^{-z^2}\,\mathrm{d}z \\
\lesssim& N_0(N+1) d^{2\alpha}(x,y).
\end{aligned}$$

Case 3– We assume that $|x_2|^{\frac{1}{2}} \leq d(x,y)$. Now we use the bound

$$|(4|x_2|a_{tr}(y))^{\frac{1}{2}} - (4|y_2|a_{tr}(y))^{\frac{1}{2}}| \lesssim |x_2 - y_2|^{\frac{1}{2}}, \tag{1.247}$$

which, after using the triangle inequality and the same tools as in the previous case, yields

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \nu(y_1 - z(4|y_2|a_{tr}(y))^{\frac{1}{2}}, 0) z((4|x_2|a_{tr}(y))^{\frac{1}{2}} - (4|y_2|a_{tr}(y))^{\frac{1}{2}}) e^{-z^2}\, \mathrm{d}z\right| \\
\lesssim & N_0(N+1) \int_{\mathbb{R}^2} |z|^{2\alpha} (|y_2 - x_2|^{\frac{2\alpha-1}{2}} + |x_2|^{\frac{2\alpha-1}{2}}) |x_2 - y_2|^{\frac{1}{2}} e^{-z^2}\, \mathrm{d}z \\
\lesssim & N_0(N+1) d^{2\alpha}(x,y).
\end{aligned}$$

Case 4– We assume that $|y_2|^{\frac{1}{2}} \leq d(x,y)$. Reusing (1.247) and again (1.365) along with (1.83), we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \nu(y_1 - z(4|y_2|a_{tr}(y))^{\frac{1}{2}}, 0) z((4|x_2|a_{tr}(y))^{\frac{1}{2}} - (4|y_2|a_{tr}(y))^{\frac{1}{2}}) e^{-z^2}\, \mathrm{d}z\right| \\
\lesssim & N_0(N+1) \int_{\mathbb{R}^2} |z|^{2\alpha} |y_2|^{\frac{2\alpha-1}{2}} |x_2 - y_2|^{\frac{1}{2}} e^{-z^2}\, \mathrm{d}z \\
\lesssim & N_0(N+1) \int_{\mathbb{R}^2} |z|^{2\alpha} d^{2\alpha-1}(x,y) |x_2 - y_2|^{\frac{1}{2}} e^{-z^2}\, \mathrm{d}z \\
\lesssim & N_0(N+1) d^{2\alpha}(x,y).
\end{aligned}$$

Combining these cases with (1.245) yields (1.242), which, when combined with (1.86), gives that $\tilde{q} + w^E$ is modelled after $\tilde{V}(\cdot, a_0)$ with $V_{int} = W_{int} - v(\cdot, a_0)$ according to a_{tr} with a modelling constant bounded by

$$M_{tr} \lesssim N_0(N+1) + \|a\|_\alpha \|W_{int} - u\|_\alpha \overset{(1.83)}{\lesssim} (N_0 + N_0^{int})(N+1). \quad (1.248)$$

By part *ii*) of Lemma 6 we know that this modelling is also according to a^{ext} and thanks to (1.173) that the new modelling constant still satisfies (1.248). We, in particular, obtain that $W^{ext} = u + \tilde{q} + w^E$ is modelled after $(v + \tilde{V})(\cdot, a_0)$, where $V(\cdot, a_0)$ has initial condition $V_{int} = W_{int} - v(\cdot, a_0)$ according to a^{ext} with modelling constant satisfying (1.72) .

ii) We now use the results of part *ii)* of Propositions 1 and 2 in combination with part *iii*) of Lemma 6 to obtain the stability result for Theorem 1.

Step 3– (Stability)

It is immediate that (1.84) and (1.89) yield (1.78). We now consider the modelling of $u_1 - u_0 + \tilde{q}_1 + w_1^E - (\tilde{q}_0 + w_0^E)$. By part *ii*) of Proposition 1 we have that $u_1 - u_0$ is modelled after $(v_1(\cdot, a_0), v_0(\cdot, a_0))$ according to (a_1^{ext}, a_0^{ext}) and $(1, -1)$ and by part *ii)* of Proposition 2 that $\tilde{q}_1 + w_1^E - (\tilde{q}_0 + w_0^E)$ is modelled after $(\tilde{V}_1(\cdot,a_0),\tilde{V}_0(\cdot,a_0))$ with $V_{int,i}=W_{int,i}-u_i$ according to $(1,-1)$ and (a_1^{ext},a_0^{ext}). Using part *iii*) of Lemma 6 the modelling of $\tilde{q}_1+w_1^E-(\tilde{q}_0+w_0^E)$ can be post-processed to give the same modelling according to $(a_{1,tr},a_{0,tr})$. Just like above, we now aim to show that $\tilde{q}_1+w_1^E-(\tilde{q}_0+w_0^E)$ is modelled after $(\tilde{V}_1(\cdot,a_0),\tilde{V}_0(\cdot,a_0))$ with $V_{int,i}=W_{int,i}-v_i(\cdot,a_0)$ according to $(1,-1)$ and $(a_{1,tr},a_{0,tr})$. After we have done this, we again apply part *iii*) of Lemma 6 to switch out $(a_{1,tr},a_{0,tr})$ for (a_1^{ext},a_0^{ext}) in the modelling.

Just like in part *i)*, showing that $\tilde{q}_1+w_1^E-(\tilde{q}_0+w_0^E)$ is modelled after $(\tilde{V}_1(\cdot,a_0),\tilde{V}_0(\cdot,a_0))$ with $V_{int,i}=W_{int,i}-v_i(\cdot,a_0)$ according to $(a_{1,tr},a_{0,tr})$ and $(1,-1)$ comes down to showing that

$$\begin{aligned}&\Big|(\tilde{V}_0(x,a_{0,tr}(y))-\tilde{V}_0(y,a_{0,tr}(y)))\\ &\qquad-(\tilde{V}_1(x,a_{1,tr}(y))-\tilde{V}_1(y,a_{1,tr}(y))-\tilde{\nu}^{\int}(y)(x-y)_1\Big| \\ &\lesssim((N_0+1)\delta N+\delta N_0(N+1))d^{2\alpha}(x,y),\end{aligned}\tag{1.249}$$

where each $V_i(\cdot,a_0)$ has the initial condition $V_{int,i}(\cdot,a_0)=u_i-v_i(\cdot,a_0)$. Here the ν is defined in the same way as (1.243), but with the ν in the definition representing the modelling of u_1-u_0 after (v_1,v_0) according to $(a_{1,tr},a_{0,tr})$ and $(1,-1)$; call the modelling constant here δM. To show (1.249) one uses the exact same argument as to show (1.242) in the previous part, but using the relation (1.84) instead of (1.83) and the bound $[\nu]_{2\alpha-1}\lesssim\delta N_0+\delta M$. We do not repeat the actual calculation here. In the end we find that $\tilde{q}_1+w_1^E-(\tilde{q}_0+w_0^E)$ is modelled after $(\tilde{V}_1(\cdot,a_0),\tilde{V}_0(\cdot,a_0))$ with $V_{int,i}=W_{int,i}-v_i(\cdot,a_0)$ according to $(a_{1,tr},a_{0,tr})$ and $(1,-1)$, where the modelling constant is bounded as

$$\delta M_{tr}\lesssim(N_0+1)\delta N+(\delta N_0+\delta N_0^{int})(N+1).$$

Notice that here we have used (1.86). By part *iii*) of Lemma 6 this upgrades to the same modelling, but according to (a_1^{ext},a_0^{ext}). The new modelling constant M satisfies (1.79). □

1.4 Construction of Singular Products for Theorem 1

In this section we prove Lemma 2, Lemma 3, Corollary 1, and Lemma 4, which are all stated in Section 1.3.

1.4.1 Construction of the Reference Products for Theorem 1

We begin by proving Lemma 2, Lemma 3, and Corollary 1 in the order in which they are presented in Section 1.3.

Proof of Lemma 2. Notice that $\partial_1^2 G$ is clearly well-defined as a distribution on $\mathbb{R}^2$ thanks to the bound (1.50) and the assumption that $\alpha \in (0,1)$. This means that for any $F \in L^\infty(\mathbb{R}^2)$ the product given by

$$F \diamond \partial_1^2 G := F \partial_1^2 G \tag{1.250}$$

is classically defined. In order to obtain (1.51) we fix $x \in \mathbb{R}^2$ and, using (1.105) and (1.50), write

$$\begin{aligned}
&\left|[F, (\cdot)_T] \diamond \partial_1^2 G(x)\right| \\
=&\left|\int_{\mathbb{R}^2} (F(x) - F(y))\psi_T(x-y)\partial_1^2 G(y)\,\mathrm{d}y\right| \\
\lesssim& C(G)\,[F]_\alpha^{loc} \left(\int_{B_1(x)} |\psi_T(x-y)|\, d^\alpha(x,y)\,(|y_2|^{\frac{\alpha-2}{2}} + |y_2|^{\frac{2\alpha-2}{2}})\,\mathrm{d}y\right. \\
&\qquad\qquad \left. + \int_{B_1^c(x)} |\psi_T(x-y)|\, d(x,y)\,(|y_2|^{\frac{\alpha-2}{2}} + |y_2|^{\frac{2\alpha-2}{2}})\,\mathrm{d}y\right) \\
\lesssim& C(G)\,[F]_\alpha^{loc}\,(T^{\frac{1}{4}})^{2\alpha-2} \times \\
&\left(\int_{-1}^{1}\int_{\mathbb{R}} |\psi_1(\hat{x}-\hat{y})|\,(d^\alpha(\hat{x},\hat{y}) + d(\hat{x},\hat{y}))\,(|\hat{y}_2|^{\frac{\alpha-2}{2}} + |\hat{y}_2|^{\frac{2\alpha-2}{2}})\,\mathrm{d}\hat{y}_1\,\mathrm{d}\hat{y}_2\right. \\
&\qquad\qquad \left. + \int_{\mathbb{R}^2} |\psi_1(\hat{x}-\hat{y})|\,(d^\alpha(\hat{x},\hat{y}) + d(\hat{x},\hat{y}))\,\mathrm{d}\hat{y}\right),
\end{aligned} \tag{1.251}$$

where we have rescaled the variables as indicated in (1.16) and used that $T \leq 1$. To handle the first term on the right-hand side of (1.251) we use that

$$p(\cdot) = \int_{\mathbb{R}} |\psi_1(x_1, \cdot)| \left(|x_1|^\alpha + |\cdot|^{\frac{\alpha}{2}} + |x_1| + |\cdot|^{\frac{1}{2}}\right) \mathrm{d}x_1 \in L^\infty(\mathbb{R}),$$

which follows from ψ_1 being a Schwarz function. In particular, this is a simple application of Morrey's inequality and the basic properties of Schwarz functions. Using this we then have that

$$\begin{aligned}
&\int_{-1}^{1}\int_{\mathbb{R}} |\psi_1(x-y)|\,(d^\alpha(x,y) + d(x,y))\,(|y_2|^{\frac{\alpha-2}{2}} + |y_2|^{\frac{2\alpha-2}{2}})\,\mathrm{d}y_1\,\mathrm{d}y_2 \\
\lesssim& \|p\| \int_{-1}^{1} (|y_2|^{\frac{\alpha-2}{2}} + |y_2|^{\frac{2\alpha-2}{2}})\,\mathrm{d}y_2 < \infty.
\end{aligned}$$

After another application of the Schwarz-ness of ψ_1 ,now to the second term on the right-hand side of (1.251), we obtain the desired (1.51).

□

Towards our proof of Corollary 1 we then prove Lemma 3, which we have actually already gotten big milage out of in the proof of Proposition 2. As we have already mentioned in Section 1.3, this proof relies mainly on the heat-kernel representation (1.58) of $V(\cdot, a_0)$. Here comes the argument:

Proof of Lemma 3. i) Fix $0 \leq k \leq 2$ and $j \geq 0$ such that $k + j \geq 1$. We use the convention that P_k and $P_{k,j}$ represent generic polynomials dependent on k or k and j respectively; even within the same line two appearances of these expressions may denote different polynomials. Changing variables, using the relations (1.59), and letting C be a generic constant that changes from line to line we then write

$$\begin{aligned}
&\partial_{a_0}^j \partial_1^k G(a_0, x_1 - y, x_2) \\
=&\partial_{a_0}^j \partial_1^k \left(\frac{e^{-x_2}}{(4\pi a_0 x_2)^{\frac{1}{2}}} e^{-z^2} \right) \\
=&C e^{-x_2} \partial_{a_0}^j \left((a_0 x_2)^{-\frac{1+k}{2}} P_k(z) e^{-z^2} \right) \\
=&C e^{-x_2} \left(P_j(a_0^{-\frac{1}{2}}) P_k(z) + P_{k,j}(z, a_0^{-\frac{1}{2}}) \right) e^{-z^2} x_2^{-\frac{1+k}{2}}.
\end{aligned} \tag{1.252}$$

Fixing a point $x \in \mathbb{R}^2_+$ and $a_0 \in [\lambda, 1]$, we then notice that

$$\begin{aligned}
&\int_{\mathbb{R}} V_{int}(y) \partial_{a_0}^j \partial_1^k G(a_0, x_1 - y, x_2) \, \mathrm{d}y \\
=&\int_{\mathbb{R}} (V_{int}(y) - V_{int}(x_1)) \partial_{a_0}^j \partial_1^k G(a_0, x_1 - y, x_2) \, \mathrm{d}y,
\end{aligned}$$

which follows easily from an integration by parts when $k > 0$ and from the observation that $\partial_{a_0}^j \int_{\mathbb{R}} G(a_0, x_2, x_1 - y) \, \mathrm{d}y = 0$ when $k = 0$ and $j > 0$. To finish we use (1.252) to calculate

$$\begin{aligned}
&\left| \int_{\mathbb{R}} (V_{int}(y) - V_{int}(x_1)) \partial_{a_0}^j \partial_1^k G(a_0, x_1 - y, x_2) \, \mathrm{d}y \right| \\
\lesssim&[V_{int}]_\alpha \int_{\mathbb{R}} |z|^\alpha x_2^{\frac{\alpha}{2}} |\partial_{a_0}^j \partial_1^k G(a_0, x_1 - y, x_2)| \, \mathrm{d}y \\
\lesssim&[V_{int}]_\alpha e^{-x_2} x_2^{\frac{\alpha-k}{2}} \int_{\mathbb{R}} |z|^\alpha \left(P_{k,j}(z, a_0^{-\frac{1}{2}}) + P_j(a_0^{-\frac{1}{2}}) P_{k,j}(z) \right) e^{-z^2} \, \mathrm{d}z \\
\lesssim&C(\lambda, \alpha) [V_{int}]_\alpha e^{-x_2} x_2^{\frac{\alpha-k}{2}}.
\end{aligned}$$

As a last remark, notice that because $\alpha \in (0, 1)$ the estimate (1.52) implies that $\partial_{a_0}^j \partial_1^k V(\cdot, a_0) \in L^1_{loc}(\mathbb{R}^2_+)$ and hence is well-defined as a distribution.

ii) For our proof of (1.53) we again fix $j \geq 0$, $x \in \mathbb{R}^2_+$, and $a_0 \in [\lambda, 1]$. The relation (1.53) then also easily follows from (1.58) and (1.252) for $k=0$:

$$\begin{aligned}
&\left|\partial_{a_0}^j V(x_1,x_2,a_0)\right| \\
\lesssim&\left|\int_{\mathbb{R}} V_{int}(y)e^{-x_2}\left(P_j(a_0^{-\frac{1}{2}})+P_j(a_0^{-\frac{1}{2}},z)\right)e^{-z^2}x_2^{-\frac{1}{2}}\,\mathrm{d}y\right| \\
\lesssim& e^{-x_2}\|V_{int}\|.
\end{aligned}$$

iii) Notice that (1.54) for $j=0$ is the classical Hölder estimate for (1.5), which follows from the heat-kernel formulation of $V(\cdot,a_0)$. To get the estimate for $1\le j\le 3$ we derive the equations satisfied by $\partial_{a_0}V(\cdot,a_0)$, $\partial_{a_0}^2V(\cdot,a_0)$, and $\partial_{a_0}^3V(\cdot,a_0)$. The equation for $\partial_{a_0}V(\cdot,a_0)$ is derived by differentiating (1.5) in terms of a_0, which yields that

$$\begin{aligned}
(\partial_2-a_0\partial_1^2+1)\partial_{a_0}V(\cdot,a_0)&=\partial_1^2V(\cdot,a_0) &&\text{in}\quad \mathbb{R}^2_+,\\
\partial_{a_0}V(\cdot,a_0)&=0 &&\text{on}\quad \partial\mathbb{R}^2_+.
\end{aligned}\tag{1.253}$$

Taking one more a_0-derivative we find that $\partial_{a_0}^2V(\cdot,a_0)$ solves

$$\begin{aligned}
(\partial_2-a_0\partial_1^2+1)\partial_{a_0}^2V(\cdot,a_0)&=2\partial_1^2\partial_{a_0}V(\cdot,a_0) &&\text{in}\quad \mathbb{R}^2_+,\\
\partial_{a_0}^2V(\cdot,a_0)&=0 &&\text{on}\quad \partial\mathbb{R}^2_+
\end{aligned}\tag{1.254}$$

and differentiating a third time gives that $\partial_{a_0}^3V(\cdot,a_0)$ solves

$$\begin{aligned}
(\partial_2-a_0\partial_1^2+1)\partial_{a_0}^3V(\cdot,a_0)&=3\partial_1^2\partial_{a_0}^2V(\cdot,a_0) &&\text{in}\quad \mathbb{R}^2_+,\\
\partial_{a_0}^3V(\cdot,a_0)&=0 &&\text{on}\quad \partial\mathbb{R}^2_+.
\end{aligned}\tag{1.255}$$

