On a formula of Coll-Gerstenhaber-Giaquinto

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Abstract

We present a unifying approach to deformations of an associative algebra A that allows to derive known formulas of Moyal-Vey (1949) and Coll-Gerstenhaber-Giaquinto (1989) from a more general point of view. Such universal deformation formulas correspond to special deformations of the comultiplication of a bialgebra.

1 Introduction

Let K be a ring containing the field $\mathbf{Q}$ of rational numbers, $K' = K[[h]]$ be the algebra of formal series on h and (A, μ_A) a K-algebra. This algebra structure extends in a natural way by K'-linearity to the algebra $A' := A[[h]]$ of power series in h with coefficients in A that we will denote by some abuse of notation also by μ_A. The aim of this paper is to study deformations of this structure.

DEFINITION 1 *A (formal)* deformation *of the K-algebra A is an algebra structure $A_h = (A', \mu_h)$ on A' with*

$$\mu_h := \mu_A + \sum_{k=1}^{\infty} h^k \varphi_k : A' \otimes A' \to A'.$$

For μ_h to be associative in first order on h, φ_1 must fulfill the property

$$\varphi_1(a_1 a_2, a_3) + \varphi_1(a_1, a_2) a_3 = \varphi_1(a_1, a_2 a_3) + a_1 \varphi_1(a_2, a_3)$$

for $a_1, a_2, a_3 \in A$, i.e. has to be a 2-cocycle in the Hochschild complex of A. Such a 2-cocycle φ_1 is called an *infinitesimal* of the deformation. We restrict ourselves to the case when the 2-cochains φ_k have the form $\varphi_k = \mu_A \circ P_k$, where $P_k : A \otimes A \to A \otimes A$ are $K-$linear maps. Given a 2-cocycle $S := P_1$ we try to define P_k for $k \geq 2$ so that μ_h is associative.

In practical applications such a 2-cocycle often appears as the product of 1-cocycles $S = D \otimes E$, where D, E are elements of a certain Lie algebra $\mathcal{G}$ acting by derivations on A. There are two famous results that describe prolongations of such 2-cocycles to associative multiplications on A_h:

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THEOREM 1 *(Moyal - Vey, [7], [3]) If the Abelian Lie algebra $\mathcal{G}$ acts on a K-algebra A by derivations, then for any element $S \in \mathcal{G} \otimes \mathcal{G}$ the composition $\mu_A \circ S$ is a 2-cocycle and the multiplication*

$$\mu_h = \mu_A \circ e^{hS}$$

is associative.

THEOREM 2 *(V.Coll, M.Gerstenhaber, A.Giaquinto, [1]) If the 2-dimensional Lie algebra $\mathcal{G}$ with generators E, D and commutator relation $[E, D] = E$ acts on the K-algebra A by derivations, then for $S = E \otimes D$ the composition $\mu_A \circ S$ is a 2-cocycle and the multiplication*

$$\mu_h = \mu_A \circ (1 + hE \otimes 1)^{1 \otimes D}$$

is associative.

Both theorems were first proved by direct calculations. For Moyal-Vey's theorem these computations are straightforward and use only the Leibniz rule, since D and E commute. The second result is less elementary. We will refer to this example as Gerstenhaber's.

Below we present a unifying approach to these results and give some generalizations of the above formulas. It is based on the notion of an admissible bialgebra action on A that allows to derive both results as partial cases of a more general principle to construct algebra deformations. More precisely, we replace the universal enveloping algebra $U_h(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ by a bialgebra B and define conditions on an element $P \in (B \otimes B)[[h]]$ such that for any admissible bialgebra action $\rho : B \to \mathrm{End}_K(A)$ the composition $\mu_h = \mu_A \circ (\rho \otimes \rho)(P)$ defines a deformation of A, i.e. we construct universal deformation formulas in the spirit of [5]. It turns out that for a consistent theory deformations of A have to be associated with deformations of the comultiplication of B leaving this way the class of universal enveloping algebras.

Different aspects of such a theory are demonstrated on Gerstenhaber's example. It turns out, that in this case on the deformed bialgebra there is a h-independent K-bialgebra structure that already exists in the first order deformation.

These investigations were stimulated by several discussions of the second author with R.-O. Buchweitz during a one month stay at the University of Toronto in december 1992 and elaborated further during several visits to the University of Leipzig.

