Some Results on the Distributions of Operator-Valued Semicircular Random Variables

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Abstract

The operator-valued free central limit theorem and operator-valued semicircular random variables were first introduced by D. Voiculescu in 1995 as operator-valued free analogues of the classical central limit theorem and normal random variables, respectively.

In 2007, R. Speicher and others showed that the operator-valued Cauchy transform of the semicircular distribution satisfies a functional equation involving the variance of the semicircular distribution.

In this thesis, we consider a non-commutative probability space (A, E_B, B) where in which A is a unital C^*-algebra, B is a C^*-subalgebra of A containing its unit and $E_B : A \to B$ is a conditional expectation. For a given $B-$valued self-adjoint semicircular random variable $s \in A$ with variance η, it is still an open question under what conditions the distribution of s has an atomic part. We provide a partial answer in terms of properties of η when B is the algebra of $n \times n$ complex matrices. In addition, we show that for a given compactly supported probability measure its associated Cauchy transform can be represented in terms of the operator-valued Cauchy transforms of a sequence of finite dimensional matrix-valued semicircular random variables in two ways. Finally, we give another representation of its Cauchy transform in terms of operator-valued Cauchy transform of an infinite dimensional matrix-valued semicircular random variable.

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Chapter 1

Introduction

This research thesis is divided in four main chapters of which the first three chapters provide required background for the presented main results in the last chapter.

Chapter 2 begins by introducing the Cauchy transform of a probability measure on $\mathbb{R}$ and studying its key properties. Then it characterizes functions which are Cauchy transforms of a probability measure in terms of simple geometrical and analytical properties. The chapter continues with dealing with the orthonormal polynomials associated with a probability measure, Jacobi coefficients, three term recurrence relations and how they relate to the convergence of the continued fraction expansion of the associated Cauchy transform of the probability measure. This self-contained chapter follows mainly [10].

Chapter 3 gives an outline of operator-valued free probability theory. It introduces operator-valued noncommutative probability spaces and random variables. Next, it defines freeness with amalgamation over a subalgebra and discusses some properties of free random variables. A Fock space type construction provides an important example of free random variables and allows one to express any operator-valued distribution as the distribution of a canonical random variable belonging to such space. Having this canonical form, one constructs the R-transform, an operator-valued free analogue of the logarithm of Fourier transform in classical probability, and proves a free central limit theorem for operator-valued random variables. The central limit is called the operator-valued free semicircular distribution and, as in the classical probability case, it is fully described by its first two moments. Finally, the concept of complete positivity of a unital linear map is introduced and as a necessary and sufficient condition, the complete positivity property of the second moment of a semicircular random variable is proved. This self contained chapter follows mainly [13], [14].

Chapter 4 begins by expressing the $R-$transform in terms of the compositional inverse of an operator-valued version of the Cauchy transform. Using the characterization of the $R-$transform of a semicircular random variable in terms of first moment and variance, one finds a quadratic functional equation satisfied by the operator-valued Cauchy transform of that semicircular random variable. While that functional equation may have many solutions, it is shown that exactly one of them is an operator-valued Cauchy transform. The presentation of this chapter follows [14], [15], [2], and [5], with the simplification in the proof of the main result of [5].

Finally, Chapter 5 is dedicated to the presentation and the proof of the main results. First, it is shown that operator-valued semicircular random variables have a certain universality property: for given arbitrary compactly supported probability measure μ, there is a $B(\ell_2)$-valued semicircular random variable s and an extremal state $\rho : B(\ell_2) \to \mathbb{C}$ so that the Cauchy transform G_μ equals to the composition of ρ with the restriction of $B(\ell_2)$-valued Cauchy transform of s to the complex upper half plane, $G_s^*|_{\mathbb{C}^+.1}$. Second, it is explored whether distributions of matrix-valued semicircular random variables can have nontrivial discrete part. This problem is expressed in the measure theoretic terms. For given $M_n(\mathbb{C})$-valued semicircular random variable s with symmetric distribution S_{μ_s} it is investigated whether the associated probability measure μ_s given by the relation $G_{\mu_s}(\xi) = tr_n \circ G_s^*(\xi.1), \quad \xi \in \mathbb{C}^+$ has an atom. It is shown that this necessarily happens if the variance of s is nilpotent, and an example is given to show that the converse does not hold. The chapter closes with several examples clarifying statements of the theorem.

Chapter 2

The Cauchy Transform and Its Continued Fraction

This chapter provides some preliminaries used in Chapters 4 and 5. In section 1, the Cauchy transform is introduced and some of its properties are studied. It is shown that three of those properties are sufficient to identify a function as a Cauchy transform. In Section 2, we discuss the idea of continued fractions and give a continued fraction representation of a category of Cauchy transforms.

2.1 The Cauchy Transform and Its Properties

Definition 2.1.1. Let μ be a probability measure on the Borel σ-algebra of $\mathbb{R}$. The associated Cauchy transform G_μ to μ is defined by:

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t}.$$

Some properties of G_μ are listed in the following proposition, [7, pp. 51-61]. In the following we denote $\mathbb{C}^+ = \{\xi \in \mathbb{C} | Im(\xi) > 0\}$, $\mathbb{C}^- = -\mathbb{C}^+$, and $Supp(\mu) = \mathbb{R} \setminus \bigcup\{U \subseteq \mathbb{R} | U \ open, \mu(U) = 0\}$.

Proposition 2.1.2. *Let $G = G_\mu$ be the Cauchy transform of a compactly supported probability measure μ on $\mathbb{R}$. Then:*

(i) G is analytic on $\mathbb{C} \setminus Supp(\mu)$,

(ii) If $Im(\xi) > 0 (< 0)$, then $Im(G(\xi)) < 0 (> 0)$,

(iii) $\lim_{\Gamma_\alpha(0) \ni \xi \to \infty} \xi . G(\xi) = 1$ *where* $\Gamma_\alpha(0) = \{\xi \in \mathbb{C}^+ : |Re(\xi)| < \alpha Im(\xi)\}$,

(iv) $|G(\xi)| \le |Im(\xi)|^{-1}$,

(v) $Im(\frac{1}{G(\xi)}) \ge Im(\xi)$,

(vi) $G(\bar{\xi}) = \overline{G(\xi)}$, *thus* $G(\xi)$ *is completely determined by its values on the* $\mathbb{C}^+$,

(vii) $\lim_{\Gamma_\alpha(r)\ni\xi\to r}(\xi - r)G(\xi) = \mu(\{r\}) \quad -\infty < r < \infty$,

(viii) For compactly supported μ *the associated* $G(\xi)$ *is analytic around infinity and* $G(\xi) = \sum_{m=0}^{\infty} \frac{\int_{\mathbb{R}} t^m d\mu(t)}{\xi^{m+1}} \quad |\xi| > \sup_{t\in Supp(\mu)} |t|$.

Proof. (i) Let $\xi_0 \in \mathbb{C} \setminus Supp(\mu)$ and $r = \frac{1}{2}dist(\xi_0, Supp(\mu)) > 0$. Then for any $\xi \in B_r(\xi_0)$, the open disc with center ξ_0 and radius r, we have $\frac{|\xi-\xi_0|}{|\xi-t|} < 1$ for all $t \in \mathbb{R}$, and hence, by uniform convergence property we have that:

$$\begin{aligned} G(\xi) &= \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t} = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - \xi_0 + \xi_0 - t} \\ &= \int_{\mathbb{R}} \frac{1}{\xi_0 - t} \frac{1}{1 - \frac{\xi-\xi_0}{t-\xi_0}} d\mu(t) = \int_{\mathbb{R}} \frac{1}{\xi_0 - t} \sum_{m=0}^{\infty} (\frac{\xi - \xi_0}{t - \xi_0})^m d\mu(t) \\ &= \sum_{m=0}^{\infty} \Big(- \int_{\mathbb{R}} (t - \xi_0)^{-m-1} d\mu(t)\Big)(\xi - \xi_0)^m \quad \xi \in B_r(\xi_0). \end{aligned}$$

(ii) Let $Im(\xi) > 0$, then by part (v),

$$\begin{aligned} Im(G(\xi)) &= \frac{1}{2i}(G(\xi) - \overline{G(\xi)}) = \frac{1}{2i} \int_{\mathbb{R}} \Big(\frac{1}{\xi - t} - \frac{1}{\bar{\xi} - t}\Big) d\mu(t) \\ &= -Im(\xi) \int_{\mathbb{R}} \frac{d\mu(t)}{(\xi - t)(\bar{\xi} - t)} \le -Im(\xi) \int_{\mathbb{R}} \frac{d\mu(t)}{(|\xi| + |t|)^2} \\ &< 0. \end{aligned}$$

Now, let $Im(\xi) < 0$, or $Im(\bar{\xi}) > 0$. Then, by above result, $Im(G(\bar{\xi})) < 0$ or by part (v), $Im(\overline{G(\xi)}) < 0$. Hence, $Im(G(\xi)) > 0$.

(iii) This is an straightforward result of the Lebesgue convergence theorem.

(iv) This part follows from the following inequality:

$$|G(\xi)| \le \int_{\mathbb{R}} \frac{d\mu(t)}{|\xi - t|} \le \int_{\mathbb{R}} \frac{d\mu(t)}{|Im(\xi)|} = |Im(\xi)|^{-1} \quad Im(\xi) > 0.$$

(v) Let $f(\xi) = \frac{1}{G(\xi)}$, $\xi \in \mathbb{C}^+$. Then by part (ii) we have:

$$Im(f(\xi)) = Im(\frac{\overline{G(\xi)}}{|G(\xi)|^2}) = -\frac{Im(G(\xi))}{|G(\xi)|^2} > 0.$$

Now, by part (iii), and Lemma 2.1.3 it follows that:

$$f(\xi) = \xi + \int_{\mathbb{R}} \frac{1+t\xi}{t-\xi} d\nu(t) + c,$$

and the assertion follows by $Im(\frac{1+t\xi}{t-\xi}) = \frac{(t^2+1)Im(\xi)}{(t-Re(\xi))^2+(Im(\xi))^2} > 0$.

(vi) It is trivial.

(vii) For fixed $-\infty < r < +\infty$, we have:

$$(\xi - r)G(\xi) - \mu(\{r\}) = \int_{\mathbb{R}} \Big(\frac{\xi - r}{\xi - t} - \chi_{\{r\}}(t)\Big) d\mu(t),$$

and

$$\lim_{\Gamma_\alpha(r)\ni\xi\to r} \Big(\frac{\xi - r}{\xi - t} - \chi_{\{r\}}(t)\Big) = 0 \quad -\infty < t < +\infty,$$

where in which $\Gamma_\alpha(r) = \{\xi \in \mathbb{C}^+ : |Re(\xi) - r| < \alpha Im(\xi)\}$. On the other hand, for any $\xi \in \Gamma_\alpha(r)$ we have:

$$\begin{aligned} \Big|\frac{\xi - r}{\xi - t} - \chi_{\{r\}}(t)\Big| &= \Big|\frac{t-r}{\xi - t} + 1 - \chi_{\{r\}}(t)\Big| \le \frac{|t-r|}{((Re(\xi)-t)^2 + (Im(\xi))^2)^{\frac{1}{2}}} + 1 \\ &\le \frac{|t - Re(\xi)| + |Re(\xi) - r|}{((Re(\xi)-t)^2 + (Im(\xi))^2)^{\frac{1}{2}}} + 1 \le \alpha + 2 \quad -\infty < t < +\infty. \end{aligned}$$

Now, the result follows by Lebesgue convergence theorem.

(viii) This follows from the analyticity of the function

$$G(\frac{1}{\xi}) = \int_{\mathbb{R}} \frac{\xi}{1 - \xi.t} d\mu(t) \quad \Big|\frac{1}{\xi}\Big| < \frac{1}{\sup_{t\in Supp(\mu)} |t|}$$

at $\xi = 0$. Moreover,

$$G(\xi) = \int_{\mathbb{R}} \sum_{m=0}^{\infty} \frac{1}{\xi}(\frac{t}{\xi})^m d\mu(t) = \sum_{m=0}^{\infty} \frac{\int_{\mathbb{R}} t^m d\mu(t)}{\xi^{m+1}}$$

for all $|\xi| > \sup_{t\in Supp(\mu)} |t|$. □

Now, we show that any Cauchy transform can be identified by the first three mentioned properties in the Proposition 2.1.2. To begin with the proof, we first need a lemma, [10, pp. 23-26].

Lemma 2.1.3. *Let $f(\xi)$ be analytic in the upper half plane $\mathbb{C}^+$, and $Im(f(\xi)) \geq 0$ for all $\xi \in \mathbb{C}^+$. Then:*

(i) there exists a bounded increasing function $\nu(t)$ such that:

$$f(\xi) = A\xi + \int_{\mathbb{R}} \frac{1+t\xi}{t-\xi} d\nu(t) + c,$$

where A and c are real constants and $A \geq 0$,

(ii)

$$\lim_{\Gamma_\alpha(0)\ni\xi\to\infty} \frac{f(\xi)}{\xi} = A.$$

Proof. (i) Consider an analytic function $g(\xi')$ in $|\xi'| < 1$ with $Im(g(\xi')) \geq 0$. Then, by Herglotz's theoremin Complex Analysis:

$$g(\xi') = i\int_0^{2\pi} \frac{e^{i\theta}+\xi'}{e^{i\theta}-\xi'} d\beta(\theta) + c,$$

where $\beta(\theta)$ is an increasing bounded function in $[0, 2\pi]$and c is a real constant. Next, the transformation $\xi' = \frac{\xi-i}{\xi+i}, \xi = i\frac{1+\xi'}{1-\xi'}$ maps $|\xi'| < 1$ conformally onto $Im(\xi) > 0$, whilst the transformation $t = -\cot(\frac{\theta}{2}), \frac{t-i}{t+i} = e^{i\theta}$, maps the unit circle onto the real axis. Now, taking $g(\xi') = f(\xi)$, it follows that:

$$\begin{aligned} f(\xi) &= g(\xi') \\ &= i\int_0^{\delta} \frac{e^{i\theta}+\xi'}{e^{i\theta}-\xi'} d\beta(\theta) + i\int_{2\pi-\delta}^{2\pi} \frac{e^{i\theta}+\xi'}{e^{i\theta}-\xi'} d\beta(\theta) + i\int_{-\cot(\frac{\delta}{2})}^{\cot(\frac{\delta}{2})} \frac{1+t\xi}{t-\xi} d\nu(t) + c, \end{aligned}$$

where $\nu(t) = \beta(-2\cot^{-1} t)$, and $\delta > 0$ is small enough. Letting $\delta \to 0$, we get:

$$\begin{aligned} f(\xi) &= i\frac{1+\xi'}{1-\xi'}(\beta(0^+) - \beta(0) + \beta(2\pi) - \beta(2\pi^-)) + \int_{\mathbb{R}} \frac{1+t\xi}{t-\xi} d\nu(t) + c \\ &= (\beta(0^+) - \beta(0) + \beta(2\pi) - \beta(2\pi^-))\xi + \int_{\mathbb{R}} \frac{1+t\xi}{t-\xi} d\nu(t) + c, \end{aligned}$$

completing the proof.

(ii) Fix $\xi_0 \in \Gamma_\alpha(0) = \{\xi \in \mathbb{C}^+ : 0 < \cot^{-1}\alpha \leq \arg\xi \leq \pi - \cot^{-1}\alpha < \pi\}$, then for any $-\infty < t < +\infty$, we have $\sin(\cot^{-1}\alpha) \leq \frac{|\xi_0 - t|}{|t|}$, and consequently $|\frac{t^2+1}{(t-\xi)(t-\xi_0)}| \leq |\frac{1+t^{-2}}{\sin^2(\cot^{-1}\alpha)}|$ for all $\xi \in \Gamma_\alpha(0)$. Now, for given $\epsilon > 0$, there is $T > 0$ such that:

$$\left|\frac{f(\xi)-f(\xi_0)}{\xi-\xi_0} - A\right| = \left|\int_{\mathbb{R}} \frac{t^2+1}{(t-\xi)(t-\xi_0)} d\nu(t)\right| \leq \int_{|t|\leq T} \left|\frac{t^2+1}{(t-\xi)(t-\xi_0)}\right| d\nu(t) + \epsilon,$$

and by Lebesgue convergence theorem it follows that $|\lim_{\Gamma_\alpha(0)\ni\xi\to\infty} \frac{f(\xi)-f(\xi_0)}{\xi-\xi_0} - A| \leq \epsilon$, and consequently

$$\lim_{\Gamma_\alpha(0)\ni\xi\to\infty} \frac{f(\xi)}{\xi} = \lim_{\Gamma_\alpha(0)\ni\xi\to\infty} \frac{\left(\frac{f(\xi)}{\xi}\right)}{\left(\frac{f(\xi)-f(\xi_0)}{\xi-\xi_0}\right)}\left(\frac{f(\xi)-f(\xi_0)}{\xi-\xi_0}\right) = \lim_{\Gamma_\alpha(0)\ni\xi\to\infty} \frac{f(\xi)-f(\xi_0)}{\xi-\xi_0} = A,$$

yielding the assertion. □

Theorem 2.1.4. *If $G(\xi)$ is analytic in the upper half plane $\mathbb{C}^+$ and $Im(G(\xi)) \leq 0$, and if in addition $\lim_{\Gamma_\alpha(0)\ni\xi\to\infty} \xi.G(\xi) = 1$, then:*

$$G(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t},$$

where in which $\mu(t)$ is a bounded increasing probability measure.

Proof. By Lemma 2.1.3,

$$-G(\xi) = A\xi + \int_{\mathbb{R}} \frac{1+t\xi}{t-\xi} d\nu(t) + c$$

where in which

$$A = \lim_{\Gamma_\alpha(0)\ni\xi\to\infty} \frac{-G(\xi)}{\xi} = -\lim_{\Gamma_\alpha(0)\ni\xi\to\infty} \xi.G(\xi) \lim_{\Gamma_\alpha(0)\ni\xi\to\infty} \frac{1}{\xi^2} = 0.$$

We now have:

$$Re(iyG(iy)) = \int_{\mathbb{R}} \frac{y^2(1+t^2)}{t^2+y^2} d\nu(t) = \int_{\mathbb{R}} \frac{1+t^2}{1+t^2y^{-2}} d\nu(t) \quad (y > 0).$$

Since the left hand side tends to a finite limit as $y \to +\infty$, we see that there exists a constant $M > 0$ such that for some $y_0 > 0$:

$$\int_{\mathbb{R}} \frac{1+t^2}{1+t^2y^{-2}} d\nu(t) < M \quad (y > y_0 > 0).$$

Now, for fixed $T > 0$ we have:

$$\int_{|t|<T} \frac{1+t^2}{1+t^2y^{-2}} d\nu(t) < M \quad (y > y_0 > 0).$$

Let $y \to +\infty$ to obtain

$$\int_{|t|<T} 1 + t^2 d\nu(t) < M \quad (T > 0).$$

and, hence, $\int_{\mathbb{R}} 1 + t^2 d\nu(t) < \infty$ implying the existence of the integral $\int_{\mathbb{R}} t^2 d\nu(t)$ and consequently existence of the integral $\int_{\mathbb{R}} t d\nu(t)$. Next, by defining the bounded increasing function $\mu(t) = \int_{-\infty}^{t} (1 + s^2) d\nu(s)$ we may write:

$$-G(\xi) = c - \int_{\mathbb{R}} t d\nu(t) + \int_{\mathbb{R}} \frac{d\mu(t)}{t - \xi} = c' + \int_{\mathbb{R}} \frac{d\mu(t)}{t - \xi}.$$

On the other hand:

$$\xi \int_{\mathbb{R}} \frac{d\mu(t)}{t - \xi} = - \int_{\mathbb{R}} d\mu(t) + \int_{\mathbb{R}} \frac{t d\mu(t)}{t - \xi},$$

and since in the area $\Gamma_\alpha(0) = \{\xi \in \mathbb{C}^+ : 0 < \cot^{-1}\alpha \leq \arg \xi \leq \pi - \cot^{-1}\alpha < \pi\}$, we have $\sin(\cot^{-1}\alpha)^{-1} \geq \frac{|t|}{|t-\xi|}$, it is easily seen that:

$$\lim_{\Gamma_\alpha(0) \ni \xi \to \infty} \xi \int_{\mathbb{R}} \frac{d\mu(t)}{t - \xi} = - \int_{\mathbb{R}} d\mu(t).$$

From above results it follows that:

$$1 = \lim_{\Gamma_\alpha(0) \ni \xi \to \infty} \xi . G(\xi) = \lim_{\Gamma_\alpha(0) \ni \xi \to \infty} (-c'\xi - \xi \int_{\mathbb{R}} \frac{d\mu(t)}{t - \xi}) = \lim_{\Gamma_\alpha(0) \ni \xi \to \infty} -c'\xi + \int_{\mathbb{R}} d\mu(t),$$

yielding $c' = 0$, and the desired result follows. □

The Cauchy transform turns out to be important in the context of free probability theory. The following notion is fundamental in non-commutative probability theory, [12, pp. 5-9].

Definition 2.1.5. A pair $(\mathcal{A}, \tau)$ consisting of a unital $*$-algebra $\mathcal{A}$ and a linear functional $\tau : \mathcal{A} \to \mathbb{C}$ with $\tau(1) = 1$, and $\tau(a^*a) \geq 0$ for all $a \in A$ is called a non-commutative probability space. Any element $a \in \mathcal{A}$ is called a non-commutative random variable.

Let $(\mathcal{A}, \tau)$ be a non-commutative probability space in which $\mathcal{A}$ is a von-Neumann algebra, and τ is a positive normal trace. For a self adjoint random variable $a \in \mathcal{A}$ we define the map $G_a : \mathbb{C}^+ \to \mathbb{C}^-$ via

$$G_a(\xi) = \sum_{m=0}^{\infty} \frac{\tau(a^m)}{\xi^{m+1}} = \tau(\frac{1}{\xi - a}).$$

One can easily check that all conditions of the Theorem 2.1.4 hold, and hence there exists a probability measure μ_a on $\mathbb{R}$, called the distribution of a, such that:

$$G_a(\xi) = \int_{\mathbb{R}} \frac{d\mu_a(t)}{\xi - t}.$$

The spectrum of $a \in A$ denoted by $\sigma(a)$ coincides with $Supp(\mu_a)$.

2.2 The Continued Fraction Representation of The Cauchy Transform

Definition 2.2.1. A sequence $\{\omega_n\}_{n=1}^{\infty}$ is called a Jacobi sequence if either $\omega_n > 0$ $(n \geq 1)$ or there exists a number $m_0 \geq 1$ such that $\omega_n > 0$ $(1 \leq n < m_0)$ and $\omega_n = 0$ $(n \geq m_0)$.

Definition 2.2.2. A probability measure μ on $\mathbb{R}$ is said to have finite moment of order $m \geq 1$ if $\int_{\mathbb{R}} |t|^m d\mu(t) < \infty$, and in this case the m^{th} moment of μ is defined by $M_m = \int_{\mathbb{R}} t^m d\mu(t)$.

We denote the set of all probability measures on $\mathbb{R}$ having finite moments of all orders by $\mathcal{B}_{fm}(\mathbb{R})$. It is trivial that any compactly supported probability measure is in this set.

For a given sequence $\{M_m\}_{m=0}^{\infty}$ $(M_0 = 1)$ of real numbers we define its associated sequence $\{\Delta_m\}_{m=0}^{\infty}$ of Hankel determinants via

$$\Delta_m = \det\begin{pmatrix} M_0 & M_1 & \cdots & M_m \\ M_1 & M_2 & \cdots & M_{m+1} \\ \cdots & \cdots & \cdots & \cdots \\ M_m & M_{m+1} & \cdots & M_{2m} \end{pmatrix} \quad m \geq 0,$$

and we denote by $\mathcal{M}$ the set of all sequences $\{M_m\}_{m=0}^{\infty}$ $(M_0 = 1)$ of the real numbers whose associated sequence of Hankel determinants $\{\Delta_m\}_{m=0}^{\infty}$ are Jacobi sequences.

Classical Moment Problem: Given a sequence $\{M_m\}_{m=0}^{\infty}$ $(M_0 = 1)$ of real numbers, find its sufficient and necessary conditions so that $\{M_m\}_{m=0}^{\infty} = \{M_m(\mu)\}_{m=0}^{\infty}$ for some $\mu \in \mathcal{B}_{fm}(\mathbb{R})$.

The following theorem of Hamburger gives a necessary and sufficient condition for the solution of the classical moment problem [7]:

Theorem 2.2.3. *The infinite sequence* $\{M_m\}_{m=0}^{\infty}$ $(M_0 = 1)$ *of the real numbers is a moment sequence of a certain* $\mu \in \mathcal{B}_{fm}(\mathbb{R})$ *if and only if* $\{M_m\}_{m=0}^{\infty} \in \mathcal{M}$.

Corollary 2.2.4. *Let $(\mathcal{A}, \tau)$ be a non-commutative probability space. Then, for any self-adjoint random variable $a \in \mathcal{A}$ there is a probability measure $\mu \in \mathcal{B}_{fm}(\mathbb{R})$ such that:*

$$\tau(a^m) = \int_{\mathbb{R}} t^m d\mu(t) \ (m \geq 1).$$

Proof. For $m \geq 0$ we set $M_m = \tau(a^m)$. Consider the associated Hankel determinant Δ_m $(m \geq 0)$. Let $m \geq 1$ be fixed and consider $x = \sum_{i=0}^{m} c_i a^i$ where $c_i \in \mathbb{C}$ $(0 \leq i \leq m)$. Then, $x \in \mathcal{A}$, and

$$0 \leq \tau(x^* x) = \sum_{i,j=0}^{m} \overline{c_i} c_j \tau(a^{i+j}) = \sum_{i,j=0}^{m} \overline{c_i} c_j M_{i+j}.$$

Since the above inequality holds for any choice of $c_i \in \mathbb{C}$ $(0 \leq i \leq m)$, it follows that the matrix $(M_{i+j}) \in M_{m+1}(\mathbb{C})$ is positive definite, so that $\Delta_m \geq 0$. To prove that the sequence $\{\Delta_m\}_{m=0}^{\infty}$ is a Jacobi sequence, assume that $\Delta_{m_0} = 0$ for some $m_0 \geq 1$. Then, there exists a choice $(c_0, c_1, \cdots, c_{m_0}) \neq (0, 0, \cdots, 0)$ such that $\sum_{i,j=0}^{m_0} \overline{c_i} c_j M_{i+j} = 0$. Then, setting $c_{m_0+1} = 0$, we obtain $\sum_{i,j=0}^{m_0+1} \overline{c_i} c_j M_{i+j} = \sum_{i,j=0}^{m_0} \overline{c_i} c_j M_{i+j} = 0$. Now, since $(c_0, c_1, \cdots, c_{m_0}, 0) \neq (0, 0, \cdots, 0)$ we have $\Delta_{m_0+1} = 0$, and accordingly $\Delta_m = 0$ $(m \geq m_0)$. □

The probability measure mentioned in above Corollary is called the distribution of the self adjoint random variable $a \in \mathcal{A}$. Indeed, it coincides with the distribution μ_a introduced in previous section.