From these equations we can read-off (1.54); of course, this presupposes the standard Schauder estimate $[g]_\alpha\lesssim[f]_{\alpha-2}$ for our definition of the negative Hölder seminorm and g solving

$$\begin{aligned}
(\partial_2-a_0\partial_1^2+1)g&=f &&\text{in}\quad \mathbb{R}^2_+,\\
g&=0 &&\text{on}\quad \partial\mathbb{R}^2_+.
\end{aligned}$$

This estimate follows from decomposing $f=\partial_2f^2+\partial_1^2f^1$ for two C^α-functions that are near optimal in the sense of Definition 2, using the standard Schauder estimates for the solutions of the initial value problems with right-hand sides f^i, the linearity of the equation, and the uniqueness of the solution g. Using the Hölder estimate that we obtain in this way, we find that

$$\begin{aligned}
&[\partial_{a_0}V(\cdot,a_0)]_\alpha\lesssim[\partial_1^2V(\cdot,a_0)]_{\alpha-2}\lesssim\|V_{int}\|_\alpha,\\
&[\partial_{a_0}^2V(\cdot,a_0)]_\alpha\lesssim[\partial_1^2\partial_{a_0}V(\cdot,a_0)]_{\alpha-2}\lesssim[\partial_{a_0}V(\cdot,a_0)]_\alpha\lesssim\|V_{int}\|_\alpha,\\
\text{and}\quad&[\partial_{a_0}^3V(\cdot,a_0)]_\alpha\lesssim[\partial_1^2\partial_{a_0}^2V(\cdot,a_0)]_{\alpha-2}\lesssim[\partial_{a_0}^2V(\cdot,a_0)]_\alpha\lesssim\|V_{int}\|_\alpha.
\end{aligned}$$

iv) For our proof of (1.55) we fix $x, y \in \mathbb{R}^2_+$ and $0 \leq j \leq 1$. We then use the triangle inequality to write

$$\begin{aligned}
&|\partial^j_{a_0} V(x_1, x_2, a_0) - \partial^j_{a_0} V(y_1, y_2, a_0)| \\
\leq &|\partial^j_{a_0} V(x_1, x_2, a_0) - \partial^j_{a_0} V(x_1, y_2, a_0)| \\
&+ |\partial^j_{a_0} V(x_1, y_2, a_0) - \partial^j_{a_0} V(y_1, y_2, a_0)|
\end{aligned} \tag{1.256}$$

and treat the two terms on the right-hand side separately. For the second term we notice that

$$\begin{aligned}
&|\partial^j_{a_0} V(x_1, y_2, a_0) - \partial^j_{a_0} V(y_1, y_2, a_0)| \\
\leq &|\partial^j_{a_0} V(x_1, y_2, a_0) - \partial^j_{a_0} V(y_1, y_2, a_0)|^{\frac{2-2\alpha}{2-\alpha}} \\
&\qquad \times |\partial^j_{a_0} V(x_1, y_2, a_0) - \partial^j_{a_0} V(y_1, y_2, a_0)|^{\frac{\alpha}{2-\alpha}} \\
\lesssim &(\|\partial^j_{a_0} V(\cdot, y_2, a_0)\|_\alpha |x_1 - y_1|^\alpha)^{\frac{2-2\alpha}{2-\alpha}} \left(|x_1 - y_1|^2 \|\partial_1^2 \partial^j_{a_0} V(\cdot, y_2, a_0)\|\right)^{\frac{\alpha}{2-\alpha}} \\
\lesssim &\|V_{int}\|_\alpha y_2^{-\frac{\alpha}{2}} d^{2\alpha}(x, y),
\end{aligned} \tag{1.257}$$

where we have used (1.52), the Hölder bound (1.54), and that $\alpha \in (0, 1)$.

We may use essentially the same argument to treat the first term on the right-hand side of (1.256). The only additional ingredient that we use is that, thanks to the equations (1.5) and (1.253), after applying (1.52) and (1.53) we have that

$$\begin{aligned}
\|\partial_2 V(\cdot, x_2, a_0)\| \leq &\|\partial_1^2 V(\cdot, x_2, a_0)\| + \|V(\cdot, x_2, a_0)\| \\
\lesssim &\|V_{int}\|_\alpha (x_2^{\frac{\alpha-2}{2}} + e^{-x_2}) \\
\lesssim &\|V_{int}\|_\alpha x_2^{\frac{\alpha-2}{2}}
\end{aligned} \tag{1.258}$$

and, similarly,

$$\begin{aligned}
&\|\partial_2 \partial_{a_0} V(\cdot, x_2, a_0)\| \\
\leq &\|\partial_1^2 V(\cdot, x_2, a_0)\| + \|\partial_1^2 \partial_{a_0} V(\cdot, x_2, a_0)\| + \|\partial_{a_0} V(\cdot, x_2, a_0)\| \\
\lesssim &\|V_{int}\|_\alpha x_2^{\frac{\alpha-2}{2}}.
\end{aligned} \tag{1.259}$$

Combining (1.258) (in the case that $j = 0$) or (1.259) (in the case that $j = 1$) with the technique of (1.257) then gives

$$\begin{aligned}
&|\partial^j_{a_0} V(x_1, x_2, a_0) - \partial^j_{a_0} V(x_1, y_2, a_0)| \\
\leq &|\partial^j_{a_0} V(x_1, x_2, a_0) - \partial^j_{a_0} V(x_1, y_2, a_0)|^{\frac{2-2\alpha}{2-\alpha}} \\
&\qquad \times |\partial^j_{a_0} V(x_1, x_2, a_0) - \partial^j_{a_0} V(x_1, y_2, a_0)|^{\frac{\alpha}{2-\alpha}} \\
\leq &(|x_2 - y_2|^{\frac{\alpha}{2}} [\partial^j_{a_0} V]_\alpha)^{\frac{2-2\alpha}{2-\alpha}} (|x_2 - y_2| \|\partial_2 \partial^j_{a_0} V(\cdot, x_2, a_0)\|)^{\frac{\alpha}{2-\alpha}} \\
\lesssim &\|V_{int}\|_\alpha x_2^{-\frac{\alpha}{2}} d^{2\alpha}(x, y)
\end{aligned} \tag{1.260}$$

Together (1.256), (1.257), (1.260) then yield (1.55).

v) Notice that when (1.5) has no massive term then (1.58) still holds, but the Green's function in (1.57) is replaced by the same without the exponential factor e^{-x_2}. Our claim then immediately follows from the above arguments.

□

Using this lemma we can then process Lemma 2 to obtain Corollary 1:

Proof of Corollary 1. This corollary follows immediately from Lemma 2 with the identification $G = \tilde{V}(\cdot, a_0)$ since (1.50) holds for $C(\tilde{V}(\cdot, a_0)) = [V_{int}]_\alpha$ by part i) of Lemma 3.

□

1.4.2 Proof of the Second Reconstruction Lemma

We move on to the proof of the second reconstruction lemma, which we use in our treatment of the linear problem in Theorem 1. As already mentioned, the proof presented here is essentially the same as in [47] and is only included in an abbreviated version in order for this thesis to be self-contained. There are small technical differences due to the loss of periodicity in the x_2-direction.

Proof of Lemma 4. In [47] it is first shown that for all dyadic multiples T of τ, i. e. $T = 2^n\tau$ for some $n \in \mathbb{N}$, the relation

$$(F\partial_1^2 u_T - \sigma_i E\,[F,(\cdot)_T] \diamond \partial_1^2 w_i) - (F\partial_1^2 u_\tau - \sigma_i E\,[F,(\cdot)_\tau] \diamond \partial_1^2 w_i)_{T-\tau} =$$
$$\sum_{t=\tau 2^i \text{ for } 0\le i\le n} \Bigg([F,(\cdot)_t]\,\partial_1^2 u_t - \sigma_i E\,[F,(\cdot)_t]\,\partial_1^2 w_{it}$$
$$-\sigma_i\,[E,(\cdot)_t]\,[F,(\cdot)_t] \diamond \partial_1^2 w_i - [\sigma_i,(\cdot)_t] E[F,(\cdot)_t] \diamond \partial_1^2 w_i \Bigg)_{T-2t}$$

holds. As the argument for this identity only relies on the semigroup property of the convolution kernel and we use the same kernel here, we use this decomposition without proof here. We then find that

$$\begin{aligned}&\left\|F\partial_1^2 u_T - \sigma_i E\,[F,(\cdot)_T] \diamond \partial_1^2 w_i - (F\partial_1^2 u_\tau - \sigma_i E\,[F,(\cdot)_\tau] \diamond \partial_1^2 w_i)_{T-\tau}\right\| \\ &\lesssim \left([F]_\alpha M + NN_i\|\sigma_i\|_\alpha\right)(T^{\frac{1}{4}})^{3\alpha-2}\end{aligned} \tag{1.261}$$

for any $\tau > 0$. In particular, using (1.21) and the triangle inequality one first writes

$$
\begin{aligned}
&\left\|F\partial_1^2 u_T - \sigma_i E\left[F,(\cdot)_T\right]\diamond\partial_1^2 w_i - (F\partial_1^2 u_\tau - \sigma_i E\left[F,(\cdot)_t\right]\diamond\partial_1^2 w_i)_{T-\tau}\right\| \\
&\lesssim \sum_{t=\tau 2^i \text{ for } 0\le i\le n} \Bigg(\left\|\left[F,(\cdot)_t\right]\partial_1^2 u_t - \sigma_i E\left[F,(\cdot)_t\right]\partial_1^2 w_{it}\right\| \\
&\qquad\qquad + \left\|\sigma_i\left[E,(\cdot)_t\right]\left[F,(\cdot)_t\right]\diamond\partial_1^2 w_i\right\| + \left\|[\sigma_i,(\cdot)_t]E[F,(\cdot)_t]\diamond\partial_1^2 w_i\right\| \Bigg).
\end{aligned}
$$

The relation (1.261) then follows from the three relations

$$
\begin{aligned}
\left\|[F,(\cdot)_t]\,\partial_1^2 u_t - \sigma_i E\left[F,(\cdot)_t\right]\partial_1^2 w_{it}\right\| &\lesssim [F]_\alpha\, M(t^{\frac14})^{3\alpha-2}, &(1.262)\\
\left\|\sigma_i\left[E,(\cdot)_t\right]\left[F,(\cdot)_t\right]\diamond\partial_1^2 w_i\right\| &\lesssim \|\sigma_i\|\,[a]_\alpha\, N N_i(t^{\frac14})^{3\alpha-2}, &(1.263)\\
\text{and}\quad \left\|[\sigma_i,(\cdot)_t]E[F,(\cdot)_t]\diamond\partial_1^2 w_i\right\| &\lesssim [\sigma_i]_\alpha\, N N_i(t^{\frac14})^{3\alpha-2}, &(1.264)
\end{aligned}
$$

which hold for any $t > 0$, and that $[a]_\alpha \le 1$ and $\alpha \in (\frac{2}{3}, 1)$.

The estimates (1.262), (1.263), and (1.264) are proven in the same way as in [47], but to maintain oversight we give their proofs here. We start with (1.262) and first write

$$
\begin{aligned}
&([F,(\cdot)_t]\,\partial_1^2 u_t - \sigma_i E\left[F,(\cdot)_t\right]\partial_1^2 w_{it})(x) \\
&= \int_{\mathbb{R}^2} \psi_t(x-y)(F(x)-F(y))(\partial_1^2 u_t(y) - \sigma_i(x)\partial_1^2 w_{it}(y,a(x)))\,\mathrm{d}y
\end{aligned}
\tag{1.265}
$$

for $x \in \mathbb{R}^2$ and then, furthermore, observe the identity

$$
\begin{aligned}
&\partial_1^2 u_t(y) - \sigma_i(x)\partial_1^2 w_{it}(y,a(x)) \\
&= \int_{\mathbb{R}^2} \partial_1^2\psi_t(y-z)(u(z) - \sigma_i(x) w_i(z,a(x)))\,\mathrm{d}z.
\end{aligned}
\tag{1.266}
$$

Using that ψ_t is a Schwarz function on $\mathbb{R}^2$ that is even in the first variable, we further process the right-hand side of (1.266) by smuggling in terms and then using the modelling of u along with (1.18):

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^2} \partial_1^2\psi_t(y-z)(u(z)-\sigma_i(x)w_i(z,a(x)))\,\mathrm{d}z\right| \\
&\le \int_{\mathbb{R}^2} |\partial_1^2\psi_t(y-z)| \\
&\qquad \times |u(z)-u(x)-\sigma_i(x)(w_i(z,a(x))-w_i(x,a(x)))-\nu(x)(z-x)_1|\,\mathrm{d}z \\
&\lesssim M\int_{\mathbb{R}^2} \partial_1^2\psi_t(y-z)(d^{2\alpha}(y,z)+d^{2\alpha}(x,y))\,\mathrm{d}z \\
&\lesssim M((t^{\frac14})^{2\alpha-2} + (t^{\frac14})^{-2} d^{2\alpha}(x,y)).
\end{aligned}
$$

Plugging this result into (1.265) and again using (1.18) we obtain (1.262).

Obtaining (1.263) and (1.264) is mainly an issue of processing notation. For (1.263) we set $\tilde{w} = [F, (\cdot)_t] \diamond \partial_1^2 w_i$ and use (1.18) to write

$$
\begin{aligned}
&\left|\sigma_i(x)\,[E, (\cdot)_t]\,[F, (\cdot)_t] \diamond \partial_1^2 w_i(x)\right| \\
=&\left|\sigma_i(x) \int_{\mathbb{R}^2} \psi_t(x-y)(\tilde{w}(y, a(x)) - \tilde{w}(y, a(y)))\,\mathrm{d}y\right| \\
\lesssim&\|\sigma_i\|\,[a]_\alpha \sup_{a_0 \in [\lambda, 1]} \left\|\frac{\partial}{\partial a_0}\tilde{w}(\cdot, a_0)\right\| (t^{\frac{1}{4}})^\alpha,
\end{aligned}
$$

which holds for all $x \in \mathbb{R}^2$. The desired estimate (1.263) then follows from the assumption (1.61). Moving on, we find that (1.264) follows from the previous argument for (1.263). In particular, now letting $\tilde{w} = E[F, (\cdot)_t] \diamond \partial_1^2 w_i$ and using (1.18) we write

$$
\begin{aligned}
\left|([\sigma_i, (\cdot)_t]E[F, (\cdot)_t] \diamond \partial_1^2 w_i)(x)\right| =& \left|\int_{\mathbb{R}^2} (\sigma_i(x) - \sigma_i(y))\psi_t(x-y)\tilde{w}(y)\,\mathrm{d}y\right| \\
\lesssim&[\sigma_i]_\alpha \|\tilde{w}\| (t^{\frac{1}{4}})^\alpha,
\end{aligned}
$$

which gives (1.264).

We now conclude the argument. Using the notation

$$\mathcal{F}^\tau = F\partial_1^2 u_\tau - \sigma_i E\,[F, (\,\cdot\,)_\tau] \diamond \partial_1^2 w_i,$$

we define $F \diamond \partial_1^2 u$ as the distributional limit of the sequence $\{\mathcal{F}^\tau\}_\tau$ as $\tau \to 0$. To see that this limit exists we first prove

$$\|\mathcal{F}^T\| \lesssim (\|F\|(M + N_i) + \|\sigma_i\| N N_i)(T^{\frac{1}{4}})^{\alpha-2}, \tag{1.267}$$

which we then combine with the rewritten (1.261)

$$\|\mathcal{F}^T - (\mathcal{F}^\tau)_{T-\tau}\| \lesssim ([F]_\alpha M + N N_i \|\sigma_i\|_\alpha)(T^{\frac{1}{4}})^{3\alpha-2} \tag{1.268}$$

to obtain

$$\|(\mathcal{F}^\tau)_T\| \lesssim (\|F\|_\alpha (M + N_i) + N N_i \|\sigma_i\|_\alpha)(T^{\frac{1}{4}})^{\alpha-2}; \tag{1.269}$$

of course, all of these relations hold for $T \leq 1$. With (1.269) in-hand we can then use the Arzelà- Ascoli Theorem in order to pass to the limit. We will go over this in detail below, but first we give the argument for (1.267). Here, we first use the triangle inequality to write

$$\|\mathcal{F}^T\| \leq \|F\partial_1^2 u_T\| + \|\sigma_i\| \sup_{a_0 \in [\lambda, 1]} \|\,[F, (\,\cdot\,)_\tau] \diamond \partial_1^2 w_i(\cdot, a_0)\|$$

and then apply (1.20) in the style of (1.116) to obtain

$$\|\partial_1^2 u_T\| \lesssim [u]_\alpha^{loc}(T^{\frac{1}{4}})^{\alpha-2}$$

Combining these two estimates with the assumptions (1.60) and (1.61) and the bound (1.107) we obtain (1.267).

