On logics with coimplication

Frank Wolter
School of Information Science
JAIST
Tatsunokuchi, Ishikawa 923-12, Japan
e-mail: wolter@jaist.ac.jp

Abstract

This paper investigates (modal) extensions of Heyting-Brouwer logic, i.e., the logic which results when the dual of implication (alias coimplication) is added to the language of intuitionistic logic. We first develop matrix as well as Kripke style semantics for those logics. Then, by extending the Gödel-embedding of intuitionistic logic into **S4**, it is shown that all (modal) extensions of Heyting-Brouwer logic can be embedded into tense logics (with additional modal operators). An extension of the Blok-Esakia-Theorem is proved for this embedding.

1 Introduction

[1] It is known that the propositional intuitionistic logic **Int** (formulated in the propositional language $\mathcal{L}_{\mathbf{Int}}$ with connectives $\wedge$, $\vee$, $\rightarrow$, $\top$, $\bot$) is determined by Heyting algebras, i.e., algebras $\langle A, \wedge, \vee, \rightarrow, \bot, \top \rangle$ such that $\langle A, \wedge, \vee, \bot, \top \rangle$ is a bounded distributive lattice and $a \rightarrow c$ is the *relative pseudo-complement* of a with respect to c. That is to say,

$$a \wedge b \leq c \Leftrightarrow b \leq a \rightarrow c, \text{ for all } b \in A.$$

(Here and in what follows we put $x \leq y \Leftrightarrow x \wedge y = x$.) While join and meet are dual to each other in distributive lattices this is not the case in Heyting algebras since there is no dual to the relative pseudo-complement. Hence this completeness result implies that the algebraic interpretations of conjunction and disjunction of intuitionistic logic are not dual to each other. Duality between conjunction and disjunction is restored, however, by adding to the language of intuitionistic logic a connective interpreted as the dual relative pseudo-complement: Denote by $\mathcal{L}_{\mathbf{HB}}$ the language obtained from $\mathcal{L}_{\mathbf{Int}}$ by adding the connective $\stackrel{\smile}{\rightarrow}$. Then the *Heyting Brouwer-logic* **HB** is defined (in the language $\mathcal{L}_{\mathbf{HB}}$)

[1] **Keywords**: Coimplication, Heyting-Brouwer-Logic, Tense Logic, Modal Logic, Blok-Esakia-Isomorphism.

as the logic determined by the *Heyting-Brouwer-algebras*[2] (*HB-algebras*, for short), i.e., Heyting algebras equipped with a new operation $\stackrel{\smile}{\to}$ satisfying for all a, b, c,

$$a \vee b \geq c \Leftrightarrow b \geq a \stackrel{\smile}{\to} c.$$

Clearly, in HB-algebras we have complete symmetry between the operators $\wedge, \top, \to$ and $\vee, \bot, \stackrel{\smile}{\to}$, respectively, and it is interesting to investigate the consequences for **HB** and its extensions and compare the results with **Int** and the lattice of super-intuitionistic logics[3] (alias intermediate logics).

In the formulation of **HB** given above the meaning of the connective $\stackrel{\smile}{\to}$ corresponding to the dual relative pseudo-complement remains unclear and its main motivation is just symmetry. For an interpretation of $\stackrel{\smile}{\to}$ Kripke type semantics is helpful[4]. Call a partially ordered set $\langle g, \lhd \rangle$ an **Int**-*frame*. With $\langle g, \lhd \rangle$ we associate the Heyting-algebra

$$\langle g, \lhd \rangle^h = \langle C(g), \cap, \cup, \to, \emptyset, g \rangle,$$

where $C(g)$ denotes the set of all cones[5] in $\langle g, \lhd \rangle$ and

$$a \to b = \{x \in g : (\forall y \in g)(x \lhd y \wedge y \in a \Rightarrow y \in b)\}.$$

Then **Int** is the logic determined by the class of Heyting-algebras of the form $\langle g, \lhd \rangle^h$. What is the meaning of $\stackrel{\smile}{\to}$ formulated in terms of $\lhd$? It will turn out that for the operation

$$a \stackrel{\smile}{\to} b = \{x \in g : (\exists y \in g)(y \lhd x \wedge y \in b - a)\}, \text{ for all } a, b \in A,$$

the algebra $\langle g, \lhd \rangle^+ = \langle C(g), \cap, \cup, \to, \stackrel{\smile}{\to}, \emptyset, g \rangle$ is a HB-algebra, and that each HB-algebra can be represented as a subalgebra of an algebra of the form $\langle g, \lhd \rangle^+$. If we interpret with Kripke [25] the points in g as points in time, at which we may have certain pieces of information, then extending the language $\mathcal{L}_{\mathbf{Int}}$ to the language $\mathcal{L}_{\mathbf{HB}}$ means that we add, in a symmetrical way, a connective talking about the past to the connectives of **Int** which talk about the future only. Truth of $\varphi \stackrel{\smile}{\to} \psi$ at a point x means that at some moment in the past (of x) ψ was known while φ was unknown[6]. A similar step has been taken in classical modal logic: We start with modal logics above **S4** with one modal operator $\Box$ talking about sets of the form $\{y : x \lhd y\}$ and get *tense logics* by adding another modal operator $\Box^{-1}$ talking about sets of the form $\{y : x \lhd^{-1} y\}$. We note that in tense logic the operator $\Box$ is mostly denoted by G (*it is always going to be*) and the operator $\Box^{-1}$ is

[2] Notice that Heyting-Brouwer-algebras are called double Heyting algebras in [22], biHeyting-algebras in [26], and Semi-Boolean algebras in [29] and [31]. We choose the name Heyting-Brouwer-algebras in order to emphazise the connection with Heyting-Brouwer logic, a term introduced by Rauszer in [29].

[3] A subset Λ of $\mathcal{L}_{\mathbf{Int}}$ is called super-intuitionistic iff it contains **Int** and is closed under modus ponens and substitutions.

[4] Consult [29] and [31] for some other reasons to study HB-logic. In [26] one may find a motivation in terms of category theory.

[5] A subset a of g is a *cone* iff $y \in a$ whenever $x \lhd y$ and $x \in a$.

[6] Obviously, also for the interpretation of **Int** as the logic of scientific research in the sense of Grzegorczyk [21], the connective $\stackrel{\smile}{\to}$ has a clear meaning. We do not see however a natural interpretation of $\stackrel{\smile}{\to}$ in terms of the interpretation of **Int** as the logic of constructive proofs, see e.g. [36] page 9.

mostly denoted by H (*it has always been*). (See e.g. [2], [3], [6] and below for information on tense logics.) The connection between **Int** and **S4** has been formalized by Gödel's [18] embedding of **Int** into **S4**. Moreover, this embedding also interprets all intermediate logics in extensions of **S4**, as has been observed by Dummett and Lemmon in [10]. [27], [4], and [11] are investigations of the structure of those embeddings and a survey of the results is given in [8]. We shall investigate in this paper the extent to which there holds a similar relation between tense logics and extensions of **HB**.

Quite often symmetry allows the introduction of elegant concepts and techniques. In the present paper we show this for the operators $\to$ and $\dot{\rightarrow}$ by taking into account modal operators based on **HB** and its extensions. That is to say, we shall develop a theory of modal logics based on extensions of **HB** and compare them with modal logics based on intermediate logics. We can indicate already why certain things become more elegant in the present case. Let us call a unary connective $\bigcirc$ $\Diamond$-like (or a *possibility*-operator) in a logic Λ iff

$$\mathbf{F}^{\Diamond} = \{\bigcirc(p \vee q) \leftrightarrow \bigcirc p \vee \bigcirc q,\ \neg \bigcirc \bot\} \tag{1}$$

is contained in Λ and let us call $\bigcirc$ $\Box$-like (or a *necessity*-operator) in a logic Λ iff

$$\mathbf{F}^{\Box} = \{\bigcirc(p \wedge q) \leftrightarrow \bigcirc p \wedge \bigcirc q,\ \bigcirc\top\} \tag{2}$$

is contained in Λ. For modal logics based on classical logic each $\Diamond$-like connective $\bigcirc$ defines a $\Box$-like connective $\Box$ via $\Box := \neg \bigcirc \neg$ satisfying moreover $\neg\Box\neg p \leftrightarrow \bigcirc p$ and vice versa. In intuitionistic logic, however, this duality does not hold, i.e., $\neg \bigcirc \neg$ does not always distribute over disjunctions when $\bigcirc$ distributes over conjunction and it is not dual to $\bigcirc$. Adding a $\Diamond$-like connective to intuitionistic logic results in modal logics which behave rather different from those obtained by adding a $\Box$-like connective to intuitionistic logic (consult e.g. [17], [7], [41], and [39] for discussions of modal logics based on intuitionistic logic and the difference between $\Diamond$- and $\Box$-like connectives based on intuitionistic logic.) The reason is - of course - the lack of duality between conjunction and disjunction in intuitionistic logic as explained above. So, by adding $\dot{\rightarrow}$ to the language $\Box$- and $\Diamond$-like operators will become dual to each other again and we shall be able to develop a quite elegant theory for modal logics based on extensions of **HB**.

The paper is organized as follows. Section 2 introduces (modal) super-HB-logics from the syntactical point of view. Sections 3 and 4 deal with matrix and Kripke semantics for them. Also in Section 4 various super-HB-logics are introduced with the help of their Kripke frames. Section 5 is concerned with duality between matrices and Kripke frames. In Section 6 the classical modal logics into which (modal) super-HB-logics are embedded, i.e. (extended) tense logics, are introduced. Sections 7 and 8, where this embedding is studied, form the main part of this paper. Section 5 is not required for those sections and Kripke semantics is used only to explain certain constructions. A reader familiar with algebraic reasoning can read the sections on the embedding without knowledge of Kripke semantics. Section 9 shows that the embedding can be used to obtain rather general results on completeness and the finite model property of (modal) super-HB-logics.

We assume that the reader has basic knowledge of algebraic as well as Kripke semantics for modal logics, tense logics, and intermediate logics. A number of proofs and derivations are left to the reader since they are straighforward combinations of techniques known from those three types of logics. Sometimes proofs are omitted since they are straightforward extensions of results obtained by C. Rauszer, who has investigated the logic **HB** itself (and the predicate logic based on **HB**) in [29], [30], and [31]. We advise the reader to have a look at [29].

Acknowledgments. The first to thank is Piotr Lukowski for it was his talk at the MLG-workshop 1995, Kanazawa, which convinced me that logics with coimplication are worth studying. I also wish to thank Hiroakira Ono for a number of helpful discussions of the subject. I am grateful to an anonymous referee for a number of helpful remarks and for pointing out an error in an earlier version of this paper.

2 Syntax

We shall introduce the logics which will be investigated in this paper. All propositional languages $\mathcal{L}$ investigated will contain the connective $\to$ and we call a set $\Lambda \subseteq \mathcal{L}$ a *$\mathcal{L}$-logic* iff it is closed under substitutions and modus ponens: $p, p \to q/q$. The following Hilbert style axiomatization of **HB** was delivered by C. Rauszer in [29]. We abbreviate $\breve{\neg} p := p \mathrel{\breve{\to}} \top$ and $\neg p := p \to \bot$. Take any set of formulas $H_1 \subseteq \mathcal{L}_{\mathbf{Int}}$ such that **Int** is the closure of H_1 under substitutions and modus ponens and put

$$H_2 = \{p \to (q \vee (q \mathrel{\breve{\to}} p)), (q \mathrel{\breve{\to}} p) \to \breve{\neg}(p \to q),$$

$$(r \mathrel{\breve{\to}} (q \mathrel{\breve{\to}} p)) \to ((p \vee q) \mathrel{\breve{\to}} p), \neg(q \mathrel{\breve{\to}} p) \to (p \to q), \neg(p \mathrel{\breve{\to}} p)\}.$$

Then **HB** is the smallest $\mathcal{L}_{\mathbf{HB}}$-logic containing $H_1 \cup H_2$ and closed under the rule

$$(\mathrm{RN}_{\neg\breve{\neg}})\ \ p/\neg\breve{\neg}p.$$

We note that **HB** can also be axiomatized by replacing the rule $(\mathrm{RN}_{\neg\breve{\neg}})$ by the axiom $\neg\breve{\neg}\top$ and the rule

$$(\mathrm{RC}_{\breve{\neg}})\ \ p \leftrightarrow q/\breve{\neg}p \leftrightarrow \breve{\neg}q.$$

A *super HB-logic* Λ is a $\mathcal{L}_{\mathbf{HB}}$-logic containing **HB**. The smallest super HB-logic containing a super-HB-logic Λ and a set of formulas Γ is denoted by $\Lambda + \Gamma$. Notice that not all super-HB-logics are closed under $(\mathrm{RN}_{\neg\breve{\neg}})$. So we call a super-HB-logic *normal* iff it is closed under $(\mathrm{RN}_{\neg\breve{\neg}})$ or, equivalently, under $(\mathrm{RC}_{\breve{\neg}})$[7]. The smallest normal super-HB-logic containing a logic Λ and a set of formulas Γ is denoted by $\Lambda \oplus \Gamma$. Notice that both the set of super-HB-logics as well as the set of modal HB-logics form complete lattices induced by the inclusion relation.

