Term Bases for Multivariate Interpolation of Hermite Type[†]

J. Apel, J. Stückrad, P. Tworzewski, T. Winiarski

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ABSTRACT

The main object of this paper is to prove that any system of $\mathcal{P} = (P_1, \ldots, P_\mu)$ of μ points with pairwise different first coordinates is universal for Hermite interpolation with respect to any elimination term order $\prec$ satisfying $\{X_1\} \ll \{X_2, \ldots, X_n\}$. Furthermore a fast algorithm for the computation of the minimal (with respect to $\prec$) term basis for Hermite interpolation is presented.

1 Introduction

In general, Hermite interpolation means that we are looking for a polynomial $F \in R$ which takes prescribed values and derivatives at given points. Let $\mathcal{P} = (P_1, \ldots, P_\mu)$ be a system of μ pairwise different points (nodes) $P_i \in \mathbb{K}^n$ and $\mathcal{D} = (D_1, \ldots, D_\mu)$ a system of μ finite Ferrers diagrams. Then we require

$$\frac{\partial^{|\alpha|} F}{\partial X^\alpha}(P_j) = b_{j,\alpha} \tag{1}$$

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for $\alpha \in D_j$ $(j = 1, \ldots, \mu)$ and given data $b_{j,\alpha} \in \mathbb{K}$.[1] All polynomials F fulfilling condition (1) form a residue class modulo an ideal $I(\mathcal{P}, \mathcal{D})$ in the polynomial ring $R = \mathbb{K}[X_1, \ldots, X_n]$ with $Krull\text{–}dim\,(R/I(\mathcal{P}, \mathcal{D})) = 0$. If we consider only polynomials F which are in a certain sense minimal with respect to a given term order $\prec$ in R then F becomes unique and can be computed effectively using, for example, Gröbner basis algorithms (see e.g. [BM82], [BW91], [MMM]). For $n = 1$ there are formulas for interpolating polynomials due to Newton, Lagrange and Hermite. These formulas have been applied in the theory of approximation (see e.g. [Fe23], [Le54], [Wi70]). The examination of the interpolation formula's structure has led to rather reach theories of polynomial interpolation for several variables and their application to the theory of approximation (see e.g. [Si62], [Wi73], [Br97]). In particular Newton's divided differences resulting in Kergin's interpolation (see e.g. [Ke78], [Ke80], [Ha82]) has been heavily investigated some years ago. It is a seemingly surprising fact that in many papers written by specialists of interpolation theory there is no information on very nice results obtained in the same time by specialists of effective methods and vice versa. In this paper we try to build a bridge between both groups. Especially, Theorem 3 connects classical and Gröbner basis methods for Hermite interpolation. A fast algorithm for the computation of the admissible Hermite term basis with respect to any elimination term order $\prec$ satisfying $\{X_1\} \ll \{X_2, \ldots, X_n\}$ follows from Theorem 2.

2 Preliminaries

Notations Let $\mathbb{K}$ be an arbitrary field of characteristic zero, $X = \{X_1, \ldots, X_n\}$ a finite set of indeterminates and $R := \mathbb{K}[X]$ the polynomial ring in X over $\mathbb{K}$. The set of terms generated by X will be denoted by $T(X)$. The elements of $T(X)$ are identified with the n-tuples of natural numbers via the exponent mapping $\exp : T(X) \to \mathbb{N}^n$ defined by $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n} \mapsto (\alpha_1, \ldots, \alpha_n) = \alpha$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we will also use the denotation $\alpha = (\alpha_1, \widetilde{\alpha})$. A total order $\prec$ on $T(X)$ is said to be admissible if it satisfies the following two conditions

i) $1 \prec X^\alpha$ for each $\alpha \in \mathbb{N}^n \backslash \{0\}$

ii) $X^\alpha \prec X^\beta \Longrightarrow X^{\alpha+\gamma} \prec X^{\beta+\gamma}$, for all $\alpha, \beta, \gamma \in \mathbb{N}^n$.

[1] We note that if all Ferrers diagrams are trivial, i.e. each of them consists of only one point, then we are in the situation of Lagrange interpolation.

By $\alpha \prec \beta \iff X^\alpha \prec X^\beta$ any order $\prec$ of $T(X)$ induces an order on $\mathbb{N}^n$ which, for simplicity, we denote also by $\prec$.

Let X be the disjoint union of Y and Z. Then an admissible term order $\prec$ is called an elimination order for Z if $t \prec s$ for all $t \in T(Y)$ and $s \in T(X) \setminus T(Y)$. In this case we write $Y \ll Z$.

The support $\mathrm{supp}(f)$ of an element $f \in R$ is defined to be the set of all terms appearing in f with non-zero coefficient. For non-zero elements f the maximal element $\mathrm{lt}_\prec(f)$ of $\mathrm{supp}(f)$ with respect to $\prec$ and the corresponding coefficient $\mathrm{lc}_\prec(f)$ are called leading term and leading coefficient, respectively, of f (w.r.t. $\prec$). Furthermore, for any non-zero polynomial f the exponent $\exp_\prec(\mathrm{lt}_\prec(f))$ of the leading term is called the leading exponent of f (w.r.t. $\prec$) and denoted by $\exp_\prec(f)$.