Notice that the equivalence (1.22) then implies that for every $\tau > 0$ there exists a decomposition

$$\mathcal{F}^\tau = \partial_1^2 \mathcal{F}^{\tau,1} + \partial_2 \mathcal{F}^{\tau,2} \tag{1.270}$$

such that the local seminorms $[\mathcal{F}^{\tau,i}]_\alpha^{loc}$ are bounded uniformly in τ and we may assume that $\mathcal{F}^{\tau,i}(0) = 0$. By the Arzelà- Ascoli theorem the distributional limit $F \diamond \partial_1^2 u$ is then well-defined and has a finite local $C^{\alpha-2}$ seminorm. By the lower semicontinuity of the L^∞-norm with respect to weak-$*$ convergence we may then pass to the limit in (1.268), which gives (1.62), and also using (1.61) gives (1.63). For the uniqueness of the distribution $F \diamond \partial_1^2 u$ satisfying (1.62) we argue by contradiction and assume that there is another such distribution h. By the triangle inequality we then find that

$$\lim_{T\to 0} \|(F \diamond \partial_1^2 u)_T - h_T\| = 0, \tag{1.271}$$

which confirms that $F \diamond \partial_1^2 u = h$. □

1.5 Equivalent Local $C^{\alpha-2}$-seminorm

In this section we give a proof of Lemma 1 that is motivated by the proof of a similar result (Lemma 5) in [36]. As a technical tool, we make use the convolution kernel $e^{-T}\psi_T$ that is associated to the semigroup of the operator $\mathcal{A} := \partial_1^4 - \partial_2^2 + 1$. We use the notational convention that $f * e^{-T}\psi_T = f_T^m$ and, as always, $f * \psi_T = f_T$.

Proof of Lemma 1. For brevity, throughout this proof we fix a convex set $\Omega \subseteq \mathbb{R}^2$ and use the notation $\|\cdot\|$ to denote $\|\cdot\|_\Omega$ and $[\,\cdot\,]_\alpha$ to denote $[\,\cdot\,]_{\alpha;\Omega}$.

Step 1– (Replacing ψ_T by $e^{-T}\psi_T$)

In this step we make the simple observation that if we show the equivalences (1.22), (1.23), and (1.24) for ψ_T replaced by $e^{-T}\psi_T$, then this yields our claim. The reason for this is that thanks to the restriction $T \leq 1$ in the supremums we are able to swallow the terms e^{-T} into the universal constants of the statements.

Step 2– (Bound for the C^α- seminorm)

We first show that

$$[f]_\alpha \lesssim \sup_{T\le 1}(T^{\frac{1}{4}})^{-\alpha}\|T\mathcal{A}f_T^m\| \tag{1.272}$$

for $\alpha \in (0,1)$. Once we have shown (1.272), (1.24) follows immediately by using Young's inequality for convolutions to notice that

$$\|T\mathcal{A}f_T^m\| = \|T\mathcal{A}\psi_{\frac{T}{2}}^m * f_{\frac{T}{2}}^m\| \overset{(1.18),(1.19)}{\lesssim} (1+T)\|f_{\frac{T}{2}}^m\| \overset{(1.21)}{\lesssim} \|f_T^m\|.$$

We now give our argument for (1.272). As the left- and right-hand sides of (1.272) scale the same if we replace f by cf for some $c\in\mathbb{R}$, we may assume that

$$\sup_{T\le 1}(T^{\frac{1}{4}})^{-\alpha}\|T\mathcal{A}f_T^m\| = 1. \tag{1.273}$$

Notice that due to the semigroup property and (1.21), the relation (1.273) implies that

$$\|T\mathcal{A}f_T^m\| = Te^{-T}\|(\mathcal{A}f_1)_{T-1}\| \lesssim Te^{-T}\|\mathcal{A}f_1\| \lesssim 1$$

when $T>1$ and combining this with (1.273) we obtain

$$\sup_{T>0}(T^{\frac{1}{4}})^{-\alpha}\|T\mathcal{A}f_T^m\| \lesssim 1. \tag{1.274}$$

For $j,l\ge 0$ and $T>0$ we may then use (1.274), the semigroup property of $e^{-T}\psi_T$, and (1.18) to write

$$\begin{aligned}\|\partial_1^j\partial_2^l\mathcal{A}f_T^m\| &= e^{-\frac{T}{2}}\|\partial_1^j\partial_2^l\psi_{\frac{T}{2}} * \mathcal{A}f_{\frac{T}{2}}^m\| \\ &\lesssim e^{-\frac{T}{2}}(T^{\frac{1}{4}})^{-j-2l}\|\mathcal{A}f_{\frac{T}{2}}^m\| \\ &\lesssim e^{-\frac{T}{2}}(T^{\frac{1}{4}})^{-j-2l+\alpha-4}.\end{aligned} \tag{1.275}$$

Continuing, we notice that by the definition of ψ_T, $e^{-T}\psi_T$ is a smooth solution of $(\partial_T+\mathcal{A})e^{-T}\psi_T=0$ and that, since $e^{-T}\psi_T$ has integrable derivatives, $\partial_T(e^{-T}\psi_T)$ is integrable, and we have that $\|f_\epsilon\|<\infty$ for all $\epsilon>0$, f_T^m is a smooth solution of $(\partial_T+\mathcal{A})f_T^m=0$. Fixing $j,l\ge 0$ and using (1.275) allows us to for all $0<t<T$ write

$$\begin{aligned}\|\partial_1^j\partial_2^l(f_t^m - f_T^m)\| &= \left\|\int_t^T \partial_1^j\partial_2^l\mathcal{A}f_s^m\,\mathrm{d}s\right\| \\ &\lesssim \int_t^T e^{-\frac{s}{2}}(s^{\frac{1}{4}})^{-j-2l+\alpha-4}\,\mathrm{d}s \\ &\lesssim ((T^{\frac{1}{4}})^{-j-2l+\alpha} + (t^{\frac{1}{4}})^{-j-2l+\alpha}).\end{aligned} \tag{1.276}$$

In the case that $j = l = 0$ this yields that

$$\|f_t^m - f_T^m\| \lesssim (T^{\frac{1}{4}})^\alpha, \tag{1.277}$$

which implies that (1.277) holds also for $t = 0$. Since in this step we have, in particular, that $\|f * \psi_1\| < \infty$, we can also write

$$\lim_{T\to\infty} \|\partial_1^j \partial_2^l f_T^m\| \lesssim e^{-(T-1)}((T-1)^{\frac{1}{4}})^{-j-2l}\|f_1\| \to 0 \text{ as } T \to \infty. \tag{1.278}$$

Fixing $0 < t < T$ and $j, l \geq 0$ such that $j + l \geq 1$, we then use the triangle inequality to write

$$\begin{aligned}\|\partial_1^j \partial_2^l f_t^m\| &\leq \|\partial_1^j \partial_2^l (f_t^m - f_T^m)\| + \|\partial_1^j \partial_2^l f_T^m\| \\ &\lesssim ((t^{\frac{1}{4}})^{-j-2l+\alpha} + (T^{\frac{1}{4}})^{-j-2l+\alpha}) + \|\partial_1^j \partial_2^l f_T^m\|,\end{aligned} \tag{1.279}$$

which after using (1.278) and (1.277) and letting $T \to \infty$ gives

$$\|\partial_1^j \partial_2^l f_t^m\| \lesssim (t^{\frac{1}{4}})^{-j-2l+\alpha}. \tag{1.280}$$

We have now already shown the key point of this argument: the function f is "close-to" the convolved function f_T by (1.277) with $t = 0$ and the convolved function has nicely behaved derivatives in the sense of (1.280).

To finish the argument for (1.272) we fix $T > 0$ and two distinct points $x, y \in \Omega$. We then find that

$$|f_T^m(y) - f_T^m(x)| \leq (\|\partial_1 f_T^m\| d(y,x) + \|\partial_2 f_T^m\| d^2(y,x)),$$

which we combine with (1.277) for $t = 0$ and (1.280) to obtain

$$\begin{aligned}|f(y) - f(x)| &\lesssim \|f - f_T^m\| + \|\partial_1 f_T^m\| d(y,x) + \|\partial_2 f_T^m\| d^2(y,x) \\ &\lesssim (T^{\frac{1}{4}})^\alpha + (T^{\frac{1}{4}})^{\alpha-1} d(y,x) + (T^{\frac{1}{4}})^{\alpha-2} d^2(y,x),\end{aligned} \tag{1.281}$$

which we may further process by setting $T^{\frac{1}{4}} = d(y,x)$. This yields that $|f(y) - f(x)| \lesssim d^\alpha(y,x)$.

Step 3– (Using the $C^{\alpha-2}$ seminorm as an upper-bound)

In this step we take $\Omega = \mathbb{R}^2$ and we show that

$$\sup_{T>0} (T^{\frac{1}{4}})^{2-\alpha} \|f_T\| \lesssim [f]_{\alpha-2} \tag{1.282}$$

for $\alpha \in (0,1)$. We decompose $f = \partial_1^2 f^1 + \partial_2 f^2$ in a way that is near optimal in the sense of Definition 2. For such a tuple (f^1, f^2) the use of (1.20) then yields

$$\sup_{T>0} (T^{\frac{1}{4}})^{2-\alpha} \|f_T\| = \sup_{T>0} (T^{\frac{1}{4}})^{2-\alpha} \|(\partial_1^2 f^1 + \partial_2 f^2)_T\| \lesssim [f]_{\alpha-2} \tag{1.283}$$

as desired.

Step 4 – A specific decomposition of f)

Towards our proof of (1.22) we assume that

$$\sup_{T\le 1}(T^{\frac14})^{2-\alpha}\|f_T^m\| = 1, \tag{1.284}$$

which we may again do by scaling. We then notice that using (1.284) we have that for $T>1$ the relation

$$\|f_T^m\| \overset{(1.19)}{=} e^{-T}\|f_1 * \psi_{T-1}\| \overset{(1.21)}{\lesssim} e^{-T}\|f_1\| \lesssim e^{-T} \tag{1.285}$$

holds. In this step we show that these observations are enough to show that

$$u = \int_0^\infty f_T^m \,\mathrm{d}T \tag{1.286}$$

is a distributional solution of

$$\mathcal{A}(u) = f \qquad\qquad \text{on} \quad \mathbb{R}^2. \tag{1.287}$$

To obtain our claim we first show that for any $t\in(0,1)$ the function

$$u^t = \int_0^\infty f_{t+T}^m \,\mathrm{d}T \tag{1.288}$$

satisfies $\mathcal{A}u^t = f_t^m$. To see this we recall from Step 2 that f_{t+T}^m solves $(\partial_T + \mathcal{A})f_{t+T}^m = 0$ on $\mathbb{R}^2$, which allows us to write

$$\int_0^\infty \partial_T f_{t+T}^m \,\mathrm{d}T = -\int_0^\infty \mathcal{A} f_{t+T}^m \,\mathrm{d}T. \tag{1.289}$$

Using that $t>0$ we process the left-hand side as

$$\int_0^\infty \partial_T f_{t+T}^m dT = f_\infty^m - f_t^m = -f_t^m, \tag{1.290}$$

where we have used that $\|f_\infty^m\| = 0$ by (1.285). For the term on the right-hand side of (1.289) we use (1.284) and (1.285) to obtain the relation

$$\int_0^\infty |\partial_1^i\partial_2^j f_{t+T}^m| \,\mathrm{d}T \lesssim (t^{\frac14})^{-(i+2j)} \int_0^1 \|f_T^m\|\mathrm{d}T \tag{1.291}$$

$$\lesssim (t^{\frac14})^{-(i+2j)} \int_0^1 (T^{\frac14})^{\alpha-2}\,\mathrm{d}T + \int_1^\infty e^{-T}\,\mathrm{d}T < \infty, \tag{1.292}$$

which means that

$$\int_0^\infty \mathcal{A} f_{t+T}^m \,\mathrm{d}T = \mathcal{A}\left(\int_0^\infty f_{t+T}^m \,\mathrm{d}T\right). \tag{1.293}$$

In particular, combining (1.289) and (1.293) we end up with

$$\mathcal{A}\left(\int_0^\infty f_{t+T}^m \,\mathrm{d}T\right) = f_t^m.$$

To show that $u^t \to u$ uniformly as $t \to 0$, we can directly estimate the difference as

$$\|u^t - u\| = \left\|\int_0^t f_T^m dT\right\| \lesssim t^{\frac{\alpha+2}{4}}. \tag{1.294}$$

Step 5– (Bounds for ∂_1^{-1} and ∂_2^{-1})

In this step we show that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is such that for every fixed $x_2 \in \mathbb{R}$ the function $f(\cdot, x_2)$ is periodic and mean-free, then

$$\|\partial_1^{-1} f\|_\alpha \lesssim \|f\|_\alpha. \tag{1.295}$$

If f is constant in the x_1-direction, then

$$[\partial_2^{-1} f]_\alpha^{loc} \lesssim \|f\|. \tag{1.296}$$

We start with our proof of (1.295). We first notice that because $f(\cdot, x_2)$ is mean-free, $\partial_1^{-1} f(\cdot, x_2)$ is again periodic. This means that to obtain (1.295) we may fix two distinct points $x, y \in \mathbb{R}^2$ such that $x_1, y_1 \leq 2$ and $d(x_1, y_1) \leq 2$ and assume that $y_1 \geq x_1$. We then notice that

$$|\partial_1^{-1} f(x_1, x_2) - \partial_1^{-1} f(x_1, y_2)| \leq \int_0^{x_1} |f(s, x_2) - f(s, y_2)| \,\mathrm{d}s \lesssim [f]_\alpha d^\alpha(x, y)$$

and

$$\begin{aligned} &|\partial_1^{-1} f(x_1, y_2) - \partial_1^{-1} f(y_1, y_2)| \\ &\leq \int_{x_1}^{y_1} |f(s, y_2)| \,\mathrm{d}s \leq |x_1 - y_1| \|f\| \lesssim \|f\| d^\alpha(x, y). \end{aligned}$$

We may also bound the L^∞-norm as

$$|\partial_1^{-1} f(x)| \leq \int_0^{x_1} |f(s, x_2)| \,\mathrm{d}s \leq \|f\|.$$

Combining the last three estimates we obtain (1.295).

The relation (1.296) is obtained in essentially the same way; since we are interested in the local Hölder seminorm we fix $x, y \in \mathbb{R}^2$ such that $d(x, y) \leq 1$

and again assume that $y_2 \geq x_2$. We then use the triangle inequality and the trait that u is constant in the x_1-direction to write

$$|\partial_2^{-1} f(x_1, x_2) - \partial_2^{-1} f(y_1, y_2)| \lesssim \int_{x_2}^{y_2} |f(y_1, s) \, \mathrm{d}s \lesssim \|f\| d^\alpha(x, y). \qquad (1.297)$$

Step 6 – (Proof of (1.22)*)*

Under the same assumptions as in Step 4 we then prove (1.22). The main idea of our argument is to take advantage of the decomposition given in Step 5, i. e.

$$f = \mathcal{A}(u) = \partial_1^2(\partial_1^2 u) + \partial_2(-\partial_2 u) + u$$

with u given as (1.286), in the sense that we use the triangle inequality to write

$$[f]_{\alpha-2}^{loc} \leq \left[\partial_1^2 u\right]_\alpha + [\partial_2 u]_\alpha + [u]_{\alpha-2}^{loc}, \qquad (1.298)$$

The first two terms on the right-hand side can be treated in the same way as in [36]. We notice that by (1.284) it holds that

$$\sup_{T \leq 1} (T^{\frac{1}{4}})^{2-\alpha} \|f_T^m\| = \sup_{T \leq 1} (T^{\frac{1}{4}})^{2-\alpha} \|(\mathcal{A}u)_T^m\| \overset{(1.21),(1.284)}{\lesssim} 1, \qquad (1.299)$$

which we process with the semigroup property and (1.18) to obtain

$$\sup_{T \leq 1} (T^{\frac{1}{4}})^{j+2l+(2-\alpha)-4} \|T(\mathcal{A}\partial_1^j \partial_2^l u^t)_T^m\| \lesssim 1 \qquad (1.300)$$

for $j, l \geq 0$. The bound $\|u\| \lesssim 1$, which we have obtained in the previous step, can then be combined with Young's inequality and (1.18) to yield

$$\|\partial_1^2 u * \psi_\epsilon\|, \|\partial_1 u * \psi_\epsilon\|, \|\partial_2 u * \psi_\epsilon\|, \|u * \psi_\epsilon\| \lesssim 1 \qquad (1.301)$$

for any $\epsilon > 0$. We estimate the first two terms on the right-hand side of (1.298) by first applying (1.272) from Step 2 (which we may do thanks to (1.301)) and then using (1.300). We find that

$$\left[\partial_1^2 u\right]_\alpha \lesssim \sup_{T \leq 1} (T^{\frac{1}{4}})^{-\alpha} \|T(\mathcal{A}\partial_1^2 u)_T^m\| \lesssim 1. \qquad (1.302)$$

The second term on the right-hand side of (1.298) is treated in the same way.

