Qualitative Analysis of Solutions to the Semiclassical Einstein Equation in homogeneous and isotropic Spacetimes

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Referat: In der vorliegenden Arbeit werden Methoden aus der Theorie der dynamischen Systeme verwendet, um das qualitative Verhalten von Lösungen der semiklassischen Einsteingleichung für Friedmann-Lamaître-Robertson-Walker Raumzeiten zu untersuchen. Es werden ausschließlich masselose und konform gekoppelte Quantenfelder betrachtet. Bei der Renormierung des Energie-Impuls-Tensors solcher Quantenfelder treten Ambiguitäten auf, die sich als freie Parameter in der semiklassischen Einsteingleichung manifestieren. Mit Hilfe der Theorie der dynamischen Systeme ist es möglich, Lösungen nach ihren qualitativen Verhalten zu klassifizieren und dadurch Argumente für oder gegen bestimmte Werte der Renormierungskonstanten herauszuarbeiten. Befindet sich das Quantenfeld im konformen Vakuumzustand, erhält man ein zweidimensionales dynamisches System. Für dieses dynamische System werden die strukturell stabilen Fälle und Bifurkationsdiagramme herausgearbeitet, sowie das globale Stabilitätsverhalten der Minkowski und De-Sitter Gleichgewichtspunkte. Mittels dieser Analyse wird das qualitative Verhalten der semiklassischen Lösungen mit dem qualitativen Verhalten der Lösungen des Λ-CDM Modells der Kosmologie verglichen. Es zeigt sich, dass das semiklassische Modell in der Lage ist das qualitative Verhalten von Lösungen des klassischen Λ-CDM Modells wiederzugeben. Weiterhin wird gezeigt, das im Vakuumfall Lösungen existieren, welche sich, im Gegensatz zu Lösungen des klassischen Λ-CDM Modells, im Allgemeinen nicht eindeutig durch ihre Anfangsdaten bestimmen lassen. Um dieses atypische Verhalten aufzulösen müssen die Trajektorien dieser Lösungen in einem dreidimensionalen Phasenraum betrachtet werden. Das entsprechende dreidimensionale dynamische System beschreibt das dynamische Verhalten der Lösungen für beliebige Quantenzustände. Für allgemeine Quantenzustände wird die lokale (Lyapunov-) Stabilität der Gleichgewichtspunkte untersucht und für eine spezielle Wahl der Renormierungskonstanten und des Quantenzustandes neue Lösungen gefunden und mit Lösungen des klassischen Λ-CDM Modells verglichen. Auch hier besteht eine qualitative Äquivalenz.

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Chapter 1

Introduction

Quantum field theory becomes significant on length scales less then the size of atoms $\sim 10^{-19}m$ [71]. In contrast, general relativity holds on scales above the Planck length $l_p := \sqrt{\hbar G/c^3} \sim 10^{-35}m$ [88]. There are many orders of magnitude in between these two length scales where both theories overlap and are valid at once. These are the scales where quantum field theory on curved spacetime (QFTCS) should be applied, i.e. the theory of quantum fields propagating in arbitrary classical Lorentzian spacetimes [14, 17, 55, 68, 79, 80, 140, 141]. One may motivate the use of QFTCS from two perspectives. First, it may be seen as semiclassical approximation to a full-fledged quantum gravity, whenever fluctuations in the stress-energy of the quantum field and curvature of the gravitational field are small. Then the coupling of the quantum fields and spacetime is achieved via the so-called *semiclassical Einstein equations* where the expectation value of the (renormalised) stress-energy tensor acts as source term curving spacetime. Studying QFTCS may shed new light on possible theories of quantum gravity, the most ambitious attempts being string theory and loop quantum gravity. In particular it gives constraints which a theory of quantum gravity should fulfil in its semiclassical limit.

On the other hand QFTCS can be seen as natural generalisation of standard quantum field theory on flat Minkowski spacetime [105]. Certainly the latter is locally a good approximation whenever the curvature is small compared to the relevant energy densities giving measurable results e.g. in particle collision experiments. However, the universe is more adequately described by a *curved* spacetime and new physical phenomena occur when curvature cannot be neglected. Most prominent among them are the Hawking effect [56, 73, 140], cosmological particle creation [43, 99] and the Fulling-Davies-Unruh effect [40, 58, 130, 140].

Due to the extreme energy and curvature involved in phenomena described by QFTCS, directly measurable results are yet not available. However, apart from the mentioned results expected in the vicinity of a collapsing black hole and the early universe, QFTCS is thought to play an important role both in the very early as well as the late universe. In the very early universe quantum fluctuations are believed to be responsible for structure formation which afterwards become blown up by an inflationary expansion phase of the universe [63, 74, 98]. Interestingly, this inflationary phase may be described by the back-reaction of a quantum field on the spacetime via the semiclassical Einstein equation [123]. In the late universe measurements suggest a large amount of “dark energy” being responsible for an acceleration of the universe [104, 114]. It is often suggested that dark energy can be identified with the “vacuum energy” of quantum fields and that this energy in turn can be modelled by a cosmological constant appearing in Einstein’s field equations. Calculations of the “vacuum energy” often don’t take the full theory of quantum fields on curved spacetime into account but rather use Minkowskian quantum field theory. These calculations often misleadingly yield the so-called “cosmological constant problem” [22, 115, 142] stating that the measured and predicted value of the cosmological constant disagree by many orders of magnitude. However, taking spacetime curvature properly into account QFTCS can indeed explain an accelerated phase of the universe as it is observed today [57, 66, 67].

If one attempts to formulate quantum field theory on curved spacetime in the same manner as one does in Minkowski spacetime one immediately faces serious problems stemming from the lack of symmetries in general spacetimes. In particular the Poincaré group is no symmetry group of a general spacetime anymore, making it impossible to define a unique vacuum state as ground state. Indeed many unitary inequivalent vacuum states could be defined making it impossible to choose a unique Hilbert space representation of the quantum field. Therefore a formulation which does not require a preferred quantum state would be much more advantageous. This is indeed possible within the framework of algebraic quantum field theory proposed by Haag and Kastler [64, 65] in Minkowski spacetime, and later generalised to quantum field theory on curved spacetimes [44–46]. The advantage of this approach is that the algebraic properties of the quantum field can be separated from its concrete Hilbert space representation by studying an abstract algebra of (field) observables. In this way a physical (quantum) system can be defined for each spacetime. A quantum field theory then associates to all reasonable spacetimes a physical quantum system in a local manner, i.e. without referring to the “surrounding”. This *locally*

and covariant quantum field theory is realised by a functorial description [29].
To obtain measurable results, (algebraic) states are needed. These correspond to positive, linear functionals on the algebra of observables. A (non-unique) Hilbert space representation can then be regained by the so-called Gel'fand-Naimark-Segal triple. It turns out that not all algebraic states are physically reasonable and one has to restrict the class of all states by the so-called Hadamard condition [89], which constrains the wave front set of the two-point function [113]. The latter can be seen as a curved spacetime generalisation of the spectrum condition of quantum field theory on Minkowski spacetime introduced by Wightman [124].
Using quantum states fulfilling the Hadamard condition one is able to define the renormalised stress energy tensor of the quantum field similar to the normal ordering prescription in Minkowski spacetime [27, 28, 77, 78]. This leads to a whole class of stress energy tensors differing by local geometric curvature terms. Each of them fulfils certain physically reasonable axioms [136, 140] and therefore serves as a candidate for the stress energy tensor of the quantum field. The non-uniqueness of this construction is expressed by a set of parameters called renormalisation constants. Once these constants are fixed, e.g. by experiment or theoretical considerations they will be fixed for any field and any spacetime.

Similar to the classical Einstein equation, where matter determines how spacetime is curved via its stress energy tensor and spacetime in turn determines how matter evolves via the Einstein tensor one postulates a *semiclassical Einstein equation* where the expectation value of the renormalised stress energy tensor in some quantum state is supposed to be responsible for spacetime curvature [140]. Due to the renormalisation freedom the resulting dynamical behaviour can be very different for different values of the renormalisation constants. It is the purpose of this thesis to classify and analyse the dependence of the dynamical evolution of spacetime upon the values of the renormalisation constants. Just like the classical Einstein equation, the semiclassical Einstein equation is too complex for being investigated in its full generality. Therefore one specialises to particular spacetime geometries obtained by demanding certain symmetries. In particular in this thesis we consider spatially isotropic and homogeneous spacetimes called *Friedmann Lamaître Robertson Walker* (FLRW) spacetimes. Despite its possible application to cosmological problems as exemplified above we take a more pragmatic view on the model discussed in this thesis. That is we assume that there are some spatial regions in the history of the universe that can be approximated by a homogeneous and isotropic spacetime if averaged on scales above the ones where QFTCS is applicable at least for some time. We further assume that within these regions one can describe the matter content by the expectation value of the (renormalised) stress energy tensor of particular free, massless and conformally coupled quantum fields (e.g. the electromagnetic field or the massless, conformally coupled scalar field). It is not hard to imagine that such conditions are met somewhere and sometime in the universe. As an example take the cosmic voids observed in the universe. However one may also consider the model investigated in this thesis as a toy model. For more realistic descriptions of the real universe one would have to consider more general spacetimes and interacting massless *and* massive quantum fields which complicates the dynamical equations tremendously. Even for classical matter models the stability and explanatory power of highly idealised geometries is still being investigated, especially by using dynamical system theory [135].

In this thesis we consider FLRW spacetimes whose dynamics is determined by a single function — the scale factor — depending on a distinct time variable. The functional dependence of the scale factor upon the time is determined by the semiclassical Einstein equation [50,107,108] which in our case reduces to a fourth order differential equation. Such equations can be transformed into a dynamical system in a standard way and hence are subject to the broad mathematical field of dynamical system theory [5, 10, 21, 62, 69, 70, 91]. Studying dynamics from the perspective of dynamical systems gives insight into the qualitative behaviour of solutions and can also help finding explicit solutions. The qualitative behaviour of solutions depending on initial conditions and parameters can be represented in phase portraits. This gives a powerful tool to analyse and classify solutions to the semiclassical Einstein equations in FLRW spacetimes. In particular many results concerning the back-reaction of quantum fields on FLRW spacetime can be summarised using this method (e.g. [38, 39, 66, 67, 123]).

In the present thesis the semiclassical Einstein equation is analysed for a massless, conformally coupled quantum field being in the conformal vacuum state and for general quantum states by means of dynamical system theory. In case the quantum field is in the conformal vacuum state one finds a Lyapunov function which gives a tool to analyse the global stability behaviour of the Minkowski equilibrium point and the de Sitter equilibrium points. Further qualitative features of solutions to the semiclassical Friedmann equation are presented by means of bifurcation theory and representative phase portraits for each topologically distinct case are drawn. The role of structural stability and bifurcations in the semiclassical model are discussed and in particular compared to the classical Λ-CDM

model. It turns out that the semiclassical model is capable to reproduce some qualitative features of the classical Λ-CDM model even for quantum fields being in the conformal vacuum state. As far as the author knows the notion of structural stability was not used so far in the literature to analyse the semiclassical Einstein equation. For specific choices of the renormalisation constants the general solution to the semiclassical Einstein equations in FLRW spacetimes is reviewed for the conformal vacuum case. Finally, it is pointed out that the semiclassical Friedmann equation for the conformal vacuum case alone does not suffice to uniquely determine solutions by its initial values. In particular, it is shown that spacetimes that change from a contracting to an expanding period or vice versa are not uniquely determined by initial conditions at the moment of change. To solve this apparent anomalous behaviour one has to study the trajectories associated to solutions of the conformal vacuum case in the phase space of solutions for general quantum states. It is shown that trajectories of the conformal vacuum case form an invariant set of the state-depending vector field associated to the dynamical system for general quantum states. This invariant set can be a closed surface, depending on the values of the renormalisation constants. It is then shown that the non-unique behaviour of solutions is caused by a non-unique projection of this surface onto the plane.
In the non-vacuum case a novel set of solutions to the Friedmann equation is found for a specific choice of the quantum state and the renormalisation constants. These also form an invariant set in the state-depending dynamical system. The explicit form of these solutions is stated and their qualitative behaviour is visualised in phase portraits. These are also qualitatively equivalent to the phase portrait one obtains in the Λ-CDM model for positive cosmological constant. Finally, the local Lyapunov stability of the Minkowski and de Sitter equilibrium points is determined for any state of the quantum field.

The present thesis is structured as follows. In Chapter 2, definitions and theorems concerning dynamical systems will be introduced for later use. Special attention will be put on the notions of *Lyapunov stability* of equilibrium points and *structural stability*. The latter can be used to find bifurcations of dynamical systems depending on parameters which gives us a tool to compare dynamical systems differing in the parameters by means of the qualitative behaviour of their solutions. Chapter 3 can be seen as brief review on the free scalar quantum field in curved spacetimes with the ultimate goal to define the expectation value of the renormalised stress energy tensor of the quantum scalar field in order to set up the semiclassical Einstein equation. For that purpose some geometric preliminaries are needed. Then the much more general framework of local and covariant quantum field theory is introduced, followed by a concrete realisation by defining the algebra of observables of the free quantum scalar field and introducing reasonable quantum states fulfilling the Hadamard condition. In chapter 4 the classical Λ-CDM model of cosmology is stated and well-known results are briefly discussed by a novel phase space representation. Finally in chapter 5 the semiclassical Einstein equation in FLRW spacetimes is discussed as a dynamical system.

This thesis will be closed by summarising the results and making some final remarks. The bibliography can be found at the end.

Chapter 2

Dynamical Systems and Ordinary Differential Equations

2.1 General Framework

In this section we introduce basic definitions and standard results of the theory of dynamical systems. It is not intended to give a complete survey on dynamical systems but rather to fix notation and present the material we will use in our later analysis. For a more comprehensive overview of the theory of general dynamical systems see for example [21]. For an Introduction to the case of ordinary differential equations seen as dynamical systems see [5, 10, 69, 70, 91].
Let us start by recalling what we mean by a dynamical system. A general definition can be found in [21]:

Definition 2.1. The triplet $(X, \mathbb{R}, \pi)$, where X is a metric space and the map $\pi : X \times \mathbb{R} \to X$ is continuous and satisfies:

(i) $\pi(x, 0) = x$, for all $x \in X$

(ii) $\pi(\pi(x, t_1), t_2) = \pi(x, t_1 + t_2)$, for all $x \in X$ and $t_1, t_2 \in \mathbb{R}$

is called *dynamical system*. The map π is called *flow*.

Consider now the autonomous differential equation

$$\dot{z} := \frac{dz}{dt} = f(z), \tag{2.1}$$

for the vector valued function (of time) $z = z(t)$ with $z : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}^n$ and some smooth function $f : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$. For ease of notation we will suppress the time-dependence of z in the further discussion. f is called a *vector field* on the *phase space* $\mathcal{D}$. The vector field f generates its associated *flow* defined by the smooth function $\varphi : \mathcal{D} \times \mathcal{I} \to \mathbb{R}^n$ satisfying the differential equation (2.1) in the sense that

$$\frac{d\varphi}{dt}(z,t)|_{t=t_0} = f(\varphi(z,t_0))$$

is fulfilled for all $z \in \mathcal{D}$ and $t_0 \in \mathcal{I}$. The flow φ satisfies the group properties (i) and (ii) of the above definition and hence the triplet $(\mathcal{D}, \mathcal{I}, \varphi)$ constitutes a dynamical system. Dynamical systems generated by vector fields not explicitly depending on time are called *autonomous dynamical systems*.

Definition 2.2. Let $\mathcal{I}^+ := [0,b)$ and $\mathcal{I}^- := (-a,0]$ with $a,b \in \mathbb{R}$. Define for each $z_0 \in \mathcal{D}$ the projections of the flow of the vector field f on $\mathcal{D}$:

$$\begin{aligned}
\gamma^+(z_0) &:= \{\varphi(z_0,t)|t \in \mathcal{I}^+\} \\
\gamma^-(z_0) &:= \{\varphi(z_0,t)|t \in \mathcal{I}^-\} \\
\gamma(z_0) &:= \gamma^+(z_0) \cup \gamma^-(z_0).
\end{aligned}$$

We call $\gamma^+(z_0)$ *positive semi-trajectory*, $\gamma^-(z_0)$ *negative semi-trajectory* and $\gamma(z_0)$ *trajectory*. The totality of all trajectories is called *phase portrait*.

For autonomous dynamical systems it is convenient to study the phase space alone rather than the flow since for each $z_0 \in \mathcal{D}$ the vector $f(z_0)$ based at z_0 is the same for all $t \in \mathcal{I}$. For linear dynamical systems $f(z) = Az$, where $A \in \mathbb{M}(n)$ is a $n \times n$ matrix, thus the general solution for initial data $\varphi(z_0,0) = z_0$ can formally be written as $\varphi(z_0,t) = e^{At}z_0$. Hence all trajectories of the phase space are globally determined by the operator e^{At} acting on an initial point z_0. Let v^i be the eigenvector belonging to the eigenvalue λ_i of A. Then v^i is also an eigenvector belonging to the eigenvalue e^{λ_i} of e^A. Thus we may choose

$$\left\{\psi^i(t) = v^i e^{\lambda_i t}\right\}$$

as basis of the space of solutions. The subspaces of the space of solutions spanned by the eigenvectors are themselves invariant under the action of the operator e^{At}. We may distinguish between three classes of eigenspaces:

Definition 2.3. Let $v_-^1, ..., v_-^{n_s}$ be the eigenvectors belonging to eigenvalues with negative real parts, $v_+^1, ..., v_+^{n_u}$ the eigenvectors belonging to eigenvalues with positive real parts and $v_0^1, ..., v_0^{n_c}$ the eigenvectors belonging to eigenvalues with zero real part, where $n_s+n_u+n_c = n$. Then define

- the *stable subspace* $E^s := span\{v_-^1, ..., v_-^{n_s}\}$
- the *unstable subspace* $E^u := span\{v_+^1, ..., v_+^{n_u}\}$ and
- the *center subspace* $E^c := span\{v_0^1, ..., v_0^{n_c}\}$.

This definition divides the eigenspaces into exponentially decaying solutions (E^s), exponentially growing solutions (E^u) and neither of both (E^c).
In the nonlinear case a general solution is in most cases not accessible and one has to approach the dynamics on phase space piece by piece. A good starting point is to try to find some special trajectories.

Definition 2.4. A point $\zeta \in \mathcal{D}$ is called *equilibrium point* if $\gamma(\zeta) = \{\zeta\}$.

Any equilibrium point ζ is a time independent solution to equation (2.1) and we have $f(\zeta) = 0$. The existence of such equilibrium points determines the behaviour of trajectories in a sufficiently small neighbourhood thereof by the theorem of Philip Hartman and David Grobman [62]:

Theorem 2.5 (Hartman-Grobman)**.** *Suppose that the autonomous dynamical system (2.1) has an equilibrium point $\zeta \in \mathcal{D} \subseteq \mathbb{R}^n$. Consider the linearised system obtained by a Taylor expansion of f where higher order terms are neglected:*

$$\dot{z} = J_f(\zeta)z,\ z \in \mathcal{D}. \tag{2.2}$$

$J_f(\zeta) := \left(\frac{\partial f_i}{\partial z_j}(\zeta)\right)_{i=1,...,n;\, j=1,...,n}$ is the Jacobian matrix of the vector field f evaluated at ζ with eigenvalues λ_i. The equilibrium is called hyperbolic *if $Re(\lambda_i) \neq 0$ for all i. Otherwise it is called* degenerate*. If ζ is hyperbolic then the flow generated by f is homeomorphic to the flow generated by $J_f(\zeta)z$ in some neighbourhood $U \subset \mathcal{D}$. The homeomorphism can be chosen such that the sense of time is preserved.*

Since the solutions to the linearised system (2.2) are in principle known, the behaviour of solutions in a small neighbourhood of the equilibrium is known too, as long as the Grobman-Hartman theorem can be applied. In two dimensions one distinguishes the following equilibrium points

Definition 2.6. Let ζ be an hyperbolic equilibrium point of the two dimensional dynamical system (2.1) with $n = 2$ and consider the linearisation $\dot{z} = J_f(\zeta)z$. Let $\lambda_1 \neq \lambda_2$ be the eigenvalues of the Jacobian $J_f(\zeta)$ evaluated at the equilibrium point. Then there are the following three cases:

- $\lambda_{1/2} \in \mathbb{R}$ and $sign(\lambda_1) = sign(\lambda_2)$ and the equilibrium is called a *node*
- $\lambda_{1/2} \in \mathbb{R}$ and $sign(\lambda_1) \neq sign(\lambda_2)$ and the equilibrium is called a *saddle point*
- $\lambda_{1/2} \in \mathbb{C}$ and $\lambda_1 = \bar{\lambda}_2$ and $Re(\lambda_{1/2}) \neq 0$ and the equilibrium is called a *focus*

If the equilibrium is degenerate the phase portrait in a small neighbourhood of such an equilibrium can be diverse and depends also on the nonlinear terms of $f(z)$. For example if there are no nonlinear terms such an equilibrium is a so-called *center*, where $\lambda_{1/2} \in \mathbb{C}$ and $\lambda_1 = \bar{\lambda}_2$ and $Re(\lambda_{1/2}) = 0$.
If an equilibrium exists we can define the nonlinear analogues of the stable and unstable subspaces E_u and E_s:

Definition 2.7. Let $\zeta \in \mathcal{D} \subseteq \mathbb{R}^n$ be an equilibrium point of the autonomous dynamical system (2.1) and let $\mathcal{U} \subseteq \mathcal{D}$ be a neighbourhood of ζ. The sets $W^s(\zeta,\mathcal{U})$ and $W^u(\zeta,\mathcal{U})$ defined by:

$$W^s_{loc}(\zeta) := \{z_0 \in \mathcal{U} | \varphi(z_0, t) \in \mathcal{U} \text{ for } t \geq 0 \text{ and } \varphi(z_0, t) \to \zeta \text{ as } t \to +\infty\}$$
$$W^u_{loc}(\zeta) := \{z_0 \in \mathcal{U} | \varphi(z_0, t) \in \mathcal{U} \text{ for } t \leq 0 \text{ and } \varphi(z_0, t) \to \zeta \text{ as } t \to -\infty\}$$

are called *local stable manifold* and *local unstable manifold*, respectively. The *global stable manifold* and *global unstable manifold*, respectively, are then given by

$$W^s(\zeta) := \bigcup_{t \leq 0} \varphi(W^s_{loc}(\zeta), t) \tag{2.3}$$
$$W^u(\zeta) := \bigcup_{t \geq 0} \varphi(W^u_{loc}(\zeta), t).$$

The existence of such trajectories is ensured by the stable manifold theorem [62]

Theorem 2.8. *Let ζ be an hyperbolic equilibrium point of the dynamical system (2.1). Then the local stable and unstable manifolds $W^s_{loc}(\zeta)$ and $W^u_{loc}(\zeta)$ exist and the eigenspaces E_s and E_u of the linearised system $x = J_f(\zeta)z$ are tangent to $W^s_{loc}(\zeta)$ and $W^u_{loc}(\zeta)$ at the equilibrium point ζ.*

Hence we may alternatively define the local stable and unstable manifold as the set of trajectories whose tangent spaces at the point ζ are the eigenspaces E_u and E_s of the linearised system. The analogue of the center subspace can not be defined in terms of the asymptotic behaviour of solutions since the direction of the flow depends on higher order terms of $f(x)$ and may be expanding or contracting. However, using the idea of theorem 2.8 the nonlinear analogue to the center subspace can be defined as follows:

Definition 2.9. Let ζ be an equilibrium of the dynamical system (2.1). Define the *center manifold* $W^c(\zeta)$ as the set of trajectories of the dynamical system (2.1) that is tangent at ζ to the eigenspace E_c of the linearised dynamical system $x = J_f(\zeta)z$.

Definition 2.10. Let $\zeta_1 \neq \zeta_2$ be equilibrium points. A trajectory is called *heteroclinic* if it is a subset of the intersection $W^s(\zeta_1) \cap W^u(\zeta_2)$. A trajectory is called *homoclinic* if it is a subset of $W^s(\zeta_1) \cap W^u(\zeta_1)$

Definition 2.11. A trajectory $\gamma(z_0) = \{\varphi(t, z_0) | t \in \mathbb{R}\}$ is called *periodic* if there exists a $T > 0$ such that $\varphi(t + T, z_0) = \varphi(t, z_0)$. The minimal T with this property is called the *period*.

A homoclinic trajectory is a periodic trajectory with $T = \infty$. For a linear, two dimensional dynamical system with centre equilibrium point, infinitely many concentric periodic trajectories are located around it [133]. For nonlinear systems only isolated periodic trajectories may exist, since the trajectories in a neighbourhood of such a periodic trajectory will be repelled or attracted due to the nonlinear terms. Such isolated periodic orbits are called *limit cycles* [69] and may be stable or unstable depending on the behaviour of orbits in their neighbourhood. In two dimensions the following theorem due to Bendixson makes a statement about the existence of periodic trajectories [69].

Theorem 2.12 (Bendixson)**.** *Consider the autonomous dynamical system (2.1) defined on a simply connected open domain $\mathcal{D} \subseteq \mathbb{R}^2$. If the divergence div$(f)$ does not change sign and div$(f) \neq 0$ almost everywhere in $\mathcal{D}$ then the dynamical system has no periodic trajectories nor homoclinic trajectories lying entirely in $\mathcal{D}$.*

Another important concept to understand the dynamics of a nonlinear dynamical system is the detection and use of symmetries. We will encounter two symmetries defined as follows:

Definition 2.13. Consider the autonomous dynamical system (2.1). An invertible map $S : \mathcal{D} \to \mathcal{D}$ is called a *symmetry* of (2.1) if:

$$\frac{d}{dt}S(z) = f(S(z)). \tag{2.4}$$

An autonomous dynamical system possessing a symmetry is called *S-equivariant.*

Many physically motivated dynamical systems possess a time-reversal symmetry.

Definition 2.14. Consider the autonomous dynamical system (2.1). An invertible map $R : \mathcal{D} \to \mathcal{D}$ is called a *reversing symmetry* of (2.1) if:

$$\frac{d}{dt}R(z) = -f(R(z)). \tag{2.5}$$

An autonomous dynamical system possessing a reversing symmetry is called *reversible.* The set $Fix(R) := \{z_0 \in \mathcal{D} | R(z_0) = z_0\}$ is called *fixed point subspace.*

In terms of the flow $\varphi(z,t)$ we have:

$$R \circ \varphi(z,t) = \varphi(z,-t) \circ R,$$

i.e. if $\varphi(z_0,t)$ is a solution to equation (2.1) for any initial point $z_0 \in D$ then $\varphi(R(z_0),-t)$ is also a solution of equation (2.1). Consequently, for each equilibrium point $\zeta \notin Fix(R)$ in the plane with eigenvalues $\lambda_{1/2}$ there exists an equilibrium point with eigenvalues $-\lambda_{1/2}$. For more results on reversible systems see [92] where also an extensive bibliography can be found.

Before discussing some notions of stability in the next section let us show how one can represent the trajectories of a two dimensional dynamical system in a *phase portrait.* For this purpose we want to plot a representative selection of the trajectories of a dynamical system in phase space. Let us take as example the damped harmonic oscillator which is led by the following differential equation

$$0 = \ddot{x} + \eta\dot{x} + \mu x, \tag{2.6}$$

where η, μ are arbitrary constants. This equation can be transformed into a two dimensional dynamical system of the form (2.1) by setting $y = \dot{x}$ and writing

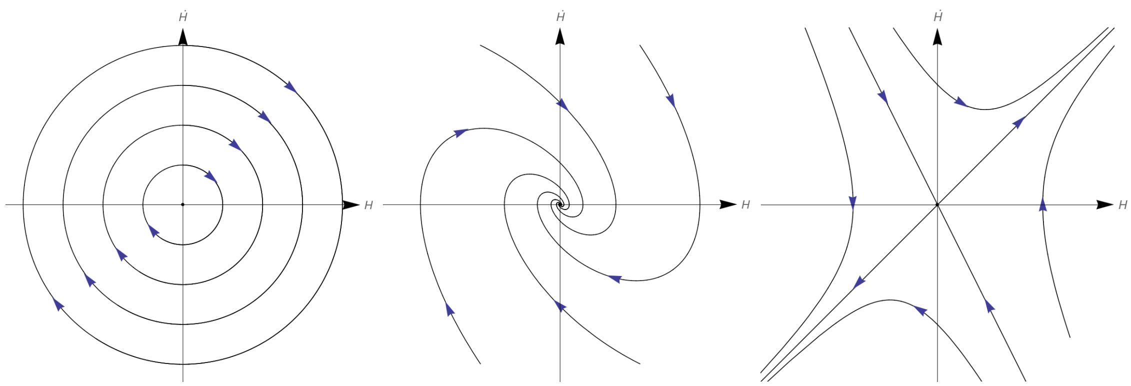
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -\eta y - \mu x \end{pmatrix}. \tag{2.7}$$

Phase portraits of two dimensional systems can easily be drawn using computer software. In this entire thesis we used the computing system *Wolfram Mathematica 10* [143] for all visualisations. We therefore will exemplarily show how to draw the phase portraits of the two dimensional system (2.7). If the damping is turned off, i.e. $\eta = 0$ and $\mu = 1$ the following Mathematica code will produce the phase portrait of this case:

```
ivp[x0_, y0_] := {x'[t] == y[t], y'[t] == -x[t], x[0] == x0,
  y[0] == y0}
{x1, y1} = {x[t], y[t]} /.
  Flatten[NDSolve[ivp[0, 0.5], {x[t], y[t]}, {t, 0, 10}]]
{x2, y2} = {x[t], y[t]} /.
  Flatten[NDSolve[ivp[0, 1], {x[t], y[t]}, {t, 0, 10}]]
{x3, y3} = {x[t], y[t]} /.
  Flatten[NDSolve[ivp[0, 1.5], {x[t], y[t]}, {t, 0, 10}]]
{x4, y4} = {x[t], y[t]} /.
  Flatten[NDSolve[ivp[0, 2], {x[t], y[t]}, {t, 0, 10}]]
Show[ParametricPlot[{{x1, y1}, {x2, y2}, {x3, y3}, {x4, y4}},
  {t, 0, 10}, PlotRange -> {{-2, 2}, {-2, 2}}, AxesStyle ->
  Arrowheads[0.05], AxesLabel -> {Style[H, Italic, Large],
  Style[OverDot[H], Italic, Large]}, PlotStyle -> {Black},
  Ticks -> None], Graphics[{PointSize[Large], Point[{{0, 0}}]}]]
```

Here the first line sets up the initial value problem (abbreviated by *ivp*) as a function of the initial values $x0$ and $y0$. Then using the command *NDSolve* one obtains numerical solutions of the equation $ivp[x0\_, y0\_]$ for specific initial values. The command *ParametricPlot* plots the solution curves for a specific time interval in two dimensional phase space and allows to modify the appearance of the phase portrait such as the size of the phase portrait (*PlotRange*), the arrowheads on the axes (*AxesStyle*), the labels on the axes (*AxesLabel*), the colour of the trajectories (*PlotStyle*) and the tick marks (*Ticks*). The equilibrium point is drawn separately as point using the command *Graphics*. All graphical elements and plots are composed by the command *Show* and the arrows of the trajectories indicating the sense of time are drawn by hand using the *Drawing Tools* in Mathematica. The result is the phase portrait in figure 2.1a.

In the figures 2.1b and 2.1c we have chosen different values of the parameters. This leads to very different dynamical behaviour as indicated by the according phase portraits. Inspecting these phase portraits suggests two different notions of stability. First one wants to know whether solutions having initial data close to equilibrium points remain close to them for all times. This would be the case in figure 2.1a and 2.1b. In the latter case trajectories are even asymptotically approaching the equilibrium point due to the damping. Mathematically this notion of stability is made precise by the definition of *Lyapunov stability*. The second notion of stability is associated with the question how a small change of the values of the involved parameters can change the qualitative behaviour of solutions. In the example of the damped oscillator the case of no damping would be unstable, since a small deviation of the particular value $\eta = 0$ changes the phase portrait dramatically. This is not the case for the phase portraits depicted in figures 2.1b and 2.1c. Mathematically the second kind of question is captured by the notions *structural stability* and *bifurcations*. In the following section we will present the relevant mathematical definitions and theorems concerning these stability concepts outlined here.



(a) $\eta = 0$, $\mu = 1$ (b) $\eta = 1$, $\mu = 1$ (c) $\eta = 1$, $\mu = -2$

Figure 2.1: Phase portraits of the damped harmonic oscillator for various values of the parameters η and μ.

2.2 Stability

2.2.1 Lyapunov Stability

Solutions to an ordinary differential equation depend smoothly on the initial conditions. Therefore it is natural to ask how the behaviour of the solutions depends on the specific value of the initial conditions chosen. Loosely speaking, one wants to know whether a solution given by the flow together with some initial conditions remains close to itself when changing the initial conditions slightly. If so, then this solution is called *Lyapunov stable*.

Physically this is motivated by the fact that measurements always inhere unavoidable errors. A state of some physical system which is not stable would be viewed as impossible to systematically detect or prepare. We want to make this idea mathematically precise:

Definition 2.15. A set $\mathcal{M}$ is called a *(positive) invariant set* if $\gamma^+(z_0) \subset \mathcal{M}$ for all $z_0 \in \mathcal{M}$.

This means that trajectories entering a positive invariant set will remain there for all future times. Although Lyapunov stability is defined for any trajectory we will consider only the case when the trajectory is an equilibrium point:

Definition 2.16. Let ζ be an equilibrium point of the autonomous dynamical system (2.1). Then ζ is called *(Lyapunov) stable* if for any neighbourhood U of ζ there is a neighbourhood $V \subset U$ of ζ which is positively invariant. ζ is called *unstable (in the sense of Lyapunov)* if it is not Lyapunov stable.

Definition 2.17. If trajectories tend towards the equilibrium point ζ we can define the *region of attraction of* ζ by the set:

$$\mathcal{A}(\zeta) := \{z_0 \in \mathcal{D} | \varphi(z_0, t) \to \zeta \text{ as } t \to +\infty\}.$$

If $\mathcal{A}(\zeta)$ is an open neighbourhood of ζ then ζ is called an *attractor*. If an equilibrium point ζ is an attractor *and* stable we call it *asymptotically stable*.

Note that an attractor is not automatically stable (see for instance page 191 of [69] for a counterexample). To analyse the stability behaviour of an equilibrium point we will use the so called *direct method of Lyapunov* (see for example [5, 69, 70, 91]). The advantage of this method over a linearization method is that we also gain insight on the region of attraction and obtain global properties of the dynamical system. The geometric idea underlying this method is similar to that of finding a first integral: One searches for a function having a minimum at the equilibrium point. If the vector field f points always tangent or inside an area bounded by a level set of that function then this area is an invariant set. Trajectories within an invariant set must either be periodic or asymptotically reach a periodic orbit or an equilibrium point. For this reason the equilibrium point is either stable or asymptotically stable. To make this precise the Lyapunov function is introduced:

Definition 2.18. Let ζ be an equilibrium point of the autonomous dynamical system (2.1). A real valued function $V \in C^1(\mathcal{G}, \mathbb{R})$ defined on the open neighbourhood $\mathcal{G} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ with $\zeta \in \mathcal{G}$ is called *Lyapunov function* for ξ if the following holds:

(i) $V(\zeta) < V(z)\,,\;$ for $z \in \mathcal{G}\backslash\{\zeta\}$

(ii) $\dot{V}(z) \equiv \langle \nabla V, f\rangle\,(z) \leq 0\,,\;$ for $z \in \mathcal{G}\backslash\{\zeta\}$,

where $\langle \cdot\,,\cdot \rangle$ is the scalar product on $\mathbb{R}^n$. If V satisfies the strict inequality $\dot{V}(z) < 0\,,\, \forall z \in \mathcal{G}\backslash\{\zeta\}$, then it is called a *strict Lyapunov function* for ζ.