Definition 2.2.5. A probability measure $\mu \in \mathcal{B}_{fm}(\mathbb{R})$ is called the solution of the determinate moment problem if μ is determined uniquely by its moment sequence $\{M_m(\mu)\}_{m=0}^{\infty}$.

In this context, we have the following Carleman's moment test:

Theorem 2.2.6. *Let the sequence $\{M_m\}_{m=0}^{\infty} \in \mathcal{M}$ satisfy the condition*

$$\sum_{m=1}^{\infty} M_{2m}^{\frac{-1}{2m}} = +\infty.$$

Then, there exists a unique $\mu \in \mathcal{B}_{fm}(\mathbb{R})$ whose moment sequence is $\{M_m\}_{m=0}^{\infty} \in \mathcal{M}$.

Corollary 2.2.7. *A compactly supported probability measure $\mu \in \mathcal{B}_{fm}(\mathbb{R})$ is a solution of the determinate moment problem.*

Proof. Let $Supp(\mu) \subseteq [-b, b]$ for some $b > 0$. Then,

$$M_{2m}^{\frac{-1}{2m}}(\mu) = (\int_{-b}^{b} t^{2m} d\mu(t))^{\frac{-1}{2m}} \geq b^{-1} \ (m \geq 1),$$

implying $\sum_{m=1}^{\infty} M_{2m}^{\frac{-1}{2m}}(\mu) = +\infty$. □

Let $\mu \in \mathcal{B}_{fm}(\mathbb{R})$ and consider the Hilbert space $L^2(\mathbb{R}, \mu)$, with the inner product:

$$\langle f, g \rangle_\mu = \int_{\mathbb{R}} \overline{f(t)} g(t) d\mu(t), \quad f, g \in L^2(\mathbb{R}, \mu).$$

We have:

Lemma 2.2.8. *Let $\mu \in \mathcal{B}_{fm}(\mathbb{R})$.*
(i) If $|Supp(\mu)| = \infty$, then the monomials $\{t^m\}_{m=0}^{\infty} \subseteq L^2(\mathbb{R}, \mu)$ are linearly independent.
(ii) If $|Supp(\mu)| = m_0 < \infty$, then the monomials $\{t^m\}_{m=0}^{m_0-1} \subseteq L^2(\mathbb{R}, \mu)$ are maximal linearly independent subset of $\{t^m\}_{m=0}^{\infty}$.

Proof. (i) Suppose for some $(c_0, c_1, \cdots, c_m) \neq (0, 0, \cdots, 0)$ $(m \geq 1)$ we have:

$$g(t) := \sum_{k=0}^{m} c_k t^k = 0, \text{ for } \mu - a.e.\ t \in \mathbb{R}.$$

Since $\mathbb{R} \setminus g^{-1}(0)$ is an open subset of $\mathbb{R}$, by definition we have:

$$Supp(\mu) = \mathbb{R} \setminus \cup \{U \subseteq \mathbb{R} : U \ open,\ \mu(U) = 0\} \subseteq \mathbb{R} \setminus (\mathbb{R} \setminus g^{-1}(0)) = g^{-1}(0),$$

implying $|Supp(\mu)| \leq |g^{-1}(0)| \leq m$, a contradiction.

(ii) Let $Supp(\mu) = \{a_m\}_{m=1}^{m_0}$, and $g(t) = 0$ $\mu - a.e.\ t \in \mathbb{R}$. Since $\mu(\mathbb{R} \setminus g^{-1}(0)) = 1$, it follows that for some polynomial $h(t)$ we have:

$$g(t) = h(t)(t - a_1) \cdots (t - a_{m_0}).$$

On the other hand, any nontrivial linear combination of $\{t^m\}_{m=0}^{m_0-1}$ is of degree less than m_0, it is a non-zero function in $L^2(\mathbb{R}, \mu)$ showing that $\{t^m\}_{m=0}^{m_0-1}$ is linearly independent in $L^2(\mathbb{R}, \mu)$. For maximality, it is sufficient to prove that $\{t^m\}_{m=0}^{m_0-1} \cup \{t^n\}$ $(n \geq m_0)$ is not linearly independent. To that end, we have:

$$t^n = h_1(t)(t - a_1) \cdots (t - a_{m_0}) + f(t),$$

for some polynomials $h_1(t)$ and $f(t)$ with $\deg(f) < m_0$. But, $h_1(t)(t - a_1) \cdots (t - a_{m_0}) = 0$ for $\mu - a.e.\ t \in \mathbb{R}$, implying $f(t) = t^n$ for $\mu - a.e.\ t \in \mathbb{R}$, and consequently, $n < m_0$, a contradiction. $\square$

By applying the Gram-Schmidt orthogonalization procedure to the sequence of monomials $\{t^m\}_{m=0}^{\infty} \subseteq L^2(\mathbb{R}, \mu)$, if $|Supp(\mu)| = \infty$, we obtain an infinite sequence of orthogonal polynomials $\{p_m(t)\}_{m=0}^{\infty}$, whilst if $|Supp(\mu)| = m_0 < \infty$, the procedure terminates in m_0 steps and we obtain a finite sequence of orthogonal polynomials $\{p_m(t)\}_{m=0}^{m_0-1}$, where in which $\deg(p_m(t)) = m$ $(m \geq 0)$, and $\langle p_m, p_n \rangle_\mu = \int_{\mathbb{R}} p_m(t) p_n(t) d\mu(t) = \delta_{m,n}$ $(m, n \geq 0)$. Since the property of being orthogonal does not change by a constant factor, we adjust p_m's to be monic polynomials, and, the sequence $\{p_m(t)\}_{m=0}^{\infty}$ obtained in this way is called the orthogonal polynomials associated with μ.

Using the idea of orthogonal polynomials, we are in the position to prove the existence of the so-called Jacobi coefficients of the given probability measure $\mu \in \mathcal{B}_{fm}(\mathbb{R})$, [7].

Theorem 2.2.9. *Let $\{p_m(t)\}_{m=0}^{\infty}$ be the orthogonal polynomials associated with given $\mu \in \mathcal{B}_{fm}(\mathbb{R})$. Then, there exists a pair of sequences $\{\alpha_m\}_{m=1}^{\infty} \subseteq \mathbb{R}$ and $\{\omega_m\}_{m=1}^{\infty} \subseteq \mathbb{R}^+$ uniquely determined by:*

$$\begin{aligned} p_0(t) &= 1, \\ p_1(t) &= t - \alpha_1, \\ tp_m(t) &= p_{m+1}(t) + \alpha_{m+1} p_m(t) + \omega_m p_{m-1}(t), \ m \geq 1 \end{aligned} \tag{2.1}$$

where in which, if $|Supp(\mu)| = \infty$, both $\{\alpha_m\}_{m=1}^{\infty}, \{\omega_m\}_{m=1}^{\infty}$ are infinite sequences, and if $|Supp(\mu)| = m_0 < \infty$, we have $\{\alpha_m\}_{m=1}^{\infty} = \{\alpha_m\}_{m=1}^{m_0}$ and $\{\omega_m\}_{m=1}^{\infty} = \{\omega_m\}_{m=1}^{m_0-1}$ with $p_{m_0} = 0$.

Proof. Suppose that $|Supp(\mu)| = \infty$. As seen above, the orthogonal polynomials $\{p_m(t)\}_{m=0}^{\infty}$ form an infinite sequence. By definition, $p_0(t) = 1$, and since $p_1(t) = t - \alpha_1$ and $\langle p_1, p_0 \rangle_\mu = 0$, we see that:

$$\alpha_1 = \int_{\mathbb{R}} t d\mu(t).$$

Let $m \geq 1$, and consider $tp_m(t)$. Since it is a monic polynomial of degree $m+1$, it will be a unique linear combination of $p_0(t), p_1(t), \cdots, p_{m+1}(t)$, say:

$$tp_m(t) = p_{m+1}(t) + \sum_{j=0}^{m} c_{m,j} p_m(t).$$

Now, we have:

$$c_{m,j}\langle p_j, p_j\rangle_\mu = \langle p_j, i_d p_m\rangle_\mu = \langle i_d p_j, p_m\rangle_\mu = 0 \quad 0 \le j \le m-2,$$

and by $\langle p_j, p_j\rangle_\mu \ne 0$, it follows that $c_{m,j} = 0$ $(0 \le j \le m-2)$. Consequently:

$$tp_m(t) = p_{m+1}(t) + c_{m,m}p_m(t) + c_{m,m-1}p_{m-1}(t) \ m \ge 1,$$

proving the first part of the assertion with $\alpha_{m+1} = c_{m,m}$ and $\omega_m = c_{m,m-1}$.

To prove the second part of the assertion, integrating Equation (2.1) with $m = 1$ yields:

$$\omega_1 = \int_{\mathbb{R}} tp_1(t)d\mu(t) = \int_{\mathbb{R}} (t-\alpha_1)p_1(t)d\mu(t) = \int_{\mathbb{R}} p_1^2(t)d\mu(t) > 0.$$

Let $m \ge 2$, then from Equation (2.1) we see that:

$$\begin{aligned} \omega_m\langle p_{m-1}, p_{m-1}\rangle_\mu &= \langle p_{m-1}, i_d p_m\rangle_\mu = \langle i_d p_{m-1}, p_m\rangle_\mu \\ &= \langle p_m + \alpha_m p_{m-1} + \omega_{m-1}p_{m-2}, p_m\rangle_\mu = \langle p_m, p_m\rangle_\mu, \end{aligned}$$

giving $\omega_m = \frac{\langle p_m, p_m\rangle_\mu}{\langle p_{m-1}, p_{m-1}\rangle_\mu} > 0$ $(m \ge 2)$, and, completing the proof for the case of $|Supp(\mu)| = \infty$. The proof of the case $|Supp(\mu)| < \infty$ is a small modification of the above proof. □

Definition 2.2.10. The pair of sequences $(\{\omega_m\}_{m=1}^\infty, \{\alpha_m\}_{m=1}^\infty)$ determined in above theorem is called the Jacobi coefficients of the probability measure $\mu \in \mathcal{B}_{fm}(\mathbb{R})$.

To calculate the Jacobi coefficients we have the following:

Corollary 2.2.11. *Let $\{p_m(t)\}_{m=0}^\infty$ be the orthogonal polynomials associated with given $\mu \in \mathcal{B}_{fm}(\mathbb{R})$. Then, the Jacobi coefficients $(\{\omega_m\}_{m=1}^\infty, \{\alpha_m\}_{m=1}^\infty)$ are calculated by:*

$$\begin{aligned} \omega_1\omega_2\cdots\omega_m &= \int_{\mathbb{R}} p_m^2(t)d\mu(t) \ m \ge 1, \\ \alpha_1 &= \int_{\mathbb{R}} td\mu(t), \\ \omega_1\cdots\omega_{m-1}\alpha_m &= \int_{\mathbb{R}} tp_{m-1}^2(t)d\mu(t) \ m \ge 2. \end{aligned}$$

Proof. Referring to the proof of the above theorem, we have:

$$\omega_1\omega_2\cdots\omega_m = \prod_{j=1}^m \frac{\langle p_j, p_j\rangle_\mu}{\langle p_{j-1}, p_{j-1}\rangle_\mu} = \frac{\langle p_m, p_m\rangle_\mu}{\langle p_0, p_0\rangle_\mu},$$

and

$$\begin{aligned}\omega_1 \cdots \omega_{m-1}\alpha_m &= \prod_{j=1}^{m}\omega_j \frac{\alpha_m}{\omega_m} = \langle p_m, p_m\rangle_\mu \frac{\alpha_m}{\omega_m} = \alpha_m \langle p_{m-1}, p_{m-1}\rangle_\mu \\ &= \langle \alpha_m p_{m-1}, p_{m-1}\rangle_\mu = \langle i_d p_{m-1}, p_{m-1}\rangle_\mu .\end{aligned}$$

□

One important application of the Jacobi coefficients of a given probability measure $\mu \in \mathcal{B}_{fm}(\mathbb{R})$, is that it enables us to calculate its moment sequence. Before stating the related result, we need some definitions:

Definition 2.2.12. Let S be a finite set. A partition ϑ of non-empty subsets ν of S is called (i) a pair partition if $|\nu| = 2$ for all $\nu \in \vartheta$, (ii) a pair partition with singletons if either $|\nu| = 2$ or $|\nu| = 1$ for all $\nu \in \vartheta$. Any element $\nu \in \vartheta$ with $|\nu| = 1$ is called a singleton.

Definition 2.2.13. Let ϑ be a pair partition with singletons of $S_m = \{1, \cdots, m\}$ $(m \geq 1)$, say

$$\vartheta = \{\{s_1\}, \cdots, \{s_{j_1}\}, \{l_1, r_1\}, \cdots, \{l_{j_2}, r_{j_2}\}\}$$

where we may assume without loss of generality that:

$$s_1 < \cdots < s_{j_1}, \; l_1 < \cdots < l_{j_2}, \quad l_1 < r_1, \cdots, l_{j_2} < r_{j_2}.$$

We call ϑ a non-crossing partition of S_m $(m \geq 1)$, if for any $1 \leq i_1, i_2 \leq j_2$:

$$[l_{i_1}, r_{i_1}] \subseteq [l_{i_2}, r_{i_2}] \; or \; [l_{i_2}, r_{i_2}] \subseteq [l_{i_1}, r_{i_1}] \; or \; [l_{i_1}, r_{i_1}] \cap [l_{i_2}, r_{i_2}] = \phi.$$

We denote $\mathcal{P}_{NCP}(m)$ and $\mathcal{P}_{NCPS}(m)$ as the set of non-crossing pair partitions of S_m $(m \geq 1)$ and that of non-crossing pair partitions with singletons of S_m $(m \geq 1)$, respectively.

Definition 2.2.14. Let $\vartheta \in \mathcal{P}_{NCPS}(m)$. The depth of $\nu \in \vartheta$ is defined by:

$$d_\vartheta(\nu) = \begin{cases} |\{\{a < b\} \in \vartheta : a < s < b\}| + 1 & \text{if } \nu = \{s\} \\ |\{\{a < b\} \in \vartheta : a < l < r < b\}| + 1 & \text{if } \nu = \{l < r\}. \end{cases}$$

As an example, for $S_9 = \{1, \cdots, 9\}$ with $\vartheta = \{\{5\}, \{1,2\}, \{3,9\}, \{4,8\}, \{6,7\}\}$ we have $d_\vartheta(\{1,2\}) = 1$, $d_\vartheta(\{4,8\}) = 2$ and $d_\vartheta(\{5\}) = 3$.

Remark 2.2.15. It is trivial that

$$\max_{\nu \in \vartheta}(d_\vartheta(\nu)) \le [\frac{m+2}{2}] \ (m \ge 1).$$

The following theorem is referred as Accardi-Bożejko theorem, [7, pp. 29-34].

Theorem 2.2.16. *Let $\mu \in \mathcal{B}_{fm}(\mathbb{R})$, let $\{M_m\}_{m=1}^\infty$ be its moment sequence with associated Jacobi coefficients $(\{\omega_m\}_{m=1}^\infty, \{\alpha_m\}_{m=1}^\infty)$. Then:*

$$M_m = \sum_{\vartheta \in \mathcal{P}_{NCPS}(m)} \Big(\prod_{\nu \in \vartheta : |\nu| = 1} \alpha_{(d_\vartheta(\nu))} \prod_{\nu \in \vartheta : |\nu| = 2} \omega_{(d_\vartheta(\nu))} \Big) \ m \ge 1.$$

It worths mentioning that from Accardi-Bożejko theorem it follows that to calculate the m^{th} moment M_m we need at most the first $[\frac{m+2}{2}]$ terms of Jacobi coefficients. We will use this result in the proof of continued fraction representation of the Cauchy transform.

Definition 2.2.17. Let $\{a_m\}_{m=1}^\infty$ and $\{b_m\}_{m=1}^\infty$ be two sequences of complex numbers. Expressions of the form:

$$CF(\{a_m\}_{m=1}^n, \{b_m\}_{m=1}^n) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n}}}} \ (n \ge 1),$$

$$CF(\{a_m\}_{m=1}^\infty, \{b_m\}_{m=1}^\infty) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}}$$

are called the n^{th} convergent of continued fraction, and the continued fraction, respectively.

To be more precise, for the map $\tau_k : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ $(k \ge 1)$ defined by $\tau_k(z) = \frac{a_k}{b_k + z}$, we have:

$$CF(\{a_m\}_{m=1}^n, \{b_m\}_{m=1}^n) = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_n(0) \ (n \ge 1)$$

and

$$CF(\{a_m\}_{m=1}^\infty, \{a_m\}_{m=1}^\infty) = \lim_{n \to \infty} \tau_1 \circ \tau_2 \circ \cdots \circ \tau_n(0).$$

Lemma 2.2.18. *Let $\{a_m\}_{m=1}^\infty$ and $\{b_m\}_{m=1}^\infty$ be two sequences of complex numbers. Define the sequences $\{A_m\}_{m=1}^\infty$ and $\{B_m\}_{m=1}^\infty$ respectively by the following recurrence relations:*

$$\begin{cases} A_{-1} = 1, \ A_0 = 0, \ A_n = b_n A_{n-1} + a_n A_{n-2} & n \ge 1 \\ B_{-1} = 0, \ B_0 = 1, \ B_n = b_n B_{n-1} + a_n B_{n-2} & n \ge 1. \end{cases}$$

Then:

$$\tau_1 \circ \tau_2 \circ \cdots \circ \tau_n(z) = \frac{A_n + A_{n-1}z}{B_n + B_{n-1}z} \ (n \geq 1), \ \ \tau_1 \circ \tau_2 \circ \cdots \circ \tau_n(0) = \frac{A_n}{B_n} \ (n \geq 1).$$

Proof. Induct on n, and use $\tau_1 \circ \tau_2 \circ \cdots \tau_n \circ \tau_{n+1}(z) = (\tau_1 \circ \tau_2 \circ \cdots \circ \tau_n)(\tau_{n+1}(z))$. □

A simple application of above lemma yields the following:

Proposition 2.2.19. *Let $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$ and $\omega_1 > 0, \cdots, \omega_n > 0$ be constant numbers. Let $\{p_k(\xi)\}_{k=1}^{\infty}$ and $\{q_k(\xi)\}_{k=1}^{\infty}$ be monic polynomials defined respectively by the following recurrence relations:*

$$p_0(\xi) = 1, \ \ p_1(\xi) = \xi - \alpha_1, p_k(\xi) = (\xi - \alpha_k)p_{k-1}(\xi) - \omega_{k-1}p_{k-2}(\xi) 2 \leq k \leq n$$

$$(2.2)$$

$$q_0(\xi) = 1, \ \ q_1(\xi) = \xi - \alpha_2, q_k(\xi) = (\xi - \alpha_{k+1})q_{k-1}(\xi) - \omega_k q_{k-2}(\xi) 2 \leq k \leq n-1.$$

Then:

$$\cfrac{1}{\xi - \alpha_1 - \cfrac{\omega_1}{\xi - \alpha_2 - \cfrac{\omega_2}{\xi - \alpha_3 - \ddots - \cfrac{\omega_{k-1}}{\xi - \alpha_k}}}} = \frac{q_{k-1}(\xi)}{p_k(\xi)} \quad 1 \leq k \leq n.$$

Theorem 2.2.20. *Let $(\{\omega_1, \cdots, \omega_{n-1}\}, \{\alpha_1, \cdots, \alpha_n\})$ be Jacobi coefficients of finite type. Define the polynomials $p_0(\xi), p_1(\xi), \cdots, p_n(\xi)$ by recurrence relations (2.2) and the measure μ on $\mathbb{R}$ by*

$$\mu = \sum_{\lambda : p_n(\lambda)=0} \|f(\lambda)\|^{-2}\delta_\lambda : \quad \|f(\lambda)\|^{-2} = (\sum_{j=1}^{n-1} \frac{p_j^2(\lambda)}{\omega_1 \cdots \omega_j})^{-1}.$$

Then $|Supp(\mu)| = n$, and $\{p_0(\xi), p_1(\xi), \cdots, p_{n-1}(\xi)\}$ form the orthogonal polynomials associated with μ. Moreover:

$$\cfrac{1}{\xi - \alpha_1 - \cfrac{\omega_1}{\xi - \alpha_2 - \cfrac{\omega_2}{\xi - \alpha_3 - \ddots - \cfrac{\omega_{n-1}}{\xi - \alpha_n}}}} = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t} \quad \xi \in \mathbb{C} \setminus \{\lambda : p_n(\lambda) = 0\}.$$

Proof. (Sketch) Define the tridiagonal Jacobi matrix

$$T = (\omega_{k-1}^{\frac{1}{2}}\delta_{k(l+1)} + \alpha_k\delta_{kl} + \omega_k^{\frac{1}{2}}\delta_{(k+1)l})_{n\times n}.$$

It follows that every eigenvalue of T is simple, hence $|\sigma(T)| = n$, and furthermore:

$$\langle e_1, (\xi - T)^{-1} e_1 \rangle_{\ell_2^n} = \cfrac{1}{\xi - \alpha_1 - \cfrac{\omega_1}{\xi - \alpha_2 - \cfrac{\omega_2}{\xi - \alpha_3 - \ddots - \cfrac{\omega_{n-1}}{\xi - \alpha_n}}}} \tag{2.3}$$

[7, pp. 45-47], and consequently, by Proposition 2.2.19,

$$\langle e_1, (\xi - T)^{-1} e_1 \rangle_{\ell_2^n} = \frac{q_{n-1}(\xi)}{p_n(\xi)}, \tag{2.4}$$

where $p_n(\xi)$ and $q_{n-1}(\xi)$ are the monic polynomials defined by the recurrence relations (2.2). On the other hand, we have $p_n(\xi) = \det(\xi - T)$, and consequently, $\sigma(T) = \{\lambda \in \mathbb{C} : p_n(\lambda) = 0\}$. Next, for any $\lambda \in \sigma(T)$, for its associated eigenvector $f(\lambda)$ we have:

$$\|f(\lambda)\|^2 = (\sum_{j=1}^{n-1} \frac{p_j^2(\lambda)}{\omega_1 \cdots \omega_j}),$$

[7, pp. 48-49]. Now, define a measure μ on $\mathbb{R}$ by

$$\mu = \sum_{\lambda \in \sigma(T)} \|f(\lambda)\|^{-2} \delta_\lambda,$$

then, by considering the finite orthonormal basis $\{\frac{f(\lambda)}{\|f(\lambda)\|} : \lambda \in \sigma(T)\}$, the spectral decomposition of T, and (2.4) it follows that the polynomials $p_0(\xi), p_1(\xi), \cdots, p_{n-1}(\xi)$ defined by the recurrence relations (2.2) are orthogonal with respect μ, and

$$\langle e_1, (\xi - T)^{-1} e_1 \rangle_{\ell_2^n} = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t}, \tag{2.5}$$

[7, p. 50]. Eventually, by (2.3) and (2.5) the desired result is proved. □

We are interested in the continued fraction representation of the Cauchy transform $G_\mu(\xi)$. For μ having a finite support, the result established in the previous theorem. For general $\mu \in \mathcal{B}_{fm}(\mathbb{R})$ we have:

Theorem 2.2.21. *Let $\mu \in \mathcal{B}_{fm}(\mathbb{R})$ and $(\{\omega_n\}_{n=1}^{\infty}, \{\alpha_n\}_{n=1}^{\infty})$ be its Jacobi coefficients. If μ is the solution of the determinate moment problem, then the Cauchy transform of it is expanded into a continued fraction*

$$G_\mu(\xi) = \cfrac{1}{\xi - \alpha_1 - \cfrac{\omega_1}{\xi - \alpha_2 - \cfrac{\omega_2}{\xi - \alpha_3 - \ddots}}} \quad Im(\xi) \neq 0.$$

Proof. The assertion for $\mu \in \mathcal{B}_{fm}(\mathbb{R})$ having a finite support, or equivalently, for $(\{\omega_n\}_{n=1}^{\infty}, \{\alpha_n\}_{n=1}^{\infty})$ being Jacobi coefficients of finite type, has already proved in the previous theorem. So, we may assume that μ has an infinite support and its Jacobi coefficient $(\{\omega_n\}_{n=1}^{\infty}, \{\alpha_n\}_{n=1}^{\infty})$ is of infinite type. For this, define polynomials $p_0(t), p_1(t), \cdots, p_n(t), \cdots$ by the recurrence relations (2.2). For each $n \geq 1$, let μ_n be the unique probability measure whose Jacobi coefficient is $(\{\omega_m\}_{m=1}^{n-1}, \{\alpha_m\}_{m=1}^{n})$. It then follows from the previous theorem that $\{p_0(t), p_1(t), \cdots, p_{n-1}(t)\}$ form the orthogonal polynomials associated with μ_n and:

$$\cfrac{1}{\xi - \alpha_1 - \cfrac{\omega_1}{\xi - \alpha_2 - \cfrac{\omega_2}{\xi - \alpha_3 - \ddots - \cfrac{\omega_{n-1}}{\xi - \alpha_n}}}} = \int_{\mathbb{R}} \frac{d\mu_n(t)}{\xi - t} \quad Im(\xi) \neq 0. \tag{2.6}$$

Now, considering the m^{th} moment of μ_n, it follows from Theorem 2.2.16 that for calculating $M_m(\mu_n)$ we need at most the first $[\frac{m+2}{2}]$ terms of Jacobi coefficients of μ_n. Hence, for a fixed m, the sequence $M_m(\mu_n)$ stays constant for all large $n > [\frac{m+2}{2}]$. Since the Jacobi coefficient of μ_n is obtained by cutting off the Jacobi coefficient of μ, the constant coincides with $M_m(\mu)$. Therefore, we have:

$$\lim_{n\to\infty} M_m(\mu_n) = M_m(\mu) \quad (m \geq 1).$$

Since μ is the solution of the determinate moment problem, it follows that that the associated sequence of probability measures μ_n $(n \geq 1)$ will be weakly convergent to the probability measure μ. Hence:

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{d\mu_n(t)}{\xi - t} = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t} \quad Im(\xi) \neq 0. \tag{2.7}$$

On the other hand, using (2.6), we have:

$$\lim_{n\to\infty}\int_{\mathbb{R}}\frac{d\mu_n(t)}{\xi - t} = \cfrac{1}{\xi-\alpha_1-\cfrac{\omega_1}{\xi-\alpha_2-\cfrac{\omega_2}{\xi-\alpha_3-\ddots}}} \quad Im(\xi)\neq 0. \tag{2.8}$$

The result now follows by (2.7) and (2.8). □

Chapter 3

The Operator-Valued Free Probability Theory

This chapter discusses the operator valued free central limit theorem and semicircular distributions, as operator-valued analogues of the classical central limit theorem and normal distributions. In section 1, the non-commutative operator-valued probability space, the free families of operator-valued random variables and their distributions are discussed. Then, using a Fock space construction, one special case of non-commutative operator-valued probability space, called the canonical probability space, is introduced in which any distribution of any random variable of the given non-commutative operator valued probability space can be identified as a distribution of an element of that canonical probability space. Next, the concept of $R-$transform is introduced and by investigating its linearization properties in the free context, we gain enough tools to study the operator-valued semicircular random variables and the operator-valued free central limit theorem. In section 2, the notion of complete positivity of a unital linear map of a given $*$-algebra into one of its unital sub$*$-algebras is introduced and in case of unital sub-C^*algebras, some general properties of a completely positive map are studied in a series of lemmas. Then, it is proved that complete positivity is a necessary and sufficient condition for a given unital linear map of that unital sub-C^*algebra into iteself, to be identified as the variance of an operator-valued semicircular random variable.