In order to treat the third term on the right-hand side of (1.298) we use Step 5. In particular, for a fixed x_2 coordinate letting P denote the projection onto the space of mean-free periodic functions in one variable, we write

$$u(x_1,x_2)=\partial_1^2\left(\partial_1^{-1}P(\partial_1^{-1}P(u))\right)+\partial_2\left(\partial_2^{-1}\int_0^1 u(s,x_2)\,\mathrm{d}s\right). \tag{1.303}$$

Applying Step 5 yields

$$\begin{aligned}\left\|\partial_1^{-1}P(\partial_1^{-1}P(u))\right\|_\alpha &\lesssim \left\|\partial_1^{-1}P(u)\right\|_\alpha+\left\|\int_0^1\partial_1^{-1}P(u)\,\mathrm{d}s\right\|\\ &\lesssim \left\|\partial_1^{-1}P(u)\right\|_\alpha \\ &\lesssim \|Pu\|_\alpha \\ &\lesssim \|u\|_\alpha\end{aligned} \tag{1.304}$$

and

$$\left[\partial_2^{-1}\int_0^1 u(s,x_2)\,\mathrm{d}s\right]_\alpha^{loc}\lesssim\left\|\int_0^1 u(s,x_2)\,\mathrm{d}s\right\|\le\|u\|. \tag{1.305}$$

To finish we notice that by Step 2 we have that

$$[u]_\alpha \overset{(1.272)}{\lesssim} \sup_{T\le 1}(T^{\frac14})^{-\alpha}\|T\mathcal{A}(u^t)_T^m\| \overset{(1.300)}{\lesssim} \sup_{T\le 1}(T^{\frac14})^2\le 1. \tag{1.306}$$

Combining the definition (1.12) with (1.303), (1.304), (1.305), (1.306), and the bound $\|u\|\lesssim 1$, we are able to bound the third term on the right-hand side of (1.298) as $[u]_{\alpha-2}^{loc}\lesssim 1$. □

Corollary 2. *Let $\alpha\in(\frac23,1)$ and $L\in(0,1)$. If the distribution f on $\mathbb{R}^2$ satisfies the relation*

$$|f(x)|\le C|x_2|^{\frac{2\alpha-2}{2}} \tag{1.307}$$

for any $x\in\mathbb{R}^2$ and some $C\in\mathbb{R}$, then we have that

$$\sup_{T\le1}(T^{\frac14})^{\frac{2-2\alpha}{2}}\|f_T\|\lesssim C. \tag{1.308}$$

Furthermore, we find that

$$[f_\tau]_\alpha^{loc}\lesssim C(\tau^{\frac14})^{-2} \tag{1.309}$$

for any $\tau\in(0,1)$ If, additionally, we know that $f=0$ on $\mathbb{R}^2_-$, then for $\tau>0$ we have that

$$[f_\tau]_{\alpha;\mathbb{R}\times(-\infty,-L]}\lesssim CL^{-\frac{\alpha+2}{2}}(\tau^{\frac14})^{2\alpha}. \tag{1.310}$$

Proof. We start by showing (1.308). For this we fix $x \in \mathbb{R}^2$ and use the growth condition (1.307) and the standard rescaling (1.16) to write

$$\begin{aligned}
&|f_T(x)| \\
\leq &C(T^{\frac{1}{4}})^{2\alpha-2} \int_{\mathbb{R}^2} |\hat{x}_2 - \hat{y}_2|^{\frac{2\alpha-2}{2}} \psi_1(\hat{y}) \,\mathrm{d}\hat{y} \\
\lesssim &C(T^{\frac{1}{4}})^{2\alpha-2} \left(\left\| \int_{\mathbb{R}} \psi_1(\hat{y}_1, \cdot) \,\mathrm{d}\hat{y}_1 \right\| \int_{\hat{x}_2-1}^{\hat{x}_2+1} |\hat{y}_2 - \hat{x}_2|^{\frac{2\alpha-2}{2}} \mathrm{d}\hat{y}_2 + \int_{\mathbb{R}^2} |\psi_1(\hat{y})| \,\mathrm{d}\hat{y} \right) \\
\lesssim &C(T^{\frac{1}{4}})^{2\alpha-2}.
\end{aligned} \tag{1.311}$$

Notice that here we have relied on ψ_1 being a Schwarz function and that $|\frac{2\alpha-2}{2}| < 1$.

The relation (1.308), in particular, implies that $[f]^{loc}_{\alpha-2} \lesssim C$ by (1.22) of Lemma 6, which we then use to prove (1.309). For this we let $f = \partial_1^2 f^1 + \partial_2 f^2$ be a near-optimal representation of f in the sense of Definition 2. We then fix $x, z \in \mathbb{R}^2$ such that $d(x,z) \leq 1$ and use an integration by parts and (1.18) to write

$$\begin{aligned}
&|f_\tau(x) - f_\tau(z)| \\
= &\left| \int_{\mathbb{R}^2} (f^1(x-y) - f^1(z-y)) \partial_1^2 \psi_\tau(y) \,\mathrm{d}y \right. \\
&\qquad \left. + \int_{\mathbb{R}^2} (f^2(x-y) - f^2(z-y)) \partial_2 \psi_\tau(y) \,\mathrm{d}y \right| \\
\lesssim &([f^1]^{loc}_\alpha + [f^2]^{loc}_\alpha)(\tau^{\frac{1}{4}})^{-2} d^\alpha(x,z) \\
\lesssim &C(\tau^{\frac{1}{4}})^{-2} d^\alpha(x,z),
\end{aligned}$$

which is (1.309).

In order to show (1.310) we use (1.24). Notice that due to the estimate (1.307) we know that $\|f_\tau * \psi_\epsilon\| < \infty$ for any $\epsilon > 0$, which means that we may apply (1.24) with $\Omega = \mathbb{R} \times (-\infty, -L]$ to obtain

$$[f_\tau]_{\alpha;\mathbb{R}\times(-\infty,-L]} \lesssim \sup_{T\leq 1} (T^{\frac{1}{4}})^{-\alpha} \|f_\tau * \psi_T\|_{\mathbb{R}\times(-\infty,-L]}. \tag{1.312}$$

To finish, we fix $x \in \mathbb{R} \times (-\infty, -L]$, $T \leq 1$ and $\tau > 0$ and first assume that

$\tau \leq T$. We then use the growth condition (1.307) to write

$$\begin{aligned}
&(T^{\frac{1}{4}})^{-\alpha}|f_\tau * \psi_T(x)| \\
&\overset{(1.19),(1.21)}{\lesssim} (\tau^{\frac{1}{4}})^{-\alpha}|f * \psi_{2\tau}(x)| \\
&\leq CL^{-\frac{\alpha+2}{2}}(\tau^{\frac{1}{4}})^{-\alpha}\int_{\mathbb{R}^2}|x_2-y_2|^{\frac{2\alpha-2}{2}}|y_2|^{\frac{\alpha+2}{2}}\psi_{2\tau}(y)\mathrm{d}y \\
&\lesssim CL^{-\frac{\alpha+2}{2}}(\tau^{\frac{1}{4}})^{2\alpha}\Bigg(\Big\|\int_{\mathbb{R}}\psi_1(\hat{y}_1,\cdot)\,\mathrm{d}\hat{y}_1\Big\|\int_{\hat{x}_2-1}^{\hat{x}_2+1}|\hat{y}_2-\hat{x}_2|^{2\alpha-2}\,\mathrm{d}\hat{y}_2 \\
&\qquad\qquad + \int_{\mathbb{R}^2}(1+|\hat{y}_2|^{\alpha+2})|\psi_1(\hat{y})|\,\mathrm{d}\hat{y}\Bigg) \\
&\lesssim CL^{-\frac{\alpha+2}{2}}(\tau^{\frac{1}{4}})^{2\alpha},
\end{aligned} \tag{1.313}$$

where we have used that f is only supported for positive times (so that we could smuggle $|\frac{y_2}{x_2}|^{\frac{\alpha+2}{2}}$ into the integral on the third line) and also that $|x_2| \geq L$. Of course, we have also used that ψ_1 is a Schwarz function and that $\alpha \in (\frac{2}{3}, 1)$. When $T \leq \tau$ we simply switch the roles of τ and T in the above calculation.

□

1.6 Appendix A: Construction of the Singular Products for the Quasilinear Problem

We now construct the second new family of reference products mentioned in Section 1.3 and also prove the first reconstruction lemma.

1.6.1 Summary

We begin our construction of the new reference products by proving the following general lemma:

Lemma 7. *Let $\alpha \in (0,1)$.*

i) Let $G \in C^\alpha(\mathbb{R}^2)$ be such that for any $x \in \mathbb{R}^2$

$$|\partial_1^k G(x)| \lesssim C(G)|x_2|^{\frac{\alpha-k}{2}} \tag{1.314}$$

holds for $k = 1, 2$ for some constant $C(G) \in \mathbb{R}$; we assume that $C(G) \geq [G]_\alpha$. Also, assume that we have a family of functions $F(\cdot, a_0) \in C^\alpha(\mathbb{R}^2)$ indexed by $a_0 \in [\lambda, 1]$ such that

$$\sup_{a_0 \in [\lambda,1]} \|F(\cdot, a_0)\|_{\alpha,2} \leq N_0 \tag{1.315}$$

for some $N_0 \in \mathbb{R}$. *Under these assumptions there exists a family of* $C^{\alpha-2}$-*distributions* $G \diamond \partial_1^2 F(\cdot, a_0)$ *such that*

$$\sup_{a_0\in[\lambda,1]} \sup_{T\le 1} (T^{\frac{1}{4}})^{2-2\alpha} \|[G, (\cdot)_T] \diamond \partial_1^2 F(\cdot, a_0)\|_2 \lesssim C(G) N_0. \tag{1.316}$$

ii) Let $i = 0, 1$ *and* G_i *and* $F(\cdot, a_0)$ *satisfy the assumptions of the previous part individually. Furthermore, assume that there is a constant* $C(G_0, G_1) \in \mathbb{R}$ *satisfying* $[G_0 - G_1]_\alpha \le C(G_0, G_1)$ *such that for any* $x \in \mathbb{R}^2$

$$|\partial_1^k (G_0 - G_1)(x)| \lesssim C(G_0, G_1) |x_2|^{\frac{\alpha-k}{2}} \tag{1.317}$$

for $k = 1, 2$. *Under these assumptions we find that the distributions defined in i) satisfy*

$$\begin{aligned}&\sup_{a_0\in[\lambda,1]} \sup_{T\le 1} \left(T^{\frac{1}{4}}\right)^{2-2\alpha} \left\|[G_0, (\,\cdot\,)_T] \diamond \partial_1^2 F(\cdot, a_0) - [G_1, (\,\cdot\,)_T] \diamond \partial_1^2 F(\cdot, a_0)\right\|_1\\ &\lesssim C(G_0, G_1) N_0.\end{aligned} \tag{1.318}$$

iii) Let $i = 0, 1$ *and* G *and* $F_i(\cdot, a_0)$ *individually satisfy the assumptions of part i). Furthermore, assume that*

$$\sup_{a_0\in[\lambda,1]} \|F_0(\cdot, a_0) - F_1(\cdot, a_0)\|_{\alpha,2} \le \delta N_0 \tag{1.319}$$

for $\delta N_0 \in \mathbb{R}$. *Under these assumptions we find that the distributions defined in i) satisfy*

$$\begin{aligned}&\sup_{a_0\in[\lambda,1]} \sup_{T\le 1} \left(T^{\frac{1}{4}}\right)^{2-2\alpha} \left\|[G, (\,\cdot\,)_T] \diamond \partial_1^2 F_0(\cdot, a_0) - [G, (\,\cdot\,)_T] \diamond \partial_1^2 F_1(\cdot, a_0)\right\|_1\\ &\lesssim C(G) \delta N_0.\end{aligned} \tag{1.320}$$

Notice that the first part of this lemma is meant to be applied with $G = \tilde{V}(\cdot, a_0)$ and $F(\cdot, a_0') = v(\cdot, a_0')$, which yields the second family of new reference products mentioned in Section 1.3; i.e. the family of distributions $\left\{\tilde{V}(\cdot, a_0) \diamond \partial_1^2 v(\cdot, a_0')\right\}$ indexed by $a_0, a_0' \in [\lambda, 1]$. In the last two parts of this lemma we address the stability of the singular products that we have constructed in part *i)* in the sense that in *ii)* we intend to set $G_i = \tilde{V}_i(\cdot, a_0)$ and $F(\cdot, a_0') = v(\cdot, a_0')$ and in *iii)* we take $G = \tilde{V}(\cdot, a_0)$ and $F_i(\cdot, a_0') = v_i(\cdot, a_0')$.

In order to show that the identifications made in the previous paragraph satisfy the necessary assumptions, we use Lemma 3 and the next lemma in which we derive a Hölder estimate for $v(\cdot, a_0)$. While the estimates for $v(\cdot, a_0)$ that we derive in the next lemma are, aside from our addition of the massive term, already contained in [47], we include them here for completeness:

Lemma 8. *Let $v(\cdot, a_0)$ solve (1.4), where the periodic right-hand side f satisfies*

$$\sup_{T\leq 1}(T^{\frac{1}{4}})^{2-\alpha}\|f_T\| \leq N_0 \tag{1.321}$$

for some $N_0 \in \mathbb{R}$. Then the bound

$$\|v(\cdot, a_0)\|_{\alpha,2} \lesssim N_0 \tag{1.322}$$

holds.

Notice that (1.321) is exactly (1.64), which is the standing assumption we have for f throughout this entire contribution. With Lemmas 3 and 8 in-hand we can then apply Lemma 7 as described above. We obtain the following:

Corollary 3 (New reference products for quasilinear problem)**.** *Let $\alpha \in (0,1)$ and $i, j = 0, 1$. Assume that we have two periodic distributions f_i on $\mathbb{R}^2$ such that independently each satisfies* (1.321) *for $N_0 \in \mathbb{R}$ and there exists $\delta N_0 \in \mathbb{R}$ such that*

$$\sup_{T\leq 1}(T^{\frac{1}{4}})^{2-\alpha}\|(f_1 - f_0)_T\| \lesssim \delta N_0. \tag{1.323}$$

Assume, furthermore, that for each pair (f_i, f_j) there exists a family of offline products $\{v_i(\cdot, a_0) \diamond_{OW} \partial_1^2 v_j(\cdot, a_0')\}$ indexed by $a_0, a_0' \in [\lambda, 1]$ satisfying

$$\sup_{a_0,a_0'\in[\lambda,1]}\sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\left\|[v_i(\cdot, a_0), (\cdot)_T] \diamond_{OW} \partial_1^2 v_j(\cdot, a_0')\right\|_{2,2} \lesssim N_0^2, \tag{1.324}$$

$$\begin{aligned}\sup_{a_0,a_0'\in[\lambda,1]}\sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\big\|&[v_1(\cdot, a_0), (\cdot)_T] \diamond_{OW} \partial_1^2 v_j(\cdot, a_0') \\ &-[v_0(\cdot, a_0), (\cdot)_T] \diamond_{OW} \partial_1^2 v_j(\cdot, a_0')\big\|_{1,1} \\ \lesssim N_0\delta N_0,&\end{aligned} \tag{1.325}$$

and

$$\begin{aligned}\sup_{a_0,a_0'\in[\lambda,1]}\sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\big\|&[v_i(\cdot, a_0), (\cdot)_T] \diamond_{OW} \partial_1^2 v_1(\cdot, a_0') \\ &-[v_i(\cdot, a_0), (\cdot)_T] \diamond_{OW} \partial_1^2 v_0(\cdot, a_0')\big\|_{1,1} \\ \lesssim N_0\delta N_0.&\end{aligned} \tag{1.326}$$

We lastly assume that we have two periodic functions $V_{int,i} \in C^{\alpha}(\mathbb{R})$ such that

$$\begin{aligned} [V_{int,i}]_{\alpha} &\leq N_0^{int} && (1.327)\\ \textit{and} \quad [V_{int,1} - V_{int,0}]_{\alpha} &\leq \delta N_0^{int} && (1.328) \end{aligned}$$

for $N_0^{int}, \delta N_0^{int} \in \mathbb{R}$. Under these assumptions, for $i, j = 0, 1$ we then find that:

i) There exists a family of distributions $\left\{\tilde{V}_i(\cdot, a_0) \diamond \partial_1^2 v_j(\cdot, a_0')\right\}$ such that

$$\sup_{a_0, a_0' \in [\lambda, 1]} \sup_{T \leq 1} (T^{\frac{1}{4}})^{2-2\alpha} \left\| \left[\tilde{V}_i(\cdot, a_0), (\cdot)_T\right] \diamond \partial_1^2 v_j(\cdot, a_0')\right\|_{2,2} \lesssim N_0 N_0^{int}. \quad (1.329)$$

ii) Defining the family of distributions

$$\begin{aligned} &(\tilde{V}_i + v_i)(\cdot, a_0) \diamond \partial_1^2 v_j(\cdot, a_0') \\ :=&\tilde{V}_i(\cdot, a_0) \diamond \partial_1^2 v_j(\cdot, a_0') + v_i(\cdot, a_0) \diamond_{OW} \partial_1^2 v_j(\cdot, a_0'), \end{aligned} \quad (1.330)$$

we then obtain the relation

$$\begin{aligned} &\sup_{a_0, a_0' \in [\lambda, 1]} \sup_{T \leq 1} (T^{\frac{1}{4}})^{2-2\alpha} \left\| \left[(\tilde{V}_i + v_i)(\cdot, a_0), (\cdot)_T\right] \diamond \partial_1^2 v_j(\cdot, a_0')\right\|_{2,2} \\ \lesssim&(N_0^{int} + N_0) N_0. \end{aligned} \quad (1.331)$$

iii) The distributions constructed in part ii) satisfy

$$\begin{aligned} &\sup_{a_0, a_0' \in [\lambda, 1]} \sup_{T \leq 1} (T^{\frac{1}{4}})^{2-2\alpha} \Big\| \left[(\tilde{V}_0 + v_0)(\cdot, a_0),\ (\,\cdot\,)_T\right] \diamond \partial_1^2 v_j(\cdot, a_0') \\ &\qquad - \left[(\tilde{V}_1 + v_1)(\cdot, a_0),\ (\,\cdot\,)_T\right] \diamond \partial_1^2 v_j(\cdot, a_0')\Big\|_{1,1} \\ \lesssim&(\delta N_0^{int} + \delta N_0) N_0 \end{aligned} \quad (1.332)$$

and

$$\begin{aligned} &\sup_{a_0, a_0' \in [\lambda, 1]} \sup_{T \leq 1} (T^{\frac{1}{4}})^{2-2\alpha} \Big\| \left[(\tilde{V}_i + v_i)(\cdot, a_0),\ (\,\cdot\,)_T\right] \diamond \partial_1^2 v_1(\cdot, a_0') \\ &\qquad - \left[(\tilde{V}_i + v_i)(\cdot, a_0),\ (\,\cdot\,)_T\right] \diamond \partial_1^2 v_0(\cdot, a_0')\Big\|_{1,1} \\ \lesssim&(N_0^{int} + N_0) \delta N_0. \end{aligned} \quad (1.333)$$

iv) Defining the further family of distributions

$$\begin{aligned} &(\tilde{V}_i + v_i)(\cdot, a_0) \diamond \partial_1^2 (\tilde{V}_j + v_j)(\cdot, a_0') \\ :=&(\tilde{V}_i + v_i)(\cdot, a_0) \diamond \partial_1^2 \tilde{V}_j(\cdot, a_0') + (\tilde{V}_i + v_i)(\cdot, a_0) \diamond \partial_1^2 v_j(\cdot, a_0'), \end{aligned} \quad (1.334)$$

where the first term on the right-hand side is defined via Corollary 1 and the second term is defined in part ii), we find that

$$\sup_{a_0,a_0'\in[\lambda,1]} \sup_{T\le 1}(T^{\frac{1}{4}})^{2-2\alpha} \left\| \left[(\tilde{V}_i+v_i)(\cdot,a_0),(\cdot)_T\right] \diamond \partial_1^2(\tilde{V}_j+v_j)(\cdot,a_0')\right\|_{2,2}$$
$$\lesssim (N_0^{int}+N_0)^2. \tag{1.335}$$