If such a function exists then we have the following theorem [69]:

Theorem 2.19. *Let ζ be an equilibrium point of the autonomous dynamical system (2.1) and let V be a (strict) Lyapunov function for ζ. Then ζ is (asymptotically) stable.*

Asymptotic stability can even be shown for somewhat relaxed conditions on the Lyapunov function, i.e. one does not always need a strict Lyapunov function as is shown in the following theorem [69, 70, 91]:

Theorem 2.20. *Suppose the autonomous dynamical system (2.1) has an equilibrium point ζ. Let $V \in C^1(\mathcal{G}, \mathbb{R})$ be a real valued function defined on the open neighbourhood $\mathcal{G} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ with $\zeta \in \mathcal{G}$. Let $\mathcal{L}(c) := \{z_0 \in \mathbb{R}^n | V(z_0) < c \in \mathbb{R}\}$ be the interior of the level sets of V containing ζ and suppose that V is a (not necessary strict) Lyapunov function in $\mathcal{L}(c)$. Let $S := \{z_0 \in \mathcal{L}(c) | \dot{V}(z_0) = 0\}$ and let $\mathcal{M}$ be the largest invariant set in S. If apart from the equilibrium point ζ no positive semi-trajectory lies entirely on S then ζ is asymptotically stable.*

If V is a strict Lyapunov function, then $\mathcal{M} = \{\zeta\}$ and we have theorem 2.19. Note that theorem 2.20 not only provides asymptotic stability of the equilibrium point but also gives an estimate of the region of attraction, namely the set $\mathcal{L}(c)$ defined as above is a subset of the region of attraction, i.e. $\mathcal{L}(c) \subseteq \mathcal{A}(\zeta)$. If $\mathcal{G}$ and $\mathcal{D}$ can be chosen such that $\mathcal{G} = \mathcal{D} = \mathbb{R}^n$ the equilibrium is called a *global attractor* and its region of attraction is the whole $\mathbb{R}^n$.
To prove instability of an equilibrium point one has to show that there exists at least one trajectory which has initial data arbitrary close to the equilibrium point but escapes from it as $t \to \infty$. This can be done by introducing the following analogue to the Lyapunov function [69, 70]:

Definition 2.21. Let ζ be an equilibrium point of the autonomous dynamical system (2.1). A real valued function $V \in C^1(\mathcal{G}, \mathbb{R})$ defined on the open neighbourhood $\mathcal{G} \subseteq \mathcal{D} \subseteq \mathbb{R}^n$ with $\zeta \in \mathcal{G}$ is called *Chetaev function* for ζ if the following holds:

(i) there is a set $\mathcal{W} \subset \mathcal{G}$ such that $V(z) < V(\zeta)$ for $z \in \mathcal{W}\backslash\{\zeta\}$,

(ii) the boundary $\partial\mathcal{W}$ of $\mathcal{W}$ is given by the surface $V(x) = V(\zeta)$ and $\zeta \in \partial\mathcal{W}$,

(iii) $\dot{V}(z) < 0\,,\ \ $ for $z \in \mathcal{W}$.

Theorem 2.22 (Chetaev). *Let ζ be an equilibrium point of the autonomous dynamical system (2.1) and let V be a Chetaev function for ζ. Then ζ is unstable.*

The above theorems and definitions are very powerful if one wants to determine the stability and global behaviour of trajectories. However they have a serious drawback, namely there is no general prescription how to find a Lyapunov function. Whereas in some physical examples an energy function serves as Lyapunov function, in most other cases one simply has to guess. However, due to the Grobman-Hartman Theorem 2.5 one is able to determine the stability of equilibrium points at least locally as long as they are hyperbolic. We have the following [5]

Theorem 2.23. *Let ζ be a hyperbolic equilibrium point of the dynamical system (2.1) and let λ_i be the eigenvalues of the Jacobian matrix $J_f(\zeta)$ evaluated at ζ. Then the equilibrium point ζ is asymptotically stable if $Re(\lambda_i) < 0$ for all i. It is unstable if at least one eigenvalue has a positive real part.*

Example 2.1. Consider the following differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{2.8}$$

where f and g are two continuously differentiable functions on some domain $\mathcal{D} \in \mathbb{R}$ and g is an odd function. Equation (2.8) is named after the french physicist Alfred-Marie Liénard [93] who first investigated such equations in 1928. It may be seen as a generalisation of a damped oscillator with damping function $f(x)$ and restoring force $g(x)$. Such equations are often encountered in the theory of non-linear oscillations. Many electrical oscillation circuits can be modelled by Liénard's equation including the famous van der Pol equation and Duffings equation [62]. Note, that in the context of the Liénard equation the function f is not the vector field used before. Also g is not the metric tensor as introduced in Chapter 3. We will still use the symbols f and g here, since the Liénard equation will be revisited in Chapter 4-6 where no confusion may arise.

By substituting $y := \dot{x}$ equation (2.8) becomes equivalent to the Liénard system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -f(x)y - g(x) \end{pmatrix} := h(x, y). \tag{2.9}$$

If $f(x)$ is an even function then the Liénard system obeys the symmetry

$$\begin{aligned} S &: \mathbb{R}^2 \to \mathbb{R}^2 \\ S &: (x, y) \mapsto (-x, -y). \end{aligned} \tag{2.10}$$

whereas when $f(x)$ is an odd function the Liénard system obeys the reversing symmetry

$$\begin{aligned} R &: \mathbb{R}^2 \to \mathbb{R}^2 \\ R &: (x, y) \mapsto (-x, y). \end{aligned} \tag{2.11}$$

Therefore in these cases the equilibrium points come in pairs and are given by

$$\mathfrak{EP} := \{(x, y) \in \mathbb{R}^2 | g(x) = 0, y = 0\}. \tag{2.12}$$

For an S-symmetric Liénard system the stability behaviour for negative and positive equilibrium points is identically. For an R-symmetric Liénard system the stability behaviour is opposite, i.e. a stable positive (negative) equilibrium point implies the existence of a unstable negative (positive) equilibrium point. The stability behaviour of the equilibrium points is obtained by using a Lyapunov function or Chetaev function, respectively. We have the following theorem:

Theorem 2.24. *Consider the Liénard system* (2.9) *with equilibrium points* $(\bar{x}, \bar{y}) \in \mathfrak{EP}$ *and let* $U(\bar{x}, \bar{y}) \in \mathbb{R}^2$ *be a neighbourhood containing* $(\bar{x}, \bar{y})$. *Denote by* g' *the derivative of* g *with respect to* x. *Then*

(i) The equilibrium point $(\bar{x}, \bar{y}) \in \mathfrak{EP}$ *is asymptotically stable if* $g'(\bar{x}) > 0$ *and* $f(x) > 0$ *for all* $x \in U(\bar{x}, \bar{y})$.

(ii) The equilibrium point $(\bar{x}, \bar{y}) \in \mathfrak{EP}$ *is a stable centre if* $g'(\bar{x}) > 0$ *and* $f(x) = 0$.

(iii) The equilibrium point $(\bar{x}, \bar{y}) \in \mathfrak{EP}$ *is unstable if* $g'(\bar{x}) < 0$ *or* $f(x) < 0$ *for all* $x \in U(\bar{x}, \bar{y})$.

Proof. The Theorem will be proven by use of Lyapunov or Chetaev functions, respectively. We define the function

$$V(x, y) := \frac{1}{2} y^2 + \int_0^x g(u) du \tag{2.13}$$

Then

$$\dot{V}(x, y) = -f(x) y^2. \tag{2.14}$$

Suppose $f(x) > 0$. Then $\dot{V}(x,y) \leq 0$, $\forall (x,y) \in \mathbb{R}^2$, where identity is only achieved on the nullcline $y = 0$, i.e. when

$$\frac{h(x,0)}{|h(x,0)|} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

However, given initial data $(x_0, 0) \notin \mathfrak{EP}$ the corresponding positive semi-trajectories always leaves the x-axis.

Now, the extremal points of $V(x,y)$ are given by the condition

$$\nabla V(x,y) \equiv \begin{pmatrix} g(x) \\ y \end{pmatrix} = 0$$

and therefore are located at the equilibrium points in $\mathfrak{EP}$. Furthermore, for $(\bar{x}, \bar{y}) \in \mathfrak{EP}$ we have

$$\begin{aligned} \left.\frac{d^2V}{dx^2}\right|_{(x,y)=(\bar{x},\bar{y})} &= \left.\frac{dg}{dx}\right|_{(x)=(\bar{x})} \equiv g'(\bar{x}) \\ \left.\left(\frac{d^2V}{dx^2}\frac{d^2V}{dy^2} - \left(\frac{d^2V}{dxdy}\right)^2\right)\right|_{(x,y)=(\bar{x},\bar{y})} &= \left.\frac{dg}{dx}\right|_{(x)=(\bar{x})} \equiv g'(\bar{x}) \end{aligned}$$

Now, there are two cases:

(i) $\boldsymbol{g'(\bar{x}) > 0}$. Then $V(x,y)$ has a minimum at $(\bar{x}, \bar{y})$. Together with equation (2.14) this implies that $V(x,y)$ is a Lyapunov function for $(\bar{x}, \bar{y})$ and from theorem 2.20 we conclude that $(\bar{x}, \bar{y})$ is an asymptotically stable equilibrium point.

(ii) $\boldsymbol{g'(\bar{x}) < 0}$. Here the function $V(x,y)$ has a saddle point at $(\bar{x}, \bar{y})$. Then there exists a domain $W := \{(x,y) \in \mathbb{R}^2 | V(x,y) < V(\bar{x}, \bar{y})\}$ with boundary $\partial W = \{(x,y) \in \mathbb{R}^2 | V(x,y) = V(\bar{x}, \bar{y})\}$ such that $(\bar{x}, \bar{y}) \in \partial W$. If N lies in W we can choose a smaller open region $\tilde{W}$ such that $N \cap \tilde{W} = \varnothing$ and $\tilde{W}$ has the same properties as W. Hence in W, $\tilde{W}$, respectively we have the strict inequality $\dot{V}(x,y) < 0$. Therefore in this case V is a Chetaev function and $(\bar{x}, \bar{y})$ is unstable.

In the case when $f(x) > 0$ choose $-V(x,y)$ as Lyapunov/Chetaev function. Then the saddle point remains a saddle point and by the same reasoning as above the equilibrium for which $g'(\bar{x}) < 0$ is unstable. The minimum is turned into a maximum and with the argument (ii) above we find that $-V(x,y)$ is again a Chetaev function for equilibrium points fulfilling $g'(\bar{x}) > 0$ and hence these equilibrium points are unstable too. Finally in the case $f(x) = 0$ the trajectories are the contour lines of the Lyapunov function (2.13), i.e. $E = V(x, y)$, where E is a constant. Since y appears as square in $V(x, y)$ and the potential is an even function it follows that the trajectories around a minimum must be closed and therefore such an equilibrium is a stable centre if $g'(\bar{x}) > 0$. □

The above theorem is rather obvious from a mechanical point of view. Equation (2.8) is the equation of a non-linear, damped oscillator located at x with potential energy $\int_0^x g(u)du$. Hence the conditions in theorem 2.24 determine whether the energy function V of the oscillator has a minimum or not. Energy functions are classical examples of Lyapunov functions.

2.2.2 Structural Stability

Modelling a physical situation involves searching for a differential equation governing the evolution of some measurable quantities. In this process one always neglects physical effects which are considered irrelevant. To justify this simplification one has to show that the qualitative prognosis of the model chosen does not depend on small changes of the differential equation itself. This situation is mathematically captured by either the notion of rough systems by Andronov and Pontryagin [9] or the notion of structurally stable systems by Peixoto [100], both of which are equivalent. However for the mathematical definition of "small" changes of a differential equation we will follow the concept underlying structural stability which is slightly different from the concept of rough systems. The basic strategy is as follows: One first takes the whole space of vector fields that define (autonomous) dynamical systems and defines a norm on it. This induces a notion of "small" perturbations (or inaccuracies) of a vector field. Then one subdivides the space of all vector fields into equivalence classes. This makes precise what qualitative features are considered to be essential. A dynamical system is then said to be structurally stable if the vector field generating it is an interior point of its equivalence class. In the following definitions we will restrict ourselves to dynamical systems on the plane. For a general introduction to structural stability see [11, 62, 70, 90, 120].

Definition 2.25. Denote by $\mathfrak{X}^k(\mathcal{D})$ the set of all C^k vector fields defined on a compact subset $\mathcal{D} \subset \mathbb{R}^2$ with smooth boundary $\partial\mathcal{D}$. Let $r = (r_1, r_2) \in \mathbb{N}_0^2$ with $|r| = r_1 + r_2 \leq k$ and let $D^r = D_{z_1}^{r_1} D_{z_2}^{r_2}$ for $z = (z_1, z_2) \in \mathbb{R}^2$. Then define the C^k-norm on $\mathfrak{X}^k(\mathcal{D})$ by:

$$\|f\|_k := \sup_{z \in \mathcal{D}} \left\{ \sum_{|r|=0}^{k} \|D^r f(z)\| \right\}$$

where $\|\cdot\|$ is any norm on $\mathbb{R}^2$ and D^r is the rth derivative. The *distance* between two vector fields $f, g \in (\mathfrak{X}^k(\mathcal{D}), \|\cdot\|_k)$ is then defined by $\|f - g\|_k$. Two vector fields g and f are called $\epsilon - C^k$-close if $\|f - g\|_k < \epsilon$ for $\epsilon > 0$.

Definition 2.26. Let $f, g \in \mathfrak{X}^k(\mathcal{D})$. f and g are called *topologically equivalent* if there exists a homeomorphism $h : \mathcal{D} \to \mathcal{D}$ such that h maps the trajectories of f onto the trajectories of g preserving the sense of direction of time.

As an example, if h is a diffeomorphism the dynamical system then f can be obtained from g by a coordinate transformation.

Definition 2.27. Let $f, g \in (\mathfrak{X}^k(\mathcal{D}), \|\cdot\|_1)$, $k \geq 1$. f is called *structurally stable* in a compact region $\mathcal{D}_0$ if there are regions V_1, $V_2 \subset \mathcal{D}$ with $\mathcal{D}_0 \subset V_1$ such that any sufficiently C^1-close (i.e. $\epsilon - C^1$-close for sufficiently small ϵ) vector field g in V_2 is topologically equivalent to f in V_1.

Usually it is furthermore required that $\langle f, n \rangle \neq 0$ for any $z \in \partial\mathcal{D}_0$ does not change its sign, where n is the normal vector on $\partial\mathcal{D}_0$. Then U and V can be chosen to coincide with $\mathcal{D}_0$. If we don't want to make this further assumption we might get problems on the boundary of $\mathcal{D}_0$. For example there might be a hyperbolic equilibrium point sitting on the boundary $\mathcal{D}_0$ which leaves the region $\mathcal{D}_0$ for any small perturbation of the vector field. In the above definition those situations are encompassed by finding regions V_1 and V_2 which essentially tell us what happens with such a hyperbolic equilibrium point if it leaves the boundary (see [90] for more discussion). Peixoto circumvents the difficulties on the boundary in his theorem by considering only flows on compact, two dimensional manifolds. He then is able to give necessary and sufficient conditions for a dynamical system to be structurally stable. Sometimes it is possible to compactify the phase space via the Poincaré sphere [103] and then use Peixotos theorem.
The following theorem is due to Andronov and Pontryagin where dynamical systems on the plane are considered [90].

Theorem 2.28 (Andronov and Pontryagin)**.** *The dynamical system induced by the vector field $f \in \mathfrak{X}^k(\mathcal{D}_0)$ is structurally stable if and only if:*

(i) It has a finite number of equilibrium points and closed trajectories which are all hyperbolic.

(ii) There are no homo- or heteroclinic trajectories connecting saddle equilibrium points,

Peixoto also showed that structural stability is a generic property of two dimensional vector fields, i.e. the set of structurally stable vector fields is open and dense in $\mathfrak{X}^k(\mathcal{D}_0)$. Therefore any perturbation of structurally unstable vector fields leads to a structurally stable vector field.
Usually dynamical systems that are motivated by physical situations depend on parameters (e.g. mass, coefficient of friction), collectively denoted by $\mu \in \mathbb{R}^m$ appearing in the vector field, i.e. $f \in C^1(\mathcal{D} \times \mathbb{R}^m, \mathbb{R}^n)$. In contrast to the smooth dependence of solutions on the initial conditions considered in the previous section the dependence on the parameters μ need not be smooth. Hence there might be critical parameter values at which the dynamical system changes its equivalence class. Such changes are subject of *bifurcation theory* [62,70]. Clearly a dynamical system generated by a vector field depending on parameters is not structurally stable for those critical parameter values. The opposite however is not true, i.e. if a system is structurally unstable for some parameter value μ_0 it need not change its equivalence class (see the example on page 119 in [62]). To encompass this situation we take the more general definition of bifurcation values, as e.g. in [62]:

Definition 2.29. Consider the autonomous dynamical system (2.1) generated by a vector field $f \in C^1(\mathcal{D} \times \mathbb{R}^m, \mathbb{R}^n)$ depending on a set of parameters $\mu \in \mathbb{R}^m$. $\mu = \mu_0$ is called a *bifurcation value* of μ if f is structurally unstable for the parameter value $\mu = \mu_0$. The set of all bifurcation values is called *bifurcation set*.

The simplest bifurcations are those involving the change of the stability behaviour of equilibrium points, referred to as *local bifurcations*.

Proposition 2.30. *Consider the autonomous dynamical system* (2.1) *generated by a vector field* $f \in C^1(\mathcal{D} \times \mathbb{R}^m, \mathbb{R}^n)$ *depending on a set of parameters* $\mu \in \mathbb{R}^m$ *and suppose there is an equilibrium point* $\zeta(\mu)$ *of the vector field* f *depending on the parameters* μ. *If there are values* $\mu = \mu_0$ *of the parameters such that the equilibrium point* $\zeta(\mu_0)$ *is degenerate in the sense of Theorem 2.5, then the dynamical system bifurcates at the bifurcation value* μ_0.

This follows immediately from the first condition of the Theorem of Andronov and Pontryagin 2.28. For the qualitative study of a dynamical system (2.1) depending on parameters it suffices to study one case for each maximally connected set of parameters for which no bifurcation occurs. This set is called *stratum* and all phase portraits within the stratum of the parameters are topologically equivalent.

Example 2.2. Suppose the Liénard system (2.9) depends on a set of parameters $\mu \in \mathbb{R}^m$ only in g but not in f. Then, a bifurcation of an equilibrium point $(\bar{x}, \bar{y}) \in \mathfrak{EP}$ occurs if

(i) $g'(\bar{x}, \mu) = 0$ or

(ii) $f(\bar{x}) = 0$ and $g'(\bar{x}, \mu) > 0$

Proof. In order to proof this we have to find those values of the parameters μ for which an equilibrium point $(\bar{x}, \bar{y}) \in \mathfrak{EP}$ becomes non-hyperbolic. The Jacobian matrix of the Liénard system is

$$J_h(x, y) = \begin{pmatrix} 0 & 1 \\ -f'(x)y - g'(x, \mu) & -f(x) \end{pmatrix},$$

and has eigenvalues evaluated at $(\bar{x}, \bar{y}) \in \mathfrak{EP}$

$$\lambda(\bar{x}, \mu)_{1/2} = -\frac{f(\bar{x})}{2} \pm \sqrt{\frac{f^2(\bar{x})}{4} - g'_x(\bar{x}, \mu)}. \tag{2.15}$$

For $f(\bar{x}) \neq 0$, the only possibility for the real part of one of the eigenvalues to become zero is

$$g'_x(\bar{x}, \mu) = 0.$$

In this case the equilibrium is non-hyperbolic and by Proposition 2.30 a bifurcation occurs. If $f(\bar{x}) = 0$ the equilibrium $(\bar{x}, \bar{y})$ bifurcates if and only if it is not a saddle point, i.e. if $g'_x(\bar{x}) > 0$ as can be seen from the form of the eigenvalues (2.15). □

Chapter 3

The Free Quantum Scalar Field in Curved Spacetimes

Quantum field theory (QFT) on curved spacetime (CS) is a theory which considers matter as quantised fields propagating in a classical, curved spacetime. One expects that classical gravity holds on scales much larger than the Planck length $l_p := \sqrt{\hbar G/c^3} \sim 10^{-35}m$, hence QFTCS should be a good approximation on scales much above the Planck length but below the size of atoms ($\sim 10^{-19}m$ [71]). A heuristic argument is that on scales less then the Planck length the associated energies would spontaneously form black holes due to the uncertainty principle [88]. Typical examples for applications of QFTCS are quantum phenomena in the vicinity of small black holes and the very early universe.

On the other hand the universe is certainly curved and QFT on Minkowski spacetime can only be an approximation of real phenomena. Hence the study of QFTCS should also provide a legitimisation for the use of QFT on Minkowski spacetime e.g. in particle accelerator experiments. In particular, since predictions of QFT on Minkowski spacetime are in agreement with experiments to highly accurate precision, QFTCS should reproduce these results.

In the present chapter we intend to review the formulation of quantum field theory on a fixed but curved background spacetime. It is desirable to do so for all plausible spacetimes at once, i.e. the construction of a physical theory on curved spacetimes should not depend on the details of the specific spacetime under consideration. Also, since experiments are restricted to more or less small spacetime regions it should also be possible to construct a theory locally, without reference to the "surrounding". This has been accomplished by incorporating the principles of locality and covariance into QFTCS in [29]. It is further convenient to use the so-called algebraic approach to QFT. Here the attention focuses on the algebraic structure of abstract observables. By using the word "abstract" we refer to the practice of merely studying the algebraic structure of the operators representing the observables rather than the action of operators upon state vectors of some Hilbert space. For the time being the latter is avoided since different observers in a curved spacetime are in general not able to agree on a preferable Hilbert space of states. This holds true even in Minkowski spactime, where non-inertial observers cannot agree on a preferred Hilbert space of states. The Unruh effect is often cited as an example to this end. Instead, in the algebraic approach, the notion of algebraic states (to be defined below) will be used, which comprise all states of all Hilber space representations (initially even including unitarily inequivalent representations). A Hilbert space representation can then be regained by the Gelfand-Naimark-Segal (GNS) construction.

3.1 Geometry and Dynamics of Spacetime

Before turning to the theory of quantum fields on curved spacetime some preparation is needed regarding the mathematics of spacetimes. In this thesis we assume that general relativity holds at all relevant scales. For a thorough introduction see e.g. [75, 96, 139]. In this section we will introduce only the relevant aspects of general relativity needed in what follows. Let us begin by making mathematically precise what we mean by a spacetime in the present thesis.

Definition 3.1. A *globally hyperbolic spacetime* is a four dimensional Lorentzian manifold (M, g), where M is an oriented and time-oriented smooth, connected and paracompact manifold obeying the Hausdorff property and g is the metric tensor with Lorentzian signature $(-, +, +, +)$. Furthermore (M, g) admits a Cauchy surface Σ.

The last property ensures global hyperbolicity, which makes such spacetimes diffeomorphic to $\mathbb{R} \times \Sigma$, where Σ may be any 3-dimensional C^∞ manifold and hence they admit a smooth time function compatible with the time-orientation [19, 20]. Global hyperbolicity is physically well motivated since it rules out certain causal pathologies like closed time-like curves and time machines and eases the mathematical formulation of a well-posed initial value problem for linear hyperbolic wave equations. Further properties desirable for spacetimes may be omitted or added (e.g. the dimension may differ from 4, or we may require a spin structure [66, 116, 132]). However for our use this definition of spacetime is sufficient.

The signature of the metric is chosen by convention ($(+,-,-,-)$ would also be possible) and throughout this thesis we use the sign convention $(+,+,+)$ for curvature quantities in the notation of [96], fixing also the sign in the definition of the Riemann tensor and in Einstein's field equation (to be introduced in due course). The Riemann tensor R_{abcd} is obtained from the covariant derivative operator ∇ by

$$R_{ab\ d}^{\ \ c}u^d = \nabla_a\nabla_b u^c - \nabla_b\nabla_a u^c,$$

for all $u^c \in TM$. Here we use abstract index notation [139], i.e. Latin indices are used as placeholders indicating the tensor type. If a specific basis is chosen Greek indices are used to denote the components of a tensor in this basis. For the spacetime (M, g) we will frequently write simply M for notational ease. Let us define some notions for later use.

Definition 3.2. Let (M, g) and (M', g') be manifolds. The map $\psi : M \to M'$ is called *isometric embedding* if ψ is a diffeomorphism such that $\psi(M) \subset M'$ is a sub-manifold and ψ is an isometry, i.e. $\psi^* g' = g|_{\psi(M)}$. Here * denotes the pull-back.

Definition 3.3. A subset $U \in M$ is called *causally convex* if for all $x, y \in U$ any causal curve connecting them lies entirely in U.

In the cosmological context spacetime is assumed to be filled with an ideal fluid representing the distribution of matter at some averaging scale. For such a fluid we can single out a particular family of observers at each point of M being at rest with respect to the fluid.

Definition 3.4. Let (M, g) be a globally hyperbolic spacetime, i.e. there exists a time parameter $t \in \mathbb{R}$ such that M can be foliated by a one-parameter family of acausal hypersurfaces $\Sigma_t = \{t\}$ such that $M \simeq \mathbb{R} \times \Sigma$. A *family of fundamental observers* is given by a vector field u^a fulfilling

$$g_{ab}u^a v^b = 0,$$

for all $v^b \in T\Sigma_t$.

A general cosmological model is then given by the triplet (M, g, u), where u is a family of fundamental observers [135]. The dynamics of the spacetime is determined by Einstein's field equations

$$G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R = 8\pi G T_{ab} - \Lambda g_{ab}, \tag{3.1}$$

where G_{ab} is the Einstein tensor, defined by a combination of the Ricci tensor $R_{ab} := R^c_{\ acb}$ and the curvature scalar $R \equiv R_a^{\ a}$. T_{ab} is the (semi-)classical stress-energy tensor. Here,

the cosmological constant Λ is written on the right hand side of Einstein's field equation to indicate its possible interpretation as a form of energy. Finally, G denotes Newton's constant and the speed of light is set equal to 1.
Since (3.1) is highly complicated to solve it is advisable to search for special solutions. This is done by restricting to certain symmetric solutions. For our purpose we restrict to homogeneous and isotropic spacetimes, called Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes.

Definition 3.5. Let (M, g, u) be a general cosmological model. The globally hyperbolic spacetime (M, g) is called

1. *(spatially) homogeneous* iff for each pair of points $p, q \in \Sigma_t$ there exists an isometry $\chi : \Sigma_t \to \Sigma_t$ such that $\chi(p) = q$.

2. *(spatially) isotropic* at each $p \in \Sigma_t$ iff for any two vectors $v^a, w^a \in T_p\Sigma_t$ and any fundamental observer u^a at p, there exists a isometry $\varphi : \Sigma_t \to \Sigma_t$ such that (i) $\varphi(p) = p$, (ii) $\varphi_*(u^a) = u^a$ and (iii) $\varphi_*(v^a) = w^a$

Hence the isometry χ assures "translation invariance" of the metric in spatial directions and φ leaves the metric invariant under "spatial rotations" about an arbitrary point. It can be shown (see e.g. [139] p.91) that, when imposing homogeneity and isotropy the metric has a certain form:

Theorem 3.6. *Let (M, g) be a homogeneous, isotropic spacetime then the line element is of the form*

$$ds^2 = -dt^2 + a^2(t)d\sigma^2 \tag{3.2}$$

where $d\sigma^2$ is the time-independent line element of the constant curvature spatial hypersurface Σ. Using spherical coordinates (r, θ, φ) on Σ one has

$$d\sigma^2 = \left[\frac{dr^2}{1 - kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right],$$

where $k = \pm 1, 0$.

The three possible values of k determine the spatial structure of Σ, i.e. $k = -1$ for hyperbolic space, $k = 0$ for flat space or $k = +1$ for elliptic space. The relevant geometric quantities for Einsteins field equations in the coordinates (t, r, θ, φ) are

$$\begin{aligned} R_{tt} &= -3\left(\dot{H} + H^2\right) \\ R_{rr} &= \frac{a^2}{1-kr}\left(\dot{H} + 3H^2 + 2\frac{k}{a^2}\right) \\ R_{\theta\theta} &= a^2 r^2 \left(\dot{H} + 3H^2 + 2\frac{k}{a^2}\right) \\ R_{\varphi\varphi} &= a^2 r^2 \sin\theta^2 \left(\dot{H} + 3H^2 + 2\frac{k}{a^2}\right) \\ R &= 6\left(\dot{H} + 2H^2 + \frac{k}{a^2}\right) \end{aligned}$$
,

where $H(t) := \dot{a}(t)/a(t)$ is the *Hubble function*. We will also need the following quantities for flat FRLW spacetimes:

$$\begin{aligned} R_{\mu\nu}R^{\mu\nu} &= 12\left(\dot{H}^2 + 3\dot{H}H^2 + 3H^4\right) \\ R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} &= 12\left(\dot{H}^2 + 2\dot{H}H^2 + 2H^4\right) \\ C_{\mu\nu\rho\sigma} := R_{\mu\nu\rho\sigma} - \left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{1}{3}Rg_{\mu[\rho}g_{\sigma]\nu} &= 0 \\ \Box R \equiv (-g)^{-\frac{1}{2}}\partial_\mu\left[(-g)^{\frac{1}{2}}g^{\mu\nu}\partial_\nu R\right] &= -6\left(\dddot{H} + 7\ddot{H}H + 12\dot{H}H^2 + 4\dot{H}^2\right), \end{aligned}$$

where $C_{\mu\nu\rho\sigma}$ are the components of the Weyl tensor and $\Box$ the d'Alembert operator. Note, that all entries of the Einstein tensor vanish except those appearing in the diagonal. The stress energy tensor has to be of the same form and therefore that of a perfect fluid $T = diag(-\rho, p, p, p)$, where ρ is the energy density and p is the isotropic pressure. As a consequence, Einstein's field equation reduce to two independent differential equations in $a(t)$, namely

$$8\pi G\rho = 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2} - \Lambda \tag{3.3}$$

$$8\pi Gp = -2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} + \Lambda. \tag{3.4}$$

The first equation is called *Friedmann's equation*. Together with appropriate initial conditions these equations determine the dynamics of the homogeneous and isotropic universe.

3.2 Locally Covariant Quantum Field Theory

A construction of a quantum field theory in curved spacetimes must respect the principle of locality and the principle of covariance. The former demands that quantum fields depend only on the local properties of the underlying spacetime. Thus incorporating the idea that quantum observables are independent of the physics outside of some spacetime region $\mathcal{U} \in M$. The latter requires the quantum fields to be local and diffeomorphism covariant, i.e. a quantum field theory should be formulated independent of the underlying spacetime (and thus for all spacetimes at once). These principles where first introduced to QFTCS to reduce the freedom in defining the expectation value of the stress energy tensor by Wald [136–138, 140] and were elaborated further in the context of renormalising interacting quantum field theories and defining Wick polynomials [27, 28, 77, 78]. A rigorous definition of a locally covariant quantum field theory was established in [29] by using the language of categories (see also [52, 54, 66, 116]). We will very briefly define the most important notions of category theory used in this context, see e.g. [95].

Definition 3.7. A *category* **Cat** consists of a class of objects denoted by $obj(\mathbf{Cat})$ and a class of morphisms $hom(\mathbf{Cat})$ between two objects called *source* and *target*. The morphisms of a catagory fulfil the following axioms. Let $\chi : A \to B$, $\psi : B \to C$ and $\varphi : C \to D$ be morphisms for $A, B, C, D \in obj(\mathbf{Cat})$

(i) The composition of morphisms is a morphism, i.e. $hom(\mathbf{Cat}) \ni \psi \circ \chi : A \to C$.

(ii) Compositions of morphisms are associative, i.e $(\varphi \circ \psi) \circ \chi = \varphi \circ (\psi \circ \chi)$.

(iii) For each $A \in obj(\mathbf{Cat})$ there is an identity morphism $id_A : A \to A$ such that $id_B \circ \chi = \chi \circ id_A$.

Definition 3.8. The *opposite category* $\mathbf{Cat^{OP}}$ of a category **Cat** is obtained by interchanging source and target of each element of $hom(\mathbf{Cat})$.

Between categories one can define the following mapping

Definition 3.9. A *covariant functor* $\mathfrak{F} : \mathbf{Cat_1} \to \mathbf{Cat_2}$ between two categories $\mathbf{Cat_1}$ and $\mathbf{Cat_2}$ has the following defining properties:

(i) $\mathfrak{F}$ assigns to each object A and each morphism φ of $\mathbf{Cat_1}$ an object $\mathfrak{F}(A)$ and a morphism $\mathfrak{F}(\varphi)$ in $\mathbf{Cat_2}$, so that if $\varphi : A \to B$, then $\mathfrak{F}(\varphi) : \mathfrak{F}(A) \to \mathfrak{F}(B)$.

(ii) $\mathfrak{F}$ preserves the identity morphism, i.e. $\mathfrak{F}(id_A) = id_{\mathfrak{F}(A)}$, for $A \in \mathbf{Cat_1}$.

(iii) $\mathfrak{F}$ preserves the composition of morphisms, i.e. $\mathfrak{F}(\psi \circ \chi) = \mathfrak{F}(\psi) \circ \mathfrak{F}(\chi)$ for all morphisms $\psi, \chi \in hom(\mathbf{Cat_1})$.

A *contravariant functor* $\mathfrak{H} : \mathbf{Cat_1} \to \mathbf{Cat_2}$ is a covariant functor $\mathfrak{F} : \mathbf{Cat_1^{OP}} \to \mathbf{Cat_2}$.

Note that the composition of functors is again a functor. The composition of two covariant (contravariant) functors is a covariant functor and the combination of a covariant and a contravariant functor is a contravariant functor. Finally we define

Definition 3.10. A *natural transformation* Φ between two functors $\mathfrak{F}$ and $\mathfrak{G}$ from $\mathbf{Cat_1}$ to $\mathbf{Cat_2}$ is a family of morphisms $\{\Phi_a\}_{A \in obj(\mathbf{Cat_1})})$ such that

(i) For each object $A \in obj(\mathbf{Cat_1})$, $\Phi_A : \mathfrak{F}(A) \to \mathfrak{G}(A)$ is in $hom(\mathbf{Cat_2})$.

(ii) For every morphism $hom(\mathbf{Cat_1}) \ni \chi : A \to B$ it holds that $\Phi_B \circ \mathfrak{F}(\chi) = \mathfrak{G}(\chi) \circ \Phi_A$.

In the framework of category theory the principle of locality is incorporated by defining the morphisms to be isomorphic embeddings fulfilling certain additional conditions. Thereby one is able to regard a (quantum) system individually without reference to a larger system ("the surrounding"). The principle of covariance enters the theory by defining the objects of the category to be any possible globally hyperbolic spacetime (in the sense defined in the last section).

Definition 3.11. We define the *category of spacetimes* **Man** to be the category whose objects $obj(\mathbf{Man})$ are globally hyperbolic spacetimes (M, g) as defined in 3.1 and whose morphisms $hom(\mathbf{Man})$ are isometric embeddings $\psi : M \to M'$ preserving the orientation and time-orientation such that $\psi(M)$ is causally convex in M' for $(M, g), (M', g') \in obj(\mathbf{Man})$.

To proceed we intend to associate to each object of **Man** a physical system. This should be done in a local way, i.e. it must be possible to enlarge the system within this framework. In particular, if a system is defined on some globally hyperbolic spacetime which can be embedded into a larger hyperbolic spacetime, then the physical system should also be embedded into the larger system associated with the larger spacetime. For this reason the algebraic approach to QFT as initiated by Haag and collaborators [64, 65] seems to be most suitable. The advantage is that one can separate the algebraic structure of observables from their representation in some Hilbert space, thus concentrating on the local aspects of the theory (the abstract algebra of observables) rather than the global aspects

(Hilbert spaces of states), which become ambiguous in a generally curved spacetime without special symmetry. For any quantum field its algebra of observables can be modelled by a unital C^*-algebra $\mathcal{A}$, i.e. a $\mathbb{C}$-vector space with multiplication operation, involution operation * fulfilling $\|A^*A\| = \|A\|^2$ for all $A \in \mathcal{A}$ and unit element $\mathbb{1}$. Certainly not all observables of $\mathcal{A}$ are physically meaningful and one has to add additional structure to the algebra of observables like canonical (anti-)commutation relations (CCR/CAR) and some dynamical law (equations of motion, e.g. Klein-Gordon equation, Dirac equation or Maxwell equation). However for the moment we will keep things rather general and state the following

Definition 3.12. The *category of topological *-algebras* **TAlg** is defined as the category whose objects in *obj*(**TAlg**) are topological C^*-algebras and whose morphisms in *hom*(**TAlg**) are continuous, unit preserving and injective *-homomorphisms.

Finally, still following [29] we are in the position to define

Definition 3.13. (i) A *locally covariant quantum field theory* is a functor

$$\mathfrak{A} : \mathbf{Man} \to \mathbf{TAlg}.$$

For any $\psi \in hom(\mathbf{Man})$ we write $\alpha_\psi := \mathfrak{A}(\psi) \in hom(\mathbf{TAlg})$.

(ii) A locally covariant quantum field theory is called *causal* if for all pairs of morphisms $hom(\mathbf{Man}) \ni \psi_i : M_i \to M$, where $i = 1, 2$ whose ranges $\psi_1(M_1)$ and $\psi_2(M_2)$ are causally separated in (M, g) it holds that:

$$[\alpha_{\psi_1}(\mathfrak{A}(M_1, g_1)), \alpha_{\psi_2}(\mathfrak{A}(M_2, g_2))] = \{0\},$$

where $[\mathcal{A}, \mathcal{B}] := \{AB - BA | A \in \mathcal{A}\ ,\ B \in \mathcal{B}\}$ for any $\mathcal{A}, \mathcal{B} \in \mathbf{TAlg}$.