Denote by $\mathcal{L}_{\mathbf{ML}}$ the language obtained from $\mathcal{L}_{\mathbf{HB}}$ by adding two modal connectives $\Box$ and $\Diamond$. The *basic modal HB-logic* **ML** is the smallest $\mathcal{L}_{\mathbf{ML}}$-logic which contains **HB**, the

[7]The distinction between normal and non-normal super-HB-logics will turn out to reflect the well-known distinction between non-normal and normal modal logics (cf. e.g. [33]). This is the reason for our terminology. Note that there is no such natural subclass of the class of intermediate logics. So we shall denote by both $\mathbf{Int} + \Gamma$ as well as by $\mathbf{Int} \oplus \Gamma$ the smallest intermediate logic containing a set Γ of formulas.

formulas in $\mathbf{F}^{\Box}$ and $\mathbf{F}^{\Diamond}$ formulated for the connectives $\Box$ and $\Diamond$, respectively, and which is closed under ($\mathrm{RC}_{\breve{\neg}}$),

$$(\mathrm{RC}_{\Box})\ \ p \leftrightarrow q / \Box p \leftrightarrow \Box q \text{ and } (\mathrm{RC}_{\Diamond})\ \ p \leftrightarrow q / \Diamond p \leftrightarrow \Diamond q.$$

A *modal HB-logic* is a $\mathcal{L}_{\mathbf{ML}}$-logic containing **ML**. A *normal modal HB-logic* is a modal HB-logic which is closed under the rules ($\mathrm{RC}_{\breve{\neg}}$), ($\mathrm{RC}_{\Box}$) and ($\mathrm{RC}_{\Diamond}$). We note that normal modal HB-logic are precisely those modal HB-logic which are closed under ($\mathrm{RN}_{\neg\breve{\neg}}$),

$$(\mathrm{RN}_{\Box})\ \ p/\Box p \text{ and } (\mathrm{RN}_{\neg\Diamond\breve{\neg}})\ \ p/\neg\Diamond\breve{\neg}p.$$

An interesting consequence of this easily proved observation is that $\mathsf{Ker}\Lambda$ is the maximal normal modal HB-logic contained in a modal HB-logic Λ. Here we put[8]

$$\mathsf{Ker}\Lambda = \{\varphi :\ s\varphi \in \Lambda \text{ for all sequences } s \in \{\neg\breve{\neg}, \Box, \neg\Diamond\breve{\neg}\}^*\},$$

where S^* denotes the set of finite strings over a set S. Similar to the definition for super-HB-logics we denote by $\Lambda + \Gamma$ ($\Lambda \oplus \Gamma$) the smallest (normal) modal HB-logic containing a modal HB-logic Λ and a set of formulas Γ. Examples of super-HB-logics and modal HB-logics will be given in the section on Kripke semantics. We note that it is certainly of interest to investigate modal logics based on super-HB-logics with only one modal operator, $\Box$ or $\Diamond$. Those logics, however, are included in our definition of modal HB-logics since we may identify them in a straightforward way with extensions of $\mathbf{ML}^{\Box} := \mathbf{ML} \oplus p \leftrightarrow \Diamond p$ and $\mathbf{ML}^{\Diamond} := \mathbf{ML} \oplus p \leftrightarrow \Box p$, respectively. Also, we shall mostly formulate our results for modal HB-logics and not for super-HB-logics since we can identify **HB** with $\mathbf{ML}^{\Box\Diamond} := \mathbf{ML}^{\Box} \oplus p \leftrightarrow \Box p$.

3 Matrix semantics

Recall some basic definitions from matrix theory (cf. e.g. [37], [5], or [9]). Consider a propositional language $\mathcal{L}$ with connectives $f_1, \ldots, f_k$. A $\mathcal{L}$-*matrix* is a structure $\mathcal{M} = \langle \mathcal{A}, F \rangle$ such that $\mathcal{A} = \langle A, f_1^{\mathcal{A}}, \ldots, f_k^{\mathcal{A}} \rangle$ is an $\mathcal{L}$-algebra and $F \subseteq A$. A valuation V in $\mathcal{M}$ is a homomorphism from the algebra of formulas $\mathcal{L}$ into $\mathcal{A}$. A formula φ is valid in a matrix $\mathcal{M}$, in symbols $\mathcal{M} \models \varphi$, if $V(\varphi) \in F$, for all valuations V. The *logic* $\mathsf{Th}\mathcal{M}$ of a matrix $\mathcal{M}$ is the set of all formulas φ which are valid in $\mathcal{M}$ and the logic of a class of matrices M is

$$\mathsf{Th}M = \bigcap\{\mathsf{Th}\mathcal{M} : \mathcal{M} \in M\}.$$

We also say that $\Lambda = \mathsf{Th}\mathcal{M}$ is *determined* by $\mathcal{M}$. Conversely, for each $\mathcal{L}$-logic Λ and each class of $\mathcal{L}$-matrices M we put

$$\mathsf{Mat}_M\Lambda = \{\mathcal{M} \in M : \mathcal{M} \models \Lambda\}.$$

We put $\mathsf{Mat}\Lambda = \mathsf{Mat}_M\Lambda$ when M is the class of all $\mathcal{L}$-matrices. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}_1$ and $\mathcal{L}_2$-algebras, respectively, and that $\mathcal{L} \subseteq \mathcal{L}_1 \cap \mathcal{L}_2$. We say that a mapping $f : \mathcal{A} \to \mathcal{B}$

[8]We take the notation $\mathsf{Ker}\Lambda$ from modal logic, see e.g. [[33], page 174] and [9].

is a $\mathcal{L}$-*homomorphism* iff $f(g^{\mathcal{A}}a_1,\ldots,a_k) = g^{\mathcal{B}}f(a_1),\ldots,f(a_k)$, for all connectives g in $\mathcal{L}$ and $a_1,\ldots,a_k \in \mathcal{A}$. A number of operations on matrices are required. Given two $\mathcal{L}$-matrices $\langle\mathcal{A},F\rangle$ and $\langle\mathcal{B},G\rangle$ we shall call $\langle\mathcal{B},G\rangle$ a *homomorphic image* of $\langle\mathcal{A},F\rangle$ if there exists a $\mathcal{L}$-homomorphism f from $\mathcal{A}$ onto $\mathcal{B}$ such that $f[F]\subseteq G$. $\langle\mathcal{B},G\rangle$ is called a *relative* of $\langle\mathcal{A},F\rangle$ if there exists a $\mathcal{L}$-homomorphism from $\mathcal{B}$ into $\mathcal{A}$ such that $f^{-1}F\subseteq G$. Also, for each ordinal α and family $\langle\mathcal{M}_i = \langle\mathcal{A}_i,F_i\rangle : i\in\alpha\rangle$ of matrices we call

$$\prod_{i=0}^{\alpha}\mathcal{M}_i := \langle\prod\langle\mathcal{A}_i :\in\alpha\rangle, \prod\langle F_i : i\in\alpha\rangle\rangle$$

the *direct product* of this family. If U is an ultrafilter on the powerset of α and $\mathcal{M}_i = \mathcal{M} = \langle\mathcal{A},F\rangle$, for all $i\in\alpha$, then

$$\prod_{i=0}^{\alpha}\mathcal{M}/U := \langle\prod\langle\mathcal{A}_i : i\in\alpha\rangle/\sim_U, \prod\langle F_i : i\in\alpha\rangle/\sim_U\rangle,$$

is the *ultrapower* of $\mathcal{M}$, where $g_1\sim_U g_2 \Leftrightarrow \{i\in\alpha : g_1(i) = g_2(i)\}\in U$. (We use, for a relation R and a congruence relation $\sim$, $R/\sim$ to denote the canonical quotient relation induced by $\sim$.) For two classes of matrices V and M we denote by $\mathsf{H}_V M$ the class of homomorphic images of matrices in M which are in V, by $\mathsf{P}M$ the class of products of families of matrices in M, by $\mathsf{R}_V M$ the class of relatives of matrices in M which are in V, and by $\mathsf{P_U}M$ the class of ultrapowers of matrices in M. The following proposition is easy to check (cf. similar results in e.g. [37] and [5]).

Proposition 1 *For all logics Λ, $\mathsf{Mat}\Lambda$ is closed under the operations $\mathsf{H}_V,\mathsf{R}_V,\mathsf{P},\mathsf{P_U}$, for all V.*

Now call an algebra $\mathcal{A} = \langle A,\wedge,\vee,\rightarrow,\ \breve{\rightarrow}\ ,\Box,\Diamond,\bot,\top\rangle$ a *modal HB-algebra* (a *ML-algebra*, for short) if the reduct without $\Box$ and $\Diamond$ is a HB-algebra and $\Box\top=\top$, $\Diamond\bot=\bot$, $\Box(a\wedge b) = \Box a\wedge\Box b$, and $\Diamond(a\vee b) = \Diamond a\vee\Diamond b$. A *filter* in a modal algebra is a filter in the underlying Heyting algebra. Call a matrix $\mathcal{M} = \langle\mathcal{A},F\rangle$ a *modal HB-matrix* (*ML-matrix*, for short) if $\mathcal{A}$ is a ML-algebra and F is a filter in A. Let us call a ML-matrix $\mathcal{M} = \langle\mathcal{A},F\rangle$ a *pointed ML-algebra* iff F is a prime filter and let us call $\mathcal{M}$ a *normal ML-matrix* iff $F = \{\top\}$. Often we shall identify the normal matrix $\langle\mathcal{A},\{\top\}\rangle$ with the algebra $\mathcal{A}$. The following is a standard result on the existence of prime filters which we shall use.

Lemma 2 *Suppose that F_1,F_2 are subsets of a Heyting-algebra $\mathcal{A}$ such that (i) F_1 is closed under $\wedge$, i.e. $b_1,b_2\in F_1\Rightarrow b_1\wedge b_2\in F_1$ holds, (ii) F_2 is closed under $\vee$, i.e. $b_1,b_2\in F_2\Rightarrow b_1\vee b_2\in F_2$ holds, and (iii) $a\not\leq b$, for all $a\in F_1$ and $b\in F_2$. Then there exists a prime filter P in $\mathcal{A}$ such that $F_1\subseteq P$ and $P\cap F_2=\emptyset$.*

Theorem 3 *(1) Each modal HB-logic Λ is determined by a class of ML-matrices $\langle\mathcal{A},D\rangle$ satisfying $\mathcal{A}\models\mathsf{Ker}\Lambda$. Conversely, each ML-matrix determines a modal HB-logic.*

(2) Each modal HB-logic Λ is determined by a class of pointed ML-algebras $\langle\mathcal{A},D\rangle$ satisfying $\mathcal{A}\models\mathsf{Ker}\Lambda$.

(3) A modal HB-logic is determined by normal ML-matrices iff it is normal. Moreover, a class of normal ML-matrices M is a variety iff there exists a normal modal HB-logic Λ such that $M = \mathsf{Mat}\Lambda$.

Proof. First observe the

Claim. Let Θ be a normal modal HB-logic. Then the relation $\sim_\Theta$ defined by $\varphi \sim_\Theta \psi \Leftrightarrow \varphi \leftrightarrow \psi \in \Theta$ defines a congruence on $\mathcal{L}_{\mathbf{ML}}$ and the quotient algebra $\mathcal{L}_{\mathbf{ML}}/\sim_\Theta$ is a ML-algebra.

For the proof observe that the corresponding result is shown for normal super-HB-logics in [29]. Now the implication $\varphi \sim_\Theta \psi \Rightarrow \Box\varphi \sim_\Theta \Box\psi$ and $\Diamond\varphi \sim_\Theta \Diamond\psi$, for all formulas φ and ψ, follows from the closure of Λ under $(\mathrm{RC}_\Box)$ and $(\mathrm{RC}_\Diamond)$. Hence $\sim_\Theta$ is a congruence relation. That $\mathcal{L}_{\mathbf{ML}}/\sim_\Theta$ is a ML-algebra follows from the condition $\mathbf{F}^\Box \cup \mathbf{F}^\Diamond \subseteq \Theta$.

Fix a modal HB-logic Λ.

(1) Form the matrix $\mathcal{M} = \langle \mathcal{A}, D\rangle$ where $\mathcal{A} := \mathcal{L}_{\mathbf{ML}}/\sim_{\mathsf{Ker}\Lambda}$ and $D := \{[\varphi] : \varphi \in \Lambda\}$ with $[\varphi] = \{\psi : \varphi \sim_{\mathsf{Ker}\Lambda} \psi\}$. By the claim above $\mathcal{A}$ is a ML-algebra and D is a filter. It is easy to show now that $\Lambda = \mathsf{Th}\mathcal{M}$ and $\mathsf{Ker}\Lambda = \mathsf{Th}\mathcal{A}$. The converse direction is left to the reader.

(2) Suppose that $\varphi \notin \Lambda$. By (1) we find a ML-matrix $\mathcal{M} = \langle \mathcal{A}, F\rangle$ validating Λ such that there is a valuation V with $V(\varphi) \notin F$. By Lemma 2 we find a prime filter P with $F \subseteq P$ and $V(\varphi) \notin P$. Hence $\langle \mathcal{A}, P\rangle$ is a pointed ML-algebra validating Λ (since $F \subseteq P$) and refuting φ.