As we will see below when we discuss Theorem 2, the reference products that we construct in *ii*) and in *iv*) are used for the treatment of the quasilinear problem. As we have explained in Section 1.4, the main step of the fixed point argument with which we obtain a solution of the quasilinear problem is the application of Theorem 1 in the case that

$$a^{ext} = a(\overline{W}), \tag{1.336}$$

where $\overline{W}$ is modelled after $(\tilde{V}+v)(\cdot,a_0)$ on $\mathbb{R}^2$ for some $\bar{a}\in C^\alpha(\mathbb{R}^2)$. The combination of the reference products from Corollary 3 and the reconstruction lemma proved below provides us with the necessary reference products assumed in the statement of Theorem 1; in particular, we obtain the family of reference products $a^{ext}\diamond\partial_1^2 v(\cdot,a_0)$ such that (1.66) is satisfied. In part *iv*) of the above corollary we construct the reference products used to define the singular product $a(W)\diamond\partial_1^2 W$ in the quasilinear problem (1.1).

We now give the first reconstruction lemma, which allows us to post-process the family of reference products provided in *ii*) above to obtain the $a^{ext}\diamond\partial_1^2 v(\cdot,a_0)$ assumed in the statement of Theorem 1, where a^{ext} is given in (1.336). We state the reconstruction lemma in a very general form and then put it into a version we actually use in Corollary 4. Up to the loss of periodicity in the x_2-direction, both the reconstruction lemma and the resulting corollary are essentially the same as the corresponding statements in [47]. Here is the reconstruction lemma:

Lemma 9 (Modified Lemma 2 of [47]). *Let $\alpha\in\left(\frac{2}{3},1\right)$. Assume that we are given a distribution h and families $\{w(\cdot,x)\}_x$ of functions and $\{w(\cdot,x)\diamond h\}_x$ of distributions both indexed by points $x\in\mathbb{R}^2$ such that the estimates*

$$[w(\cdot,x)]_\alpha \le N, \tag{1.337}$$
$$[w(\cdot,x)-w(\cdot,x')]_\alpha \le N d^\alpha(x,x'), \tag{1.338}$$
$$\sup_{T\le 1}(T^{\frac{1}{4}})^{2-\alpha}\|h_T\| \le N_0, \tag{1.339}$$
$$\sup_{T\le 1}(T^{\frac{1}{4}})^{2-2\alpha}\|\left[w(\cdot,x),(\cdot)_T\right]\diamond h]\| \le NN_0, \ \textit{and} \tag{1.340}$$

$$\sup_{T\le 1}(T^{\frac{1}{4}})^{2-2\alpha}\|\,[w(\cdot,x),(\cdot)_T]\diamond h-[w(\cdot,x'),(\cdot)_T]\diamond h\|\le NN_0d^{\alpha}(x,x') \tag{1.341}$$

hold for all $x,x'\in\mathbb{R}^2$ *and for some constants* $N,N_0\in\mathbb{R}$.

For $u\in L^\infty(\mathbb{R}^2)$ *such that there exists a function* ν *and constant* $M\in\mathbb{R}$ *ensuring that*

$$|u(y)-u(x)-(w(y,x)-w(x,x))-\nu(x)(y-x)_1|\le Md^{2\alpha}(x,y) \tag{1.342}$$

for any $x,y\in\mathbb{R}^2$, *there exists a unique distribution* $u\diamond h$ *satisfying*

$$\lim_{T\to 0}\|\,[u,(\cdot)_T]\diamond h-E_{tr}\,[w,(\cdot)_T]\diamond h-\nu\,[x_1,(\cdot)_T]\,h\|=0, \tag{1.343}$$

where E_{tr} *denotes evaluation of a function of* (x,y) *at* (x,x). *Furthermore, this distribution has finite local* $C^{\alpha-2}$ *seminorm and satisfies*

$$\sup_{T\le 1}(T^{\frac{1}{4}})^{2-2\alpha}\|\,[u,(\cdot)_T]\diamond h\|\le (M+N)N_0. \tag{1.344}$$

In this lemma we assume that all functions and distributions are periodic in the x_1*-direction.*

Just as for the other reconstruction lemma, Lemma 4, the proof of Lemma 9 follows from Otto and Weber's arguments without any substantial changes. In order to use this lemma we make various choices of h and $\{w(\cdot,x)\}_x$. In particular, as indicated above, we plan to use this lemma in two places: 1) To obtain a family of distributions $a^{ext}\diamond\partial_1^2 v(\cdot,a_0)$ from the reference products $(v+\tilde{V})(\cdot,a_0)\diamond\partial_1^2 v(\cdot,a_0')$ when $a^{ext}\in C^\alpha(\mathbb{R}^2)$ is modelled after $(v+\tilde{V})(\cdot,a_0)$ and 2) To define the product $a(W)\diamond\partial_1^2 W$ in (10) assuming that W is modelled after $(v+\tilde{V})(\cdot,a_0)$. These applications are summarized in the following corollary.

Corollary 4 (Modified Lemma 3 and Corollary 1 of [47])**.** *Let* $\alpha\in(\frac{2}{3},1)$. *All functions and distributions are assumed to be periodic in the* x_1*-direction. Under the same assumptions as in Corollary 2, we obtain that for any* $i,j=0,1$ *the following points hold:*

i) If $u\in C^\alpha(\mathbb{R}^2)$ *is modelled after* $(v_i+\tilde{V}_i)(\cdot,a_0)$ *according to* a_i, σ_i, *and* ν_i *such that* $\|a_i\|_\alpha\le 1$ *and* $\|\sigma_i\|_\alpha\le 1$ *with modelling constant* $M\in\mathbb{R}$, *then for every* $a_0'\in[\lambda,1]$ *it is possible to construct a unique distribution* $u\diamond\partial_1^2 v_j(\cdot,a_0')$ *such that*

$$\lim_{T\to 0}\Big\|[u,(\cdot)_T]\diamond\partial_1^2 v_j(\cdot,a_0')-\sigma_iE_i\left[(v_i+\tilde{V}_i)(\cdot,a_0),(\cdot)_T\right]\diamond\partial_1^2 v_j(\cdot,a_0') \\ -\nu_i\,[x_1,(\cdot)_T]\,\partial_1^2 v_j(\cdot,a_0')\Big\|=0, \tag{1.345}$$

where E_i denotes the evaluation of a function depending on (x, a_0) at $(x, a_i(x))$. The distributions $u \diamond \partial_1^2 v_j(\cdot, a_0')$ are constructed to have finite local $C^{\alpha-2}$ semi-norm and, furthermore, satisfy

$$\sup_{a_0\in[\lambda,1]} \sup_{T\le 1} (T^{\frac14})^{2-2\alpha} \big\|[u, (\cdot)_T] \diamond \partial_1^2 v_j(\cdot, a_0)\big\|_2 \le (N_0^{int} + N_0 + M)N_0. \quad (1.346)$$

and

$$\begin{aligned}&\sup_{a_0\in[\lambda,1]} \sup_{T\le 1} (T^{\frac14})^{2-2\alpha} \big\|[u, (\cdot)_T] \diamond \partial_1^2 v_1(\cdot, a_0) - [u, (\cdot)_T] \diamond \partial_1^2 v_0(\cdot, a_0)\big\|_1 \\ \lesssim&(N_0^{int} + N_0 + M)\delta N_0.\end{aligned} \quad (1.347)$$

ii) Let $i = 0, 1$. Assume that we have two functions $u_i \in C^\alpha(\mathbb{R}^2)$ each modelled after $(v_i + \tilde V_i)(\cdot, a_0)$ according to a_i and σ_i, which again satisfy $\|a_i\| \le 1$ and $\|\sigma_i\| \le 1$, such that also the difference $u_1 - u_0$ is modelled after $((v_1+\tilde V_1)(\cdot, a_0), (v_0+\tilde V_0)(\cdot, a_0))$ according to (a_1, a_0) and $(\sigma_1, -\sigma_0)$ with modelling constant $\delta M \in \mathbb{R}$. For the singular products $u_j \diamond \partial_1^2 (v_i + \tilde V_i)(\cdot, a_0)$ constructed in part i) we then have that

$$\begin{aligned}&\sup_{a_0\in[\lambda,1]} \sup_{T\le 1} (T^{\frac14})^{2-2\alpha} \sup_{a_0\in[\lambda,1]} \big\|[u_1, (\cdot)_T] \diamond \partial_1^2 v_i(\cdot, a_0) - [u_0, (\cdot)_T] \diamond \partial_1^2 v_i(\cdot, a_0)\big\|_1 \\ \lesssim&N_0(\delta M + (N_0 + N_0^{int})(\|a_1 - a_0\|_\alpha + \|\sigma_1 - \sigma_0\|_\alpha) + \delta N_0 + \delta N_0^{int}).\end{aligned} \quad (1.348)$$

1.6.2 Proofs

We start with the proof of Lemma 7. Here is the argument:

Proof of Lemma 7. i) Notice that, in comparison to the situation in Lemma 2, we do not have an analogue of (1.50) for $F(\cdot, a_0)$. To remedy the situation, we symbolically apply Leibniz' rule with the goal of moving the derivatives "∂_1" off of F and onto G:

$$\begin{aligned}&G\partial_1^2 F(\cdot, a_0) \\ \text{“}=\text{”}&\partial_1^2(F(\cdot, a_0)G) - 2\partial_1 F(\cdot, a_0)\partial_1 G - F(\cdot, a_0)\partial_1^2 G \\ \text{“}=\text{”}&-2(\partial_1(F(\cdot, a_0)\partial_1 G) - F(\cdot, a_0)\partial_1^2 G) + \partial_1^2(F(\cdot, a_0)G) - F(\cdot, a_0)\partial_1^2 G \\ \text{“}=\text{”}&\partial_1^2(F(\cdot, a_0)G) - 2\partial_1(F(\cdot, a_0)\partial_1 G) + F(\cdot, a_0)\partial_1^2 G.\end{aligned} \quad (1.349)$$

This heuristic calculation motivates the definition

$$G \diamond \partial_1^2 F(\cdot, a_0) := \partial_1^2(F(\cdot, a_0)G) - 2\partial_1(F(\cdot, a_0)\partial_1 G) + F(\cdot, a_0)\partial_1^2 G. \quad (1.350)$$

We must check that the terms on the right-hand side of (1.350) are well-defined. To begin, we notice that thanks to the assumption (1.314) the third term on the right-hand side was shown to make sense as a distribution in Lemma 2. For the first term we remark that the product $F(\cdot, a_0)G$ is clearly classical and of class C^α. Moving to the second term, we find that, thanks to (1.314) for $i = 1$ and the bound $\|F(\cdot, a_0)\| \leq N_0$, the classical product $F(\cdot, a_0)\partial_1 G \in L^1_{loc}(\mathbb{R}^2)$.

For (1.316) we first use our definition (1.350) to write (over two pages):

$$\begin{aligned}
&\left|[G, (\,\cdot\,)_T] \diamond \partial_1^2 F(\cdot, a_0)(x)\right| \\
&= \left|\int_{\mathbb{R}^2} (G(x)\partial_1^2 F(y, a_0) - G(y) \diamond \partial_1^2 F(y, a_0))\psi_T(x-y)\,\mathrm{d}y\right| \\
&= \left|\int_{\mathbb{R}^2} (G(x) - G(y))F(y, a_0)\partial_1^2\psi_T(x-y)\,\mathrm{d}y\right. \\
&\qquad - 2\int_{\mathbb{R}^2} F(y, a_0)\partial_1 G(y)\partial_1\psi_T(x-y)\,\mathrm{d}y \qquad (1.351) \\
&\qquad \left. - \int_{\mathbb{R}^2} F(y, a_0)\partial_1^2 G(y)\psi_T(x-y)\,\mathrm{d}y\right| \\
&= \left|\int_{\mathbb{R}^2} (G(x) - G(y))(F(y, a_0) - F(x, a_0))\partial_1^2\psi_T(x-y)\mathrm{d}y\right. \\
&\qquad - 2\int_{\mathbb{R}^2} (F(y, a_0) - F(x, a_0))\partial_1 G(y)\partial_1\psi_T(x-y)\,\mathrm{d}y \\
&\qquad - \int_{\mathbb{R}^2} (F(y, a_0) - F(x, a_0))\partial_1^2 G(y)\psi_T(x_1-y)\,\mathrm{d}y \\
&\qquad + \int_{\mathbb{R}^2} (G(x) - G(y))F(x, a_0)\partial_1^2\psi_T(x-y)\,\mathrm{d}y \\
&\qquad - 2\int_{\mathbb{R}^2} F(x, a_0)\partial_1 G(y)\partial_1\psi_T(x-y)\,\mathrm{d}y \\
&\qquad \left. - \int_{\mathbb{R}^2} F(x, a_0)\partial_1^2 G(y)\psi_T(x-y)\,\mathrm{d}y\right|
\end{aligned}$$

for $a_0 \in [\lambda, 1]$ and $x \in \mathbb{R}^2$. Here we have used multiple applications of integration by parts in which the boundary terms vanish because ψ_T is Schwarz function.