(iii) A locally covariant quantum field theory fulfils the *time-slice axiom* if for all morphisms $hom(\mathbf{Man}) \ni \psi : M \to M'$ for which $\psi(M)$ contains a Cauchy surface in (M', g') it holds that:

$$\alpha_\psi(\mathfrak{A}(M, g)) = \mathfrak{A}(M', g').$$

Causality as defined in (*ii*) ensures that causally separated regions of spacetime cannot influence each other. The time slice axiom states that an algebra of observables is completely determined by the algebra of observables of any subregion containing a Cauchy surface.
As mentioned above, the elements of the algebras $\mathcal{A}$ are yet too abstract so as to represent

objects of physical interest. However we know that quantum fields are operator-valued distributions, i.e. functionals from the space of test functions on (M, g) denoted by $C_0^\infty(M)$ to the *-algebra $\mathcal{A}$ (see [124, 140]). These are the elements we want to consider in the algebras of observables. The spaces of test functions may be seen in a categorical way as follows.

Definition 3.14. Consider the category of globally hyperbolic spacetimes **Man** and define the functor

$$\mathfrak{D} : \mathbf{Man} \to \mathbf{Test},$$

where **Test** is the *category of test function spaces* whose objects consist of all spaces $C_0^\infty(M)$ of compactly supported, smooth functions on M, where $(M, g) \in obj(\mathbf{Man})$ and whose morphisms are the push-forwards ψ_* of the isometric embeddings $\psi \in hom(\mathbf{Man})$, i.e. $\psi_*(f) = f \circ \psi^{-1}$ for all $f \in C_0^\infty(M)$.

Observing that both, **TAlg** as well as **Test** are subcategories of the *category of topological spaces* **Top** whose objects are topological spaces and whose morphisms are continuous functions, we are led to the following functorial point of view on locally covariant quantum fields.

Definition 3.15. A *locally covariant quantum field* Φ is a natural transformation between suitable functors $\mathfrak{D}$ and $\mathfrak{A}$.

In order to produce physically measurable numbers, i.e. expectation values one has to define states upon which the observables act on.

Definition 3.16. An *algebraic state* for a given topological C^*-algebra of observables $\mathcal{A}$ is defined as a functional

$$\omega : \mathcal{A} \to \mathbb{C},$$

being

(i) linear: $\omega(c_1 A + c_2 B) = c_1\omega(A) + c_2\omega(B)$,

(ii) positive: $\omega(A^*A) \geq 0$ and

(iii) normalised: $\omega(A^*A) = 1$,

for all $A, B \in \mathcal{A}$ and $c_1, c_2 \in \mathbb{C}$. The space of states is denoted by $\mathcal{A}^*_{\geq,N}$.

Picking out one specific state ω one is able to construct a Hilbert space representation of the algebra of observables via the GNS construction, i.e. there exists a triplet $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ consisting of a Hilbert space $\mathcal{H}_\omega$ a representation π_ω of the algebra $\mathcal{A}$ in this Hilbert space and a unit vector $\Omega_\omega \in \mathcal{H}_\omega$ such that for all $A \in \mathcal{A}$ we have $\omega(A) = \langle \Omega_\omega | \pi_\omega(A) | \Omega_\omega \rangle$. The converse is also true. Note that this construction depends on the specific choice of the state ω, which is used to define the inner product of $\mathcal{H}_\omega$. Taking a different state could lead to a unitarily inequivalent Hilbert space representation. For more details see [64].
From the definition of states we see that these are obtained by taking the subalgebra of positive and normalised elements of the dual to the algebra of observables. As such it is a convex space. In the categorical framework we make the

Definition 3.17. The *category of states* **Sta** is the category whose objects $obj(\mathbf{Sta})$ are the set of states $\mathcal{A}^*_{\geq,N}$ and morphisms $hom(\mathbf{Sta}) \ni \alpha^*_\psi : \mathcal{B}^*_{\geq,N} \to \mathcal{A}^*_{\geq,N}$ being the dual maps of the morphisms $hom(\mathbf{Talg}) \ni \alpha_\psi : \mathcal{A} \to \mathcal{B}$, i.e.

$$(\alpha^*_\psi \omega_{\mathcal{B}})(A) = \omega_{\mathcal{B}}(\alpha_\psi(A)),$$

where $A \in \mathcal{A}$ and $\omega_{\mathcal{B}} \in \mathcal{B}^*_{\geq,N}$.

Taking the dual of objects of a category together with taking the dual of its morphisms constitutes a contravariant functor. In particular $\mathfrak{T} : \mathbf{TAlg} \to \mathbf{Sta}$ is a contravariant functor. Hence the composite functor $\mathfrak{S} := \mathfrak{T}\mathfrak{A} : \mathbf{Man} \to \mathbf{Sta}$ is a contravariant functor. We will call $\mathfrak{S}$ the *state space* for the locally covariant QFT given by $\mathfrak{A}$.

3.3 The Algebra of Observables of the Klein-Gordon Field

Before investigating the quantum Klein-Gordon field, let us state the following well known properties of normally hyperbolic operators and about uniqueness and existence of solutions to the associated Cauchy problem in globally hyperbolic spacetimes (see e.g. [15])

Theorem 3.18. *Let P be a normally hyperbolic operator (of scalar type) on a globally hyperbolic spacetime (M, g), i.e. in local coordinates,*

$$P := -g^{\mu\nu}\partial_\mu\partial_\nu + A^\mu\partial_\mu + B,$$

for $A^\mu, B \in C^\infty(M)$. Then

(i) *Let $f \in C_0^\infty(M)$, Σ a Cauchy surface, $u_0, u_1 \in C_0^\infty(M)$ some smooth initial data and n the future directed normal unit vector of Σ. Then the* Cauchy problem

$$Pu = f \ , \ u|_\Sigma = u_0 \ , \ \nabla_n u|_\Sigma = u_1,$$

has a unique solution $u \in C^\infty$ with support

$$supp(u) \subset J(supp(f) \cup supp(u_0) \cup supp(u_1)),$$

where $J(U)$ is the set of all points in M which can be connected to any point of $U \subset M$ by a future or past directed causal curve.

(ii) *There exist unique* retarded/advanced fundamental solutions $E^\pm : C_0^\infty(M) \to C^\infty(M)$ *of P with support $supp(E^\pm) \subset J^\pm(supp(f))$ such that*

$$PE^\pm(f) = E^\pm(Pf) = f,$$

for all $f \in C_0^\infty(M)$.

(iii) *The continuous map $E := E^- - E^+ : C_0^\infty(M) \to C^\infty(M)$ is called the* causal propagator *and has the following properties: (a) For all solutions u of $Pu = 0$ with initial data $u_0, u_1 \in C_0^\infty(M)$ there exists a test function $f \in C_0^\infty(M)$ such that $u = E(f)$ and (b) for every $f \in C_0^\infty(M)$ satisfying $E(f) = 0$ there exists a test function $h \in C_0^\infty(M)$ such that $f = Ph$.*

The Klein-Gordon operator is defined by

$$P_{KG} := \nabla_\mu \nabla^\mu + m^2 + \xi R$$

where ∇ is the covariant derivative of the metric tensor g, m is the mass, R is the scalar curvature and ξ is a constant describing the coupling of the scalar field to gravity. The latter is said to be *minimal* if $\xi = 0$ and *conformal* if $\xi = 1/6$. P_{KG} is a normally hyperbolic operator and hence has the properties stated in Theorem 3.18.

Observables of a classical theory are functions from phase space to the real numbers. Since solutions are unique by the above theorem, any point of the phase space (serving as initial data) gives rise to a unique solution and hence the phase space can be identified with the space of solutions $\mathscr{S}(M)$. Therefore an observable may also be seen as a function from the space of solutions to the real numbers. The space of solutions of a classical Hamiltonian theory — e.g. $\mathscr{S}_{KG}(M)$ for the Klein Gordon field on M — is naturally equipped with a bi-linear, non-degenerate symplectic form, which is well-defined if one of its entries has space-like compact support

$$\Omega : \mathscr{S}_{KG}(M) \times \mathscr{S}_{KG}(M) \to \mathbb{R}.$$

In particular $\Omega(\phi_0, \cdot)$ is a linear map on the space of solutions for any fixed solution $\phi_0 \in \mathscr{S}_{KG}(M)$ and thus defines (classical) linear observables of the theory. These are the fundamental observables of the theory since $\Omega(\phi_0, \cdot)$ are linear combinations of the solutions and its derivatives and any further observable may be expressed by polynomials of $\Omega(\phi_0, \cdot)$.

For any solution $\phi \in \mathscr{S}_{KG}(M)$ and any test function $f \in C_0^\infty(M)$ the propagator E and the symplectic form fulfil the following identity [140]:

$$\Omega(E(f), \phi) = \int f(x)\phi(x)d\mu_g(x), \tag{3.5}$$

where $d\mu_g(x) = \sqrt{-\det(g_{\mu\nu}(x))}d^4x$.

The space of distributions forms a vector space dual to the space of (compactly supported) smooth test functions [14]. From now on we denote by $\mathcal{E}'(M)$ the space of distributions on M (or $\mathbb{R}^n$ if the M is replaced accordingly) dual to the space of smooth functions $C^\infty(M)$ and by $\mathcal{D}'(M)$ the space of distributions dual to the compactly supported smooth function space $C_0^\infty(M)$.

By Theorem 3.18 and equation (3.5) we may define the bi-distribution

$$\mathscr{E}(f, h) := \int f(x)[Eh](x)d\mu_g(x)$$

Let us call the bi-distribution $\mathscr{E} \in \mathcal{D}'(M \times M)$ for later convenience *commutator distribution*.

To pass to a quantum theory the classical observables are replaced by algebra-valued distributions, i.e. distributions with range in a *-algebra. For the Klein-Gordon field the latter can be defined as follows:

Definition 3.19. Define the *off-shell Borchers Uhlmann *-algebra* $\mathcal{T}(C_0^\infty(M))$ by the tensor algebra over the space of compactly supported test functions $C_0^\infty(M)$ on (M, g), i.e. we define the direct sum

$$\mathcal{T}(C_0^\infty(M)) := \bigoplus_{k=0}^{\infty}(C_0^\infty(M))^{\otimes k} \simeq \bigoplus_{k=0}^{\infty} C_0^\infty(M^{\times k}),$$

where $(C_0^\infty(M))^{\otimes 0} := \mathbb{C}$. Elements of $\mathcal{T}(C_0^\infty(M))$ can be viewed as sequences $\{f^{(n)}\}_{n\in\mathbb{N}}$ with finitely many non-zero entries $f^{(n)}(x_1, ..., x_n) \in C_0^\infty(M^{\times n})$ and $f^{(0)} \in \mathbb{C}$. In addition the *-operation acting on $\{f^{(n)}\} \in \mathcal{T}(C_0^\infty(M))$ is defined by $\{f^{(n)}\}^* = \{(f^{(n)})^*\}$ with entries $(f^{(n)}(x_1,...,x_n))^* := \overline{f^{(n)}}(x_n,...,x_1)$.

As usual for tensor algebras addition and scalar multiplication is defined element-wise. The unit element is given by the sequence whose entries are $f^{(0)} = 1$ and $f^{(n)} = 0$ for all $n > 0$. Multiplication is given by the isomorphism

$$\mathcal{T}(C_0^\infty(M)) \otimes \mathcal{T}(C_0^\infty(M)) = \bigoplus_{k,l=0}^{\infty} (C_0^\infty(M))^{\otimes k} \otimes (C_0^\infty(M))^{\otimes l} \to \bigoplus_{k,l=0}^{\infty} (C_0^\infty(M))^{\otimes k+l},$$

i.e. for any sequences $\{f^{(n)}\}, \{h^{(n)}\} \in \mathcal{T}(C_0^\infty(M))$ the product $\{f^{(n)}\}\{h^{(n)}\} = \{j^{(n)}\} \in \mathcal{T}(C_0^\infty(M))$ has entries

$$j^{(n)}(x_1,...x_n) = \sum_{i+j=n} f^{(i)}(x_1,...,x_i)h^{(j)}(x_{i+1},...,x_n).$$

As was shown in [29] $\mathfrak{A}(M,g) = \mathcal{T}(C_0^\infty(M))$ defines a covariant functor between the category **Man** and the category **TAlg** and thus forms a locally covariant QFT. Furthermore

$$\Phi_{(M,g)}(f) := \{f^{(n)}\}, \tag{3.6}$$

where $\{f^{(n)}\} \in \mathcal{T}(C_0^\infty(M))$ is such that $f^{(1)} = f \in C_0^\infty(M)$ and all others vanish, defines a locally covariant quantum field. In this way $\mathcal{T}(C_0^\infty(M))$ is the algebra whose elements are finite linear combinations of products of locally covariant quantum fields $\Phi_{(M,g)}(f)$ with *-operation $(\Phi_{(M,g)}(f))^* = \Phi_{(M,g)}(\bar{f})$ such that

$$(\Phi_{(M,g)}(f_1)...\Phi_{(M,g)}(f_n))^* = \Phi_{(M,g)}(\bar{f}_n)...\Phi_{(M,g)}(\bar{f}_1),$$

for $f, f_i \in C_0^\infty(M)$.
The algebra $\mathcal{T}(C_0^\infty(M))$ is called *off-shell* since no equations of motion are so far imposed on its elements. Furthermore, to obtain a quantum theory canonical (anti-) commutation relations (CCR/CAR) have to be imposed. Therefore it is convenient to define the *-algebra of observables of the Klein-Gordon field by

Definition 3.20. The *(on-shell) Borchers-Uhlmann algebra* $\mathcal{A}_{BU}$ is defined by the quotient algebra

$$\mathcal{A}_{BU}(M) := \mathcal{T}(C_0^\infty(M)) \Big/ \mathcal{J}(M),$$

where $\mathcal{J}(M)$ is the ideal generated by the set $\mathcal{G}(M) := \mathcal{G}_{KG}(M) \cup \mathcal{G}_{CCR}(M)$, where

$$\mathcal{G}_{KG}(M) := \Big\{\{P_{KG}f^{(n)}\} \in \mathcal{T}(C_0^\infty(M)) \Big| f^{(1)} = f \text{ and } f^{(n)} = 0 \text{ for all } n \neq 1\Big\}.$$

and

$$\mathcal{G}_{CCR}(M) := \Big\{\{f^{(n)}\}\{h^{(n)}\} - \{h^{(n)}\}\{f^{(n)}\} - i\mathscr{E}(f,h)\mathbb{1} \in \mathcal{T}(C_0^\infty(M))\Big| \\ f^{(1)} = f\ ,\ h^{(1)} = h \text{ and } f^{(n)} = h^{(n)} = 0 \text{ for all } n \neq 1\Big\}.$$

The ideal generated by $\mathcal{G}(M)$ is

$$\mathcal{J}(M) = \left\{\sum_{i=1}^{m}\{f_i^{(n)}\}\{k_i^{(n)}\}\{h_i^{(n)}\}\middle|\{f_i^{(n)}\},\{h_i^{(n)}\} \in \mathcal{T}(C_0^\infty(M)) \text{ and } \{k_i^{(n)}\} \in \mathcal{G}(M)\right\}$$

and elements of the Borchers Uhlmann algebra are equivalence classes

$$[\{f^{(n)}\}] := \Big\{\{f^{(n)}\} + \{j^{(n)}\}\Big|\{j^{(n)}\} \in \mathcal{J}(M)\Big\}.$$

In particular we have $\{f^{(n)}\} \sim \{f^{(n)}\} + \{j^{(n)}\}$ and hence in the Borchers Uhlmann algebra elements of the ideal are set to zero. By the definition of the ideal this can be the case if and only if the elements of the generating set $\mathcal{G}(M)$ are zero, i.e. replacing the definition (3.6) of the locally covariant quantum field now by

$$\Phi_{(M,g)}(f) \equiv \Phi(f) := [\{f^{(n)}\}], \tag{3.7}$$

where we omit from now on the index indicating the spacetime dependence for notational ease. Again $\{f^{(n)}\} \in \mathcal{T}(C_0^\infty(M))$ is such that $f^{(1)} = f \in C_0^\infty(M)$ and all others vanish. We thus obtain the CCR for the locally covariant quantum field, i.e.

$$[\Phi(f), \Phi(h)] = i\mathscr{E}(f,h)\mathbb{1}, \tag{3.8}$$

for all $f, h \in C_0^\infty(M)$. Due to the support properties of $\mathscr{E}$ the CCR ensure causality as defined in definition 3.13. Moreover the equations of motion are automatically fulfilled in the distributional sense, i.e.

$$\Phi(P_{KG}f) = 0, \tag{3.9}$$

for all $f \in C_0^\infty(M)$. Due to the existence and uniqueness of solutions of the Klein-Gordon equation ensured by Theorem 3.18 this implies that also the time-slice axiom is fulfilled. The algebraic structure of the Borchers-Uhlmann algebra is obtained from algebraic structure of the off-shell algebra by means of the natural homomorphism $[\cdot] : \mathcal{T}(C_0^\infty(M)) \to \mathcal{A}_{BU}(M)$. In particular setting $\mathfrak{A}(M,g) = \mathcal{A}_{BU}(M)$ we obtain a locally covariant QFT of the Klein-Gordon field (cf. [116]).

To sum things up, we defined the algebra of field observables of the Klein-Gordon field to be the Borchers Uhlmann algebra $\mathcal{A}_{BU}(M)$, consisting of finite linear combinations of finite products of the locally covariant quantum field $\Phi(f)$ being subject to the following conditions:

(**A**) **Linearity**: $\Phi(c_1 f_1 + c_2 f_2) = c_1\Phi(f_1) + c_2\Phi(f_2)$, for all $c_1, c_2 \in \mathbb{C}$,

(**B**) **Hermiticity**: $(\Phi(f))^* = \Phi(\bar{f})$,

(**C**) **CCR**: $[\Phi(f), \Phi(h)] = i\mathscr{E}(f, h)\mathbb{1}$,

(**D**) **Klein-Gordon equation**: $\Phi(P_{KG} f) = 0$.

Using the GNS construction the representation of observables of the Borchers Uhlmann algebra in some Hilbert space yields unbounded operators. To encompass the associated difficulties one may prefer to consider C^*-algebras instead. An example for such an algebra is the Weyl algebra which is obtained by taking the exponentiated version of the field operators formally written as $\exp\{i\Phi(f)\}$ as observables. In this thesis it is sufficient and convenient to consider the Borchers Uhlmann algebra.

3.4 States for the Klein-Gordon Field Observables and Microlocal Analysis

Expectation values are obtained by algebraic states acting on the field observables. The expectation value of a general element of the Borchers-Uhlmann algebra yields a finite sum (over n) of distributions of the form

$$f_1 \otimes ... \otimes f_n \mapsto \omega_n(f_1, ..., f_n) := \omega(\Phi(f_1)...\Phi(f_n)) \in \mathbb{C}$$

called *n-point distributions*, where $f_i \in C_0^\infty(M)$ for $i = 1, ..., n$. Hence the action of a state on a generic element of the Borchers Uhlmann algebra $\mathcal{A}_{BU}(M)$ is completely determined if all n-point distributions are known. Of special interest are states corresponding to ground states in the Fock space representation obtained via the GNS construction. These can be defined as follows [89].

Definition 3.21. A state ω is called *quasi-free* if all odd n-point distributions vanish and all even n-point distributions are entirely determined by the 2-point function via

$$\omega_n(f_1, ...f_n) = \sum_{\sigma_n \in \mathscr{P}} \prod_{i=1}^{n/2} \omega_2\left(f_{\sigma_n(2i-1)}, f_{\sigma_n(2i)}\right),$$

where $\mathscr{P}$ is the set of ordered permutations of the set $\{1, ..., n\}$ with

(i) $\sigma_n(2i-1) < \sigma_n(2i)$ for $1 \leq i \leq \frac{n}{2}$,

(ii) $\sigma_n(2i-1) < \sigma_n(2i+1)$ for $1 \leq i \leq \frac{n}{2}$.

From now on we will limit ourselves to quasi-free states, but actually this is no restriction of generality for the purpose of our investigation.
Due to the properties **A-D** of the quantum scalar field Φ the 2-point distribution has to fulfil similar conditions. Bilinearity is guaranteed by the distributional nature of ω_2. Additionally the antisymmetrised 2-point distribution must equal the commutator distribution, i.e.

$$\omega_2(f,h) - \omega_2(h,f) = i\mathscr{E}(f,h),$$

and the Klein Gordon equation must be fulfilled in both arguments, i.e.

$$\omega_2(P_{KG}f,h) = 0 = \omega_2(f,P_{KG}h).$$

Hence, the determination of a state is equivalent to choosing bi-solutions of the Klein Gordon equation which are positive and have the correct antisymmetric part. It turns out that not all such bi-solutions give rise to physically reasonable quantum states. In particular states not fulfilling the so-called Hadamard condition, which will be introduced shortly, are ignored. Let us be more precise in characterising the two-point distributions. For this purpose we want to study the singular behaviour of distributions which can be studied within the context of microlocal analysis. For a detailed discussion see for example [14, 25, 82].
Let us begin by defining the Fourier transform $\hat{f}(k)$ of a function $f \in L^1(\mathbb{R}^n)$ by

$$\hat{f}(k) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} dx e^{-ik\cdot x} f(x).$$

The Fourier transform of a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is then defined by

$$\hat{u}(f) := u(\hat{f}).$$

Theorem 3.22. *A distribution $u \in \mathcal{D}'(\mathbb{R})^n$ is a smooth function if and only if for all $N \in \mathbb{N}_0$ there exists a constant C_N such that*

$$|\hat{u}(k)| \leq \frac{C_N}{(1+|k|)^N} \tag{3.10}$$

for all k. If (3.10) *holds, then $\hat{u}$ is said to be of* rapid decay*.*

This gives us a tool to determine the singular behaviour of a distribution.

Definition 3.23. Let $u \in \mathcal{D}'(\mathbb{R}^n)$. A point $(x_0, k_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ is called *regular directed* if there exist

(i) a function $f \in C_0^\infty(\mathbb{R}^n)$ with $f(x_0) \neq 0$.

(ii) a conic neighbourhood of k_0, i.e. a subset $V(k_0) \subset \mathbb{R}^n$ such that the ball $B_\epsilon(k_0) := \{k \in \mathbb{R}^n | |k - k_0| < \epsilon\}$ is contained in $V(k_0)$ and for any $k \in V(k_0)$, all αk are contained in $V(k_0)$ for $\alpha > 0$.

(iii) a constant C_N for all $N \in \mathbb{N}_0$ such that

$$|\widehat{fu}(k)| \leq \frac{C_N}{(1 + |k|)^N},$$

for all $k \in V(k_0)$.

The complement in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ of the set of regular directed points is called the *wavefront set* $WF(u)$ of u.

Thus, the wavefront set gives not only information about the singular support, i.e. points of $\mathbb{R}^n$ for which there is no neighbourhood U s.t. $u|_U \in C^\infty(U)$ but also the directions in which the distribution fails to be a smooth function. Also this is a local definition, since the test-function f in the above definition is used to localise the distribution to some arbitrarily small neighbourhood of x_0. Therefore the concept of wavefront sets can naturally be extended to curved spacetime. Since the wavefront set of a distribution transforms covariantly (see [14]) the wavefront set is a closed conic subset of the cotangent bundle T^*M [81].
It can be shown that the wavefront set of a solution of partial differential operators lie within its characteristic set [48, 51, 82]. As a consequence the wavefront set of any bi-solution F of the Klein-Gordon operator P_{KG} reads

$$WF(F) \subset \mathscr{N} \times \mathscr{N}, \tag{3.11}$$

where

$$\mathscr{N} := \{(q, \xi) \in T^*M \Big| g^{\mu\nu}(q)\xi_\mu \xi_\nu = 0\},$$

In Minkowski spacetime $\mathbb{M}$ it is reasonable to postulate the *spectrum condition* saying that the spectrum of the energy-momentum operator lies in the closed forward light cone $\overline{V^+} := \{x \in \mathbb{R}^4 \Big| (x^0)^2 - (x_1)^2 - (x^2)^2 - (x^3)^2 > 0, x^0 \geq 0\}$ [64,124]. This places additional constraints on the two point distribution, namely the Fourier transformed 2-point distribution must fulfil $\hat{\omega}_2(\xi, \xi') = 0$ whenever ξ or ξ' does not lie in the spectrum of the energy-momentum tensor, hence

$$supp(\hat{\omega}_2) \subset \{(\xi, \xi') \in \mathbb{M} \setminus \{0\} \times \mathbb{M} \setminus \{0\} \Big| \xi \in \overline{V^+}\,,\ \xi + \xi' = 0\} \cup \{0\}.$$

Now let us define the sets

$$\mathcal{N}^{\pm} := \{(q, \xi) \in \mathcal{N} \Big| \xi \text{ is future } (+)/ \text{ past}(-)\text{directed}\}.$$

Then the support properties of $\hat{\omega}_2$ imply that the wavefront set of the 2-point distribution must be contained in the set $\mathcal{N}^+ \times \mathcal{N}^-$ because $\hat{\omega}_2$ is of rapid decay in any region not contained in $supp(\hat{\omega}_2)$. Then we say [51]

Definition 3.24. A quasi-free state ω fulfills the *microlocal spectrum condition* μSC if

$$WF(\omega_2) \subset \mathcal{N}^+ \times \mathcal{N}^-, \tag{3.12}$$

The latter statement can be seen as a microlocal version of the spectrum condition. But it contains more information, namely the singularity behaviour of the 2-point distribution. In particular it tells us that the 2-point distribution is smooth for space-like separated points.

As the above discussion shows the μSC holds in Minkowski spacetime. But it is strongly suggested that it is also meaningful in general curved spacetimes. In the context of local and covariant quantum field theory one should always be able to embed a spacetime into a larger spacetime. Hence for any curved spacetime under consideration one can choose a larger spacetime which contains a static region in its past. It was shown by [59, 60] that the singularity structure of the 2-point distribution in the static past is preserved under the Cauchy evolution in any globally curved spacetime. This "rigidity argument" in fact can be used on several occasions to induce properties from QFT on Minkowski spacetime to QFTCS [55]. A similar argument was given by the local-to-global theorem of Radzikowski [112], which shows that if a distribution fulfills the μSC locally, then it does fulfil the μSC also globally. Hence by the same way of reasoning in a local and covariant QFT the spectrum condition should always hold. If this is the case then the wavefront set of a bi-solution (3.11) fulfilling the μSC reads [48, 51, 82]

$$WF(\omega_2) = \{(q, \xi; q', -\xi') \in \mathcal{N}^+ \times \mathcal{N}^- \Big| (q, \xi) \sim (q', \xi')\} \tag{3.13}$$

where the equivalence relation $\sim$ here is defined by saying $(q, \xi) \sim (q', \xi')$ if there is a light-like geodesic connecting q with q' such that ξ and ξ' are parallel transports of each

other.
It was shown by Radzikowski [113] that the 2-point distribution of a state fulfilling the μSC can be written in a specific form known as Hadamard form [89].

Theorem 3.25. *If the wavefront set of the 2-point distribution $\omega_2 \in \mathcal{D}'(M \times M)$ fulfils the μSC, then for any $x_0 \in M$ there exists a geodesically convex neighbourhood $\mathscr{O}$ of x_0 such that ω_2 on $\mathscr{O} \times \mathscr{O}$ is of the form*

$$\omega_2(f,h) = \lim_{\epsilon \to 0} \iint\limits_{\mathscr{O}\times\mathscr{O}} \Omega_{\epsilon,\omega}(x,y) f(x) h(y) d\mu_g(x) d\mu_g(y) \tag{3.14}$$

$$\equiv \lim_{\epsilon \to 0} \iint\limits_{\mathscr{O}\times\mathscr{O}} \left[H_\epsilon(x,y) + W_\omega(x,y)\right] f(x) h(y) d\mu_g(x) d\mu_g(y) \tag{3.15}$$

where H_ϵ is the Hadamard parametrix *given by*

$$H_\epsilon(x,y) := \frac{1}{8\pi^2}\left[\frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y)\ln\left\{\frac{\sigma_\epsilon(x,y)}{\lambda^2}\right\}\right], \tag{3.16}$$

where V is given by the series expansion

$$V(x,y) = \sum_{n=0}^{\infty} v_n(x,y)\sigma^n(x,y),$$

and $U, v_n, W_\omega \in C^\infty(M \times M)$. For a time function t on M and $\sigma \in C^\infty(M \times M)$ being half of the squared geodesic distance:

$$\sigma_\epsilon(x,y) := \sigma(x,y) + 2i\epsilon(t(x) - t(y)) + \epsilon^2.$$

λ is a free parameter defining the length scale.

Note that for light-like separated points x and y and for $x \neq y$ the kernel of the 2-point distribution is singular, which corresponds just to the singularity structure observed for the 2-point distribution in Minkowski spacetime. Also the singularity structure of the Hadamard parametrix is independent of the chosen time function t. The coefficients U, V and W_ω are determined by the requirement that ω_2 is a bi-solution to the Klein-Gordon equation. In general the above series does not converge. However, since one is ultimately interested in the coincident limit $y \to x$ where $\sigma_\epsilon(x,y)$ vanishes it suffices to determine the singularity structure of the 2-point distribution with arbitrary precision ($n \geq 1$ in the series expansion of V).
If we split the 2-point distribution into a symmetric and antisymmetric part

$$\omega_2^\pm(f,h) := \frac{1}{2}(\omega_2(f,h) \pm \omega_2(h,f)),$$

then the Hadamard condition is equivalently a condition for the singularity structure of the symmetric part ω_2^+ of the 2-point distribution since by the commutation relations the antisymmetric part $\omega_2^- = i\mathscr{E}$. We write

$$\omega_2^+(f,h) = \lim_{\epsilon\to 0} \iint\limits_{\mathscr{O}\times\mathscr{O}} \Omega^+_{\epsilon,\omega}(x,y) f(x) h(y) d\mu_g(x) d\mu_g(y).$$

3.5 Expectation values of the Stress Energy Tensor

The stress-energy tensor of a classical scalar field ϕ is given by [139]:

$$T^{class}_{\mu\nu}[\phi] := \nabla_\mu\phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \left[\nabla_\rho\phi \nabla^\rho \phi + m^2\phi^2\right] + \xi \left[G_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right]\phi^2, \quad (3.17)$$

where $G_{\mu\nu} := R_{\mu\nu} - 1/2 g_{\mu\nu} R$ is the Einstein tensor. In order to obtain a quantised version of the stress energy tensor we would like to replace the classical field ϕ in (3.17) by the symbols Φ. For this we have to define new field observables not yet contained in the free field algebra $\mathcal{A}_{BU}(M)$, namely powers of the quantum field. In particular we have to make sense of objects as for example the square "Φ^2". This can consistently be done by a procedure similar to the definition of Wick polynomials in Minkowski spacetime [27,28,77,78]. However since we are primarily interested in the expectation value of the stress-energy tensor rather then the stress-energy operator we follow a different path. Following [97, 140] we may rewrite the classical stress-energy tensor by a point-splitting procedure by defining:

$$T^{class}_{\mu\nu}[\phi](x) := \lim_{y\to x} D_{\mu\nu}(x,y)\phi(x)\phi(y), \quad (3.18)$$

where $D_{\mu\nu}(x,y)$ is the symmetric differential operator given by

$$\begin{aligned} D_{\mu\nu}(x,y) :&= \frac{1}{6} g_{\mu\nu}\left(P_{KG(x)} + P_{KG(y)}\right) + \frac{1}{2}\left(\delta_\nu^{\nu'}(x,y) \nabla_{(x)\mu} \nabla_{(y)\nu'} + \delta_\mu^{\mu'}(x,y) \nabla_{(x)\mu'} \nabla_{(y)\nu'}\right) \\ &- \frac{1}{2} g_{\mu\nu}(x)\left(g^{\gamma\alpha}(x)\delta^\alpha_{\gamma'}(x,y) \nabla_{(x)\gamma} \nabla_{(y)\gamma'} + m^2\right) + \xi\Big[G_{\mu\nu} + \frac{1}{2} g_{\mu\nu}\left(\Box_{(x)} + \Box_{(y)}\right) \\ &- \frac{1}{2}\left(\nabla_{(x)\mu} \nabla_{(x)\nu} - \delta^\nu_{\nu'}(x,y)\delta^\mu_{\mu'}(x,y) \nabla_{(y)\mu'} \nabla_{(y)\nu'}\right)\Big]. \end{aligned} \quad (3.19)$$

The differential operators ∇, $\Box$ and P_{KG} appearing in the definition of $D_{\mu\nu}(x,y)$ act with respect to the variable written in the index. $\delta(x,y)$ is the operator of parallel transport from T_yM to T_xM, i.e. if $v \in T_yM$ then it can be parallel transported to the point x resulting in a vector $u \in T_xM$ with coordinates

$$u^\mu = \delta^\mu_{\mu'}(x,y)v^{\mu'}.$$

The first term in the definition of $D_{\mu\nu}(x,y)$ is added to ensure conservation of the quantised stress-energy tensor but does not change the stress-energy tensor for classical solutions [97]. Define the symmetric symbol

$$F_2(f,h) := \frac{1}{2}\left(\Phi(f)\Phi(h) + \Phi(h)\Phi(f)\right).$$

Since acting with an differential operator upon a distribution is well defined (see Chapter 4 of [14]) the symbols

$$\widetilde{T}_{\mu\nu}(f,h) := [D_{\mu\nu}F_2](f,h), \tag{3.20}$$

as well as F_2 are well defined as elements of the algebra $\mathcal{A}_{BU}(M)$. Taking the expectation value of $\widetilde{T}$ yields

$$\omega(\widetilde{T}(f,h)) = (D_{\mu\nu}\omega_2^+)(f,h). \tag{3.21}$$

If ω fulfils the microlocal spectrum condition $\omega(\widetilde{T})$ is again singular for $x = y$ and light-like separated points. However from theorem 3.25 we immediately observe that for two states ω and $\widetilde{\omega}$ fulfilling the μSC the kernel of $\omega_2 - \widetilde{\omega}_2$ is smooth. Hence the difference $\omega(\widetilde{T}) - \widetilde{\omega}(\widetilde{T})$ is also represented by a smooth kernel for any two states ω and $\widetilde{\omega}$ fulfilling the μSC. In particular defining $H^+_\epsilon(x,y) := 1/2(H_\epsilon(x,y) + H_\epsilon(y,x))$, the limits

$$\omega(:\Phi^2:(x)) := \lim_{y\to x}[\Omega^+_{\epsilon,\omega}(x,y) - H^+_\epsilon(x,y)]. \tag{3.22}$$

and

$$\omega(:T_{\mu\nu}:(x)) := \lim_{y\to x} D_{\mu\nu}(x,y)[\Omega^+_{\epsilon,\omega}(x,y) - H^+_\epsilon(x,y)] \tag{3.23}$$

are finite. Taking these coincident limits after subtracting off the singularity from the 2-point distribution ensures that the result is a smooth function. Furthermore the higher orders of $\sigma(x,y)$ appearing in $\Omega^+_{\epsilon,\omega}(x,y)$ and $H^+_\epsilon(x,y)$ vanish in the coincident limit $y \to x$ and the originally divergent series given by (3.14) becomes convergent.

$\omega(:T_{\mu\nu}:(x))$ defined by equation (3.23) is a good candidate for defining the kernel of the expectation value of the renormalised stress-energy tensor because it has some physical preferable properties which one expects for such a quantity, namely $\omega(:T_{\mu\nu}:(x))$

1. is locally covariant, i.e. given an isometric embedding $\chi : (M, g) \to (M', g')$ there exists a injective $*$-morphism $\alpha_\chi : \mathcal{A}_{BU}(M) \to \mathcal{A}_{BU}(M')$. If two states $\omega : \mathcal{A}_{BU}(M) \to \mathbb{C}$ and $\omega' : \mathcal{A}_{BU}(M') \to \mathbb{C}$ fulfil $\omega = \omega' \circ \alpha_\chi$ then
$$\omega'(: T_{\mu\nu} : (x')) = \chi_* \omega(: T_{\mu\nu} : (x)).$$
2. is covariantly conserved, i.e.
$$\nabla^\mu \omega(: T_{\mu\nu} : (x)) = 0.$$
3. vanishes for the vacuum $\omega = \omega^{vac}$ in Minkowski spacetime.