The notions of leading term and leading exponent are extended to subsets F of polynomials by defining $\mathrm{lt}_\prec(F) := \{\mathrm{lt}_\prec(f) | 0 \neq f \in F\}$ and $\exp_\prec(F) := \{\exp_\prec(f) | 0 \neq f \in F\}$, respectively. Moreover, we introduce the notations $\Delta_\prec(F) := \exp_\prec(F) + \mathbb{N}^n$ and $\mathcal{D}_\prec(F) := \mathbb{N}^n \setminus \Delta_\prec(F)$. An arbitrary subset $D \subseteq \mathbb{N}^n$ will be called a Ferrers diagram if there exists an ideal $I \subseteq R$ such that $D = \mathcal{D}_\prec(I)$. For any Ferrers diagram D there exist a uniquely determined monomial ideal I such that $D = \mathcal{D}_\prec(I)$ for any admissible order $\prec$. If $\{t_1, \ldots, t_s\} \subset T(X)$ generates this monomial ideal I then we say that the system $\{\alpha_1 = \exp(t_1), \ldots, \alpha_s = \exp(t_s)\}$ determines D. In the case $D \neq \mathbb{N}^n$, i.e. I is not the zero ideal, $\Delta(I) = \mathbb{N}^n \setminus D$ is the monoid ideal of $\mathbb{N}^n$ generated by $\{\alpha_1, \ldots, \alpha_s\}$. Since, in general, it will be clear from the context which order is meant we will omit the index $\prec$ and we write e.g. $\mathrm{lt}(f)$ instead of $\mathrm{lt}_\prec(f)$.

About interpolation. In general, Hermite interpolation means that we are looking for a polynomial $F \in R$ which takes prescribed values and derivatives at given points. Let $\mathcal{P} = (P_1, \ldots, P_\mu)$ be a system of μ points (nodes) $P_i \in \mathbb{K}^n$ and $\mathcal{D} = (D_1, \ldots, D_\mu)$ a system of μ Ferrers diagrams. Then we require

$$\frac{\partial^{|\alpha|} F}{\partial X^\alpha}(P_j) = b_{j,\alpha} \tag{2}$$

for $\alpha \in D_j$, $j = 1, \ldots, \mu$ and given data $b_{j,\alpha} \in \mathbb{K}$. There can be applied Gröbner basis techniques (c.f. [Bu65], [BWK], [CLO]) in order to solve this problem for arbitrary given data $\mathcal{P}$, $\mathcal{D}$ and $b_{j,\alpha}$ (c.f. [BW91]). To each pair (P_j, D_j) we assign the ideal

$$I_j = I(P_j, D_j) = \left\{ F \in R \;\middle|\; \frac{\partial^{|\alpha|} F}{\partial X^\alpha}(P_j) = 0 \text{ for all } \alpha \in D_j \right\}$$

which is generated by the set of all $(X - P_j)^\beta$, where β ranges over a determining system of D_j. Using Gröbner basis techniques we can compute a generating set of the intersection ideal $I(\mathcal{P}, \mathcal{D}) = \bigcap_{j=1}^{\mu} I_j$ (see [BWK], [CLO]). If the residue classes of the subset $B \subset R$ form a $\mathbb{K}$-vector space basis of $R/I(\mathcal{P}, \mathcal{D})$ then if for a given data $b_{j,\alpha}$ there exists a solution then there exists also a uniquely determined linear combination F of B satisfying the conditions (2) of the interpolating polynomial. From any Gröbner basis of I with respect to a fixed admissible term order $\prec$ one can directly read off the set $D = \mathcal{D}_\prec(I)$ obtaining a suitable basis $B_\prec = \{X^\beta \mid \beta \in D\}$. Such a basis $B_\prec$ has the nice property that the support of the polynomial F contains the smallest (with respect to $\prec$) terms among all possible interpolating polynomials and that F can be obtained by Gröbner reduction. For example, if one is looking for an interpolating polynomial F of minimal degree then one can use a total degree order as $\prec$.

In order to guaranty the existence of a solution for any data $b_{j,\alpha}$ we have to assume that all Ferrers diagrams D_j are finite. In this case we have

$$d := \dim_{\mathbb{K}} R/I(\mathcal{P}, \mathcal{D}) = \sum_{j=1}^{\mu} \#(D_j) \tag{3}$$

and calculation of F needs only the solution of a finite system of linear equations in finitely many indeterminates corresponding to the elements of a vector space basis B.

Admissible interpolation systems From now we fix an admissible term order $\prec$, a number μ of nodes and a system $\mathcal{D} = (D_1, \ldots, D_\mu)$ of finite Ferrers diagrams. Let d be as defined in (3) and

$$\mathfrak{F}_d = \{D \subset \mathbb{N}^n \mid \#(D) = d \text{ and } D \text{ is a Ferrers diagram}\} \quad . \tag{4}$$

The set $V = \{(P_1, \ldots, P_\mu) \in (\mathbb{K}^n)^\mu \mid P_i = P_j \text{ for some } i \neq j\}$ is an algebraic subset of $(\mathbb{K}^n)^\mu$ of codimension n. Therefore, $\Omega = (\mathbb{K}^n)^\mu \setminus V$ is open in the Zariski topology. We consider the mapping

$$\Phi_{\mathcal{D},\prec} \,:\, \Omega \longrightarrow \mathfrak{F}_d$$

defined by $\mathcal{P} \mapsto \mathcal{D}_\prec(I(\mathcal{P}, \mathcal{D}))$.

Theorem 1 *There exists a Zariski open set $\Omega' \subseteq \Omega$ and $\mathbf{D}_{\mathcal{D},\prec} \in \mathfrak{F}_d$ such that*

$$\Phi_{\mathcal{D},\prec}(\mathcal{P}) = \mathbf{D}_{\mathcal{D},\prec}$$

for any system of points $\mathcal{P} \in \Omega'$.