Some of the ideas were considered in the articles [9], [10].

2 Admissible bialgebra actions on algebras

Let (B, μ_B, Δ_B) be a bialgebra as defined, for example, in [2]. Here μ_B denotes the algebra multiplication and $\Delta_B : B \longrightarrow B \otimes B$ the comultiplication. We use the standard notion where an integer index of an operator, acting on a tensor product, denotes the tensor cofactor, on which the operator acts. For example,

$$\Delta_i : C^{\otimes n} \to C^{\otimes(n+1)} : c_1 \otimes \ldots \otimes c_n \mapsto c_1 \otimes \ldots \otimes c_{i-1} \otimes \Delta(c_i) \otimes c_{i+1} \otimes \ldots \otimes c_n,$$

$$\mu_{A,23} : A^{\otimes 3} \to A^{\otimes 2} : a_1 \otimes a_2 \otimes a_3 \mapsto a_1 \otimes \mu_A(a_2 \otimes a_3) = a_1 \otimes a_2 a_3.$$

For $b \in B$ we use the Sweedler notation $\Delta(b) = \sum b_{(1)} \otimes b_{(2)}$ and $\Delta_1\Delta(b) = \Delta_2\Delta(b) = \sum b_{(1)} \otimes b_{(2)} \otimes b_{(3)}$ if we need to exploit their special structure as elements of $B \otimes B$ resp. $B \otimes B \otimes B$.

For a K-coalgebra C there is a notion of cohomology groups $H^n(K, C)$ as explained e.g. in [6]. They are the homologies of the complex

$$0 \longrightarrow C \longrightarrow \ldots \stackrel{\delta}{\longrightarrow} C^{\otimes k} \longrightarrow \ldots$$

where for $S \in C^{\otimes k}$ the coboundary formula is defined as

$$\delta S = 1 \otimes S + \sum_{i=1}^{k} (-1)^i \Delta_i S + (-1)^{k+1} S \otimes 1.$$

Especially, a 1-cocycle $X \in C$ fulfills the condition $\Delta(X) = X_1 + X_2$. For a 2-cocycle $S \in C \otimes_K C$ we get $\Delta_2(S) + S_{23} = \Delta_1(S) + S_{12}$.

For any two left modules $(M, \nu_M), (N, \nu_N)$ over the *algebra* B the tensor product $M \otimes_K N$ has a natural structure as left module over the algebra B defined by

$$\nu_{M\otimes N} : B \otimes (M \otimes N) \stackrel{\Delta_1}{\longrightarrow} (B \otimes B) \otimes (M \otimes N) \stackrel{S_{23}}{\longrightarrow} (B \otimes M) \otimes (B \otimes N) \stackrel{\nu_M \otimes \nu_N}{\longrightarrow} M \otimes N.$$

If a bialgebra B acts on a K-algebra A by $\rho : B \longrightarrow \mathrm{End}_K(A)$ we have the natural condition that $\mu_A : A \otimes A \longrightarrow A$ is a B-module homomorphism, i.e.

$$\begin{array}{ccc} A \otimes A & \stackrel{\mu_A}{\longrightarrow} & A \\ {\scriptstyle (\rho\otimes\rho)\circ\Delta_B(b)}\downarrow & & \downarrow{\scriptstyle \rho(b)} \\ A \otimes A & \stackrel{\mu_A}{\longrightarrow} & A \end{array}$$

commutes, where $A \otimes A$ is equipped with the B-module structure just defined. For $b \in B$ and $a_1, a_2 \in A$ we get the following condition:

$$\rho_A(b)(\mu_A(a_1 \otimes a_2)) = \mu_A(\rho_{A\otimes A}(b)(a_1 \otimes a_2))$$

or more explicitly

$$\rho(b)(a_1 \cdot a_2) = \sum \rho(b_{(1)})(a_1) \cdot \rho(b_{(2)})(a_2)$$

This can be written as

$$\forall b \in B \quad \rho(b) \circ \mu_A = \mu_A \circ (\rho \otimes \rho) \circ \Delta_B(b) \tag{1}$$

Definition 2 : *A homomorphism $\rho : B \to End_K(A)$ of K-algebras, that satisfies the compatibility condition (1) between μ_A and Δ_B is called an* admissible action *of the bialgebra (B, μ_B, Δ_B) on the K-algebra A.*

For the sake of simplicity we often will omit the symbol of the action ρ. Then, for example, (1) has the form

$$b \circ \mu_A = \mu_A \circ \Delta_B(b) \quad \text{or} \quad b(a_1 a_2) = \sum b_{(1)}(a_1) b_{(2)}(a_2)$$

for $a_1, a_2 \in A, b \in B$.