Together with the point-splitting procedure outlined above formally defining $\omega(: T_{\mu\nu} : (x))$ these properties constitute Wald's requirements for the renormalisation of the expectation value of the stress-energy tensor [136,140]. It was also shown by Wald that $\omega(: T_{\mu\nu} : (x))$ is not the only candidate for defining the expectation value of the renormalised stress-energy tensor. There is the freedom to add a conserved local curvature term of the form

$$t_{\mu\nu} := r_1 I_{\mu\nu} + r_2 J_{\mu\nu} + r_3 G_{\mu\nu} + r_4 g_{\mu\nu}, \tag{3.24}$$

where $G_{\mu\nu}$ is the Einstein tensor and

$$I_{\mu\nu} := \frac{1}{\sqrt{|\det(g)|}} \frac{\delta}{\delta g^{\mu\nu}} \int d\mu_g(x) R^2(x) = \nabla_\mu \nabla_\nu R + R R_{\mu\nu} - g_{\mu\nu}\left(\Box R + \frac{1}{4} R^2\right) \tag{3.25}$$

$$\begin{aligned} J_{\mu\nu} &:= \frac{1}{\sqrt{|\det(g)|}} \frac{\delta}{\delta g^{\mu\nu}} \int d\mu_g(x) \left(R_{\rho\sigma} R^{\rho\sigma}\right)(x) \\ &= \frac{1}{2}(\nabla_\mu \nabla_\nu R - \Box R_{\mu\nu}) + R^{\rho\sigma} R_{\rho\mu\sigma\nu} - \frac{1}{4} g_{\mu\nu}(\Box R + R_{\rho\sigma} R^{\rho\sigma}). \end{aligned} \tag{3.26}$$

The constants r_i, $i = 1, ..., 4$ are undetermined renormalisation constants. These can be determined by comparison with experiments or by imposing additional constraints on the expectation values of the renormalised stress-energy tensor. Since neither of both are at hand, one might obtain many possible solutions to the semiclassical Einstein equations depending on the renormalisation constants. It is the purpose of this thesis to classify these solutions in the context of cosmology.

To this avail we need an explicit form of the expectation value of the renormalised stress-energy tensor. Hence we need to compute the coefficients of the Hadamard parametrix H_ϵ. This is done by formally inserting H_ϵ into the Klein-Gordon equation. The following theorem summarises the relevant results [61, 97]:

Theorem 3.26. *1. Let $H_\epsilon(x,y)$ be the Hadamard parametrix given by equation* (3.16)*. Then the coefficients $U(x,y), V(x,y)$ are obtained by the recursion relations*

$$
\begin{aligned}
0 &= 2\,\nabla_\mu U\sigma^\mu + (\Box_{(x)}\sigma - 4)U \\
0 &= -P_{KG(x)}U + 2\,\nabla_\mu v_0\sigma^\mu + (\Box_{(x)}\sigma - 2)v_0 \\
0 &= -P_{KG(x)}v_n + 2(n+1)\,\nabla_\mu v_{n+1}\,\nabla^\mu\,\sigma + (n+1)(\Box_{(x)}\sigma + 2n)v_{n+1}.
\end{aligned}
$$

Choosing the initial value $U(x) := \lim_{y\to x} U(x,y) = 1$ one obtains:

$$
\begin{aligned}
v_1(x) := \lim_{y\to x} v_1(x,y) &= \frac{m^4}{8} + \frac{m^2}{4}\left(\xi - \frac{1}{6}\right)R + \frac{1}{8}\left(\xi - \frac{1}{6}\right)^2 R^2 \\
&- \frac{1}{24}\left(\xi - \frac{1}{5}\right)\Box R - \frac{1}{720}R_{\mu\nu}R^{\mu\nu} + \frac{1}{720}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}
\end{aligned}
\tag{3.27}
$$

2. For $H_\epsilon(x,y)$ the following identities hold:

$$
\begin{aligned}
\lim_{y\to x} P_{KG(x)}H_\epsilon(x,y) = \lim_{y\to x} P_{KG(y)}H_\epsilon(x,y) &= -\frac{6}{(2\pi)^2}v_1(x) \\
\lim_{y\to x} P_{KG(x)}\,\nabla^\mu_{(y)}\,H_\epsilon(x,y) = \lim_{y\to x} \nabla^\mu_{(x)}P_{KG(y)}H_\epsilon(x,y) &= -\frac{2}{(2\pi)^2}\nabla^\mu_{(x)}v_1(x)
\end{aligned}
$$

3. The trace of the expectation value of $\omega(:T_{\mu\nu}:(x))$ defined by equation (3.23) *fulfils*

$$
g^{\mu\nu}\omega(:T_{\mu\nu}:(x)) = \frac{1}{8\pi^2}\left(\left[3\left(\xi - \frac{1}{6}\right)\Box - m^2\right]\omega(:\Phi^2:(x)) + 2v_1(x)\right) \tag{3.28}
$$

4. Changing the length scale $\lambda \to \lambda' > 0$ in the Hadamard parametrix (3.16) *used to define $\omega(:T_{\mu\nu}:(z))$ by equation* (3.23) *yields:*

$$
\omega(:T_{\mu\nu}:(x)) - \omega(:T'_{\mu\nu}:(x)) = \ln\left\{\frac{\lambda'}{\lambda}\right\} t_{\mu\nu}(x),
$$

where $t_{\mu\nu}$ is the symmetric tensor given by equation (3.24) *and the $\omega(:T'_{\mu\nu}:(x))$ is the expectation value of the renormalised stress-energy tensor obtained by using λ' int the Hadamard parametrix.*

For the massless ($m = 0$) and conformally invariant ($\xi = 1/6$) scalar field the trace of the stress energy tensor becomes state independent and non-vanishing. The latter fact is referred to as *trace anomaly*, since in the classical and Minkowski case the trace vanishes. In FLRW spacetime we explicitly have

$$
g^{\mu\nu}\omega(:T_{\mu\nu}:) = \frac{A_S}{2880\pi^2}\Box R + \frac{1}{2880\pi^2}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}\right) - r_3 R + r_4. \tag{3.29}
$$

where $A_S := 1 - 2880\pi^2(3r_1 + r_2)$.

In the present chapter we have defined a quantum field theory for the scalar field and constructed its expectation values of the renormalised stress-energy tensor. In principle one need not restrict to the scalar field but can consistently define a quantum field theory for other more physical fields like the Maxwell field [46, 53, 106, 117] or the Dirac field [39,45,66,116,132]. However things become slightly more complicated since one has to take care of additional structure of the spacetime and gauge symmetries. The regularisation of the expectation values of the stress-energy tensor can then be done in a similar way as in the present section. The trace of the expectation value of the renormalised stress-energy tensor for the Maxwell field takes a similar form as for the scalar field (3.28). Following [1, 2, 26] in FLRW spacetimes one explicitly has for the free Maxwell field

$$g^{\mu\nu}\omega(:T^M_{\mu\nu}:) = \frac{A_M}{2880\pi^2}\Box R + \frac{62}{2880\pi^2}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}\right) - r_3 R + r_4, \tag{3.30}$$

where $A_M := -18 - 2880\pi^2(3r_1 + r_2)$.

Finally in the case of a massless Dirac field in FLRW spacetimes one calculates [39]

$$g^{\mu\nu}\omega(:T^D_{\mu\nu}:) = \frac{A_D}{2880\pi^2}\Box R + \frac{11}{2880\pi^2}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}\right) - r_3 R + r_4, \tag{3.31}$$

with $A_D := -6 - 2880\pi^2(3r_1 + r_2)$.

Inspecting the terms in equations (3.29), (3.30) and (3.31) one observes that these differ only by different values of the constants. In the spirit of structural stability of dynamical systems different values of parameters might change the qualitative behaviour of solutions crucially. Below will will investigate the influence of these parameters in detail.

Chapter 4

The Lambda-CDM Cosmological Model

In this section we want to study the mathematics of a homogeneous and isotropic spacetime filled with classical matter. We expect that there should exist some regions in the actual universe that can be modeled in such a way at least for some time. In the context of cosmology it is even supposed that the whole observable universe can be modeled by a homogeneous, isotropic spacetime to a good approximation. It is questionable if this is true and it is not clear which properties of the oversimplified model are robust when using more complicated and realistic models.

4.1 Dynamical Behavior of Classical Matter: A Phase Space Representation

In this section we want to study the phase space of the standard Λ-CDM model. For this purpose we consider equations (3.3) and (3.4) for several matter types, i.e.

$$\rho = \sum_A \rho^A$$
$$p = \sum_A p^A$$

where A denotes the various matter types. As is well known, Einstein's field equations imply energy conservation, which for a FLRW spacetime takes the form

$$\dot{\rho}^A = -3H(\rho^A + p^A). \tag{4.1}$$

Vice versa — given appropriate initial conditions — the energy conservation equation (4.1) together with Friedmann's equation (3.3) is dynamically equivalent to the full Einstein equations (3.4) and (3.3). To see this one just has to differentiate equation (3.3) with respect to the time t, then Raychaudhuri's equation (3.4) is obtained by a linear combination of the resulting equation and Friedmann's equation. In order to find a solution one has to specify an equation of state, which depends on the type of matter A considered. In the classical case one usually takes the equation of state:

$$p^A = (\gamma_A - 1)\rho^A, \tag{4.2}$$

where γ_A is a constant, depending on the type of matter. This equation of state comprises the three most important matter types considered in standard cosmology, namely:

1. dust: $\gamma_{dust} = 1$
2. radiation: $\gamma_{rad} = 4/3$
3. dark energy: $\gamma_{de} = 0$.

Fixing initial conditions $\rho^A(t_0) = \rho_0^A$ and $a(t_0) = a_0$ one can solve equation (4.1) by using the equation of state (4.2) to get the evolution of the energy density of matter type A:

$$\rho^A(t) = \rho_0^A \left(\frac{a(t)}{a_0}\right)^{-3\gamma_A}. \tag{4.3}$$

Note that we assume conservation of energy for each matter type individually. This means in particular that the matter types can not be transformed into each other. All matter components together have to fulfill Friedmann's equation (3.3).
To proceed one usually defines the density parameter of matter type A by

$$\Omega^A(t) := \frac{\rho^A(t)}{\rho_c(t)} = \frac{8\pi G\rho^A(t)}{3H^2(t)}.$$

Here ρ_c is the critical energy density, i.e. the total energy density required to make the universe flat. Note that this definition is meaningful only if $H \neq 0$, i.e. if we don't consider the Minkowski universe. In [49] the density parameters of dust, radiation and dark energy (modelled as cosmological constant) are used as dynamical variables and the properties of the corresponding dynamical system are studied. In [49] the density parameters are parametrised by the scale factor a and not by time t. However a need not be (strictly) monotonically increasing, i.e. we can not specify the direction of the flow in the phase

portrait. There might be universes that expand for some finite time (a increases) and afterwards contract (a decreases) and for these we have to switch the direction of a trajectory somewhere in between the trajectory.
To encompass these difficulties and to incorporate the Minkowski solution in our considerations we take a different approach to study the dynamics of the Λ-CDM model. For this purpose we consider Friedmann's equation (3.3) for classical matter, radiation and cosmological constant in a flat universe. Using (4.3) we find

$$H^2 = k_1 a^{-4} + k_2 a^{-3} + k_3, \tag{4.4}$$

where $k_1 := 8\pi G \rho_0^{rad} a_0^4$, $k_2 := 8\pi G \rho_0^{dust} a_0^3$ and $k_3 := \Lambda$. Differentiating equation (4.4) with respect to the time t and eliminating either k_2 or k_1 gives:

$$k_1 a^{-4} = -2\dot{H} - 3H^2 + 3k_3 \tag{4.5}$$

$$k_2 a^{-3} = 2\dot{H} + 4H^2 - 4k_3. \tag{4.6}$$

Differentiating again equation (4.5) or (4.6) gives

$$0 = \ddot{H} + 7\dot{H}H + 6H\left(H^2 - k_3\right), \tag{4.7}$$

which is a Liénard type equation for $H = H(t)$. Equation (4.7) is equivalent to the original Friedmann equation by imposing initial data $(a_0, H_0, \dot{H}_0)$ fulfilling (4.4) and one of the equations (4.5) or (4.6). Note that for a non-flat universe (4.7) would contain a term proportional to Ha^{-2} which could be eliminated by differentiating (4.7) once again leading to a third order differential equation.
A first integral of equation (4.7) can be obtained from (4.5) and (4.6) and reads:

$$\varphi(H, \dot{H}) = k_1^3 \left(\dot{H} + 2H^2 - 2k_3\right)^4 + \frac{1}{2} k_2^4 \left(\dot{H} + \frac{3}{2}H^2 - \frac{3}{2}k_3\right)^3 = 0. \tag{4.8}$$

For $k_3 = 0$ $\varphi(H, \dot{H})$ correspond to the first integrals stated in [31,32,85]. The case $k_3 \neq 0$ is included in [110]. These authors also give the general solutions in that case. However, for the moment we will not consider these since they are quite complicated. Note also that we have the solutions from the beginning parametrised by a. To understand their qualitative behavior we will instead investigate the phase portrait given by the curves (4.8), i.e. we consider the Liénard system defined by the vector field

$$h(H, \dot{H}) = \begin{pmatrix} \dot{H} \\ -\dot{H} f(H) - g(H) \end{pmatrix}, \tag{4.9}$$

where

$$f(H) = 7H \tag{4.10}$$

$$g(H) = 6H(H^2 - k_3). \tag{4.11}$$

First note that the Liénard system obeys a reversing symmetry

$$R : (H, \dot{H}) \mapsto (-H, \dot{H}). \tag{4.12}$$

Taking into account that, with $H \equiv \dot{a}(t)/a(t)$, the reversing symmetry R is solely a reversion of time. In particular, the stress energy tensor is invariant under time reversal (and so is the 00-component of the Einstein tensor $G_{00} = -3H^2$). The equilibrium points of $h(H, \dot{H})$ are

$$\mathfrak{EP} = \left\{(0,0)\,,\left(\pm\sqrt{k_3}, 0\right)\right\}, \tag{4.13}$$

which correspond to the Minkowski solution and the de Sitter solutions.

The first integrals given by equation (4.8) can also be seen as contour lines of a Lyapunov function. Then we have the following Lyapunov function:

$$V(H, \dot{H}) = \frac{\left(H + 2H^2 - 2k_3\right)^4}{\left(\dot{H} + \frac{3}{2}H^2 - \frac{3}{2}k_3\right)^3}. \tag{4.14}$$

If $k_3 < 0$ then $V(H, \dot{H})$ has a local minimum at $(0, 0)$. Since the contour lines of the Lyapunov function are first integrals of the Liénard system we obviously have $\dot{V} \equiv \langle \nabla V, h \rangle = 0$. Hence $(0, 0)$ is stable if $k_3 < 0$. If $k_3 > 0$ then $(0, 0)$ becomes a saddle point. At the de Sitter equilibrium $(-\sqrt{k_3}, 0)$ the Lyapunov function has a local maximum for the negative sign and therefore $(-\sqrt{k_3}, 0)$ is an unstable equilibrium. Similarly at the positive de Sitter point $(+\sqrt{k_3}, 0)$ the Lyapunov function has a local minimum and therefore implies a stable de Sitter equilibrium.

The curve $\dot{H} = -\frac{3}{2}H^2 + \frac{3}{2}k_3$ forms an asymptote of the Lyapunov function and therefore divides the phase space into two regions which are not connected by any trajectory defined as follows

$$A := \left\{(H, \dot{H}) \in \mathbb{R}^2 | \dot{H} > -\frac{3}{2}H^2 + \frac{3}{2}k_3\right\}$$
$$B := \left\{(H, \dot{H}) \in \mathbb{R}^2 | \dot{H} \leq -\frac{3}{2}H^2 + \frac{3}{2}k_3\right\}.$$

We find that $V(H, \dot{H}) > 0$ for all $(H, \dot{H}) \in A$ and $V(H, \dot{H}) < 0$ for all $(H, \dot{H}) \in B$. The contour lines of $V(H, \dot{H})$ have to fulfil $V(H, \dot{H}) = -k_2/(2k_1^3)$. Hence a positive value of the Lyapunov function would require either k_1 or k_2 to be negative, which by the definition of these constants implies a negative energy density of the radiation or matter component. Hence, classically the region A is not considered to be of physical interest. This can already be seen from equation (4.5). Analogously, by equation (4.6) the complement of the region

$$C := \left\{(H, \dot{H}) \in \mathbb{R}^2 | \dot{H} \geq -2H^2 + 2k_3\right\}$$

is classically not realistic since it requires a negative energy density k_2. The only physically relevant region in phase space is therefore $B \cap C$.

The local behaviour near the equilibrium points is determined by the eigenvalues of the Jacobian matrix of h which read:

$$\lambda_{1/2}(H) = \frac{1}{2}\left(-f(H) \pm \sqrt{f(H)^2 - 4g'(H)}\right), \tag{4.15}$$

where prime denotes the derivative with respect to H. Depending on the value of k_3 we have three cases:

(i) $\mathbf{k_3 < 0}$: Here the de Sitter equilibrium points do not exist and the Minkowski equilibrium has purely negative eigenvalues. Hence it is a centre or focus. A focus, however is forbidden due to the reversing symmetry R. The Minkowski equilibrium lies in the interior of the region A. Since the Lyapunov function $V(H, \dot{H})$ is monotonically increasing within A and has a minimum at the centre $(0, 0)$ we conclude that except for the Minkowski equilibrium the region A consists of periodic trajectories encircling the equilibrium point. The phase portrait of this case is visualised in figure 4.1a. The solid trajectory corresponds to a universe filled with dust and dark energy and separates the phase space into region A, which is above the solid trajectory and the region B below this trajectory. The dashed trajectory corresponds to a universe filled with radiation and dark energy. The region in between these two trajectories is the physically accessible region $B \cap C$. Note that all such universes start with an expanding period until they reach a critical time after which they contract forever. Both periods are symmetric due to the reversing symmetry. This behaviour is due to the negative value of the cosmological constant k_3. After sufficiently long time the inverse scale factor appearing in equation (4.4) will become very small and the cosmological constant will dominate the evolution. A negative cosmological constant acts as an attracting force. Such a scenario is called *big crunch*.

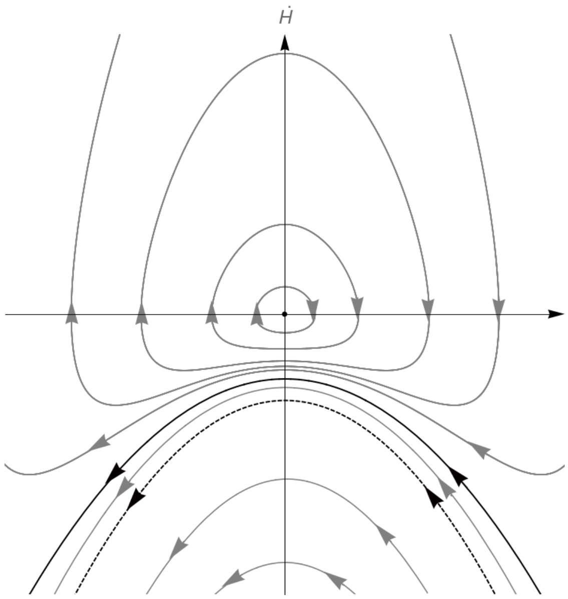
(ii) $\mathbf{k_3 = 0}$: In this case the Minkowski equilibrium lies exactly on the boundary of A, i.e. $\dot{H} = -\frac{3}{2}H^2$ runs through $(0, 0)$. As a consequence the periodic trajectories turn into homoclinic trajectories starting at $(0,0)$ in the asymptotic past and asymptotically reaching $(0,0)$ again in the future. Since there are infinitely many homoclinic orbits, this case must be structurally unstable by the theorem of Andronov and Pontryagin. The phase portrait of this situation is shown in figure 4.1b. The universes filled with dust and dark energy, radiation and dark energy, respectively are highlighted in the same way as for $k_3 < 0$. Again the physically accessible region lies in between these two trajectories. As a consequence any expanding universe will asymptotically reach the Minkowski universe, i.e. it will approach a vacuum universe.

(iii) $\mathbf{k_3 > 0}$: Inspecting the eigenvalues of the Jacobian of h we find that for $k_3 > 0$ the Minkowski equilibrium is an unstable saddle point whereas the negative de Sitter equilibrium is an unstable node and the positive de Sitter equilibrium is an asymptotically stable node. The stable and unstable manifolds of the Minkowski saddle point are those integral curves $\varphi(H, \dot{H}) = 0$ which contain $(H, \dot{H}) = (0,0)$ for all $k_3 > 0$. Hence we must find those values of k_1 and k_2 which fulfil $0 = \varphi(0,0)$ and thus obtain the stable and unstable manifolds of the Minkowski equilibrium:

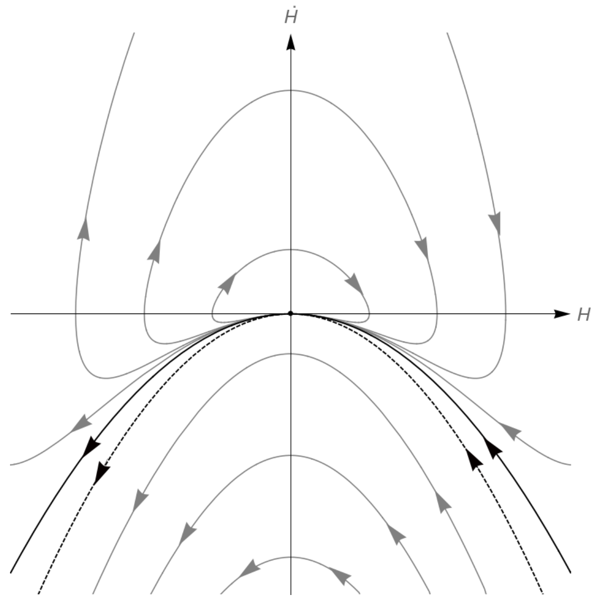
$$\left(\dot{H} + 2H^2 - 2k_3\right)^4 + \frac{128}{27}k_3\left(\dot{H} + \frac{3}{2}H^2 - \frac{3}{2}k_3\right)^3 = 0. \tag{4.16}$$

These correspond to the red coloured trajectories of figure 4.1c. There is a slightly non-smooth behaviour of the red trajectories near the de Sitter equilibrium points caused by numerical errors of the computer software used. The stable manifold forms the boundary of the region of attraction of the asymptotically stable de Sitter equilibrium $(\sqrt{k_3}, 0)$, i.e. all trajectories with initial data above the stable manifold will asymptotically reach the positive de Sitter solution. The physically accessible regions are again those below the solid trajectory but above the dashed trajectory. As a consequence the big crunch scenario is excluded for any physical trajectory in this scenario. Also the Minkowski universe is not accessible anymore. This is due to the repelling force coming from a positive cosmological constant which, for $H > 0$, leads to an exponentially expanding universe represented by the positive attracting de Sitter equilibrium. The nine-year data of the WMAP [18] favours a positive cosmological constant. The deceleration parameter defined by $q := -\left(1 + \frac{\dot{H}}{H^2}\right)$ can be obtained by Raychaudhuri's equation (3.4). In terms of the density parameters this is equivalent to $q = \sum_A \left(\frac{3}{2}\gamma_A - 1\right)\Omega^A$. Then the WMAP data implies that today's value $q(t_0) \approx -0.578$. Hence $\dot{H} < 0$ and we are in the lower half plane of the phase portrait 4.1c. Also we a very close to the solid trajectory today, corresponding to a universe filled with dust and dark energy but our trajectory still lies below that line due to a small amount of radiation. Since the amount of dark energy is measured to be much larger than the amount of dust we are also close to the positive de Sitter equilibrium of the phase portrait 4.1c. All in all the universe today is very close to the red trajectory approaching the positive de Sitter equilibrium from the right hand side.

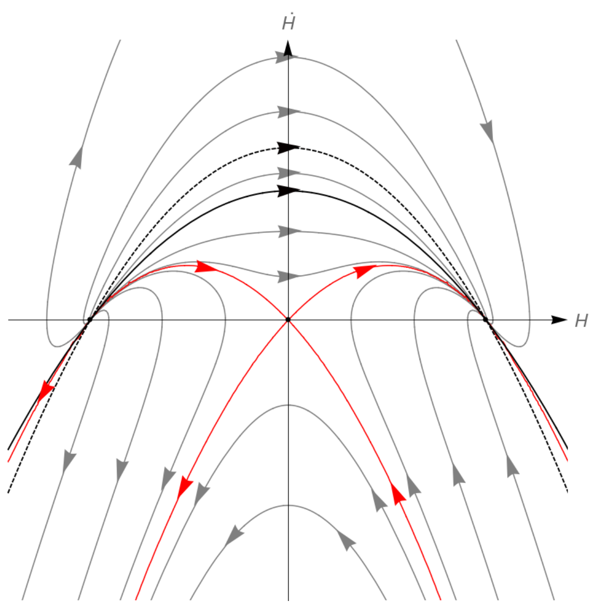
Note that in all three cases those physical trajectories that have a period of expansion asymptotically reach $H \to \infty$ as $t \to -\infty$. Therefore any such universe has a singularity in its past, as the scale factor approaches zero. This is the content of the singularity theorems of Hawking and Penrose [75] and is referred to as *big bang*. However, such behaviour is usually not considered physical, and one should not take this as a prediction that the universe started with an initial singularity. One should rather conclude that the classical model of the universe fails in the asymptotic past and a better theory has to be found for this regime. Finally, the above considerations indicate that there is a local bifurcation at the bifurcation value $k_3 = 0$. This is indeed the case as can be shown by inspecting the eigenvalues (4.15) and using Proposition 2.30. Also note that for $k_3 < 0$ the region A consists of infinitely many periodic trajectories. According to the theorem of Andronov and Pontryagin 2.28 the Liénard system is thus structurally unstable within this region. Hence if one would reject structurally unstable vector fields as good models for reality, then this already implies a positive cosmological constant on theoretical grounds which is in accord with the measurements of the WMAP. The bifurcation diagram is shown in figures 4.1 and the phase portraits are plotted using the computer software Wolfram Mathematica and a code similar to the one used at the end of section 2.1.



(a) Phase portrait of the dynamical system (4.9) for $k_3 = -1$.



(b) Phase portrait of the dynamical system (4.9) for $k_3 = 0$.



(c) Phase portrait of the dynamical system (4.9) for $k_3 = 2$.

Figure 4.1: Representative phase portraits for various values of the cosmological constant k_3. The coordinates of the H-axis range from -2 to 2 and for the $\dot{H}$-axis from -6.5 to 6.5.

4.2 The generalised Friedmann equation

In the previous section we have seen that the Friedmann equation (4.4) can be written as a second order, nonlinear differential equation of Liénard type (4.7). Here we want to generalise this equation by allowing more values of the constants, i.e. we consider the equation:

$$0 = \ddot{H} + p\dot{H}H + qH^3 - rH. \tag{4.17}$$

The name "generalised Friedmann equation" applies only in the context of this theses and is not the name of this equation as known in the literature. In fact Chandrasekar and Co-workers [31, 32, 85, 110] intensively studied equation (4.17) under the name "modified Emden type equation". Here we will reconsider this equation in the context of cosmology. In particular we will use the fact that the Hubble function $H = \dot{a}/a$ is a function of the scale factor a. The general solution of equation (4.17) can be obtained by transforming the Liénard equation into an Abel equation [72, 85]. This is done by making the ansatz

$$\dot{H} := \frac{qH^2 - r}{pw(H)}. \tag{4.18}$$

Using the chain rule for $w(H) = w(H(t))$ and plugging the ansatz into equation (4.17) one obtains the following Abel differential equation determining the function $w(H)$:

$$\frac{dw}{dH} = \frac{p^2 H}{qH^2 - r} w(w^2 + w + \alpha\frac{q}{p^2}). \tag{4.19}$$

For later use we introduce the parameter α which can be any real number. For the case of the generalised Friedmann equation considered here we have $\alpha = 2$. Equation (4.19) is separable and hence can be integrated, i.e.:

$$I(w) := \int \frac{dw}{w(w^2 + w + \alpha\frac{q}{p^2})} = \int \frac{p^2 H}{qH^2 - r} dH = \frac{p^2}{2q}\left(\ln\{qH^2 - r\} + \ln c\right), \tag{4.20}$$

where ln denotes the natural logarithm and $c := (qH_0^2 - r)^{-1}$ is a constant of integration and $H_0 := H(0)$. The integration of the left hand side yields three distinct dynamical regimes, depending on the value of $\alpha q/p^2$, namely

$$2\frac{\alpha q}{p^2} I(w) = \ln\left\{\frac{w^2}{w^2 + w + \alpha\frac{q}{p^2}}\right\} - \begin{cases} \frac{1}{\rho}\ln\left\{\frac{w+\frac{1}{2}(1-\rho)}{w+\frac{1}{2}(1+\rho)}\right\} & \text{if } 4\alpha q < p^2 \\ -\frac{1}{w+\frac{1}{2}} & \text{if } 4\alpha q = p^2 \\ \frac{2}{\delta}\arctan\left\{\frac{2}{\delta}\left(w + \frac{1}{2}\right)\right\} & \text{if } 4\alpha q > p^2 \end{cases}, \tag{4.21}$$

where $\rho := \sqrt{1-4\alpha\frac{q}{p^2}}$ and $\delta := \sqrt{4\alpha\frac{q}{p^2}-1}$. Together with (4.20) equation (4.21) gives a first integral of the generalised Friedmann equation (4.17) for $\alpha = 2$, as follows:

$$\boldsymbol{4\alpha q < p^2:}\ \tilde{c}_1^{-\frac{\alpha}{2}\rho} = \left(qH^2 - r\right)^{\left(\frac{\alpha}{2}-1\right)\rho} \frac{\left(\frac{p}{2}(1-\rho)\dot{H} + qH^2 - r\right)^{\frac{1+\rho}{2}}}{\left(\frac{p}{2}(1+\rho)\dot{H} + qH^2 - r\right)^{\frac{1-\rho}{2}}} \tag{4.22}$$

$$\boldsymbol{4\alpha q = p^2:}\ \tilde{c}_1^{-\frac{\alpha}{2}} = \left(qH^2 - r\right)^{\left(\frac{\alpha}{2}-1\right)} \left(\frac{p}{2}\dot{H} + qH^2 - r\right) \exp\left\{-\frac{\frac{p}{2}\dot{H}}{\frac{p}{2}\dot{H} + qH^2 - r}\right\} \tag{4.23}$$

$$\boldsymbol{4\alpha q > p^2:}\ \tilde{c}_1^{-\frac{\alpha}{2}\delta} = \left(qH^2 - r\right)^{\left(\frac{\alpha}{2}-1\right)\delta} \left(\frac{p}{2}(1+\delta)\dot{H} + qH^2 - r\right)^{\frac{\delta}{2}} \left(\frac{p}{2}(1-\delta)\dot{H} + qH^2 - r\right)^{\frac{\delta}{2}} \times$$
$$\times \exp\left\{\arctan\left\{\frac{\frac{p}{2}\dot{H} + qH^2 - r}{\frac{p}{2}\delta\dot{H}}\right\}\right\}. \tag{4.24}$$

The solutions of the generalised Friedmann equation can be found by using $dt = (aH)^{-1}da$ in equation (4.19), which then becomes

$$p^{-1}a\frac{dw}{da} = w^2 + w + \alpha\frac{q}{p^2}. \tag{4.25}$$

Integrating this equation and combining it with the Abel equation (4.19) yields

$$c^{\frac{\alpha}{2}}\left(qH^2 - r\right)^{\frac{\alpha}{2}} = \tilde{a}^{-\frac{p}{2}} w \left(w^2 + w + \alpha\frac{q}{p^2}\right)^{-\frac{1}{2}}, \tag{4.26}$$

where $\tilde{a} := a/a_0$ for initial values $a(0) =: a_0$. To obtain the solution H parametrised by the scale factor a one has to insert $w = w(a)$ into equation (4.26). From equation (4.25) we find that

$$w(\tilde{a}) = \frac{1}{2}(f_\alpha(\tilde{a}) - 1), \tag{4.27}$$

where

$$\boldsymbol{4\alpha q < p^2:}\ f_\alpha(\tilde{a}) = \rho\frac{\tilde{c}_2 + \tilde{a}^{p\rho}}{\tilde{c}_2 - \tilde{a}^{p\rho}}, \tag{4.28}$$

$$\boldsymbol{4\alpha q = p^2:}\ f_\alpha(\tilde{a}) = \left(\ln\{\tilde{a}^{-\frac{p}{2}}\} - \frac{\tilde{c}_2}{2}\right)^{-1}, \tag{4.29}$$

$$\boldsymbol{4\alpha q > p^2:}\ f_\alpha(\tilde{a}) = \delta\tan\left\{\delta\ln\{\tilde{a}^{\frac{p}{2}}\} - \tilde{c}_2\right\}, \tag{4.30}$$

and $\tilde{c}_2$ is a constant of integration depending on the initial value $w(0) = w_0$ and which is different for each case. Finally, inserting these explicit expressions of $w(\tilde{a})$ we obtain the solutions

$$\boldsymbol{4\alpha q < p^2:}\ \left(qH^2 - r\right)^{\frac{\alpha}{2}} = c_1\tilde{a}^{-\frac{p}{2}(1-\rho)} + c_2\tilde{a}^{-\frac{p}{2}(1+\rho)}, \tag{4.31}$$

$$\boldsymbol{4\alpha q = p^2:}\ \left(qH^2 - r\right)^{\frac{\alpha}{2}} = \tilde{a}^{-\frac{p}{2}}\left(c_1 \ln\{\tilde{a}^{-\frac{p}{2}}\} + c_2\right), \tag{4.32}$$

$$\boldsymbol{4\alpha q > p^2:}\ \left(qH^2 - r\right)^{\frac{\alpha}{2}} = c_1\tilde{a}^{-\frac{p}{2}}\left(\cos\left\{\delta \ln\{\tilde{a}^{\frac{p}{2}}\}\right\} - c_2 \sin\left\{\delta \ln\{\tilde{a}^{\frac{p}{2}}\}\right\}\right), \tag{4.33}$$

where c_1 and c_2 depend on the initial data $H_0, \dot{H}_0$ and differ for each case. Equations (4.31)-(4.33) can again be seen as Friedmann equations of the form $H^2 = \rho_1 + \rho_2 + r/q$ (in the case of $\alpha = 2$) with two forms of matter described by energy densities ρ_1 and ρ_2 and cosmological constant $\rho_0 = r/q$. If we again assume that the three matter components can not be converted into each other we obtain the equations of state by the conservation equation (4.1). For each case we have

$$8q < p^2: \quad p_0 = -\rho_0\,,\ p_{1/2} = -\left(\frac{p}{6}(\rho \pm 1) + 1\right)\rho_{1/2} \tag{4.34}$$

$$8q = p^2: \quad p_0 = -\rho_0\,,\ p_1 = \left[\left(\frac{p}{6} - 1\right)\rho_1 - \frac{p}{6}\rho_2\right]\,,\ p_2 = \left(\frac{p}{6} - 1\right)\rho_2 \tag{4.35}$$

$$8q > p^2: \quad p_0 = -\rho_0\,,\ p_1 = \left[\left(\frac{p}{6} - 1\right)\rho_1 - \frac{p}{6}\rho_2\right]\,,\ p_2 = -\left[\left(\frac{p}{6} + 1\right)\rho_2 + \frac{p}{6}\delta^2\rho_1\right] \tag{4.36}$$

The classical Friedmann equation (4.7) is contained in the first case. In the second case the second matter component interacts with the first on but not vice versa. In the third case both, matter component 1 and 2 interact with each other.
The stability properties of equation (4.17) are obtained by investigating

$$g(H) := qH^3 - rH$$

of the associated Liénard system. The equilibrium points are again two de Sitter equilibrium points and one Minkowski equilibrium:

$$\left\{(0,0)\,,\left(\sqrt{\frac{r}{q}},0\right),\left(-\sqrt{\frac{r}{q}},0\right)\right\},$$

and

$$g'\left(\pm\sqrt{\frac{r}{q}}\right) = 2r$$
$$g'(0) = -r.$$

According to Theorem 2.24 the positive de Sitter equilibrium point is asymptotically stable if $r < 0$ and unstable if $r > 0$. By the reversing symmetry for the negative de Sitter

equilibrium point one finds the opposite behaviour. The Minkowski solution is stable for $r > 0$ and unstable for $r < 0$.
According to Example 2.2 the de Sitter equilibrium points bifurcate at the bifurcation value $r = 0$. Furthermore, according to example 2.2 there is a bifurcation at $p = 0$ since a change of sign reverses the direction of trajectories. Note that as $q > 0$ approaches zero the de Sitter equilibrium points are shifted towards infinity and cease to exist for $q < 0$. Hence as q is varied through the value $q = 0$ the number of equilibrium points changes and therefore a bifurcation occurs according to the theorem of Andronov and Pontryagin. The bifurcation analysis shows that we can alter the coefficients in the classical Friedmann equation (4.7) without changing the qualitative behaviour of the solutions, as long as we remain in the same stratum of the parameters $\{p, q, r\}$. This is remarkable since as we have seen, different values of the parameters correspond to quite different equations of state (4.34) - (4.36). In particular, when fixing $r = k_3$ then the Λ-CDM model is topologically equivalent to the generalised Friedmann equation as long as p and q are positive. We will use this fact later to show that the qualitative behaviour of solutions of the classical Λ-CDM model are actually similar to the qualitative behaviour of solutions of the semiclassical model considered in Chapter 5.