3.1 Voiculescu's Operator-Valued Free Central Limit Theorem

Definition 3.1.1. Let A be a unital $*$-algebra. Let B denote a fixed unital $*$-subalgebra of A over $\mathbb{C}$. A linear map $E_B : A \to B$ is called a conditional expectation if it satisfies the following conditions:

(i) $E_B(b_1 a b_2) = b_1 E_B(a) b_2$ for all $a \in A,\ b_1, b_2 \in B$, and $E_B(1) = 1$,

(ii) $E_B(a^* a) \geq 0$, for all $a \in A$,

It is noteworthy that the condition $E_B(1) = 1$ is equivalent to being a projection onto B.

Example 3.1.2. Let $A \subseteq B(H)$ be a finite von Neumann algebra and $\tau : A \to \mathbb{C}$ be a faithful normal trace, that is, a bounded linear functional such that $\tau(ab) = \tau(ba)$ for all $a, b \in A$, $\tau(a^* a) = 0$ if and only if $a = 0$, and for any net $\{a_i\}$ of self-adjoint elements of A with $a_i \nearrow a$ we have $\tau(a_i) \nearrow \tau(a)$. By considering the inner product $\langle a, b\rangle = \tau(ab^*)$ for all $a, b \in A$ on A, we denote the completion of A with respect to the norm $\|a\|_2 = \tau(a^* a)^{\frac{1}{2}}$, for all $a \in A$, by $L^2(A, \tau)$, [11, pp. 37-42].

(1) If B is separable in the $\|.\|_2$ norm containing the sequence $(b_n)_{n=1}^{\infty}$ as the orthonormal basis of $L^2(B, \tau)$, then:

$$E_B(x) = \sum_{n=1}^{\infty} \langle x, b_n\rangle b_n = \sum_{n=1}^{\infty} \tau(x b_n^*) b_n \qquad x \in A.$$

(2) If B be a finite dimensional abelian subalgebra of A with minimal projections $f_j (1 \leq j \leq n)$, then:

$$E_B(x) = \sum_{j=1}^{n} \tau(f_j)^{-1} \tau(x f_j) f_j \qquad x \in A,$$

(In fact, $E_B(x) = \sum_{j=1}^{n} \lambda_j f_j$. To find λ_j, we have

$$\tau(x f_j) = \langle x, f_j\rangle = \langle E_B(x), f_j\rangle = \langle \sum_{j=1}^{n} \lambda_j f_j, f_j\rangle = \lambda_j \tau(f_j)$$

for all $1 \le j \le n$.).

(3) If $B = M_n(\mathbb{C})$ for some $n \ge 1$, and $\tau((a_{ij})) = \frac{\sum_{i=1}^n a_{ii}}{n}$, then:

$$E_B(x) = \sum_{i=1}^n \sum_{j=1}^n n\tau(E_{ji}x)E_{ij},$$

where $E_{ij} = (\delta_{(i,j)(k,l)})$ for all $1 \le i, j \le n$ (in fact, let $E_B(x) = \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij}(x)E_{ij}$, then using

$$\tau(aE_B(b)) = \tau(E_B(a)E_B(b)) = \tau(E_B(a)b) \quad \text{for all } a, b \in A$$

we have:

$$\begin{aligned} \tau(E_{ij}x) &= \tau(E_B(E_{ij})x) = \tau(E_{ij}E_B(x)) = \tau(E_{ij}\sum_{k=1}^n \sum_{l=1}^n \alpha_{kl}(x)E_{kl}) \\ &= \sum_{k=1}^n \sum_{l=1}^n \alpha_{kl}(x)\tau(E_{ij}E_{kl}) = \sum_{l=1}^n \alpha_{jl}(x)\tau(E_{ij}E_{jl}) = \sum_{l=1}^n \alpha_{jl}(x)\tau(E_{il}) \\ &= \alpha_{ji}(x)\tau(E_{ii}) = \alpha_{ji}(x)\frac{1}{n}.). \end{aligned}$$

Definition 3.1.3. A triple (A, E_B, B) as in Definition 3.1.1 is called a $B-$ valued non-commutative probability space. An element $a \in A$ is called a $B-$ valued random variable.

Throughout this chapter we assume that (A, E_B, B) is a given fixed $B-$ valued non-commutative probability space.

Definition 3.1.4. Let (A, E_B, B) be as in Definition 3.1.3, and $B \subseteq A_i \subseteq A (i \in I)$ be subalgebras. The family $\{A_i\}_{i\in I}$ is called free if

$$E_B(a_{i_1}a_{i_2}\cdots a_{i_n}) = 0 \qquad i_1 \ne i_2 \ne \cdots \ne i_n, a_{i_j} \in A_{i_j}, E_B(a_{i_j}) = 0, (1 \le j \le n)$$

We call the family $\{X_i\}_{i\in I}$ of subsets of A (elements $\{a_i\}_{i\in I}$ of A) free if the corresponding family of subalgebras $\{\langle X_i \cup B\rangle\}_{i\in I}$ ($\{\langle\{a_i\} \cup B\rangle\}_{i\in I}$) is free.

Proposition 3.1.5. *Let (A, E_B, B) be as in Definition 3.1.3, and $B \subseteq A_i \subseteq A (i \in I)$ be subalgebras such that the family $\{A_i\}_{i\in I}$ is free and A is generated by $\cup_{i\in I} A_i$. Then E_B is completely determined by the $E_B|_{A_i} (i \in I)$.*

Proof. Let $a \in A$. Then, by hypothesis $a = \sum_{m:\text{finite}} \alpha_m a_{mi_1} \cdots a_{mi_n}$ implying $E_B(a) = \sum_{m:\text{finite}} \alpha_m E_B(a_{mi_1} \cdots a_{mi_n})$. So, it is sufficient to prove that we can compute $E_B(a_{i_1} \cdots a_{i_n})$'s whenever $a_{i_j} \in A_{i_j} (1 \le j \le n)$. Define:

$$k = \min\{s \in \mathbb{N}_0 | E_B(a_{i_j}) = 0 \quad s+1 \le j \text{ and } \; i_{s+1} \ne i_{s+2} \ne \cdots \ne i_n\}.$$

We prove the assertion by applying strong induction on k. If $k = 0$, then by definition of freeness $E_B(a_{i_1} \cdots a_{i_n}) = 0$. Assume our assertion has been established up to certain k. Then, for $k+1$ write:

$$\begin{aligned} E_B(a_{i_1} \cdots a_{i_n}) &= E_B(a_{i_1} \cdots a_{i_k} (E_B|_{A_{i_{k+1}}}(a_{i_{k+1}})) a_{i_{k+2}} \cdots a_{i_n}) \\ &+ E_B(a_{i_1} \cdots a_{i_k} (a_{i_{k+1}} - E_B|_{A_{i_{k+1}}}(a_{i_{k+1}})) a_{i_{k+2}} \cdots a_{i_n}) \end{aligned}$$

if $i_k \neq i_{k+1}$, and

$$\begin{aligned} E_B(a_{i_1} \cdots a_{i_n}) &= E_B(a_{i_1} \cdots a_{i_{k-1}} (E_B|_{A_{i_k}}(a_{i_k} a_{i_{k+1}})) a_{i_{k+2}} \cdots a_{i_n}) \\ &+ E_B(a_{i_1} \cdots a_{i_{k-1}} (a_{i_k} a_{i_{k+1}} - E_B|_{A_{i_k}}(a_{i_k} a_{i_{k+1}})) a_{i_{k+2}} \cdots a_{i_n}) \end{aligned}$$

if $i_k = i_{k+1}$, so that the induction hypothesis applies. □

Definition 3.1.6. Let $B\langle X\rangle = \langle \{b_1 X b_2 X \cdots b_{n-1} X b_n | n \in \mathbb{N}, b_j \in B, 1 \leq j \leq n\} \cup \{1_B\}\rangle$ be the algebra freely generated by B and an indeterminate X, (A, E_B, B) be as in Definition 3.1.3, $a \in A$ be a $B-$ valued random variable, and $\tau_a : B\langle X\rangle \to A$ be the unique algebra homomorphism such that $\tau_a(b) = b$ $(b \in B)$ and $\tau_a(X) = a$. Then, the conditional expectation:

$$\begin{aligned} \mu_a &: B\langle X\rangle \to B \\ \mu_a(P(X)) &= E_B \circ \tau_a(P(X)) \end{aligned}$$

is called the distribution of a. Next, quantities such as:

$$\mu_a(1 X b_1 \cdots X b_n X 1) = E_B(a b_1 \cdots a b_n a)$$

are called moments. The zeroth moment m_0 is by definition $E_B(1) = 1$, the first moment m_1 is $E_B(a)$, the second moment is the linear map $b \to E_B(aba)$, and in general the $(n+1)^{th}$ moment is the $n-$linear map $(b_1, \cdots, b_n) \to E_B(a b_1 \cdots a b_n a)$. Thus, we can view the distribution of $a \in A$ as the set of multilinear maps $\{m_n : B^n \to B | m_n(b_1, \cdots, b_n) = E_B(a b_1 \cdots a b_n a)\}_{n=0}^{\infty}$ with the above conventions for $n = 0, 1$.

Finally, we define:

$$\Sigma_B = \{\mu | \mu : B\langle X\rangle \to B \quad \text{conditional expectation}\}.$$

As an example, for $a = 0$ we have that $\mu_0 = \mathcal{X}_B.P_B$, where P_B denotes the projection onto B.

Definition 3.1.7. Let $S_n (n \geq 1)$ denote the symmetric group and put:

$$\begin{aligned} S_0 &= 1 \\ S_1(b) &= b \quad (b \in B) \\ S_n(b_1, \cdots, b_n) &= \sum_{\sigma \in S_n} b_{\sigma(1)} X \cdots b_{\sigma(n-1)} X b_{\sigma(n)}. \quad (b_i \in B, n \geq 2) \end{aligned}$$

We define

$$SB\langle X \rangle = \text{lin}\{S_n(b, \cdots, b) | b \in B\}_{n=0}^{\infty}.$$

Lemma 3.1.8. $SB\langle X \rangle = lin\{S_n(b_1, \cdots, b_n) | b_i \in B, 0 \leq i \leq n\}_{n=0}^{\infty}$.

Proof. It is clear that $SB\langle X \rangle \subseteq \text{lin}\{S_n(b_1, \cdots, b_n) | b_i \in B, 0 \leq i \leq n\}_{n=0}^{\infty}$. To prove the opposite inclusion, for $b_i \in B$ and $1 \leq i \leq n$ we have:

$$\begin{aligned} S_n((b_i)_{i=1}^n) &= S_n\Big((\sum_{k=1}^n b_k)_{i=1}^n \Big) \\ &- \sum_{A_{n-1} \subseteq \{b_k\}_{k=1}^n : |A_{n-1}| = n-1} S_n\Big((\sum_{b_k \in A_{n-1}} b_k)_{i=1}^n \Big) \\ &+ \cdots \\ &+ (-1)^s \sum_{A_{n-s} \subseteq \{b_k\}_{k=1}^n : |A_{n-s}| = n-s} S_n\Big((\sum_{b_k \in A_{n-s}} b_k)_{i=1}^n \Big) \\ &+ \cdots \\ &+ (-1)^{n-2} \sum_{A_2 \subseteq \{b_k\}_{k=1}^n : |A_2| = 2} S_n\Big((\sum_{b_k \in A_2} b_k)_{i=1}^n \Big) \\ &+ (-1)^{n-1} \sum_{A_2 \subseteq \{b_k\}_{k=1}^n : |A_1| = 1} S_n\Big((\sum_{b_k \in A_1} b_k)_{i=1}^n \Big). \end{aligned}$$

We note that the number of summands on the left hand side is $n!$ whilst on the right hand side is $\sum_{s=0}^{n-1} \binom{n}{n-s} (n-s)^n (-1)^s$. It is known these two numbers are equal. Thus, $S_n(b_1, \ldots, b_n) \in SB\langle X \rangle$ and hence the desired inclusion is established. □

Lemma 3.1.9. *Let* $\widetilde{SB}\langle X \rangle = \mathbb{C}X + X(SB\langle X \rangle)X$. *Then:*

$$B(SB\langle X \rangle)B = B + B(\widetilde{SB}\langle X \rangle)B.$$

Proof. We have:

$$\begin{aligned} BS_n(b,\cdots,b)B &= B(\sum_{\sigma\in S_n} bX\cdots bXb)B = (Bb)X(\sum_{\sigma\in S_n} bX\cdots bXb)X(bB) \\ &\in B + B(X(SB\langle X\rangle)X)B \subseteq B + B(\widetilde{SB}\langle X\rangle)B, \end{aligned}$$

implying $B(SB\langle X\rangle)B \subseteq B + B(\widetilde{SB}\langle X\rangle)B$. To prove the other inclusion, by considering:

$$\begin{aligned} &(n-k)(n-k-1)XS_{n-2}(b_1,\cdots,b_k,1,\cdots,1)X = \\ &\quad S_n(b_1,\cdots,b_k,1,\cdots,1) \\ &\quad -(n-k)kbXS_{n-2}(b_1,\cdots,b_{k-1},1,\cdots,1)X \\ &\quad -(n-k)kXS_{n-2}(b_1,\cdots,b_{k-1},1,\cdots,1)Xb \\ &\quad -k(k-1)bXS_{n-2}(b_1,\cdots,b_{k-2},1,\cdots,1)Xb, \end{aligned}$$

for $n \geq k+1, n \geq 3$, and $b_1 = \cdots = b_k = b$, and

$$n(n-1)XS_{n-2}(1,\cdots,1)X = S_n(1,\cdots,1) \qquad n \geq 3,$$

and strong induction for $1 \leq k \leq n-2, n \geq 3$ we have:

$$XS_{n-2}(b_1,\cdots,b_k,1,\cdots,1)X \in B(SB\langle X\rangle)B \qquad b_1 = \cdots = b_k = b.$$

But, we have $X = \frac{1}{2}S_2(1,1) \in B(SB\langle X\rangle)B$, hence $B(SB\langle X\rangle)B \supseteq B + B(\widetilde{SB}\langle X\rangle)B$. □

Corollary 3.1.10. *For any $\mu \in \Sigma_B$, $\mu|_{SB\langle X\rangle}$ is completely determined by $\mu|_{\widetilde{SB\langle X\rangle}}$ and conversely.*

Definition 3.1.11. We define $S\Sigma_B = \{S_\mu | S_\mu = \mu|_{B(SB\langle X\rangle)B}, \mu \in \Sigma_B\}$. Let $a \in A$ be a random variable. Then, S_{μ_a} is called the symmetric distribution of a, and $\mu_a(S(b_1,\cdots,b_n))$ or $\mu_a(XS(b_1,\cdots,b_n)X)$ are called symmetric moments of a.

Definition 3.1.12. Let $a_1, a_2 \in A$ be two $B-$ valued random variables. Then, it follows by Proposition 3.1.5, that $\mu_{a_1+a_2}$ and $\mu_{a_1a_2}$ depend only on μ_{a_1} and μ_{a_2}. Thus, there are well defined binary operations $\boxplus$ and $\boxtimes$ on Σ_B (called free additive convolution and free multiplicative convolution, respectively), such that:

$$\mu_{a_1+a_2} = \mu_{a_1} \boxplus \mu_{a_2} \qquad \mu_{a_1a_2} = \mu_{a_1} \boxtimes \mu_{a_2}.$$

It is noteworthy that for $a_1 = a$ and $a_2 = 0$, we have that $\mu_a = \mu_a \boxplus \mu_0$.

Now, we construct one special operator-valued non-commutative probability space called the canonical probability space. Looking at B as a right B-module, let $\mathcal{X}_n(B) = \mathcal{L}(B^{\otimes n}, B)$ be the set of all $n-$ linear $B-$ valued maps of B^n into B (the $\otimes$ and linearity are over $\mathbb{C}$) and $\mathcal{X}_0(B) = B$. Let further, $\mathcal{X}(B) = \oplus_{n\geq 0}\mathcal{X}_n(B) \subseteq \prod_{n\geq 0}\mathcal{X}_n(B) = \overline{\mathcal{X}}(B)$ with its natural $B-$ module structure and obvious grading. Fix $\xi \in \mathcal{X}_n(B)$ and define the endomorphism:

$$\lambda(\xi) : \mathcal{X}(B) \to \mathcal{X}(B)$$

by $\lambda(\xi)(\eta)(b_1 \otimes \cdots \otimes b_{n+k}) = \eta(b_{n+1}\xi(b_1 \otimes \cdots \otimes b_n) \otimes b_{n+2} \otimes \cdots \otimes b_{n+k})$ if $deg(\eta) = k > 0$ and $\lambda(\xi)(\eta) = \xi\eta$ if $deg(\eta) = k = 0$ ($\eta \in B$). Next, fix $b \in B$ and define:

$$\lambda^*(b) : \mathcal{X}(B) \to \mathcal{X}(B)$$

via $\lambda^*(b)(\eta)(b_1 \otimes \cdots \otimes b_{k-1}) = \eta(b \otimes b_1 \otimes \cdots \otimes b_{k-1})$if $deg(\eta) = k > 0$ and $\lambda^*(b)(\eta) = 0$ if $deg(\eta) = k = 0(\eta \in B)$.

Definition 3.1.13. We define $A(B)$ as the generated algebra of endomorphisms $\lambda(\xi), \lambda^*(b)$ of the right $B-$module $\mathcal{X}(B)$, that is:

$$A(B) = \langle \{\lambda(\xi) | \xi \in \mathcal{X}_n(B), n \geq 0\} \cup \{\lambda^*(b) | b \in B\}\rangle.$$

Lemma 3.1.14.

$$\begin{aligned}
(i)\ \lambda(\xi_1)\lambda(\xi_2) &= \lambda(\lambda(\xi_1)\xi_2) \\
(ii)\ \lambda^*(b)\lambda(\xi) &= \begin{cases} \lambda(\lambda^*(b)\xi) & \text{if } deg(\xi) > 0 \\ \lambda^*(b\xi) & \text{if } deg(\xi) = 0. \end{cases}
\end{aligned}$$

Proof. (i) Let $deg(\xi_i) = n_i, (i = 1, 2)$ and $deg(\eta) = k > 0$. Then:

$$\begin{aligned}
\lambda(\xi_1)\lambda(\xi_2)\eta(b_1 \otimes \cdots \otimes b_{n_1+n_2+k}) &= \lambda(\xi_2)\eta(b_{n_1+1}\xi_1(b_1 \otimes \cdots \otimes b_{n_1}) \otimes b_{n_1+2} \otimes \cdots \otimes b_{n_1+n_2+k}) \\
&= \eta(b_{n_1+n_2+1}\xi_2(b_{n_1+1}\xi_1(b_1 \otimes \cdots \otimes b_{n_1}) \otimes b_{n_1+2} \otimes \cdots \otimes b_{n_1+n_2}) \\
&\quad \otimes b_{n_1+n_2+2} \otimes \cdots \otimes b_{n_1+n_2+k}) \\
&= \eta(b_{n_1+n_2+1}\lambda(\xi_1)\xi_2(b_1 \otimes \cdots \otimes b_{n_1+n_2}) \\
&\quad \otimes b_{n_1+n_2+2} \otimes \cdots \otimes b_{n_1+n_2+k}) \\
&= \lambda(\lambda(\xi_1)\xi_2)\eta(b_1 \otimes \cdots \otimes b_{n_1+n_2+k}).
\end{aligned}$$

(ii) Let $deg(\xi) = n$, and $deg(\eta) = k > 0$. Then:

$$\begin{aligned}
\lambda^*(b)\lambda(\xi)\eta(b_1 \otimes \cdots \otimes b_{n+k-1}) &= \lambda(\xi)\eta(b \otimes b_1 \otimes \cdots \otimes b_{n+k-1}) \\
&= \eta(b_n \xi(b \otimes b_1 \otimes \cdots \otimes b_{n-1}) \otimes b_{n+1} \otimes \cdots \otimes b_{n+k-1}) \\
&= \eta(b_n \lambda^*(b)\xi(b_1 \otimes \cdots \otimes b_{n-1}) \otimes b_{n+1} \otimes \cdots \otimes b_{n+k-1}) \\
&= (\lambda(\lambda^*(b)\xi)\eta)(b_1 \otimes \cdots \otimes b_{n+k-1}).
\end{aligned}$$

Next, let $deg(\xi) = 0$ and $deg(\eta) = k > 0$. Then:

$$\begin{aligned}
\lambda^*(b)\lambda(\xi)\eta(b_1 \otimes \cdots \otimes b_{k-1}) &= \lambda(\xi)\eta(b \otimes b_1 \otimes \cdots \otimes b_{k-1}) \\
&= \eta(b\xi \otimes b_1 \otimes \cdots \otimes b_{k-1}) \\
&= \lambda^*(b\xi)\eta(b_1 \otimes \cdots \otimes b_{k-1}),
\end{aligned}$$

completing the proof. □

Lemma 3.1.15. *The linear map $\gamma : (\oplus_{n\geq 0}\mathcal{X}_n(B)) \otimes (\oplus_{k\geq 0}B^{\otimes k}) \to A(B)$ given by*

$$\gamma(\sum_{finite:(i,j)} \xi_{i,j} \otimes (b_{i1,j} \otimes \cdots \otimes b_{ik_i,j})) = \sum_{finite:(i,j)} \lambda(\xi_{i,j})\lambda^*(b_{i1,j})\cdots\lambda^*(b_{ik_i,j})$$

is a bijection.

Proof. Since $\gamma(\xi \otimes 1) = \lambda(\xi)$, where $1 \in B^{\otimes 0} = \mathbb{C}$, and $\lambda^*(b) = \lambda(1)\lambda^*(b) = \gamma(1 \otimes b)$, it follows that $Range(\gamma)$ contains all $\lambda(\xi)'s$ and all $\lambda^*(b)'s$. Next, by Lemma 3.1.14 we have that :

$$\begin{aligned}
\gamma(\xi \otimes (b_1 \otimes \cdots \otimes b_k))\gamma(\xi' \otimes (b_1' \otimes \cdots \otimes b_{k'}')) &= \lambda(\xi)(\prod_{i=1}^{k} \lambda^*(b_i))\lambda(\xi')(\prod_{i=1}^{k'} \lambda^*(b_i')) \\
&= \lambda(\xi)\lambda(\prod_{i=1}^{k} \lambda^*(b_i)\xi') \prod_{i=1}^{k'} \lambda^*(b_i') \\
&= \lambda(\lambda(\xi)\prod_{i=1}^{k} \lambda^*(b_i)\xi') \prod_{i=1}^{k'} \lambda^*(b_i') \\
&= \gamma((\lambda(\xi)\prod_{i=1}^{k} \lambda^*(b_i)\xi') \otimes (b_1' \otimes \cdots \otimes b_{k'}')),
\end{aligned}$$

implying that $Range(\gamma)$ is an algebra and so that γ is onto.