Proof: The assertion follows directly from the following

Lemma 1 *There exists an algorithm based on Gröbner bases for the computation of an algebraic subset $W \subsetneqq (\mathbb{K}^n)^\mu$ such that $\Phi_{\mathcal{D},\prec}$ is constant on $\Omega \setminus W$.*

Proof: Let $\mathcal{P} = (P_1, \ldots, P_\mu) \in \Omega$ be a system of pairwise different points of $\mathbb{K}^n$ and $\mathcal{D} = (D_1, \ldots, D_\mu)$ a system of finite Ferrers diagrams. The ideals $I_j = I(P_j, D_j)$ are pairwise comaximal, hence

$$I = \bigcap_{j=1}^{\mu} I_j = \prod_{j=1}^{\mu} I_j \ . \tag{5}$$

Let $P_j = (p_{j,1}, \ldots, p_{j,n})$ and let

$$\left\{\alpha_{j,1} = (\alpha_{j,1,1}, \ldots, \alpha_{j,1,n}), \ldots, \alpha_{j,k_j} = (\alpha_{j,k_j,1}, \ldots, \alpha_{j,k_j,n})\right\} \subset \mathbb{N}^n$$

be a finite generating set of the monoid ideal $\Delta_j = \mathbb{N}^n \setminus D_j$, i.e. $\Delta_j = \bigcup_{i=1}^{k_j} (\alpha_{j,i} + \mathbb{N}^n)$. Then the set

$$\{(X_1 - p_{j,1})^{\alpha_{1,j,i}} \cdot \ldots \cdot (X_n - p_{j,n})^{\alpha_{n,j,i}} \ \mid \ 1 \le i \le k_j\} \tag{6}$$

generates I_j. Using (6) and (5) we can construct a generating set $F_{\mathcal{P},\mathcal{D}}$ of I.

We introduce new indeterminates $P = \{p_{j,i} \ \mid \ 1 \le j \le \mu, 1 \le i \le n\}$ for the coordinates of the points and consider the ideal J generated by $F_{\mathcal{P},\mathcal{D}}$ in the ring $\mathbb{K}[P, X]$. Let $\sqsubset$ denote an arbitrary admissible term order on $T(P, X)$ such that $\sqsubset|_{T(X)} = \prec$ and $P \ll X$. Using Buchberger's algorithm we compute the reduced Gröbner basis $G_{\mathcal{P},\mathcal{D}}$ of J with respect to $\sqsubset$. Let $C \in \mathbb{K}[P]$ be the least common multiple of all coefficients appearing in the elements of $G_{\mathcal{P},\mathcal{D}}$ considered under the natural isomorphism $\mathbb{K}[P, X] \simeq \mathbb{K}[P][X]$.

Let $\varphi : \mathbb{K}[P, X] \to \mathbb{K}[X]$ be a homomorphism such that $\varphi(X_i) = X_i$ and $\varphi(p_{j,i}) \in \mathbb{K}$ for all $1 \le j \le \mu$ and $1 \le i \le n$. By well-known facts on Gröbner bases under specializations we deduce that $\varphi(C) \ne 0$ implies that $\varphi(G_{\mathcal{P},\mathcal{D}})$ is a Gröbner basis of I with respect to $\prec$ (see [CLO]). Hence, the zero set of the ideal generated by C in $\mathbb{K}[P]$ satisfies the conditions of the algebraic set W. □

This completes also the proof of Theorem 1. □

Definition 1 *A system $\mathcal{P}$ of nodes is called admissible for Hermite interpolation with respect to $\prec$ and $\mathcal{D}$ if*

$$\mathcal{D}_\prec(I(\mathcal{P}, \mathcal{D})) = \Phi_{\mathcal{D},\prec}(\mathcal{P}) = \mathbf{D}_{\mathcal{D},\prec} \quad .$$

Furthermore, the Ferrers diagram $\mathbf{D}_{\mathcal{D},\prec}$ *is called the* admissible Hermite term basis *with respect to* $\prec$ *and* $\mathcal{D}$.

In the special case that all Ferrers diagrams D_j, $j = 1, \ldots, \mu$, consist only of the zero vector of $\mathbb{N}^n$ we have Lagrange interpolation and the characterization of admissible nodes and term bases can be found from Theorem 3 in [ASTW].

For a system of points $\mathcal{P} = (P_1, \ldots, P_\mu) \in (\mathbb{K}^n)^\mu$, a system of finite Ferrers diagrams $\mathcal{D} = (D_1, \ldots, D_\mu)$ and a permutation π of $\{1, \ldots, \mu\}$ we introduce the denotations $\mathcal{P}_\pi = \big(P_{\pi(1)}, \ldots, P_{\pi(\mu)}\big)$ and $\mathcal{D}_\pi = \big(D_{\pi(1)}, \ldots, D_{\pi(\mu)}\big)$.