Chapter 5

Semiclassical Cosmological Models

Now we turn to the case of quantum matter. Here we consider the massless, conformally coupled scalar field, the electromagnetic field and the massless, conformally coupled Dirac field entering Einstein's field equation via the renormalised expectation value of its stress energy tensor. We will proceed analogously to the classical case. To solve the conservation equation we need an equation of state. This is given via the expectation value of the trace of the stress energy tensor. In FLRW-spacetimes we have:

$$-\rho_\omega^{sc} + 3p_\omega^{sc} = g^{\mu\nu}\omega(: T_{\mu\nu} :). \tag{5.1}$$

Using equations (3.29), (3.30) and (3.31) we find

$$\begin{aligned} g^{\mu\nu}\omega(: T_{\mu\nu} :) = &- \frac{6A}{2880\pi^2}\left(\dddot{H} + 7\ddot{H}H + 4\dot{H}^2 + 12\dot{H}H^2\right) - \frac{4B}{2880\pi^2}\left(\dot{H}H^2 + H^4\right) \\ &- 6r_3\left(\dot{H} + 2H^2\right) - \frac{4D}{2880\pi^2}, \end{aligned} \tag{5.2}$$

where

$$A = \begin{cases} 1 - 2880\pi^2(3r_1 + r_2) & \text{massless, conformally coupled scalar field} \\ -18 - 2880\pi^2(3r_1 + r_2) & \text{electromagnetic field} \\ -6 - 2880\pi^2(3r_1 + r_2) & \text{massless Dirac field} \end{cases}$$

$$\frac{B}{3} = \begin{cases} 1 & \text{massless, conformally coupled scalar field} \\ 62 & \text{electromagnetic field} \\ 11 & \text{massless Dirac field} \end{cases}$$

and

$$D = -2880\pi^2 r_4.$$

r_i, $i = 1, ..., 4$ are the yet undetermined renormalisation constants. Using the equation of state (5.1) and the equation of conservation (4.1) one obtains the relation:

$$g^{\mu\nu}\omega(:T_{\mu\nu}:) = -\left(\frac{1}{H}\frac{d}{dt} + 4\right)\rho^{sc}_{\omega} = -\left(a\frac{d}{da} + 4\right)\rho^{sc}_{\omega}, \tag{5.3}$$

where in the last identity $Hdt = a^{-1}da$ was used. The energy density can be given explicitly by solving equation (5.3)(see [3, 41]):

$$\rho^{sc}_{\omega} = \frac{c_{\omega}}{a^4} + \frac{1}{2880\pi^2}\left(D + 3A\left(2\ddot{H}H + 6\dot{H}H^2 - \dot{H}^2\right) + BH^4\right) + 3r_3H^2. \tag{5.4}$$

Hence the energy density of a massless, conformally coupled quantum field (scalar, Maxwell or Dirac) has a radiation term proportional to a^{-4} and quantum corrections including higher order derivatives of the Hubble function $H(t)$. Particularly the renormalisation constant D serves as a cosmological constant term. The radiation term is determined by a constant c_{ω} depending on the quantum state ω of the field relative to some reference quantum state. To see this take for example the conformal vacuum state [23, 42] denoted by vac and define:

$$\rho^{sc}_{\omega} - \rho^{sc}_{vac} =: W(a). \tag{5.5}$$

Since the trace is state independent in the massless case, equation (5.3) gives

$$\left(a\frac{d}{da} + 4\right)W(a) = 0, \tag{5.6}$$

which can be solved to give

$$W(a) = \frac{c_{\omega}}{a^4}. \tag{5.7}$$

Hence the semiclassical energy density in some state ω is given by:

$$\rho^{sc}_{\omega} = \rho^{sc}_{vac} + \frac{c_{\omega}}{a^4}, \tag{5.8}$$

with $c_{vac} = 0$. This is consistent with the renormalised expectation value of the stress energy tensor of the quantum scalar field in the conformal vacuum found in [42].
If the mass m of the quantum field is non-vanishing, a state-dependent extra term enters the trace $g^{\mu\nu}\omega(:T_{\mu\nu}:)$. For the conformally coupled, massive scalar field this term would read

$$\mathcal{Z}_{\omega,m} = -\frac{m^2}{8\pi^2}\omega(:\varphi^2:) \tag{5.9}$$

and a similar term would appear for the massive Dirac field. Assume a quantum state $\tilde{\omega}$ such that

$$\widetilde{\omega}(:\varphi^2:) = \frac{8\pi^2 c_{\widetilde{\omega},m}}{m^2} a^{-3}.$$

An approximate example of such a quantum state $\tilde{\omega}$ was examined in [67]. Then we can even arrive at the energy density of a massless, conformally coupled scalar field in quantum state ω together with massive, conformally coupled scalar field in quantum state $\tilde{\omega}$, i.e.

$$\rho^{sc}_{\omega,\widetilde{\omega}} = \frac{c_\omega}{a^4} + \frac{c_{\widetilde{\omega},m}}{a^3} + \frac{D_m}{2880\pi^2} + \text{quantum corrections}, \tag{5.10}$$

where this time the cosmological constant term depends on the mass, i.e.

$$D_m = -2880\pi^2 \left(\frac{m^4}{128\pi^2} + r_4 \right)$$

From equation (5.10) the quantum corrections can explicitly be given, since the first three terms correspond to the energy density of classical dust, radiation and cosmological constant. Furthermore we see that the quantum vacuum energy is not merely a cosmological constant term but contains higher order derivatives of the Hubble function $H(t)$, which puts the so-called cosmological constant problem [22, 115, 142] into a different light. The main focus of this thesis will be to analyse the dynamics of the quantum corrections including the cosmological constant term. If we now plug the energy density $\rho^{sc}_{\omega,\widetilde{\omega}}$ into Friedmann's equation we may use the renormalisation freedom we have to find some distinguished classical solutions that are still solutions to the semiclassical Friedmann equation. Let

$$C := \frac{360\pi\,(1 - 8\pi G r_3)}{G},$$

then:

1. The Milne universe $a(t) \sim t$ is a solution to the semiclassical Friedmann equation if and only if $A = \frac{B}{9}$.

2. The dust universe $a(t) \sim t^{\frac{2}{3}}$ is a solution to the semiclassical Friedmann equation if and only if $A = \frac{4B}{27}$.

3. The de Sitter solutions $a(t) \sim e^{H^{\pm}t}$ where $H^{\pm} = \sqrt{\frac{3}{2}\frac{C}{B}\left(1 \pm \sqrt{1 - \frac{4}{9}\frac{BD_m}{C^2}}\right)}$ are solutions to the semiclassical Friedmann equation as long as the terms in the square roots are non-negative.

4. The classical solution of a universe filled with dust and dark energy, i.e. $3H^2 = \frac{p^\pm}{a^3} + (q^\pm)^2$, where $p^\pm = \frac{2880\pi^2 c_{\omega,m}^{\sim}}{C - \frac{2}{3}B(q^\pm)^2}$ and $q^\pm = \sqrt{\frac{3}{2}\frac{C}{B}\left(1 \pm \sqrt{1 - \frac{4}{9}\frac{BD_m}{C^2}}\right)}$ is a solution to the semiclassical Friedmann equation if and only if $A = \frac{4B}{27}$.

Note that neither the classical radiation solution nor the solution of a universe filled with classical radiation and dark energy is a solution to the semiclassical Friedmann equation. These little calculations can already be seen in the light of stability arguments. Namely, under which values of A, B, C and D_m do the classical solutions still exist when quantum corrections are added. This however does not answer the question of Lyapunov stability or structural stability.

Often the renormalisation constants A, C are fixed at the beginning for diverse reasons. One of the reasons for setting $A = 0$ is that, besides simplifying the dynamical equations considerably, higher order derivatives could change the dynamical behaviour of semiclassical solutions tremendously compared to the solutions for classical matter [136]. We will show that this must not be the case. Also we will show, that one can not neglect A even if it is small, because this would indeed change the dynamical behaviour of solutions completely. The constant C is often set equal to Newtons constant G because it appears in front of the Einstein tensor (e.g. in [67, 108]) and the value of Newtons constant is fixed. However for the author of the present thesis it is not clear whether C is indeed Newtons constant. In fact, as the de Sitter solutions of the semiclassical Friedmann equation show, for $D = 0$ the constant C acts as a cosmological constant compared to the classical de Sitter solution. In particular C appears in the argument of the exponential function of the scale factor. Hence a measurement of the constant C must not influence the value of Newtons constant. In this thesis we will therefore take all renormalisation constants serious and allow any value.

From now on we will restrict to the case of a massless, conformally coupled quantum field. Then the energy density is given by equation (5.4) and the dynamics of the spacetime is given by Friedmann's equation (3.3), i.e.

$$0 = \ddot{H}H - \frac{1}{2}\dot{H}^2 + 3\dot{H}H^2 + \frac{1}{6}\frac{B}{A}H^4 - \frac{1}{2}\frac{C}{A}H^2 + \frac{1}{6}\frac{D}{A} + \frac{c_\omega}{2}a^{-4} \tag{5.11}$$

The state dependent term can be eliminated by differentiating equation (5.11) with respect to the time t and inserting it again in the resulting equation in virtue of equation (5.3). This gives the trace of Einstein's field equation

$$0 = \dddot{H} + 7\ddot{H}H + 4\dot{H}^2 + \left(\frac{2}{3}\frac{B}{A} + 12\right)\dot{H}H^2 - \frac{C}{A}\dot{H} + \frac{2}{3}\frac{B}{A}H^4 - 2\frac{C}{A}H^2 + \frac{2}{3}\frac{D}{A} \tag{5.12}$$

Conversely, the Friedmann equation (5.11) is a first integral of equation (5.12). Equation (5.12) is a non-linear, third order, ordinary differential equation. The initial value $\ddot{H}_0$ is not free but determined by Friedmann's equation at some fixed initial time t_0. Hence any solution of equation (5.12) determined by a set of initial data $\mathcal{I} = \{H_0, \dot{H}_0, a_0, c_\omega, A, B, C, D\}$ together with Friedmann's equation at the fixed initial time t_0 corresponds to a model of a perfectly homogeneous and isotropic, flat universe filled with a massless, conformally coupled quantum field in quantum state ω. Remember that the constant B specifies whether the quantum field is a scalar field, an electromagnetic field or a massless Dirac field and the constants A, C and D are reformulations of the free renormalisation constants $r_1 \dots r_4$. Before we pass to the analysis of equations (5.11) and (5.12) in terms of dynamical system theory let us go one step back and consider the trace of Einsteins field equation for a massive, non-conformally coupled scalar field. From equation (3.28) we find

$$-3\left(\frac{1}{6} - \xi\right)\Box\Omega(t) + m^2\Omega(t) = F(H(t), \dot{H}(t), \ddot{H}(t), \dddot{H}(t)), \tag{5.13}$$

where $F(H(t), \dot{H}(t), \ddot{H}(t), \dddot{H}(t)) = F(t)$ abbreviates the geometric part of the trace of the semiclassical Einstein equation and can be seen as a function depending only on the time t. $\Omega(t) := \omega(:\, \Phi^2 :\, (t))$ describes the state-depending part. In FLRW spacetime the d'Alembert operator becomes:

$$\Box = \frac{d}{dt^2} + 3H(t)\frac{d}{dt}.$$

Hence we may rewrite equation (5.13):

$$\ddot{\Omega}(t) + 3H(t)\dot{\Omega}(t) + \alpha\Omega(t) = \beta F(t), \tag{5.14}$$

where

$$\alpha := \frac{2m^2}{6\xi - 1},$$

and

$$\beta := \frac{2}{6\xi - 1}.$$

Hence $\Omega(t)$ fulfils the differential equation of an driven and damped oscillator, where the driving force is expressed by $F(t)$ and the damping is $H(t)$ and hence both depend on the spacetime. By substituting

$$q(t) := a(t)^{\frac{3}{2}}\Omega(t),$$

one can transform equation (5.14) such that no damping occurs:

$$\ddot{q}(t) + G(t)q(t) = \beta a^{\frac{3}{2}}(t)F(t), \tag{5.15}$$

with

$$G(t) := \alpha - \frac{3}{2}\left(\dot{H}(t) + \frac{3}{2}H(t)^2\right).$$

To close the differential equation (5.14) (or (5.15)) one needs information about the expectation value of the Wick square Ω as a function of t. For an analysis in the context of dynamical systems one would like to have an additional differential equation for Ω in terms of its time derivatives, which can also depend on $H(t)$ and its derivatives. In [50] the Klein-Gordon equation is used to express the Ω and its derivatives. This however leads to an infinite dimensional dynamical system [129], where the infinite degrees of freedom stem from the infinite modes (one mode for each k). This makes the dynamical system analysis quite complicated. Another way to determine a differential equation for Ω could be to restrict to a certain class of states.

Instead of specifying the state and therefore $\Omega(t)$ one can read equation (5.14) (or (5.15)) also the other way around. Given a specific FLRW spacetime in terms of $H(t)$ equation (5.14) (or (5.15)) determines the possible states allowed in that spacetime. These states may be further restricted by physical considerations. As an example consider de Sitter spacetime $H(t) = H = \mathit{constant}$. Then also $F(t) = F = \mathit{constant}$ and the solution of equation (5.14) can be written as

$$\Omega(t) = c_1 \exp\left\{\left(-\frac{3}{2}H - \sqrt{\frac{9}{4}H^2 - \alpha}\right)t\right\} + c_2 \exp\left\{\left(-\frac{3}{2}H + \sqrt{\frac{9}{4}H^2 - \alpha}\right)t\right\} + \frac{4\rho_\omega^{sc}}{m^2}, \tag{5.16}$$

where c_1 and c_2 are appropriate initial conditions and equation (5.3) was used to express $F = 4\rho_\omega^{sc}$. This expression of $\Omega(t)$ has also been found in [47]. Note that the *Bunch Davies vacuum* [30] can be recovered by setting the initial conditions such that $c_1 = c_2 = 0$.

5.1 Dynamical System Analysis for the Conformal Vacuum

Let us begin by analysing the solutions to the trace equation (5.12) corresponding to a universe filled with a conformally coupled, massless quantum field being in the conformal vacuum state. These correspond to the set of solution one obtains by setting $c_\omega = 0$ in (5.11), [42]. Thus, one obtains the second order non-linear differential equation in $H = H(t)$

$$0 = \ddot{H}H - \frac{1}{2}\dot{H}^2 + 3\dot{H}H^2 + \frac{1}{6}\frac{B}{A}H^4 - \frac{1}{2}\frac{C}{A}H^2 + \frac{1}{6}\frac{D}{A}. \tag{5.17}$$

This equation was investigated on several occasions in the literature (e.g. [13,123]). Other than in the classical vacuum the Minkowski universe is not the only solution. In fact, in most cases (i.e. whenever $D \neq 0$) the Minkowski solution doesn't even exist. In exchange, the semiclassical Friedmann equation allows a rich diversity of dynamical behaviour even in the conformal vacuum quantum state. This ability of the vacuum to act as a gravitational source is referred to as *vacuum fluctuations*.
Due to the non-linearity we are not able to find the general solution to equation (5.17). To understand the qualitative behaviour of the solutions without knowing them explicitly it is therefore advisable to use the theory of dynamical systems. If we exclude the line $H = 0$ of the phase space we may formulate the dynamical equation (5.17) as two dimensional dynamical system defined by the vector field

$$f(H, \dot{H}) := \begin{pmatrix} \dot{H} \\ \frac{1}{2}\frac{\dot{H}^2}{H} - 3\dot{H}H - \frac{1}{6}\frac{B}{A}H^3 + \frac{1}{2}\frac{C}{A}H - \frac{1}{6}\frac{D}{A}H^{-1} \end{pmatrix}. \tag{5.18}$$

Note that by considering the vector field (5.18) we lost information about the second derivative of $H(t)$. This causes the non-unique behaviour of Proposition 5.2 in the next Subsection. The vector field (5.18) obeys an reversing symmetry, i.e. f is invariant under the reversing symmetry

$$R : (H, \dot{H}) \mapsto (-H, \dot{H}).$$

The fixed point subspace of R is $Fix(R) = \{(H, \dot{H}) \in \mathbb{R}^2 | H = 0\}$. We may therefore divide the phase space into the two regions

$$\mathbb{H}^2_+ := \{(H, \dot{H}) \in \mathbb{R}^2 | H > 0\}$$
$$\mathbb{H}^2_- := \{(H, \dot{H}) \in \mathbb{R}^2 | H < 0\}$$

Then the trajectories in region $\mathbb{H}^2_+$ are completely determined by the trajectories in region $\mathbb{H}^2_-$ by the symmetry R. We may classify trajectories of (5.18) as follows

Definition 5.1. (i) The set of *expanding trajectories* $\mathcal{E} := \{\gamma(t) \in \mathbb{H}^2_+ |$ for all $t \in \mathbb{R}\}$.

(ii) The set of *contracting trajectories* $\mathcal{C} := \{\gamma(t) \in \mathbb{H}^2_- |$ for all $t \in \mathbb{R}\}$.

(iii) The *Minkowski equilibrium point* $\mathcal{M} = \{\gamma(t) = (0, 0)$ for all $t\} \subset Fix(R)$

(iv) The set of *reversing trajectories* $\mathcal{R} := \{\gamma \in \mathbb{R}^2 | \gamma \cap Fix(R) \neq \varnothing$ and $\gamma \not\subseteq Fix(R)\}$.

Contracting trajectories are obtained from the expanding trajectories using the symmetry: $R(\mathcal{E}) = \mathcal{C}$. The Minkowski equilibrium point exists if and only if $D = 0$. If this is the case, then $\mathcal{M} \in Fix(R)$ and we have $R(\mathcal{M}) = \mathcal{M}$. Reversing trajectories may switch between an expanding and an contracting period arbitrary often. This is not the case for smooth vector fields [92, 131], where reversing trajectories can only cross $Fix(R)$ exactly once or exactly twice. In the latter case such a trajectory would be periodic. In our case this must not be the case, since trajectories must not be unique at $Fix(R)$. Hence to clarify this situation we have to pay particular attention to reversing trajectories $\mathcal{R}$ and the Minkowski equilibrium point $\mathcal{M}$ since the vector field (5.18) is no longer defined for $H = 0$.

5.1.1 Reversing Universes

In the following proposition we will make use of the one-to-one correspondence between solutions of equation (5.17) with initial conditions $H(0) = H_0$ and $\dot{H}(0) = \dot{H}_0$ here denoted by $H(t) = H(t, H_0, \dot{H}_0)$ and trajectories in the plane arising from solutions by $\gamma = \gamma(t) = (H(t), \dot{H}(t))^T$.

Proposition 5.2. *Consider equation* (5.17) *and let $\mathcal{R}$ be the set of reversing trajectories crossing the fixed point subspace $Fix(R)$ and $\mathcal{M}$ the Minkowski equilibrium point. Define the point*

$$\mathcal{P}_{\pm} := \left(0, \pm\sqrt{\frac{1}{3}\frac{D}{A}}\right) \in Fix(R).$$

Then the following holds.

(i) If $\frac{D}{A} < 0$, then $\mathcal{R} = \varnothing$.

(ii) Let $\frac{D}{A} \geq 0$. Then there are infinitely many trajectories $\gamma \in \mathcal{R}$ such that $\gamma \cap Fix(R) = \mathcal{P}_{\pm}$. Each trajectory $\gamma \in \mathcal{R}$ starting in $\mathbb{H}^2_+$ runs through $\mathcal{P}_-$ and each trajectory $\gamma \in \mathcal{R}$ starting in $\mathbb{H}^2_-$ runs through $\mathcal{P}_+$.

(iii) The Minkowski equilibrium point exists if and only if $D = 0$. Then $\mathcal{P}_+ = \mathcal{P}_- = \mathcal{M}$, i.e. all trajectories $\gamma \in \mathcal{R}$ run through $\mathcal{M}$.

Proof. Let $H(t) = H(t, H_0, \dot{H}_0)$ be a solution of equation (5.17) with initial data $(H_0, \dot{H}_0)$ and suppose that there is a time $t_R = t_R(H_0, \dot{H}_0)$ such that $H(t_R, H_0, \dot{H}_0) = 0$. Then by equation (5.17) we must have:

$$\left(\dot{H}(t_R)\right)^2 = \frac{1}{3}\frac{D}{A},$$

and the associated trajectory must reach the point

$$\gamma(t_R) = \left(0, \pm\sqrt{\frac{1}{3}\frac{D}{A}}\right) \equiv P \in Fix(R)$$

at the time t_R. If $\frac{D}{A} < 0$ no such trajectory exists since this would require $\dot{H} \in \mathbb{C}$ which proves part (i) of the Proposition.
If $\frac{D}{A} \geq 0$ then any trajectory $\gamma \in \mathcal{R}$ must run through $P_\pm$. Now consider smooth functions $a_\epsilon(\dot{H}_0)$ and $b_\epsilon(\dot{H}_0)$ fulfilling

(a) $a_\epsilon(\dot{H}_0) > 0$ and $b_\epsilon(\dot{H}_0) < 0$ for all $\epsilon > 0$ and all $\dot{H}_0 \in \mathbb{R}$.

(b) $a_\epsilon, b_\epsilon \to 0$ as $\dot{H}_0 \to \pm\infty$.

(c) for any given $\dot{H}_0$, it holds that $a_\epsilon(\dot{H}_0) - b_\epsilon(\dot{H}_0) \leq \epsilon$.

Then define a region "close" to the fixed point subspace $Fix(R)$ by $U_\epsilon := \{(H_0, \dot{H}_0) \in \mathbb{R}^2 | b_\epsilon(\dot{H}_0) \leq H_0 \leq a_\epsilon(\dot{H}_0), \epsilon > 0\}$. For an illustration see Figure 5.1. For initial conditions

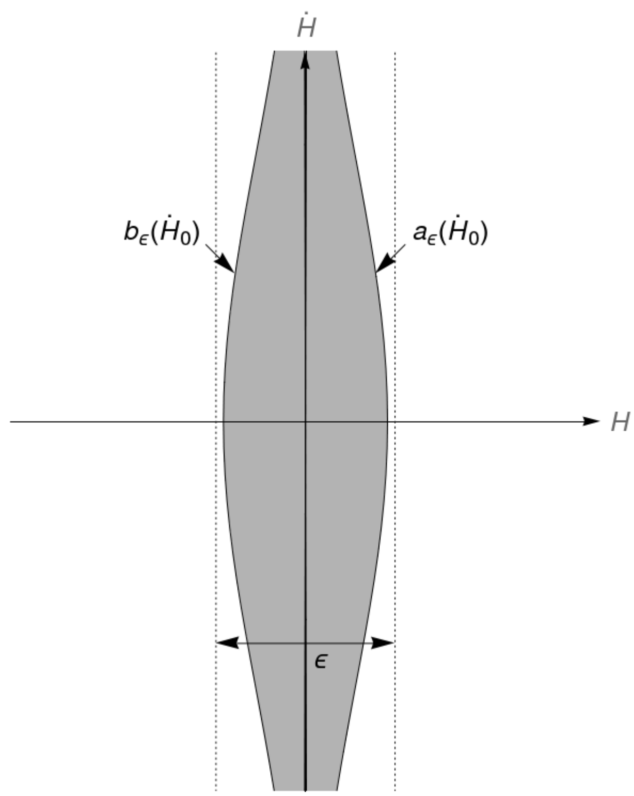


Figure 5.1: The region U_ϵ around the $\dot{H}$-axis

$(H_0, \dot{H}_0) \in U_\epsilon$, $t \in I \subseteq \mathbb{R}$ such that $H(t, H_0, \dot{H}_0) \in U_\epsilon$ and $\epsilon > 0$ sufficiently small, equation (5.17) can be approximated by:

$$0 = 2\ddot{H}_\epsilon H_\epsilon - \dot{H}_\epsilon^2 + \frac{1}{3}\frac{D}{A}. \tag{5.19}$$

Let us justify this approximation. First note that we can make U_ϵ small enough, such that no equilibrium lies within $U_\epsilon \backslash Fix(R)$. Then the dynamical system (5.18) is structurally stable in any subregion of $\mathbb{H}^2_\pm$ by the theorem of Andronov and Pontryagin 2.28 and therefore any ϵ-close vector field is topologically equivalent to the dynamical system (5.18). In particular the vector field

$$f_\epsilon(H, \dot{H}) := \begin{pmatrix} \dot{H} \\ \frac{1}{2}\frac{\dot{H}^2}{H} - \frac{1}{6}\frac{D}{A}H^{-1} \end{pmatrix}. \tag{5.20}$$

arising from equation (5.19) will be topologically equivalent to the dynamical system (5.18) in any subregion of $\mathbb{H}^2_\pm$ as long as we are close enough to $H = 0$, which we can achieve by choosing ϵ accordingly. Further note, that the difference between the vector field (5.18) and the vector field f_ϵ vanishes in the in the limit of $H \to 0$ and hence both vector fields are equal at $H = 0$.

We are interested in the behaviour of solutions to equation (5.19) as long as these don't leave the region U_ϵ. The general solution of equation (5.19) is

$$H_\epsilon(t, H_0, \dot{H}_0) = \Omega(H_0, \dot{H}_0)t^2 + \dot{H}_0 t + H_0, \tag{5.21}$$

where

$$\Omega(H_0, \dot{H}_0) := \frac{\dot{H}_0^2 - \frac{1}{3}\frac{D}{A}}{4H_0},$$

with initial conditions $(H_0, \dot{H}_0) \in U_\epsilon$. Any solution (5.21) passes $H_\epsilon = 0$ at some time t_R:

$$0 = H_\epsilon(t_R) = \Omega(H_0, \dot{H}_0)t_R^2 + \dot{H}_0 t_R + H_0.$$

Hence:

$$t_R^\pm(H_0, \dot{H}_0) = \frac{-\dot{H}_0 \pm \sqrt{\frac{1}{3}\frac{D}{A}}}{2\Omega(H_0, \dot{H}_0)}$$

and

$$\begin{aligned} \dot{H}_\epsilon(t_R^\pm) &= 2\Omega(H_0, \dot{H}_0)t_R^\pm + \dot{H}_0 \\ &= \pm\sqrt{\frac{1}{3}\frac{D}{A}}. \end{aligned}$$

Hence all solution starting in U_ϵ will reach the points $P_\pm$ at finite times $t_R^\pm$ as long as $\frac{D}{A} \geq 0$. Without loss of generality we can choose solutions (5.21) starting in U_ϵ with $\dot{H}_0 > 0$ since any such solution must approach $+\sqrt{\frac{1}{3}\frac{D}{A}}$. Then $t_R^+ > 0$ if and only if $H_0 < 0$. Hence trajectories $\gamma \in \mathcal{R}$ starting in $\mathbb{H}^2_-$ must reach $+\sqrt{\frac{1}{3}\frac{D}{A}}$. Analogously for $\dot{H}_0 < 0$ trajectories $\gamma \in \mathcal{R}$ starting in $\mathbb{H}^2_+$ must reach $-\sqrt{\frac{1}{3}\frac{D}{A}}$. This proves part (ii) of the proposition.
Finally it is easy to show that $H = 0$ is a solution of equation (5.17) if and only if $D = 0$. Part (ii) of the above proposition entails that all solutions must run through the Minkowski equilibrium $\mathcal{M}$ if $D = 0$. □

First of all note that the Minkowski solution cannot be stable since all reversing solutions run through the Minkowski equilibrium. Furthermore, Proposition 5.2 entails that solutions $H(t)$ to the semiclassical Friedmann equation (5.17) that change their sign, and the Minkowski solution, if these exist are in general not uniquely determined by the initial conditions $(H_0, \dot{H}_0)$. In particular the initial conditions $(H_0, \dot{H}_0) = \mathcal{P}_\pm$ do not uniquely determine solutions and the continuation of a solution reaching $H = 0$ is ambiguous. This was also realised by [13] without dissolving the problem of non-uniqueness. As we will show in due course the anomalous behaviour at $H = 0$ can be explained by studying the more general three dimensional dynamical system obtained from the trace (5.12).

5.1.2 Equilibrium Points and their Stability

Let us consider the dynamical system defined by the vector field (5.18) exclusively on $\mathbb{H}^2_+$. We perform the following transformation

$$\begin{pmatrix} H \\ \dot{H} \end{pmatrix} \in \mathbb{H}^2_+ \mapsto \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{L}^2_+ := \{(x,y) \in \mathbb{R}^2 | x > 0\}, \tag{5.22}$$

where

$$\begin{aligned} x &= H^{\frac{1}{2}} \\ y &= \dot{x} \equiv \frac{1}{2}\dot{H}H^{-\frac{1}{2}}. \end{aligned} \tag{5.23}$$

Then, the dynamical system associated to the vector field (5.18) becomes a Liénard system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(x,y) = \begin{pmatrix} y \\ f(x)y - g(x) \end{pmatrix}, \tag{5.24}$$

where

$$\begin{aligned} f(x) &= 3x^2 \\ g(x) &= \frac{1}{12}\frac{B}{A}x^5 - \frac{1}{4}\frac{C}{A}x + \frac{1}{12}\frac{D}{A}x^{-3}. \end{aligned} \tag{5.25}$$

The equilibrium points of the vector field (5.24) in $\mathbb{L}^2_+$ all correspond to de Sitter equilibrium points $H \neq 0$ and we will denote them as

$$\begin{aligned} dS :=& \{(x,y) \in \mathbb{L}^2_+ | g(x) = 0, y = 0\} \\ =& \{(x_+, 0), (x_-, 0)\}, \end{aligned}$$

where

$$x_\pm := H_\pm^{\frac{1}{2}} := \left[\frac{3}{2}\frac{C}{B}\left(1 + \Delta^\pm\right)\right]^{\frac{1}{4}}, \ \Delta^\pm := \pm\sqrt{1 - \frac{4}{9}\frac{BD}{C^2}}. \tag{5.26}$$

By the transformation (5.22) these equilibrium points of the Liénard system transform to the equilibrium points $H_\pm \in \mathbb{H}^2_+$ and by the reversing symmetry R to the additional equilibrium points $-H_\pm \in \mathbb{H}^2_-$ of the original system. Note that the equilibrium point x_- might become zero when choosing the renormalisation constants appropriately. This solution corresponds to the Minkowski equilibrium $\mathcal{M}$ by the substitution (5.22). Finally, note that for $D = 0$ the de Sitter equilibrium points $H = \sqrt{3\frac{C}{B}}$. Hence, compared to the classical case, here C plays the role of a cosmological constant for the de Sitter solution. The actual cosmological constant D appearing in the semiclassical Einstein equations gives rise to a second de Sitter solution.

The stability behaviour of the equilibrium points can be shown by means of the Lyapunov function

$$V(x,y) := \frac{1}{2}y^2 + \int_0^x g(u)du. \tag{5.27}$$

Then, using theorem 2.24 the stability of the equilibrium points in dS depends on the sign of

$$g'(x_\pm) = \frac{C}{A}\Delta^\pm,$$

and therefore on the signs of C and A. The various cases are listed in Tables 5.1 and 5.2. For completeness we included the Minkowski equilibrium $\mathcal{M}$. However, remember that only the behaviour in a neighbourhood of $\mathcal{M}$ for $x > 0$ corresponds to the relevant behaviour in the $(H, \dot{H})$-phase space. As a consequence, if $\mathcal{M}$ is an unstable saddle point, then it will also be unstable in the $(H, \dot{H})$-phase space. However, if it is asymptotically stable it need not be stable in the $(H, \dot{H})$-phase space. In fact it will turn out that the

Minkowski equilibrium is never stable in the $(H, \dot{H})$-phase space.
The Lyapunov function $V(x, y)$ can be used to estimates the regions of attraction of the asymptotically stable equilibrium points.

Table 5.1: $C > 0$: Stability behaviour of the equilibrium points $(x_\pm, 0)$ for different values of the renormalisation constants A and D.

	$A > 0$	$A < 0$
$D < 0$	$(x_+, 0)$ is asymptotically stable	$(x_+, 0)$ is an unstable saddle point
$D = 0$	$(x_+, 0)$ is asymptotically stable and $(x_-, 0) = (0, 0)$ is an unstable saddle point	$(x_+, 0)$ is an unstable saddle point and $(x_-, 0) = (0, 0)$ is asymptotically stable
$0 < D < \frac{9}{4}\frac{C^2}{B}$	$(x_+, 0, 0)$ is asymptotically stable and $(x_-, 0, 0)$ is an unstable saddle point	$(x_+, 0)$ is an unstable saddle point and $(x_-, 0)$ is asymptotically stable
$D = \frac{9}{4}\frac{C^2}{B}$	$g'(x_+) = 0 = g'(x_-)$ and therefore no conclusion can be drawn	
$D > \frac{9}{4}\frac{C^2}{B}$	-	-

Table 5.2: $C < 0$: Stability behaviour of the equilibrium points $(x_\pm, 0)$ for different values of the renormalisation constants A and D.

	$A > 0$	$A < 0$
$D < 0$	$(x_+, 0)$ is asymptotically stable	$(x_+, 0)$ is an unstable saddle point
$D = 0$	$(x_-, 0, 0) = (0, 0)$ is asymptotically stable	$(x_-, 0) = (0, 0)$ is an unstable saddle point
$D > 0$	-	-

5.1.3 Structural Stability, Bifurcation Sets and Phase Portraits

Even without knowing the explicit form of solutions of the semiclassical Friedmann equation (5.17) — at least in most cases — we already gathered a large amount of qualitative information thereof. Obviously the properties of solutions depend on the initial data *and* on the values of the renormalisation constants. If the latter are free then there are various possibilities for the dynamical behaviour of the universe even in this very simplified scenario. In order to distinguish and discuss the solutions of equation (5.17) it is convenient to

study their strata, i.e. those intervals of the renormalisation constants for which the phase portraits are topologically equivalent. Unfortunately the vector field determining the dynamical behaviour is not continuous for $H = 0$. Even worse the vector field is not defined on $H = 0$ if $D \neq 0$ except for the two points $\mathcal{P}^{\pm}$ defined in proposition 5.2. Hence all the theorems concerning structural stability and bifurcations cannot be applied to regions of the phase space containing $H = 0$. To encompass these problems we will discuss the effect of varying the renormalisation constants on the qualitative behaviour of the phase portrait for $\mathbb{H}^2_{\pm}$ and on reversing solutions $\mathcal{R}$ and the Minkowski solution $\mathcal{M}$ separately.
Only for this section we will introduce an additional parameter, say $E \in (-1, 1)$, such that equation (5.17) reads

$$0 = A\ddot{H}H + AE\dot{H}^2 + 3A\dot{H}H^2 + \frac{1}{6}BH^4 - \frac{1}{2}CH^2 + \frac{1}{6}D. \tag{5.28}$$

The reason is to make contact to the generalised Friedmann equation (4.17), where $E = D = 0$ in (5.28). The similarity between the semiclassical Friedmann equation of the vacuum case and the generalised Friedmann equation can be made even more dramatic by introducing the variable $v = v(H)$ and making the ansatz $v(H)H^2 := -(2/3)\dot{H}$. Then in both, the classical and semiclassical case v fulfills an Abel differential equation of the second kind [109]

$$0 = v'v + H^{-1}(c_1 v^2 + c_2 v + c_3) + c_4 H^{-3} + c_5 H^{-5}, \tag{5.29}$$

where prime denotes the derivative with respect to H. The parameters of either equation appear as c_i, $i = 1, ..., 5$. In particular for the semiclassical and the Λ-CDM model we have:

$$\{c_1, c_2, c_3, c_4, c_5\} = \begin{cases} \{2, -\frac{14}{3}, 4, -\frac{8}{3}k_3, 0\} & \Lambda\text{-CDM} \\ \{\frac{3}{2}, -2, \frac{1}{9}\frac{B}{A}, -\frac{2}{9}\frac{C}{A}, -\frac{2}{27}\frac{D}{A}\} & \text{semiclassical} \end{cases}. \tag{5.30}$$

Hence the question whether classical and semiclassical solutions behave qualitative alike can be answered by analysing the qualitative behaviour of solutions of equation (5.29) depending on the parameters c_i. Since we already have shown that the Λ-CDM model is topologically equivalent to the generalised Friedmann equation for according values of the parameters it remains to compare the semiclassical equation (5.28) with the generalised Friedmann equation (4.17).
Furthermore we allow the parameter B to vary in order to compare the different values of the scalar field, the electromagnetic field and the Dirac field and to relate the semiclassical model considered here to certain $f(R)$-theories considered in [16], where $B = 0$ in equation

(5.28).