For injectivity, we prove that if $\alpha \neq 0$, then $\gamma(\alpha) \neq 0$. Let

$$\alpha = \sum_{k_0 \leq k \leq k_1} \sum_{i \in I_k} \xi_{i,k} \otimes \nu_{i,k} \neq 0,$$

where $\xi_{i,k} \in \oplus_{n\geq 0}\mathcal{X}_n(B)$ and $\nu_{i,k} \in B^{\otimes k}$ for $i \in I_k$. Since, $\alpha \neq 0$, we may assume the $\nu'_{i,k}s$ are linearly independent and the $\xi'_{i,k}s$ are non-zero. Then, fixing $i_0 \in I_{k_0}$ there is $\eta \in \mathcal{X}_{k_0}(B)$ such that $\eta(\nu_{i_0,k_0}) = 1 \in B$ and $\eta(\nu_{i,k_0}) = 0, i \in I_{k_0} - \{i_0\}$. Let $\eta' \in \mathcal{X}_{k_0}(B)$ be defined by:

$$\eta' : B^{\otimes k_0} \to B$$
$$\eta'(b_1 \otimes \cdots \otimes b_{k_0}) = \eta(b_{k_0} \otimes \cdots \otimes b_1).$$

Then, we have that:

$$\gamma(\alpha)\eta' = \gamma(\sum_{k_0\leq k\leq k_1}\sum_{i\in I_{k_0}} \xi_{i,k_0} \otimes \nu_{i,k_0})\eta' = \sum_{i\in I_{k_0}} \lambda(\xi_{i,k_0})\eta(\nu_{i,k_0}) = \lambda(\xi_{i_0,k_0})1 = \xi_{i_0,k_0} \neq 0,$$

implying $\gamma(\alpha) \neq 0$. □

Lemma 3.1.16. *Let $B = \mathcal{X}_0(B)$ be identified with the subalgebra of $A(B)$ via $\lambda : \mathcal{X}_0(B) \to A(B)$. Then, the linear map:*

$$\begin{aligned} \varepsilon_B &: A(B) \to B \\ \varepsilon_B(\sum_{finite} \gamma(\xi_{n_i} \otimes \nu_{k_i})) &= \begin{cases} 0 & \text{if } \min_i(n_i + k_i) > 0, \\ \sum_{i:n_i+\nu_i=0} \xi_{n_i}\nu_{n_i} & \text{if } \min_i(n_i + k_i) = 0, \end{cases} \end{aligned}$$

where $\xi_{n_i} \in \mathcal{X}_{n_i}(B)$ and $\nu_{k_i} \in B^{\otimes k_i}$, is a conditional expectation.

Proof. According to [14] we need only to check condition (i) of Definition 3.1.1. Let $a \in A(B)$. Then, by Lemma 3.1.15, and a linearity argument, it is sufficient to consider $a = \gamma(\xi_n \otimes \nu_k)$. Let $b'_1, b'_2 \in B$, $\nu_k = (b_1 \otimes \cdots \otimes b_k)$ and denote $\lambda^*(\nu_k) = \lambda^*(b_1)\cdots\lambda^*(b_k)$. Then, we have two cases:

Case(i): $n + k > 0$:

$$\begin{aligned} \varepsilon_B(\lambda(b'_1)\gamma(\xi_n \otimes \nu_k)\lambda(b'_2)) &= \varepsilon_B(\lambda(b'_1)\lambda(\xi_n)\lambda^*(\nu_k)\lambda(b'_2)) \\ &= \varepsilon_B(\lambda(\lambda(b'_1)\xi_n)\lambda^*(\nu_k b'_2)) \\ &= \varepsilon_B(\gamma(\lambda(b'_1)\xi_n \otimes \nu_k b'_2)) \\ &= 0 = b'_1\varepsilon_B(\gamma(\xi_n \otimes \nu_k))b'_2. \end{aligned}$$

Case(ii): $n+k=0$:

$$\begin{aligned}\varepsilon_B(\lambda(b_1)\gamma(\xi_0\otimes\nu_0)\lambda(b_2)) &= \varepsilon_B(\lambda(b_1')\lambda(\xi_0\nu_0)\lambda(b_2'))\\ &= \varepsilon_B(\lambda(b_1'\xi_0\nu_0 b_2'))\\ &= b_1'\xi_0\nu_0 b_2'\\ &= b_1'\varepsilon_B(\gamma(\xi_0\otimes\nu_0))b_2'.\end{aligned}$$

Finally, for any $b\in B$,

$$\varepsilon_B(\lambda(b))=\varepsilon_B(\gamma(b\otimes 1))=b.1=b.$$

□

Let $\overline{A}(B)\supseteq A(B)$ be the algebra acting on $\overline{\mathcal{X}}(B)$ such that the map:

$$\overline{\gamma}:\overline{\mathcal{X}}(B)\otimes(\oplus_{k\geq 0}B^{\otimes k})\to\overline{A}(B)$$

which naturally extends

$$\gamma:\mathcal{X}(B)\otimes(\oplus_{k\geq 0}B^{\otimes k})\to A(B)$$

via

$$\begin{aligned}\overline{\gamma}((\xi_n)_{n\geq 0}\otimes(b_1\otimes\cdots\otimes b_k)) &= \sum_{n\geq 0}\gamma(\xi_n\otimes(b_1\otimes\cdots\otimes b_k))\\ &= \sum_{n\geq 0}\lambda(\xi_n)\lambda^*(b_1)\cdots\lambda^*(b_k),\end{aligned}$$

is a bijection and the multiplication of the formal sums which constitutes $\overline{A}(B)$ is also determined by Lemma 3.1.14. Next, let $(.)_0$ denote the component of degree zero and $1\in B=\mathcal{X}_0(B)\subseteq\overline{\mathcal{X}}(B)$. Then:

$$\begin{aligned}\varepsilon_B &: \overline{A}(B)\to B\\ \varepsilon_B(T) &= (T1)_0\end{aligned}$$

is the extension of $\varepsilon_B:A(B)\to B$.

Let $B=B_1\oplus B_2, \overline{\mathcal{X}}(B_j)=\prod_{n\geq 0}\mathcal{X}_n(B_j), i_j:B_j\to B(i_1(b_1)=b_1\oplus 0, i_2(b_2)=0\oplus b_2), pr_j: B\to B_j(pr_j(b_1\oplus b_2)=b_j)$ for $(j=1,2)$. Define the injections:

$$\begin{aligned}\mathcal{X}_j &: \overline{\mathcal{X}}(B_j)\otimes(\oplus_{k\geq 0}B_j^{\otimes k})\to\overline{\mathcal{X}}(B)\otimes(\oplus_{k\geq 0}B^{\otimes k})\\ \mathcal{X}_j((\xi_n)_{n\geq 0}\otimes\nu_k) &= (\xi_n\circ pr_j^{\otimes n})_{n\geq 0}\otimes(i_j^{\otimes k}(\nu_k))\end{aligned}$$

where for $\xi_n : \prod_{i=1}^n B_j \to B$ and $pr_j^{\otimes n} : B^{\otimes n} \to B_j^{\otimes n}$ we have that $\xi_n \circ pr_j^{\otimes n} : B^{\otimes n} \to B$ for $j = 1, 2$, and $\xi_0 \circ pr_j^{\otimes 0} = \xi_0$.

Lemma 3.1.17. *The maps:*

$$\begin{aligned} h_j &: \overline{A}(B_j) \to \overline{A}(B) \\ h_j \circ \overline{\gamma} &= \overline{\gamma} \circ \mathcal{X}_j \quad (j = 1, 2) \end{aligned}$$

are homomorphisms, $h_j(\lambda(b)) = \lambda(b)$ *for* $b \in B = \mathcal{X}_0(B_j) = \mathcal{X}_0(B)$ *and* $\varepsilon_B \circ h_j = \varepsilon_{B_j} \quad (j = 1, 2)$.

Proof. Fix $j = 1, 2$, and let $T_i \in \overline{A}(B_j) \quad (i = 1, 2)$. Since $\overline{\gamma}$ is bijection, there are unique $U_i \in \overline{\mathcal{X}}(B_j) \otimes (\oplus_{k \geq 0} B_j^{\otimes k})$ with $\overline{\gamma}(U_i) = T_i \quad (i = 1, 2)$. Then, by Lemma 3.1.14:

$$\begin{aligned} h_j(T_1 T_2) &= h_j(\overline{\gamma}(U_1)\overline{\gamma}(U_2)) \\ &= h_j(\overline{\gamma}((\xi_n)_{n\geq 0} \otimes (b_1 \otimes \cdots \otimes b_k))\overline{\gamma}((\xi_m')_{m\geq 0} \otimes (b_1' \otimes \cdots \otimes b_{k'}'))) \\ &= h_j(\sum_{n\geq 0}\sum_{m\geq 0} \gamma(\xi_n \otimes (b_1 \otimes \cdots \otimes b_k))\gamma(\xi_m' \otimes (b_1' \otimes \cdots \otimes b_{k'}'))) \\ &= \sum_{n\geq 0}\sum_{m\geq 0} h_j(\gamma(\xi_n \otimes (b_1 \otimes \cdots \otimes b_k))\gamma(\xi_m' \otimes (b_1' \otimes \cdots \otimes b_{k'}'))) \\ &= \sum_{n\geq 0}\sum_{m\geq 0} h_j \circ \gamma((\lambda(\xi_n)\lambda^*(b_1)\cdots\lambda^*(b_k)\xi_m') \otimes (b_1' \otimes \cdots \otimes b_{k'}')) \\ &= \sum_{n\geq 0}\sum_{m\geq 0} \gamma \circ \mathcal{X}_j((\lambda(\xi_n)\lambda^*(b_1)\cdots\lambda^*(b_k)\xi_m') \otimes (b_1' \otimes \cdots \otimes b_{k'}')) \\ &= \sum_{n\geq 0}\sum_{m\geq 0} \gamma((\lambda(\xi_n)\lambda^*(b_1)\cdots\lambda^*(b_k)\xi_m' pr_j^{\otimes deg(\xi_n)+\deg(\xi_m')-k}) \otimes i_j^{\otimes k'}(b_1' \otimes \cdots \otimes b_{k'}')) \\ &= \sum_{n\geq 0} \gamma((\xi_n \otimes pr_j^{deg(\xi_n)}) \otimes (i_j^{\otimes k}(b_1 \otimes \cdots \otimes b_k))) \sum_{m\geq 0} \gamma((\xi_m' \otimes pr_j^{deg(\xi_m')}) \otimes (i_j^{\otimes k'}(b_1' \otimes \cdots \otimes b_{k'}'))) \\ &= \sum_{n\geq 0} \gamma \circ \mathcal{X}_j(\xi_n \otimes (b_1 \otimes \cdots \otimes b_k)) \sum_{m\geq 0} \gamma \circ \mathcal{X}_j(\xi_m' \otimes (b_1' \otimes \cdots \otimes b_{k'}'))) \\ &= \sum_{n\geq 0} h_j \circ \gamma(\xi_n \otimes (b_1 \otimes \cdots \otimes b_k)) \sum_{m\geq 0} h_j \circ \gamma(\xi_m' \otimes (b_1' \otimes \cdots \otimes b_{k'}'))) \\ &= h_j(\sum_{n\geq 0} \gamma(\xi_n \otimes (b_1 \otimes \cdots \otimes b_k)))h_j(\sum_{m\geq 0} \gamma(\xi_m' \otimes (b_1' \otimes \cdots \otimes b_{k'}'))) \\ &= h_j(\overline{\gamma}((\xi_n)_{n\geq 0} \otimes (b_1 \otimes \cdots \otimes b_k))h_j(\overline{\gamma}((\xi_m)_{n\geq 0} \otimes (b_1 \otimes \cdots \otimes b_k)) \\ &= h_j(\overline{\gamma}(U_1))h_j(\overline{\gamma}(U_2)) \\ &= h_j(T_1)h_j(T_2) \end{aligned}$$

and extend by linearity to finite summations.

Next, fix $j = 1, 2$ and let $b \in B = \mathcal{X}_0(B_j) = \mathcal{X}_0(B)$. Then, for $\lambda(b) = \gamma(\xi \otimes 1) \in A(B)$ we have that:

$$\begin{aligned} h_j(\lambda(b)) &= h_j(\overline{\gamma}(\xi \otimes 1)) = h_j \circ \overline{\gamma}(\xi \otimes 1) \\ &= \overline{\gamma} \circ \mathcal{X}_j(\xi \otimes 1) = \overline{\gamma}(\xi \otimes 1) = \lambda(b). \end{aligned}$$

For the last part, fix $j = 1, 2$ and let $\xi_0 \in \mathcal{X}_0(B) = \mathcal{X}(B_j) = B$ and $\nu_0 \in B^{\otimes 0} = B_j^{\otimes 0} = \mathbb{C}$. Then, by $\overline{\gamma}(\xi_0 \otimes \nu_0) = \gamma(\xi_0 \otimes \nu_0) = \xi_0 \nu_0 \in B$ and $\mathcal{X}_j(\xi_0 \otimes \nu_0) = (\xi_0 \otimes pr_j^{\otimes 0}) \otimes (i_j^{\otimes 0} \nu_0) = \xi_0 \otimes \nu_0$ it follows that:

$$\begin{aligned} \varepsilon_B \circ h_j(\gamma(\xi_0 \otimes \nu_0)) &= \varepsilon_B(h_j \circ \gamma(\xi_0 \otimes \nu_0)) = \varepsilon_B(\gamma \circ \mathcal{X}_j(\xi_0 \otimes \nu_0)) \\ &= \varepsilon_B(\gamma(\xi_0 \otimes \nu_0)) = \gamma(\xi_0 \otimes \nu_0) = \xi_0 \nu_0 = \varepsilon_{B_j}(\gamma(\xi_0 \otimes \nu_0)). \end{aligned}$$

Finally, by bijectivity of $\overline{\gamma}$, injectivity of $\mathcal{X}_j$ and the equation $h_j \circ \overline{\gamma} = \overline{\gamma} \circ \mathcal{X}_j$ it follows that h_j is injective and since $\mathcal{X}_j$ preserves the degree it follows that h_j preserves the degree, as well. Hence, $\ker(\varepsilon_{B_j}) = \ker(\varepsilon_B \circ h_j)$ implying $\varepsilon_{B_j} = \varepsilon_B \circ h_j$. □

Now, considering the canonical operator-valued noncommutative probability space $(\overline{A}(B), \varepsilon_B, B)$, we have:

Proposition 3.1.18. *If $B = B_1 \oplus B_2$ then with h_1, h_2 defined as above, the pair of subalgebras $(h_j(\overline{A}(B)))_{j=1}^2$ is B–free in $(\overline{A}(B), \varepsilon_B, B)$.*

Proof. We write:

$$\begin{aligned} \overline{\mathcal{X}}(B) &= \prod_{n \geq 0} \mathcal{X}_n(B) \\ &= B \oplus \prod_{n \geq 1} \mathcal{L}((B_1 \oplus B_2) \otimes B^{\otimes n-1}, B) \\ &= B \oplus \prod_{n \geq 1} \mathcal{L}(B_1 \otimes B^{\otimes n-1}, B) \oplus \prod_{n \geq 1} \mathcal{L}(B_2 \otimes B^{\otimes n-1}, B) \end{aligned}$$

where $\mathcal{L}(B_j \otimes B^{\otimes n-1}, B)$is identified as a subspace of $\mathcal{L}(B^{\otimes n}, B)$ via: $\eta_n \to \eta_n \circ (pr_j \otimes id_B^{\otimes n-1})$. Next, put $\Gamma_j = \prod_{n \geq 1} \mathcal{L}((B_j) \otimes B^{\otimes n-1}, B) \quad (j = 1, 2)$.

We claim that $T \in h_1(\overline{A}(B_1))(T \in h_2(\overline{A}(B_2)))$ and $\varepsilon_B(T) = 0$ it follows $T(\Gamma_2 \oplus B) \subseteq \Gamma_1(T(\Gamma_1 \oplus B) \subseteq \Gamma_2)$.

Indeed, let $T \in h_1(\overline{A}(B_1))$ such that $\varepsilon_B(T) = 0$. It is trivial that $T(B) \subseteq \Gamma_1$, hence it remains to show that $T(\Gamma_2) \subseteq \Gamma_1$. By assumption, T has the form of $T = \sum_{n\geq 0} \lambda(\xi_n)\lambda^*(b_{n1})\cdots\lambda^*(b_{nk_n})$ where $\xi_n \in \mathcal{X}_{deg\xi_n}(B_1)$ and $b_{nj} \in B_1$ for $1 \leq j \leq k_n$. Passing to monomial, it is sufficient to prove the case for $T = \lambda(\xi)\lambda^*(b_1)\cdots\lambda^*(b_k)$ where $\xi \in \mathcal{X}_{deg\xi}(B_1)$ and $b_i \in B_1$ for all $1 \leq i \leq k$. Let $\eta = (\eta_n)_{n\geq 1} \in \Gamma_2$ where $\eta_n \in \mathcal{L}(B_2 \otimes B^{deg\eta_n - 1}, B) \subseteq \mathcal{L}(B^{deg\eta_n}, B) = \mathcal{X}_{deg\eta_n}(B)$, then:

$$T\eta = (\lambda(\xi)\lambda^*(b_1)\cdots\lambda^*(b_k)\eta_n)_{n\geq 1}.$$

Now we have several cases:

Case(i): If $\deg\eta_n - k < 0$ then, $\lambda(\xi)\lambda^*(b_1)\cdots\lambda^*(b_k)\eta_n = 0$ and hence

$$\lambda(\xi)\lambda^*(b_1)\cdots\lambda^*(b_k)\eta_n = 0 \in \mathcal{L}(B_1 \otimes B^{\otimes n-1}, B).$$

Case(ii): If $\deg\eta_n - k = 0$ then, $\lambda(\xi)\lambda^*(b_1)\cdots\lambda^*(b_k)\eta_n = \eta_{00} \in B$ and hence

$$\lambda(\xi)\lambda^*(b_1)\cdots\lambda^*(b_k)\eta_n = \xi\eta_{00} \in \mathcal{X}_{deg\xi}(B_1) \subseteq \mathcal{L}(B_1 \otimes B^{\otimes deg\xi - 1}, B).$$

Case(iii): If $\deg\eta_n - k > 0$ then,

$$\begin{aligned}
\lambda(\xi)\lambda^*(b_1)\cdots\lambda^*(b_k)\eta_n(b_1^{'} \otimes \cdots \otimes b_{deg\xi+deg\eta_n-k}^{'}) &= \lambda^*(b_1)\cdots\lambda^*(b_k)\eta_n(b_{deg\xi+1}^{'}\xi(b_1^{'} \otimes \cdots \otimes b_{deg\xi}^{'}) \\
&\quad \otimes b_{deg\xi+2}^{'} \otimes \cdots \otimes b_{deg\xi+deg\eta_n-k}^{'}) \\
&= \eta_n(b_k \otimes \cdots \otimes b_1 \otimes b_{deg\xi+1}^{'}\xi(b_1^{'} \otimes \cdots \otimes b_{deg\xi}^{'}) \\
&\quad \otimes b_{deg\xi+2}^{'} \otimes \cdots \otimes b_{deg\xi+deg\eta_n-k}^{'}),
\end{aligned}$$

so that $\lambda(\xi)\lambda^*(b_1)\cdots\lambda^*(b_k)\eta_n \in \mathcal{L}(B_1 \otimes B^{deg\xi+deg\eta_n-k-1}, B)$, implying $T\eta \in \Gamma_1$.

Next, let $T_j \in h_1(\overline{A}(B_1))$and $S_j \in h_2(\overline{A}(B_2))$ and $\varepsilon_B(T_j) = \varepsilon_B(S_j) = 0 \ \ (1 \leq j \leq n)$. Then:

$$T_1 1 \in \Gamma_1, S_1T_1 1 \in \Gamma_2, T_2S_1T_1 1 \in \Gamma_1, S_2T_2S_1T_1 1 \in \Gamma_2, \cdots, S_nT_n\cdots S_1T_1 1 \in \Gamma_2,$$

so that $\varepsilon_B(S_nT_n\cdots S_1T_1) = 0$. □

We should mention that similar to the above procedure, if we replace B with the right B-module $M = B^m$ $(m \geq 1)$, we can construct the associated canonical operator-valued noncommutative probability space $(\overline{A}(M), \varepsilon_M, B)$.

Definition 3.1.19. Elements of $\overline{A}(B^1)$ of the form

$$a = \lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_n) \quad \xi_n \in \mathcal{X}_n(B^1)$$

are called canonical.

Proposition 3.1.20. *Given a distribution* $\mu \in \Sigma_B$*, there is a unique canonical element*

$$a = \lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_n) \quad \xi_n \in \mathcal{X}_n(B^1)$$

such that $\mu = \mu_a$*. Here,* ξ_n*'s are denoted by* $\xi_n = R_{n+1}(\mu)$ $(n \geq 0)$.

Proof. Since we want $\mu = \mu_a$, it follows that $\mu(X) = \mu_a(X) = \varepsilon_{B^1}(a) = \xi_0$, and consequently, we define $\xi_0 = \mu(X)$. Let $P(X) = (\prod_{i=1}^n Xb_i)X \in B\langle X\rangle$. Then:

$$\begin{aligned}
\mu(P(X)) &= \mu_a(P(X)) \\
&= \varepsilon_{B^1}\Big((\prod_{i=1}^n a\lambda(b_i))a\Big) \\
&= \varepsilon_{B^1}\Big((\prod_{i=1}^n(\lambda^*(1) + \sum_{m\geq 0}\lambda(\xi_m))\lambda(b_i))(\lambda^*(1) + \sum_{m\geq 0}\lambda(\xi_m))\Big) \\
&= \varepsilon_{B^1}\Big((\prod_{i=1}^n \lambda^*(1)\lambda(b_i))\lambda(\xi_n)\Big) + E_n(\xi_0,\cdots,\xi_{n-1})(b_1\otimes\cdots\otimes b_n) \\
&= \varepsilon_{B^1}\Big((\prod_{i=1}^n \lambda^*(b_i))\lambda(\xi_n)\Big) + E_n(\xi_0,\cdots,\xi_{n-1})(b_1\otimes\cdots\otimes b_n) \\
&= \xi_n(b_n\otimes\cdots\otimes b_1) + E_n(\xi_0,\cdots,\xi_{n-1})(b_1\otimes\cdots\otimes b_n),
\end{aligned}$$

where $E_n(\xi_0,\cdots,\xi_{n-1}) \in \mathcal{L}(B^{\otimes n}, B)$ depends only on $\xi_0,\cdots,\xi_{n-1}$. So we define ξ_n inductively and uniquely as:

$$\xi_n(b_1\otimes\cdots\otimes b_n) = \mu(Xb_nXb_{n-1}\cdots Xb_1X) - E_n(\xi_0,\cdots,\xi_{n-1})(b_n\otimes\cdots\otimes b_1) \quad (n\geq 1).$$

□

Definition 3.1.21. The canonical form of all B−valued random variables with distribution μ, is the unique canonical element in the proposition 3.1.20 such that $\mu = \mu_a$.

It is noteworthy that when $a = \lambda^*(1)$, we have $\mu_a = \mu_0$, i.e. the distribution of the 0 random variable.

Proposition 3.1.22. *Let* $a_k = \lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_{n,k}) \quad k = 1,2,3$ *be canonical elements. Then,* $\mu_{a_3} = \mu_{a_1} \boxplus \mu_{a_2}$ *if and only if* $\xi_{n,3} = \xi_{n,1} + \xi_{n,2} \quad n = 0,1,\cdots$.

Proof. Let $\mu_{a_3} = \mu_{a_1} \boxplus \mu_{a_2} = \mu_{a_1+a_2}$. Then:

$$\begin{aligned} \xi_{0,3} &= \mu_{a_3}(X) = \mu_{a_1+a_2}(X) = \varepsilon_{B^1}(a_1 + a_2) \\ &= \varepsilon_{B^1}(a_1) + \varepsilon_{B^1}(a_2) = \mu_{a_1}(X) + \mu_{a_2}(X) = \xi_{0,1} + \xi_{0,2}. \end{aligned}$$

Assume the assertion holds up to positive integer $n-1$. Then, with E_n from the proof of Proposition 3.1.20:

$$\begin{aligned} \xi_{n,3}(b_1 \otimes \cdots \otimes b_n) &= \mu_{a_3}(Xb_nXb_{n-1}\cdots Xb_1X) - E_n(\xi_{0,3},\cdots,\xi_{n-1,3})(b_n \otimes \cdots \otimes b_1) \\ &= \mu_{a_1+a_2}(Xb_nXb_{n-1}\cdots Xb_1X) - E_n(\xi_{0,1}+\xi_{0,2},\cdots,\xi_{n-1,1}+\xi_{n-1,2})(b_n \otimes \cdots \otimes b_1) \\ &= (\xi_{n,1}(b_1 \otimes \cdots \otimes b_n) + E_n(\xi_{0,1},\cdots,\xi_{n-1,1})(b_n \otimes \cdots \otimes b_1) \\ &+ (\xi_{n,2}(b_1 \otimes \cdots \otimes b_n) + E_n(\xi_{0,2},\cdots,\xi_{n-1,2})(b_n \otimes \cdots \otimes b_1)) \\ &- E_n(\xi_{0,1}+\xi_{0,2},\cdots,\xi_{n-1,1}+\xi_{n-1,2})(b_n \otimes \cdots \otimes b_1) \\ &= (\xi_{n,1}+\xi_{n,2})(b_1 \otimes \cdots \otimes b_n), \end{aligned}$$

implying $\xi_{n,3} = \xi_{n,1} + \xi_{n,2}$, and completing the proof.