Remark: There exists a Zariski open set of systems $\mathcal{P}$ of nodes which for any permutation π of $\{1, \ldots, \mu\}$ are admissible for Hermite interpolation with respect to $\prec$ and $\mathcal{D}_\pi$. An algebraic set $\widetilde{W}$ of $(\mathbb{K}^n)^\mu$ defining such an open set $\Omega \setminus \widetilde{W}$ can be obtained from the algebraic set W introduced in Lemma 1 by computing

$$\widetilde{W} = \{\mathcal{P} \in (\mathbb{K}^n)^\mu \ \mid \ \mathcal{P}_\pi \in W \text{ for some permutation } \pi\} \ .$$

Definition 2 *A system* $\mathcal{P}$ *of* μ *points is called universal for Hermite interpolation with respect to* $\prec$ *if it is admissible for Hermite interpolation with respect to* $\mathcal{D}$ *and* $\prec$ *for any system* $\mathcal{D}$ *of* μ *finite Ferres diagrams.*

In general, it is a very hard requirement to be a universal system and it is difficult to formulate properties of the set formed by all universal systems of μ points with respect to a given admissible term order $\prec$. The only direct consequence of Theorem 1 is that it is dense in the usual topology if $\mathbb{K}$ is the field of real or complex numbers.

In the next section we will show that for particular admissible term orders there exist Zariski open sets of universal systems for Hermite interpolation.

3 Hermite interpolation with respect to elimination orders

From now we assume that $\prec$ is an elimination order satisfying $\{X_1\} \ll \{X_2, \ldots, X_n\}$. Furthermore, all Ferrers diagrams $D_1, \ldots, D_\mu$ of the system $\mathcal{D}$ have to be finite.

Lemma 2 *Let* $\mathcal{P} = (P_1, \ldots, P_\mu) \in \Omega$ *be a system of points* P_j *with pairwise different first coordinates. If* $(l_j, \widetilde{\beta}) \in \Delta_\prec(I(P_j, D_j))$ *for* $j = 1, \ldots, \mu$ *then* $(l_1 + \cdots + l_\mu, \widetilde{\beta}) \in \Delta_\prec(I(\mathcal{P}, \mathcal{D}))$.

Proof: Let $P_j = (p_{j,1}, \ldots, p_{j,n})$ and $\widetilde{\beta} = (\beta_2, \ldots, \beta_n)$. Since $p_{j,1} \neq p_{k,1}$ for $j \neq k$ there exist univariate polynomials ψ_i in X_1 such that

$$\begin{aligned} \psi_i(p_{j,1}) &= p_{j,i}, \quad i = 1, \ldots, \mu \text{ and } j = 1, \ldots, n \\ \frac{d^k \psi_i}{dX_1^k}(p_{j,1}) &= 0, \quad i = 1, \ldots, \mu,\ j = 1, \ldots, n \text{ and } k = 1, 2, \ldots, d \quad , \end{aligned}$$

where d was defined in (3). The polynomial

$$F(X) := (X_2 - \psi_2(X_1))^{\beta_2} \cdot \ldots \cdot (X_n - \psi_n(X_1))^{\beta_n} (X_1 - p_{1,1})^{l_1} \cdot \ldots \cdot (X_1 - p_{\mu,1})^{l_\mu} \tag{7}$$

belongs to the intersection ideal $I(\mathcal{P}, \mathcal{D})$ and satisfies $\exp_\prec(F) = (l_1 + \cdots + l_\mu, \widetilde{\beta})$. $F \in I(\mathcal{P}, \mathcal{D})$ can be verified by observing $\frac{\partial^{|\alpha|} F}{\partial X^\alpha}(P_j) = 0$ for all $j = 1, \ldots, \mu$ and $\alpha \in D_j$. $\exp_\prec(F) = \beta$ follows immediately from $\{X_1\} \ll \{X_2, \ldots, X_n\}$. □

Theorem 2 *For systems $\mathcal{P} = (P_1, \ldots, P_\mu)$ of points with pairwise different first coordinates we have*

$$\Delta_\prec(I(\mathcal{P}, \mathcal{D})) = \left\{ (l_1 + \cdots + l_\mu, \widetilde{\beta}) \ \middle|\ (l_j, \widetilde{\beta}) \in \Delta_\prec(I_j),\ j = 1, \ldots, \mu \right\} . \tag{8}$$

Proof: Let $\Delta = \left\{ (l_1 + \cdots + l_\mu, \widetilde{\beta}) \ \middle|\ (l_j, \widetilde{\beta}) \in \Delta_\prec(I_j),\ j = 1, \ldots, \mu \right\}$. From Lemma 2 it follows $\Delta \subseteq \Delta_\prec(I(\mathcal{P}, \mathcal{D}))$.

For each $\widetilde{\alpha} \in \mathbb{N}^{n-1}$ and $1 \leq j \leq \mu$ let $m_{j,\widetilde{\alpha}}$ denote the number of n-tuples of the form $(\alpha_1, \widetilde{\alpha})$ belonging to the Ferrers diagram $D_j = \mathbb{N}^n \setminus \Delta_\prec(I_j)$. Since all Ferrers diagrams D_j are finite it follows that all numbers $m_{j,\widetilde{\alpha}}$ are finite and almost all of them are zero. Furthermore, $m_{j,\widetilde{\alpha}}$ is the minimal natural number α_1 with the property $(\alpha_1, \widetilde{\alpha}) \in \Delta_\prec(I_j)$. Hence, $(\alpha_1, \widetilde{\alpha}) \notin \Delta$ if and only if $\alpha_1 < m_{1,\widetilde{\alpha}} + \cdots + m_{\mu,\widetilde{\alpha}}$. Consequently,

$$\#(D) = \sum_{\widetilde{\alpha} \in \mathbb{N}^{n-1}} m_{1,\widetilde{\alpha}} + \cdots + m_{\mu,\widetilde{\alpha}} = \sum_{j=1}^{\mu} \sum_{\widetilde{\alpha} \in \mathbb{N}^{n-1}} m_{j,\widetilde{\alpha}} = \sum_{j=1}^{\mu} d_j = d \quad , \tag{9}$$

where $D = \mathbb{N}^n \setminus \Delta$. Since $\dim_{\mathbb{K}}(R/I) = d$ if follows $\#(D_\prec(I(\mathcal{P}, \mathcal{D}))) = d = \#(D)$ and, therefore, $\Delta_\prec(I(\mathcal{P}, \mathcal{D})) = \Delta$. □