(3) One direction follows from (1) and the fact that $\mathsf{Ker}\Lambda = \Lambda$ iff Λ is normal. For the other direction it suffices to observe that $\neg\breve{\neg}\top = \Box\top = \neg\Diamond\breve{\neg}\top = \top$ holds in all ML-algebras. The second part of (3) follows from the fact that for all equations $\varphi = \psi$ with $\varphi, \psi \in \mathcal{L}_{\mathbf{ML}}$ and all ML-algebras $\mathcal{A}$, we have $\mathcal{A} \models \varphi = \psi \Leftrightarrow \mathcal{A} \models \varphi \leftrightarrow \psi = \top$. $\dashv$

Note that by the second part of (3) we know that the lattice of normal modal HB-logics containing $\mathbf{ML}^\Diamond$ ($\mathbf{ML}^\Box$) is isomorphic to the lattice of subvarieties of the variety of ML-algebras validating the equation $a = \Box a$ ($a = \Diamond a$). Certainly those two lattices of varieties are isomorphic to each other (by duality) and we conclude:

Corollary 4 *The lattice of normal modal HB-logics containing* $\mathbf{ML}^\Diamond$ *is isomorphic to the lattice of normal modal HB-logics containing* $\mathbf{ML}^\Box$.

Recall that this result on the equivalence of *possibility*-operators and *necessity*-operators is in contrast with the situation for modal logics based on intermediate logics (cf. e.g. [39] and [17]).

4 Kripke semantics

Having established a simple and standard algebraic completeness result for modal HB-logics we now develop Kripke semantics. A *HB-frame* is a structure $\mathcal{G} = \langle g, \lhd, A\rangle$ such that $\langle g, \lhd\rangle$ is a partial ordering and A is a set of cones with respect to $\langle g, \lhd\rangle$ which is closed under intersection, union and the operations $\rightarrow$ and $\breve{\rightarrow}$ introduced above. We extend the notion of a HB-frame to the notion of a *ML-frame* by saying that a structure $\mathcal{G} = \langle g, \lhd, R_\Box, R_\Diamond, A\rangle$ is a ML-frame if the reduct $\langle g, \lhd, A\rangle$ is an HB-frame and A is closed under the modal operations

$$\Box a := \{x \in g : (\forall y \in g)(xR_\Box y \Rightarrow y \in a)\},$$

$$\Diamond a := \{x \in g : (\exists y \in g)(xR_\Diamond y \wedge y \in a)\},$$

and $R_\Box$ and $R_\Diamond$ satisfy

$$\lhd \circ R_\Box \circ \lhd = R_\Box, \tag{3}$$

$$\lhd^{-1} \circ R_\Diamond \circ \lhd^{-1} = R_\Diamond. \tag{4}$$

We call $\mathcal{G}$ a *full frame* iff A consists of all cones in $\langle g, \lhd\rangle$. In this case we often write $g = \langle g, \lhd, R_\Box, R_\Diamond\rangle$ instead of $\mathcal{G}$. We note that the set of all cones is always closed under the operations $\Box$ and $\Diamond$, since $\lhd \circ R_\Box \subseteq R_\Box$ and $\lhd^{-1} \circ R_\Diamond \subseteq R_\Diamond$ follow from (3) and (4), repectively[9].

With each ML-frame $\mathcal{G}$ we associate the ML-algebra $\mathcal{G}^+$ defined by

$$\mathcal{G}^+ = \langle A, \cap, \cup, \rightarrow, \breve{\rightarrow}, \Box, \Diamond, \emptyset, g\rangle.$$

(We leave the straightforward proof that $\mathcal{G}^+$ is a ML-algebra to the reader. One may also combine proofs of similar results from [29] and [42].) A *filtered ML-frame* is a structure $\mathcal{F} = \langle \mathcal{G}, F\rangle$ such that $\mathcal{G}$ is a ML-frame and F is a filter in $\mathcal{G}^+$. A *pointed ML-frame* is a structure $\langle \mathcal{G}, x\rangle$ such that $\mathcal{G}$ is a ML-frame and $x \in g$. We identify $\langle \mathcal{G}, x\rangle$ with the filtered ML-frame $\langle \mathcal{G}, \{a \in \mathcal{G}^+ : x \in a\}\rangle$ and we identify a ML-frame $\mathcal{G}$ with the filtered ML-frame $\langle \mathcal{G}, \{g\}\rangle$. The ML-matrix corresponding to a filtered frame $\mathcal{F} = \langle \mathcal{G}, F\rangle$ is $\mathcal{F}^+ = \langle \mathcal{G}^+, F\rangle$. Filtered frames form a semantics for modal HB-logics in the standard way: A *valuation* in a filtered frame is a homomorphism from $\mathcal{L}_{\mathbf{ML}}$ into $\mathcal{G}^+$. Now all the semantic notation can be translated from matrices to filtered frames, e.g., a formula is *valid* in $\langle \mathcal{G}, F\rangle$ iff it is valid in $\langle \mathcal{G}^+, F\rangle$. We are going to show that all modal HB-logics are determined by filtered frames. To this end we asscociate with each ML-algebra $\mathcal{A}$ its *Stone representation* $\mathcal{A}_+ = \langle A_+, \lhd, R_\Box, R_\Diamond, \beta[A]\rangle$, where A_+ denotes the set of prime filters in $\mathcal{A}$ and, for $X, Y \in A_+$,

$$X \lhd Y \Leftrightarrow X \subseteq Y,$$

$$XR_\Box Y \Leftrightarrow (\forall a \in A)(\Box a \in X \Rightarrow a \in Y),$$

$$XR_\Diamond Y \Leftrightarrow (\forall a \in A)(a \in Y \Rightarrow \Diamond a \in X),$$

[9] It is possible to work with frames satisfying only these weaker conditions on the connection between $\lhd$, $R_\Box$ and $R_\Diamond$. However, for the investigation of embeddings into tense logics with modal operators the stronger conditions (3) and (4) will be quite useful.

$$\beta[A] = \{\beta(a) : a \in A\}, \text{ where } \beta(a) = \{X : a \in X\}.$$

We leave it to the reader to prove that $\mathcal{A}_+$ is a ML-frame and that $(\mathcal{A}_+)^+ \simeq \mathcal{A}$. (The proof is standard by using Lemma 2. The reader may also combine proofs of similar results from [19], [7], [42], [29].) Clearly the converse, i.e. $(\mathcal{G}^+)_+ \simeq \mathcal{G}$, only holds for a special class of frames: We call - following [42] - a ML-frame $\mathcal{G}$ *descriptive* iff

$$(\forall x, y)(x \lhd y \Leftrightarrow (\forall a \in A)(x \in a \Rightarrow y \in a)),$$

$$(\forall x, y)(xR_\Box y \Leftrightarrow (\forall a \in A)(x \in \Box a \Rightarrow y \in a)),$$

$$(\forall x, y)(xR_\Diamond y \Leftrightarrow (\forall a \in A)(y \in a \Rightarrow x \in \Diamond a)),$$

and for every $\mathcal{X} \subseteq A$ and $\mathcal{Y} \subseteq \{g - a : a \in A\}$ we have $\bigcap(\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$ whenever $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property.

Let us also put, for each ML-matrix $\mathcal{M} = \langle \mathcal{A}, F\rangle$, $\mathcal{M}_+ := \langle \mathcal{A}_+, \{\beta(a) : a \in F\}\rangle$. The proof of the following theorem is left to the reader who may consult again [19], [7], [42] for proofs of similar results.

Theorem 5 *$(\mathcal{M}_+)^+ \simeq \mathcal{M}$, for all ML-matrices $\mathcal{M}$. $(\mathcal{F}^+)_+ \simeq \mathcal{F}$ iff $\mathcal{F}$ is descriptive, for all filtered ML-frames $\mathcal{F}$. (Here $\langle \mathcal{G}, F\rangle$ is called descriptive iff $\mathcal{G}$ is descriptive.)*

We are in a position now to state the completeness result for Kripke-semantics.

Corollary 6 *(1) Each modal HB-logic is determined by a class of filtered descriptive ML-frames.*

(2) Each modal HB-logic is determined by a class of pointed descriptive ML-frames.

(3) A modal HB-logic is normal iff it is determined by a class of ML-frames.

Proof. (1) Consider a modal HB-logic Λ. Λ is determined by a class of modal HB-matrices M, by Theorem 3 (1). Hence, by Theorem 5, Λ is determined by $M_+ = \{\mathcal{M}_+ : \mathcal{M} \in M\}$.

(2) Λ is even determined by a class of pointed ML-algebras M', by Theorem 3 (2). Certainly $\mathcal{M}_+$ is a descriptive pointed ML-frame whenever $\mathcal{M}$ is a pointed ML-algebra. Thus, by Theorem 5, $(M')_+ = \{\mathcal{M}_+ : \mathcal{M} \in M'\}$ is as required.

(3) Left to the reader. $\dashv$

We call a normal modal HB-logic *complete* iff it is determined by its full filtered frames. It is time to introduce some examples. Certainly **HB** is the logic determined by all full HB-frames and **ML** is the logic determined by all full ML-frames.

1.1 The logic **Tree** $:= \mathbf{HB} \oplus \neg((p \stackrel{\smile}{\rightarrow} q) \wedge (q \stackrel{\smile}{\rightarrow} p))$ is determined by the class of full frames $\langle g, \lhd \rangle$ which are trees.[10]

Notice that **Tree** is a *conservative extension* of **Int**, i.e., $\mathbf{Tree} \cap \mathcal{L}_{\mathbf{Int}} = \mathbf{Int}$, since **Int** is determined by the class of trees. It seems that for the semantic interpretations of **Int** in terms of information structures (cf. [25]) as well as for the interpretation as the logic of scientific research (cf. [21]) the extension of **Int** to **Tree** is more natural than the one given by **HB**.

1.2 The logic **LIN** $:= \mathbf{Tree} \oplus (p \rightarrow q) \vee (q \rightarrow p)$ is determined by the class of linear orderings.

1.3 $\mathbf{HB}^C := \mathbf{HB} + p \vee \neg p$ is the logic determined by the pointed frames $\langle \langle g, \lhd \rangle, x \rangle$ in which x is $\lhd$-final, i.e. $\{y : x \lhd y\} = \{x\}$.

Maybe this logic can be interpreted by extending Grzegorczyk's interpretation [21] of **Int** to the situation in which scientific research has come to an end.

1.4 The logic $\mathbf{HB} \oplus p \vee \neg p$ coincides with classical logic.

The last two examples show the difference between logics defined by $\oplus$ (i.e. taking closure also under $p/\neg\stackrel{\smile}{\neg}p$) and logics defined by using $+$.

1.5 The *normal Heyting-Brouwer-extension* Λ^B of an intermediate logic Λ is the smallest normal super-HB-logic containing Λ. For a class of Heyting-algebras M denote by M^B the class of all those HB-algebras whose reducts without the dual relative pseudocomplement are in M. It is easily shown that $\Lambda^B = \mathsf{Th}(\{\mathcal{A} : \mathcal{A} \models \Lambda\}^B)$. It is an interesting open problem whether Λ^B is always a conservative extension of Λ. However, note that this is true for complete intermediate logics Λ because for each Heyting algebra of the form $\langle g, \lhd \rangle^h \in M$ we have $\langle g, \lhd \rangle^+ \in M^B$.

1.6 Call a logic Λ formulated in the language $\mathcal{L}_{\Box\Diamond}$ of intuitionistic logic with two new modal operators $\Box$ and $\Diamond$ a *normal intermediate modal logic* if it contains **Int**, $\mathbf{F}^\Box$, $\mathbf{F}^\Diamond$, and is closed under ($\mathrm{RC}_\Box$) and ($\mathrm{RC}_\Diamond$). We denote the smallest normal intermediate logic by $\mathbf{IntK}_{\Box\Diamond}$. Normal intermediate modal logics have been investigated in e.g. [7], [28], [15], [41], [42], [39]. Omitting the closure conditions concerning $\stackrel{\smile}{\rightarrow}$ for A, ML-frames $\langle g, \lhd, R_\Box, R_\Diamond, A \rangle$ form a complete semantics for those logics (cf. [42]). Now we denote by Λ^B the smallest modal HB-logic containing a normal intermediate modal logic Λ. Again for complete Λ it is easily seen that Λ^B is a conservative extension of Λ. One may prove completeness results for Λ^B for a number of interesting systems Λ, e.g., the systems introduced by Fischer-Servi (see [15], [16]). We note only that the logics $\mathbf{ML} \oplus \Diamond^n \Box^m p \rightarrow \Box^k \Diamond^l p$, for $n, m, l, k \in \omega$, are determined by the full ML-frames satisfying

$$(xR^n_\Diamond y \wedge xR^k_\Box z) \Rightarrow (\exists v)(yR^m_\Box v \wedge zR^l_\Diamond v).$$

[10] For the proof the following observation is useful: For a formula $\varphi \in \mathcal{L}_{\mathbf{Int}}$ denote by $\varphi^d \in \mathcal{L}_{\mathbf{HB}}$ the formula which results when $\rightarrow$, $\wedge$, and $\vee$ are replaced in φ by $\stackrel{\smile}{\rightarrow}$, $\vee$ and $\wedge$, respectively. It is readily checked that if $\mathbf{HB} \oplus \varphi$ is determined by a class of frames M, then $\mathbf{HB} \oplus \neg\varphi^d$ is determined by $\{\langle g, \lhd^{-1} \rangle : \langle g, \lhd \rangle \in M\}$. Now it is well-known that $\mathbf{HB} \oplus (p \rightarrow q) \vee (q \rightarrow p)$ is determined by the class of converse trees.