Conversely, let $\xi_{n,3} = \xi_{n,1} + \xi_{n,2} \quad n = 0,1,\cdots$. Then, by Proposition 3.1.18 for $M = B^2$ and $a = h_1(a_1) + h_2(a_2)$ we have $\mu_a = \mu_{a_1} \boxplus \mu_{a_2}$. But:

$$\begin{aligned} a &= \lambda^*(1 \oplus 0) + \sum_{n\geq 0} \lambda(\xi_{n,1} \circ pr_1) + \lambda^*(0 \oplus 1) + \sum_{n\geq 0} \lambda(\xi_{n,2} \circ pr_2) \\ &= \lambda^*(1 \oplus 1) + \sum_{n\geq 0} \lambda(\xi_{n,1} \circ pr_1 + \xi_{n,2} \circ pr_2), \end{aligned}$$

$$\varepsilon_{B^2}(\lambda(\beta_n \circ pr_k^{\otimes n})) = \varepsilon_{B^2}(h_k(\lambda(\beta_n))) = \varepsilon_{B^2} \circ h_k(\lambda(\beta_n)) = \varepsilon_{B^1}(\lambda(\beta_n)) \quad (k = 1,2),$$

where $\beta_n \in \mathcal{X}_n(B \oplus 0)or\mathcal{X}_n(0 \oplus B), pr_1^{\otimes n} : (B^2)^{\otimes n} \to (B \oplus 0)^{\otimes n}$, and $pr_2^{\otimes n} : (B^2)^{\otimes n} \to (0 \oplus B)^{\otimes n}, (n \geq 0)$, and

$$\begin{aligned} \varepsilon_{B^2}(\lambda^*(1 \oplus 1)) &= \mu_{\lambda^*(1\oplus 1)}(X) = (\mu_{\lambda^*(1\oplus 0)} \boxplus \mu_{\lambda^*(0\oplus 1)})(X) \\ &= (\mu_0 \boxplus \mu_0)(X) = \mu_0(X) = \mu_{\lambda^*(1)}(X) = \varepsilon_{B^1}(\lambda^*(1)), \end{aligned}$$

implying:

$$\begin{aligned}
\mu_a\Big(\prod_{i=1}^{n}(Xb_i)X\Big) &= \varepsilon_{B^2}\Big(\prod_{i=1}^{n}(a\lambda(b_i))a\Big)\\
&= \sum_{S_{ij}=\lambda^*(1\oplus 1),\lambda(\beta_n\circ pr_k^{\otimes n}):k=1,2,\ n\in\mathbb{N}} \varepsilon_{B^2}(S_{i1}\lambda(b_1)\cdots S_{in}\lambda(b_n)S_{i(n+1)})\\
&= \sum_{S'_{ij}=\lambda^*(1),\lambda(\beta_n):\ n\in\mathbb{N}} \varepsilon_{B^1}(S'_{i1}\lambda(b_1)\cdots S'_{in}\lambda(b_n)S'_{i(n+1)})\\
&= \varepsilon_{B^1}\Big(\prod_{i=1}^{n}(a_3\lambda(b_i))a_3\Big)\\
&= \mu_{a_3}\Big(\prod_{i=1}^{n}(Xb_i)X\Big).
\end{aligned}$$

Thus, $\mu_a(P(X)) = \mu_{a_3}(P(X))$ for all $P(X)\in B\langle X\rangle$and hence, $\mu_a = \mu_{a_3}$. □

Corollary 3.1.23.

$$R_n(\mu_1 \boxplus \mu_2) = R_n(\mu_1) + R_n(\mu_2) \quad n = 1, 2, \cdots,$$

for all $\mu_1, \mu_2 \in \Sigma_B$.

Proof. By Proposition 3.1.20, there are canonical elements a_i with $\mu_{a_i} = \mu_i$ $(i = 1, 2)$. Hence, by Proposition 3.1.22:

$$R_n(\mu_1 \boxplus \mu_2) = R_n(\mu_{a_1} \boxplus \mu_{a_2}) = R_n(\mu_{a_1}) + R_n(\mu_{a_2}) = R_n(\mu_1) + R_n(\mu_2) \quad n = 1, 2, \cdots .$$

□

Lemma 3.1.24. *Let* $c \in \mathbb{C}$ *and* $a \in A$ *with associated distribution* $\mu_a \in \Sigma_B$. *Then for any* $P(X) = (\prod_{i=1}^{n} Xb_i)X \in B\langle X\rangle$ *we have:*

$$\mu_{ca}(P(X)) = c^{n+1}\mu_a(P(X)).$$

Proof.

$$\begin{aligned}
\mu_{ca}(P(X)) &= E_B\circ\tau_{ca}(Xb_1\cdots Xb_nX) = E_b(cab_1\cdots cab_nca)\\
&= c^{n+1}E_B(ab_1\cdots ab_na) = c^{n+1}E_B\circ\tau_a(Xb_1\cdots Xb_nX)\\
&= c^{n+1}\mu_a(P(X)).
\end{aligned}$$

□

Remark 3.1.25. Let $c \in \mathbb{C}$, $\xi_n^{(a)}$ $(n \geq 0)$ appeared in the canonical form of $a \in A$, and $E_n(\xi_0^{(a)}, \cdots, \xi_{n-1}^{(a)})$ $(n \geq 1)$ be the $n-$linear map defined in the proof of the Proposition 3.1.20. Then:

$$E_n(c\xi_0^{(a)}, \cdots, c^n\xi_{n-1}^{(a)}) = c^{n+1}E_n(\xi_0^{(a)}, \cdots, \xi_{n-1}^{(a)}) \quad (n \geq 1).$$

Proposition 3.1.26. *Let $a \in A$ be a $B-$valued random variable with the canonical form*

$$\lambda^*(1) + \sum_{n \geq 0} \lambda(\xi_n^{(a)}),$$

and let $c \in \mathbb{C}$. Then, the canonical form of ca is

$$\lambda^*(1) + \sum_{n \geq 0} \lambda(c^{n+1}\xi_n^{(a)}).$$

Proof. Let ca have the canonical form

$$\lambda^*(1) + \sum_{n \geq 0} \lambda(\xi_n^{(ca)}).$$

We prove the assertion by showing that $\xi_n^{(ca)} = c^{n+1}\xi_n^{(a)}$ $(n \geq 0)$. Let $n = 0$, then $\xi_0^{(ca)} = \mu_{ca}(X) = c\mu_a(X) = c\xi_0^{(a)}$. Assume the equality holds up to positive integer $n-1$, then using the proof of Proposition 3.1.20, Lemma 3.1.24, and Remark 3.1.25 it follows that:

$$\begin{aligned}
\xi_n^{(ca)}(b_1 \otimes \cdots \otimes b_n) &= \mu_{ca}(Xb_n \cdots Xb_1X) - E_n(\xi_0^{(ca)}, \cdots, \xi_{n-1}^{(ca)})(b_n \otimes \cdots \otimes b_1) \\
&= c^{n+1}\mu_a(Xb_n \cdots Xb_1X) - E_n(c\xi_0^{(a)}, \cdots, c^n\xi_{n-1}^{(a)})(b_n \otimes \cdots \otimes b_1) \\
&= c^{n+1}\Big(\mu_a(Xb_n \cdots Xb_1X) - E_n(\xi_0^{(a)}, \cdots, \xi_{n-1}^{(a)})(b_n \otimes \cdots \otimes b_1)\Big) \\
&= c^{n+1}\xi_n^{(a)}(b_1 \otimes \cdots \otimes b_n),
\end{aligned}$$

proving the desired assertion. □

Now, as an immediate consequence of the properties of the canonical form of random variables, we can state and prove the $B-$ valued central limit theorem, as before we follow [14].

Definition 3.1.27. A random variable $s \in A$ is called $B-$semicircular if its canonical form is

$$\lambda^*(1) + \lambda(\eta_0) + \lambda(\eta).$$

The distribution of the $B-$ semicircular random variable is called the $B-$ semicircular distribution. As an immediate consequence of the proof of Proposition 3.1.20 the distribution of semi-circular random variable is seen to be entirely determined by its first two moments:

$$\begin{aligned} E_B(s) &= E_B \circ \tau_s(X) = \mu_s(X) \\ &= \mu_{\lambda^*(1)+\lambda(\eta_0)+\lambda(\eta)}(X) = \eta_0 \in B, \\ E_B(sbs) &= E_B \circ \tau_s(XbX) = \mu_s(XbX) \\ &= \mu_{\lambda^*(1)+\lambda(\eta_0)+\lambda(\eta)}(XbX) = \eta_0 b \eta_0 + \eta(b) \quad b \in B. \end{aligned}$$

A $B-$ semicircular random variable s is called centered if $E_B(s) = 0$. In this case $E_B(sbs) = \eta(b)$ for all $b \in B$. Furthermore, for $m \geq 1$ we have that:

$$\begin{aligned} E_B(s^{2m-1}) &= E_B \circ \tau_s\Big(\prod_{i=1}^{2m-2} X1.X\Big) = \mu_s\Big(\prod_{i=1}^{2m-2} X1.X\Big) \\ &= \varepsilon_{B^1}\Big(\prod_{i=1}^{2m-1} (\lambda^*(1) + \lambda(\eta))\Big) = \varepsilon_{B^1}\Big(\sum_{U_i=\lambda^*(1),\lambda(\eta):i=1,\cdots,2m-1} U_1 \cdots U_i \cdots U_{2m-1}\Big) \\ &= \sum_{U_i=\lambda^*(1),\lambda(\eta):i=1,\cdots,2m-1} \varepsilon_{B^1}\big(U_1 \cdots U_i \cdots U_{2m-1}\big) = \sum_{U_i=\lambda^*(1),\lambda(\eta):i=1,\cdots,2m-1} 0 = 0. \end{aligned}$$

Definition 3.1.28. Let B be a Banach algebra and $\mu, \mu_n : B\langle X\rangle \to B$ $(n \in \mathbb{N})$ be $B-$ valued distributions. We say μ_n convergence pointwise to μ if

$$\lim_{n\to\infty} \|\mu_n(P(X)) - \mu(P(X))\| = 0,$$

for all $P(X) \in B\langle X\rangle$.

Lemma 3.1.29. *Let $\mu_m : B\langle X\rangle \to B$ $(m \in \mathbb{N})$ be a sequence of $B-$ valued distributions with corresponding sequence of canonical forms $\lambda^*(1) + \sum_{n\geq 0} \lambda(\xi_{n,m})$ $(m \in \mathbb{N})$. Then, the following two conditions are equivalent:*

(i) There are constants $C_k(0 \leq k \leq n)$ such that:

$$\sup_{m\in\mathbb{N}} \|\mu_m(Xb_1X \cdots b_kX)\| \leq C_k\|b_1\| \cdots \|b_k\| \quad b_k \in B,\ 0 \leq k \leq n,$$

(ii) There are constants $D_k(0 \leq k \leq n)$ such that:

$$\sup_{m\in\mathbb{N}} \|\xi_{k,m}(b_1 \otimes \cdots \otimes b_k)\| \leq D_k\|b_1\| \cdots \|b_k\| \quad b_k \in B,\ 0 \leq k \leq n.$$

Proof. Referring to the proof of Proposition 3.1.20, the assertion follows from induction and the fact that:

$$\xi_{k,m}(b_1 \otimes \cdots \otimes b_k) = \mu_m(Xb_k \cdots Xb_1X) - E_{k,m}(\xi_{0,m}, \cdots, \xi_{k-1,m})(b_k \otimes \cdots \otimes b_1)$$

for all $b_k \in B,\ 0 \le k \le n, m \ge 1$. □

Lemma 3.1.30. *Let B be a Banach algebra, $\mu, \mu_m : B\langle X\rangle \to B$ $(m \in \mathbb{N})$ be $B-$ valued distributions so that the equivalent conditions of the previous lemma hold. Then, the following assertions are equivalent:*

(i) $\lim_{m\to\infty} \mu_m(Xb_1X \cdots b_kX) = \mu(Xb_1X \cdots b_kX)$ $b_k \in B,\ 0 \le k \le n,$

(ii) $\lim_{m\to\infty} \xi_{k,m}(b_1 \otimes \cdots \otimes b_k) = \xi_k(b_1 \otimes \cdots \otimes b_k)$ $b_k \in B,\ 0 \le k \le n.$

Proof. Referring to the proof of Proposition 3.1.20, the assertion follows from strong induction after n and the relations:

$$\xi_{k,m}(b_1 \otimes \cdots \otimes b_k) = \mu_m(Xb_k \cdots Xb_1X) - E_{k,m}(\xi_{0,m}, \cdots, \xi_{k-1,m})(b_k \otimes \cdots \otimes b_1)$$

for $b_k \in B,\ 0 \le k \le n, m \ge 1$ and

$$\xi_k(b_1 \otimes \cdots \otimes b_k) = \mu(Xb_k \cdots Xb_1X) - E_k(\xi_0, \cdots, \xi_{k-1})(b_k \otimes \cdots \otimes b_1)$$

for $b_k \in B,\ 0 \le k \le n.$ □

Theorem 3.1.31. *(Free Central Limit Theorem) Let B be a Banach algebra and a_m $(m \in \mathbb{N})$ be a sequence of free B-valued random variables in the non-commutative operator valued probability space (A, E_B, B)such that:*

(i) $E_B(a_m) = 0, m \in \mathbb{N},$

(ii) there is a bounded linear map $\eta : B \to B$ such that

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} E_B(a_m b a_m)}{n} = \eta(b),\ \ b \in B,$$

(iii) there are constants C_k $(k \in \mathbb{N})$ such that

$$\sup_{m\in\mathbb{N}} \|E_B(a_m b_1 a_m \cdots b_k a_m)\| \le C_k \|b_1\| \cdots \|b_k\|\ \ (k \in \mathbb{N}).$$

Then the distributions of $S_n = \frac{\sum_{m=1}^{n} a_m}{\sqrt{n}}$ $(n \in \mathbb{N})$ *convergence pointwise to the semicircular distribution with canonical form* $\lambda^*(1) + \lambda(\eta)$.

Proof. Let $\lambda^*(1) + \sum_{k\geq 0} \lambda(\xi_{k,m})$ be the canonical form of a_m $(m \in \mathbb{N})$. By Lemma 3.1.29 and Lemma 3.1.30 we have:

$$\begin{aligned} \xi_{0,m} &= 0, \;\; m \in \mathbb{N} \\ \lim_{n\to\infty} \frac{\sum_{m=1}^{n} \xi_{1,m}(b)}{n} &= \eta(b), \;\; b \in B \\ \sup_{m\in\mathbb{N}} \| \, \xi_{k,m}(b_1 \otimes \cdots \otimes b_m) \, \| &\leq C_k \|b_1\| \cdots \|b_k\|, b_k \in B, k \in \mathbb{N}. \end{aligned}$$

Let $\lambda^*(1) + \sum_{k\geq 0} \lambda(\eta_{k,n})$ be the canonical form of the random variable S_n $(n \geq 1)$. Then, by induction , Proposition 3.1.22, and Proposition 3.1.26 for $c = n^{\frac{-1}{2}}$ it follows that

$$\eta_{k,n} = n^{\frac{-(k+1)}{2}} (\xi_{k,1} + \cdots + \xi_{k,n}) \;\; k \geq 0, \;\; n \in \mathbb{N}.$$

Since:

$$\begin{aligned} \eta_{0,n} &= 0 \\ \lim_{n\to\infty} \eta_{1,n}(b) &= \eta(b), b \in B \\ \|\eta_{k,n}(b_1 \otimes \cdots \otimes b_k)\| &\leq C_k \|b_1\| \cdots \|b_k\| n^{\frac{-(k-1)}{2}}, b_k \in B, k \in \mathbb{N}, \end{aligned}$$

by Lemma 3.1.29, and Lemma 3.1.30 it follows that the distributions of S_n $(n \in \mathbb{N})$ convergence pointwise to the B−semicircular distribution of $\lambda^*(1) + \lambda(\eta)$. □

3.2 The Complete Positivity Property of the Second Moment

This section deals with the necessary and sufficient condition of the second moment of a given random variable $a \in A$ ensuring it to be semicircular. Here, by $M_n(A)$ we denote the algebra of all $n \times n$ matrices with entries from A, equipped with the canonical $*$ structure. The conjugate of $\alpha = (a_{ij})_{i,j=1}^{n} \in M_n(A)$ will be $\alpha^* = (a_{ji}^*)_{i,j=1}^{n}$. Throughout this section, we assume that A to be a unital C^*-algebra and B is its unital C^*-subalgebra containing its unit, [13, pp. 41-44].

Definition 3.2.1. Let A_1, A_2 be two unital C^*-algebras. A unital linear map $\Phi: A_1 \to A_2$ is positive if $\Phi(aa^*) \geq 0$ for all $a \in A_1$. Moreover, it is completely positive, if for each $n \geq 1$, the map

$$\Phi_n : M_n(A_1) \to M_n(A_2)$$
$$\Phi_n((a_{ij})_{i,j=1}^n) = (\Phi(a_{ij}))_{i,j=1}^n$$

is positive.

Lemma 3.2.2. *(1) Let $\alpha \in M_n(A)$ $(n \geq 1)$. Then, $\alpha\alpha^* = \sum_{k=1}^n (a_{k,i}a_{k,j}^*)_{i,j=1}^n$ for some $a_{k,i} \in A$ $(1 \leq i, k \leq n)$.*

(2) Let $\Phi : A \to B$ be a unital linear map. Then, the following statements are equivalent:

(a) The map $\Phi : A \to B$ is completely positive.

(b) For each $n \in \mathbb{N}$, and all $a_1, \cdots, a_n \in A$, the matrix $(\Phi(a_i a_j^*))_{i,j=1}^n$ is positive.

(c) For each $n \in \mathbb{N}$, and all $a_1, \cdots, a_n \in A$, there exist elements $b_{k,i} \in B$ $(1 \leq i, k \leq n)$ such that $\Phi(a_i a_j^*) = \sum_{k=1}^n b_{k,i} b_{k,j}^*$ for all $1 \leq i, j \leq n$.

Proof. (1) Writing $\alpha = (a_{ij})_{i,j=1}^n$ we have:

$$\alpha\alpha^* = \Big(\sum_{k=1}^n a_{ik}a_{jk}^*\Big)_{i,j=1}^n = \sum_{k=1}^n (a_{ik}a_{jk}^*)_{i,j=1}^n = \sum_{k=1}^n (a_{k,i}a_{k,j}^*)_{i,j=1}^n.$$

(2) Let part (a) hold. By part (1) for $a_{1,i} = a_i, a_{k,i} = 0$ where $2 \leq k \leq n, 1 \leq i \leq n$, it follows that $(a_i a_j^*)_{i,j=1}^n \in M_n(A)$ is positive. Hence, by complete positivity of Φ, part (b) is proved. Next, let part (b) hold. Consider the matrix $(\Phi(a_i a_j^*))_{i,j=1}^n \in M_n(B)$. Then, by part (b) it is positive and hence by part (1):

$$\Big(\Phi(a_i a_j^*)\Big)_{i,j=1}^n = \sum_{k=1}^n (b_{k,i}b_{k,j}^*)_{i,j=1}^n = \Big(\sum_{k=1}^n b_{k,i}b_{k,j}^*\Big)_{i,j=1}^n,$$

and by comparing the corresponding entries in both matrices, part (c) is proved. Finally. let part (c) hold. For any $\alpha \in M_n(A)$ $(n \geq 1)$, using part (1) we have: $\alpha\alpha^* = \sum_{k=1}^n (a_{k,i}a_{k,j}^*)_{i,j=1}^n$ for some $a_{k,i} \in A$ $(1 \leq k, i \leq n)$. Thus, part (c) implies:

$$\Phi_n(\alpha\alpha^*) = \sum_{k=1}^n (\Phi(a_{k,i}a_{k,j}^*))_{i,j=1}^n = \sum_{k=1}^n \sum_{l=1}^n (b_{k,l,i}b_{k,l,j}^*)_{i,j=1}^n \geq 0,$$

proving the positivity property of Φ_n, and hence part (a). $\square$

As one of key properties of the conditional expectation, the positivity property and complete positivity property coincide. Before proving that we need the following lemma, [8].

Lemma 3.2.3. *Let B be a unital C^*-algebra and $b_{ij} \in B$ $(1 \le i, j \le n)$. Then, the following statements are equivalent:*

(a) The matrix $(b_{ij})_{i,j=1}^n \in M_n(B)$ is positive.

(b) We have: $\sum_{i,j=1}^n b_i b_{ij} b_j^* \ge 0$ for all $b_1, \cdots, b_n \in B$.

Proposition 3.2.4. *Let B be unital C^*-algebra, A be a unital $*-$ algebra containing B. Then, any conditional expectation $E_B : A \to B$ is completely positive.*

Proof. Let $n \in \mathbb{N}$ and $a_i \in A$ $(1 \le i \le n)$. Then:

$$\sum_{i,j=1}^{n} b_i E_B(a_i a_j^*) b_j^* = \sum_{i,j=1}^{n} E_B(b_i a_i (b_j a_j)^*) = E_B\Big((\sum_{i=1}^{n} b_i a_i)(\sum_{j=1}^{n} b_j a_j)^* \Big) \ge 0.$$

Hence, by Lemma 3.2.3 the matrix $(E_B(a_i a_j^*))_{i,j=1}^n \in M_n(B)$ is positive, and therefore, by Lemma 3.2.2, E_B is completely positive. □

Theorem 3.2.5. *Let B be a unital C^*-algebra. A unital linear map $\eta : B \to B$ is the second moment of a $B-$ semicircular random variable if and only if it is completely positive.*

Proof. Assume that $s \in A$ is a B-valued semicircular random variable. Let $\eta(b) = E_B(sbs)$ for all $b \in B$. Then, for

$$(\eta(b_i b_j^*))_{i,j=1}^n = (E_B(s b_i b_j^* s))_{i,j=1}^n = \Big(E_B((s b_i)(s b_j)^*) \Big)_{i,j=1}^n \in M_n(B) \quad (n \ge 1),$$

an application of Proposition 3.2.4, and two applications of of Lemma 3.2.2, respectively, show that η is completely positive.

Conversely, let the unital linear map $\eta : B \to B$ be completely positive. We need to find a semicircular random variable with second moment η. We will construct the distribution of such a random variable as the distribution of the sum of 'creation' and 'annihilation' operators on a 'degenerate' Fock space. The degenerate Fock space $\mathcal{F}$ is the $B-B-$ bimodule

$\mathcal{F} = B \oplus BXB$ equipped with a $B-$valued inner product:

$$\begin{aligned}\langle , \rangle \quad &: \quad \mathcal{F} \times \mathcal{F} \to B \\ \langle b_0 + b_1 X b_2, \tilde{b_0} + \tilde{b_1} X \tilde{b_2} \rangle \quad &= \quad b_0^* \tilde{b_0} + b_2^* \eta(b_1^* \tilde{b_1}) \tilde{b_2}.\end{aligned}$$

Now, define a creation operator l^* and an annihilation operator l via:

$$\begin{aligned}&l^* : \mathcal{F} \to \mathcal{F} &\qquad &l : \mathcal{F} \to \mathcal{F} \\ &l^*(b_0 + b_1 X b_2) = 1 X b_0 &\qquad &l(b_0 + b_1 X b_2) = \eta(b_1) b_2.\end{aligned}$$

Next, as a result of positivity of η we have $\eta(b^*) = \eta(b)^*$ for all $b \in B$. Consequently, the operators l and l^* are adjoints with respect to $\langle , \rangle$; i.e., for $f_1 = b_0 + b_1 X b_2$ and $f_2 = \tilde{b_0} + \tilde{b_1} X \tilde{b_2}$ we have that:

$$\begin{aligned}\langle l f_1, f_2 \rangle \quad &= \quad \langle \eta(b_1) b_2 + 0X0, \tilde{b_0} + \tilde{b_1} X \tilde{b_2} \rangle = (\eta(b_1) b_2)^* \tilde{b_0} \\ &= \quad b_2^* \eta(b_1^* .1) \tilde{b_0} = \langle b_0 + b_1 X b_2, 0 + 1 X \tilde{b_0} \rangle \\ &= \quad \langle f_1, l^* f_2 \rangle.\end{aligned}$$

Now, we identify all elements $b \in B$ with their corresponding left multiplication operators $l_b : \mathcal{F} \to \mathcal{F}$, and also we consider b and b^* as adjoints in this context. We consider our $B-$valued non-commutative probability space (A, E_B, B), where $A = \langle \{ l, l^*, l_b : b \in B \} \rangle$, and $E_B : A \to B$ defined via: $E_B(a) = \langle 1, a1 \rangle$. It is worth mentioning that since $\langle f_1 b_1, f_2 b_2 \rangle = b_1^* \langle f_1, f_2 \rangle b_2$ for all $f_1, f_2 \in \mathcal{F}$ and all $b_1, b_2 \in B$, we have:

$$E_B(b_1 a b_2) = \langle 1, b_1 a b_2 1 \rangle = b_1 \langle 1, a1 \rangle b_2 = b_1 E_B(a) b_2$$

for all $a \in A$ and all $b_1, b_2 \in B$. Let $s := l^* + l \in A$. Then:

$$E_B(sbs) = \langle 1, (l^* + l) b (l^* + l) 1 \rangle = \eta(b) \quad b \in B.$$

So, it remains to show that E_B is positive. Let $a \in A$, then writing $a^* 1 \in \mathcal{F}$ in the form $a^* 1 = b + \sum_{i=1}^n b_i X \tilde{b_i}$ where $b, b_i, \tilde{b_i} \in B$ $(1 \leq i \leq n)$, we have:

$$\begin{aligned}E_B(aa^*) \quad &= \quad \langle 1, aa^* 1 \rangle = \langle a^* 1, a^* 1 \rangle \\ &= \quad \langle b + \sum_{i=1}^n b_i X \tilde{b_i}, b + \sum_{j=1}^n b_j X \tilde{b_j} \rangle \\ &= \quad b^* b + \sum_{i,j=1}^n \tilde{b_i}^* \eta(b_i^* b_j) \tilde{b_j}.\end{aligned}$$

Now, by complete positivity of η and Lemma 3.2.2, we have:

$$\eta(b_i^* b_j) = \sum_{k=1}^{n} b_{k,i}^* b_{k,j} \quad b_{k,i} \in B \quad (1 \leq i,k \leq n).$$

Hence:

$$\begin{aligned} E_B(aa^*) &= b^*b + \sum_{i,j=1}^{n} \tilde{b_i}^* \Big(\sum_{k=1}^{n} b_{k,i}^* b_{k,j} \Big) \tilde{b_j} \\ &= b^*b + \sum_{i,j=1}^{n} \sum_{k=1}^{n} \tilde{b_i}^* b_{k,i}^* b_{k,j} \tilde{b_j} \\ &= b^*b + \sum_{k=1}^{n} \Big(\sum_{i=1}^{n} b_{k,i} \tilde{b_i} \Big)^* \Big(\sum_{j=1}^{n} b_{k,j} \tilde{b_j} \Big) \\ &\geq 0, \end{aligned}$$

completing the proof. □

We close this section by stating M. D. Choi's representation of completely positive maps from a finite dimensional matrix algebra to another one, [1].