As a direct consequence of the above theorem we obtain:

Corollary 1 *Let $\prec$ be an elimination order for $\{X_2, \ldots, X_n\}$. Then any system $\mathcal{P} \in (\mathbb{K}^n)^\mu$ of points with pairwise different first coordinates is universal for Hermite interpolation with respect to $\prec$.*

Furthermore, Theorem 2 provides a fast algorithm for the computation of a (minimal) Gröbner basis of the intersection ideal $I(\mathcal{P},\mathcal{D}) = \bigcap_{j=1}^{\mu} I(P_j, D_j)$ with respect to $\prec$ for a given system $\mathcal{P}$ of points with pairwise different first coordinates and a given system $\mathcal{D}$ of finite Ferrers diagrams.

Let B_j denote the minimal generating set of the monoid ideal $\Delta_{\prec}(I_j) = \mathbb{N}^n \setminus D_j$, $j = 1, \ldots, \mu$. According to Lemma 2 the element $\beta(\alpha_1, \ldots, \alpha_\mu) = \left(\sum_{j=1}^{\mu} \alpha_{j,1}, \max_{1\le j\le\mu} \alpha_{j,2}, \ldots, \max_{1\le j\le\mu} \alpha_{j,\mu}\right)$ belongs to $\Delta_{\prec}(I(\mathcal{P},\mathcal{D}))$ for any elements $\alpha_1 = (\alpha_{1,1}, \ldots, \alpha_{1,n}) \in B_1, \ldots, \alpha_\mu = (\alpha_{\mu,1}, \ldots, \alpha_{\mu,n}) \in B_\mu$. Moreover, using formula (7) there can be constructed a polynomial $F_u \in I$ with the property $\exp_{\prec}(F_u) = \beta(u)$ for an arbitrary given $u \in B_1 \times \cdots \times B_\mu$. Theorem 2 ensures that $G = \{F_u \mid u \in B_1 \times \cdots \times B_\mu\}$ is a Gröbner basis of $I = I(\mathcal{P},\mathcal{D})$ with respect to $\prec$. It is easy to construct a minimal subset $U \subseteq B_1 \times \cdots \times B_\mu$ such that $G' = \{\beta(\alpha_1, \ldots, \alpha_\mu) \mid (\alpha_1, \ldots, \alpha_\mu) \in U\}$ generates the monoid ideal $\Delta_{\prec}(I)$. From well-known facts on Gröbner bases it follows that $G = \{F_u \mid u \in U\}$ is a minimal Gröbner basis of I with respect to $\prec$.

The above algorithm can be divided in two steps. The first step is purely combinatorial and computes a generating set of the monoid ideal $\Delta_{\prec}(I)$ which of course also describes the Hermite term basis $\mathbb{N}^n \setminus \Delta_{\prec}(I)$. As a byproduct step 1 produces the information how the generating set of $\Delta_{\prec}(I)$ can be lifted to a Gröbner basis of I in a second step.

A different approach for computing the Ferrers diagram $\mathbb{N}^n \setminus \Delta_{\prec}(I)$ using combinatorics in the special case that $\prec$ is the lexicographical order was developed by Cerlienco and Mureddu in [CM95]. The computation of a Gröbner basis with respect to any term order $\prec$ can be performed also using an algorithm due to Marinari, Möller and Mora (see [MMM]). Both algorithms have in common that they are passing through the whole Ferrers diagram $\mathbb{N}^n \setminus \Delta_{\prec}(I)$. In contrary, our method computes directly a generating set of $\Delta_{\prec}(I)$ and a minimal Gröbner basis of I. Since the minimal generating set of Δ is much smaller than the Ferrers diagram $\mathbb{N}^n \setminus \Delta$ in most interesting cases it can be expected that our algorithm should have a better average computing time behaviour. Moreover, the algorithms of Cerlienco-Mureddu and Marinari-Möller-Mora work in the more general context of arbitrary systems $\mathcal{P}$ of pairwise different points. But from the point of view of interpolation theory only systems $\mathcal{P}$ which are admissible for Hermite interpolation are of interest since only admissibility will guarantee continuity of the solution of the interpolation problem. The exploitation of the admissibility of $\mathcal{P}$ results in a speed up of the algorithm.

While the algorithms of Cerlienco-Mureddu and Marinari-Möller-Mora have polynomial complexity in n and $dim_{\mathbb{K}}(R/I)$ (see [MMM]) our algorithm is exponential in the number μ of points. A comparison of both complexities is impossible since there is no polynomial relationship between $dim_{\mathbb{K}}(R/I)$ and μ.

The algorithm sketched above can be improved by removing such tuples of the set $B_1 \times \cdots \times B_\mu$ for which it can be proved that they will not contribute to the minimal generating set of $\Delta_\prec(I)$. For instance, if the same Ferrers diagram $D_i = D_j$ is associated to the points P_i and P_j then only those tuples of $B_1 \times \cdots \times B_\mu$ with equal i-th and j-th component need to be considered.