We can generalise the transformation (5.22) and substitute

$$x := H^{E+1} \tag{5.31}$$
$$y := \dot{x} \equiv (E+1)\dot{H}H^{E}, \tag{5.32}$$

to obtain again a Liénard equation from (5.28). For $A \neq 0$ we have

$$0 = \ddot{x} + f(x)\dot{x} + g(x), \tag{5.33}$$

where

$$f(x) := 3x^{\frac{1}{E+1}}, \tag{5.34}$$
$$g(x) := (E+1)\left(\frac{1}{6}\frac{B}{A}x^{\frac{E+3}{E+1}} - \frac{1}{2}\frac{C}{A}x + \frac{1}{6}\frac{C}{A}x^{\frac{E-1}{E+1}}\right). \tag{5.35}$$

The equilibrium points of the Liénard system are

$$x_{\pm} = H_{\pm}^{E+1}, \tag{5.36}$$

where $H_{\pm}$ is defined in equation (5.26). We have the following

Proposition 5.3. *On $\mathbb{L}_+^2 := \{(x,y) \in \mathbb{R}^2 | x > 0\}$ consider the Liénard system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = h(x,y) = \begin{pmatrix} y \\ -f(x)y - g(x) \end{pmatrix}, \tag{5.37}$$

arising from equation (5.33) *with $f(x)$ and $g(x)$ defined by equations* (5.34) *and* (5.35). *Let $A \neq 0$ such that we can define $B_A := \frac{B}{A}$, $C_A := \frac{C}{A}$ and $D_A := \frac{D}{A}$ and collectively denote the free parameters of the system as $\mu := (B_A, C_A, D_A, E)$, where $E \in (-1,1)$. Then $\mu_0 := \left(B_A, C_A, \frac{9}{4}\frac{C_A^2}{B_A}, E\right)$ is the bifurcation set of the parameter μ.*

Proof. We have to show, that the phase portrait of the Liénard system, resulting from the Liénard equation (5.33), is structurally stable for any value μ except for μ_0. First of all, observe that

$$g'(x_{\pm}) = C_A \Delta^{\pm} \equiv \pm\sqrt{C_A^2 - \frac{4}{9}B_A D_A},$$

which becomes zero if and only if $D_A = \frac{9}{4}\frac{C_A^2}{B_A}$. According to Example 2.2 we conclude that this is indeed a bifurcation value of the free parameters. In all other cases the de

Sitter equilibrium points dS given by equation (5.26) are hyperbolic. The divergence of the vector field is

$$div(h(x,y)) = -3x^{\frac{1}{E+1}},$$

and never changes sing for any $x \in \mathbb{L}^2_+$. Hence, from the theorem of Bendixson 2.12 there are no closed trajectories lying entirely in $\mathbb{L}^2_+$. In particular there are no homoclinic trajectories lying entirely in $\mathbb{L}^2_+$. There are also no heteroclinic trajectories in $\mathbb{L}^2_+$ since there is maximally only one saddle point. Therefore, from the theorem of Andronov and Pontryagin we deduce that the vector field $h(x,y)$ is structurally stable as long as $\mu \neq \mu_0$.

□

According to Proposition 5.3 there is only one bifurcation in $\mathbb{L}^2_+$ caused by the collision of a saddle point and a node. In particular, there is no bifurcation associated with any value of E whatsoever. Hence, for all values $E \in (-1,1)$ the according phase portraits are topologically equivalent. By the reversing symmetry this holds also for the regions $\mathbb{H}^2_\pm$ and any value $E \in (-1,1)$. In particular the semiclassical Friedmann equation with parameter values $\mu_1 = \left(B_A, C_A, 0, -\frac{1}{2}\right)$ is topologically equivalent to the generalised Friedmann equation (4.17) with values $\mu_2 = (B_A, C_A, 0, 0)$ in $\mathbb{H}^2_\pm$. Hence the semiclassical model of the universe considered here is capable to reproduce the qualitative features in $\mathbb{H}^2_+$ or $\mathbb{H}^2_-$ of the Λ-CDM model considered in chapter 4 once the renormalisation constants are chosen appropriately. This means in particular that expanding and contracting universes of the semiclassical model have the same asymptotically behaviour and the same stability behaviour as in the Λ-CDM model. However, any value E deviating from zero leads to the non-unique behaviour at $H = 0$. As a consequence reversing trajectories do either not exist at all for $E \neq 0$ or are non-unique at $H = 0$ in contrast to reversing solutions of the generalised Friedman equation. Furthermore the cosmological constant of the Λ-CDM model can be identified with the renormalisation constant C_A. Unfortunately this does not explain the nature of the cosmological constant since C_A is itself a free parameter of the quantum theory.

There are further critical values of the renormalisation constants changing the qualitative behaviour of solutions. However, these are not covered by the theory of bifurcations considered here. One of the reasons is that a change of the value of a renormalisation constants involves a higher order perturbation. This is the case when A is varied through zero. If $A = 0$ the higher order terms in (5.28) vanish and the following algebraic equation (in the x-variables) results:

$$0 = g(x). \tag{5.38}$$

To discuss structural stability for $A = 0$ within the space of all two dimensional vector fields we have to differentiate this equation twice with respect to time to obtain the two dimensional dynamical system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = y \begin{pmatrix} 1 \\ -\frac{g''(x)}{g'(x)} y \end{pmatrix} . \tag{5.39}$$

Obviously there are infinitely many equilibrium points on the line $y = 0$. Therefore, by the theorem of Andronov and Pontryagin, any region containing this line is structurally unstable. Note that $A = 0$ is also a critical value for the state dependent Friedmann equation. This can be seen by considering the trace equation (5.12) for $A = 0$ which leads to a first order differential equation in H. By differentiating this equation again with respect to the time t we obtain again a vector field having an infinite number of equilibrium points on the $H = 0$ axis. This tremendous qualitative change of behaviour of solutions by varying A through zero motivated Wald to formulate his fifth axiom, excluding higher derivative terms (see [136] and in particular footnote 6 on page 9 therein): Because the higher order terms can produce solutions that do not have the same stability types as the classical solutions in some classical limit we must exclude them. This however need not be true and depends on the concrete form of the higher order terms. In particular in the cosmological context considered here we have already seen the close similarity between the semiclassical equation (5.28) and the (generalised) Friedmann equation (4.17) if $A \neq 0$. Both share the same equilibrium points which have the same stability behaviour. In fact, choosing $A = 0$ leads to dynamical behaviour of the semiclassical solutions which is qualitatively completely different to the classical behaviour. Furthermore, a slight deviation of $A = 0$ leads to completely different behaviour of semiclassical solutions requiring a fine tuning of $A = 0$. Hence the stability argument in fact can be seen as an argument *against* the value $A = 0$.

Further note that the renormalisation constant A depends on the quantum field considered. In contrast the renormalisation constants have to be fixed for all fields and geometries at once. In particular the constants r_1 and r_2 can be chosen such that $A = 0$ only for one choice of matter content. If for example the electromagnetic field alone is considered, then one is indeed able to fix the renormalisation constants r_1 and r_2 such that $A = 0$ in the semiclassical Friedmann equation. However then one is not allowed to change r_1, r_2 in a different situation. In particular if one considers for example a situation where the electromagnetic field *and* the massless scalar field enters the semiclassical Friedmann equation then $A \neq 0$ for the same choice of values of r_1 and r_2. Hence, if A is supposed to vanish, then in *any* physical situation the vacuum fluctuations of *all* fields have to be considered since otherwise higher derivative terms would appear and change the qualitative behaviour of solutions radically.

A completely different point of view compared to Wald's fifth axiom is taken in [101, 102, 119]. Here the qualitative change at $A = 0$ is used to improve Starobinski's "graceful exit" scenario [123]. Starobinski uses the instability of the de Sitter equilibrium point for $A < 0$ to explain the ending of an inflationary phase of the universe (modelled by the de Sitter universe). The above authors propose a process which alters the value of A such that an initially positive value becomes negative and therefore performing a phase transition through the value $A = 0$. This turns the stable de Sitter equilibrium point into an unstable saddle point thus leading to an end of the inflationary phase.

Another perturbation which is not considered in the context of bifurcation theory is associated with the change of D when considering trajectories running through $H = 0$. The behaviour of these kind of solutions is summarised in proposition 5.2. Accordingly the value $D = 0$ causes the equilibrium points $-H_-$ and H_- to collide to form the unstable Minkowski equilibrium. For any value $D \neq 0$ this Minkowski solution ceases to exist. Furthermore the sign of D_A determines whether reversing solutions $\mathcal{R}$ exist. If $A > 0$ then reversing solutions only exist for $D > 0$ whereas if $A < 0$ then reversing solutions exist if and only if $D < 0$. For the associated vector field of equation (5.28) these perturbations correspond to adding terms proportional to H^{-1} in the second component. For smooth perturbations at $H = 0$ the behaviour near $H = 0$ is not affected.

Finally we allow B to vary in order to explore how the phase portrait of equation (5.28) changes. First of all note that there is no bifurcation in $\mathbb{H}^2_\pm$ for any value B according to proposition 5.3. Furthermore, the behaviour at $H = 0$ does not depend on the value of B. Hence, once A is chosen such that its sign does not change for various fields we conclude that the dynamical behaviour of an isotropic and homogeneous spacetime does not depend on whether we consider it to be filled with a massless scalar quantum field ($B = 3$) or an electromagnetic quantum field ($B = 186$) or with the Dirac field ($B = 33$) all three being in the conformal vacuum. However care has to be taken when the values of r_1 and r_2 are such that the sign of A changes when considering a different field. Then the dynamical behaviour for different quantum fields is not topologically equivalent. Therefore one has to be careful when modelling radiation by a massless scalar field and the Dirac field by a massive scalar field as was done for example in [57].

Special attention has to be put to the case $B = 0$, since here the number of equilibrium points changes as B changes sign. For the equilibrium points $H_\pm$ we find:

$$\lim_{B\to 0} H_+ = \infty, \tag{5.40}$$

for $C > 0$ and

$$\lim_{B\to 0} H_- = \sqrt{\frac{1}{3}\frac{D}{C}}. \tag{5.41}$$

Hence the equilibrium point H_+ is pushed to infinity and ceases to exist when $B < 0$. Therefore the number of equilibrium points change while varying B through zero resulting in topologically inequivalent phase portraits. However, as can be seen from Proposition 5.3, if one considers a sufficiently small compact region around H_-, then there is no bifurcation for $B = 0$. In [16] certain (classical) $f(R)$-theories are considered. On FLRW spacetimes these result in the dynamical equation

$$0 = \ddot{H}H - \frac{1}{2}\dot{H}^2 + 3\dot{H}H^2 - \lambda_1 H^2 + \lambda_2$$

for arbitrary constants λ_i, $i = 1, 2$. Obviously this equation corresponds to the semiclassical Einstein equation considered in this thesis with $B = 0$. Hence we may conclude that locally the semiclassical model also reproduces the $f(R)$-theories considered by Barrow and Ottewill [16]. However globally this is not the case since the number of equilibrium points changes.

The latter observation and the comparison of the semiclassical model with the classical model shows how powerful dynamical system analysis and in particular bifurcation theory can be. It not only singles out structurally unstable cases but also makes it possible to compare completely different matter models by the qualitative behaviour of their solutions. Let us give two further examples. First consider the generalised Chaplygin gas considered in [144]. In this case the classical Friedmann equation reads

$$H^2 = \kappa\left(M + Na^{-3(1+\alpha)}\right)^{\frac{1}{1+\alpha}},$$

which after differentiating becomes

$$0 = \ddot{H}H + 2\alpha\dot{H}^2 + 3(1+\alpha)\dot{H}H^2.$$

This corresponds to the case when $B = C = D = 0$ and therefore is in general not able to reproduce the semiclassical case which was found for a more general case already in [144]. Another example can be found in [94] where an imperfect fluid in FLRW spacetimes was considered. In this case the dynamical equation may be written as

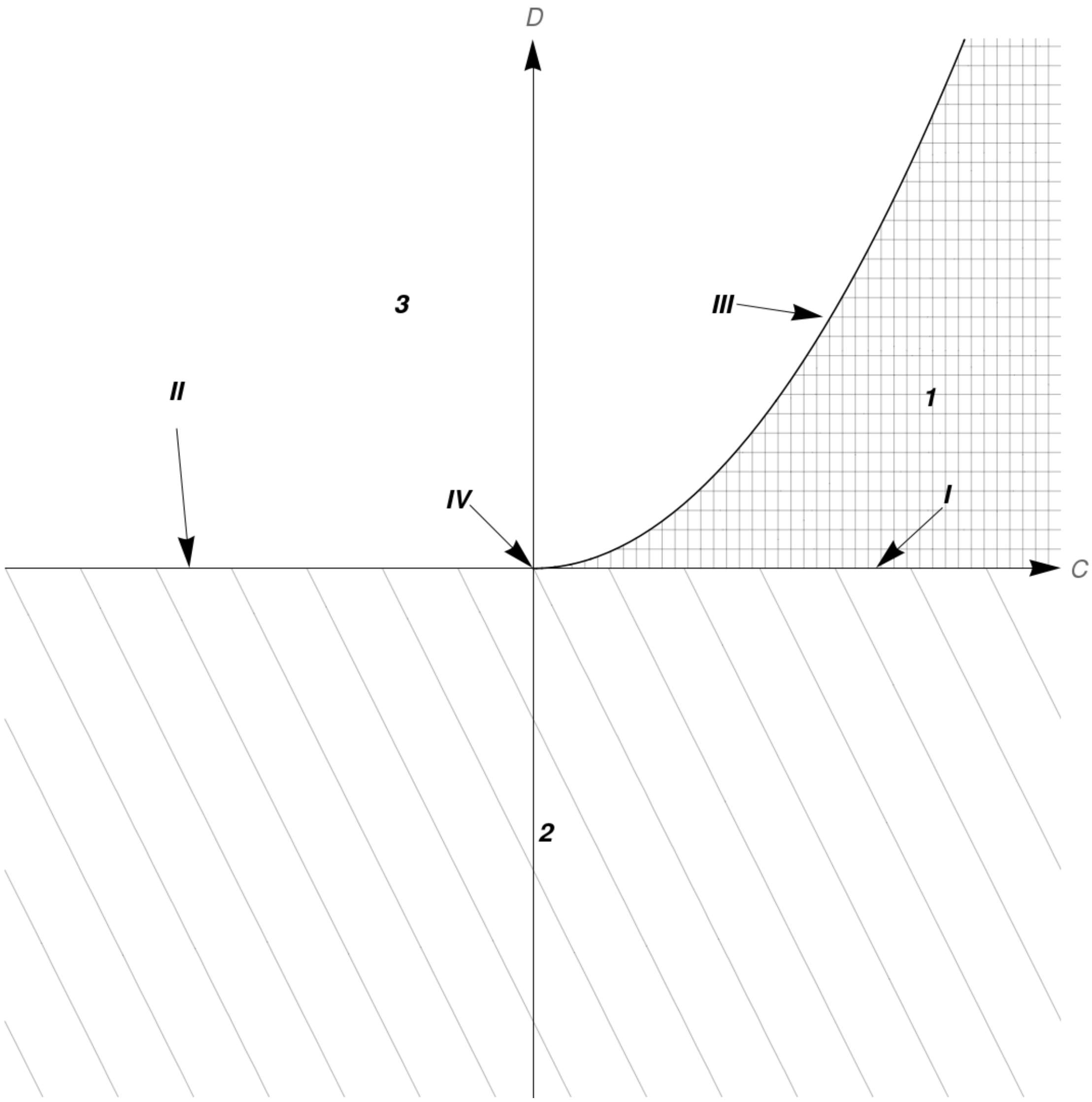
$$0 = \ddot{H}H + \zeta\dot{H}^2 + \eta\dot{H}H^2 + \lambda H^4,$$

which corresponds to the case $C = D = 0$.

Since we fixed our spacetime in the beginning to be of FLRW form, the perturbations as defined for the notion of structural stability do not allow perturbations of the spacetime itself. That is perturbations considered here can only depend upon the scale factor. Hence they cover only perturbations of the renormalised expectation value of the stress energy tensor but not perturbations of the geometry. Such perturbations could for example come from small masses appearing in the equations or small deviations from the conformal coupling or by including small effects of interaction into the stress-energy tensor or allowing other states than the conformal vacuum or by including further fields — provided all of these perturbations can be expressed as functions of H. Including perturbations of the geometry itself can complicate the problem tremendously. For example an anisotropic universe filled with a quantum scalar field would enlarge the dimension of the phase space of the isotropic case by a factor 6 which complicates particularly the discussion of structural stability. However, the discussion of Lyapunov stable equilibrium points and other qualitative features of solutions in anisotropic or non-homogeneous spacetimes is still an interesting task (for the classical analogue see [135] and references therein).

We close this section by summarising all the critical values of the renormalisation constants A, C and D changing the topology of the phase portraits found above. These may be visualised in the 3 dimensional parameter space with axis A, C and D. However there is only one change of the topology for A, namely at $A = 0$. Therefore it suffices to draw a 2 dimensional slice as in Figure 5.2. The three open domains 1-3 are locally structurally stable except for the case $A = 0$, where a small change of A inverts the stability behaviour of the equilibrium points, i.e. a saddle point is converted into a stable focus and vice versa. Whenever a renormalisation constant changes its value and crosses one of the sets $I - III$ a change of the topology class of the vector field occurs.

Figure 5.2: Bifurcation diagram in all three cases for A. Here the representative value $B = 3$ for the scalar field is chosen for visualisation

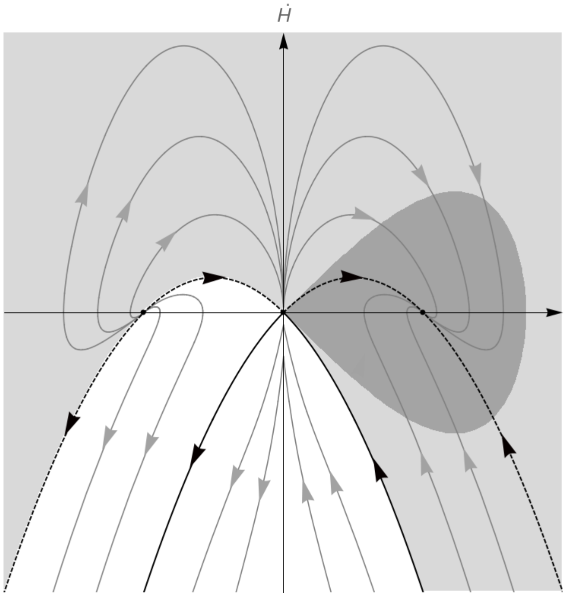


Finally we will draw the phase portraits in the $(H, \dot{H})$-plane of the individual cases shown in Figure 5.2 for $A \neq 0$. In the following list an upper $+$ indicates that $A > 0$ and an upper $-$ indicates $A < 0$. The Minkowski equilibrium $\mathcal{M}$, if it exists, is always unstable. Also it would be possible for trajectories to evolve from a state near the contracting de Sitter equilibrium point towards the expanding de Sitter equilibrium point in its asymptotic future in the cases 1^+, I^+ and III^+. In some cases we find a class of solutions, all running through the same points, where they change sign. There are three cases: either the solutions change once, twice or three times their sign. The first case occurs for $I^\pm$ and

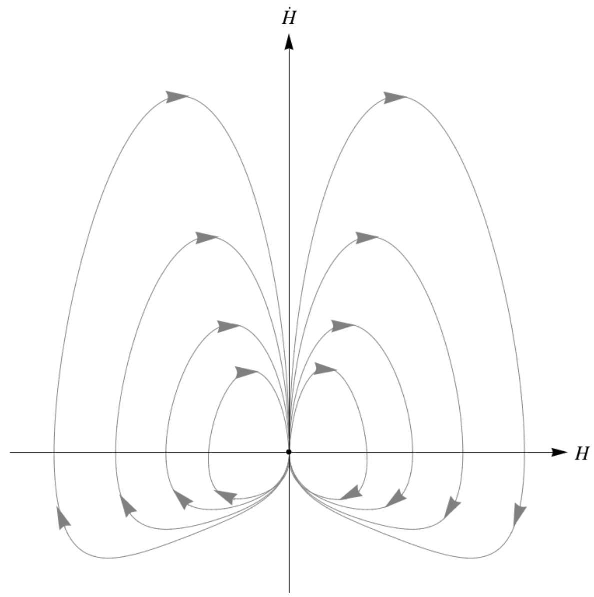
$II^{\pm}$. In all four cases these solutions run through the Minkowski equilibrium point.

The second case describes expanding solutions that become contracting for some period of time, change their sign a second time to become again an expanding solution and then either asymptotically reach a de Sitter equilibrium (in the cases 1^+ and III^+) or go to infinity (for 2^-). Also the time reversal of such scenarios is possible. See Figure 5.7 for the special case $A = \frac{4B}{27}$, $0 < D < \frac{9}{4}\frac{C^2}{B}$. For this case we can find the general solution of the semiclassical equation [33, 34, 36, 111]. The phase portrait lies within the equivalence class 1^+. The plot (a) in figure 5.7 represents a trajectory changing its sign twice. If we interpret H^2 as the energy density of the quantum field then along such a solution curve all the energy of the quantum field is used to drive the expansion of the universe. When the energy density reaches zero the universe begins to contract and the quantum field thereby gains some amount of energy back until it reaches zero a second time. From there on the universe expands again leading to a final de Sitter stage while the energy density of the quantum field reaches its final constant value. Note that this scenarios occurs for the conformal vacuum of the quantum field. Hence, similar to the Casimir effect, the quantum field is able to transfer energy to its surrounding (here the gravitational field, in the Casimir effect the conducting plates) even in the vacuum state causing the dynamical behaviour of the system. However this loss of energy is always bounded, i.e. one cannot extract an arbitrarily large amount of energy from the quantum field.

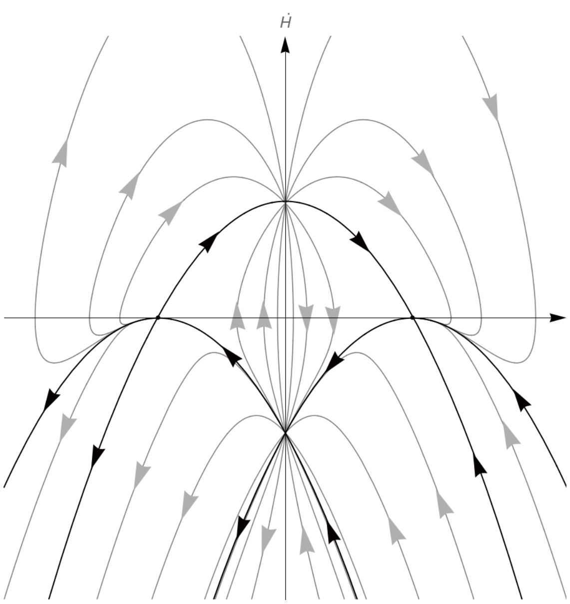
The third case occurs for 3^+, where expanding solutions of the last kind intersect themselves again after changing their sign twice instead of converging or diverging. This intersection occurs at the first point of transition after which the solution ultimately becomes a contracting solution going to $-\infty$ as $t \to \infty$. The regions of attraction for the asymptotically stable de Sitter equilibrium points are shaded in light grey and the estimate for the region of attraction by means of the Lyapunov function is shaded in dark grey.



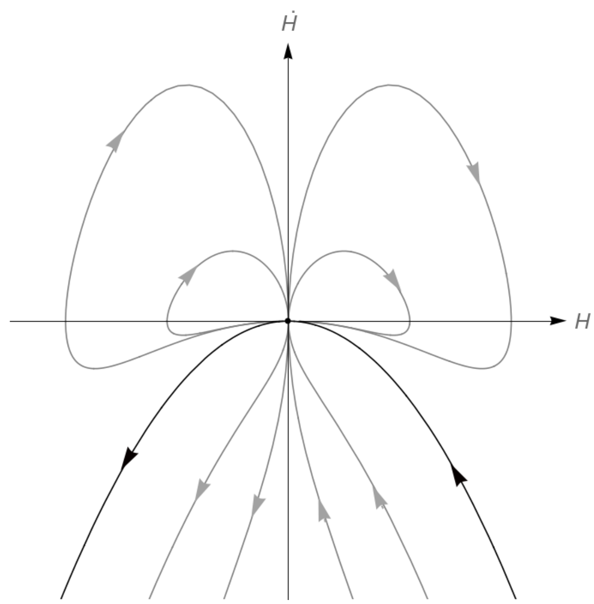
(a) Typical phase portrait of the region I^+ for the values $4B = 27A$, $4C = 9A$, $D = 0$.



(b) Typical phase portrait of the region II^+ for the values $4B = 27A$, $C = 3A$, $D = 0$.

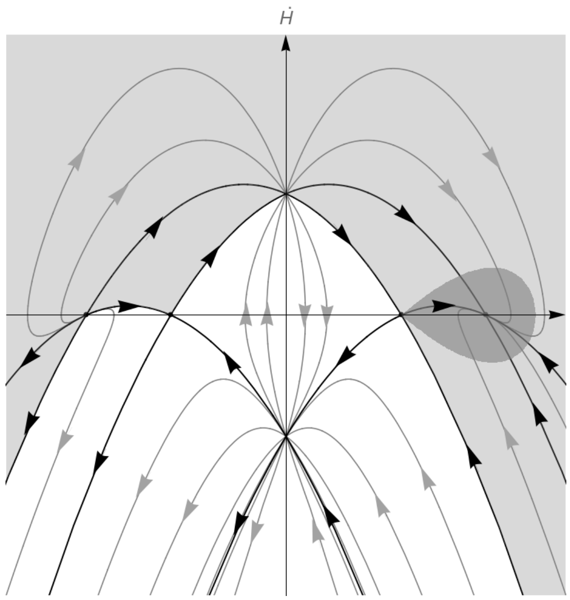


(c) Typical phase portrait of the region III^+ for the values $4B = 27A$, $\sqrt{6}C = 9A$, $2D = 9A$.

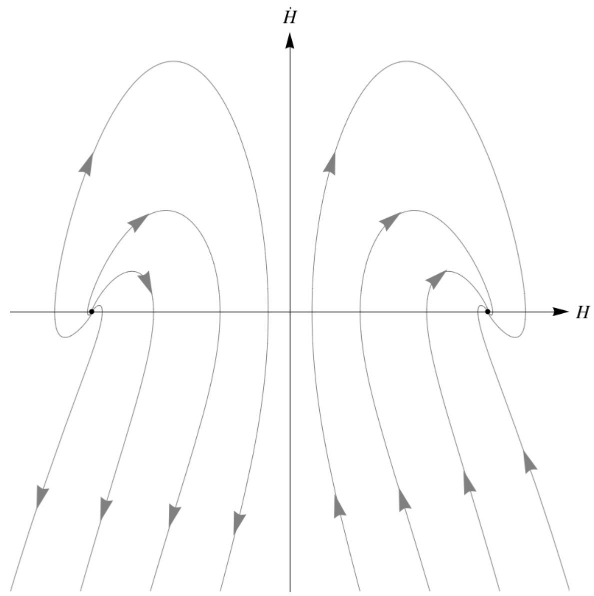


(d) Typical phase portrait of the region IV^+ for the values $4B = 27A$, $C = D = 0$.

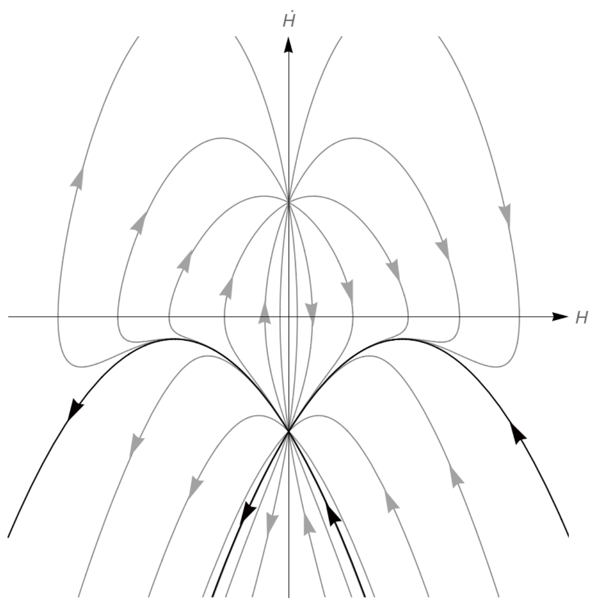
Figure 5.3: Bifurcations of C and D for arbitrary $A > 0$.



(a) Typical phase portrait of the region 1^+ for the values $4B = 27A$, $4C = 3A$, $64D = 9A$.

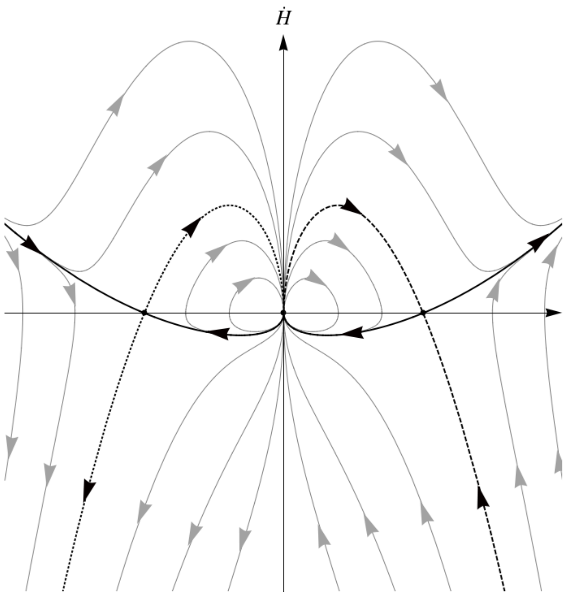


(b) Typical phase portrait of the region 2^+ for the values $4B = 27A$, $C = 3A$, $D = -9A$.

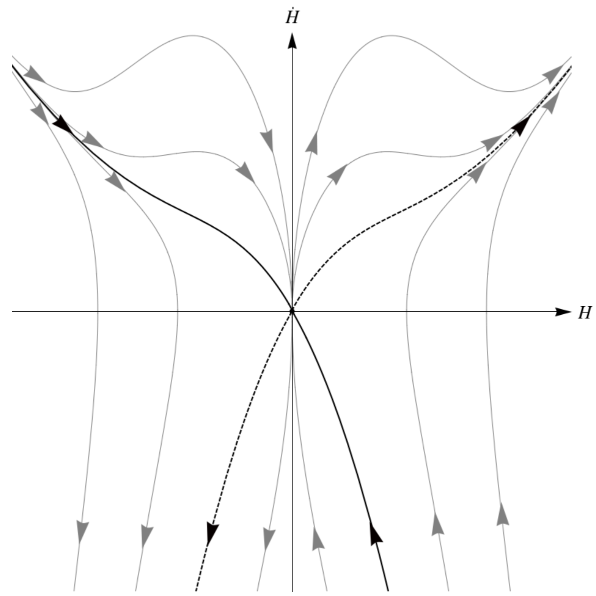


(c) Typical phase portrait of the region 3^+ for the values $4B = 27A$, $4C = 9A$, $2D = 9A$.

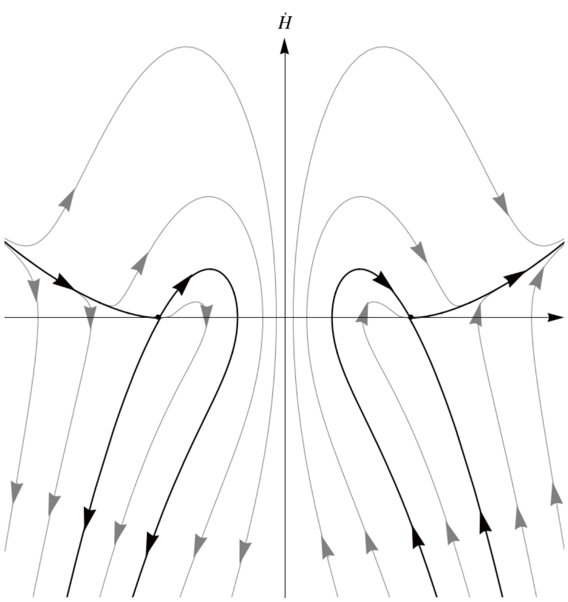
Figure 5.4: Structurally stable cases of C and D for arbitrary $A > 0$.



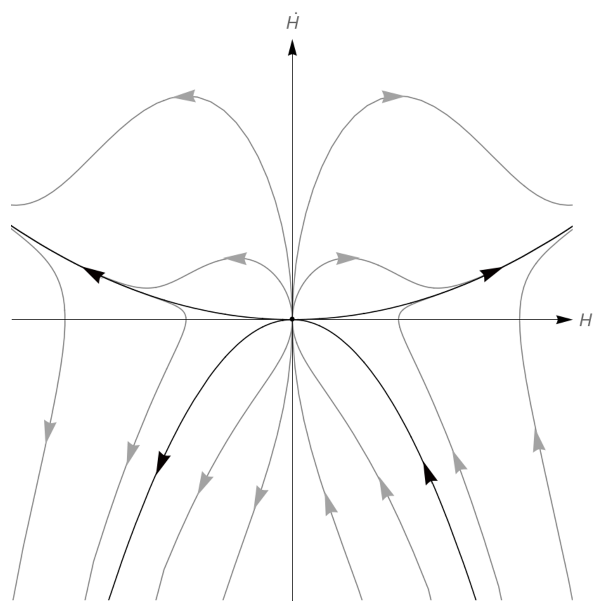
(a) Typical phase portrait of the region I^- for the values $4B = 27A$, $4C = 9A$, $D = 0$



(b) Typical phase portrait of the region II^- for the values $4B = 27A$, $C = 3A$, $D = 0$

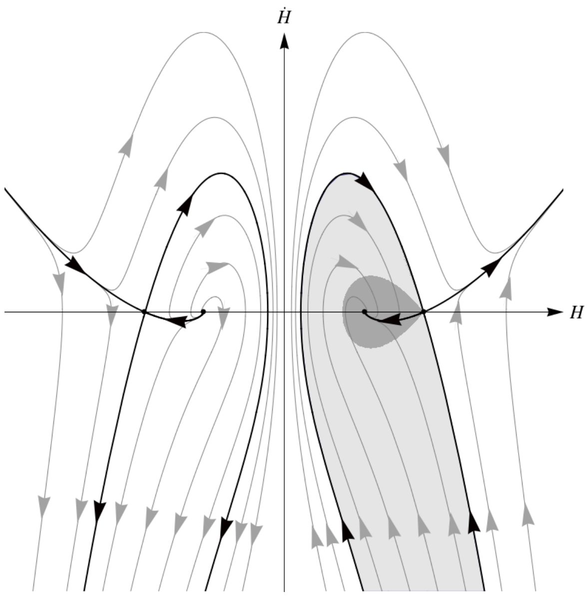


(c) Typical phase portrait of the region III^- for the values $4B = 27A$, $\sqrt{6}C = 9A$, $2D = 9A$

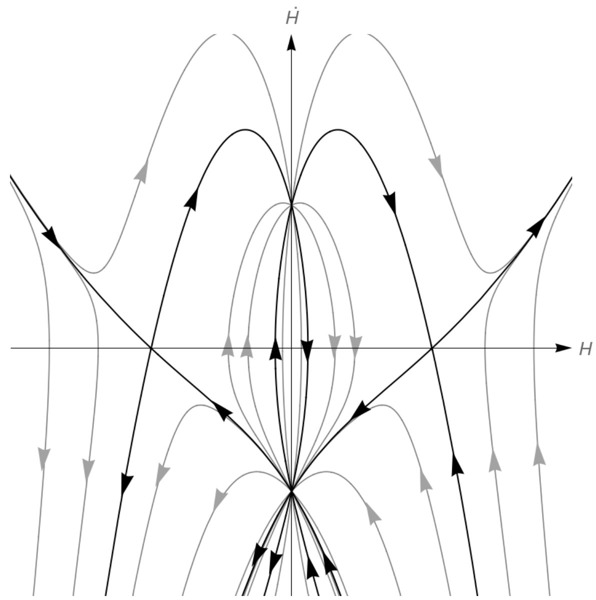


(d) Typical phase portrait of the region IV^- for the values $B = 7A$ and $C = D = 0$

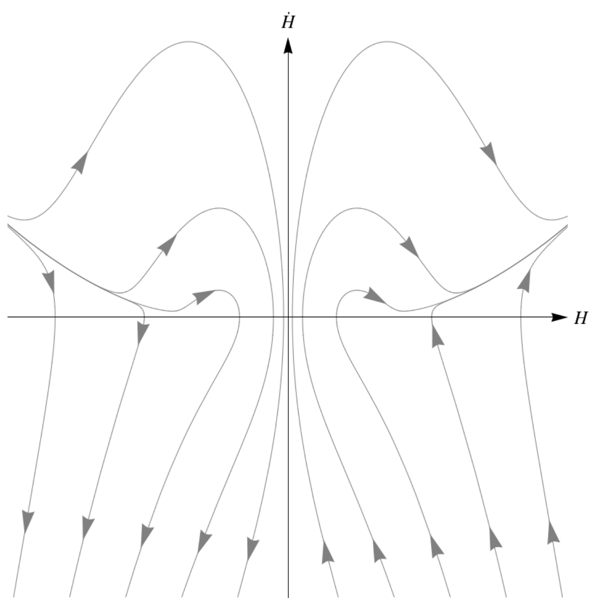
Figure 5.5: Bifurcations of C and D for arbitrary $A < 0$.