Theorem 3.2.6. *Let $\eta : M_{n_1 \times n_1}(\mathbb{C}) \to M_{n_2 \times n_2}(\mathbb{C})$ be a completely positive linear map. Then there exist $a_j \in M_{n_1 \times n_2}(\mathbb{C})$ $(1 \leq j \leq n_1 n_2)$, such that*

$$\eta(a) = \sum_{j=1}^{n_1 n_2} a_j^* a a_j$$

for all $a \in M_{n_1 \times n_1}(\mathbb{C})$.

Proof. Let $E_{l_1 l_2} \in M_{n_1 \times n_1}(\mathbb{C})$ be defined by $E_{l_1 l_2} = (\delta_{(s,t),(l_1,l_2)})$, $1 \leq l_1, l_2 \leq n_1$. Since η is a n_1−positive map, it follows that $\big(\eta(E_{l_1 l_2})\big) \in M_{n_1 n_2 \times n_1 n_2}(\mathbb{C})$ is positive. Now, by spectral resolution theorem we have $\big(\eta(E_{l_1 l_2})\big) = \sum_{j=1}^{n_1 n_2} \lambda_j v_j v_j^*$ where $v_j \in \mathbb{C}^{n_1 n_2}$ and $\lambda_j \geq 0$, $1 \leq j \leq n_1 n_2$. By absorbing λ_j in v_j we have that:

$$\big(\eta(E_{l_1 l_2})\big) = \sum_{j=1}^{n_1 n_2} v_j v_j^*. \tag{3.1}$$

Next, writing the vector space $\mathbb{C}^{n_1 n_2}$ in the form $\mathbb{C}^{n_1 n_2} = \bigoplus_{i=1}^{n_1} \mathbb{C}^{n_2}$, considering $p_i \in M_{n_1 n_2 \times n_1 n_2}(\mathbb{C})$ as the projection on the i^{th} copy of $\mathbb{C}^{n_2}$, $1 \leq i \leq n_1$, and Equality (3.1) we have:

$$\eta(E_{l_1 l_2}) = p_{l_1}.\big(\eta(E_{l_1 l_2})\big).p_{l_2} = \sum_{j=1}^{n_1 n_2} p_{l_1} v_j (p_{l_2} v_j)^*, \tag{3.2}$$

for all $1 \leq l_1, l_2 \leq n_1$. Now, define $a_j^* \in M_{n_1 \times n_1}(\mathbb{C})$ by

$$a_j^* e_l \quad = \quad p_l v_j \quad 1 \leq j \leq n_1 n_2, 1 \leq l \leq n_1, \tag{3.3}$$

and from (3.2) it follows that:

$$\eta(E_{l_1 l_2}) \quad = \quad \sum_{j=1}^{n_1 n_2} a_j^* e_{l_1} e_{l_2}^* a_j = \sum_{j=1}^{n_1 n_2} a_j^* E_{l_1 l_2} a_j \quad 1 \leq l_1, l_2 \leq n_1 n_2. \tag{3.4}$$

Accordingly, by linear extension of the Equality (3.4) the desired result follows. □

Chapter 4

A Special Functional Equation for Some Operator-Valued Cauchy Transforms

This chapter discusses the operator-valued Cauchy transform of a random variable and an associated functional equation to it for the case of semi-circular random variables. In the section 1, the operator-valued Cauchy transform of a random variable is introduced, then using a general implicit function theorem, we introduce the operator-valued R-transform of a random variable satisfying a special functional equation involving its operator-valued Cauchy transform. In section 2, we discuss sufficient conditions under which the mentioned operator-valued functional equation involving the operator-valued Cauchy transform of the semicircular random variable has a unique solution.

4.1 Some Analytic Transforms in Operator-Valued Settings

Given an operator-valued noncommutative probability space (A, E_B, B), we recall that any $a \in A$ can be written as $a = Re(a) + i.Im(a)$ where $Re(a)$ and $Im(a)$ are self-adjoint elements. We define $H^+(A) = \{a \in A | Im(a) > 0\}$, where $Im(a) > 0$ means $Im(a) > \epsilon.1$ for some $\epsilon > 0$, and similarly $H^+(B)$. Then the operator-valued Cauchy transform of $a \in A$ is defined via:

$$G_a^*(b) \quad = \quad E_B((b-a)^{-1}).$$

We notice that if $\|b^{-1}\| < \|a\|^{-1}$ (and hence $\|b\| > \|a\|$), so that $\|ab^{-1}\| < 1$, then we have:

$$G_a^*(b) = E_B((b-a)^{-1}) = \sum_{n=0}^{\infty} b^{-1} E_B((ab^{-1})^n).$$

This is an analytic function with Fréchet derivative

$$(D_b G_a^*(b))(h) = -E_B((b-a)^{-1}.h.(b-a)^{-1}), \quad (h \in B).$$

Next, by Lemma 3.1.8 from Chapter 3 and the fact that the n^{th} derivative of the function

$$b^{-1}.G_a^*(b^{-1}).b^{-1} = 1 + \sum_{n=0}^{\infty} m_n(b, \cdots, b)$$

in $b = 0$ gives exactly the symmetric part of m_n, it follows that the symmetric distribution of $a_1 \in A$ equals to the symmetric distribution of $a_2 \in A$ if and only if $G_{a_1}^* = G_{a_2}^*$.

Let S_n $(n \geq 1)$ be the symmetric group. The subspace of symmetric $n-$linear maps of $\mathcal{X}_n(B)$ is defined as

$$S\mathcal{X}_n(B) = \{\xi_n \in \mathcal{X}_n(B) | \xi_n(b_1 \otimes \cdots \otimes b_n) = \xi_n(b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(n)}) \text{ for all } \ \sigma \in S_n\}.$$

If $\xi_n \in \mathcal{X}_n(B)$, we denote by $S\xi_n \in S\mathcal{X}_n(B)$ the element defined by

$$S\xi_n(b_1 \otimes \cdots \otimes b_n) = \sum_{\sigma \in S_n} \frac{1}{n!} \xi_n(b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(n)})$$

such that $S\xi_n(b^{\otimes n}) = \xi_n(b^{\otimes n})$. Next, we represent the elements of $S\mathcal{X}(B) = \prod_{n\geq 0} S\mathcal{X}_n(B) \subseteq \prod_{n\geq 0} \mathcal{X}_n(B) = \overline{\mathcal{X}}(B)$ by $\sum_{n\geq 0} \xi_n$.

Definition 4.1.1. Let $S_\mu \in S\Sigma_B$. We define:

$$G_{S_\mu} : B \to B \qquad\qquad \Gamma_{S_\mu} : B \to B$$

$$G_{S_\mu}(b) = \sum_{n\geq 0} \mu(b(Xb)^n) \qquad\qquad \Gamma_{S_\mu}(b) = \sum_{n\geq 0} \mu((Xb)^n X).$$

Here, by conditional expectation properties of μ it follows that:

$$G_{S_\mu}(b) = \mu(b) + \sum_{n\geq 1} \mu(b(Xb)^n) = b + \sum_{n\geq 1} b\mu((Xb)^{n-1}X)b = b + b\Gamma_{S_\mu}(b)b.$$

It should be mentioned that for $a \in A$ we have

$$G_{S_{\mu_a}}(b) = G_a^*(b^{-1}), \quad \|b\| < \|a\|^{-1}.$$

Next, since the Frechet differential of $G_{S_{\mu_a}}$ at $b = 0$ is the identity map, $G_{S_{\mu_a}}$ is invertible with respect to composition in a neighborhood of zero. Before introducing the $R-$ transform of a random variable $a \in A$, we need a general implicit function theorem due to T. H. Hildebrandt and L. M. Graves, [6].

Theorem 4.1.2. *Let X, Y and Z be normed linear spaces, of which X and Z are also complete. Suppose that $G : X \times Y \to Z$ is of class $C^{(n)}$ on a region $R \subseteq X \times Y$, and let (x_0, y_0) be an initial solution of the equation*

$$G(x,y) \quad = \quad 0 \tag{4.1}$$

at which the partial differential $\frac{\partial}{\partial y} G(x_0, y_0)$ has an inverse. Then in a sufficiently small neighborhood of y_0 in Y the equation (4.1) *has a unique solution $y = y(x)$ of $C^{(n)}$ class defined on a neighborhood of x_0 in X.*

Now, the $R-$ transform of a is introduced in the following theorem, [2].

Theorem 4.1.3. *There is a unique $B-$valued analytic function R_a, defined in a neighborhood of* 0 *in B, such that*

$$G^{-1}_{S_{\mu_a}}(b) = (1 + bR_a(b))^{-1} b = b(1 + R_a(b)b)^{-1}, \tag{4.2}$$

where the inverse in the left hand side is considered as invertibility with respect to composition.

Proof. First, the uniqueness is clear by power series expansion. Second, the right-most equality in (4.2) holds for and analytic function R_a. Hence, it remains to find a function R_a, such that

$$G_{S_{\mu_a}}((1 + bR_a(b))^{-1} b) = b.$$

But,

$$\begin{aligned} G_{S_{\mu_a}}((1 + bR_a(b))^{-1} b) \quad &= \quad (1 + bR_a(b))^{-1} b \\ &+ \quad (1 + bR_a(b))^{-1} b \Gamma_{S_{\mu_a}}\Big((1 + bR_a(b))^{-1} b\Big)(1 + bR_a(b))^{-1} b, \end{aligned}$$

so it will suffice to find R_a so that any of the following hold:

$$(1 + bR_a(b))^{-1} + (1 + bR_a(b))^{-1} b \Gamma_{S_{\mu_a}}\Big((1 + bR_a(b))^{-1} b\Big)(1 + bR_a(b))^{-1} = 1,$$

$$1 + b\Gamma_{S_{\mu_a}}\Big((1 + bR_a(b))^{-1} b\Big)(1 + bR_a(b))^{-1} = 1 + bR_a(b),$$

$$\Gamma_{S_{\mu_a}}\Big((1 + bR_a(b))^{-1} b\Big)(1 + bR_a(b))^{-1} = R_a(b). \tag{4.3}$$

However, $R_a(0) = E(a)$ is a solution of equation (4.3) at $b = 0$, and the Frechet differential of the function $x \mapsto \Gamma_{S_{\mu_a}}\Big((1+bx)^{-1}b\Big)(1+bx)^{-1} - x$ at $b = 0$ is the negative of the identity map, hence is invertible. Now, the existence of R_a is guaranteed by Theorem 4.1.2. □

It should be mention that by [14], we have:

$$R_a(b) = \sum_{n\geq 0} SR_{n+1}(S_{\mu_a})(b^{\otimes n})$$

where $SR_{n+1}(S_{\mu_a})$ $(n \geq 0)$ are given by the canonical element with distribution S_{μ_a}.

Corollary 4.1.4. *If $a_1, a_2 \in A$ are free over B, then:*

$$\begin{aligned} R_{a_1+a_2}(b) &= \sum_{n\geq 0} SR_{n+1}(S_{\mu_{a_1+a_2}})(b^{\otimes n}) = \sum_{n\geq 0} SR_{n+1}(S_{\mu_{a_1}} \boxplus S_{\mu_{a_2}})(b^{\otimes n}) \\ &= \sum_{n\geq 0} SR_{n+1}(S_{\mu_{a_1}})(b^{\otimes n}) + \sum_{n\geq 0} SR_{n+1}(S_{\mu_{a_2}})(b^{\otimes n}) = R_{a_1}(b) + R_{a_2}(b). \end{aligned}$$

Corollary 4.1.5. *Let $b(1 + R_a(b)b)^{-1} = G^{-1}_{S_{\mu_a}}(b)$ as in Theorem 4.1.3. Replacing b with $G_{S_{\mu_a}}(b^{-1})$ in the equation and considering $G_{S_{\mu_a}}(b^{-1}) = G^*_a(b)$, we get $G^*_a(b)(1+R_a(G^*_a(b))G^*_a(b))^{-1} = b^{-1}$, yielding:*

$$bG^*_a(b) = 1 + R_a(G^*_a(b))G^*_a(b).$$

In particular, for the semicircular element $s \in A$ with the first moment $D = E_B(s) \in B$ and the variance $\eta(b) = E_B(sbs)$ we have that:

$$bG^*_s(b) = 1 + (D + \eta(G^*_s(b)))G^*_s(b).$$

4.2 The Functional Equation of the Operator-Valued Cauchy Transform of a Semicircular Random Variable

Let A be a unital C^*-algebra, B be a C^*-subalgebra of A, and $E_B : A \to B$ be a conditional expectation. For a given self-adjoint semicircular random variable $s \in A$ with a completely positive linear map $\eta : B \to B$ as its variance, its operator-valued Cauchy transform G^*_s maps

$H^+(B)$ into $H^-(B)$. Indeed, let $b \in H^+(B)$ where $b = Re(b) + i.Im(B)$ with $Im(b) > \epsilon.1$ for some $\epsilon > 0$. Then as in [15] we have that:

$$\begin{aligned}(b-s)^{-1} &= (Re(b) - s + i.Im(b))^{-1}\\ &= \left((Im(b))^{\frac{1}{2}}\left((Im(b))^{\frac{-1}{2}}(Re(b)-s)(Im(b))^{\frac{-1}{2}} + i\right)(Im(b))^{\frac{1}{2}}\right)^{-1}.\end{aligned}$$

Here, since $Re(b) - s$ is self adjoint, it follows that $(Im(b))^{\frac{-1}{2}}(Re(b)-s)(Im(b))^{\frac{-1}{2}}$ is self-adjoint too. But, $-i \notin \sigma\left((Im(b))^{\frac{-1}{2}}(Re(b)-s)(Im(b))^{\frac{-1}{2}}\right) \subseteq \mathbb{R}$, implying that $(Im(b))^{\frac{-1}{2}}(Re(b)-s)(Im(b))^{\frac{-1}{2}} + i$ is invertible and, consequently , $\left((Im(b))^{\frac{1}{2}}\left((Im(b))^{\frac{-1}{2}}(Re(b)-s)(Im(b))^{\frac{-1}{2}} + i\right)(Im(b))^{\frac{1}{2}}\right)$ is invertible. Hence, by equation above $(b-s)^{-1}$ is well-defined. Next, we have:

$$\begin{aligned}Im((b-s)^{-1}) &= Im\left(\left((Im(b))^{\frac{1}{2}}\left((Im(b))^{\frac{-1}{2}}(Re(b)-s)(Im(b))^{\frac{-1}{2}} + i\right)(Im(b))^{\frac{1}{2}}\right)^{-1}\right)\\ &= (Im(b))^{\frac{-1}{2}}.Im\left(\left((Im(b))^{\frac{-1}{2}}(Re(b)-s)(Im(b))^{\frac{-1}{2}} + i\right)^{-1}\right).(Im(b))^{\frac{-1}{2}}\\ &= (Im(b))^{\frac{-1}{2}}.Im\left(\frac{(Im(b))^{\frac{-1}{2}}(Re(b)-s)(Im(b))^{\frac{-1}{2}} - i}{\left((Im(b))^{\frac{-1}{2}}(Re(b)-s)(Im(b))^{\frac{-1}{2}}\right)^2 + 1}\right).(Im(b))^{\frac{-1}{2}}\\ &= -(Im(b))^{-1}.\left(\left((Im(b))^{\frac{-1}{2}}(Re(b)-s)(Im(b))^{\frac{-1}{2}}\right)^2 + 1\right)^{-1}\\ &< 0,\end{aligned}$$

and by complete positivity of E_B it follows that:

$$Im(G_s^*(b)) = Im(E_B((b-s)^{-1})) = E_B(Im((b-s)^{-1})) < 0.$$

By Corollary 4.1.5 of section 1, the operator-valued Cauchy transform operator-valued Cauchy transform of the semicircular random variable, $G_s^* : H^+(B) \to H^-(B)$ satisfies the the equation:

$$b.G_s^*(b) = 1 + (D + \eta(G_s^*(b))).G_s^*(b) \quad Im(b) > 0 \tag{4.4}$$

where $D = E_B(s) \in B$ is self-adjoint. We want to show that this equation for the analytic $G_s^* : H^+(B) \to H^-(B)$ together with its asymptotic condition:

$$\lim_{b^{-1}\to 0} b.G_s^*(b) = 1 \tag{4.5}$$

uniquely determines the G_s^*. For that aim, we need a special fixed point theorem. Before proceeding to its statement, we need some definitions, [4], [3].

Definition 4.2.1. Let $(B_i, \|.\|_i)$ $i = 1,2$, be two complex Banach spaces and $D_i \subseteq B_i$ $i = 1,2$, be two bounded domains.

(i) A subset $D_2' \subseteq B_2$ lies strictly in $D_2 \subseteq B_2$, and we write $D_2' \subseteq D_2$ strictly, if there is $\epsilon > 0$ such that for all $b_2' \in D_2'$:

$$D(B_2, b_2', \epsilon) = \{b_2 \in B_2 | \|b_2 - b_2'\|_2 < \epsilon\} \subseteq D_2.$$

(ii) Let $H^\infty(D_1)$ be the Banach space of all bounded holomorphic functions on D_1, and Γ be the set of all curves in D_1 with piecewise continuous derivative. Define:

$$\begin{aligned} \alpha &: D_1 \times B_1 \to \mathbb{R}_0^+ \\ \alpha(b_1', b_1) &= \sup\{|(Df(b_1'))b_1| | f \text{ is in the unit ball of } H^\infty(D_1)\}, \end{aligned}$$

and set $L(\gamma) = \int_0^1 \alpha(\gamma(t), \gamma'(t))dt$ $\gamma \in \Gamma$. The Caratheodory-Reiffen Finsler metric (CRF-metric) ρ is defined as follows:

$$\begin{aligned} \rho &: D_1 \times D_1 \to \mathbb{R}_0^+ \\ \rho(b_1', b_1'') &= \inf\{L(\gamma) | \gamma \in \Gamma : \gamma(0) = b_1', \gamma(1) = b_1''\}. \end{aligned}$$

Regarding the CRF-metric ρ we have:

Proposition 4.2.2. *Let ρ be the CRF-metric. Then, there exists a constant $m > 0$ such that $\rho(b_1', b_1'') \geq m\|b_1' - b_1''\|_1$ for all $b_1', b_1'' \in D_1$.*

Proof. Since D_1 is bounded, $d = diam(D_1) < \infty$ and we take $m = \frac{1}{d}$. For given $b_1' \in D_1$ and $b_1 \in B_1$ define:

$$\begin{aligned} f_l &: D_1 \to \mathbb{C} \\ f_l(b_1'') &= ml(b_1'' - b_1') \end{aligned}$$

where $l \in B_1^*$ with $\|l\| = 1$. Then, f_l is in the unit ball of $H^\infty(D_1)$, and $Df_l(b_1')b_1 = ml(b_1)$. Hence, $\frac{1}{m}\alpha(b_1', b_1) \geq |l(b_1)|$ for all $l \in B_1^*$ with $\|l\| = 1$, and, consequently, by a corollary of the Hahn-Banach theorem, $\frac{1}{m}\alpha(b_1', b_1) \geq \|b_1\|$. Now, by integrating from both sides the assertion is proved. □

The following fixed point theorem is due to C. J. Earle and R. S. Hamilton, [4].

Theorem 4.2.3. *Let D be a non-empty domain in a complex Banach space B, and let $h : D \to D$ be a bounded holomorphic function. If $h(D) \subseteq D$ strictly, then h is a strict contraction in the CRF-metric ρ, and thus has a unique fixed point in D. Furthermore, for any $b_0 \in D$, the sequence $(h^{\circ n}(b_0))_{n=1}^{\infty}$ converges in norm to this fixed point.*

Using above mentioned theorem we can state and prove the following theorem which has previously appeared in [5]:

Theorem 4.2.4. *Let A be a unital C^*-algebra, B a C^*-subalgebra of A, and $s \in A$ be a self-adjoint B-valued semicircular random variable with first moment $D \in B$ and variance $\eta : B \to B$. Then its associated operator-valued Cauchy transform $G_s^* : H^+(B) \to H^-(B)$ is the unique solution of the functional equation* (4.4) *together with asymptotic condition* (4.5)*.*

Proof. Fix $b \in H^+(B)$ where $Im(b) > \epsilon.1$ for some $\epsilon > 0$. Then the equation (4.4) holds if and only if $b - (D + \eta(G_s^*(b))) = G_s^{*-1}(b)$. Now, define:

$$F_b : H^+(B) \to H^+(B)$$
$$F_b(w) = b - (D + \eta(w^{-1})).$$

Let $r = 2(\| b \| + \| D \| + \| \eta \| \epsilon^{-1})$, and define :

$$D_{r,\epsilon} = (H^+(B) + \epsilon.i) \cap \{w \in B | \; \| w \| < r\}.$$

Let $w \in D_{r,\epsilon}$, then by $Im(w) > \epsilon.1$ and using properties of C^*-algebra and functional calculus it follows that:

$$\begin{aligned}
\| w^{-1} \| &= \| (Re(w) + i.Im(w))^{-1} \| \\
&= \| (Im(w))^{\frac{-1}{2}}\Big((Im(w))^{\frac{-1}{2}}.Re(w).(Im(w))^{\frac{-1}{2}} + i\Big)^{-1}(Im(w))^{\frac{-1}{2}} \| \\
&\leq \| \Big((Im(w))^{\frac{-1}{2}}.Re(w).(Im(w))^{\frac{-1}{2}} + i\Big)^{-1} \| \| (Im(w))^{\frac{-1}{2}} \|^2 \\
&\leq 1. \| (Im(w))^{-1} \| \\
&\leq \epsilon^{-1},
\end{aligned}$$

yielding:

$$\begin{aligned}\| F_b(w) \| &= \| b - (D + \eta(w^{-1})) \| \leq \| b \| + \| D \| + \| \eta \| \| w^{-1} \| \\ &\leq \| b \| + \| D \| + \| \eta \| \epsilon^{-1} = \frac{r}{2}.\end{aligned}$$

Next, since $\| w \| < r$, by complete positivity of η it follows that:

$$\begin{aligned}Im(F_b(w)) &= Im(b) - Im(\eta(w^{-1})) > \epsilon - \eta(Im(w^{-1})) \\ &= \epsilon + \eta(\frac{Im(w)}{Re^2(w) + Im^2(w)}) = \epsilon + \eta\Big((Re^2(w)Im^{-1}(w) + Im(w))^{-1}\Big) \\ &\geq \epsilon + \eta\Big((\| Re^2(w)Im^{-1}(w) + Im(w) \|)^{-1}.1\Big) \\ &\geq \epsilon + \Big(\| Re(w) \|^2 \| Im^{-1}(w) \| + \| Im(w) \| \Big)^{-1} \| \eta \| \\ &\geq \epsilon + \frac{\| \eta \|}{r^2.\epsilon^{-1} + r}.\end{aligned}$$

Accordingly, $F_b(w) \in D_{\frac{r}{2}, \epsilon + \frac{\|\eta\|}{r^2.\epsilon^{-1}+r}}$, implying:

$$F_b(D_{r,\epsilon}) \subseteq D_{\frac{r}{2}, \epsilon + \frac{\|\eta\|}{r^2.\epsilon^{-1}+r}} \subsetneq D_{r,\epsilon} \quad \text{strictly,}$$

for all $\epsilon > 0$, $r \geq 2(\| b \| + \| D \| + \| \eta \| \epsilon^{-1})$ and fixed $b \in H^+(B)$. Now, by Theorem 4.2.3, we conclude that there is a unique $w_b \in D_{r,\epsilon}$ such that $F_b(w_b) = w_b$, and furthermore, for any $w_0 \in D_{r,\epsilon}$ the sequence $(F_b^{\circ n}(w_0))_{n=0}^{\infty}$ converges to this fixed point. Hence, $F_b(w_b) = w_b$ if and only if

$$w_b = b - (D + \eta(w_b^{-1})) \tag{4.6}$$

where w_b is a unique element in $D_{r,\epsilon}$ satisfying this equation for all $r \geq 2(\| b \| + \| D \| + \| \eta \| \epsilon^{-1})$. On the other hand, since $Im(G_s^{-1}(b)) = Im(b - (D + \eta(G_s(b)))) > \epsilon$, we have $G_s^*(b)^{-1} \in H^+(B) + i.\epsilon$, and consequently, for some large enough $r \geq 2(\| b \| + \| D \| + \| \eta \| \epsilon^{-1})$, we have:

$$G_s^*(b)^{-1} = b - (D + \eta(G_s^*(b)). \tag{4.7}$$

By comparison of equations (4.6) and (4.7), we conclude that $w_b^{-1} = G_s^*(b)$ so that $G_s^*(b)$ is unique. □

Chapter 5
Main Results

This last chapter is divided into two sections. In section 1, using the continued fraction representation of the Cauchy transform of a compactly supported probability measure, we give two of its representations in terms of sequences of finite dimensional matrix-valued Cauchy transforms of semicircular random variables. The section closes with a representation of the mentioned Cauchy transform of the compactly supported probability measure in terms of an infinite dimensional matrix-valued Cauchy transform of a semicircular random variable. In section 2, the existence of atoms of distributions of finite dimensional matrix valued semicircular random variables are discussed. Using M. D. Choi's representation of completely positive maps from a finite dimensional matrix algebra to another one, we give some sufficient conditions on the variance of a semicircular random variable such that its associated probability measure has atom.

5.1 Representation of the Cauchy Transform Using Semicircular Random Variables

This section deals with some representations of the the Cauchy transform of a probability measure in terms of operator-valued Cauchy transforms of semicircular distributions. The proof of the following three results are based on the Theorem 4.2.4 of Chapter 4. Indeed, given D and η as in that theorem, we are guaranteed that for each $b \in H^{+}(B)$ we shall find a unique $G_s^*(b) \in H^{-}(B)$ so that:

$$b - (D + \eta(Gs^*(b))) \;=\; G_s^*(b)^{-1} \quad \text{and} \quad Im(G_s^*(b)^{-1}) \geq Im(b) \tag{5.1}$$

and, hence, $G^*_s(b)$ is defined as its operator-valued Cauchy transform of the B-valued semicircular random variable with the first moment D and variance η. In our following three results we shall verify the conditions (5.1) for $b = \xi.1 \quad \xi \in \mathbb{C}^+$ and $G^*_s(b)$ a diagonal matrix of complex analytic functions. Theorem 4.2.4 of Chapter 4 will guarantee us that there is a semicircular random variable s with first moment D and variance η so that $G^*_s(b)$ is the restriction of the operator-valued Cauchy transform of s to $\mathbb{C}^+.1$.