We then treat the terms on the right-hand side of (1.351) separately. The first term is easily handled using (1.18):

$$\left|\int_{\mathbb{R}^2} (G(x) - G(y))(F(y, a_0) - F(x, a_0))\partial_1^2\psi_T(x-y)\,\mathrm{d}y\right|$$

$$
\begin{aligned}
&\leq [G]_\alpha \, [F(\cdot, a_0)]_\alpha \int_{\mathbb{R}^2} |\partial_1^2 \psi_T(x-y)| \, d^{2\alpha}(x,y) \, \mathrm{d}y \\
&\lesssim [G]_\alpha N_0 (T^{\frac{1}{4}})^{2\alpha - 2}.
\end{aligned}
$$

For the second term we use (1.314) for $i = 1$ and (1.18) to obtain

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^2} (F(y, a_0) - F(x, a_0)) \partial_1 G(y) \partial_1 \psi_T(x-y) \, \mathrm{d}y \right| \\
\lesssim & C(G) \, [F(\cdot, a_0)]_\alpha \int_{\mathbb{R}^2} |x_2|^{\frac{\alpha-1}{2}} d^\alpha(x,y) |\partial_1 \psi_T(x-y)| \, \mathrm{d}y \\
\lesssim & C(G) N_0 (T^{\frac{1}{4}})^{2\alpha-2}.
\end{aligned} \tag{1.352}
$$

Finally, the last term is treated in essentially the same way as the second, but using (1.314) for $i = 2$. In particular, we may write

$$
\begin{aligned}
&\left| \int_{\mathbb{R}^2} (F(y, a_0) - F(x, a_0)) \partial_1^2 G(y) \psi_T(x-y) \, \mathrm{d}y \right| \\
\lesssim & C(G) [F(\cdot, a_0)]_\alpha \int_{\mathbb{R}^2} |x_2|^{\frac{\alpha-2}{2}} d^\alpha(x,y) |\psi_T(x-y)| \, \mathrm{d}y \\
\lesssim & C(G) N_0 (T^{\frac{1}{4}})^{2\alpha-2}.
\end{aligned} \tag{1.353}
$$

Notice that the integrals in the second lines of (1.352) and (1.353) are treated in basically the same way as in (1.311). To finish our analysis of the right-hand side of (1.351), we lastly notice that a couple of integration by parts, where once again the boundary terms vanish due to the Schwarz-ness of the convolution kernel, show that the last three terms on the right-hand side actually cancel each other. Combining all of these observations and using that by assumption $[G]_\alpha \leq C(G)$ we find that

$$
\sup_{a_0 \in [\lambda, 1]} \sup_{T \leq 1} \left\| [G, (\,\cdot\,)_T] \diamond \partial_1^2 F(\cdot, a_0) \right\| \lesssim N_0 C(G) (T^{\frac{1}{4}})^{2\alpha-2}. \tag{1.354}
$$

To obtain a similar result for higher parameter derivatives as claimed in (1.316) for $\partial_{a_0}^j$ with $j = 1, 2$, we notice that due to our definition of $G \diamond \partial_1^2 F(\cdot, a_0)$ we have that

$$
\begin{aligned}
&\partial_{a_0}^j G \diamond \partial_1^2 F(\cdot, a_0) \\
&= \partial_1^2 (G \partial_{a_0}^j F(\cdot, a_0)) - 2\partial_1 (\partial_{a_0}^j F(\cdot, a_0) G) + \partial_{a_0}^j F(\cdot, a_0) \partial_1^2 G.
\end{aligned} \tag{1.355}
$$

Thanks to our assumption (1.315), which includes two parameter derivatives, we then find that the exact same argument as above gives (1.354) with the desired parameter derivatives included.

ii) The proof of this part is a simple matter of noticing that the definition (1.350) for $G \diamond \partial_1^2 F$ is linear in G. In particular, we notice that

$$G_0 \diamond \partial_1^2 F - G_1 \diamond \partial_1^2 F = (G_0 - G_1) \diamond \partial_1^2 F, \tag{1.356}$$

which by the linearity of the convolution then implies that

$$\begin{aligned} &[G_0,\, (\,\cdot\,)_T] \diamond \partial_1^2 F(\cdot, a_0) - [G_1,\, (\,\cdot\,)_T] \diamond \partial_1^2 F(\cdot, a_0) \\ =& [G_0 - G_1,\, (\,\cdot\,)_T] \diamond \partial_1^2 F(\cdot, a_0). \end{aligned} \tag{1.357}$$

By assumption (1.317) on top of the assumptions that we carry over from part *i)*, we may then apply the result of *i)* to obtain the desired bound (1.318). In fact, we get the bound for one more parameter derivative than we claim.

iii) For this part we use the linearity of the definition of the product $G \diamond \partial_1^2 F$ in F. In particular, we have that

$$G \diamond \partial_1^2 F_0 - G \diamond \partial_1^2 F_1 = G \diamond \partial_1^2 (F_0 - F_1), \tag{1.358}$$

which again, when combined with the linearity of the convolution, implies that

$$\begin{aligned} &[G,\, (\,\cdot\,)_T] \diamond \partial_1^2 F_0(\cdot, a_0) - [G,\, (\,\cdot\,)_T] \diamond \partial_1^2 F_1(\cdot, a_0) \\ =& [G,\, (\,\cdot\,)_T] \diamond \partial_1^2 (F_0 - F_1)(\cdot, a_0). \end{aligned} \tag{1.359}$$

Since we have carried over the assumptions of *i)* and have added the assumption (1.319), we may apply part *i)* to obtain the desired (1.320). Once again, we actually get the bound for one more parameter derivative than we need. □

In order to apply this lemma with the proper identifications in proof of Corollary 4 we use Lemma 3 and Lemma 8, the latter of which we prove below. As already mentioned, this argument is standard and essentially already contained in [47], but we give it here for completeness.

Proof of Lemma 8. Since f is periodic, we may interpret the assumption (1.321) in terms of Lemma 9 of [47]. In particular, we find that there exists a decomposition of f,

$$f = \partial_1^2 f^1 + \partial_2 f^2 + c, \tag{1.360}$$

such that for $i = 1, 2$ the f^i are periodic of vanishing average,

$$c = \int_{\mathbb{T}^2} f \,\mathrm{d}x, \tag{1.361}$$

and $[f^1]_\alpha + [f^2]_\alpha + |c| \lesssim N_0$. In particular, notice that, as we have already pointed out in the introduction, (1.361) is guaranteed by (1.360) with the choice of test function 1. The exact same argument applied to (1.4) yields that

$$\int_{\mathbb{T}^2} v(\cdot, a_0)\,\mathrm{d}x = \int_{\mathbb{T}^2} f\,\mathrm{d}x. \tag{1.362}$$

To finish obtaining (1.322) we let $v^i(\cdot, a_0)$ be the C^α- solution of

$$(\partial_2 - a_0\partial_1^2 + 1)v^i(\cdot, a_0) = f^i \qquad \text{on} \quad \mathbb{R}^2$$

for $i = 1, 2$. The classical Hölder estimate for these equations then gives that $[v^i(\cdot, a_0)]_{\alpha+2} \lesssim [f^i]_\alpha$. Using this we obtain that

$$[\partial_1^2 v^1(\cdot, a_0)]_\alpha + [\partial_2 v^2(\cdot, a_0)]_\alpha \lesssim N_0.$$

By the uniqueness of C^α-solutions to (1.4) we know that $v(\cdot, a_0) = \partial_1^2 v^1(\cdot, a_0) + \partial_2 v^2(\cdot, a_0) + c$, which yields that $[v(\cdot, a_0)]_\alpha \lesssim N_0$.

For the higher parameter derivatives we emulate the argument from part *iii)* of Lemma 3. In particular, differentiating (1.4) in terms of a_0 gives that

$$(\partial_2 - a_0\partial_1^2 + 1)\partial_{a_0} v(\cdot, a_0) = \partial_1^2 v(\cdot, a_0) \qquad \text{on} \quad \mathbb{R}^2, \tag{1.363}$$

which by the same arguments as above yields

$$[\partial_{a_0} v(\cdot, a_0)]_\alpha \lesssim [\partial_1^2 v(\cdot, a_0)]_{\alpha-2} \leq [v(\cdot, a_0)]_\alpha \lesssim N_0.$$

Differentiating in terms of a_0 again we find that $\partial_{a_0}^2 v(\cdot, a_0)$ solves

$$(\partial_2 - a_0\partial_1^2 + 1)\partial_{a_0}^2 v(\cdot, a_0) = 2\partial_1^2\partial_{a_0} v(\cdot, a_0) \qquad \text{on} \quad \mathbb{R}^2, \tag{1.364}$$

which again yields that

$$[\partial_{a_0}^2 v(\cdot, a_0)]_\alpha \lesssim [\partial_1^2\partial_{a_0} v(\cdot, a_0)]_{\alpha-2} \lesssim [\partial_{a_0} v(\cdot, a_0)]_\alpha \lesssim N_0.$$

Like before, testing the equations (1.363) and (1.364) with the constant function 1 yields that $\partial_{a_0} v(\cdot, a_0)$ and $\partial_{a_0}^2 v(\cdot, a_0)$ have vanishing average. Combining all of these observations we obtain the desired (1.322).

□

We are now ready to post-process Lemma 7 to get Corollary 3.

Proof of Corollary 4. i) This part follows from *i)* of Lemma 7 with the identifications $G = \partial_{a_0}^l \tilde{V}_i(\cdot, a_0)$ and $F(\cdot, a_0') = \partial_{a_0'}^j v_j(\cdot, a_0')$ for $l, j = 0, 1, 2$. The assumption (1.315) holds thanks to Lemma 8 and the assumption (1.314) with $C(\partial_{a_0}^l \tilde{V}_i(\cdot, a_0)) \lesssim N_0^{int}$ is verified by part $i)$ of Lemma 3. The claim then follows from the observation that by the definition (1.350), we have that $\partial_{a_0}^l \partial_{a_0'}^j G(\cdot, a_0), (\cdot)_T] \diamond F(\cdot, a_0') = [\partial_{a_0}^l G(\cdot, a_0), (\cdot)_T] \diamond \partial_{a_0'}^j F(\cdot, a_0')$.

ii), iv) These parts are immediate consequences of the triangle inequality and the assumption (1.324) on the offline products borrowed from [47]. For part *iv)* we also apply Corollary 1 with $F = (\tilde{V}_i + v_i)(\cdot, a_0)$.

iii) We start by showing (1.332). For this we notice that by the definition (1.330) we have that

$$
\begin{aligned}
&\Big[(\tilde{V}_0 + v_0)(\cdot, a_0),\, (\,\cdot\,)_T\Big] \diamond \partial_1^2 v_j(\cdot, a_0') - \Big[(\tilde{V}_1 + v_1)(\cdot, a_0),\, (\,\cdot\,)_T\Big] \diamond \partial_1^2 v_j(\cdot, a_0') \\
&= \Big[(\tilde{V}_0 - \tilde{V}_1)(\cdot, a_0),\, (\,\cdot\,)_T\Big] \diamond \partial_1^2 v_j(\cdot, a_0') \\
&\quad + [v_0(\cdot, a_0),\, (\,\cdot\,)_T] \diamond \partial_1^2 v_j(\cdot, a_0') - [v_1(\cdot, a_0),\, (\,\cdot\,)_T] \diamond \partial_1^2 v_j(\cdot, a_0').
\end{aligned}
$$

The relation (1.332) then follows from the triangle inequality, the assumption (1.325), and part *ii)* of Lemma 7 with $G_i = \tilde{V}_i(\cdot, a_0)$ for $i = 0, 1$, $C(\tilde{V}_0(\cdot, a_0), \tilde{V}_1(\cdot, a_0)) = [V_{int,0} - V_{int,1}]_\alpha$, and $F(\cdot, a_0) = v_j(\cdot, a_0)$.

Obtaining (1.333) is essentially the same argument. Again by the definition (1.330) we can write

$$
\begin{aligned}
&\Big[(\tilde{V}_i + v_i)(\cdot, a_0),\, (\,\cdot\,)_T\Big] \diamond \partial_1^2 v_1(\cdot, a_0') - \Big[(\tilde{V}_i + v_i)(\cdot, a_0),\, (\,\cdot\,)_T\Big] \diamond \partial_1^2 v_0(\cdot, a_0') \\
&= \Big[\tilde{V}_i(\cdot, a_0),\, (\,\cdot\,)_T\Big] \diamond \partial_1^2 (v_1 - v_0)(\cdot, a_0') \\
&\quad + [v_i(\cdot, a_0),\, (\,\cdot\,)_T] \diamond \partial_1^2 v_1(\cdot, a_0') - [v_i(\cdot, a_0),\, (\,\cdot\,)_T] \diamond \partial_1^2 v_0(\cdot, a_0').
\end{aligned}
$$

The relation (1.333) is then obtained via the triangle inequality using (1.326) for the second term on the right-hand side and part $iii)$ of Lemma 7 with $G = \tilde{V}_i(\cdot, a_0)$, $F_1(\cdot, a_0) = v_1(\cdot, a_0)$, and $F_0(\cdot, a_0) = v_0(\cdot, a_0)$. For the assumption (1.319) we notice that $v_1 - v_0$ solves (1.4) with right-hand side $f_1 - f_0$. □

Having finished our construction of the reference products we then move on to the proof of the reconstruction lemma. As already mentioned a couple of times, it is not difficult to adapt the proof of Otto and Weber to our setting and the majority of the below argument is taken straight from [47]

Proof of Lemma 9. To begin, one notices that as a result of the modelling the bound

$$[\nu]_{2\alpha-1} \lesssim M + N \tag{1.365}$$

holds. To see this we fix $x, y \in \mathbb{R}^2$ and, using the notation $l_x(y) = \nu(x)y_1$, rewrite (1.342) as

$$|(u - w(\cdot, x) - l_x)(y) - (u - w(\cdot, x) - l_x)(x)| \leq Md^{2\alpha}(x, y).$$

Using the triangle inequality we see that for a third point $y' \in \mathbb{R}^2$ this gives

$$\begin{aligned} &|(u - w(\cdot, x) - l_x)(y) - (u - w(\cdot, x) - l_x)(y'))| \\ &\leq M(d^{2\alpha}(x, y) + d^{2\alpha}(x, y')). \end{aligned} \tag{1.366}$$

Introducing a fourth point $x' \in \mathbb{R}^2$, we then use (1.338) and another application of the triangle inequality to write

$$\begin{aligned} &|(u - w(\cdot, x') - l_x)(y) - (u - w(\cdot, x') - l_x)(y'))| \\ \leq &M(d^{2\alpha}(x, y) + d^{2\alpha}(x, y')) + Nd^{\alpha}(y, y')d^{\alpha}(x, x'). \end{aligned} \tag{1.367}$$

Replacing x by x' in (1.366) and the taking the difference between this and (1.367) gives that

$$\begin{aligned} &|(l_x - l_{x'})(y) - (l_x - l_{x'})(y')| \\ &\leq M \left(d^{2\alpha}(x', y) + d^{2\alpha}(x', y') + d^{2\alpha}(x, y) + d^{2\alpha}(x, y')\right) \\ &+ Nd^{\alpha}(y, y')d^{\alpha}(x', x) \end{aligned} \tag{1.368}$$

We then take $y = x$ and $y' = x + (R, 0)$ for some $R > 0$ to obtain

$$|\nu(x) - \nu(x')|R \lesssim M(R^{2\alpha} + d^{2\alpha}(x', x)) + NR^{\alpha}d^{\alpha}(x', x). \tag{1.369}$$

To finish the argument for (1.365), we let $R = d(x', x)$. Using the modelling assumption we also obtain an L^{∞}-bound for ν. In particular, the triangle inequality gives that

$$|\nu(x)(x - y)_1| \leq Md^{2\alpha}(x, y) + |u(x) - u(y) - (w(x, x) - w(y, x))|,$$

which, after exploiting the periodicity of u and $w(\cdot, x)$ in the x_1-direction, gives that

$$\|\nu\| \leq M. \tag{1.370}$$

Having obtained the bounds for ν, the proof in [47] then continues in much the same fashion as the proof of Lemma 4. In particular, for any $\tau, T > 0$ such that $T = 2^n\tau$ for some $n \in \mathbb{N}$, it is shown that

$$\begin{aligned} &(uh_T - E_{tr}[w, (\cdot)_T] \diamond h - \nu [x_1, (\cdot)_T] h) \\ &\qquad - (uh_\tau - E_{tr}[w, (\cdot)_\tau] \diamond h - \nu [x_1, (\cdot)_\tau] h)_{T-\tau} \\ &= \sum_{t=\tau 2^i \text{ for } 0 \leq i \leq n} \Bigg(([u, (\cdot)_t] - E_{tr}[w, (\cdot)_t] - \nu [x_1, (\cdot)_\tau])h_t \\ &\qquad - [\nu, (\cdot)_t][x_1, (\cdot)_t] h - [E_{tr}, (\cdot)_t][w, (\cdot)_t] \diamond h \Bigg)_{T-2t}. \end{aligned} \tag{1.371}$$

As in the proof of Lemma 4, for this decomposition we reference Step 3 of the proof of Lemma 2 in [47] and do not give the argument here. The relation (1.371) is then used to show that

$$\begin{aligned} \|uh_T - E_{tr}\,[w,(\cdot)_T] \diamond h - \nu\,[x_1,(\cdot)_T]\,h \\ - (uh_\tau - E_{tr}\,[w,(\cdot)_\tau] \diamond h - \nu\,[x_1,(\cdot)_\tau]\,h)_{T-\tau}\| \\ \lesssim (M+N)N_0(T^{\frac{1}{4}})^{3\alpha-2} \end{aligned} \tag{1.372}$$

for $\tau < T \le 1$. In particular, (1.372) is obtained from (1.371) after using (1.21) and the three estimates

$$\|([u,(\cdot)_t] - E_{tr}\,[w,(\cdot)_t] - \nu\,[x_1,(\cdot)_\tau])h_t\| \lesssim MN_0(t^{\frac{1}{4}})^{3\alpha-2}, \tag{1.373}$$

$$\|\,[\nu,(\cdot)_t]\,[x_1,(\cdot)_t]\,h\| \lesssim (M+N)N_0(t^{\frac{1}{4}})^{3\alpha-2}, \tag{1.374}$$

$$\text{and } \|\,[E_{tr},(\cdot)_t]\,[w,(\cdot)_t] \diamond h\| \lesssim NN_0(t^{\frac{1}{4}})^{3\alpha-2}, \tag{1.375}$$

which hold for all $t > 0$. Notice that in this step, just as in the corresponding step of the proof of Lemma 4, one critically uses that $\alpha \in (\frac{2}{3}, 1)$

The estimates (1.373), (1.374), and (1.375) are proven as in [47] without any changes necessary. The first estimate relies on the modelledness assumption (1.341) in the sense that for all $x \in \mathbb{R}^2$ the left-hand side can be rewritten and bounded as

$$\begin{aligned} &|([u,(\cdot)_t] - E_{tr}\,[w,(\cdot)_t] - \nu\,[x_1,(\cdot)_t])h_t(x)| \\ =&\left|\int_{\mathbb{R}^2} (u(x) - u(y) - (w(x,x) - w(y,x)) - \nu(x)(x-y)_1) \right. \\ &\qquad\qquad \left. \times\, \psi_t(x-y)h_t(x)\,\mathrm{d}y\right| \\ \le& M\|h_t\| \int_{\mathbb{R}^2} |\psi_t(x-y)| d^{2\alpha}(x,y) \\ \overset{(1.339),(1.18)}{\lesssim}& MN_0(t^{\frac{1}{4}})^{3\alpha-2}. \end{aligned}$$

For the relation (1.374) one uses Lemma 10 of [47], which gives us that

$$\|[x_1,(\cdot)_t]h\| \lesssim t^{\frac{1}{4}}\|h_t\| \overset{(1.339)}{\le} N_0(t^{\frac{1}{4}})^{\alpha-1}.$$

For $x \in \mathbb{R}^2$ we then combine this with Young's inequality to give

$$\begin{aligned} |\,[\nu,(\cdot)_t]\,[x_1,(\cdot)_t]\,h(x)| \lesssim& \|[x_1,(\cdot)_t]h\| \int_{\mathbb{R}^2} |\nu(x) - \nu(y)|\,|\psi_t(x-y)|\,\mathrm{d}y \\ \lesssim& [\nu]_{2\alpha-1}\,\|\,[x_1,(\cdot)_t]\,h\| \int_{\mathbb{R}^2} d^{2\alpha-1}(x,y)|\psi_t(x-y)|\,\mathrm{d}y \\ \overset{(1.18)}{\lesssim}& (M+N)N_0(t^{\frac{1}{4}})^{3\alpha-2}, \end{aligned}$$

where we have applied (1.365). Lastly, (1.375) is obtained using (1.340) by writing

$$\begin{aligned}
&|[E_{tr}, (\cdot)_t]\,[w, (\cdot)_t] \diamond h(x)| \\
\leq &\int_{\mathbb{R}^2} |\psi_t(x-y)|\,|[w(\cdot,x), (\cdot)_t] \diamond h(y) - [w(\cdot,y), (\cdot)_t] \diamond h(y)|\,\mathrm{d}y \\
\lesssim &N N_0 (t^{\frac{1}{4}})^{2\alpha-2} \int_{\mathbb{R}^2} |\psi_t(x-y)| d^\alpha(x,y)\,\mathrm{d}y \\
\overset{(1.18)}{\lesssim} &N N_0 (t^{\frac{1}{4}})^{3\alpha-2}
\end{aligned}$$

for every $x \in \mathbb{R}^2$.