(a) Typical phase portrait of the region 1^- for the values $4B = 27A$, $4C = 3A$, $64D = 9A$



(b) Typical phase portrait of the region 2^- for the values $4B = 27A$, $C = 3A$, $D = 9A$



(c) Typical phase portrait of the region 3^- for the values $4B = 27A$, $4C = 9A$, $4D = 9A$

Figure 5.6: Structurally stable cases of C and D for arbitrary $A < 0$.

5.1.4 Solutions for Quantum Fields in the Conformal Vacuum

When $A = \frac{4}{27}B$ the the semiclassical Friedmann vacuum equation (5.17) can be solved exactly [4, 33–36, 111]. Following [36] we may differentiate equation (5.17) with respect to time t and substitute $H = \frac{2}{3}\frac{\dot{s}}{s}$, which transforms the resulting equation into a fourth order linear differential equation

$$\dot{z} = Az,$$

where $z := (s, \dot{s}, \ddot{s}, \dddot{s})$ and

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{81}{16}\frac{D}{B} & 0 & \frac{27}{4}\frac{C}{B} & 0 \end{pmatrix},$$

which can be solved in terms of the eigenvalues $\lambda_{1/2} = \frac{3}{2}\sqrt{\frac{3}{2}\frac{C}{B}}\sqrt{1 \pm \sqrt{1 - \frac{4}{9}\frac{BD}{C^2}}}$, $\lambda_{2/3} = -\lambda_{1/2}$ and associated eigenvectors v_1, v_2, v_3, v_4 of the matrix A. Then the general solution of the linear equation is

$$z(t) = \sum_{i=1}^{4} c_i v_i e^{\lambda_i t},$$

where c_i are constants of integration.
Another way to find the solutions is as follows. First define new variables

$$u_\pm := \frac{\dot{H}}{H} + \frac{3}{2}H + \frac{1}{3}\left(\omega_\pm^2 - \omega^2\right) H^{-1}, \tag{5.42}$$

where $\omega_\pm^2 := \omega^2 \pm \frac{9}{2}\sqrt{\frac{D}{B}}$ and $\omega^2 = \frac{27}{4}\frac{C}{B}$. Equation (5.17) then transforms into the two equations

$$0 = 2\dot{u}_\pm + u_\pm^2 - \omega_\pm^2 - \frac{9}{4}\left(1 - \frac{4}{27}\frac{B}{A}\right) H^2. \tag{5.43}$$

Note that using equations (5.42) and (5.43) the original equation (5.17) can be transformed into a dynamical system which is a mixture of a gradient and a Hamiltonian system, i.e.

$$\begin{pmatrix} \dot{u}_+ \\ \dot{u}_- \end{pmatrix} = -\nabla V(u_+, u_-) + \mathscr{X}_{\mathcal{H}}(u_+, u_-) \tag{5.44}$$

where

$$V(u_+, u_-) := \frac{1}{6}(u_+^3 + u_-^3) - \frac{1}{2}(\omega_+^2 u_+ + \omega_-^2 u_-^2)$$

and the Hamiltonian vector field $\mathscr{X}_{\mathcal{H}} := \left(\frac{\partial \mathcal{H}}{\partial u_-}, -\frac{\partial \mathcal{H}}{u_+}\right)^T$ is obtained from the Hamiltonian

$$\mathcal{H}(u_+, u_-) = \frac{1}{8}\left(1 - \frac{4}{27}\frac{B}{A}\right)\frac{\left(\omega_+^2 - \omega_-^2\right)^2}{u_+ - u_-}.$$

The pure gradient system, i.e. when $A = \frac{4}{27}B$ decouples and can be solved explicitly. In this case equations (5.43) turn into the separable differential equations

$$0 = 2\dot{u}_\pm + u_\pm^2 - \omega_\pm^2, \tag{5.45}$$

which is an example of the logistic equation with a harvesting term known from certain population growth models [24]. The equilibrium points of equations (5.45) are $u_+ = \pm\omega_+$ and $u_- = \pm\omega_-$. It can be easily checked that these equilibrium points correspond to the stable, unstable, centre manifold, respectively of the original equation (5.17).

Let us define

$$W_\pm(t) := u_{\pm 0} \sin\left\{\frac{\sqrt{-\omega_\pm^2}}{2} t\right\} + \sqrt{-\omega_\pm^2} \cos\left\{\frac{\sqrt{-\omega_\pm^2}}{2} t\right\}, \tag{5.46}$$

and

$$\varphi_\pm := \frac{\pi}{\sqrt{-\omega_\pm^2}}. \tag{5.47}$$

Note that $W_\pm(t)$ is periodic, i.e.

$$W_\pm(t + 4n\varphi_\pm) = W_\pm(t), \tag{5.48}$$

for $n \in \mathbb{Z}$ and $W_\pm(t_n) = 0$ whenever

$$t_n = -\frac{2}{\sqrt{-\omega_\pm^2}} \arctan\left\{\frac{\sqrt{-\omega_\pm^2}}{u_{\pm 0}}\right\} - 2n\varphi, \tag{5.49}$$

for $n \in \mathbb{Z}$. Furthermore the identity

$$(W_\pm(t))^2 + (W_\pm(t + \varphi))^2 = u_{\pm 0}^2 - \omega_\pm^2 \tag{5.50}$$

holds.

The general solution to equations (5.45) depends on the sign of $\omega_\pm^2$ and reads

$$\begin{aligned}
\boldsymbol{\omega_\pm^2 > 0:} \quad & u_\pm(t) &&= \omega_\pm \tanh\left\{\tfrac{\omega_\pm}{2} t + \operatorname{arctanh}\left\{\tfrac{u_{\pm 0}}{\omega_\pm}\right\}\right\} \\
& &&= -\omega_\pm \tfrac{1+\kappa_\pm e^{\omega_\pm t}}{1-\kappa_\pm e^{\omega_\pm t}} \\
\boldsymbol{\omega_\pm^2 = 0:} \quad & u_\pm(t) &&= \tfrac{u_{\pm 0}}{1+\frac{u_{\pm 0}}{2} t} \\
\boldsymbol{\omega_\pm^2 < 0:} \quad & u_\pm(t) &&= \sqrt{-\omega_\pm^2} \tan\left\{-\tfrac{\sqrt{-\omega_\pm^2}}{2} t + \arctan\left\{\tfrac{u_{\pm 0}}{\sqrt{-\omega_\pm^2}}\right\}\right\} \\
& &&= \sqrt{-\omega_\pm^2}\, \tfrac{W_\pm(t+\varphi_\pm)}{W_\pm(t)},
\end{aligned} \tag{5.51}$$

for initial values

$$u_{\pm}(0) = u_{\pm 0} \equiv \frac{\dot{H}_0}{H_0} + \frac{3}{2}H_0 + \frac{1}{3}\left(\omega_{\pm}^2 - \omega^2\right)H_0^{-1}$$

where $H_0 := H(0)$ and $\dot{H}_0 := \dot{H}(0)$ and

$$\kappa_{\pm} := \frac{u_{\pm 0} + \omega_{\pm}}{u_{\pm 0} - \omega_{\pm}}.$$

Equations (5.51) together with the substitution (5.42) provide time-dependent first integrals of the semiclassical Friedmann equation (5.17) for each case of $A = \frac{4}{27}B$. Using equation (5.42) these can be written as a Riccati type equation [109]:

$$\dot{H} = -\frac{3}{2}H^2 + u_{\pm}(t)H - \frac{1}{3}\left(\omega_{\pm}^2 - \omega^2\right), \tag{5.52}$$

whose solutions yield the final solutions to the semiclassical Friedmann equation (5.17). For the case $D = 0$ we have $u_+(t) = u_-(t) \equiv u(t)$ and $\omega_+ = \omega_- = \omega$ and equation (5.52) simplifies even further to a Bernoulli type differential equation [109] where the constant term in equation (5.52) vanishes. The general solution of the Bernoulli differential equation is

$$H(t) = \frac{F(t)}{C_0 + \frac{3}{2}\int_0^t F(s)ds}, \tag{5.53}$$

where

$$F(t) := \exp\left\{\int_0^t u(r)dr\right\}, \tag{5.54}$$

and C_0 is a constant of integration depending on the initial conditions.
If $D > 0$ we don't have to solve the Riccati equation (5.52) since in this case we have two independent first integrals (5.51) — one for ω_+ and one for ω_-. The solution $H(t)$ of the semiclassical Friedmann equation may then be written as a combination of two first integrals such that $\dot{H}$ is eliminated, i.e.:

$$H(t) = \frac{1}{3}\frac{\omega_+^2 - \omega_-^2}{u_+(t) - u_-(t)}. \tag{5.55}$$

Equations (5.53) and (5.55) provide the solutions in the cases *I*-*IV* and 1 and 3 of Figure 5.2 for positive $A = \frac{4}{27}B$.

Case 1

In this case both, ω_+^2 and ω_-^2 are positive. Using the first equation of (5.51) we may eliminate the time t to obtain one time-independent first integral, comparable to the first integral of the generalised Friedmann equation (4.22) (for $8q < p$)

$$I(H,\dot{H}) = \left(\frac{u_+(H,\dot{H}) + \omega_+}{u_+(H,\dot{H}) - \omega_+}\right)^{\omega_-}\left(\frac{u_-(H,\dot{H}) + \omega_-}{u_-(H,\dot{H}) - \omega_-}\right)^{-\omega_+}, \tag{5.56}$$

where $u_{\pm}(H, \dot{H})$ is given by equation (5.42). With equation (5.56) the trajectories can be drawn directly (see Figure 5.7). On the other hand we may use the first equation of (5.51) to eliminate $\dot{H}$ to obtain the solution

$$H(t) = -\frac{1}{3}\frac{\left(\omega_+^2 - \omega_-^2\right)\left(1 - \kappa_- e^{\omega_- t}\right)\left(1 - \kappa_+ e^{\omega_+ t}\right)}{\omega_+\left(1 - \kappa_- e^{\omega_- t}\right)\left(1 + \kappa_+ e^{\omega_+ t}\right) - \omega_-\left(1 + \kappa_- e^{\omega_- t}\right)\left(1 - \kappa_+ e^{\omega_+ t}\right)}. \qquad (5.57)$$

Depending on the initial conditions the solutions (5.57) can behave very different. The qualitatively diverse cases are depicted in Figure 5.7 together with their according trajectories in the phase space. These have been discussed in previous subsection.

Case *III*

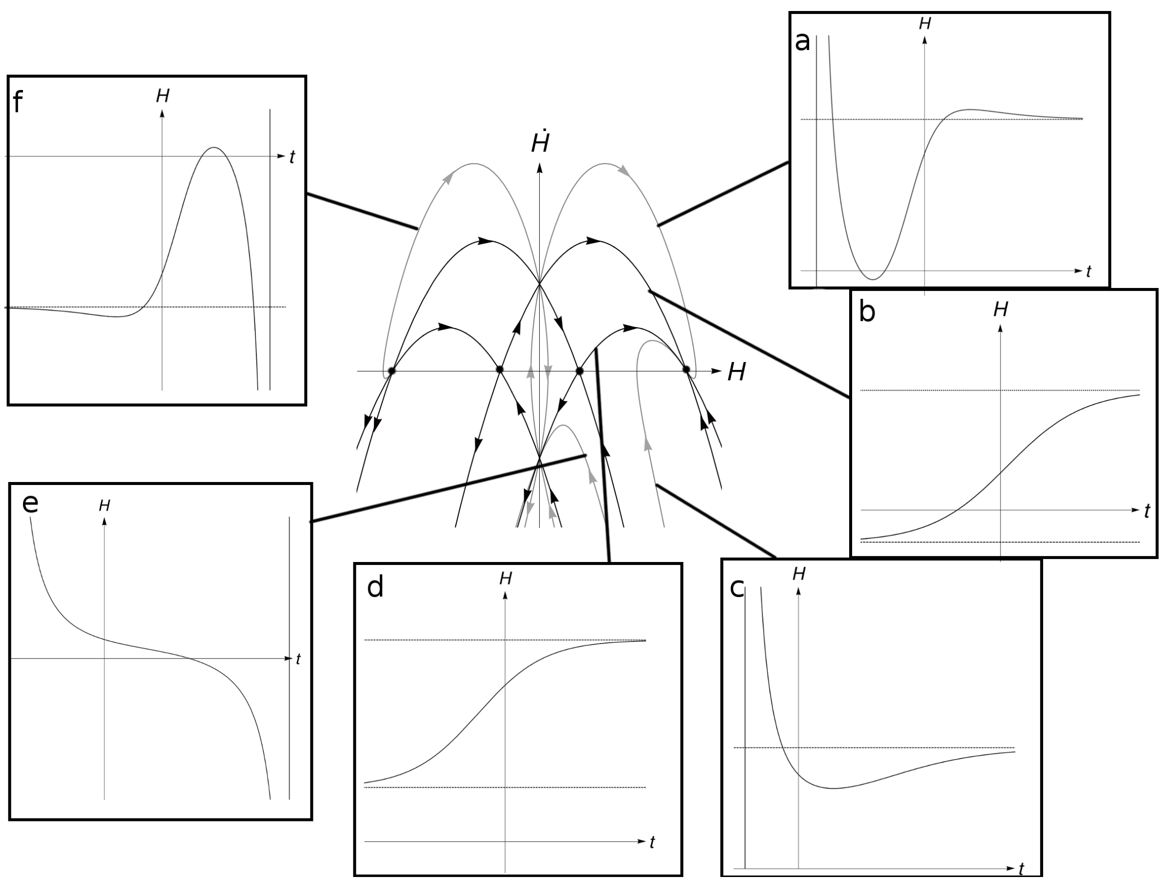


Figure 5.7: Some trajectories in the case 1 for the values $27A = 4B$, $C = A$, $12D = A$ (i.e. $\omega_+ = \sqrt{1/2}$ and $\omega_- = \sqrt{3/2}$). Initial data are chosen such that for trajectory (a) $u_+ = 61/60$ and $u_- = 101/60$, (b) $u_+ = 5/3 + \sqrt{1/2}$ and $u_- = \sqrt{1/2}$, (c) $u_+ = 41/60$ and $u_- = 1/60$, (d) $u_+ = -2/3 + \sqrt{3/2}$ and $u_- = \sqrt{3/2}$, (e) $u_+ = -41/30$ and $u_- = 3/10$ and (f) $u_+ = -61/60$ and $u_- = 101/60$.

Here $\omega_+^2 > 0$ and $\omega_-^2 = 0$. The two first integrals are therefore given by the first and the last equation of (5.51). Eliminating t gives the time-independent first integral

$$I(H,\dot{H}) = \frac{u_+(H,\dot{H}) + \omega_+}{u_+(H,\dot{H}) - \omega_+} \exp\left\{-2\frac{\omega_+}{u_-(H,\dot{H})}\right\}, \tag{5.58}$$

which may be compared to the case $8q = p$ of the first integrals (4.22) of the generalised Friedmann equation. Eliminating $\dot{H}$ and solving for H gives

$$H(t) = -\frac{2}{3}\frac{\omega^2\left(\frac{u_{-0}}{2}t + 1\right)\left(1 - \kappa_+ e^{\sqrt{2}\omega t}\right)}{\sqrt{2}\omega\left(\frac{u_{-0}}{2}t + 1\right)\left(1 + \kappa_+ e^{\sqrt{2}\omega t}\right) + u_{-0}\left(1 - \kappa_+ e^{\sqrt{2}\omega t}\right)}. \tag{5.59}$$

Case 3

Here ω_-^2 becomes negative while ω_+^2 remains positive. By eliminating the time in the first and the last equation of (5.51) we obtain the time-independent first integral

$$\sqrt{-\omega_-^2}\,\frac{u_-(H,\dot{H}) - u_{-0}}{u_{-0}u_-(H,\dot{H}) - \omega_-^2} = \tan\left\{-\frac{1}{2}\frac{\sqrt{-\omega_-^2}}{\omega_+}\ln\left\{\frac{1}{\kappa_+}\frac{u_+(H,\dot{H}) + \omega_+}{u_+ - \omega_+}\right\}\right\} \tag{5.60}$$

Eliminating $\dot{H}$ and solving for H yields the solution

$$H(t) = -\frac{1}{3}\frac{\left(\omega_+^2 - \omega_-^2\right)\left(1 - \kappa_+ e^{\omega_+ t}\right)W_-(t)}{\omega_+\left(1 + \kappa_+ e^{\omega_+ t}\right)W_-(t) + \sqrt{-\omega_-^2}\left(1 - \kappa_+ e^{\omega_+ t}\right)W_-(t+\varphi)}, \tag{5.61}$$

where $W_-(t)$ and φ_- are given by equations (5.46) and (5.47).

Case *I*

In this case we have $\omega_+^2 = \omega_-^2 \equiv \omega^2 > 0$. The solution is obtained by integrating the first equation of (5.51) twice and using then equation (5.53). We have:

$$H(t) = \frac{\omega\left(1 - \kappa e^{\omega t}\right)^2}{4\left(C_0 + \frac{\omega}{2}t\right)\kappa e^{\omega t} + \left(1 + \kappa e^{\omega t}\right)\left(1 - \kappa e^{\omega t}\right)}, \tag{5.62}$$

with

$$C_0 = \frac{\omega}{H_0}\left(\frac{u_0 H_0 - \omega^2}{u_0^2 - \omega^2}\right). \tag{5.63}$$

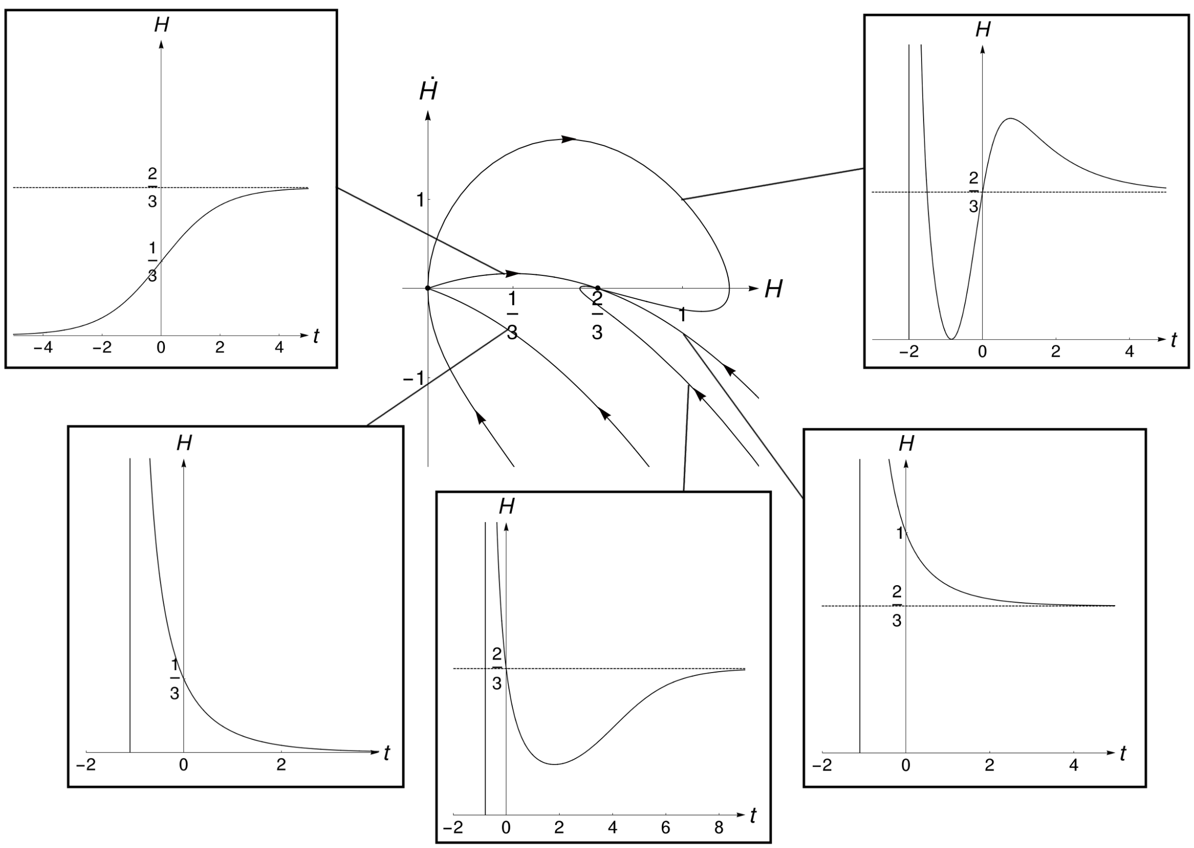
From this solution and the first equation in (5.51) we obtain an algebraic expression for the trajectories in the phase space corresponding to a time-independent first integral:

$$I(H,\dot{H}) = \frac{u(H,\dot{H}) + \omega}{u(H,\dot{H}) - \omega}\exp\left\{-2\frac{\omega}{H}\frac{u(H,\dot{H})H - \omega^2}{\left(u(H,\dot{H})\right)^2 - \omega^2}\right\}. \tag{5.64}$$

Positive solutions together with their associated trajectories are shown in Figure 5.8.

Case *II*

Figure 5.8: Phase portrait of representative solutions to the semiclassical Friedmann equation with $A = \frac{4}{27}B$ and $D = 0$. C is chosen here such that $\omega = 1$. The equilibrium points are the Minkowski equilibrium point $(0,0)$, and the two de Sitter equilibrium points $(\pm\frac{2}{3}, 0)$. Here only the positive half $H > 0$ is shown.



Here $\omega_+ = \omega_- \equiv \omega < 0$. Again we integrate the third equation of (5.51) twice. By equation (5.53) the solution in this case reads

$$H(t) = \frac{\sqrt{-\omega^2}(W(t))^2}{\sqrt{-\omega^2}\bar{C}_0 + \frac{3}{4}\sqrt{-\omega^2}\,(u_0^2 - \omega^2)\,t - \frac{3}{2}W(t)W(t+\varphi)} \tag{5.65}$$

with

$$\bar{C}_0 = \frac{\frac{3}{2}u_0 - \omega^2}{H_0}. \tag{5.66}$$

A time-independent first integral of this case is obtained from the solution and the third equation of (5.51):

$$I(H,\dot{H}) = \frac{\sqrt{-\omega^2}}{3H}\frac{\frac{3}{2}u(H,\dot{H}) - \omega^2}{(u(H,\dot{H}))^2 - \omega^2} + \arctan\left\{\frac{u(H,\dot{H})}{\sqrt{-\omega^2}}\right\} \tag{5.67}$$

Due to the periodicity (5.48) of $W(t)$ the solutions (5.65) fulfil

$$H(t + 4\varphi) = \frac{(W(t))^2}{3\pi\left(u_0^2 - \omega^2\right)H(t) + (W(t))^2}H(t). \tag{5.68}$$

If $H(t) > 0$ the fraction in front of $H(t)$ becomes smaller than 1. On the other hand if $H(t) < 0$ this fraction becomes greater than 1. Hence, after a period of 4φ, trajectories starting at some initial point $H_0 > 0$, $\dot{H}_0 = 0$ at $t = 0$ will be at the point $H(4\varphi) < H_0$, $\dot{H}_0 = 0$ and therefore trajectories starting in $\mathbb{H}^2_\pm$ will move towards the origin of the phase space. Due to Equation (5.49), $H(t)$ becomes zero after each time t_n, $n \in \mathbb{Z}$ and therefore runs infinitely often through the Minkowski equilibrium point. The solution is plotted for representative initial conditions $H_0 > 0$ and $\dot{H}_0 = 0$ in Figure 5.9. In phase space the trajectories having initial data $H > 0$ would form spirals running through the Minkowski point after each time t_n, given by equation (5.49). The trajectory associated with the solution drawn in Figure 5.9 would thus look as the trajectory depicted in Figure 5.10.

Case *IV*

Here $\omega_+ = \omega_- \equiv \omega = 0$. Using equation (5.53) the solution is

$$H(t) = \frac{u_0\left(1 + \frac{u_0}{2}t\right)^2}{\tilde{C}_0 + \left(1 + \frac{u_0}{2}t\right)^3} \tag{5.69}$$

where

$$\tilde{C}_0 = \frac{u_0}{H_0} - 1. \tag{5.70}$$

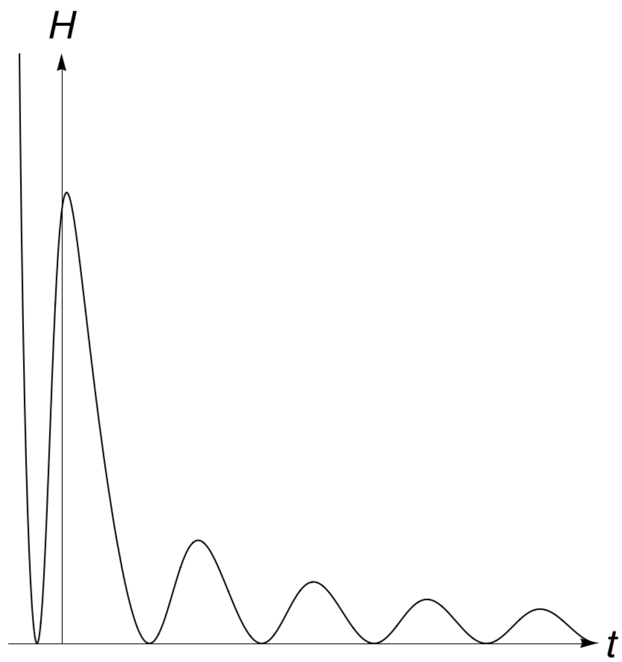
A time-independent first integral of this case is obtained from the solution and the second equation of (5.51):

$$I(H,\dot{H}) = \frac{\left(u(H,\dot{H})\right)^3 H}{u(H,\dot{H}) - H}. \tag{5.71}$$

In this particular case ($C = D = 0$) we may also find a solution for all values of A. Consider the semiclassical Friedmann equation (5.17) for the case $C = D = 0$, i.e.

$$0 = \ddot{H}H - \frac{1}{2}\dot{H}^2 + p\dot{H}H^2 + qH^4, \tag{5.72}$$

Figure 5.9: Typical solution $H(t)$ in the case $27A = 4B$, $C = -A$ and $D = 0$ for initial values such that $\bar{C}_0 = 7/2$ and $u_0 = 6/5$



where we used the notation of the generalised Friedmann equation (4.17) (with vanishing cosmological constant $r = 0$) to abbreviate the constants, i.e. $p = 3$ and $q = B/(6A)$. Similar to the case of the generalised Friedmann equation (4.17) we may substitute

$$v := \frac{q}{p}\frac{H^2}{\dot{H}} \tag{5.73}$$

to transform the semiclassical equation into the Abel equation

$$\frac{dv}{dH} = \frac{p^2}{q}\frac{v}{H}\left(v^2 + v + \frac{3}{2}\frac{q}{p}\right). \tag{5.74}$$

The solution to this equation was given in Section 4.4. Accordingly, H parametrised by the scale factor $\tilde{a} = a/a_0$ is

$$\boldsymbol{6q < p^2:}\ (\tilde{a}H)^{\frac{3}{2}} = c_1\tilde{a}^{\frac{3}{2}\rho} + c_2\tilde{a}^{-\frac{3}{2}\rho} \tag{5.75}$$

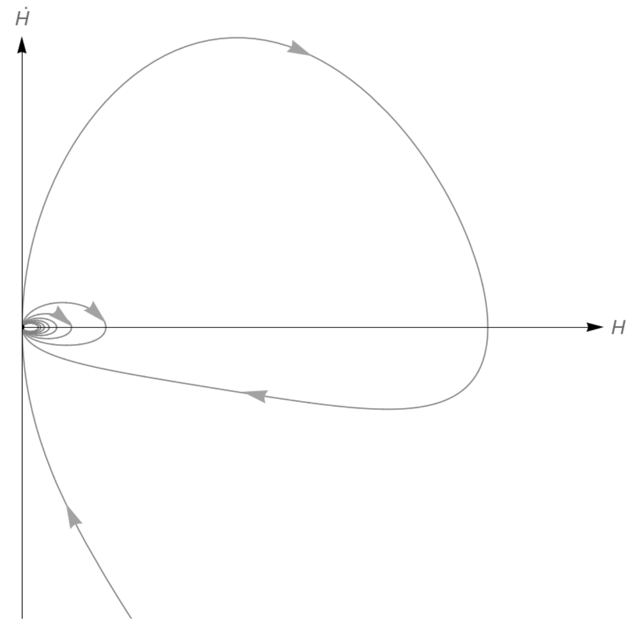
$$\boldsymbol{6q = p^2:}\ (\tilde{a}H)^{\frac{3}{2}} = c_1\ln\{\tilde{a}^{-\frac{3}{2}}\} + c_2 \tag{5.76}$$

$$\boldsymbol{6q > p^2:}\ (\tilde{a}H)^{\frac{3}{2}} = c_1\cos\left\{\delta\ln\{\tilde{a}^{\frac{3}{2}}\}\right\} + c_2\sin\left\{\delta\ln\{\tilde{a}^{\frac{3}{2}}\}\right\}, \tag{5.77}$$

where c_1 and c_2 are constants of integration.

Case 2

Figure 5.10: Typical trajectory for the case when $27A = 4B$, $C = -A$ and $D = 0$ with initial conditions $(H_0, \dot{H}_0) = (0.8, 0)$.



This case has to be investigated in a different manner since here

$$\omega_{\pm}^2 = \omega^2 + i\left(\pm\frac{9}{2}\sqrt{-\frac{D}{B}}\right), \tag{5.78}$$

with $i := \sqrt{-1}$ being the imaginary unit, is always imaginary for $D < 0$. In particular $\omega_{\pm}^2$ will never become zero. Hence the complex solution to Equation (5.45) reads

$$\begin{aligned} u_{\pm}(t) &= \omega_{\pm}\tanh\left\{\frac{\omega_{\pm}}{2}t + \operatorname{arctanh}\left\{\frac{u_{\pm 0}}{\omega_{\pm}}\right\}\right\} \\ &= \omega_{\pm}\frac{\omega_{\pm}\tanh\left\{\frac{\omega_{\pm}}{2}t\right\} + u_{\pm 0}}{\omega_{\pm} + u_{\pm 0}\tanh\left\{\frac{\omega_{\pm}}{2}t\right\}}, \end{aligned} \tag{5.79}$$

where $\omega_{\pm}$, $u_{\pm 0} \in \mathbb{C}$. For convenience we set

$$\omega_{\pm} = \omega_1^{\pm} + i\omega_2^{\pm},$$

then

$$\left(\omega_1^{\pm}\right)^2 - \left(\omega_2^{\pm}\right)^2 = \omega^2 \tag{5.80}$$

$$\omega_1^{\pm}\omega_2^{\pm} = \pm\frac{9}{4}\sqrt{-\frac{D}{B}}, \tag{5.81}$$

and

$$u_{\pm} = u + \frac{2}{3} i \omega_1^{\pm} \omega_2^{\pm} H^{-1}, \tag{5.82}$$

where $u := \dot{H}/H + 3/2H$. With these definitions the real and imaginary parts of $u_{\pm}(t)$ are obtained:

$$Re\left\{u_{\pm}(t)\right\} = \frac{\omega_1^{\pm}\left(C_1 \sinh\left\{\omega_1^{\pm} t\right\} + C_2^{\pm} \cosh\left\{\omega_1^{\pm} t\right\}\right) + \omega_2^{\pm}\left(C_3 \sin\left\{\omega_2^{\pm} t\right\} + C_4^{\pm} \cos\left\{\omega_2^{\pm} t\right\}\right)}{C_1 \cosh\left\{\omega_1^{\pm} t\right\} + C_2^{\pm} \sinh\left\{\omega_1^{\pm} t\right\} - C_3 \cos\left\{\omega_2^{\pm} t\right\} + C_4^{\pm} \sin\left\{\omega_2^{\pm} t\right\}} \tag{5.83}$$

$$Im\left\{u_{\pm}(t)\right\} = \frac{\omega_2^{\pm}\left(C_1 \sinh\left\{\omega_1^{\pm} t\right\} + C_2^{\pm} \cosh\left\{\omega_1^{\pm} t\right\}\right) - \omega_1^{\pm}\left(C_3 \sin\left\{\omega_2^{\pm} t\right\} + C_4^{\pm} \cos\left\{\omega_2^{\pm} t\right\}\right)}{C_1 \cosh\left\{\omega_1^{\pm} t\right\} + C_2^{\pm} \sinh\left\{\omega_1^{\pm} t\right\} - C_3 \cos\left\{\omega_2^{\pm} t\right\} + C_4^{\pm} \sin\left\{\omega_2^{\pm} t\right\}}, \tag{5.84}$$

where

$$\begin{aligned} C_1 =& H_0 \left|u_{+0}\right| + \left|\omega_+^2\right| \\ C_2^{\pm} =& \omega_1^{\pm}\left(2H_0 u_{+0} - \frac{2}{3}\left(\omega_+^2 - \left|\omega_+^2\right|\right)\right) \\ C_3 =& H_0 \left|u_{+0}\right| - \left|\omega_+^2\right| \\ C_4^{\pm} =& \omega_2^{\pm}\left(2H_0 u_{+0} - \frac{2}{3}\left(\omega_+^2 + \left|\omega_+^2\right|\right)\right), \end{aligned}$$

with $|z|$ the absolute value of a complex number $z \in \mathbb{C}$. Note that the constants $C_1 \ldots C_4^{\pm}$ are real numbers. Also note that in the definitions of $C_1 \ldots C_4^{\pm}$ we could replace all the $+$ by $-$ without changing the values of these constants.

From Equation (5.82) we find that the the functions $H(t)$ have to fulfil

$$\dot{H} = -\frac{3}{2} H^2 + Re\{u_{\pm}(t)\} H + i\left(Im\{u_{\pm}(t)\} H - \frac{2}{3} \omega_1^{\pm} \omega_2^{\pm}\right). \tag{5.85}$$

Since we are looking for real solutions, $H(t)$ has to fulfil the following two equations

$$\dot{H} = -\frac{3}{2} H^2 + Re\{u_{\pm}(t)\} H \tag{5.86}$$

$$H = \frac{2}{3} \omega_1^{\pm} \omega_2^{\pm} \left(Im\{u_{\pm}(t)\}\right)^{-1}. \tag{5.87}$$

Further, since from equation (5.82) $Re\{u_+(t)\} = Re\{u_-(t)\}$ (and $Im\{u_+(t)\} = -Im\{u_-(t)\}$) we may take either the $+$ sign or the $-$ sign. From the second Equation (5.87) we immediately find that

$$H(t) = \frac{2}{3} \omega_1^{+} \omega_2^{+} \frac{C_1 \cosh\left\{\omega_1^{+} t\right\} + C_2^{+} \sinh\left\{\omega_1^{+} t\right\} - C_3 \cos\left\{\omega_2^{+} t\right\} + C_4^{+} \sin\left\{\omega_2^{+} t\right\}}{\omega_2^{+}\left(C_1 \sinh\left\{\omega_1^{+} t\right\} + C_2^{+} \cosh\left\{\omega_1^{+} t\right\}\right) - \omega_1^{+}\left(C_3 \sin\left\{\omega_2^{+} t\right\} + C_4^{+} \cos\left\{\omega_2^{+} t\right\}\right)}, \tag{5.88}$$

which is also obtained by solving the Bernoulli equation (5.86) (the integration constant C_0 in the solution (5.53) turns out to be zero in this case).