(Here, as usual $\Box^n$ denotes a string of n boxes. The same applies to $\Diamond^n$ and R^n.) This can be proved similarly to the proof of correponding results in [42].

5 Duality

One of the most important steps in the development of classical modal logic was the introduction of p-morphisms, generated subframes and the observation that they are the duals of the algebraic notions of subalgebras and homomorphisms, respectively. In [42] an analogous result is proved for modal logics based on intuitionistic logic. It turned out, however, that in the intuitionistic case those duals are non-standard and not as natural as in the classical case. We show now that in the presence of duals of relative pseudo-complements we have the canonical and natural concepts again. Let $\mathcal{G} = \langle g, \lhd, R_\Box, R_\Diamond, A\rangle$ be a ML-frame and W a non-empty subset of g such that

$$(\forall x, y)(x \in W \wedge xSy \Rightarrow y \in W) \text{ for } S \in \{\lhd, \lhd^{-1}, R_\Box, R_\Diamond\}. \tag{5}$$

Then certainly

$$\langle W, \lhd\restriction W, R_\Box \restriction W, R_\Diamond \restriction W, \{W \cap a : a \in A\}\rangle$$

is a ML-frame as well and is called a *generated subframe* of $\mathcal{G}$. If $\mathcal{H} = \langle h, \lhd', S_\Box, S_\Diamond, B\rangle$ is another ML-frame then a mapping $f : g \to h$ onto h is called a *p-morphism* iff

$$(\forall x, y \in g)(xRy \Rightarrow f(x)Sf(y)), \tag{6}$$

$$(\forall x \in g, y \in h)(f(x)Sy \Rightarrow (\exists z \in g)(xRy \wedge f(z) = y), \tag{7}$$

for all $(R, S) \in \{(\lhd, \lhd'), (\lhd^{-1}, \lhd'^{-1}), (R_\Box, S_\Box), (R_\Diamond, S_\Diamond)\}$, and such that $f^{-1}b \in A$, for all $b \in B$.

Theorem 7 *Let $\mathcal{G} = \langle g, \lhd, R_\Box, R_\Diamond, A\rangle$ and $\mathcal{H} = \langle h, \lhd', S_\Box, S_\Diamond, B\rangle$ be ML-frames.*

(1) If $\mathcal{H}$ is a generated subframe of $\mathcal{G}$ then the mapping f defined by

$$f(a) = a \cap h, \text{ for all } a \in A,$$

is a homomorphism from $\mathcal{G}^+$ onto $\mathcal{H}^+$.

(2) If $f : \mathcal{G} \to \mathcal{H}$ is a p-morphism, then f^+ defined by

$$f^+b = f^{-1}b, \text{ for all } b \in B,$$

is an embedding of $\mathcal{H}^+$ into $\mathcal{G}^+$.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are ML-algebras.

(3) If f is a homomorphism from $\mathcal{A}$ onto $\mathcal{B}$, then the mapping f_+ defined by

$$f_+X = f^{-1}X, \text{ for all prime filters } X \text{ in } \mathcal{B},$$

is an isomorphism of $\mathcal{B}_+$ onto a generated subframe of $\mathcal{A}_+$.

(4) If $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ then the mapping f defined by

$$f(X) = X \cap B, \text{ for all prime filters } X \text{ in } \mathcal{A},$$

is a p-morphism from $\mathcal{A}_+$ onto $\mathcal{B}_+$.

Proof. The proof is quite similar to proofs of corresponding results in e.g. [42] and [9]. We shall sketch, however, some steps of the proof in order to keep the paper reasonably self-contained. (1) and (2) are easy and left to the reader. (3) and (4) are similar, so we shall check (4) only. Suppose that $\mathcal{A}_+ = \langle g, \lhd, R_\Box, R_\Diamond, A\rangle$ and $\mathcal{B}_+ = \langle h, \lhd', S_\Box, S_\Diamond, B\rangle$ and that $\mathcal{B}$ is a subalgebra of $\mathcal{A}$. Now $f^{-1}b \in A$, for all $b \in B$, is clear and condition (6) is straightforward for all the relations. Hence it remains to prove (7) for $\lhd$, $\lhd^{-1}$, $R_\Box$, and $R_\Diamond$. We start with $\lhd^{-1}$. Suppose that $f(X)(\lhd')^{-1}Y$ for a $Y \in h$ and a $X \in g$. We construct a $Z \in g$ with (a) $f(Z) = Y$ and (b) $X \lhd^{-1} Z$. It suffices to show that

$$F_1 = Y \text{ and } F_2 = \{c \vee d : c \in (A - X), d \in (B - Y)\}$$

satisfy the conditions (i), (ii), (iii) of Lemma 2 because then, by Lemma 2, there exists a $Z \in g$ with $F_1 \subseteq Z$ and $F_2 \cap Z = \emptyset$. It is easily shown that such a Z is as required. Now, (i) and (ii) are obvious and it remains to show (iii). Assume that there are $a \in F_1$ and $b \in F_2$ with $a \leq b$. We may assume that $b = c \vee d$ with $c \in A - X$ and $d \in B - Y$. Hence, by the definition of the dual relative pseudocomplement $d \stackrel{\smile}{\rightarrow} a \leq c$. Now $c \notin X$, hence $d \stackrel{\smile}{\rightarrow} a \notin X$. We know $d \stackrel{\smile}{\rightarrow} a \in B$ and $X \cap B \supseteq Y$. Hence $d \stackrel{\smile}{\rightarrow} a \notin Y$. We also know $d \notin Y$. Note that, in general

Fact. For all prime filters P and all c_1, c_2, c_3, $((c_1 \stackrel{\smile}{\rightarrow} c_2 \notin P \wedge c_1 \notin P) \Rightarrow c_2 \notin P)$.

(We note that this is just the dual statement to the well-known fact that prime filters are "closed under modus ponens".) Applying this fact we conclude that $a \notin Y$, which is a contradiction.

Let us now consider $R_\Diamond$. Suppose that $f(X)S_\Diamond Y$ for a $Y \in h$ an $X \in g$. We construct a $Z \in g$ with (a) $f(Z) = Y$ and (b) $XR_\Diamond Z$. Again it suffices to show that

$$F_1 = Y \text{ and } F_2 = \{c \vee d : c \in A \wedge \Diamond c \notin X, d \in (B - Y)\}$$

satisfy the conditions (i), (ii), (iii) of Lemma 2 because then a $Z \in g$ with $F_1 \subseteq Z$ and $F_2 \cap Z = \emptyset$ is as required. Again, (i) and (ii) are obvious and we show (iii). Assume that there are $a \in F_1$ and $b \in F_2$ with $a \leq b$ and that $b = c \vee d$ with $\Diamond c \notin X$ and $d \in B - Y$. By the definition of the dual relative pseudocomplement we conclude $d \stackrel{\smile}{\rightarrow} a \leq c$. By monotonicity of $\Diamond$ we get $\Diamond(d \stackrel{\smile}{\rightarrow} a) \leq \Diamond c$. From $\Diamond c \notin X$ we get $\Diamond(d \stackrel{\smile}{\rightarrow} a) \notin X$. This means, since $f(X)S_\Diamond Y$, $d \stackrel{\smile}{\rightarrow} a \notin Y$. We also know that $d \notin Y$ and get from the fact above that $a \notin Y$. We have derived a contradiction. The cases $\lhd$ and $R_\Box$ are similar and left to the reader. $\dashv$

6 Tense Logics

We shall introduce the logics into which super-HB-logics and modal HB-logics shall be embedded. But first we require some notation and basic results on polymodal quasi-normal logics (consult e.g [33], [5] and [9] for information on monomodal quasi-normal logics). Proofs will be omitted since they are straightforward extensions of the monomodal case. We introduce the notation with some care, however, since we do not know of any investigation of polymodal quasi-normal logic. Denote the propositional modal language with n modal operators $\bigcirc_1, \ldots, \bigcirc_n$ by $\mathcal{L} = \mathcal{L}_{\{\bigcirc_1,\ldots\bigcirc_n\}}$. We shall fix this language for this section. A *normal modal logic* (*nm-logic*, for short) is a $\mathcal{L}$-logic containing classical propositional logic, the axioms $\mathbf{F}^{\Box}$, formulated for all the operators $\bigcirc_1, \ldots, \bigcirc_n$, and closed under $(\mathrm{RN}_{\bigcirc_i})$ $p/\bigcirc_i p$, for all $1 \leq i \leq n$. The smallest nm-logic containing a nm-logic Λ and a set of formulas Γ is denoted by $\Lambda \oplus \Gamma$ and the smallest nm-logic in the language $\mathcal{L}$ is denoted by $\mathbf{K}_{\{\bigcirc_1,\ldots,\bigcirc_n\}}$. nm-logics Λ_1 and Λ_2 formulated in languages $\mathcal{L}_1, \mathcal{L}_2$ with different modal operators are combined by forming the *fusion* $\Lambda_1 \otimes \Lambda_2$, i.e., the smallest nm-logic in the language $\mathcal{L}_1 \cup \mathcal{L}_2$ containing $\Lambda_1 \cup \Lambda_2$. (See [24] for an investigation of the properties of fusions.)

For a set $S \subseteq \{\bigcirc_1, \ldots \bigcirc_n\}$ we call a $\mathcal{L}$-logic Λ a *S-normal modal logic* (*S-logic*, for short) iff Λ contains $\mathbf{K}_{\{\bigcirc_1\ldots,\bigcirc_n\}}$ and is closed under $(\mathrm{RN}_{\bigcirc_i})$, for all $\bigcirc_i \in S$. The smallest S-logic containing a S-logic Λ and a set of formulas Γ is denoted by $\Lambda +_S \Gamma$. The following proposition generalizes a well-known characterization from monomodal logic (cf. [32]).

Proposition 8 *Suppose that Λ is a nm-logic in the language $\mathcal{L}_{\bigcirc_1,\ldots,\bigcirc_n}$ and $\Gamma \subseteq \mathcal{L}_{\bigcirc_1,\ldots,\bigcirc_n}$ such that for all $\varphi \in \Gamma$ and $\bigcirc_i$, $1 \leq i \leq n$, there exists $\psi \in \Gamma$ such that $\psi \to \bigcirc_i\varphi \in \Lambda$. Then $\Lambda +_S \Gamma$ is normal, for all $S \subseteq \{\bigcirc_1, \ldots, \bigcirc_n\}$.*

Proof. Certainly it suffices to prove the Proposition for $S = \emptyset$. Recall that $\varphi \in \Lambda +_\emptyset \Gamma$ iff φ is derivable from $\Lambda \cup \Gamma$ by using modus ponens and substitutions. Hence closure of $\Lambda +_\emptyset \Gamma$ under the rules $(\mathrm{RN}_{\bigcirc_i})$, $1 \leq i \leq n$, follows inductively from:

1. $\varphi \in \Lambda \cup \Gamma \Rightarrow \bigcirc_i\varphi \in \Lambda +_\emptyset \Gamma$, for all $1 \leq i \leq n$.
2. $\bigcirc_i\varphi, \bigcirc_i(\varphi \to \psi) \in \Lambda +_\emptyset \Gamma \Rightarrow \bigcirc_i\psi \in \Lambda +_\emptyset \Gamma$, for all $1 \leq i \leq n$.
3. $\bigcirc_i\varphi \in \Lambda +_\emptyset \Gamma \Rightarrow s(\bigcirc_i\varphi) \in \Lambda +_\emptyset \Gamma$, for all substitutions s and $1 \leq i \leq n$.