The first two propositions deal with the finite dimensional matrix valued representations of the Cauchy transform.

Proposition 5.1.1. *Let μ be a probability measure with compact support in $\mathbb{R}$. Then there exists a sequence s_n $(n \geq 1)$ of self-adjoint operator valued semicircular random variables with associated operator valued Cauchy transforms $G^*_{s_n} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ $(n \geq 1)$ such that the Cauchy transform $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ is represented as:*

$$G_\mu(\xi) = \lim_{n\to\infty} \langle G^*_{s_n}(\xi.1_n)e_n, e_n\rangle_{\ell_2^n} \qquad Im(\xi) > 0.$$

Proof. Let

$$G_\mu(\xi) = \cfrac{1}{\xi - \alpha_1 - \cfrac{\omega_1}{\xi - \alpha_2 - \cfrac{\omega_2}{\xi - \alpha_3 - \ddots - \cfrac{\omega_{n-1}}{\xi - \alpha_n - \cfrac{\omega_n}{\xi - \alpha_{n+1} - \ddots}}}}}$$

be the continued fraction representation of G_μ. Fix positive integer $n \geq 1$, then define $b = \xi.1_n$, $D_n = (\alpha_{n+1-k}\delta_{kl})^n_{k,l=1}$ and the completely positive map η_n via :

$$\eta_n : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$
$$\eta_n\big((a_{kl})^n_{k,l=1}\big) = (\omega^{\frac{1}{2}}_{n-k+1}\delta_{k(l+1)})^n_{k,l=1}(a_{kl})^n_{k,l=1}(\omega^{\frac{1}{2}}_{n-k}\delta_{(k+1)l})^n_{k,l=1}.$$

Then, for the self-adjoint semicircular element s_n with operator valued Cauchy transform $G^*_{s_n}$ satisfying the functional equation:

$$bG^*_{s_n}(b) = 1 + (D_n + \eta_n(G^*_{s_n}(b)))G^*_{s_n}(b),$$

we have $G^*_{s_n}(b) = (g_{n,k}(\xi)\delta_{kl})^n_{k,l=1}$ where in which:

$$g_{n,k}(\xi) = \cfrac{1}{\xi - \alpha_1 - \cfrac{\omega_1}{\xi - \alpha_2 - \cfrac{\omega_2}{\xi - \alpha_3 - \ddots - \cfrac{\omega_{k-1}}{\xi - \alpha_k}}}} \qquad 1 \le k \le n \quad Im(\xi) > 0.$$

Accordingly:

$$\begin{aligned} G_\mu(\xi) &= \lim_{n\to\infty} g_{n,n}(\xi) \\ &= \lim_{n\to\infty} \langle G^*_{s_n}(\xi.1_n)e_n, e_n\rangle_{\ell_2^n} \qquad Im(\xi) > 0. \end{aligned}$$

□

Proposition 5.1.2. *Let μ be a probability measure with compact support in $\mathbb{R}$. Then there exists a sequence s_n $(n \ge 1)$ of self-adjoint operator valued semicircular random variables with associated operator valued Cauchy transforms $G^*_{s_n} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ $(n \ge 1)$ such that the Cauchy transform $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ is represented as:*

$$G_\mu(\xi) = \lim_{n\to\infty} \langle G^*_{s_n}(\xi.1_n)e_1, e_1\rangle_{\ell_2^n} \qquad Im(\xi) > 0.$$

Proof. Let

$$G_\mu(\xi) = \cfrac{1}{\xi - \alpha_1 - \cfrac{\omega_1}{\xi - \alpha_2 - \cfrac{\omega_2}{\xi - \alpha_3 - \ddots - \cfrac{\omega_{n-1}}{\xi - \alpha_n - \cfrac{\omega_n}{\xi - \alpha_{n+1} - \ddots}}}}}$$

be the continued fraction representation of G_μ. Fix positive integer $n \ge 1$, then define $b = \xi.1_n$, $D_n = (\alpha_k\delta_{kl})^n_{k,l=1}$ and the completely positive map η_n via :

$$\begin{aligned} &\eta_n : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \\ &\eta_n\big((a_{kl})^n_{k,l=1}\big) = (\omega_k^{\frac{1}{2}}\delta_{(k+1)l})^n_{k,l=1}(a_{kl})^n_{k,l=1}(\omega_{k-1}^{\frac{1}{2}}\delta_{k(l+1)})^n_{k,l=1}. \end{aligned}$$

Then, for the self-adjoint semicircular element s_n with operator valued Cauchy transform $G^*_{s_n}$ satisfying the functional equation:

$$bG^*_{s_n}(b) = 1 + (D_n + \eta_n(G^*_{s_n}(b)))G^*_{s_n}(b),$$

we have $G^*_{s_n}(b) = (g_{n,n-k+1}(\xi)\delta_{kl})^n_{k,l=1}$ where in which:

$$g_{n,n-k+1}(\xi) = \cfrac{1}{\xi - \alpha_1 - \cfrac{\omega_1}{\xi - \alpha_2 - \cfrac{\omega_2}{\xi - \alpha_3 - \ddots - \cfrac{\omega_{n-k+1}}{\xi - \alpha_{n-k+1}}}}} \qquad 1 \leq k \leq n \quad Im(\xi) > 0.$$

Accordingly:

$$\begin{aligned} G_\mu(\xi) &= \lim_{n\to\infty} g_{n,n}(\xi) \\ &= \lim_{n\to\infty} \langle G^*_{s_n}(\xi.1_n)e_1, e_1\rangle_{\ell_2^n} \qquad Im(\xi) > 0. \end{aligned}$$

□

The following theorem deals with the infinite dimensional matrix valued representation of the Cauchy transform. Here, we denote by $B(\ell_2(\mathbb{N}))$ the space of bounded operators on the separable Hilbert space $\ell_2(\mathbb{N})$, and we consider the orthonormal basis $\{e_n\}_{n=1}^\infty$ of $\ell_2(\mathbb{N})$ given by $e_n = \{\delta_{mn}\}_{m=1}^\infty$ $(n \geq 1)$.

Theorem 5.1.3. *Let μ be a compactly supported probability measure in $\mathbb{R}$. Then there exist a self-adjoint $B(\ell_2(\mathbb{N}))$−valued semicircular random variable s and an state $\rho : B(\ell_2(\mathbb{N})) \to \mathbb{C}$ given by $\rho(T) = \langle T(e_1), e_1\rangle_{\ell_2(\mathbb{N})}$ such that the Cauchy transform $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ is represented as:*

$$G_\mu(\xi) = \rho(G^*_s(\xi.1)) \qquad Im\xi > 0.$$

Proof. Let

$$G_\mu(\xi) = \cfrac{1}{\xi - \alpha_1 - \cfrac{\omega_1}{\xi - \alpha_2 - \cfrac{\omega_2}{\xi - \alpha_3 - \ddots - \cfrac{\omega_{n-1}}{\xi - \alpha_n - \cfrac{\omega_n}{\xi - \alpha_{n+1} - \ddots}}}}}$$

be the continued fraction representation of G_μ. Define $b = \xi.1 \in B(\ell_2(\mathbb{N}))$, $D = (\alpha_k\delta_{kl})_{k,l=1}^\infty \in B(\ell_2(\mathbb{N}))$, and the completely positive map η is given by [9, Theorem 4.1]:

$$\eta : B(\ell_2(\mathbb{N})) \to B(\ell_2(\mathbb{N}))$$
$$\eta((a_{kl})_{k,l=1}^\infty) = (\omega_k^{\frac{1}{2}}\delta_{(k+1)l})_{k,l=1}^\infty (a_{kl})_{k,l=1}^\infty (\omega_{k-1}^{\frac{1}{2}}\delta_{k(l+1)})_{k,l=1}^\infty.$$

Note that both D and η are bounded in their respected norms; in fact using equation (2.1) of Chapter 2 we have:

$$\begin{aligned}
\sup_m(|\alpha_m| + \omega_m) &= \sup_m\Big(|\int_{\mathbb{R}} t.p_{m-1}^2(t)d\mu(t)| + \int_{\mathbb{R}} t.p_m(t)p_{m-1}(t)d\mu(t)\Big) \\
&\leq \sup_m\Big(\Big|\sup_{t\in Supp(\mu)}|t|(\int_{\mathbb{R}} p_m^2(t)d\mu(t))\Big| + \sup_{t\in Supp(\mu)}|t|(\int_{\mathbb{R}}|p_m(t)p_{m-1}(t)|d\mu(t))\Big) \\
&\leq \sup_m\Big(\sup_{t\in Supp(\mu)}|t| + \sup_{t\in Supp(\mu)}|t|(\int_{\mathbb{R}} p_m^2(t)d\mu(t)\int_{\mathbb{R}} p_{m-1}^2(t)d\mu(t))^{\frac{1}{2}}\Big) \\
&\leq 2\sup_{t\in Supp(\mu)}|t| \\
&< \infty.
\end{aligned}$$

Now, for the self-adjoint semicircular element s with operator valued Cauchy transform G_s^* satisfying the functional equation:

$$bG_s^*(b) = 1 + (D + \eta(G_s^*(b)))G_s^*(b),$$

we have

$$G_s^*(b) = (x_{kl}\delta_{kl})_{k,l=1}^{\infty}$$

where in which:

$$x_{nn} = \cfrac{1}{\xi - \alpha_n - \cfrac{\omega_n}{\xi - \alpha_{n+1} - \cfrac{\omega_{n+1}}{\xi - \alpha_{n+2} - \cfrac{\omega_{n+2}}{\xi - \alpha_{n+3} - \cfrac{\omega_{n+3}}{\xi - \alpha_{n+4} - \ddots}}}}} \quad n \geq 1.$$

Consequently, for the state $\rho : B(\ell_2(\mathbb{N})) \to \mathbb{C}$ defined by:

$$\rho(T) = \langle T(e_1), e_1\rangle_{\ell_2(\mathbb{N})}$$

the assertion follows. □

5.2 Atoms of Distributions of Matrix-Valued Semicircular Random Variables

Let A be a unital C^*-algebra, B be a unital C^*-subalgebra of $n \times n$ complex matrices, $E_B : A \to B$ be a conditional expectation, and $G^* : \mathbb{H}^+(M_n(\mathbb{C})) \to \mathbb{H}^-(M_n(\mathbb{C}))$ be the operator valued Cauchy transform of the self- adjoint semicircular random variable $s \in A$. Define a map $G\colon \mathbb{C}^+ \to \mathbb{C}^-$ via:

$$G(\xi) = tr_n(G^*(\xi.1)) = tr_n \circ E_B(\frac{1}{\xi.1 - s}),$$

where $tr_n(.)$ denotes the normalized trace on $n \times n$ complex matrices. Referring to the note after theorem 2.1.4 of Chapter 2 for the special case of positive normal trace $\tau : A \to \mathbb{C}$ defined by $\tau(a) = tr_n \circ E_B(a)$, we observe that there is a probability measure $\mu = \mu_s$ on $\mathbb{R}$, which we call a semicircular distribution, such that:

$$tr_n(G^*(\xi.1)) = G(\xi) = G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t} \quad \xi \in \mathbb{C}^+.$$

Note that in the functional equation (5.1) if $n = 2$, $D = 0$ and $\eta(a) = a$, then using $d\mu(t) = \frac{-1}{\pi} \lim_{y \downarrow 0} Im G_\mu(t + iy)$ it follows that $d\mu(t) = \frac{1}{2\pi}\sqrt{4 - t^2}$ $|t| \leq 2$, the Wigner semicircular distribution.

Proposition 5.2.1. *Let $G^* : \mathbb{H}^+(M_n(\mathbb{C})) \to \mathbb{H}^-(M_n(\mathbb{C}))$ be the operator-valued Cauchy transform satisfying the functional equation $bG^*(b) = 1 + \eta(G^*(b))G^*(b)$, where in which $b = \xi.1 \in M_n(\mathbb{C})$, and the completely positive map η is given by:*

$$\begin{aligned} &\eta : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \\ &\eta(a) = \sum_{j=1}^{n^2} a_j a a_j^* \quad a_j a_j^* = a_j^* a_j \ (1 \leq j \leq n^2), \quad a_{j_1} a_{j_2} = a_{j_2} a_{j_1} \ (1 \leq j_1, j_2 \leq n^2). \end{aligned}$$

Then:

(i) the associated probability measure μ to G^* has no atoms if and only if the matrix $\eta(1)$ is invertible,

(ii) the only possible atom of the associated probability measure μ to G^* is $x = 0$ with possible values $\frac{k}{n}$ $(1 \leq k \leq n)$.

Proof. Let $a_j = (a_{j,kl})_{k,l=1}^n$ for $(1 \leq j \leq n^2)$ and $G^*(b) = (x_{kl})_{k,l=1}^n$. Since all the matrices a_j $(1 \leq j \leq n^2)$ are normal and commute with each other, there is an unitary matrix $u = (u_{kl})_{k,l=1}^n \in M_n(\mathbb{C})$ such that:

$$(a_u)_j := u^* a_j u = (\alpha_{j,k}\delta_{kl})_{k,l=1}^n \quad 1 \leq j \leq n^2.$$

Also:

$$G_u^{**}(b) := u^*G^*(b)u = (x'_{kl})_{k,l=1}^n$$

where in which $x'_{kl} = x'_{kl}(u_{s,t}, x_{s,t} :\ 1 \le s,t \le n)$. Since the matrix u is unitary, it follows that:

$$tr_n(G^*(b)) \;=\; tr_n(G_u^{**}(b)) = \frac{\sum_{k=1}^n x'_{kk}}{n}. \tag{5.2}$$

Besides:

$$\begin{aligned} bG_u^{**}(b) &= u^*(bG^*(b))u = u^*(1+\eta(G^*(b))G^*(b))u \\ &= 1 + u^*\Big(\sum_{j=1}^{n^2} a_jG^*(b)a_j^*G^*(b)\Big)u \\ &= 1 + \sum_{j=1}^{n^2}(u^*a_ju)(u^*G^*(b)u)(u^*a_ju)^*(u^*G^*(b)u) \\ &= 1 + \sum_{j=1}^{n^2}(a_u)_jG_u^{**}(b)(a_u)_j^*G_u^{**}(b). \end{aligned} \tag{5.3}$$

Next, we notice that:

$$u^*\eta(1)u \;=\; \sum_{j=1}^{n^2}(u^*a_ju)(u^*a_j^*u) = \sum_{j=1}^{n^2}(a_u)_j(a_u)_j^* = \Big((\sum_{j=1}^{n^2}|\alpha_{j,k}|^2)\delta_{kl}\Big)_{k,l=1}^n,$$

and by defining $W := \{k \in \{1,2,\cdots,n\} : \sum_{j=1}^{n^2}|\alpha_{j,k}|^2 = 0\}$ it follows that $\eta(1)$ is invertible if and only if $W = \phi$.

To find $tr_n(G^*(b))$ in terms of b, using equations (5.2) and (5.3) it is sufficient to find $tr_n(G_u^{**}(b))$ in terms of b or equivalently x'_{kk} $(1 \le k \le n)$ in terms of ξ. For that, substituting the introduced matrices in the equation (5.3), we have that:

$$
\begin{aligned}
(\xi x'_{k,l})_{k,l=1}^n &= (\xi\delta_{k,l})_{k,l=1}^n (x'_{k,l})_{k,l=1}^n \\
&= bG_u^{**}(b) \\
&= 1 + \sum_{j=1}^{n^2} (a_u)_j G_u^{**}(b)(a_u)_j^* G_u^{**}(b) \\
&= 1 + \sum_{j=1}^{n^2} (\alpha_{j,k}\delta_{kl})_{k,l=1}^n (x'_{kl})_{k,l=1}^n (\overline{\alpha_{j,k}}\delta_{kl})_{k,l=1}^n (x'_{kl})_{k,l=1}^n \\
&= 1 + \sum_{j=1}^{n^2} (\alpha_{j,k} x'_{kl})_{k,l=1}^n (\overline{\alpha_{j,k}} x'_{kl})_{k,l=1}^n \\
&= 1 + \sum_{j=1}^{n^2} \Big(\sum_{s=1}^{n} \alpha_{j,k}\overline{\alpha_{j,s}} x'_{ks} x'_{sl} \Big)_{k,l=1}^n \\
&= (\delta_{k,l})_{k,l=1}^n + \Big(\sum_{s=1}^{n}\sum_{j=1}^{n^2} \alpha_{j,k}\overline{\alpha_{j,s}} x'_{ks} x'_{sl} \Big)_{k,l=1}^n \\
&= \Big(\delta_{k,l} + \sum_{s=1}^{n}\sum_{j=1}^{n^2} \alpha_{j,k}\overline{\alpha_{j,s}} x'_{ks} x'_{sl} \Big)_{k,l=1}^n,
\end{aligned}
$$

yielding the following system of n^2 equations:

$$
\begin{aligned}
\Big(\sum_{j=1}^{n^2} |\alpha_{j,1}|^2\Big) {x'_{11}}^2 + \sum_{s=2}^{n} \Big(\sum_{j=1}^{n^2} \alpha_{j,1}\overline{\alpha_{j,s}} \Big) x'_{1s} x'_{s1} - \xi x'_{11} + 1 &= 0 \\
\sum_{s=1}^{n} \Big(\sum_{j=1}^{n^2} \alpha_{j,1}\overline{\alpha_{j,s}} \Big) x'_{1s} x'_{s2} - \xi x'_{12} &= 0 \\
&\vdots \\
\sum_{s=1}^{n} \Big(\sum_{j=1}^{n^2} \alpha_{j,1}\overline{\alpha_{j,s}} \Big) x'_{1s} x'_{sn} - \xi x'_{1n} &= 0 \\
&\vdots \\
\sum_{s=1}^{n} \Big(\sum_{j=1}^{n^2} \alpha_{j,k}\overline{\alpha_{j,s}} \Big) x'_{ks} x'_{s1} - \xi x'_{k1} &= 0 \\
&\vdots \\
\sum_{s=1}^{k-1} \Big(\sum_{j=1}^{n^2} \alpha_{j,k}\overline{\alpha_{j,s}} \Big) x'_{ks} x'_{sk} + \Big(\sum_{j=1}^{n^2} |\alpha_{j,k}|^2\Big) {x'_{kk}}^2 + \sum_{s=k+1}^{n} \Big(\sum_{j=1}^{n^2} \alpha_{j,k}\overline{\alpha_{j,s}} \Big) x'_{ks} x'_{sk} - \xi x'_{kk} + 1 &= 0 \\
&\vdots \\
\sum_{s=1}^{n} \Big(\sum_{j=1}^{n^2} \alpha_{j,k}\overline{\alpha_{j,s}} \Big) x'_{ks} x'_{sn} - \xi x'_{kn} &= 0
\end{aligned}
$$

$$\vdots$$

$$\sum_{s=1}^{n}\Big(\sum_{j=1}^{n^2}\alpha_{j,n}\overline{\alpha_{j,s}}\Big)x'_{ns}x'_{s1}-\xi x'_{n1} = 0$$

$$\vdots$$

$$\sum_{s=1}^{n}\Big(\sum_{j=1}^{n^2}\alpha_{j,n}\overline{\alpha_{j,s}}\Big)x'_{ns}x'_{s(n-1)}-\xi x'_{n(n-1)} = 0$$

$$\vdots$$

$$\sum_{s=1}^{n-1}\Big(\sum_{j=1}^{n^2}\alpha_{j,n}\overline{\alpha_{j,s}}\Big)x'_{ns}x'_{sn}+\Big(\sum_{j=1}^{n^2}|\alpha_{j,n}|^2\Big)x'^{\,2}_{nn}-\xi x'_{nn}+1 = 0. \tag{5.4}$$

Solving the system of equations (5.4) it follows that:

$$x'_{kl}=0\ (1\le k\ne l\le n)$$

and

$$x'_{kk}=\frac{1}{\xi}\chi_W(k)+\frac{\xi-\sqrt{\xi^2-4(\sum_{j=1}^{n^2}|\alpha_{j,k}|^2)}}{2(\sum_{j=1}^{n^2}|\alpha_{j,k}|^2)}(1-\chi_W(k)) \qquad 1\le k\le n,$$

is one of possible solutions, yielding:

$$tr_n(G^*(b)) = \frac{1}{n}\Big(\frac{|W|}{\xi}+\sum_{k\notin W}\frac{\xi-\sqrt{\xi^2-4(\sum_{j=1}^{n^2}|\alpha_{j,k}|^2)}}{2(\sum_{j=1}^{n^2}|\alpha_{j,k}|^2)}\Big).$$

A simple calculation shows that

$$\lim_{\xi\to\infty}\xi.tr_n(G^*(\xi.1))=1, \qquad Im\xi>0$$

where we put $\xi=yi$ $(y>0)$ and let $y\to+\infty$. It follows that the given solution x'_{kl} $(1\le k,l\le n)$ to (5.4) is the only acceptable one. Next, to find the atoms of a probability measure μ such that

$$tr_n(G^*(b))=\int_{\mathbb{R}}\frac{1}{\xi-t}d\mu(t),$$

fix $-\infty<x<\infty$, then:

$$\begin{aligned}\mu(\{x\}) &= \lim_{\Gamma_\alpha(x)\ni\xi\to x}(\xi-x)\int_{\mathbb{R}}\frac{1}{\xi-t}d\mu(t)=\lim_{\Gamma_\alpha(x)\ni\xi\to x}(\xi-x)tr_n(G^*(\xi.1))\\ &= \lim_{y\downarrow 0}(iy)tr_n(G^*((x+iy).1))=0 \ \text{ if } \ x\ne 0, \ \frac{|W|}{n} \ \text{ if } \ x=0.\end{aligned}$$

Thus, μ has no atoms if and only if $W = \phi$ or equivalently $\eta(1)$ is invertible, proving (i). In addition, the number of possible atoms of μ is at most one at $x = 0$ with possible values $\frac{|W|}{n}$ $(1 \leq |W| \leq n)$, proving (ii). □

Remark 5.2.2. In the above Proposition, the assumption of normality of the matrices a_j $(1 \leq j \leq n^2)$ is necessary. To see this, let $n = 2$, $0 \neq |\alpha| \neq |\beta| \neq 0$ and define:

$$a_1 = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}, \quad a_j = 0 \ (2 \leq j \leq 4).$$

Then, $a_1 a_1^* = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix} \neq \begin{pmatrix} |\beta|^2 & 0 \\ 0 & |\alpha|^2 \end{pmatrix} = a_1^* a_1$ and $\eta(1) = a_1 a_1^*$ is invertible. Let

$$b = \xi 1 = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} \quad \text{and} \quad G^*(b) = \begin{pmatrix} z & v \\ y & w \end{pmatrix}.$$

Under these conditions we are interested in finding $G(b)$ in terms of b or finding z, v, y, w in terms of ξ. Now, substituting above matrices in the given equation, we have that:

$$\begin{aligned}
\begin{pmatrix} \xi z & \xi v \\ \xi y & \xi w \end{pmatrix} &= \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}\begin{pmatrix} z & v \\ y & w \end{pmatrix} \\
&= b.G^*(b) \\
&= 1 + \eta(G^*(b))G^*(b) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}\begin{pmatrix} z & v \\ y & w \end{pmatrix}\begin{pmatrix} 0 & \overline{\beta} \\ \overline{\alpha} & 0 \end{pmatrix}\begin{pmatrix} z & v \\ y & w \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \alpha.y & \alpha.w \\ \beta.z & \beta.v \end{pmatrix}\begin{pmatrix} \overline{\beta}.y & \overline{\beta}.w \\ \overline{\alpha}.z & \overline{\alpha}.v \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \alpha.\overline{\beta}.y^2 + |\alpha|^2.w.z & \alpha.\overline{\beta}.y.w + |\alpha|^2.w.v \\ |\beta|^2.z.y + \beta.\overline{\alpha}.v.z & |\beta|^2.z.w + \beta.\overline{\alpha}.v^2 \end{pmatrix} \\
&= \begin{pmatrix} \alpha.\overline{\beta}.y^2 + |\alpha|^2.w.z + 1 & \alpha.\overline{\beta}.y.w + |\alpha|^2.w.v \\ |\beta|^2.z.y + \beta.\overline{\alpha}.v.z & |\beta|^2.z.w + \beta.\overline{\alpha}.v^2 + 1 \end{pmatrix}.
\end{aligned}$$

Consequently, we have the following equalities:

$$\begin{aligned}
\alpha.\overline{\beta}.y^2 + |\alpha|^2.w.z + 1 - \xi z &= 0\\
\alpha.\overline{\beta}.y.w + |\alpha|^2.w.v - \xi v &= 0\\
|\beta|^2.z.y + \beta.\overline{\alpha}.v.z - \xi y &= 0\\
|\beta|^2.z.w + \beta.\overline{\alpha}.v^2 + 1 - \xi w &= 0.
\end{aligned}$$

Let $v = 0 = y$, to obtain:

$$\begin{aligned}
|\alpha|^2.w.z - \xi z + 1 &= 0\\
|\beta|^2.z.w - \xi w + 1 &= 0,
\end{aligned}$$

yielding:

$$z = \frac{\xi^2 - |\alpha|^2 + |\beta|^2 \pm \sqrt{(\pm|\alpha|^2 + |\beta|^2 - \xi^2)^2 - 4.|\alpha|^2\xi^2}}{2\xi|\beta|^2}$$

and,

$$w = \frac{\xi^2 + |\alpha|^2 - |\beta|^2 \pm \sqrt{(-|\alpha|^2 + |\beta|^2 - \xi^2)^2 - 4.|\alpha|^2\xi^2}}{2\xi|\alpha|^2}.$$