Let us consider two special cases for the system $\mathcal{D}$ of Ferrers diagrams. First we assume that for each $1 \leq i \leq \mu$ there exists $m_i \in \mathbb{N}$ such that $D_i = \{(\alpha_1, \ldots, \alpha_n) \mid 0 \leq \alpha_1, \ldots, \alpha_n < m_i\}$. Let $\{m_1, \ldots, m_\mu\} = \{k_1, \ldots, k_r\}$, where $k_1 > k_2 > \cdots > k_r$. Furthermore, let μ_j, $j = 1, \ldots, r$, be the number of Ferrers diagrams D_i, $i = 1, \ldots, \mu$, such that $m_i = k_j$. In particular, $\sum_{j=1}^r \mu_j = \mu$. By $\mathfrak{e}_j$ we denote the j-th unit vector of $\mathbb{N}^n$. Then we can write down the minimal generating set of $\Delta_\prec(I)$ immediately in the form

$$\left\{\left(\sum_{i=1}^{r} \mu_i k_i\right) \mathfrak{e}_1\right\} \cup \left\{\left(\sum_{i=1}^{l-1} \mu_i k_i\right) \mathfrak{e}_1 + k_l \mathfrak{e}_j \;\middle|\; 2 \leq j \leq n, 1 \leq l \leq r\right\} \qquad (10)$$

according to Theorem 2.

By (7) and (10) it follows that the ideal I is generated by polynomials $g_1(X_1), X_2^{k_1} - g_2(X_1, X_2), \ldots, X_n^{k_1} - g_n(X_1, X_n)$, where the degree of the polynomials $g_j(X_1, X_j)$ in X_j is less than k_1 for all $2 \leq j \leq n$, and possibly additional polynomials depending only on X_1 and one of the variables $X_2, \ldots, X_n$ and having degree less than k_1 in the second variable. This statement can be considered as a straight forward generalization of the Shape Lemma (see [GM89, Proposition 1.6]).

From the practical point of view the most interesting case of interpolation of Hermite type consists in $D_i = \{(\alpha_1, \ldots, \alpha_n) \mid 0 \leq \sum_{j=1}^n \alpha_j < m_i\}$, where $m_i \in \mathbb{N}$, for all $i = 1, \ldots, \mu$. This is the second particular case which we will discuss. W.l.o.g. assume $m_1 \geq m_j$ for all $1 < j \leq \mu$. Then the minimal generating set of $\Delta_\prec(I)$ consists of all tuples $(\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n$ such that $\beta := \beta_2 + \cdots + \beta_n \leq m_1$ and $\beta_1 = \sum_{j=1}^\mu (m_j \dot{-} \beta)$, where

$$a \dot{-} b := \begin{cases} a - b & : \quad a > b \\ 0 & : \quad \text{otherwise} \end{cases} .$$

In both investigated special cases the complexity order cannot be improved since it is just the number of elements belonging to the reduced

Gröbner basis of I. Application of (7) again yields a generalisation of the Shape Lemma to a subclass of ideals of Krull dimension zero containing more than only radical ideals.

4 Term bases for Hermite interpolation

The investigation of the case that $\prec$ has not the elimination property $\{X_j\} \ll X \setminus \{X_j\}$ for some $1 \leq j \leq n$ is much more delicate. In general, we cannot read off directly the admissible Hermite term basis $\mathbf{D}_{\mathcal{D},\prec}$ with respect to $\prec$ and a given system $\mathcal{D}$ of finite Ferrers diagrams. Lemma 1 provides a method for the computation of $\mathbf{D}_{\mathcal{D},\prec}$ using Buchberger's algorithm for the computation of Gröbner bases. Though the worst case complexity of Buchberger's algorithm is very high, in our particular case we are in a better situation because only linear algebra is needed for the Gröbner basis calculation.

In contrary to Lemma 1 now we consider $I(\mathcal{P},\mathcal{D})$ as an ideal of $\mathbb{K}(P)[X]$ rather than of $\mathbb{K}[P,X]$, where $P = \{p_{j,i} \mid 1 \leq j \leq \mu, 1 \leq i \leq n\}$ are the variables introduced for the coordinates of the points belonging to $\mathcal{P}$. The Gröbner basis of $I(\mathcal{P},\mathcal{D}) \subset \mathbb{K}(P)[X]$ with respect to $\prec$ provides the admissible Hermite term basis $\mathbf{D}_{\mathcal{D},\prec}$ with respect to $\prec$ and $\mathcal{D}$. However, in general it will not carry all information about admissibility of $\mathcal{P}$. For methods to compute the missing conditions we refer to [CLO].

For any $\prec$ the ideal of $\mathbb{K}(P)[X]$ generated by the leading terms of the elements of the generating set $F_{\mathcal{P},\mathcal{D}}$ of $I(\mathcal{P},\mathcal{D})$ resulting from (5) and (6) is zero-dimensional. Therefore, the Gröbner basis calculation takes place in a finite dimensional $\mathbb{K}(P)$-vector space and requires only linear algebra. But also more efficient methods for computing a Gröbner basis of $I(\mathcal{P},\mathcal{D})$ are known, e.g. the algorithm of Marinari, Möller and Mora mentioned already in the previous section can be applied. Taking into account the presumed admissibility of $\mathcal{P}$ there is a further possibility. Let $\prec'$ be an arbitrary elimination order for $\{X_2,\ldots,X_n\}$. Since there exists a Zariski non empty open set of systems $\mathcal{P} \in \Omega$ which are admissible for Hermite interpolation with respect to $\mathcal{D}$ and both orders $\prec$ and $\prec'$ we can do the following: first we compute a Gröbner basis of $I(\mathcal{P},\mathcal{D})$ with respect to $\prec'$ using the methods presented in the previous section and second we apply the FGLM-algorithm (see [FGLM]) for the transformation to a Gröbner basis with respect to $\prec$.