This definition generalizes to bialgebras the concept of *actions of universal enveloping algebras* induced by Lie algebra of derivations. Indeed, given an algebra A and a Lie algebra $\mathcal{G}$ acting on A, the universal enveloping algebra $B = U(\mathcal{G})$ has a natural bialgebra structure with comultiplication Δ defined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X \qquad \text{for } X \in \mathcal{G}.$$

Then the action of B on A is admissible iff for $X \in \mathcal{G}$ and $a_1, a_2 \in A$

$$X(a_1 \cdot a_2) = \mu_A\left((X \otimes 1 + 1 \otimes X)(a_1 \otimes a_2)\right) = (Xa_1)a_2 + a_1(Xa_2),$$

i.e. X is a derivation of A.

Note that an action of a bialgebra B on a K-algebra A is defined by the action of the generators of B on the generators of A.

Examples:

1. The left action of A on itself is an admissible bialgebra action, if we define $\Delta(L_a) = L_a \otimes 1$ for the left action $L_a \in \mathrm{End}_K(A)$ of $a \in A$. Analogously the right action of A^{op} on A is an admissible bialgebra action wrt. $\Delta(R_a) = 1 \otimes R_a$.

This may be extended to an admissible action of the enveloping algebra $A^e := A \otimes_K A^{op}$ on A, where the comultiplication is given by the rule $\Delta(x \otimes y) = (x \otimes 1) \otimes (1 \otimes y)$. If $A^e = \mathrm{End}_K(A)$, e.g. for a matrix algebra $M_n(K)$, this construction allows to introduce an admissible bialgebra structure on the whole algebra of endomorphisms $\mathrm{End}_K(A)$.

2. The natural action of the bialgebra $B = K[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$ on $A = K[x_1, \ldots, x_n]$ is an admissible bialgebra action, since B is the universal enveloping algebra of an Abelian Lie algebra acting on A by derivations.

3. It induces an action of the Weyl algebra $W = A \otimes_K B$ on A that is an admissible bialgebra action on A with respect to the natural bialgebra structure on W obtained by scalar extension $K \longrightarrow A$ from B. More precisely, the multiplication on W is induced by the commutation rules

$$\frac{\partial}{\partial x_i} \cdot x_j = \delta_{ij} + x_j \cdot \frac{\partial}{\partial x_i}$$

and the comultiplication by the corresponding rules on A and B

$$\Delta(x_i) = x_i \otimes 1 \quad \text{and} \quad \Delta(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i} \otimes 1 + 1 \otimes \frac{\partial}{\partial x_i}.$$

4. This may be generalized to arbitrary Lie algebras $\mathcal{G}$ acting on A by derivations. Indeed, the admissible bialgebra action of the universal enveloping algebra $B = U(\mathcal{G})$ on A described above may be extended to an admissible bialgebra action on $W = A \otimes_K B = A[\mathcal{G}]$ on A, if we extend multiplication and comultiplication on W by the following rules:

$$X \cdot a = X(a) + a \cdot X, \quad \Delta(a) = a \otimes 1, \quad \Delta(X) = X \otimes 1 + 1 \otimes X.$$

Here and below $a \in A$ and $X \in \mathcal{G}$ are identified with their images in W under the embeddings $A \to A \otimes 1 \subset W$ and $\mathcal{G} \to 1 \otimes \mathcal{G} \subset W$.

5. This may be generalized once more: For any admissible action of a bialgebra B on an algebra A there is a natural bialgebra structure on the K-module $W := A^e \otimes B$

extending those of A^e and B. As above we have only to define the product $b \cdot (x \otimes y)$ for $b \in B, (x \otimes y) \in A^e$. As easily seen the correct rule is

$$b \cdot (x \otimes y) = \sum (b_{(1)}(x) \otimes b_{(3)}(y)) \cdot b_{(2)},$$

where $\Delta_1\Delta(b) = \Delta_2\Delta(b) = \sum b_{(1)} \otimes b_{(2)} \otimes b_{(3)}$ are obtained from the comultiplication rule on B. Again admissibility of the B-action on A guarantees that the given rules define a bialgebra structure on W and the natural action of W on A is admissible.