Thus,

$$\begin{aligned}
tr_2(G^*(b)) &= \frac{z}{2} + \frac{w}{2}\\
&= \frac{\xi^2 - |\alpha|^2 + |\beta|^2 \pm \sqrt{(-|\alpha|^2 + |\beta|^2 - \xi^2)^2 - 4.|\alpha|^2\xi^2}}{4\xi|\beta|^2}\\
&\quad + \frac{\xi^2 + |\alpha|^2 - |\beta|^2 \pm \sqrt{(-|\alpha|^2 + |\beta|^2 - \xi^2)^2 - 4.|\alpha|^2\xi^2}}{4\xi|\alpha|^2}\\
&= \frac{|\alpha|^2\xi^2 - |\alpha|^2|\alpha|^2 + |\alpha|^2|\beta|^2 \pm |\alpha|^2\sqrt{(-|\alpha|^2 + |\beta|^2 - \xi^2)^2 - 4.|\alpha|^2\xi^2}}{4\xi|\beta|^2|\alpha|^2}\\
&\quad + \frac{|\beta|^2\xi^2 + |\beta|^2|\alpha|^2 - |\beta|^2|\beta|^2 \pm |\beta|^2\sqrt{(-|\alpha|^2 + |\beta|^2 - \xi^2)^2 - 4.|\alpha|^2\xi^2}}{4\xi|\alpha|^2|\beta|^2}\\
&= \frac{(|\alpha|^2 + |\beta|^2)\Big(\xi^2 \pm \sqrt{(-|\alpha|^2 + |\beta|^2 - \xi^2)^2 - 4.|\alpha|^2\xi^2}\Big) - (|\alpha|^2 - |\beta|^2)^2}{4\xi|\alpha|^2|\beta|^2}.
\end{aligned}$$

Now, to verify the condition

$$\lim_{\xi\to\infty} \xi.tr_2(G^*(\xi.1)) = 1 \qquad Im\xi > 0,$$

we put $\xi = yi$ $(y > 0)$ and let $y \to +\infty$. It follows that the positive sign is acceptable. Next, we are looking for a probability measure μ such that

$$tr_2(G^*(b)) = \int_{\mathbb{R}} \frac{1}{\xi - t} d\mu(t).$$

To find the atoms of μ fix $-\infty < x < \infty$, then:

$$\begin{aligned}
\mu(\{x\}) &= \lim_{\Gamma_\alpha(x)\ni\xi\to x} (\xi - x)\int_{\mathbb{R}} \frac{1}{\xi - t} d\mu(t) = \lim_{\Gamma_\alpha(x)\ni\xi\to x} (\xi - x) tr_2(G^*(\xi.1)) \\
&= \lim_{y\downarrow 0} (iy) tr_2(G^*((x+iy).1)) = \lim_{y\downarrow 0} \frac{iy}{4(x+iy)|\alpha|^2|\beta|^2} \times \Bigg((|\alpha|^2 + |\beta|^2) \\
& \quad \Big((x+iy)^2 + \sqrt{(-|\alpha|^2 + |\beta|^2 - (x+iy)^2)^2 - 4.|\alpha|^2 (x+iy)^2} \Big) - (|\alpha|^2 - |\beta|^2)^2 \Bigg) \\
&= 0 \qquad \text{if} \quad (x \neq 0), \ \frac{1}{2}\Big(1 - |\frac{\beta}{\alpha}|^{2sgn(1-|\frac{\beta}{\alpha}|)}\Big) \quad \text{if} \quad (x = 0).
\end{aligned}$$

□

Theorem 5.2.3. *Let $G_s^* : \mathbb{H}^+(M_n(\mathbb{C})) \to \mathbb{H}^-(M_n(\mathbb{C}))$ be the operator valued Cauchy transform satisfying the functional equation $bG_s^*(b) = 1 + \eta(G_s^*(b))G_s^*(b)$, $b \in H^+(M_n(\mathbb{C}))$, where $\eta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a nilpotent completely positive map. Then the associated probability measure μ to G_s^* has at least one atom .*

Proof. Let $\eta, \cdots, \eta^{m-1} \neq 0$ and $\eta^m = 0$, for some $m \geq 1$. Writing the functional equation in the form of $b - G_s^*(b)^{-1} = \eta(G_s^*(b))$ for $\|b^{-1}\| << \infty$ it follows that:

$$\eta^{m-1}(b - G_s^*(b)^{-1}) = \eta^m(G_s^*(b)) = 0 \qquad Im(b) > 0, \quad \|b^{-1}\| << \infty.$$

Now, if $\ker(\eta^{m-1}) = 0$, then $G_s^*(b) = b^{-1}$ and it follows that

$$\begin{aligned}
\mu(\{0\}) &= \lim_{\Gamma_\alpha(0)\ni\xi\to 0} (\xi - 0)\int_{\mathbb{R}} \frac{1}{\xi - t} d\mu(t) = \lim_{\Gamma_\alpha(0)\ni\xi\to 0} (\xi - 0) tr_n(G_s^*(\xi.1)) \\
&= \lim_{y\downarrow 0} (iy) tr_n(G_s^*(iy.1)) = \lim_{y\downarrow 0} (iy)(\frac{1}{iy}) \\
&= 1,
\end{aligned}$$

proving the assertion. Hence, we assume $\ker(\eta^{m-1}) \neq 0$. Pick $0 \neq c \in \ker(\eta^{m-1}) \cap M_n^+(\mathbb{C})$ with $\|c\| = 1$, then by the Schwarz inequality for completely positive maps [9, p. 40], it follows that:

$$\eta^{m-1}(c^{\frac{1}{2^n}})^* \eta^{m-1}(c^{\frac{1}{2^n}}) \leq \|\eta^{m-1}(1)\| \eta^{m-1}(c^{\frac{1}{2^{n-1}}}) \qquad n \geq 1,$$

and by induction we conclude that $\eta^{m-1}(c^{\frac{1}{2^n}}) = 0$ $(n \geq 1)$. By defining:

$$p :=^{s.o.t} \lim_{n\to\infty} c^{\frac{1}{2^n}},$$

it follows that p is a projection in $\ker(\eta^{m-1})$.

Claim (1): There exists a unique projection $1 \neq q \in \ker(\eta^{m-1})$ such that for any projection $p \in \ker(\eta^{m-1})$ we have: $p \leq q$.

We showed that there is at least one projection p in $\ker(\eta^{m-1})$. Let p_1, p_2 be two projections in $\ker(\eta^{m-1})$. Then :

$$\eta^{m-1}((p_1+p_2)^{\frac{1}{2^n}})^*\eta^{m-1}(p_1+p_2{}^{\frac{1}{2^n}}) \leq \|\eta^{m-1}(1)\|\eta^{m-1}((p_1+p_2)^{\frac{1}{2^{n-1}}}) \qquad n \geq 1,$$

and by induction it follows that $\eta^{m-1}((p_1+p_2)^{\frac{1}{2^n}}) = 0$ $(n \geq 1)$. Now, define $p_3 :=^{s.o.t} \lim_{n\to\infty}(p_1+p_2)^{\frac{1}{2^n}}$, then p_3 is a projection in $\ker(\eta^{m-1})$. On the other hand

$$(p_1+p_2)^{\frac{1}{2^n}} \geq p_1, p_2 \qquad (n \geq 1),$$

yielding $p_3 \geq p_1, p_2$. Next, using the maximality argument and by repeating this process there will be a unique maximal projection q in $\ker(\eta^{m-1})$ such that for any other projection p in it, we have $p \leq q$. Finally, if $q = 1$, then using the same Schwarz inequality as above, and the canonical decomposition of elements of $M_n(\mathbb{C})$ into its positive elements it follows that $\eta^{m-1} = 0$, a contradiction.

Claim(2): For any $0 \neq c \in \ker(\eta^{m-1}) \cap M_n^+(\mathbb{C})$, we have $cq = qc = c$.

Indeed, since $c \in M_n^+(\mathbb{C})$, by spectral theorem we have:

$$c = \sum_{k=1}^{N} \lambda_k p_k$$

where the projections p_k's satisfy $\sum_{k=1}^{N} p_k = 1$, $p_{k_1}p_{k_2} = 0$ $(1 \leq k_1 \neq k_2 \leq N)$ and $\lambda_k \geq 0$ $(1 \leq k \leq N)$. Now, define:

$$r :=^{s.o.t} \lim_{n\to\infty} c^{\frac{1}{2^n}}.$$

Then it follows that $r = \sum_{\lambda_k \neq 0} p_k$, yielding $p_k \leq r$ $(\lambda_k \neq 0)$. On the other hand, by definition of q we have $r \leq q$ and hence $p_k \leq q$ $(\lambda_k \neq 0)$. But all of $p_k \leq q$ $(\lambda_k \neq 0)$, and q are projections and, consequently, $p_k q = q p_k = p_k$ $(\lambda_k \neq 0)$. Now, by multiplying all sides by λ_k $(1 \leq k \leq N)$, and taking summation the claim is proved.

Next, take $c_0 = \frac{1}{i}(G_s^*(b)^{-1} - b)$ where $b = iy.1$ $(y > 0)$. Then using the fact that s is centered, $E_B(s^{2m-1}) = 0$ $\ m \geq 1$, by

$$\begin{aligned} Re(G_s^*(b)) &= Re\Big(\sum_{m=0}^{\infty} b^{-1} E_B((sb^{-1})^m)\Big) = Re\Big(\sum_{m=0}^{\infty} i^{m+1}(-y)^{m+1} E_B(s^m)\Big) \\ &= \sum_{m=1}^{\infty} (-y^2)^m E_B(s^{2m-1}) = 0, \quad y > \|s\| \end{aligned}$$

it follows that

$$G_s^*(b)^{-1} - b = \Big(Re(G_s^*(b)) + i.Im(G_s^*(b))\Big)^{-1} - b = i.\Big(- Im(G_s^*(b))^{-1} - \frac{b}{i}\Big),$$

and, hence $c_0 = \frac{1}{i}(G_s^*(b)^{-1} - b) = Im((G_s^*(b)^{-1} - b)) \geq 0$. Now, by claim (2) for $c = c_0$ we have that:

$$G_s^*(b)(1-q) = b^{-1}(1-q) = (1-q)G_s^*(b) \text{ and } G_s^*(b)q = qG_s^*(b),$$

and hence:

$$\begin{aligned} G_s^*(b) &= (1-q)G_s^*(b)(1-q) + qG_s^*(b)q \\ &= b^{-1}(1-q) + qG_s^*(b)q \qquad b = iy.1 \ (y > 0), \|b^{-1}\| << \infty. \end{aligned}$$

Now, applying analytic continution for the complex function $tr_n \circ G_s^*|_{\mathbb{C}^+.1} : \mathbb{C}^+ \to \mathbb{C}^-$ we conclude that:

$$tr_n \circ G_s^*(\xi.1) = tr_n(\xi^{-1}(1-q) + qG_s^*(\xi.1)q) \quad Im\xi > 0,$$

and, consequently:

$$\begin{aligned} \mu(\{0\}) &= \lim_{\Gamma_\alpha(0) \ni \xi \to 0} (\xi - 0) \int_{\mathbb{R}} \frac{1}{\xi - t} d\mu(t) = \lim_{\Gamma_\alpha(0) \ni \xi \to 0} (\xi - 0) tr_n(G_s^*(\xi.1)) \\ &= \lim_{y \downarrow 0} (iy) tr_n((iy)^{-1}(1-q) + qG_s^*(iy.1)q) = tr_n(1-q) \\ &> 0, \end{aligned}$$

completing the proof. □

Before stating a more concrete and special case of the above theorem, we need a lemma:

Lemma 5.2.4. *Let η be a completely positive map defined by*

$$\eta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$
$$\eta(a) = \sum_{j=1}^{n^2} a_j a a_j^* \qquad a_j \in M_n(\mathbb{C}) \ (1 \le j \le n^2).$$

Then:

(i) if the map η is nilpotent, then all matrices a_j $(1 \le j \le n^2)$ are nilpotent,

(ii) if all matrices a_j $(1 \le j \le n^2)$ are nilpotent and commute with each other, then the map η is nilpotent.

Proof. (i) Let $\eta^m = 0$ for some $m \ge 1$. Then, it follows that:

$$\begin{aligned}
\sum_{1 \le j_1, \cdots, j_m \le n^2} (a_{j_1} \cdots a_{j_m} a)(a_{j_1} \cdots a_{j_m} a)^* &= \sum_{1 \le j_1, \cdots, j_m \le n^2} (a_{j_1} \cdots a_{j_m}) a a^* (a_{j_m}^* \cdots a_{j_1}^*) \\
&= \sum_{j_1=1}^{n^2} \cdots \sum_{j_m=1}^{n^2} (a_{j_1} \cdots a_{j_m}) a a^* (a_{j_m}^* \cdots a_{j_1}^*) \\
&= \sum_{j_1=1}^{n^2} a_{j_1} \Big(\cdots \Big(\sum_{j_m=1}^{n^2} a_{j_m} a a^* a_{j_m}^* \Big) \cdots \Big) a_{j_1}^* \\
&= \eta^m (a a^*) \\
&= 0
\end{aligned}$$

for all $a \in M_n(\mathbb{C})$. Consequently, by positivity of all elements of the form $(a_{j_1} \cdots a_{j_m} a)(a_{j_1} \cdots a_{j_m} a)^*$ it follows that

$$(a_{j_1} \cdots a_{j_m} a)(a_{j_1} \cdots a_{j_m} a)^* = 0 \ \ (1 \le j_1, \cdots, j_m \le n^2),$$

for all $a \in M_n(\mathbb{C})$. On the other hand , $M_n(\mathbb{C})$ is a C^* algebra and hence:

$$a_{j_1} \cdots a_{j_m} a = 0 \ \ (1 \le j_1, \cdots, j_m \le n^2),$$

for all $a \in M_n(\mathbb{C})$, or equivalently:

$$a_{j_1} \cdots a_{j_m} = 0 \ \ (1 \le j_1, \cdots, j_m \le n^2).$$

Now, take $j_1 = \cdots = j_m = j$ where $1 \le j \le n^2$ and the desired result is proved.

(ii) Since $a_j \ (1 \le j \le n^2)$ are nilpotent, it follows that $a_j^n = 0 \ (1 \le j \le n^2)$. Put $m = n^3$, then by commutativity of the these matrices it follows that:

$$a_{j_1} \cdots a_{j_m} = \prod_{p=0}^{n-1} \prod_{q=1}^{n^2} a_{j_{pn^2+q}} = a^n_{j_{(j_1,\cdots,j_m)}} \prod_{j_{pn^2+q} \ne j_{(j_1,\cdots,j_m)}} a_{j_{pn^2+q}} = 0,$$

for all $1 \le j_1, \cdots, j_m \le n^2$. Consequently,

$$\eta^m(a) = \sum_{1 \le j_1,\cdots,j_m \le n^2} (a_{j_1} \cdots a_{j_m}) a (a^*_{j_m} \cdots a^*_{j_1}) = 0$$

for all $a \in M_n(\mathbb{C})$. □

Considering above lemma , for a category of nilpotent $\eta' s$ we have:

Corollary 5.2.5. *Let $G^* : \mathbb{H}^+(M_n(\mathbb{C})) \to \mathbb{H}^-(M_n(\mathbb{C}))$ be the operator valued Cauchy transform satisfying the functional equation $bG^*(b) = 1 + \eta(G^*(b))G^*(b)$, $b \in H^+(M_n(\mathbb{C}))$, and the completely positive map η is given by:*

$$\eta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$
$$\eta(a) = \sum_{j=1}^{n^2} a_j a a_j^* \qquad a_j^n = 0 \ (1 \le j \le n^2), \quad a_{j_1} a_{j_2} = a_{j_2} a_{j_1} \ (1 \le j_1, j_2 \le n^2).$$

Then the associated probability measure μ to G^* has at least one atom at $x = 0$.

Remark 5.2.6. In the above theorem, the converse of the assertion does not hold. To see this, we refer to Remark 5.2.2.

Remark 5.2.7. In the above theorem, the assumption of nilpotency of the map η is necessary. To see this, consider the above corollary and let $n = 2$, $|\alpha| = |\beta| \ne 0$ and define:

$$a_1 = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}, \quad a_j = 0 \ (2 \le j \le 4).$$

Then, $a_1^2 = \begin{pmatrix} \alpha\beta & 0 \\ 0 & \alpha\beta \end{pmatrix} \ne 0$. Let

$$b = \xi 1 = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} \quad \text{and} \quad G^*(b) = \begin{pmatrix} z & v \\ y & w \end{pmatrix}.$$

Then, similar to computations of the previous remark, we conclude that:

$$w = z = \frac{\xi - \sqrt{\xi^2 - 4|\alpha|^2}}{2|\alpha|^2}.$$

Thus,

$$tr_2(G^*(b)) \quad = \quad \frac{z + w}{2} = \frac{\xi - \sqrt{\xi^2 - 4|\alpha|^2}}{2|\alpha|^2}.$$

This is the Cauchy transform of the standard semicircular law of Wigner which has no atom.

□

Remark 5.2.8. In the above theorem , for given nilpotent map η the associated probability measure to the operator valued Cauchy transform $G^*(b)$ may not be purely atomic. To see this, let $n = 3$ and consider the following completely positive map:

$$\eta : M_3(\mathbb{C}) \to M_3(\mathbb{C})$$

$$\eta(\begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix}) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^*,$$

where in which $\eta^3 = 0$. Replacing $b = \xi.1$ and $G^*(b) = \begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix}$ in the given functional equation, it follows that:

$$\begin{pmatrix} \xi r & \xi s & \xi t \\ \xi u & \xi v & \xi w \\ \xi x & \xi y & \xi z \end{pmatrix} = \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi \end{pmatrix}\begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix} = bG^*(b) = 1 + \eta(G^*(b)).G^*(b)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} r & s & t \\ u & v & w \\ x & y & z \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} u+x & v+y & w+z \\ x & y & z \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ r & s & t \\ r+u & s+v & t+w \end{pmatrix}$$

$$= \begin{pmatrix} 1+(v+y)r+(w+z)(r+u) & (v+y)s+(w+z)(s+v) & (v+y)t+(w+z)(t+w) \\ yr+z(r+u) & 1+ys+z(s+v) & yt+z(t+w) \\ 0 & 0 & 1 \end{pmatrix},$$

yielding the following system of equations:

$$\begin{aligned} vr+yr+wr+wu+zr+zu+1-\xi.r &= 0 \\ vs+ys+ws+wv+zs+zv-\xi.s &= 0 \\ vt+yt+wt+w^2+zt+zw-\xi.t &= 0 \\ yr+zr+zu-\xi.u &= 0 \\ ys+zs+zv-\xi.v+1 &= 0 \\ yt+zt+zw-\xi.w &= 0 \\ 0-\xi.x &= 0 \\ 0-\xi.y &= 0 \\ 1-\xi.z &= 0. \end{aligned}$$

Thus, $x = y = 0$ and $z = \frac{1}{\xi}$. Putting these values in above system of equations and considering the following Groebner basis program in Mathematica (with ξ replaced by c)

$Clear[c, r, s, t, u, v, w, x, y, z];$
$poly = \{v*r+w*r+w*u+z*r+z*u+1-c*r, v*s+w*s+w*v+z*s+z*v-c*s, v*t+w*t+w^2+z*t+z*w-c*t, z*r+z*u-c*u, z*s+z*v-c*v+1, z*t+z*w-c*w, 1-c*z\};$
$B = GroebnerBasis[poly, \{s, t, w, u, x, y, r, v, z\}]$

we obtain the following output:
$\{-1+cz, -1+c^2+cv-c^3v-v^2+c^2v^2, -c-v+c^2v-cv^2+z+v^2z, -c+c^3+r-v+2c^2v-c^4v, c+u+v-c^2v, w, t, c+s+v-c^2v\}.$

It follows that:

$$z = \frac{1}{\xi} \quad v = \frac{\xi-\sqrt{\xi^2-4}}{2} \quad r = (\xi^2-1)(\frac{\xi-\sqrt{\xi^2-4}}{2})^3.$$

Hence,

$$tr_3(G^*(b)) = \frac{1}{3}\Big(\frac{1}{\xi} + \frac{\xi - \sqrt{\xi^2-4}}{2} + (\xi^2-1)(\frac{\xi - \sqrt{\xi^2-4}}{2})^3\Big),$$

and a simple calculation shows that

$$\lim_{\xi\to\infty} \xi.tr_3(G^*(\xi.1)) = 1 \qquad Im\xi > 0,$$

where we put $\xi = yi$ $(y > 0)$ and let $y \to +\infty$. It follows that the given values for z, v and r are acceptable. Next, to find the atoms of a probability measure μ such that

$$tr_3(G^*(b)) = \int_{\mathbb{R}} \frac{1}{\xi - t} d\mu(t),$$

fix $-\infty < x' < \infty$, then:

$$\begin{aligned} \mu(\{x'\}) &= \lim_{\Gamma_\alpha(x')\ni\xi\to x'} (\xi - x')\int_{\mathbb{R}} \frac{1}{\xi - t} d\mu(t) = \lim_{\Gamma_\alpha(x')\ni\xi\to x'} (\xi - x')tr_3(G^*(\xi.1)) \\ &= \lim_{y\downarrow 0}(iy)tr_3(G^*((x' + iy).1)) = 0 \ \text{ if } \ x' \neq 0, \ \frac{1}{3} \ \text{if} \ \ x' = 0, \end{aligned}$$

showing that μ is not purely atomic.

□

Remark 5.2.9. Any probability measure μ on $\mathbb{R}$ whose support is a finite set can be realized as a component of a (scalar-valued) semicircular distribution μ_s of some matrix-valued semicircular random variable s with nilpotent variance. (If μ and ν are probability measures on $\mathbb{R}$, we shall say that ν is a component of μ if there exists a finite family $\{\nu_1, \ldots, \nu_n\}$ so that $\nu \in \{\nu_1, \ldots, \nu_n\}$ and $\mu = \sum_{j=1}^n \alpha_j \nu_j$ for some $\alpha_1, \ldots, \alpha_n \in [0,1]$ satisfying $\alpha_1 + \cdots + \alpha_n = 1$.) Indeed, assume $|Supp(\mu)| = n < \infty$, and let

$$G_\mu(\xi) = \cfrac{1}{\xi - \alpha_1 - \cfrac{\omega_1}{\xi - \alpha_2 - \cfrac{\omega_2}{\xi - \alpha_3 - \ddots \xi - \alpha_{n-1} - \cfrac{\omega_{n-1}}{\xi - \alpha_n}}}}$$

be the continued fraction representation of G_μ. Then as in the proof of Proposition 5.1.2, define $b = \xi.1_n$, $D_n = (\alpha_k \delta_{kl})_{k,l=1}^n$ and the nilpotent completely positive map η_n via:

$$\eta_n : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

$$\eta_n\big((a_{kl})_{k,l=1}^n\big) = (\omega_k^{\frac{1}{2}}\delta_{(k+1)l})_{k,l=1}^n (a_{kl})_{k,l=1}^n (\omega_{k-1}^{\frac{1}{2}}\delta_{k(l+1)})_{k,l=1}^n.$$

Then for the self-adjoint semicircular element s_n with operator-valued Cauchy transform $G^*_{s_n}$ satisfying the functional equation:

$$bG^*_{s_n}(b) = 1 + (D_n + \eta_n(G^*_{s_n}(b)))G^*_{s_n}(b),$$

we have

$$G_\mu(\xi) = \langle G^*_{s_n}(\xi.1_n)e_1, e_1\rangle_{\ell_2^n} \quad Im\xi > 0.$$

Next, let $\mu_k \ \ (1 \leq k \leq n)$ be a finite purely atomic probability measure on $\mathbb{R}$ with associated Cauchy transform:

$$G_{\mu_{n-(k-1)}}(\xi) = \cfrac{1}{\xi - \alpha_1 - \cfrac{\omega_1}{\xi - \alpha_2 - \cfrac{\omega_2}{\xi - \alpha_3 - \ddots \xi - \alpha_{k-1} - \cfrac{\omega_{k-1}}{\xi - \alpha_k}}}} \quad (1 \leq k \leq n).$$

Note that $\mu = \mu_1$. Then by proof of Proposition 5.1.2, we have

$$G_{\mu_k}(\xi) = \langle G^*_{s_n}(\xi.1_n)e_k, e_k\rangle_{\ell_2^n} \quad (1 \leq k \leq n),\ Im\xi > 0.$$

Consequently, we have:

$$G_{\mu_{s_n}}(\xi) = tr_n(G^*_{s_n}(\xi.1)) = \frac{1}{n}\sum_{k=1}^{n}\langle G^*_{s_n}(\xi.1)e_k, e_k\rangle_{\ell_2^n} = \frac{1}{n}\sum_{k=1}^{n} G_{\mu_k}(\xi) \quad Im\xi > 0,$$

and by $d\nu(t) = \frac{-1}{\pi}\lim_{y\downarrow 0} ImG_\nu(t+iy)$ it follows that:

$$\mu_{s_n} = \frac{1}{n}\sum_{k=1}^{n}\mu_k,$$

proving the desired result.

Chapter 6

Summary and Future Work

In this thesis we proved some new results about the distributions of operator-valued semicircular random variables.

The first result shows that the Cauchy transform of any compactly supported probability measure can be realized as a restriction to scalars of composition of an extremal state and an operator-valued Cauchy transform of a semicircular random variable with values in $B(H)$ for some separable Hilbert space H. Moreover, we give a constructive method to find the mentioned semicircular random variable using the Jacobi coefficients associated to that given compactly supported probability measure.

The second result deals with the regularity property of distributions of $M_n(\mathbb{C})$−valued semicircular random variables. We show that such semicircular distributions have nonzero discrete part when the associated variance to the semicircular random variable is nilpotent. It is still an open question to find necessary and sufficient conditions for such semicircular random variables so that their distributions have nonzero discrete part.

The last result discusses on the covering property of distributions of $M_n(\mathbb{C})$−valued semicircular random variables. Whilst we show that any finitely supported probability measure can be component of one of them, it still remains an open question to find whether any compactly supported probability measure is a component of distribution of a semicircular random variable or not.

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