A different approach towards interpolation is to prescribe a set $C \subset \mathbb{N}^n$, which in all interesting cases is a Ferrers diagram, and to ask for $\mathcal{D}$ such that C becomes the Hermite term basis with respect to $\prec$ and $\mathcal{D}$. The following theorem provides a method for checking whether a given C is Hermite term basis with respect to $\prec$ and $\mathcal{D}$. In general this method will be faster than computing the Hermite term basis with respect to $\prec$ and $\mathcal{D}$ and comparing it with C. Moreover, the theorem allows to derive conditions on $\mathcal{D}$ for given C and $\prec$. Before we present the theorem we need to introduce some notation. Let $\mathcal{P} = (P_1, \ldots, P_\mu)$ a system of points, $\mathcal{D} = (D_1, \ldots, D_\mu)$ a system of finite Ferrers diagrams and $d = \sum_{i=1}^{\mu} \#(D_i)$. To each pair $1 \leq j \leq \mu$ and $\alpha \in D_j$ we associate the functional $\widetilde{\varphi_{j,\alpha}} : \mathbb{K}[X] \to \mathbb{K}$ defined by

$$\forall F \in \mathbb{K}[X] \ : \ \widetilde{\varphi_{j,\alpha}}(F) = \frac{\partial^{|\alpha|} F}{\partial X^\alpha}(P_j) \quad .$$

Let $t_1, \ldots, t_d \in T(X)$ be arbitrary terms considered as polynomial functions, $C = \{\exp(t_1), \ldots, \exp(t_d)\}$ the set of their exponent vectors, and $\varphi_1, \ldots, \varphi_d$ an arbitrary fixed enumeration of the functionals $\widetilde{\varphi_{j,\alpha}}$, $1 \leq j \leq \mu, \alpha \in D_j$. Then by definition

$$\mathit{det}\,(C, \mathcal{D}, \mathcal{P}) := \mathit{det}(\varphi_i(t_k))_{i,k=1,\ldots,d} \quad .$$

The sets of terms and functionals are uniquely determined by C, $\mathcal{D}$ and $\mathcal{P}$. Furthermore, changing the enumerations of terms or functionals will effect at most the sign of $\mathit{det}\,(C, \mathcal{D}, \mathcal{P})$. Therefore, writing $\mathit{det}\,(C, \mathcal{D}, \mathcal{P}) \neq 0$ without claiming the enumerations can be understand with respect to arbitrary enumerations. Finally, by $D_\prec(I)^{(d)}$ we denote the set consisting of the d smallest elements of $D_\prec(I)$ with respect to $\prec$.

Theorem 3 *For any subideal $I \subseteq I(\mathcal{D}, \mathcal{P})$ we have*

$$D_\prec\left(I(\mathcal{D}, \mathcal{P})\right) = D_\prec(I)^{(d)} \iff \mathit{det}\left(D_\prec(I)^{(d)}, \mathcal{D}, \mathcal{P}\right) \neq 0\ . \qquad (11)$$

Proof: For any set $C \in \mathbb{N}^n$ consisting of d elements we have $\mathit{det}\,(C, \mathcal{D}, \mathcal{P}) \neq 0$ if and only if $\{f \in \mathbb{K}[X] \ \mid \ \forall t \in \operatorname{supp}(f) \ : \ \exp(t) \in C\} \cap I(\mathcal{P}, \mathcal{D}) = \{0\}$.
($\Leftarrow$) Assume $D_\prec\left(I(\mathcal{D}, \mathcal{P})\right) = D_\prec(I)^{(d)}$. From the theory of Gröbner basis we deduce that the terms X^α, $\alpha \in D_\prec(I)^{(d)}$, are linearly independent modulo the ideal $I(\mathcal{D}, \mathcal{P})$. Hence, it follows $\mathit{det}\left(D_\prec(I)^{(d)}, \mathcal{D}, \mathcal{P}\right) \neq 0$.
($\Rightarrow$) Assume $\mathit{det}\left(D_\prec(I)^{(d)}, \mathcal{D}, \mathcal{P}\right) \neq 0$.

Suppose that there exists $\alpha \in D_\prec\left(I(\mathcal{D}, \mathcal{P})\right) \setminus D_\prec(I)$. From the theory of Gröbner bases it follows the existence of $f \in \mathbb{K}[X]$ such that $X^\alpha - f \in I$ and $\exp_\prec(f) \prec \alpha$. Hence, $D_\prec\left(I(\mathcal{D}, \mathcal{P})\right) \ni \exp_\prec\left(X^\alpha - f\right) = \alpha$ from what we deduce $X^\alpha - f \notin I(\mathcal{D}, \mathcal{P})$ in contradiction to $I \subseteq I(\mathcal{D}, \mathcal{P})$. Consequently, $D_\prec\left(I(\mathcal{D}, \mathcal{P})\right) \subseteq D_\prec(I)$.