(2.) and (3.) are clear. Condition (1.) is trivial for $\varphi \in \Lambda$. Suppose now $\varphi \in \Gamma$ and $1 \leq i \leq n$. Then there exists $\psi \in \Gamma$ such that $\psi \to \bigcirc_i\varphi \in \Lambda$. Hence $\bigcirc_i\varphi \in \Lambda +_\emptyset \Gamma$. $\dashv$

A modal $\mathcal{L}$-algebra is an algebra $\mathcal{A} = \langle A, \wedge, \vee, -, \bigcirc_1, \ldots, \bigcirc_n, \bot, \top\rangle$ such that the reduct $\langle A, \wedge, \vee, -, \bot, \top\rangle$ is a boolean algebra and so that

$$\bigcirc_i(a \wedge b) = \bigcirc_i a \wedge \bigcirc_i b, \;\; \bigcirc_i \top = \top,$$

for all $1 \leq i \leq n$. For $S \subseteq \{\bigcirc_1, \dots \bigcirc_n\}$, we call a matrix $\mathcal{M} = \langle \mathcal{A}, F \rangle$ a *S-matrix* iff F is a boolean filter which is closed under $a/\bigcirc_i a$, for all $\bigcirc_i \in S$. If F is an ultrafilter, then we call $\langle \mathcal{A}, F \rangle$ a *pointed $\mathcal{L}$-algebra* and if $F = \{\top\}$, then we call $\langle \mathcal{A}, F \rangle$ a *normal matrix* (which we shall often identify with the algebra $\mathcal{A}$). Generalising the notation from [33] we put, for a S-logic Λ,

$$\mathsf{Ker}\Lambda = \{\varphi : s\varphi \in \Lambda \text{ for all sequences } s \in \{\bigcirc_1, \dots, \bigcirc_n\}^*\}.$$

and realize that $\mathsf{Ker}\Lambda$ is the maximal normal logic contained in Λ. For a modal $\mathcal{L}$-algebra $\mathcal{A}$ and two ultrafilters X and Y in $\mathcal{A}$ we put

$$XR_iY \Leftrightarrow (\forall a \in A)(\bigcirc_i a \in X \Rightarrow a \in Y).$$

Now we have the following result on algebraic semantics for S-logics.

Theorem 9 *(1) Each S-logic Λ is determined by a class of S-matrices $\langle \mathcal{A}, D \rangle$ such that $\mathcal{A} \models \mathsf{Ker}\Lambda$. (2) A modal logic Λ is a S-logic iff it is determined by a class M of pointed algebras $\langle \mathcal{A}, F \rangle$ satisfying $\mathcal{A} \models \mathsf{Ker}\Lambda$ and, for all $1 \leq i \leq n$:*

$$(\langle \mathcal{A}, X \rangle \in M \wedge XR_iY \wedge \bigcirc_i \in S) \Rightarrow \langle \mathcal{A}, Y \rangle \in M.$$

(3) A S-logic is normal iff it is determined by normal matrices.

Proof. The proof is similar to the proof of Theorem 3, see also [5] or [9] for proofs in the monomodal case. $\dashv$

On the other hand we have Kripke-type semantics for S-logics. Recall that a $\mathcal{L}$-frame is a structure $\mathcal{G} = \langle g, R_1, \dots, R_n, A \rangle$ such that A is closed under intersection and complements and

$$\bigcirc_i a := \{y \in g : (\forall x \in g)(yR_ix \Rightarrow x \in a)\}.$$

With $\mathcal{G}$ we can associate the $\mathcal{L}$-algebra $\mathcal{G}^+ = \langle A, \cap, \cup, -, \bigcirc_1, \dots, \bigcirc_n, \emptyset, g \rangle$. Conversely, for each classical $\mathcal{L}$-algebra $\mathcal{A}$ we find a *descriptive* $\mathcal{L}$-frame $\mathcal{A}_+$ such that $\mathcal{A} \simeq (\mathcal{A}_+)^+$. Following [19] we call a $\mathcal{L}$-frame $\mathcal{G} = \langle g, R_1, \dots, R_n, A \rangle$ *descriptive* iff

$$x = y \Leftrightarrow (\forall a \in A)(x \in a \Leftrightarrow y \in a),$$

$$xR_iy \Leftrightarrow (\forall a \in A)(x \in \bigcirc_i a \Rightarrow y \in a),$$

and if $\bigcap U = \{x\}$, for some x, for each ultrafilter U in A. One can show that a $\mathcal{L}$-frame $\mathcal{G}$ is descriptive iff $\mathcal{G} \simeq (\mathcal{G}^+)_+$. A *S-frame* is a pair $\langle \mathcal{G}, F \rangle$ such that $\langle \mathcal{G}^+, F \rangle$ is a S-matrix. A pair $\langle \mathcal{G}, x \rangle$ with $x \in g$ is a *pointed frame*[11]. We certainly have the following reformulation of Theorem 9 in terms of frames.

Theorem 10 *(1) Each S-logic Λ is determined by a class of descriptive S-frames $\langle \mathcal{G}, D \rangle$ such that $\mathcal{G} \models \mathsf{Ker}\Lambda$. (2) A logic Λ is a S-logic iff it is determined by a class M of descriptive pointed frames $\langle \mathcal{G}, x \rangle$ satisfying $\mathcal{G} \models \mathsf{Ker}\Lambda$ and, for $1 \leq i \leq n$:*

$$(\langle \mathcal{G}, x \rangle \in M \wedge xR_iy \wedge \bigcirc_i \in S) \Rightarrow \langle \mathcal{G}, y \rangle \in M.$$

(3) A S-logic is normal iff it is determined by frames.

[11] Again, we identify the pointed frame $\langle \mathcal{G}, x \rangle$ with the filtered frame $\langle \mathcal{G}, \{a \in \mathcal{G}^+ : x \in a\} \rangle$.

We are in a position now to introduce the logics into which the logics with coimplication shall be embedded. The language of tense logic is $\mathcal{L}_{\{G,H\}}$ and the minimal tense logic we deal with in this paper is

$$\mathbf{S4}.t := \mathbf{K}_{\{G,H\}} \oplus p \to GPp \oplus p \to HFp \oplus Gp \to p \oplus Gp \to GGp.$$

(Here and in what follows $F := \neg G \neg$ and $P := \neg H \neg$.) The $\emptyset$-modal logics containing $\mathbf{S4}.t$ will be called *tense logics*. We note that this notion of a tense logic is not standard. Mostly in the literature the condition $Gp \to p$ (corresponding to reflexivity of the frames) is omitted and only normal logics are considered (cf. [2], [6], [20], [38]). A $\mathcal{L}_{\{G,H\}}$-algebra validating $\mathbf{S4}.t$ will be called a *tense algebra*[12]. Let us call an element a of the form $a = Gb$ *G-open* and let us call an element a of the form $a = Pb$ *P-closed*. Then it is readily checked that tense algebras are precisely those $\mathcal{L}_{\{G,H\}}$-algebras in which G is an interior operator, P is a closure operator, and in which the G-open elements are precisely the P-closed ones.[13]

A S-matrix $\langle \mathcal{A}, F \rangle$, $S \subseteq \{G, H\}$, in which $\mathcal{A}$ is a tense algebra will be called a *S-tense matrix*. We note that in all $\{G\}$-tense matrices we have $Ga \in F \Leftrightarrow a \in F$, for all $a \in A$. (This follows from $Ga \leq a$, for all $a \in A$.) This means that F is determined by its G-closed elements. Let us now call a frame $\langle g, R, R^{-1}, A \rangle$ a *tense frame* iff R is a quasi-ordering[14]. It follows immediately from Theorem 10 that tense logics Λ are determined by classes of descriptive pointed tense frames $\langle \mathcal{G}, x \rangle$ such that $\mathcal{G} \models \mathsf{Ker}\Lambda$.

Recall now the Grzegorzcyk-axiom

$$\mathbf{grz} = \Box(\Box(p \to \Box p) \to p) \to p.$$

The logic $\mathbf{Grz} = \mathbf{K}_{\{\Box\}} \oplus \mathbf{grz}$ is known to be the maximal normal modal logic into which $\mathbf{Int}$ is embedded by Gödels interpretation (cf. [8]) and plays a major role in investigations on the relation between extensions of $\mathbf{S4}$ and intermediate logics (cf. e.g. [8], [4], [11].) An analogous role will be played here by the tense logic

$$\mathbf{Grz}.t := \mathbf{S4}.t \oplus \mathbf{grz}^G \oplus \mathbf{grz}^H.$$

Here, for a monomodal formula φ we denote by $\varphi^{\bigcirc}$ the translation of φ into the language with the operator $\bigcirc$. Notice that $\mathbf{Grz}$ is the logic determined by the finite partial orderings (cf. e.g. [13]) and that the same is true for $\mathbf{Grz}.t$, see Corollary 27 below.

Tense logics will interpret super-HB-logics. To interpret modal HB-logics we need two more modal operators and obtain the language $\mathcal{L}_{\mathbf{TL}} = \mathcal{L}_{\{G,H,\Box_1,\Box_2\}}$. The *basic extended tense logic* is $\mathbf{TL} := \mathbf{S4}.t \otimes \mathbf{K}_{\{\Box_1,\Box_2\}}$ and, for $S \subseteq \{G, H, \Box_1, \Box_2\}$, an *extended S-tense logic* is a $\mathcal{L}_{\mathbf{TL}}$-logic containing $\mathbf{TL}$ which is closed under $p/\bigcirc p$, for $\bigcirc \in S$. $\mathcal{L}_{\mathbf{TL}}$-algebras in which the reduct without $\Box_1$ and $\Box_2$ is a tense algebra will be called *TL-algebras* and

[12] Some authors call tense algebras bi-topological boolean algebras, e.g. Rauszer in [29].

[13] An operator C is called an interior operator iff $Ca \leq a$, $Ca = CCa$, and $C(a \wedge b) = Ca \wedge Cb$, for all a, b. The formulation for closure operators is dual.

[14] Consult [35] for the first introduction and investigation of tense frames. There full tense frames were called second order frames and tense frames were called first order frames.

$\{G\}$-matrices based on TL-algebras will be called *TL-matrices.* Correspondingly, we call frames $\langle g, R, R^{-1}, R_1, R_2, A\rangle$ in which R is a quasi-ordering *TL-frames.* In TL-matrices there is no connection between the tense operators and the modal operators. In order to simulate the conditions (3) and (4) for $R_\Box$ and $R_\Diamond$ on the classical side we introduce

$$\mathbf{mix} = \{G\Box_1 Gp \leftrightarrow \Box_1 p,\ P\Diamond_2 Pp \leftrightarrow \Diamond_2 p\}$$

and put

$$\mathbf{Mix} = \mathbf{TL} \oplus \mathbf{mix},\ \ \mathbf{GMix} = \mathbf{Grz}.t \oplus \mathbf{mix}.$$

One can easily prove

Proposition 11 *The full frames $\langle g, R, R^{-1}, R_1, R_2\rangle$ validating* **Mix** *are precisely those full TL-frames in which the following versions of (3) and (4) hold.*

$$R \circ R_1 \circ R = R_1 \ \textit{and}\ R^{-1} \circ R_2 \circ R^{-1} = R_2.$$

The full frames validating **GMix** *are precisely the* **Mix***-frames without infinite R-chains.*

7 Embeddings

We extend the Gödel translation of $\mathcal{L}_{\mathbf{Int}}$ into $\mathcal{L}_{\{G\}}$ to a translation t from $\mathcal{L}_{\mathbf{ML}}$ into $\mathcal{L}_{\mathbf{TL}}$ as follows:

$$\begin{array}{rcl} t(p) & = & Gp \\ t(\varphi \circ \psi) & = & t(\varphi) \circ t(\psi),\ \text{for}\ \circ \in \{\wedge, \vee\} \\ t(\varphi \to \psi) & = & G(t(\varphi) \to t(\psi)) \\ t(\varphi \stackrel{\smile}{\to} \psi) & = & P(\neg t(\varphi) \wedge t(\psi)) \\ t(\Box\varphi) & = & G\Box_1 t(\varphi) \\ t(\Diamond\varphi) & = & P\Diamond_2 t(\varphi) \end{array}$$

Note that everything we shall show below for this mapping t holds also if we manipulate t by putting $t'(p) = Fp$, or $t'(\varphi \circ \psi) = G(t(\varphi) \circ t(\psi))$, or $t'(\varphi \circ \psi) = P(t(\varphi) \circ t(\psi))$, for $\circ \in \{\wedge, \vee\}$. This follows immediately from the following easily proved but important observation.

Lemma 12 *For all $\varphi \in \mathcal{L}_{\mathbf{ML}}$,*

$$t(\varphi) \leftrightarrow Gt(\varphi) \in \mathbf{TL}\ \textit{and}\ t(\varphi) \leftrightarrow Pt(\varphi) \in \mathbf{TL}.$$

Proof. By an easy induction using the fact that $Gp \leftrightarrow PGp \in \mathbf{TL}$ and $Pp \leftrightarrow GPp \in \mathbf{TL}$. $\dashv$

In order to analyse the translation t we shall transform ML-matrices into TL-matrices and vice versa. For a TL-algebra $\mathcal{B} = \langle B, \wedge, \vee, G, H, \Box_1, \Box_2, \bot, \top\rangle$ define another TL-algebra $\mathcal{B}^{\mathbf{mix}}$ by putting

$$\mathcal{B}^{\mathbf{mix}} = \langle B, \wedge, \vee, G, H, G\Box_1 G, H\Box_2 H, \bot, \top\rangle.$$

It is easy to show that $\mathcal{B}^{\mathbf{mix}}$ is a TL-algebra as well. Moreover

Lemma 13 *(1)* $\mathcal{B}^{\mathbf{mix}} \models \mathbf{mix}$.

(2) $\langle \mathcal{B}, F\rangle \models t(\varphi) \Leftrightarrow \langle \mathcal{B}^{\mathbf{mix}}, F\rangle \models t(\varphi)$*, for each TL-matrix* $\langle \mathcal{B}, F\rangle$ *and* $\varphi \in \mathcal{L}_{\mathbf{ML}}$.