Remark: The only place we found in the literature, where a condition similar to (1) was considered, is the following general result [5, Lemma 9.2.]:

> *Let V be any vector space, $\mu : k\langle V\rangle \otimes k\langle V\rangle \longrightarrow k\langle V\rangle$ be the multiplication in the free (=tensor) algebra $k\langle V\rangle$, $\mathcal{C} \subset \mathrm{End}(k\langle V\rangle)$ be a subspace of the linear endomorphisms, and $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$ be a linear map such that $\mu\Delta f(a \otimes b) = f(a\,b)$ for all $f \in \mathcal{C}, a, b \in k\langle V\rangle$. Then Δ is coassociative, i.e. $\mathcal{C}$ is a coalgebra.*

If moreover C is a subalgebra of $\mathrm{End}(k\langle V\rangle)$ and Δ a k-algebra homomorphism the natural action of the bialgebra $(\mathcal{C}, \mu_C, \Delta)$ on $k\langle V\rangle$ is admissible.

3 Deformations of algebras with an admissible bialgebra action

The main idea of this section is the observation that for both formulae considered in the introduction the deformed multiplication has the form $\mu_h = \mu_A \circ P$ for a certain element $P \in (U(\mathcal{G}) \otimes U(\mathcal{G}))[[h]]$.

So let B be a bialgebra with an admissible action on A as in the last section. As above the bialgebra structure extends to $B' = B[[h]]$ by K'-linearity in such a way that (B', μ_B, Δ_B) acts admissible on (A', μ_A). Below we consider the question, how deformations of the algebra structure on A are related to those of the bialgebra B.

Let's consider the condition that must be fulfilled by an element $P = 1 + \sum_{i=1}^{\infty} h^i P_{(i)} \in B' \otimes_{K'} B' = (B \otimes_K B)[[h]]$ for $\mu_h = \mu_A \circ P$ to be associative.

$$0 = \mu_h \circ (\mu_{h,12} - \mu_{h,23}) = \mu \circ P \circ (\mu_{12} \circ P_{12} - \mu_{23} \circ P_{23})$$

Since B acts admissible we get $P \circ \mu_{12} = \mu_{12} \circ \Delta_1(P), P \circ \mu_{23} = \mu_{23} \circ \Delta_2(P)$ and altogether

$$0 = \mu \circ \mu_{12} \circ (\Delta_1(P)P_{12} - \Delta_2(P)P_{23})$$

Hence

$$\Delta_1(P)P_{12} - \Delta_2(P)P_{23} = 0 \qquad (2)$$

is a sufficient condition for P to make μ_h associative for any admissible B-action on A.

This yields already a proof of the following generalization of the Moyal-Vey formula.

Theorem 3 *If the commutative bialgebra B acts admissible on A then for any 2-cocycle $S \in B \otimes B$ the multiplication*

$$\mu_h = \mu_A \circ e^{hS}$$

is associative.

PROOF: Indeed, for $P = e^{hS}$ condition (2) is equivalent to

$$e^{h\Delta_1(S)} \circ e^{hS_{12}} = e^{h\Delta_2(S)} \circ e^{hS_{23}}$$

and finally to

$$\Delta_1(S) + S_{12} = \Delta_2(S) + S_{23}. \quad \square$$

EXAMPLE : Let us consider the commutative bialgebra B with the free generators $E_i, D^i, L_i^j \in C,\ i,j = 1,\ldots,n$ and the comultiplication that using the matrix notation

$$\mathbf{E} = (\, E_1 \quad E_2 \quad \ldots \quad E_n \,),\ \mathbf{D} = \begin{pmatrix} D^1 \\ D^2 \\ \vdots \\ D^n \end{pmatrix},\ \mathbf{L} = (\, L_i^j \,),$$

may be written in the following form

$$\Delta(\mathbf{E}) = \mathbf{E}_1\mathbf{L}_2 + \mathbf{E}_2,\ \Delta(\mathbf{D}) = \mathbf{D}_2 + \mathbf{L}_1\mathbf{D}_2,\ \Delta(\mathbf{L}) = \mathbf{L}_1\mathbf{L}_2.$$

Then the 2-cochain $S = \mathbf{E}_1\mathbf{D}_2 = \sum_{i=1}^n E_i \otimes D^i$ is a cocycle and the power series $P = e^{hS}$ satisfies the equation (2).