Now, assume the existence of $\alpha \in D_{\prec}(I(\mathcal{D},\mathcal{P})) \setminus D_{\prec}(I)^{(d)}$. Then from $D_{\prec}(I(\mathcal{D},\mathcal{P})) \subseteq D_{\prec}(I)$ it follows $\beta \prec \alpha$ for all $\beta \in D_{\prec}(I)^{(d)}$. The inequality $det\left(D_{\prec}(I)^{(d)},\mathcal{D},\mathcal{P}\right) \neq 0$ implies that the terms X^{β}, $\beta \in D_{\prec}(I)^{(d)}$, are linearly independent modulo the ideal $I(\mathcal{D},\mathcal{P})$. Since the $\mathbb{K}$-vector space dimension of $R/I(\mathcal{P},\mathcal{D})$ is equal to $d = \#\left(D_{\prec}(I)^{(d)}\right)$ it follows that the set $B = \left\{X^{\beta} + I(\mathcal{D},\mathcal{P}) \mid \beta \in D_{\prec}(I)^{(d)}\right\}$ is a vector space basis of $R/I(\mathcal{P},\mathcal{D})$. Hence, the residue class $X^{\alpha}+I(\mathcal{D},\mathcal{P})$ is a linear combination of the elements of B. Therefore, there exist $c_{\beta} \in \mathbb{K}$, $\beta \in D_{\prec}(I)^{(d)}$, such that $f = X^{\alpha} + \sum_{\beta \in D_{\prec}(I)^{(d)}} c_{\beta} X^{\beta} \in I(\mathcal{D},\mathcal{P})$. It follows $\Delta_{\prec}(I(\mathcal{D},\mathcal{P})) \ni \exp_{\prec}(f) = \alpha$ in contradiction to $\alpha \in D_{\prec}(I(\mathcal{D},\mathcal{P})) = \mathbb{N}^n \setminus \Delta_{\prec}(I(\mathcal{D},\mathcal{P}))$. Hence, $D_{\prec}(I(\mathcal{D},\mathcal{P})) \subseteq D_{\prec}(I)^{(d)}$ and the trivial observation $\#\left(D_{\prec}(I(\mathcal{D},\mathcal{P}))\right) = d = \#\left(D_{\prec}(I)^{(d)}\right)$ completes the proof. □

As an obvious consequence we have

Corollary 2 *Let $\mathcal{D} = (D_1,\ldots,D_{\mu})$ be a system of finite Ferrers diagrams, $\mathcal{P} = (P_1,\ldots,P_{\mu})$ a system of points with indeterminate coordinates $P = \{p_{j,i} \mid 1 \leq j \leq \mu, 1 \leq i \leq n\}$, and $C \in \mathfrak{F}_d$ a Ferrers diagram consisting of $d = \sum_{i=1}^{\mu} \#(D_i)$ elements. Furthermore, fix an arbitrary enumeration of C and the functionals $\varphi_{j,\alpha}$. Then $\mathbf{D}_{\mathcal{D},\prec} = C$ if and only if the polynomial $det\,(C,\mathcal{D},\mathcal{P})$ in the indeterminates P is not identical zero.*

The particular case that I is a radical ideal and $D_j = \{(0,\ldots,0)\}$ for all $1 \leq j \leq \mu$ was investigated in [ASTW] and corresponds to interpolation of Lagrange type on algebraic sets. In that situation the question for admissible systems of points has always a positive solution and, moreover, any admissible system of μ points can be extended to an admissible system of $\mu+1$ points. Note, if $\mathcal{D}$ is of Lagrange type then the determinants $det(D_{\prec}(I)^{(d)},\mathcal{P})$ used in [ASTW, Theorem 3] and $det\left(D_{\prec}(I)^{(d)},\mathcal{D},\mathcal{P}\right)$ introduced in this paper refer to transposed matrices if the elements of $\mathcal{P}$ and $D_{\prec}(I)^{(d)}$ are enumerated in the same way. Moreover, while $D_{\prec}(I)^{(d)}$ was a subset of $T(X)$ in [ASTW] now it consists of the corresponding exponent vectors.

In the Hermite case the situation is much more delicate. For instance, if $\prec$ is an elimination order for $\{X_2,\ldots,X_n\}$, $I \cap \mathbb{K}[X_1] = \{0\}$, and some Ferrers diagram D_j contains an element $(\alpha_1,\ldots,\alpha_n)$ such that $\{\alpha_2,\ldots,\alpha_n\} \neq \{0\}$ then $det\left(D_{\prec}(I)^{(d)},\mathcal{D},\mathcal{P}\right)$ is identical zero in the coordinates of $\mathcal{P}$.

In the case that the term order $\prec$ is degree compatible and $I = \{0\}$ Hakopian characterized some situations of the type that there exist nonnegative integers $k_1,\ldots,k_{\mu},k$ such that D_j consists of the exponent vectors of all terms of degree smaller or equal than k_j and $\sum_{j=1}^{\mu} \binom{n+k_j}{k_j} = \binom{n+k}{k}$ (see [Ha82]).

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Authors addresses.

J. Apel, J. Stückrad
Fakultät für Mathematik und Informatik, Universität Leipzig,
Augustusplatz 10–11, D-04109 Leipzig, Germany

apel@informatik.uni-leipzig.de
stueckrad@mathematik.uni-leipzig.de

P. Tworzewski, T. Winiarski
Instytut Matematyki, Uniwersytet Jagielloński, Reymonta 4,
30-059 Kraków, Poland

tworzews@im.uj.edu.pl
winiarsk@im.uj.edu.pl