Proof. (1) follows immediately from the condition that both G and H are interior operators. (2) Let V be a valuation of $\mathcal{B}$ and V' be a valuation of $\mathcal{B}^{\mathbf{mix}}$ so that V and V' coincide on the propositional variables. It can be shown by induction that $V(t(\varphi)) = V'(t(\varphi))$, for all $\varphi \in \mathcal{L}_{\mathbf{ML}}$. (2) follows immediately. $\dashv$

Remark 1 At the level of TL-frames $\mathcal{G} = \langle g, R, R^{-1}, R_1, R_2, A\rangle$, forming $\mathcal{B}^{\mathbf{mix}}$ corresponds to the operation $\mathcal{G} \mapsto \mathcal{G}^{\mathbf{mix}}$ defined by

$$\mathcal{G}^{\mathbf{mix}} = \langle g, R, R^{-1}, R \circ R_1 \circ R, R^{-1} \circ R_2 \circ R^{-1}, A\rangle,$$

since it is easily shown that $(\mathcal{G}^+)^{\mathbf{mix}} = (\mathcal{G}^{\mathbf{mix}})^+$.

Now consider a TL-algebra $\mathcal{B} = \langle B, \wedge, \vee, G, H, \Box_1, \Box_2, \bot, \top\rangle$ validating **mix**. We define a ML-algebra $\rho\mathcal{B}$ by putting

$$\rho\mathcal{B} := \langle \rho B, \wedge, \vee, \to, \stackrel{\smile}{\to}, \Box, \Diamond, \bot, \top\rangle,$$

where

$$a \to b := G(-a \vee b), \quad a \stackrel{\smile}{\to} b := P(-a \wedge b),$$
$$\Box b := \Box_1 b, \quad \Diamond b := \Diamond_2 b,$$

and $\rho B := \{Gb : b \in B\}$. In other words, ρB is the set of all G-open sets (or equivalently, the set of all P-closed sets). Thus $a \to b$ and $a \stackrel{\smile}{\to} b$ are well defined. Also, since $\mathcal{B} \models \mathbf{mix}$, we know that $\Box b, \Diamond b \in \rho B$. Now it is easily shown that $\rho\mathcal{B}$ is a ML-algebra. For an arbitrary TL-algebra $\mathcal{B}$ let $\rho\mathcal{B} := \rho(\mathcal{B}^{\mathbf{mix}})$ and for a TL-matrix $\mathcal{M} = \langle \mathcal{B}, F\rangle$ let $\rho\mathcal{M} = \langle \rho\mathcal{B}, \rho F\rangle$, where $\rho F := \{Gb : b \in F\}$. Certainly ρF is a filter and so $\rho\mathcal{M}$ is a ML-matrix.

Remark 2 At the level of TL-frames $\mathcal{G} = \langle g, R, R^{-1}, R_1, R_2, A\rangle$ validating **mix** the operation ρ corresponds to the following construction. Define an equivalence relation $\sim$ on g by putting $x \sim y$ iff xRy and yRx. Now form the quotient frame

$$\mathcal{F} = \langle g/\sim, R/\sim, R^{-1}/\sim, R_1/\sim, R_2/\sim, A/\sim\rangle.$$

From $\mathcal{F}$ we form the ML-frame $\rho\mathcal{G} = \langle g/\sim, \lhd, R_\Box, R_\Diamond, \{Ga : a \in A/\sim\}\rangle$, where $\lhd = R/\sim$, $R_\Box = R_1/\sim$, $R_\Diamond = R_2/\sim$. Then one can show $(\rho\mathcal{G})^+ \simeq \rho(\mathcal{G}^+)$.

For an extended $\{G\}$-tense logic (a $\{G\}$-tense logic) Λ we call the modal HB-logic (the super-HB-logic)

$$\rho\Lambda := \{\varphi \in \mathcal{L}_{\mathbf{ML}} : t(\varphi) \in \Lambda\}$$

the *modal HB-fragment* (*HB-fragment*) of Λ. Conversely, we say that an extended $\{G\}$-tense logic (a $\{G\}$-tense logic) Λ is a *companion* of a modal HB-logic (super-HB-logic) Θ iff $\rho\Lambda = \Theta$.

Lemma 14 *(1) For all TL-matrices $\mathcal{M}$ and all $\varphi \in \mathcal{L}_{\mathbf{ML}}$, $\mathcal{M} \models t(\varphi) \Leftrightarrow \rho\mathcal{M} \models \varphi$.*

(2) If an extended $\{G\}$-tense logic (a $\{G\}$-tense logic) Λ is determined by a class of TL-matrices $\mathcal{K}$ then $\rho\Lambda$ is determined by the class $\rho\mathcal{K} = \{\rho\mathcal{M} : \mathcal{M} \in \mathcal{K}\}$.

(3) If an extended $\{G\}$-tense logic (a $\{G\}$-tense logic) Λ is complete, then $\rho\Lambda$ is complete.

(4) If Λ is a normal (extended) tense logic, then $\rho\Lambda$ is a normal logic.

Proof. (1) is easily proved by induction on the subformulas of φ by using Lemma 13. (One may also use Remark 2.) (2) Suppose that Λ is determined by $\mathcal{K}$. We have $\rho\mathcal{M} \models \varphi$, for all $\varphi \in \rho\Lambda$ and $\mathcal{M} \in \mathcal{K}$, by (1). Conversely, suppose that $\varphi \notin \rho\Lambda$. Then $t(\varphi) \notin \Lambda$. Hence there exists $\mathcal{M} \in \mathcal{K}$ with $\mathcal{M} \not\models t(\varphi)$. Thus, by (1), $\rho\mathcal{M} \not\models \varphi$. (3) By (2) it suffices to show that for each full filtered TL-frame $\langle \mathcal{G}, F\rangle$ there exists a full filtered ML-frame $\langle \mathcal{H}, G\rangle$ such that $\rho\langle \mathcal{G}, F\rangle \simeq \langle \mathcal{H}^+, G\rangle$. But this is the contents of Remark 1 and Remark 2 above. (4) follows from (2), Theorem 3 (3), Theorem 9 (3), and the fact that $\rho F = \{\top\}$ whenever $F = \{\top\}$. $\dashv$

Corollary 15 *The mapping ρ preserves completeness, the finite model property and decidability.*

Having defined a mapping ρ from the class of TL-matrices into the class of ML-matrices we are now going to define a mapping σ in the opposite direction. We shall need the notion of a *free Boolean extension*. Suppose that $\mathcal{A}$ is a Boolean algebra and D is a subset of A. The Boolean algebra generated by D in $\mathcal{A}$ is denoted by $[D]_{BL}$. For a bounded distributive lattice $\mathcal{D} = \langle D, \wedge, \vee, \bot, \top\rangle$ there always exists a (uniquely determined) Boolean algebra $\mathcal{A} = \langle A, \vee, \wedge, -, \bot, \top\rangle$ such that $[D]_{BL} = \mathcal{A}$ and such that, for each homomorphism (for the signature $\vee, \wedge, \bot$ and $\top$) $f : \mathcal{D} \to \mathcal{B}$, $\mathcal{B}$ a Boolean algebra, there exists a unique Boolean homomorphism $h : \mathcal{A} \to \mathcal{B}$ with $h \restriction D = f$. (Consult [1] for more information.) We denote the set A by σD.

Remark 3 Clearly, for each $a \in \sigma D$ there are $a_i, b_i, c_i, d_i \in A$, $1 \leq i \leq n$, such that $a = \bigwedge_{i=1}^{n}(-a_i \vee b_i) = \bigvee_{i=1}^{n}(-c_i \wedge d_i)$.

Consider a ML-algebra $\mathcal{A} = \langle A, \wedge, \vee, \to, \breve{\to}, \Box, \Diamond, \bot, \top\rangle$ and take the free Boolean extension $\sigma\langle A, \wedge, \vee, \bot, \top\rangle = \langle \sigma A, \wedge, \vee, -, \bot, \top\rangle$ of the distributive lattice $\langle A, \wedge, \vee, \bot, \top\rangle$. We extend this Boolean algebra to a TL-algebra $\sigma\mathcal{A}$ by defining the operations $\Box_1, \Diamond_2, G$, and P as follows: Let $a \in \sigma A$ and take $a_i, b_i, c_i, d_i \in A$, $1 \leq i \leq n$, such that

$$a = \bigwedge_{i=1}^{n} (-a_i \vee b_i) = \bigvee_{i=1}^{n} (-c_i \wedge d_i).$$

Now we put

$$Ga := \bigwedge_{i=1}^{n} (a_i \rightarrow b_i), \quad Pa := \bigvee_{i=1}^{n} (c_i \breve{\rightarrow} d_i).$$

$$\Box_1 a = \Box Ga, \quad \Diamond_2 a = \Diamond Pa.$$

The operators G as well as P are well defined. For G this follows immediately from the easily proved fact that $\bigwedge_{i=1}^{n}(a_i \rightarrow b_i) \not\leq a \rightarrow b$ implies $\bigwedge_{i=1}^{n}(-a_i \vee b_i) \not\leq -a \vee b$, for all $a_i, b_i, a, b \in A$. The argument for P is dual. If $\mathcal{M} = \langle \mathcal{A}, F \rangle$ is a modal matrix then we put $\sigma\mathcal{M} = \langle \sigma\mathcal{A}, \sigma F \rangle$, where $\sigma F = \{b \in \sigma A : Gb \in F\}$.

Remark 4 The construction of $\sigma\mathcal{A}$ corresponds to the following operation on ML-frames $\mathcal{G} = \langle g, \lhd, R_\Box, R_\Diamond, A \rangle$. Namely, for $\sigma\mathcal{G} = \langle g, R, R^{-1}, R_1, R_2, \sigma A \rangle$ such that $R = \lhd$, $R_1 = R_\Box$ and $R_2 = R_\Diamond$ and $\sigma A = \{\bigcap_{i=1}^{n}(-a_i \cup b_i) : a_i, b_i \in A, n \in \omega\}$ it follows easily that $(\sigma\mathcal{G})^+ \simeq \sigma(\mathcal{G}^+)$.

Lemma 16 *Let $\mathcal{M} = \langle \mathcal{A}, F \rangle$ be a ML-matrix and $\mathcal{M}'$ be a TL-matrix.*

(1) $\sigma\mathcal{M}$ is a TL-matrix. Moreover, $\sigma\mathcal{M}$ is the only TL-matrix such that $\rho\sigma\mathcal{M} \simeq \mathcal{M}$.

*(2) $\sigma\mathcal{A} \models$ **GMix**.*

(3) $\sigma\rho\mathcal{M}' \in \mathsf{R}\mathcal{M}'$.

(4) $\mathcal{M} \models \varphi \Leftrightarrow \sigma\mathcal{M} \models t(\varphi)$, for all $\varphi \in \mathcal{L}_{\mathbf{ML}}$.

Proof. The proof is a straightforward combination of known proofs from [29] and [42]. The reader may, however, easily prove this herself by using Remark 4. $\dashv$

For each modal HB-logic $\Theta = \mathbf{ML} + \Gamma$ put $\sigma\Theta := \mathbf{GMix} +_{\{G\}} t(\Gamma)$ and $\tau\Theta := \mathbf{TL} +_{\{G\}} t(\Gamma)$. Both τ and σ are well-defined, by Lemma 16. Correspondingly, for each super-HB-logic $\Theta = \mathbf{HB} + \Gamma$ put $\sigma\Theta := \mathbf{Grz}.t +_{\{G\}} t(\Gamma)$ and $\tau\Theta := \mathbf{S4}.t +_{\{G\}} t(\Gamma)$.

Lemma 17 *If Θ is a normal super-HB-logic, then both $\tau\Theta$ as well as $\sigma\Theta$ are normal tense logics. Correspondingly for normal modal HB-logics.*

Proof. We prove the second part. Suppose that Θ is normal. Then $\tau\Theta = \mathbf{TL} +_{\{G\}} t(\Theta)$ and $\sigma\Theta = \mathbf{GMix}.t +_{\{G\}} t(\Theta)$. By Proposition 8, it suffices to show that for each $\psi \in t(\Theta)$ and $\bigcirc \in \{G, H, \Box_1, \Box_2\}$ there exists a $\psi' \in t(\Theta)$ such that $\psi' \rightarrow \bigcirc\psi \in \mathbf{TL}$. For

$\bigcirc = G$ this follows from Lemma 12. For the other three operators observe that $\neg\breve{\neg}\varphi$, $\Box\varphi$, $\neg\Diamond\breve{\neg}\varphi \in \Theta$ whenever $\varphi \in \Theta$. Hence $GHGt(\varphi)$, $G\Box_1 Gt(\varphi)$, $GH\Box_2 HGt(\varphi) \in t(\Theta)$ whenever $t(\varphi) \in t(\Theta)$. Moreover,

$$GHGp \to Hp,\ G\Box_1 Gp \to \Box_1 p,\ GH\Box_2 HGp \to \Box_2 p \in \mathbf{TL}.$$

$\dashv$

It follows that for each representation $\Theta = \mathbf{ML} \oplus \Gamma$ we have $\tau\Theta = \mathbf{TL} \oplus t(\Gamma)$ and $\sigma\Theta = \mathbf{GMix} \oplus t(\Gamma)$.