If B acts admissible on an algebra A this yields an explicit formula for a deformation of A that doesn't fit into the frame of theorem 1.

The solution $P = e^{hS}$ of (2) described in theorem 3 is expressed as an exponential function. Since $f(x,h) = e^{hx}$ is the solution of the differential equation $\frac{\partial f}{\partial h} = x \cdot f$ with initial condition $f(x,0) = 1$ the expression

$$S_h := P^{-1}\frac{\partial P}{\partial h} \in (B \otimes B)[[h]]$$

also may play a crucial role for other applications. Note that the power series P is uniquely defined by S_h but their connection may be more difficult to describe than in theorem 3. Since $S_h|_{h=0} = P_{(1)}$ coincides with the element $S \in B \otimes B$ defined in the introduction, S_h is a deformation of S (in a sense to be specified).

Under certain additional assumptions the condition (2) may be reformulated as a condition on S_h. For example, if $S = S_h$ does not depend on h we get $P = \exp(hS)$ and (2) may be reformulated as

$$e^{h\Delta_1(S)} \cdot e^{hS_{12}} - e^{h\Delta_2(S)} \cdot e^{hS_{23}} = 0.$$

If the exponents mutually commute, i.e. $[\Delta_1(S), S_{12}] = [\Delta_2(S), S_{23}] = 0$ we can rewrite this equation as

$$e^{h(\Delta_1(S)+S_{12})} - e^{h(\Delta_2(S)+S_{23})} = 0$$

that holds for any 2-cocycle S. Thus we proved the following generalization of the previous theorem.

THEOREM 4 *Let S be a 2-cocycle of a (of a not necessarily commutative) bialgebra B and*

$$[\Delta_1(S), S_{12}] = [\Delta_2(S), S_{23}] = 0.$$

Then $P = \exp(hS)$ satisfies the eq. (2). $\square$

4 A first proof of Gerstenhaber's formula

With some more effort we also may prove Gerstenhaber's formula. By (2) we only have to show

$$\psi(E_1+E_2,D_3)\,\psi(E_1,D_2)=\psi(E_1,D_2+D_3)\,\psi(E_2,D_3) \tag{3}$$

for $\psi(E,D)=P=(1+hE_1)^{D_2}$.

To see this lets first collect several helpful identities :

Lemma 1 *For $f,g\in K[x][[h]]$ we get*

1. $E^n f(D)=f(D+n)E^n$,

2. $[D,f(E)]=-x\frac{\partial}{\partial x}f(x)\mid_{x=E}$,

3. $f(E)D=(D+E\frac{\partial}{\partial E}\ln f(E))\cdot f(E)$,

4. $f(E)g(D)=g(D+E\frac{\partial}{\partial E}\ln f(E))\cdot f(E)$ *(note that $g(D+E\frac{\partial}{\partial E}\ln f(E))$ is a function with non commuting arguments !),*

5.

$$e^{hED}=\sum_{k=0}^{\infty}h^kE^k\binom{D}{k},$$

where

$$\binom{x}{k}:=\frac{x(x-1)\dots(x-k+1)}{k!},\quad \textit{such that}\quad (1+hx)^y=\sum_{k=0}^{\infty}h^kx^k\binom{y}{k}.$$

6. $f(E)e^{\alpha D}=e^{\alpha D}f(e^\alpha E)$ *and* $e^{\alpha D}f(E)=f(\frac{E}{e^\alpha})e^{\alpha D}$.

In particular

7. $(1+hx)^Df(E)=f(\frac{E}{1+hx})(1+hx)^D$.

Proof: These formulas may be proved immediately by straightforward computations. 1. – 5. follow almost directly from the commutation rule $[E,D]=E$ and linearity. To prove 6. we obtain from 1. for $f=\sum a_kx^k$

$$f(E)e^{\alpha D}=\sum_{k=0}^{\infty}a_kE^ke^{\alpha D}=\sum_{k=0}^{\infty}a_ke^{\alpha(D+k)}E^k=\sum_{k=0}^{\infty}a_ke^{\alpha D}(e^{\alpha k}E^k)$$

$$=e^{\alpha D}\sum_{k=0}^{\infty}a_k(e^\alpha E)^k=e^{\alpha D}f(e^\alpha E).\quad\square$$