5.2 Dynamical System for general states

So far, we have studied the dynamical behaviour of solutions only for the conformal vacuum quantum state. Here we want to study solutions to equation (5.12) from the point of view of dynamical systems. To this extend we write equation (5.12) as a three dimensional dynamical system

$$\dot{w} = \mathscr{X}(w)$$

by taking $w = (H, \dot{H}, \ddot{H})^T \in \mathbb{R}^3$ as phase space variables and $\mathscr{X} : \mathbb{R}^3 \to \mathbb{R}^3$. The vector field determining the dynamical behaviour reads

$$\mathscr{X} = \begin{pmatrix} \dot{H} \\ \ddot{H} \\ -7\ddot{H}H - 4\dot{H}^2 - 12\dot{H}H^2 - \frac{2B}{3A}\left(\dot{H}H^2 + H^4\right) + \frac{C}{A}\left(\dot{H} + 2H^2\right) - \frac{2D}{3A} \end{pmatrix} \tag{5.89}$$

for the massless, conformally coupled quantum field being in a general state ω.
From the dynamical system (5.89) we immediately see that trajectories obey again a reversing symmetry

$$\begin{aligned} R &: \mathbb{R}^3 \to \mathbb{R}^3 \\ R &: (H, \dot{H}, \ddot{H}) \mapsto (-H, \dot{H}, -\ddot{H}). \end{aligned} \tag{5.90}$$

The fixed point subspace of R is $Fix(R) = \{(H, \dot{H}, \ddot{H}) \in \mathbb{R}^3 | H = \ddot{H} = 0\}$. We may therefore divide the phase space into the four regions

$$\begin{aligned} \mathbb{H}^3_{++} &:= \{(H, \dot{H}, \ddot{H}) \in \mathbb{R}^3 | H > 0,\ \ddot{H} > 0\} \\ \mathbb{H}^3_{+-} &:= \{(H, \dot{H}, \ddot{H}) \in \mathbb{R}^3 | H > 0,\ \ddot{H} < 0\} \\ \mathbb{H}^3_{-+} &:= \{(H, \dot{H}, \ddot{H}) \in \mathbb{R}^3 | H < 0,\ \ddot{H} > 0\} \\ \mathbb{H}^3_{--} &:= \{(H, \dot{H}, \ddot{H}) \in \mathbb{R}^3 | H < 0,\ \ddot{H} < 0\} \end{aligned}$$

Then the trajectories in region $\mathbb{H}^3_{-+}$ ($\mathbb{H}^3_{--}$) are completely determined by the trajectories in region $\mathbb{H}^3_{+-}$ ($\mathbb{H}^3_{++}$) by the symmetry R. Since the vector field $\mathscr{X}$ is continuous the phase portrait is completely determined by the trajectories in the region

$$\mathbb{H}^3_{+} := \mathbb{H}^3_{++} \cup \mathbb{H}^3_{+-} \cup \{(H, \dot{H}, \ddot{H}) \in \mathbb{R}^3 | H > 0,\ \ddot{H} = 0\}.$$

The trajectories in $\mathbb{H}^3_-$ are obtained by the symmetry R. For continuous vector fields the behaviour of trajectories traversing the fixed point subset $Fix(R)$ is uniquely determined. We may classify trajectories similar as before in *expanding trajectories* $\mathcal{E}$, *contracting trajectories* $\mathcal{C}$, the *Minkowski equilibrium point* $\mathcal{M}$ and *reversing trajectories* $\mathcal{R}$. Let us again restrict to the phase space region $\mathbb{H}^3_+$ and perform the transformation

$$\begin{pmatrix} H \\ \dot{H} \\ \ddot{H} \end{pmatrix} \in \mathbb{H}^3_+ \mapsto \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{L}^3_+ := \{(x,y,z) \in \mathbb{R}^3 | x > 0\}, \tag{5.91}$$

where

$$\begin{aligned} x &= H^{\frac{1}{2}} \\ y &= \dot{x} \equiv \frac{1}{2}\dot{H}H^{-\frac{1}{2}} \\ z &= 2x^3\left(\dot{y} + f(x)y + g(x)\right) \equiv \ddot{H}H - \frac{1}{2}\dot{H}^2 + 3\dot{H}H^2 + \frac{1}{6}\frac{B}{A}H^4 - \frac{1}{2}\frac{C}{A}H^2 + \frac{1}{6}\frac{D}{A} \end{aligned} \tag{5.92}$$

with

$$\begin{aligned} f(x) &= 3x^2 \\ g(x) &= \frac{1}{12}\frac{B}{A}x^5 - \frac{1}{4}\frac{C}{A}x + \frac{1}{12}\frac{D}{A}x^{-3}. \end{aligned} \tag{5.93}$$

Then equation (5.12) becomes

$$0 = \dot{z} + 4x^2 z \tag{5.94}$$

and the dynamical system (5.89) reads

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \mathscr{Y}(x,y,z) = \begin{pmatrix} y \\ \frac{1}{2}x^{-3}z - f(x)y - g(x) \\ -4x^2 z \end{pmatrix}. \tag{5.95}$$

The equilibrium points all lie in the $z = 0$-plane and are identical to the equilibrium points (5.26). The local Lyapunov stability of the de Sitter equilibrium points is easily obtained by inspecting the eigenvalues of the Jacobian

$$J\mathscr{Y}(x,y,z) = \begin{pmatrix} 0 & 1 & 0 \\ -3/2x^{-4}z - f'(x)y - g'(x) & -f(x) & \frac{1}{2}x^{-3} \\ -8xz & 0 & -4x^2 \end{pmatrix}, \tag{5.96}$$

where $'$ denotes the derivative with respect to x. The eigenvalues of the Jacobian matrix evaluated at a de Sitter equilibrium point $(x_\pm, 0, 0) \in dS$ are then

$$\lambda_1(x_\pm) := -4x_\pm^2 \tag{5.97}$$

$$\lambda_{2/3}(x_\pm) := -\frac{1}{2}f(x_\pm) \pm \sqrt{\frac{1}{4}f(x_\pm)^2 - g'(x_\pm)}. \tag{5.98}$$

The real parts of λ_1 and λ_3 are always negative. The real part of λ_2 is negative if and only if $g'(x_\pm) > 0$. Using Theorem 2.23 we obtain the Lyapunov stability listed in Table 5.1 and 5.2 with $z = 0$.

All bifurcations observed in subsection 5.1.3 are also bifurcations of the general system since these are all associated to a change of the number of equilibrium points or a collision of equilibrium points resulting in a non-hyperbolic equilibrium point.

Since Friedmann's equation (5.11) is a first integral of equation (5.12) it is clear that the set

$$\mathscr{V} := \left\{ (H, \dot{H}, \ddot{H}) \in \mathbb{R}^3 \,\middle|\, \rho = -\ddot{H}H + \frac{1}{2}\left(\dot{H} - 3H^2\right)^2 - \alpha\left(H^2 - \beta\right)^2 \right\}, \tag{5.99}$$

with

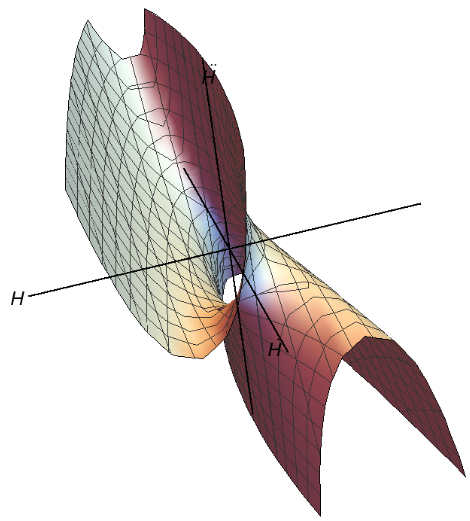
$$\begin{aligned} \rho &:= \frac{1}{6}\frac{D}{A} - \beta^2 \\ \alpha &:= \frac{1}{6}\frac{B}{A} + \frac{9}{2} \\ \beta &:= \frac{1}{4\alpha}\frac{C}{A}. \end{aligned}$$

is an invariant set of the vector field $\mathscr{X}$. The defining equation in (5.99) is obtained by rewriting (5.17). For any trajectory γ with $\gamma \cap \mathscr{V} \neq \varnothing$ we have $\gamma \subset \mathscr{V}$, i.e. trajectories with initial data $\{H_0, \dot{H}_0, \ddot{H}_0\} \in \mathscr{V}$ will stay in $\mathscr{V}$ for all times. Furthermore, from the first integral (5.11) we observe that expanding trajectories $\mathcal{E}$ asymptotically reach $\mathscr{V}$. This is a consequence of the fact that $H(t) > 0$ implies a monotonically increasing scale factor $a(t)$ which causes the state-depending term to decrease. Physically this behaviour is plausible: an expanding universe filled with some matter will look like a vacuum universe in its asymptotic future (compare [76]). Trajectories associated to solutions in $\mathcal{C}$ consequently depart from the vacuum surface $\mathscr{V}$, which can easily be proven by using the reversing symmetry. Hence $\mathscr{V}$ is an attractor for $H > 0$ and a repeller for $H < 0$.

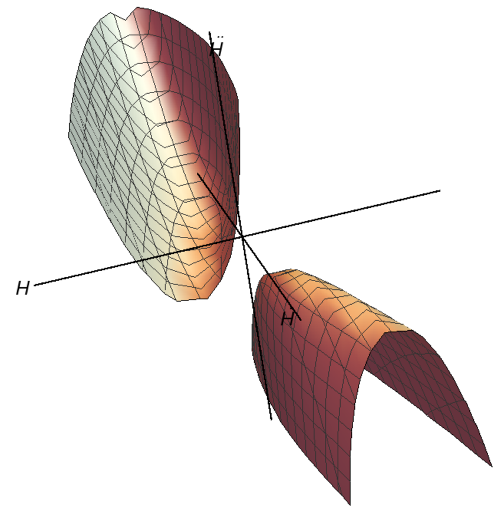
Depending on the values of renormalisation constant this surface can behave very different in three dimensional space. Note that at $H = 0$ we have $1/2\dot{H}^2 - \alpha\beta^2 = \rho$ for all

$\ddot{H}$. Therefore all reversing trajectories lying on the invariant set $\mathscr{V}$ cross one of the lines $L_\pm := \{(H, \dot{H}, \ddot{H}) \in \mathbb{R}^3 | H = 0 \ , \ \dot{H} = \pm\sqrt{\frac{1}{3}\frac{D}{A}}\}$. The projection of this line onto the $\ddot{H} = 0$-plane is identical to the point $\mathcal{P}_\pm$ defined in Proposition 5.2. Therefore the projection from the invariant set (5.99) onto the $\ddot{H} = 0$-plane is not unique at $H = 0$, which explains the non-uniqueness at $H = 0$ for the two-dimensional dynamical system (5.18). Therefore any trajectory running through the points $\mathcal{P}_\pm$ in fact is above or below the $\ddot{H} = 0$-plane. In particular, no trajectory runs through the Minkowski equilibrium and the Minkowski equilibrium must be unstable.

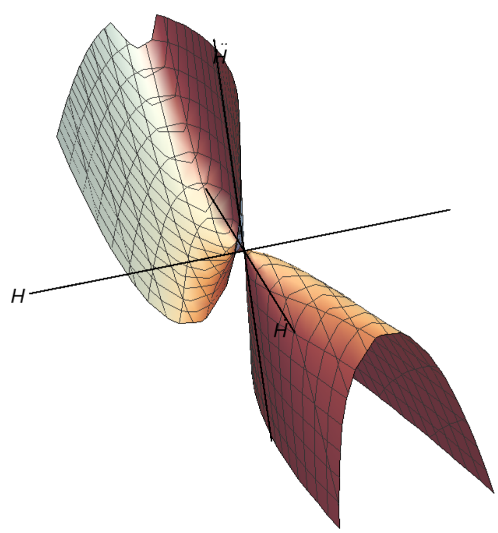
From the form of the defining equation in (5.99) we see, that the invariant manifold can be discussed similarly to the hyperboloid with its three distinct cases. Hence depending on the renormalisation constants we have two disconnected sheets, two sheets being connected at a single point or a tube-like surface. In the first case trajectories of the one sheet cannot pass to the other sheet. These cases correspond to the cases 2^+, 1^-, 3^- and III^- of Figures 5.3 - 5.6. One of the sheet therefore lies completely in the region $\mathbb{H}^3_+$, whereas the other one lies completely in $\mathbb{H}^3_-$. In the second case, the point where the two sheets are connected is identical to the Minkowski equilibrium point. Hence trajectories cannot pass from one to the other sheet neither. Theses cases correspond to the cases I^+, II^+, IV^+, I^-, II^- and IV^- of Figures 5.3 - 5.6. The line $L_+ = L_-$ now is identical to the $\ddot{H}$-axis. One sheet now lies completely in $\mathbb{H}^3_+ \cup L_+$ and the other sheet lies completely in $\mathbb{H}^3_- \cup L_+$. The trajectories crossing the line L_+ are all projected onto the Minkowski equilibrium in the phase portraits I^+, II^+, IV^+, I^-, II^- and IV^- of Figures 5.3 - 5.6. Finally if the sheets are connected trajectories may pass from one to the other. Also the sheets cross the $\ddot{H}$-axis and therefore reversing trajectories are allowed. This is the case for III^+, 1^+, 3^+ and 2^- of Figures 5.3 - 5.6. The three distinct cases for the surface $\mathscr{V}$ are depicted in figure 5.11 for the values $\alpha = \beta = 1$.



(a) Invariant manifold with $\rho = -0.5$.



(b) Invariant manifold with $\rho = -1.5$.



(c) Invariant manifold with $\rho = -1$.

Figure 5.11: The invariant manifold $\mathscr{V}$ for the values $\alpha = \beta = 1$.

Other invariant sets of the vector field $\mathscr{X}$ of the form $\Sigma := \{\left(H, \dot{H}, \ddot{H}\right) \in \mathbb{R}^3 \mid 0 =$

$\sigma(H,\dot{H},\ddot{H})\}$ can be obtained by solving

$$\langle \nabla\sigma, \mathscr{X}\rangle = 0, \tag{5.100}$$

i.e. the vector field $\mathscr{X}$ is always normal to the gradient of the surface Σ. We make the ansatz

$$\sigma(H,\dot{H},\ddot{H}) = \ddot{H} + P(H)\dot{H} + Q(H). \tag{5.101}$$

Then, by a straightforward calculation one finds

Proposition 5.4. *Let* $A = \frac{4B}{27}$ *and* $D = \frac{81}{100}\frac{C^2}{B}$*. Then, the ansatz* (5.101) *solves equation* (5.100) *and*

$$\Sigma_\pm := \Big\{ \big(H,\dot{H},\ddot{H}\big) \in \mathbb{H}^3_+ | 0 = \sigma_\pm(H,\dot{H},\ddot{H}) = \ddot{H} + P_\pm(H)\dot{H} + Q_\pm(H)\Big\}.$$

where

$$\begin{aligned} P_\pm(H) &= 4H + k_\pm \\ Q_\pm(H) &= \frac{3}{2}H^3 + \frac{k_\pm}{2}H^2 - \frac{3}{2}k_\pm^2 H - \frac{k_\pm^3}{2} \end{aligned}$$

and $k_\pm := \pm\sqrt{\frac{27}{10}\frac{C}{B}}$ *are two invariant manifolds of the vector field* $\mathscr{X}$.

Now define the following invariant manifolds lying in the vacuum invariant manifold $\mathscr{V}$ for $A = 4B/27$ and $D = (81C^2)/(100B)$

$$\Gamma_1^\pm := \Big\{(H,\dot{H},\ddot{H}) \in \mathscr{V} | \dot{H} = -\frac{3}{2}H^2 - 2k_\pm H - \frac{k_\pm^2}{2}\Big\}, \tag{5.102}$$

$$\Gamma_2 := \Big\{(H,\dot{H},\ddot{H}) \in \mathscr{V} | \dot{H} = -\frac{1}{2}H^2 + \frac{k_\pm^2}{2}\Big\}. \tag{5.103}$$

Then the invariant manifolds $\Sigma_\pm$ and $\mathscr{V}$ intersect as follows

$$\Sigma_\pm \cap \mathscr{V} = \Gamma_1^\pm \cup \Gamma_2, \qquad \Sigma_\pm \cap \Sigma_\mp = \Gamma_2 \tag{5.104}$$

5.2.1 Solutions for Quantum Fields in a specific Quantum State

The defining equations

$$0 = \ddot{H} + (4H + k_\pm)\dot{H} + \frac{3}{2}H^3 + \frac{k_\pm}{2}H^2 - \frac{3}{2}k_\pm^2 H - \frac{k_\pm^3}{2} \tag{5.105}$$

of $\Sigma_\pm$ can be solved. For this purpose let us first define a new variable $v(t)$ by

$$\dot{H} = -\frac{3}{2}H^2 - 2k_\pm H - \frac{k_\pm^2}{2} + v(t). \tag{5.106}$$

Then plugging (5.106) into Equation (5.105) we get

$$0 = \dot{v} + v\left(H - k_\pm\right). \tag{5.107}$$

The solutions of equation (5.107) can be written in terms of the scale factor $a = a(t)$ and the time t and read

$$v(t) = v_0 \left(\frac{a}{a_0}\right)^{-1} e^{k_\pm t}, \tag{5.108}$$

for initial conditions $v_0 := v(0)$ and $a_0 := a(0)$. Using equation (5.106) we obtain

$$v_0 = \left(\dot{H} + \frac{3}{2}H^2 + 2k_\pm H + \frac{k_\pm^2}{2}\right)\frac{a}{a_0}e^{k_\pm t}. \tag{5.109}$$

Similarly, defining another variable $w(t)$ by

$$\dot{H} = -\frac{1}{2}H^2 + \frac{k_\pm^2}{2} + w(t). \tag{5.110}$$

and plugging (5.110) into Equation (5.105) we obtain

$$0 = \dot{w} + w\left(3H + k_\pm\right). \tag{5.111}$$

The solutions to Equation (5.111) are

$$w(t) = w_0 \left(\frac{a}{a_0}\right)^{-3} e^{-k_\pm t}, \tag{5.112}$$

for initial conditions $w_0 := w(0)$ and $a_0 := a(0)$. Re-substitution leads to

$$w_0 = \left(\dot{H} + \frac{1}{2}H^2 - \frac{k_\pm^2}{2}\right)\left(\frac{a}{a_0}\right)^3 e^{-k_\pm t}. \tag{5.113}$$

Combining equations (5.109) and (5.113) to eliminate the scale factor a we obtain the time-dependent first integral

$$I(t, H, \dot{H}) = \frac{\dot{H} + \frac{1}{2}H^2 - \frac{k_\pm^2}{2}}{\left(\dot{H} + \frac{3}{2}H^2 + 2k_\pm H + \frac{k_\pm^2}{2}\right)^3} e^{-4k_\pm t} = \frac{w_0}{v_0^3}. \tag{5.114}$$

If we use equations (5.109) and (5.113) to eliminate the time t we obtain

$$\frac{v_0 w_0 a_0^4}{a^4} = \dot{H}^2 + 2(H + k_\pm)\dot{H}H + \frac{3}{4}H^4 + k_\pm H^3 - \frac{k_\pm}{2}H^2 - k_\pm^3 H - \frac{k_\pm}{4}. \tag{5.115}$$

The left-hand side of this equation is identical to what we obtain if we plug equation (5.105) into the first integral (5.11) and solve for $c_\omega a^{-4}$. Hence this shows, that equation (5.105) indeed defines a set of solutions for the quantum field being in an excited state $c_\omega = v_0 w_0 a_0^4$.
Equations (5.109) and (5.113) can also be used to eliminate $\dot{H}$. Then we obtain

$$(H + k_\pm)^2 = v_0 \left(\frac{a}{a_0}\right)^{-1} e^{k_\pm t} - w_0 \left(\frac{a}{a_0}\right)^{-3} e^{-k_\pm t}. \tag{5.116}$$

Note that for $k_\pm = 0$ ($C = 0$) the solution (5.116) becomes (4.31) as should be the case. The substitution

$$\tau := e^{k_\pm t} \tag{5.117}$$

$$y := \tau^{\frac{3}{2}} a^{\frac{3}{2}}, \tag{5.118}$$

transforms Equation (5.116) into

$$(y')^2 = \frac{9}{4}\frac{v_0}{k_\pm^2} y^{\frac{4}{3}} - \frac{9}{4}\frac{w_0}{k_\pm^2}, \tag{5.119}$$

where $'$ denotes the derivative with respect to τ. Differentiating equation (5.119) with respect to τ leads to an Emden-Fowler equation of the form [109]

$$y'' = \frac{3}{2}\frac{v_0}{k_\pm^2} y^{1/3}. \tag{5.120}$$

The solutions of equation (5.119) is

$$\tau = \pm\frac{2}{3}k_\pm \int\limits_{y_0}^{y} \frac{d\xi}{\sqrt{v_0 \xi^{\frac{4}{3}} - w_0}}, \tag{5.121}$$

for initial conditions $y_0 = a_0^{\frac{3}{2}}$. This determines the scale factor $a(t)$.
To understand the qualitative behaviour of the solutions (5.116) it is advisable to study the corresponding trajectories in phase space. Thereby we will be able to draw phase portraits visualising the qualitative behaviour of solutions depending on the initial values. The vector field associated to equation (5.105) reads

$$f(H, \dot{H}) = \begin{pmatrix} \dot{H} \\ -P(H)\dot{H} - Q(H) \end{pmatrix}, \tag{5.122}$$

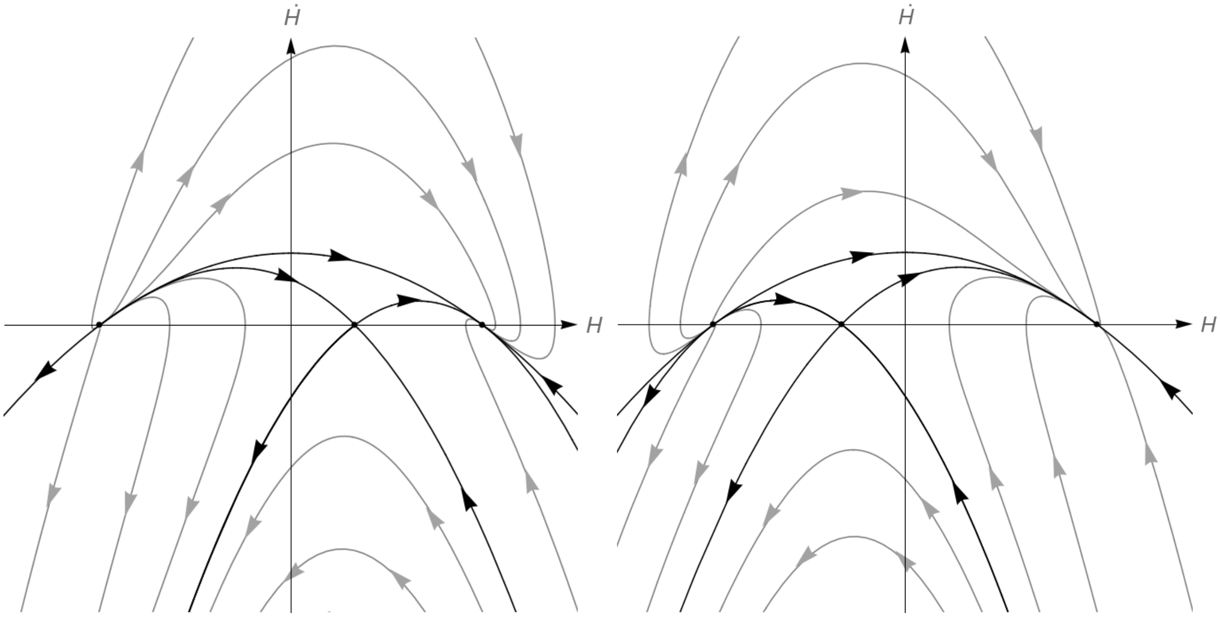
and is an example for a Liénard system. Its equilibrium points are

$$\left\{(k_\pm, 0), (-k_\pm, 0), \left(-\frac{k_\pm}{3}, 0\right)\right\},$$

and correspond to de Sitter universes or in the case $k_+ = k_- = 0$ to the Minkowski universe. Using the Lyapunov function

$$V(H, \dot{H}) := \frac{1}{2}\dot{H}^2 + \int_0^H Q(u)du, \tag{5.123}$$

one finds that $(k_\pm, 0)$ is asymptotically stable for the $+$ sign and unstable for $-$. Therefore the behaviour of $(-k_\pm, 0)$ is just the opposite. $(-k_\pm/3, 0)$ is an unstable saddle point. Due to Theorem 2.12 there are no periodic nor homo- or heteroclinic trajectories. Note that the number of equilibrium points and their stability behaviour is identical to the case of the Λ-CDM model with positive cosmological constant. Therefore both cases are topologically equivalent to Λ-CDM model with positive cosmological constant. The two cases for $k_+ \neq 0$ and $k_- \neq 0$ are shown in Figure 5.12. These phase portraits are projections of the vector field $\mathscr{X}$ restricted to $\Sigma_\pm$ projected on the $(\ddot{H} = 0)$-plane. The discussion of the trajectories can be taken over from the discussion of the Λ-CDM model with positive cosmological constant. However, this time the unstable saddle equilibrium point does no longer correspond to a Minkowski universe. The invariant manifold $\Gamma_1^\pm$ corresponds to the unstable manifolds of the de Sitter equilibrium points which are drawn as solid curve in figure 5.12. Γ_2 corresponds to the solid curves connecting the asymptotically stable de Sitter equilibrium points in both figures 5.12a and 5.12b. As was mentioned before, trajectories belonging to $\mathcal{E}$ approach the vacuum invariant manifold $\mathscr{V}$. In particular those lying in $\Sigma_\pm$ asymptotically reach the positive de Sitter equilibrium points. For $C = 0$ $(k_\pm = 0)$ the equation defining $\Sigma_+ = \Sigma_-$ becomes a generalised Friedmann equation considered in section 4.2. The according phase portrait is shown in figure 4.1b.



(a) Phase portrait of the vector field (5.122) for $k_+ = 2$.

(b) Phase portrait of the vector field (5.122) for $k_- = -2$.

Figure 5.12: Phase portraits of the two cases for the vector field (5.122)

Chapter 6

Discussion and Conclusion

In the present thesis we have analysed the qualitative behaviour of solutions to the semi-classical Einstein equations in homogeneous and isotropic spacetimes. To keep things manageable we specialised to the cases where spacetime is filled with a massless, conformally coupled scalar field, with the electromagnetic field or with the massless, conformally coupled Dirac field all being free fields. The analysis was undertaken by using dynamical system analysis. For this purpose Einstein's field equation for the cases at hand was rewritten as a dynamical system. For general quantum states this leads to a three dimensional dynamical system depending on free parameters with the Hubble function $H(t)$ and its first two derivatives as phase space variables. In general a trajectory (and hence a solution) will depend on the set of data $(H_0, \dot{H}_0, \ddot{H}_0, A, B, C, D)$ or alternatively on the data set $(H_0, \dot{H}_0, a_0, A, B, C, D, c_\omega)$, where A, B, C and D are free constants produced by the renormalisation process. The constant c_ω can be seen either as constant of integration or as a constant determining the quantum state of the field. In this thesis, solutions where classified by their qualitative behaviour according to these sets of initial data. On general grounds we observed that the dynamical system obeys a reversing symmetry, i.e. when $H(t)$ is a solution then $-H(-t)$ is a solution too. This symmetry can be used to classify solutions into contracting, expanding and reversing solutions. Reversing symmetry appears already in the classical case.

If the quantum field is in the conformal vacuum state one obtains a second order differential equation. Here special attention has to be taken to reversing trajectories since these are in general not uniquely determined by initial conditions $(H_0, \dot{H}_0)$. In particular, it was shown that all reversing trajectories have to run through either of the two points $\mathcal{P}^{\pm}$

defined in proposition 5.2. The non-uniqueness can be resolved by analysing the trajectories corresponding to the conformal vacuum case in the three dimensional phase space. Then, it was shown that these trajectories form an invariant set $\mathscr{V}$ of the state-dependent vector field. The invariant set $\mathscr{V}$ forms a complicated surface in three-dimensional space. Since, depending on the values of the renormalisation constants $\mathscr{V}$ can be a closed surface the projection of the state-dependent vector field restricted to the invariant set $\mathscr{V}$ on the H-$\dot{H}$-plane is not unique. In particular, it was shown that all trajectories running through the points $\mathcal{P}^{\pm}$ in fact have a unique value of $\ddot{H}$. We therefore where able to extend the results of [13], where the non-unique behaviour was observed but not discussed. One is able to circumvent these kind of discussion by restricting the renormalisation constants such that $A > 0$ and $D < 0$ or $A < 0$ and $D > 0$ or $A = 0$. In the first two cases reversing trajectories do not exist at all. Uniqueness and existence of solutions of the latter case where shown in a more general setting by [108]. In contrast to [108] in the present thesis uniqueness of solutions for the conformally coupled, massless quantum field was shown for any value of the renormalisation constants but vanishing mass.

In the case of a general quantum state only the local (Lyapunov-) stability of equilibrium points (i.e. the Minkowski and the de Sitter spacetime) was analysed. It turned out that the stability of de Sitter spacetime crucially depends on the values of the renormalisation constants, whereas Minkowski spacetime is unstable for any choice of renormalisation constants if it exists. Depending on the physical situation the stability of de Sitter spacetime might be preferable or not. For example, Starobinsky [123] argues that instability of de Sitter spacetime may explain the termination of a de Sitter like inflation of the very early universe. On the other hand an asymptotically stable de Sitter universe could explain the observed acceleration of the universe today [114].
Instability of Minkowski spacetime has been an issue in the literature for quite a time [83,84,118,121,122,125–128]. Two possibilities to argue where proposed in the article [83]: either Minkowski spacetime is really unstable, but on time-scales much larger than those in terrestrial experiments or semiclassical theory does not accurately describe nature. In fact, as was shown in Chapter 4 Minkowski spacetime is already unstable in the classical case where the cosmological constant is positive or zero, and structurally unstable for negative cosmological constant. Note that this result is in no contradiction to the result of [37] since different notions of "stability" are considered. In [37] it was shown that small perturbations of the Minkowski solution are still solutions of the Einstein-Vacuum equation (i.e. when

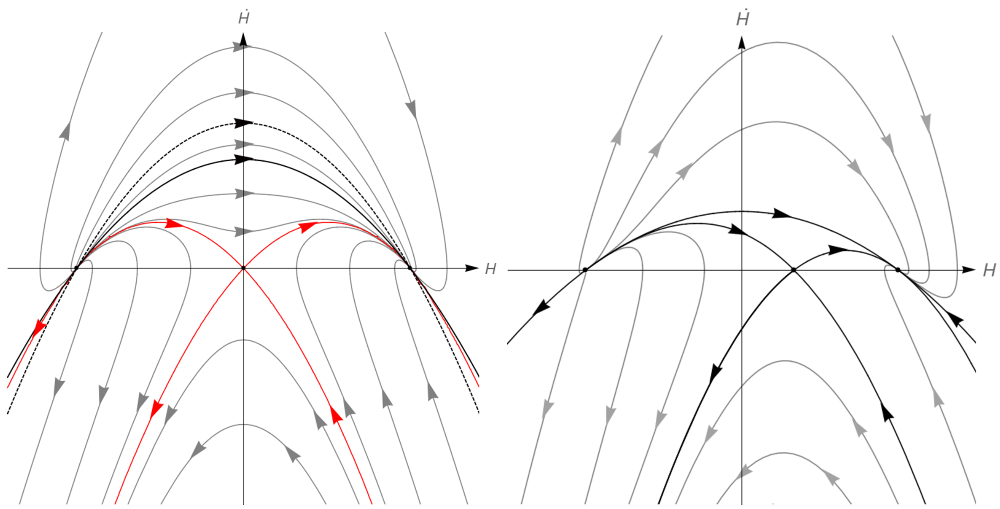
$T_{\mu\nu} = 0$) as long as these perturbations are constructed from strongly asymptotic initial data. In our case the entire spacetime is filled with a fluid which does not fulfil the requirements of [37]. Therefore (Lyapunov-) instability (in the sense presented in this thesis) of Minkowski spacetime seems not to be an argument to identify a theory as “nonphysical”.

In case the quantum field is in the conformal vacuum state we were able to find a Lyapunov function to investigate the global stability of equilibrium points, i.e. we obtained information about the region of attraction of the asymptotically stable de Sitter equilibrium. A similar kind of analysis has been done already in [13] by transforming the semiclassical Friedmann equation into a Liénard system as we did and discussing its potential. Besides [13], there is a vast amount of literature concerned with the qualitative behaviour of solutions to the semiclassical Friedmann equation including the stability of the equilibrium points, e.g. [6–8, 12, 38, 86, 87, 123, 134]. In the present work we were able to subsume all of these results. We pointed out that the qualitative behaviour of solutions depends crucially on the values of the renormalisation parameters. Using bifurcation theory and in particular bifurcation diagrams, we presented a classification together with representative phase portraits for each equivalence class. We furthermore identified the structurally stable cases. Following [62] one may take the point of view that “the only properties of a dynamical system (or a family of dynamical systems) which are *physically relevant* are those which are preserved under perturbations of the system”. Structural stability might therefore help to find “good” models. This however depends on the physical situation one considers. In particular the notion of structural stability does not distinguish between physical motivated perturbations and purely mathematical perturbations. If a dynamical system is structurally stable then it is stable with respect to *any* perturbation. Further it does not distinguish between physically irrelevant regions of spacetime and physically relevant regions and therefore does not take into account that perturbations might preserve the topological structure of the phase portrait in relevant regions of phase space. In this thesis we have proven structural stability of the semiclassical Friedmann equation with respect to perturbations that can be written as a function of the Hubble function and its first derivative, i.e. perturbations of the semiclassical Friedmann equation of the form $\epsilon F(H, \dot{H})$ for some small $\epsilon > 0$. Physically this might be relevant if for example small interaction terms are included or small mass terms. Then the qualitative behaviour of the perturbed and unperturbed system remains preserved. However the “stability dogma” [62] is arguable. Further we find more than one structurally stable case, hence there does not

seem to be an inherent argument to choose one of the cases as the correct one to describe nature. For the analysis in this thesis the significance of structurally stable models is not certain.
There is the following interesting observation which might suggest a certain choice of renormalisation constants. If one contrasts the phase portrait of a homogeneous and isotropic universe filled with classical matter and positive cosmological constant with the phase portrait of case 1^+ (i.e. figure 5.4a), one finds that the qualitative behaviour of solutions looks quite similar, the only qualitative difference being the non-smooth behaviour at $H = 0$. Hence, even if the quantum field is in the conformal vacuum state, the semiclassical model is able to reproduce the qualitative dynamical behaviour of the classical model to a certain extent. As is suggested by comparing the phase portraits of the conformal vacuum case 5.4a to the state dependent cases 5.12a and 5.12b (see Figure 6.1), this equivalence is preserved when considering a specific excited quantum state. Note that in the semiclassical case the unstable saddle point does not correspond to a Minkowski universe anymore but represents a de Sitter universe. The connection of two de Sitter equilibrium points, one being unstable might serve as an explanation for a early inflationary phase *and* an accelerated expanding late universe. Similarly to the Starobinsky model [123] a trajectory might have been close to the unstable de Sitter universe in its past undergoing a inflationary phase and then evolving towards another asymptotically stable de Sitter equilibrium explaining the accelerated expansion observed today. However this remains to be shown and calls for a more rigorous analysis (i.e. the massive, state-dependent case).
Using bifurcation theory we where further able to contrast the various (massless) fields and even certain $f(R)$-theories with each other from a qualitative point of view. We further discussed the presence of higher order derivative terms. For an extensive discussion of bifurcations and structural stability see the discussion in subsection 5.3.3.

Finally we gave a new derivation of solutions for the special case $A = 4B/27$ in the conformal vacuum. These solutions where already found by [31–36, 85]. We further found new solutions on the state-dependent invariant sets $\Sigma_\pm$ presented in section 5.2.1.

The model considered here is not exhaustive in several respects. First of all we specialised to massive and conformally coupled fields. If this was not the case the trace of the stress energy tensor would become state-dependent. To consider the resulting dynamical equation as dynamical system one would have to consider the term $\omega(:\Phi^2:)$ more closely. It would be preferable to have $\omega(:\Phi^2:)$ in a form such that it depends on the Hubble



(a) classical case (b) excited quantum state

Figure 6.1: Comparison of the phase portraits of classical matter and positive cosmological constant, the case 1^+ for a specific excited quantum state

function and its derivatives and possibly the scale factor. Then it would be easy to transform the dynamical equation into a dynamical system as was done in the present thesis. Another way to discuss the massive case is to consider $\omega(:\Phi^2:)$ as a function of the time t. Then we may treat the dynamical system similar to the forced Duffing oscillator as was done e.g. in [62]. For this purpose one would need more qualitative information about the behaviour of the forcing term $\omega(:\Phi^2:)(t)$.

Finally, techniques of infinite dimensional dynamical systems [129] may be applied to the coupled system of the trace equation and the modes appearing in $\omega(:\Phi^2:)$.

In this thesis the state dependent case was discussed only locally by the stability of its equilibrium points and the two invariant manifolds. With more effort, one could gain more information about this three dimensional dynamical system. Although no general theorems are available concerning the structural stability for dynamical systems of dimension larger than two, one can try to find further bifurcations of the state-dependent case and draw representative phase portraits.

As was also remarked in Subsection 5.1.3, we did not consider geometric perturbations. It would be interesting to depart from the very special case of spatial homogeneous and isotropic models to more general geometries as for example homogeneous and anisotropic models. Then the dynamical system would increase in dimension by a factor of three. It would be interesting if the invariant manifold of spatial homogeneous and isotropic

spacetimes would be stable within this dynamical system and in particular if the stability behaviour of Minkowski and de Sitter spacetimes remains the same. If so, quantum fields would be able to smooth out derivations from FLRW spacetimes.
Finally the effect of interacting quantum fields on the back-reaction problem need to be analysed in depth.

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