Theorem 18 *Suppose that Θ is a modal HB-logic (super-HB-logic). Then $\rho\Lambda = \Theta$, for each extended $\{G\}$-tense logic ($\{G\}$-tense logic) Λ in the interval $[\tau\Theta, \sigma\Theta]$.*

Proof. Assume $\Theta = \mathbf{ML} + \Gamma$. Suppose that $\mathcal{M} \models \tau\Theta$. Then $\mathcal{M} \models t(\Gamma)$. Hence, by Lemma 14 (1), $\rho\mathcal{M} \models \Gamma$. It follows that $\rho\mathcal{M} \models \Theta$. Conversely, suppose that $\mathcal{M} \models \Theta$. Then $\sigma\mathcal{M} \models \sigma\Theta$ (by Lemma 16 (2) and (4)) and $\rho\sigma\mathcal{M} \simeq \mathcal{M}$ (by Lemma 16 (1)). Thus Θ is determined by $\{\rho\mathcal{M} : \mathcal{M} \models \Lambda\}$, for each $\Lambda \in [\tau\Theta, \sigma\Theta]$. The Theorem follows with Lemma 14. $\dashv$

The most important consequence of this Theorem is that each normal modal HB-logic can be embedded in a *natural* way into a *normal* classical modal logic with four modal operators. So we have indeed (modulo the Blok-Esakia-Theorem to be proved below) a reduction of modal HB-logics to classical modal logics which belong to the mainstream of research on modal logic. This is far from true for modal logics based on intuitionistic logic with a $\Diamond$-like operator (called IM-logics in what follows). It is proved in [42] that IM-logics are embedded into classical modal logics in a natural way. Those classical modal logics are, however, not quasi-normal but only monotonic (i.e., we do not have $\Diamond(p \vee q) \to (\Diamond p \vee \Diamond q)$ since the interpretation of $\Diamond$ cannot be forced to be *possibility*-like). This is, we believe, not only a technical obstacle for the embeddings introduced in [42] but also of philosophical interest. We proceed with some examples.

2.1 We certainly have $\rho\mathbf{S4}.t = \rho\mathbf{Grz}.t = \mathbf{HB}$ and $\rho\mathbf{TL} = \rho\mathbf{Mix} = \rho\mathbf{GMix} = \mathbf{ML}$.

2.2 $\rho(\mathbf{S4}.t +_{\{G\}} Gp \leftrightarrow p) = \mathbf{HB}^C$, see Example 1.2.

2.3 For each set of formulas $\Gamma \subseteq \mathcal{L}_{\mathbf{HB}}$ we have $t(\Gamma) \subseteq \mathcal{L}_{\{G,H\}}$. Now, for $\mathbf{S4}.t \subseteq \Theta \subseteq \mathbf{Grz}.t$,

$$\rho((\Theta \oplus t(\Gamma)) \otimes \mathbf{K}_{\{\Box_1\Box_2\}}) = \mathbf{ML} \oplus \Gamma.$$

$$\rho((\Theta \oplus t(\Gamma)) \otimes (\mathbf{K}_{\{\Box_1\}} \oplus \Box_1 p \to p) \otimes \mathbf{K}_{\{\Box_2\}}) = \mathbf{ML} \oplus \Gamma \oplus \Box p \to p.$$

$$\rho((\Theta \oplus t(\Gamma)) \otimes \mathbf{K}_{\{\Box_1\}} \otimes (\mathbf{K}_{\{\Box_2\}} \oplus \Box_2 p \to p)) = \mathbf{ML} \oplus \Gamma \oplus p \to \Diamond p.$$

We show the third part. Put $\Lambda = (\Theta \oplus t(\Gamma)) \otimes \mathbf{K}_{\{\Box_1\}} \otimes (\mathbf{K}_{\{\Box_2\}} \oplus \Box_2 p \to p)$. By Theorem 18 it suffices to show that

$$\mathbf{TL} \oplus t(\Gamma) \oplus t(p \to \Diamond p) \subseteq \Lambda \subseteq \mathbf{GMix} \oplus t(\Gamma) \oplus t(p \to \Diamond p).$$

Now $t(p \to \Diamond p)$ is, modulo **TL**, deductively equivalent with $Pp \to P\Diamond_2 Pp$ and, modulo **Mix**, it is deductively equivalent with $Pp \to \Diamond_2 p$. The inclusions follow immediately.

2.4 Define a mapping ρ_G from the lattice of monomodal nm-logics containing **S4** (formulated in the language $\mathcal{L}_{\{G\}}$) into the lattice of intermediate logics by putting $\rho_G\Lambda = \{\varphi \in \mathcal{L}_{\mathbf{Int}} : t(\varphi) \in \Lambda\}$ (For information on ρ_G consult the survey [8], where ρ_G is denoted by ρ.) Also, define the *minimal tense extension* $\Lambda.t$ of a normal extension Λ of **S4** to be the smallest normal tense logic containing Λ. The mapping $\Lambda \mapsto \Lambda.t$ has been investigated in e.g. [38] and [40]. Now we have, for all normal logics containing **S4**,

$$\rho(\Lambda.t) = (\rho_G\Lambda)^B.$$

(See Example 1.5 for the definition of $(-)^B$.) The proof of this observation is left to the reader. We derive e.g. that $\rho\mathbf{S5}.t$ coincides with classical logic and that $\rho\mathbf{S4.3}.t = \mathbf{HB} \oplus (p \to q) \vee (q \to p)$ (by using that $\rho_G\mathbf{S5}$ is classical logic and that $\rho_G\mathbf{S4.3} = \mathbf{Int} + (p \to q) \vee (q \to p)$, cf. [8]). We leave it to the reader to compute more examples.

2.5 We have

$$\rho(\mathbf{Mix} \oplus \Diamond_2^n \Box_1{}^m p \to \Box_1{}^k \Diamond_2^l p) = \mathbf{ML} \oplus \Diamond^n \Box^m p \to \Box^k \Diamond^l p,$$

for all $m, n, l, k \in \omega$. The (easy) proof is left to the reader.

8 A Blok-Esakia-Theorem

Independently, Blok [4] and Esakia [11] have proved the fundamental result that the lattice of normal modal logics containing **Grz** and the lattice of intermediate logics are isomorphic (cf. also [8]). Here we shall prove an analogous result for logics with coimplication and tense logics.

Proposition 19 *Let $\mathcal{A}$ be a ML-algebra, $\mathcal{B}$ be a TL-algebra validating* **mix** *and suppose that $f : \mathcal{A} \to \rho\mathcal{B}$ is a $\mathcal{L}_{\mathbf{ML}}$-homomorphism. Then there exists a (uniquely determined) $\mathcal{L}_{\mathbf{TL}}$-homomorphism $h : \sigma\mathcal{A} \to \mathcal{B}$ with $h \restriction A = f$.*

Proof. There exists a unique Boolean homomorphism $h : \sigma\mathcal{A} \to \mathcal{B}$ extending f, by the definition of the free Boolean extension. It remains to show for $a \in \sigma A$

$$h(\bigcirc a) = \bigcirc h(a), \text{ for } \bigcirc \in \{\Box_1, \Diamond_2, G, P\}.$$

We show this for P and $\Diamond_2$. Take $a_i, b_i \in A$, $1 \leq i \leq n$, such that $a = \bigvee_{i=1}^n (-a_i \wedge b_i)$ and compute as follows:

$$h(Pa) = f(P \bigvee_{i=1}^n (-a_i \wedge b_i)) \quad = \quad f(\bigvee_{i=1}^n (a_i \stackrel{\smile}{\to} b_i))$$

$$\begin{aligned} &= \bigvee_{i=1}^{n} (f(a_i) \dot{\to} f(b_i)) \\ &= P \bigvee_{i=1}^{n} (-h(a_i) \wedge h(b_i)) \;=\; Ph(a). \end{aligned}$$

For the following computation for $\Diamond_2$ note that we use $h(Pa) = Ph(a)$.

$$\begin{aligned} h(\Diamond_2 a) = f(\Diamond Pa) &= \Diamond f(Pa) \\ &= \Diamond h(Pa) \\ &= \Diamond Ph(a) = \Diamond_2 h(a). \end{aligned}$$

⊣

We denote by ML the class of ML-matrices and by TL the class of TL-matrices. The following result characterizes definable classes of ML-algebras by means of closure conditions with respect to operators. Similar results are well-known from the literature on matrices (cf. [37], [5],[8]). We sketch the proof, however, since the result is crucial for the proof of the Blok-Esakia-Theorem.

Theorem 20 *Suppose that $M \subseteq \mathsf{ML}$ and $\Lambda = \mathsf{Th}M$. Then $M = \mathsf{Mat}_{\mathsf{ML}}\Lambda$ iff M is closed under P, R_{ML}, and H_{ML}.*

Proof. Only one directions remains to be shown, by Proposition 1. Suppose that $\mathcal{M} = \langle \mathcal{A}, D \rangle \models \Lambda$ and that M is closed under P, R_{ML}, and H_{ML}. We show that $\mathcal{M} \in M$. Take a set X of cardinality $\max\{|A|, \aleph_0\}$ and denote by $\mathsf{Fr}(X)$ the free modal HB-algebra with free generating set X. Take a homomorphism f from $\mathsf{Fr}(X)$ onto $\mathcal{A}$ and put $E := f^{-1}D$. Clearly we have $\mathcal{M} \in \mathsf{H}_{\mathsf{ML}}\mathcal{M}'$, for $\mathcal{M}' = \langle \mathsf{Fr}(X), E \rangle$ and we are done if $\mathcal{M}' \in M$. To this end let $\mathcal{I}$ be a set of matrices which contains an isomorphic copy of each matrix in M of cardinality $\leq \max\{|A|, \aleph_0\}$. For each matrix $\langle \mathcal{B}, P_{\mathcal{B}} \rangle \in M$ and each mapping $h : X \to B$ denote by $\mathcal{B}[h]$ the algebra generated by $\{h(x) : x \in X\}$ in $\mathcal{B}$. We have $\langle \mathcal{B}[h], P_{\mathcal{B}} \cap B[h] \rangle \in \mathsf{R}_{\mathsf{ML}}M$, for all such matrices $\langle \mathcal{B}, P_{\mathcal{B}} \rangle$ and mappings h. Denote by T the collection of all those h. Then

$$\langle \mathcal{H}, D' \rangle = \prod \langle \langle \mathcal{B}[h], P_{\mathcal{B}} \cap B[h] \rangle : h \in T \rangle$$

belongs to M. Define a homomorphism $g : \mathsf{Fr}(X) \to \mathcal{H}$ by putting

$$g(x) := \langle h(x) : h \in T \rangle, \text{ for } x \in X.$$

It remains to show that $g^{-1}D' \subseteq E$, for then $\mathcal{M}' \in \mathsf{R}_{\mathsf{ML}}\langle \mathcal{H}, D' \rangle$, by Proposition 1. Let $g(a) \in D'$, for some $a \in \mathsf{Fr}(X)$. There exists a formula $\varphi \in \mathcal{L}_{\mathbf{ML}}$ and $x_1, \ldots, x_k \in X$ with $a = \varphi(x_1, \ldots, x_k)$. Now, by the definition of $\langle \mathcal{H}, D' \rangle$, $g(a) \in D'$ means that

$$\varphi(h(x_1), \ldots, h(x_n)) \in P_{\mathcal{B}} \cap B[h],$$

for all $h \in T$, which is equivalent with $\varphi \in \Lambda$. But then $f(\varphi(x_1, \ldots, x_k)) \in D$ since $\mathcal{M} \models \Lambda$. This means that $a = \varphi(x_1, \ldots, x_n) \in E$. ⊣

Theorem 21 *Suppose that Λ is an extended $\{G\}$-tense logic and $M = \mathsf{Mat}_{\mathsf{TL}}\Lambda$. Then ρM is closed under the operations H_{ML}, P, and R_{ML}. Thus $\mathsf{Mat}_{\mathsf{ML}}\rho\Lambda = \rho M$.*

Proof. The second part follows from the first part with Theorem 20 and Lemma 14 (2). So it remains to show the closure conditions for ρM. Closure of ρM under P is obvious. For closure under homomorphic images assume that $\langle \mathcal{A}, D\rangle \in M$ and that $\langle \mathcal{B}, E\rangle \in \mathsf{ML}$ such that there is a homomorphism f from $\rho\mathcal{A}$ onto $\mathcal{B}$ with $E \supseteq f[\rho D]$. By Proposition 19 there exists a unique homomorphism h from $\sigma\rho\mathcal{A}$ onto $\sigma\mathcal{B}$ such that $h \restriction \rho A = f$. We show that $\sigma E \supseteq h[\sigma\rho D]$. Let $a \in \sigma\rho D$. Then $Ga \in \sigma\rho D$ and so $Ga \in \rho D$. We know $f[\rho D] \subseteq E$ and derive

$$Gh(a) = h(Ga) = f(Ga) \in E.$$

So $Gh(a) \in \sigma E$ and therefore $h(a) \in \sigma E$. Hence $\sigma\langle \mathcal{B}, E\rangle$ is a homomorphic image of $\sigma\rho\langle \mathcal{A}, D\rangle$. We have $\sigma\rho\langle \mathcal{A}, D\rangle \in \mathsf{R}\langle \mathcal{A}, D\rangle$, by Lemma 16. Hence $\sigma\rho\langle \mathcal{A}, D\rangle \in M$ and we conclude that $\sigma\langle \mathcal{B}, E\rangle \in M$. Now $\langle \mathcal{B}, E\rangle \in \rho M$ follows from $\rho\sigma\langle \mathcal{B}, E\rangle \simeq \langle \mathcal{B}, E\rangle$.