There is a more rigid result than theorem 2:

THEOREM 5 *A power series* $f(x,y) \in K[x,y][[h]]$ *with* $f(0,y)=1,\ f_x(0,y)=h\,y$ *satisfies (3) iff* $f=\psi$, *i.e.*

$$f(x,y) = (1+hx)^y = \sum_{k=0}^{\infty} h^k x^k \binom{y}{k}.$$

PROOF: Replacing in (3) the commuting variables E_1, D_3 by x resp. y and the remaining non commuting D_2, E_2 by D, E we have to solve the equation

$$f(x+E,y)f(x,D) = f(x,D+y)f(E,y).$$

We will solve this functional equation transforming it to a differential equation for f. Take the first derivative with respect to x

$$f_x(x+E,y)f(x,D) + f(x+E,y)f_x(x,D) = f_x(x,D+y)f(E,y)$$

and set $x=0$. Then $(f(0,y)=1,\ f_x(0,y)=hy)$

$$f_x(E,y) + f(E,y)hD = h(D+y)f(E,y)$$

or

$$f_x(E,y) = h[D, f(E,y)] + hyf(E,y). \tag{4}$$

Lemma 1 yields

$$[D, f(E,y)] = -E\frac{\partial}{\partial E}f(E,y) = -Ef_x(E,y).$$

Substituting this expression in (4) we get an equation in E only.

$$f_x(E,y) = -hEf_x(E,y) + hyf(E,y).$$

Its integral with respect to the initial conditions yields $f(x,y) = (1+hx)^y$ and vice versa. $\square$

5 A bialgebra deformation

An extended version of the condition (2) is contained in the following theorem:

THEOREM 6 *Assume that the bialgebra* (B, μ_B, Δ_B) *acts admissibly on* A *and* $P \in 1 + h(B \otimes B)[[h]]$ *satisfies condition (2). Then for* $A' = A[[h]]$ *and* $B' = B[[h]]$

1. $A_h = (A', \mu_h = \mu_A \circ P)$ *is a* K'*-algebra.*

2. $B_h = (B', \mu_B, \Delta_h)$ *with* $\Delta_h(b) := P^{-1}\Delta_B(b)P$ *is a bialgebra.*

3. *The natural (*K'*-linear) bialgebra action of* B_h *on* A_h *is admissible.*

4. $S_h = P^{-1}\frac{\partial P}{\partial h}$ *is a 2-cocycle of the coalgebra* (B', Δ_h).

Proof: 2. One has only to prove the coassociativity of Δ_h, i.e.

$$\forall\; b \in B' \qquad \Delta_{h,1} \circ \Delta_h(b) = \Delta_{h,2} \circ \Delta_h(b).$$

With the definition of Δ_h the left hand side of this expression expands as

$$P_{12}^{-1} \Delta_{B,1}(P^{-1} \Delta_B(b) P) P_{12}.$$

Applying the multiplicativity of Δ we finally obtain

$$\left(P_{12}^{-1} \Delta_{B,1}(P^{-1})\right) \, (\Delta_{B,1}(\Delta_B(b))) \, (\Delta_{B,1}(P) P_{12})$$

and analogously for the right hand side

$$\left(P_{23}^{-1} \Delta_{B,2}(P^{-1})\right) \, (\Delta_{B,2}(\Delta_B(b))) \, (\Delta_{B,2}(P) P_{23}) \, .$$

But the left and right bracket terms are equal by (2) and its inverse whereas the middle bracket terms are the same by the coassociativity of Δ_B.

3. We have only to prove that condition (1) is fulfilled, i.e.

$$b \circ \mu_h = b \circ \mu_A \circ P = \mu_h \circ \Delta_h(b) = \mu_A \circ \Delta_B(b) \circ P.$$

But this follows immediately from (1) for B.