The proof of this lemma is then finished in a manner very similar to Lemma 4. In particular, we introduce the notation

$$\mathcal{F}^\tau = u h_\tau - E_{tr}\,[w, (\cdot)_\tau] \diamond h - \nu\,[x_1, (\cdot)_\tau]\,h$$

with which (1.372) becomes

$$\sup_{T\leq 1} (T^{\frac{1}{4}})^{2-3\alpha} \|\mathcal{F}^T - (\mathcal{F}^\tau)_{T-\tau}\| \lesssim (M+N) N_0. \tag{1.376}$$

Using the restriction $T \leq 1$, the assumptions (1.339) and (1.340), the bound (1.370), and Lemma 10 of [47] we notice that

$$\begin{aligned}
\sup_{T\leq 1} (T^{\frac{1}{4}})^{2-\alpha} \|\mathcal{F}^T\| =& \sup_{T\leq 1} (T^{\frac{1}{4}})^{2-\alpha} \|u h_T - E_{tr}\,[w, (\cdot)_T] \diamond h - \nu\,[x_1, (\cdot)_T]\,h\| \\
\lesssim& (\|u\| + \|\nu\|) \sup_{T\leq 1} (T^{\frac{1}{4}})^{2-\alpha} \|h_T\| + N N_0 \\
\lesssim& (\|u\| + M + N) N_0.
\end{aligned} \tag{1.377}$$

Combining this with (1.21) and (1.376) we find that

$$\sup_{T\leq 1} (T^{\frac{1}{4}})^{2-\alpha} \|(\mathcal{F}^\tau)_T\| \lesssim (\|u\| + M + N) N_0. \tag{1.378}$$

By Lemma 1 the bound (1.378) yields that the F^τ have uniformly bounded local $C^{\alpha-2}$ seminorms and the argument then finishes exactly as in Lemma 4. □

Moving on to the proof of Corollary 4 we note again that every part of the statement is an application of the previous lemma with different choices for the family $\{w(\cdot,x)\}_x$ and the distribution h. For each part we must specify these identifications and then check that the assumptions (1.337)-(1.342) hold for some constants $N, N_0 \in \mathbb{R}$. In the same fashion as the reconstruction lemma, the proof of this corollary is also essentially the same as in [47]. Since this corollary is quite long and we do not actually handle the quasilinear problem in this thesis, in some places we only give an outline of the proof without giving the detailed calculations.

Proof of Corollary 4. For use throughout this proof, we fix points $x, x', y, y' \in \mathbb{R}^2$ and reintroduce the notation

$$a_i^t = ta_i(x) + (1-t)a_i(x'). \tag{1.379}$$

i) We set

$$\begin{aligned} w(\cdot, x) &= \sigma_i(x)(v_i + \tilde{V}_i)(\cdot, a_i(x)), \\ h &= \partial_1^2 v_j(\cdot, a_0'), \\ \text{and} \quad w(\cdot, x) \diamond h &= \sigma_i(x)(v_i + \tilde{V}_i)(\cdot, a_i(x)) \diamond \partial_1^2 v_j(\cdot, a_0') \end{aligned}$$

and check that the assumptions (1.337)-(1.341) hold. We go down the laundry list:

- For the assumption (1.337) we may simply write

$$\sup_{x\in\mathbb{R}^2} \Big[\sigma_i(x)(v_i + \tilde{V}_i)(\cdot, a_i(x))\Big]_\alpha \lesssim \|\sigma\|(N_0 + N_0^{int}) \leq N_0 + N_0^{int}$$

by (1.54) applied to $V_i(\cdot, a_0)$, (1.322) of Lemma 8, and the assumption that $\|\sigma_i\| \leq 1$.

- For (1.338) we obtain

$$\begin{aligned} &\frac{|w(y,x) - w(y,x') - (w(y',x) - w(y',x'))|}{d^\alpha(y,y')} \\ \lesssim &\|\sigma\| \frac{|a_i(x) - a_i(x')|}{d^\alpha(y,y')} \left| \int_0^1 (\partial_{a_0} v_i(y, a_i^t) - \partial_{a_0} v_i(y', a_i^t)) \,\mathrm{d}t \right| \\ &+ \frac{|\sigma(x) - \sigma(x')|}{d^\alpha(y,y')} |v_i(y, a(x')) - v_i(y', a(x'))| \\ &+ \textit{same expression in } \tilde{V}_i \\ \lesssim &(N_0 + N_0^{int}) d^\alpha(x,x'), \end{aligned}$$

where we have used Lemmas 3 and 8 and the assumptions on σ_i and a_i.

- For (1.339) we notice that

$$\sup_{T\leq 1} (T^{\frac{1}{4}})^{2-\alpha} \|(\partial_1^2 v_j(\cdot, a_0'))_T\| \overset{(1.20)}{\lesssim} [v_j(\cdot, a_0')]_\alpha \overset{(1.322)}{\lesssim} N_0.$$

• The assumption (1.340) is verified in ii) of Corollary 3 with right-hand side $(N_0^{int} + N_0)N_0$.

• We lastly verify the assumption (1.341) for which we write

$$\begin{aligned}
&\sup_{a_0\in[\lambda,1]} \sup_{T\leq 1}(T^{\frac14})^{2-2\alpha} \Big\|\sigma_i(x) \Big[(v_i+\tilde{V}_i)(\cdot,a_i(x)),(\cdot)_T\Big] \diamond \partial_1^2 v_j(\cdot,a_0') \\
&\qquad\qquad -\sigma_i(x') \Big[(v_i+\tilde{V}_i)(\cdot,a_i(x')),(\cdot)_T\Big] \diamond \partial_1^2 v_j(\cdot,a_0')\Big\| \\
&\lesssim \Bigg([a_i]_\alpha \sup_{a_0,a_0'\in[\lambda,1]} \sup_{T\leq 1}(T^{\frac14})^{2-2\alpha} \Big\|\Big[(v_i+\tilde{V}_i)(\cdot,a_0),(\cdot)_T\Big] \diamond \partial_1^2 v_j(\cdot,a_0')\Big\|_{1,0} \\
&+[\sigma_i]_\alpha \sup_{a_0,a_0'\in[\lambda,1]} \sup_{T\leq 1}(T^{\frac14})^{2-2\alpha} \Big\|\Big[(v_i+\tilde{V}_i)(\cdot,a_0),(\cdot)_T\Big] \diamond \partial_1^2 v_j(\cdot,a_0')\Big\|\Bigg)\, d^\alpha(x,x') \\
&\lesssim (N_0^{int}+N_0)N_0 d^\alpha(x,x').
\end{aligned} \tag{1.380}$$

Here we have used the results of Corollary 3 and our assumptions on a_i and σ_i.

Combining all of these calculations we find that we may set the constants in Lemma 9 as $N = c(N_0^{int} + N_0)$ and $N_0 = cN_0$ for some large enough $c \in \mathbb{R}$, where the N_0 on the left-hand side is the constant in Lemma 9 and the N_0 on the right-hand side come from (1.321). The application of Lemma 9 then yields (1.346) without the two parameter derivatives. It still remains to show (1.346) for the indicated parameter derivatives. We start with the case of one parameter derivative and, setting up for another application of Lemma 9, let

$$\begin{aligned}
w(\cdot,x) &= \sigma_i(x)(v_i+\tilde{V}_i)(\cdot,a_i(x)), \\
h &= \partial_1^2 v_j(\cdot,a_0^+) - \partial_1^2 v_j(\cdot,a_0^-), \\
\text{and}\quad w(\cdot,x)\diamond h &= \sigma_i(x)\Big((v_i+\tilde{V}_i)(\cdot,a_i(x))\diamond \partial_1^2 v_j(\cdot,a_0^+) \\
&\qquad -(v_i+\tilde{V}_i)(\cdot,a_i(x))\diamond \partial_1^2 v_j(\cdot,a_0^-)\Big)
\end{aligned}$$

for any $a_0^-, a_0^+ \in \mathbb{R}$. We must again check the assumptions of Lemma 9, which is made easier by the fact that our choice of $w(\cdot,x)$ has not changed. Here is the list of assumptions that have not been proved above:

• We start by checking the assumption (1.339). In particular, we use Lemma 8, the equivalence in Lemma 1, and the assumption (1.321) to write

$$\sup_{T\leq 1}(T^{\frac14})^{2-\alpha}\|(\partial_1^2 v_j(\cdot,a_0^+) - \partial_1^2 v_j(\cdot,a_0^-))_T\| \lesssim N_0|a_0^+ - a_0^-|.$$

• The relation (1.340) follows from part $ii)$ of Corollary 3 and our assumption $\|\sigma_i\| \leq 1$ as

$$\begin{aligned}
&\sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\|[w(\cdot,x),(\cdot)_T]\diamond h\| \\
\leq & |a_0^+ - a_0^-| \sup_{a_0,a_0'\in[\lambda,1]} \sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\left\|\left[(v_i+\tilde{V}_i)(\cdot,a_0),(\cdot)_T\right]\diamond\partial_1^2 v_j(\cdot,a_0')\right\|_{0,1} \\
\lesssim & |a_0^+ - a_0^-|(N_0^{int}+N_0)N_0.
\end{aligned}$$

• For the assumption (1.341) we write

$$\begin{aligned}
&\sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\|[w(\cdot,x),(\cdot)_T]\diamond h - [w(\cdot,x'),(\cdot)_T]\diamond h\| \\
&\lesssim \|\sigma\||a_0^+ - a_0^-||a(x)-a'(x)|\times \\
&\qquad \sup_{a_0,a_0'\in[\lambda,1]} \sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\left\|\left[(v_i+\tilde{V}_i)(\cdot,a_0),(\cdot)_T\right]\diamond\partial_1^2 v_j(\cdot,a_0')\right\|_{1,1} \\
&+ [\sigma]_\alpha d^\alpha(x,x')|a_0^+ - a_0^-|\times \\
&\qquad \sup_{a_0,a_0'\in[\lambda,1]} \sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\left\|\left[(v_i+\tilde{V}_i)(\cdot,a_0'),(\cdot)_T\right]\diamond\partial_1^2 v_j(\cdot,a_0)\right\|_{0,1} \\
&\lesssim |a_0^+ - a_0^-|d^\alpha(x,x')(N_0^{int}+N_0)N_0.
\end{aligned}$$

After these calculations we can set $N = c(N_0^{int}+N_0)$ and $N_0 = cN_0|a_0^+ - a_0^-|$ for $c \in \mathbb{R}$ large enough and apply Lemma 9. In this way, we obtain a distribution $u \diamond (\partial_1^2 v_j(\cdot,a_0^+) - \partial_1^2 v_j(\cdot,a_0^-))$ such that

$$\begin{aligned}
&\sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\left\|[u,(\cdot)_T]\diamond(\partial_1^2 v_j(\cdot,a_0^+) - \partial_1^2 v_j(\cdot,a_0^-))\right\| \\
\lesssim & |a_0^+ - a_0^-|(N_0^{int}+N_0+M)N_0.
\end{aligned} \tag{1.381}$$

To finish showing that (1.346) holds for one parameter derivative, we notice that due to the built-in linearity of our definition of the product $w(\cdot,x)\diamond h(\cdot)$, the property $x(\partial_1^2 v_j(\cdot,a_0^+) - \partial_1^2 v_j(\cdot,a_0^-)) = x\partial_1^2 v_j(\cdot,a_0^+) - x\partial_1^2 v_j(\cdot,a_0^-)$, and the uniqueness in Lemma 4 it holds that

$$u \diamond (\partial_1^2 v_j(\cdot,a_0^+) - \partial_1^2 v_j(\cdot,a_0^-)) = u \diamond \partial_1^2 v_j(\cdot,a_0^+) - u \diamond \partial_1^2 v_j(\cdot,a_0^-).$$

Plugging this into (1.381) we obtain (1.346) for one parameter derivative.

Obtaining (1.346) for two parameter derivative is a similar argument. In particular, it follows from applying Lemma 9 with

$$w(\cdot,x) = \sigma_i(x)(v_i+\tilde{V}_i)(\cdot,a_i(x)),$$

$$h = (\partial_1^2 v_j(\cdot, a_0^{++}) - \partial_1^2 v_j(\cdot, a_0^{+-})) - (\partial_1^2 v_j(\cdot, a_0^{-+}) - \partial_1^2 v_j(\cdot, a_0^{--})), \text{ and}$$
$$\begin{aligned} w(\cdot,x) \diamond h = \sigma_i(x) \Big(& (v_i + \tilde{V}_i)(\cdot, a_i(x)) \diamond \partial_1^2 v_j(\cdot, a_0^{++}) \\ & - (v_i + \tilde{V}_i)(\cdot, a_i(x)) \diamond \partial_1^2 v_j(\cdot, a_0^{+-}) \\ & - \Big((v_i + \tilde{V}_i)(\cdot, a_i(x)) \diamond \partial_1^2 v_j(\cdot, a_0^{-+}) \\ & -(v_i + \tilde{V}_i)(\cdot, a_i(x)) \diamond \partial_1^2 v_j(\cdot, a_0^{--}) \Big) \Big) \end{aligned}$$

for any $a_0^{++}, a_0^{+-}, a_0^{-+}, a_0^{--} \in \mathbb{R}$ such $a_0^{++} - a_0^{+-} = a_0^{-+} - a_0^{--}$. Again, $w(\cdot, x)$ has not changed, which reduces the number of assumptions of Lemma 9 that we must check. In this part we often make use of the following estimate

$$\begin{aligned} & \big| (v_j(x, a_0^{++}) - v_j(y, a_0^{++})) - (v_j(x, a_0^{+-}) - v_j(y, a_0^{+-})) \\ & \quad -(v_j(x, a_0^{-+}) - v_j(y, a_0^{-+}) - (v_j(x, a_0^{--}) - v_j(y, a_0^{--}))) \big| \\ & \le \sup_{a_0} [v_j(\cdot, a_0)]_{\alpha,2} |a_0^{++} - a_0^{+-}| |a_0^{-+} - a_0^{--}| \\ & \quad + \sup_{a_0} [v_j(\cdot, a_0)]_{\alpha,1} |a_0^{++} - a_0^{+-} - (a_0^{-+} - a_0^{--})| \\ & \le \sup_{a_0} [v_j(\cdot, a_0)]_{\alpha,2} |a_0^{++} - a_0^{+-}| |a_0^{-+} - a_0^{--}| \end{aligned} \tag{1.382}$$

which follows from the fundamental theorem of calculus and the condition that $a_0^{++} - a_0^{+-} = a_0^{-+} - a_0^{--}$. Here is the list of assumptions:

- We start by checking (1.339):

$$\begin{aligned} & \sup_{T \le 1} (T^{\frac14})^{2-\alpha} \| \big((\partial_1^2 v_j(\cdot, a_0^{++}) - \partial_1^2 v_j(\cdot, a_0^{+-})) \\ & \qquad\qquad -(\partial_1^2 v_j(\cdot, a_0^{-+}) - \partial_1^2 v_j(\cdot, a_0^{--})) \big)_T \| \\ & \lesssim [v_j(\cdot, a_0)]_{\alpha,2} |a_0^{++} - a_0^{+-}| \, |a_0^{-+} - a_0^{--}| \\ & \overset{(1.322)}{\lesssim} N_0 \, |a_0^{++} - a_0^{+-}| \, |a_0^{-+} - a_0^{--}|, \end{aligned}$$

where we have used (1.382) and (1.18).