4. From (2) and the definition of Δ_h we obtain

$$P_{12} \Delta_{h,1}(P) - P_{23} \Delta_{h,2}(P) = 0.$$

Since $\frac{\partial}{\partial h} P = P\, S_h$ the derivative of (2) yields

$$\Delta_1(P S_h) P_{12} + \Delta_1(P) P_{12} S_{h,12} = \Delta_2(P S_h) P_{23} + \Delta_2(P) P_{23} S_{h,23}.$$

Note that further

$$\Delta_1(P S_h) P_{12} = \Delta_1(P) \Delta_1(S_h) P_{12} = \Delta_1(P) P_{12} \Delta_{h,1}(S_h)$$

and also

$$\Delta_2(P S_h) P_{23} = \Delta_2(P) P_{23} \Delta_{h,2}(S_h).$$

Altogether we obtain

$$\Delta_1(P) P_{12} \cdot \delta_h(S) = 0.$$

Hence $\delta_h(S) = 0$ since the first cofactor is invertible. $\square$

This theorem shows that our approach to algebra deformations through admissible bialgebra actions is a very natural one. It does not only allow to formulate a condition on P that implies the associativity of $\mu_h = \mu_A \circ P$ but also yields a deformation of the coalgebra structure on B in such a way that the deformation process may be iterated. Its this point where we leave the original setting of (universal enveloping algebras of) Lie algebras acting by derivations, since the deformed comultiplication rule is usually more difficult.

Let's explain these changes on Gerstenhaber's example. For $P = (1+hE_1)^{D_2}$ we get as new comultiplication

$$\Delta_h(E) = P^{-1}\Delta_B(E)P = (1+hE_1)^{-D_2}(E_1+E_2)(1+hE_1)^{D_2}$$

Applying the rules collected in lemma 1 we get

$$\Delta_h(E) = E_1 + E_2(1+hE_1)^{-D_2+1}(1+hE_1)^{D_2} = E_1 + (1+hE_1)E_2.$$

In the same way we obtain

$$\Delta_h(D) = P^{-1}\Delta(D)P = (1+hE_1)^{-D_2}D_1(1+hE_1)^{D_2} + D_2.$$

Since

$$[(1+hE_1)^{-D_2}, D_1] = E_1\frac{\partial}{\partial E_1}(1+hE_1)^{-D_2} = -hD_2E_1(1+hE_1)^{-D_2-1}$$

we get

$$(1+hE_1)^{-D_2}D_1 = (D_1 - hD_2E_1(1+hE_1)^{-1})(1+hE_1)^{-D_2}$$

and finally

$$\Delta_h(D) = D_1 + D_2 - \frac{hE_1D_2}{(1+hE_1)} = D_1 + (1+hE_1)^{-1}D_2.$$

Note that moreover

$$\Delta_h(1+hE) = (1+hE_1)(1+hE_2),$$

i.e. the B-cocycle E may be lifted to the B_h-cocycle $\ln(1+hE)$. Since

$$S_h = P^{-1}\frac{\partial P}{\partial h} = (1+hE_1)^{-D_2}\cdot E_1\,D_2\cdot(1+hE_1)^{D_2-1} = L_1^{-1}\,E_1\,D_2.$$

we get $\delta_h(D) = h\,S_h$, i.e. the B-cocycle D is not liftable. S is a bialgebra analog of a jump cocycle as defined for algebras in [4].

Remark: Over $K'[h^{-1}]$ the bialgebra B_h may be generated by D and $L = 1+hE$ with the following relations

$$\Delta(D) = D_2 + L_1^{-1}D_2,\ \Delta(L) = L_1L_2,\ [L,D] = L-1.$$

There is a K-bialgebra $\tilde{B} = K\langle D, L, L^{-1}\rangle$ with the same relations. If we extend it trivially to $\tilde{B}' = \tilde{B}[[h]]$ there is a bialgebra homomorphism

$$f_h : \tilde{B}' \longrightarrow B_h \quad \text{via} \quad L \mapsto 1+hE.$$

This K-algebra will be considered in the next section.

6 Another derivation of Gerstenhaber's formula

From the above considerations we can extract the conditions on D, L that are necessary for Gerstenhaber's formula to be fulfilled. This way we get the following generalization:

THEOREM 7 *Let $\tilde{B}$ be a K'-bialgebra and $L, D \in \tilde{B}$ such that L^{-1} exists and the following relations are fulfilled*

$$L - 1 \in h\tilde{B},\ [L, D] = L - 1,\ \Delta(L) = L_1 L_2,\ \Delta(L) = D_1 + L_1^{-1} D_2.$$

Then the power series

$$P = L_1^{-D_2} = \exp(-\ln L_1 \cdot D_2)$$

satisfies eq.(2).

Note that D and L may depend (almost) arbitrarily on h.

PROOF: For our P eq.(2) has the form

$$(L_1 L_2)^{-D_3} \cdot L_1^{-D_2} = L_1^{-D_2 - L_2^{-1} D_3} \cdot L_2^{-D_3}$$

or

$$L_1^{-D_3} \cdot L_2^{-D_3} \cdot L_1^{-D_2} = L_1^{-D_2 - L_2^{-1} D_3} \cdot L_2^{-D_3} \tag{5}$$

Here only L_2 and D_2 don't commute. In order to exchange the two factors $L_2^{-D_3}$ and $L_1^{-D_2}$ in the left hand side we introduce the element $E := L - 1$. Then $[E, D] = E$ and by lemma 1 we have

$$f(E)g(D) = g(D + E\frac{\partial}{\partial E} \ln f(E)) \cdot f(E)$$

for $f, g \in K[x][[h]]$. Since

$$f(E_2) = L_2^{-D_3} = (1 + E_2)^{-D_3} \quad \text{and} \quad E_2 \frac{\partial}{\partial E_2} \ln f(E_2) = -E_2 L_2^{-1} D_3$$

the left hand side of (5) may be written as

$$(L_1)^{-D_3} \cdot L_1^{-(D_2 - E_2 L_2^{-1} D_3)} \cdot L_2^{-D_3}.$$

Comparing this with the right hand side of (5) we see that the exponents of L_1 are equal. □

Substituting $L = 1 + hE$ we get a new proof of Gerstenhaber's formula. The special form of L may be derived in the following way: Assume that only L depends on h. We get

$$S_h = P^{-1} \frac{\partial P}{\partial h} = L_1^{-1} \frac{\partial L_1}{\partial h} D_2$$

and the choice of S_h as the jump cocycle $\frac{1}{h}\delta D$ gives us

$$L_1^{-1} \frac{\partial L_1}{\partial h} D_2 = \frac{1}{h} L_1^{-1}(L_1 - 1) D_2,$$

i.e. $\frac{\partial L}{\partial h} = \frac{1}{h}(L - 1)$. Its solution is $L = 1 + hE$ with $E = L|_{h=0}$.

7 First order deformations

In this section let $K' = K[h]/(h^2)$, $B' = B \otimes_K K'$, and $P = 1 \otimes 1 + hS$ for $S \in B' \otimes_{K'} B'$. Our considerations so far may be transferred to this setting to obtain a theory of *first order deformations*. This linearizes all problems and, e.g., eq. (2) is equivalent to the condition $\delta(S) = 0$. Thus there is a one-to-one correspondence between 2-cocycles of the coalgebra B and solutions P of (2).

The new comultiplication in B_h defined by theorem 6

$$\Delta_h(b) = P^{-1} \cdot \Delta_B(b) \cdot P = (1 - hS)\Delta_B(b)(1 + hS)$$

yields

$$\Delta_h(b) = \Delta_B(b) + h[\Delta_B(b), S]$$

and for the new coboundary operator δ_h of B_h

$$\delta_h S = \delta S - h[\Delta_{B,1}(S), S_{12}] + h[\Delta_{B,2}(S), S_{23}].$$

Hence S may not be a B_h-cocycle. To prolongate the deformation to the next order S has to be changed into $S_h = S + hS'$ such that

$$\delta S' = [\Delta_{B,1}(S), S_{12}] - [\Delta_{B,2}(S), S_{23}].$$

Let's apply this construction to Gerstenhaber's example. The first order deformation of $B = U(\mathcal{G})$ generated by the 2-cocycle $S = E_1 D_2$ yields the $K[h]/(h^2)$-algebra B_h generated by two elements E, D with the following relations

$$[E, D] = E, \quad \Delta_h(E) = E_1 + E_2 + hE_1E_2, \quad \Delta_h(D) = D_1 + D_2 - hE_1D_2$$

and

$$\delta_h(S) = -2hE_1E_2D_3$$

For the 2-cochain $E_1^2 D_2 = E^2 \otimes D \in B \otimes B$ we get

$$\delta(E^2 \otimes D) = \delta(E^2) \otimes D = -2E_1E_2D_3.$$

Thus the B-cocycle S may be lifted to the B_h-cocycle

$$S_h = E_1D_2 - hE_1^2D_2 = (1 - hE_1)E_1D_2.$$

Hence B_h has the $K-$bialgebra structure considered in theorem 7. Thus we derived Gerstenhaber's formula already at the first order deformation step.

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