- The assumption (1.340) follows from the calculation

$$\begin{aligned} & \sup_{T \le 1} (T^{\frac14})^{2-2\alpha} \| \sigma_i(x) [w(\cdot, x), (\cdot)_T] \diamond h \| \\ \lesssim & |a_0^{++} - a_0^{+-}| \, |a_0^{-+} - a_0^{--}| \|\sigma\| \times \\ & \qquad \sup_{a_0, a_0' \in [\lambda, 1]} \sup_{T \le 1} (T^{\frac14})^{2-2\alpha} \left\| \Big[(v_i + \tilde{V}_i)(\cdot, a_0), (\cdot)_T \Big] \diamond \partial_1^2 v_j(\cdot, a_0') \right\|_{0,2} \\ \lesssim & |a_0^{++} - a_0^{+-}| \, |a_0^{-+} - a_0^{--}| N_0 (N_0^{int} + N_0). \end{aligned}$$

Here we have again used (1.382), but now in combination with *ii)* of Corollary 3.

• Lastly, we check (1.341) by writing

$$\begin{aligned}
&\sup_{T\le 1}(T^{\frac14})^{2-2\alpha}\|[w(\cdot,x),(\cdot)_T]\diamond h-[w(\cdot,x'),(\cdot)_T]\diamond h\|\\
\lesssim&|a_0^{++}-a_0^{+-}|\,|a_0^{-+}-a_0^{--}|[a]_\alpha d^\alpha(x,x')\times\\
&\qquad\sup_{a_0,a_0'\in[\lambda,1]}\sup_{T\le 1}(T^{\frac14})^{2-2\alpha}\left\|[(v_i+\tilde V_i)(\cdot,a_0),(\cdot)_T]\diamond\partial_1^2 v_j(\cdot,a_0')\right\|_{1,2}\\
&+|a_0^{++}-a_0^{+-}|\,|a_0^{-+}-a_0^{--}|[\sigma]_\alpha d^\alpha(x,x')\times\\
&\qquad\sup_{a_0,a_0'\in[\lambda,1]}\sup_{T\le 1}(T^{\frac14})^{2-2\alpha}\left\|[(v_i+\tilde V_i)(\cdot,a_0),(\cdot)_T]\diamond\partial_1^2 v_j(\cdot,a_0')\right\|_{0,2}\\
\lesssim&|a_0^{++}-a_0^{+-}|\,|a_0^{-+}-a_0^{--}|d^\alpha(x,x')(N_0^{int}+N_0)N_0.
\end{aligned}$$

With these bounds in-hand we again apply Lemma 9, now with $N=c(N_0+N_0^{int})$ and $N_1=cN_0|a_0^{++}-a_0^{+-}|\,|a_0^{-+}-a_0^{--}|$ for large enough $c\in\mathbb{R}$. To obtain (1.346) for two parameter derivatives one then follows the same steps as in the case of only one parameter derivative above.

To obtain (1.347) without parameter derivative we use Lemma 9, now with the identifications

$$\begin{aligned}
w(\cdot,x)&=\sigma_i(x)(v_i+\tilde V_i)(\cdot,a_i(x)),\\
h&=(\partial_1^2 v_1-\partial_1^2 v_0)(\cdot,a_0),\\
\text{and}\quad w(\cdot,x)\diamond h&=\sigma_i(x)((v_i+\tilde V_i)(\cdot,a_i(x))\diamond\partial_1^2 v_1(\cdot,a_0)\\
&\qquad-(v_i+\tilde V_i)(\cdot,a_i(x))\diamond\partial_1^2 v_0(\cdot,a_0)).
\end{aligned}\tag{1.383}$$

Since our choice of $w(\cdot,x)$ is the same as in the previous part, this reduces the list of assumptions that must be checked. For the last time, here is the list of remaining assumptions:

• We start with (1.339) and notice that $(v_1-v_0)(\cdot,a_0)$ solves (1.4) with right-hand side f_1-f_0. By Lemma 8 this implies that

$$\sup_{T\le 1}(T^{\frac14})^{2-\alpha}\|((\partial_1^2 v_1-\partial_1^2 v_0)(\cdot,a_0))_T\|\lesssim\delta N_0.$$

• The assumption (1.340) is verified in part *iii*) of Corollary 3 with right-hand side given by $(N_0^{int}+N_0)\delta N_0$.

• To finish, we check (1.341) similarly to (1.380), again using part *iii)* of Corollary 3. The right-hand side that one obtains in this calculation is $(N_0^{int} + N_0)\delta N_0 d^\alpha(x,x')$.

All combined, we find that we may take $N = c(N_0^{int} + N_0)$ and $N_1 = c\delta N_0$ for large enough $c \in \mathbb{R}$ in our application of Lemma 9. The relation (1.347) for one parameter derivatives follows in the same way as in part *i)*, but now taking advantage of the version of (1.333) involving one parameter derivative in a_0'.

ii) For this part we would like to apply Lemma 9 with

$$\begin{aligned} w(\cdot,x) &= \sigma_1(x)(v_1+\tilde{V}_1)(\cdot,a_1(x)) - \sigma_0(x)(v_0+\tilde{V}_0)(\cdot,a_0(x)),\\ h &= \partial_1^2 v_i(\cdot,a_0),\\ \text{and}\quad w(\cdot,x)\diamond h &= \sigma_1(x)(v_1+\tilde{V}_1)(\cdot,a_0(x))\diamond \partial_1^2 v_i(\cdot,a_0)\\ &\quad - \sigma_0(x)(v_0+\tilde{V}_0)(\cdot,a_0(x))\diamond \partial_1^2 v_i(\cdot,a_0). \end{aligned}$$

Here is our list of assumptions:

• We start by checking (1.337):

$$\begin{aligned} [w(\cdot,x)]_\alpha \lesssim &\|\sigma_0\| \sup_{a_0\in[\lambda,1]} \Bigg([(v_0-v_1)(\cdot,a_0)]_\alpha + \Big[(\tilde{V}_0-\tilde{V}_1)(\cdot,a_0)\Big]_\alpha \\ &\qquad + |a_0(x)-a_1(x)| \Big([v_0(\cdot,a_0)]_{\alpha,1} + \Big[\tilde{V}_0(\cdot,a_0)\Big]_{\alpha,1}\Big)\Bigg) \\ &+ \|\sigma_0-\sigma_1\| \sup_{a_0\in[\lambda,1]} \Big[(v_1+\tilde{V}_1)(\cdot,a_0)\Big]_\alpha \\ \lesssim &\delta N_0 + \delta N_0^{int} + (\|a_1-a_0\|_\alpha + \|\sigma_1-\sigma_0\|_\alpha)(N_0+N_0^{int}). \end{aligned}$$

• For the assumption (1.338) we calculate

$$\begin{aligned} &\frac{|w(y,x)-w(y,x')-(w(y',x)-w(y',x'))|}{d^\alpha(y,y')} \\ &\lesssim \|\sigma_1\| \frac{|a_1(x)-a_1(x')|}{d^\alpha(y,y')} \left| \int_0^1 \partial_{a_0}(v_1-v_0)(y,a_1^t) - \partial_{a_0}(v_1-v_0)(y',a_1^t)\,\mathrm{d}t \right| \\ &+ \frac{\|\sigma_1\|}{d^\alpha(y,y')} \Big| v_0(y,a_0(x)) - v_0(y,a_0(x')) - (v_0(y',a_0(x)) - v_0(y',a_0(x'))) \\ &\qquad - (v_0(y,a_1(x)) - v_0(y,a_1(x')) - (v_0(y',a_1(x)) - v_0(y',a_1(x')))) \Big| \end{aligned}$$

$$
\begin{aligned}
&+ (\sigma_1-\sigma_0)(x)v_0(y,a_0(x)) - (\sigma_1-\sigma_0)(x')v_0(y,a_0(x'))\\
&- ((\sigma_1-\sigma_0)(x)v_0(y',a_0(x)) - (\sigma_1-\sigma_0)(x')v_0(y',a_0(x')))\\
&+ \textit{analogous expression in } \tilde{V}_1 \textit{ and } \tilde{V}_0.
\end{aligned}
$$

Ultimately, we will be able to bound the right-hand side of this by

$$
(\delta N_0 + \delta N_0^{int} + (\|a_1-a_0\|_\alpha + \|\sigma_1-\sigma_0\|_\alpha)(N_0+N_0^{int}))d^\alpha(x,x').
$$

To bound the second term on the right-hand side of the above expression we use (1.382), which gives

$$
\begin{aligned}
&|(v_0(y,a_0(x)) - v_0(y,a_0(x'))) - (v_0(y',a_0(x)) - v_0(y',a_0(x')))\\
&\qquad -((v_0(y,a_1(x)) - v_0(y,a_1(x'))) - (v_0(y',a_1(x)) - v_0(y',a_1(x'))))|\\
&\lesssim \sup_{a_0\in[\lambda,1]} [\partial_{a_0}^2 v_0(\cdot,a_0)]_\alpha |a_1^s(x) - a_0^s(x')||a_1(x)-a_1(x')| d^\alpha(y,y')\\
&\quad + \sup_{a_0\in[\lambda,1]} [\partial_{a_0} v_0(\cdot,a_0)]_\alpha |a_0^s(x) - a_0^s(x') - (a_1^s(x) - a_1^s(x'))| d^\alpha(y,y')\\
&\lesssim N_0\|a_1-a_0\|_\alpha d^\alpha(x,x') d^\alpha(y,y').
\end{aligned}
$$

We perform the analogous calculation for the term involving $\tilde{V}_1$ and $\tilde{V}_0$.

• Continuing, we notice that the assumption (1.339) of Lemma 9 is verified with right-hand side N_0 in part *i)*.

• For (1.340) we use the triangle inequality to write

$$
\begin{aligned}
&\sup_{a_0\in[\lambda,1]}\sup_{T\le 1}(T^{\frac14})^{2-2\alpha}\Big\|\sigma_0(x)[(v_0+\tilde{V}_0)(\cdot,a_0(x)),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0)\\
&\qquad\qquad -\sigma_1(x)[(v_1+\tilde{V}_1)(\cdot,a_1(x)),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0)\Big\|\\
&\le \|\sigma_0\| \sup_{a_0,a_0'\in[\lambda,1]}\sup_{T\le 1}(T^{\frac14})^{2-2\alpha}\times\\
&\Big(\|[v_0(\cdot,a_0),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0')\|_{1,0}\|a_1-a_0\|\\
&\qquad + \|[(v_0-v_1)(\cdot,a_0),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0')\|\\
&\qquad\qquad + \textit{analogous expression in } \tilde{V}_0 \textit{ and } \tilde{V}_1\Big)\\
&+\|\sigma_0-\sigma_1\| \sup_{a_0,a_0'\in[\lambda,1]}\sup_{T\le 1}(T^{\frac14})^{2-2\alpha}\Big\|[(v_1+\tilde{V}_1)(\cdot,a_0),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0')\Big\|\\
&\lesssim N_0((N_0+N_0^{int})(\|a_1-a_0\|+\|\sigma_1-\sigma_0\|)+\delta N_0+\delta N_0^{int}).
\end{aligned}
$$

- We finish by showing the assumption (1.341). Here, we calculate

$$
\begin{aligned}
&\sup_{a_0'\in[\lambda,1]}\sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\big\|[w(\cdot,x),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0') \\
&\qquad\qquad\qquad\qquad - [w(\cdot,x'),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0')\big\| \\
\lesssim&\|\sigma_0\||a_0(x)-a_0(x')|\sup_{T\leq 1}(T^{\frac{1}{4}})^{2-2\alpha}\times \\
&\sup_{a_0,a_0'\in[\lambda,1]}\|[v_0(\cdot,a_0),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0') - [v_1(\cdot,a_0),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0')\|_{1,0} \\
&+\sup_{a_0'\in[\lambda,1]}\|[v_1(\cdot,a_1(x)),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0') - [v_1(\cdot,a_0(x)),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0') \\
&\quad + [v_1(\cdot,a_0(x')),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0')) - [v_1(\cdot,a_1(x')),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0')\| \\
&+\|\sigma_1-\sigma_0\|[a_1]_\alpha d^\alpha(x,x')\sup_{a_0,a_0'\in[\lambda,1]}\|[v_1(\cdot,a_0),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0')\|_{1,0} \\
&+[\sigma_1-\sigma_0]_\alpha d^\alpha(x,x')\sup_{a_0,a_0'\in[\lambda,1]}\|[v_1(\cdot,a_0),(\cdot)_T]\diamond\partial_1^2 v_i(\cdot,a_0')\| \\
&+\textit{analogous expression in } \tilde{V}_1 \textit{ and } \tilde{V}_0 \\
\lesssim&N_0((\|a_1-a_0\|_\alpha+|\sigma_1-\sigma_0\|_\alpha)(N_0+N_0^{int})+\delta N_0+\delta N_0^{int})d^\alpha(x,x').
\end{aligned}
$$

Combining the above find that we may apply Lemma 9 with $N_1 = cN_0$ and $N = c(\delta N_0 + \delta N_0^{int} + (\|a_1 - a_0\|_\alpha + \|\sigma_1 - \sigma_0\|_\alpha)(N_0 + N_0^{int}))$. Just like in part *i)* it remains to verify (1.348) for one parameter derivative. However, as the technique is the same as in *i)* we do not include the argument here. □

1.7 Appendix B: Proof of Theorem 2

Proof of Theorem 2. i) We apply part *i)* Theorem 1 with $a^{ext} = a(\overline{W})$. In order to check the assumptions, we start by noticing that

$$[a(\overline{W})]_\alpha \leq \|a'\|[\overline{W}]_\alpha \ll 1$$

and $a(\overline{W})$ is periodic in the x_1-direction since $\overline{W}$ is. For the assumptions (1.66) and (1.67) we must apply part *i)* of Corollary 4 with $u = a(\overline{W})$. Using Lemma 1 of [47] as stated in Section 1.1, we have that $a(\overline{W})$ is modelled after $(v+\tilde{V})(\cdot,a_0)$ according to $\bar{a}$ and $\mu = a'(\overline{W})$ with a modelling constant bounded as

$$M_{a^{ext}} \lesssim \|a'\|M + \|a''\|[\overline{W}]_\alpha^2, \tag{1.384}$$

where M corresponds to the modelling of $\overline{W}$, and such that

$$\|\mu\|_\alpha \leq \max(\|a'\|, \|a''\|[\overline{W}]_\alpha) \leq 1.$$

By Corollary 4 part $i)$ we then obtain a $C^{\alpha-2}(\mathbb{R}^2)$ family of distributions $\{a(\overline{W}) \diamond \partial_1^2 v(\cdot, a_0)\}$ satisfying (1.66) with right-hand side given by $(N_0^{int} + N_0 + M_{a^{ext}})N_0$. Applying part $i)$ of Theorem 1 then yields the result.

$ii)$ We apply part $ii)$ Theorem 1 with $a_0^{ext} = a(\overline{W}_0)$ and $a_1^{ext} = a(\overline{W}_1)$. In order to check the assumptions, we again use Lemma 1 of [47], but now also part $ii)$. In particular, we find that

$$\|a(\overline{W}_0) - a(\overline{W}_1)\|_\alpha \lesssim \|a'\|\|\overline{W}_1 - \overline{W}_0\|_\alpha + \|a''\|\|\overline{W}_1 - \overline{W}_0\| \max_i [\overline{W}_i]_\alpha.$$

Using the bounds from the previous part, we find that part $i)$ of Corollary 4 also yields (1.75) with right-hand side $(N_0^{int} + N_0 + M_{a^{ext}})\delta N_0$. For the assumption (1.76), we must use part $ii)$ of Corollary 4. For this we notice that $a(\overline{W}_1) - a(\overline{W}_0)$ is modelled after $((v_1 + \tilde{V}_1)(\cdot, a_0), (v_0 + \tilde{V}_0)(\cdot, a_0)$ according to $(\bar{a}_1, \bar{a}_0)$ and $(a'(\overline{W}_1), -a'(\overline{W}_0))$ with modelling constant $\delta M_{a^{ext}}$ bounded as

$$\begin{aligned}\delta M_{a^{ext}} \lesssim & \|a'\|\delta M + \|a''\| \max_i [\overline{W}_i]_\alpha [\overline{W}_1 - \overline{W}_0]_\alpha \\ & + \frac{1}{2}\|a'''\|\|\overline{W}_1 - \overline{W}_0\| \max_i [\overline{W}_i]_\alpha^2 + \|a''\|\|\overline{W}_1 - \overline{W}_0\| \max_i M_i,\end{aligned}$$

where δM denotes the modelling constant associated to $\overline{W}_1 - \overline{W}_0$ and M_i corresponds to the modelling of $\overline{W}_i$. Applying part $ii)$ of Corollary 4 along with the observation

$$\|a'(\overline{W}_1) - a'(\overline{W}_0)\|_\alpha \lesssim \|a''\|\|\overline{W}_1 - \overline{W}_0\|_\alpha + \|a'''\|\|\overline{W}_1 - \overline{W}_0\| \max_i [\overline{W}_i]_\alpha,$$

we find that (1.76) holds with right-hand side

$$N_0(\delta M_{a^{ext}} + N_0(\|\bar{a}_1 - \bar{a}_0\|_\alpha + \|a'(\overline{W}_1) - a'(\overline{W}_0)\|_\alpha) + \delta N_0 + \delta N_0^{int}).$$

Applying part $ii)$ of Theorem 1 then yields the result. $\square$

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Curriculum vitae

Name: Claudia Caroline Raithel
Date of Birth: August 31, 1990
Place of Birth: Munich, Germany

Educational Background:

Sept. 2014 - Sept. 2018: PhD at the Max-Planck Institute for Mathematics in the Sciences. Advisor: Felix Otto.

2011 - 2013: Master's degree in Mathematics at the University of Texas at Austin. Advisor: Thomas Chen.

2007 - 2011: Bachelor's degree in Honor's Mathematics and Interdisciplinary Physics at the University of Michigan, Ann Arbor.

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