For closure under R_{ML} assume that $\langle \mathcal{A}, D\rangle \in M$ and that $\langle \mathcal{B}, E\rangle$ is a ML-matrix such that f is a homomorphism from $\mathcal{B}$ into $\rho\mathcal{A}$ with $f^{-1}\rho D \subseteq E$. By Proposition 19 there is a unique homomorphism h from $\sigma\mathcal{B}$ into $\mathcal{A}$ with $h \restriction A = f$. We show that $h^{-1}D \subseteq \sigma E$. Suppose that $c \in h^{-1}D$. From $h(c) \in D$ we conclude $Gh(c) \in \rho D$. We have $f(Gc) = h(Gc) = Gh(c)$. It follows $Gc \in E$ because $f^{-1}\rho D \subseteq E$. But then $c \in \sigma E$. We have shown that $\sigma\langle \mathcal{B}, E\rangle \in \mathsf{R}\langle \mathcal{A}, D\rangle$. So $\sigma\langle \mathcal{B}, E\rangle \in M$ and $\langle \mathcal{B}, E\rangle \simeq \rho\sigma\langle \mathcal{B}, E\rangle \in \rho M$. $\dashv$

Theorem 22 *Suppose that $\mathcal{M} = \langle \mathcal{A}, D\rangle$ is a TL-matrix with $\mathcal{A} \models$ **GMix**.t. Then*

$$\mathsf{Th}\mathcal{M} = \mathsf{Th}\sigma\rho\mathcal{M}.$$

Proof. By Lemma 16 (3) $\sigma\rho\mathcal{M} \in \mathsf{R}\mathcal{M}$. Thus it suffices to show that $\mathsf{Th}\mathcal{M} \supseteq \mathsf{Th}\sigma\rho\mathcal{M}$. First we observe that it suffices to prove this for finitely generated matrices $\mathcal{M}$. (We call a matrix $\langle \mathcal{B}, E\rangle$ *finitely generated* if the algebra $\mathcal{A}$ is finitely generated.) For suppose that there is a matrix $\mathcal{M} = \langle \mathcal{A}, D\rangle$ such that $\mathsf{Th}\mathcal{M} \not\supseteq \mathsf{Th}\sigma\rho\mathcal{M}$. There exist $\varphi \in \mathsf{Th}\sigma\rho\mathcal{M}$ and a valuation V of $\mathcal{M}$ with $V(\varphi) \notin D$. Denote by $\mathcal{A}'$ the subalgebra of $\mathcal{A}$ generated by $\{V(\psi) : \psi \text{ is a subformula of } \varphi\}$ and put $D' = D \cap A'$. Then $V'(\varphi) \notin D'$, for the restriction V' of V to the variables in φ. Thus $\langle \mathcal{A}', D'\rangle \not\models \varphi$ but $\varphi \in \mathsf{Th}\sigma\rho\langle \mathcal{A}', D'\rangle$ since $\sigma\rho\langle \mathcal{A}', D'\rangle \in \mathsf{R}\sigma\rho\mathcal{M}$. So we shall restrict attention to finitely generated matrices.

Claim. Suppose that $\langle \mathcal{A}, D\rangle$ is finitely generated and assume that $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ such that $\rho A \subseteq B$ and such that $\mathcal{A} = [B \cup \{c\}]_{BL}$, for a c in A. Then $\langle \mathcal{A}, D\rangle \in \mathsf{RP}_{\mathsf{U}}\langle \mathcal{B}, D \cap B\rangle$.

Proof. We extend the proof of Lemma 7.6 in [4]. Some notation is needed. Enumerate the elements of B by $B = \{b_0, b_1, \ldots\}$. Let U be a non-principal ultrafilter on ω. Put, for $g \in \prod_{i=0}^{\omega} B$, $[g] := \{g' : g \sim_U g'\}$. Also put, for $b \in B$, $\hat{b} = \langle b, b, \ldots\rangle \in \prod_{i=0}^{\omega} B$. Now it is well known that the mapping

$$f : \langle \mathcal{B}, D \cap B\rangle \to \prod_{i=0}^{\omega}\langle \mathcal{B}, D \cap B\rangle/U, \text{ defined by putting } f(b) = [\hat{b}],$$

is a matrix-embedding, i.e. f is a 1-1 homomorphism from the algebra $\mathcal{B}$ into $\prod_{i=0}^{\omega} \mathcal{B}/U$ and $b \in D \cap B \Leftrightarrow [\hat{b}] \in \prod_{i=0}^{\omega} D \cap B/U$, for all $b \in B$. Put

$$\hat{c} := \langle \bigvee \langle b_i : b_i \leq c,\ i \leq n \rangle \rangle_{n=0}^{\omega}.$$

It is proved in Lemma 7.6 of [4] that there exists a (uniquely determined) $\mathcal{L}_{\{G\}}$-homomorphism h from $\mathcal{A}$ into $\prod_{i=0}^{\omega} \mathcal{B}/U$ which extends f and so that $h(c) = [\hat{c}]$. By dualising the proof in [4] it can be shown that $h(Pa) = Ph(a)$, for all $a \in A$, i.e., that h is an $\mathcal{L}_{\{G,H\}}$-homomorphism. We proceed by proving that h is a $\mathcal{L}_{\mathbf{TL}}$-homomorphism. But

$$h(\Box_1 a) = h(G\Box_1 Ga) = f(G\Box_1 Ga) = G\Box_1 f(Ga) = G\Box_1 h(Ga) = G\Box_1 Gh(a) = \Box_1 h(a),$$

since $\rho A \subseteq B$. The proof for $\Diamond_2$ is similar and left to the reader. We now show that $D \supseteq h^{-1}[\prod_{i=0}^{\omega}(B \cap D)/\sim_U]$. Suppose that $h(a) \in \prod_{i=0}^{\omega}(B \cap D)/\sim_U$. Then, since $\prod_{i=0}^{\omega}\langle \mathcal{B}, B \cap D\rangle/U$ is a TL-matrix, $h(Ga) \in \prod_{i=0}^{\omega}(B \cap D)/\sim_U$. Thus $f(Ga) \in \prod_{i=0}^{\omega}(B \cap D)/\sim_U$. It follows that $Ga \in B \cap D$ because f is a matrix-embedding. Hence $a \in D$. We have proved that $\langle \mathcal{A}, D\rangle \in \mathsf{R}\prod_{i=0}^{\omega}\langle \mathcal{B}, D \cap B\rangle/U$.

Now assume that $\mathcal{M} = \langle \mathcal{A}, D\rangle$ and $\mathcal{A}$ is finitely generated by $\{a_1, \ldots, a_k\}$. Define a sequence of subalgebras of $\mathcal{A}$ by putting $\mathcal{A}_0 = \sigma\rho\mathcal{A}$,

$$\mathcal{A}_{i+1} = [A_i \cup \{a_{i+1}\}]_{BL},$$

for $0 \leq i < k$. We show that $\mathsf{Th}\langle \mathcal{A}_i, A_i \cap D\rangle = \mathsf{Th}\langle \mathcal{A}_{i+1}, A_{i+1} \cap D\rangle$, for $0 \leq i < k$, from which we get

$$\mathsf{Th}\sigma\rho\mathcal{M} = \mathsf{Th}\langle \mathcal{A}_0, A_0 \cap D\rangle = \mathsf{Th}\langle \mathcal{A}_k, A_k \cap D\rangle = \mathsf{Th}\mathcal{M}.$$

Certainly it suffices to show that $\langle \mathcal{A}_{i+1}, A_{i+1} \cap D\rangle \in \mathsf{RP_U}\langle \mathcal{A}_i, A_i \cap D\rangle$. But this is the contents of the claim above. $\dashv$

Theorem 23 *(1) σ is an isomorphism from the lattice of modal HB-logics onto the lattice of extended $\{G\}$-tense logics containing* **GMix**. *(2) The restriction of σ to the lattice of normal modal HB-logics is an isomorphism onto the lattice of normal extended tense logics containing* **GMix**. *(3) The restriction of σ to the lattice of super-HB-logics is an isomorphism onto the lattice of $\{G\}$-tense logics containing* **Grz**.*t*. *(4) The restriction of σ to the lattice of normal super-HB-logics is an isomorphism onto the lattice of normal tense logics containing* **Grz**.*t*.

Proof. (1) We show that σ is onto. The other conditions are easy and left to the reader. Certainly it suffices to show that $\sigma\rho\Lambda = \Lambda$, for all extended $\{G\}$-tense logics Λ containing **GMix**. We have $\sigma\rho\Lambda \subseteq \Lambda$, by definition. Conversely, suppose that $\Lambda \not\subseteq \sigma\rho\Lambda$. There exists an $\mathcal{M} \in \mathsf{Mat_{TL}}\sigma\rho\Lambda$ such that $\mathcal{M} \not\models \Lambda$. We have $\rho\mathcal{M} \models \rho\sigma\rho\Lambda$ and $\rho\sigma\rho\Lambda = \rho\Lambda$. By Theorem 21 there exists $\mathcal{M}' \in \mathsf{TL}$ with $\rho\mathcal{M}' \simeq \rho\mathcal{M}$ and $\mathcal{M}' \models \Lambda$. But then $\sigma\rho\mathcal{M}' \simeq \sigma\rho\mathcal{M}$ and therefore $\sigma\rho\mathcal{M} \models \Lambda$. Hence, by Theorem 22, $\mathcal{M} \models \Lambda$. We have a contradiction. (2) follows with Lemma 17. (3) and (4) are proved analogously. $\dashv$

The following result follows immediately from the proof above.

Theorem 24 *If a modal HB-logic Λ is determined by a class of matrices M, then $\sigma\Lambda$ is determined by $\sigma M = \{\sigma\mathcal{M} : \mathcal{M} \in M\}$. Hence σ reflects and preserves the finite model property.*

9 Applications

We are now going to list a number of results on super-HB-logics and modal-HB-logics which follow from known results in tense logic and polymodal logic by using the embedding studied above[15]. The list is far from complete and we encourage the reader to transfer more results from e.g. [38] and [40].

Theorem 25 *Suppose that $\Gamma \subseteq \mathcal{L}_{\mathbf{Int}}$ is a set of disjunction free formulas. Then $\mathbf{HB} \oplus \Gamma$ has the finite model property. $\mathbf{HB} \oplus \Gamma$ is a conservative extension of $\mathbf{Int} + \Gamma$.*

Proof. Suppose that $\Gamma \subseteq \mathcal{L}_{\mathbf{Int}}$ is disjunction free. Then, by a result of [43], $\mathbf{S4} \oplus t(\Gamma)$ is a cofinal subframe logic whose frames form a first order definable class. It is proved in [38], that in this case the minimal tense extension $(\mathbf{S4} \oplus t(\Gamma)).t$ of $\mathbf{S4} \oplus t(\Gamma)$ also has the finite model property. Now $\rho(\mathbf{S4} \oplus t(\Gamma)) = \mathbf{HB} \oplus \Gamma$ and ρ preserves the finite model property. Hence $\mathbf{HB} \oplus \Gamma$ has the finite model property. $\mathbf{HB} \oplus \Gamma$ is a conservative extension of $\mathbf{Int} + \Gamma$ since $\mathbf{Int} + \Gamma$ is complete. $\dashv$

Theorem 26 *Define $wd_n = \bigvee\langle p_i \to \bigvee\langle p_j : j \neq i\rangle : 0 \leq i \leq n\rangle$. Then all logics of the form $\mathbf{HB} \oplus wd_n \oplus \Gamma$ with $\Gamma \subseteq \mathcal{L}_{\mathbf{Int}}$ are complete. $\mathbf{HB} \oplus wd_n \oplus \Gamma$ is a conservative extension of $\mathbf{Int} + wd_n + \Gamma$.*

Proof. It suffices to show that all minimal tense extensions of normal modal logic containing $\mathbf{S4} \oplus t(wd_n)$ are complete (because ρ preserves completeness). But $\mathbf{K4} \oplus t(wd_n)$ coincides with the logic of width n from [12] and it is shown in [40] that all minimal tense extensions of logics of finite width are complete. The second part follows from the completeness of all logics of the form $\mathbf{Int} + wd_n + \Gamma$, which follows from the completeness of all logics of the form $\mathbf{K4} \oplus t(wd_n) \oplus \Gamma$ (cf. [12]). $\dashv$

Corollary 27 $\mathbf{Grz}.t$ *has the finite model property.*

[15] We shall always transfer from tense logic and polymodal logic to super-HB-logic and modal HB-logic, respectively. There is, however, at least one mathematically interesting transfer result in the other direction: It is readily checked that the lattice of congruences of an HB-algebra $\mathcal{A}$ is isomorphic to the lattice of congruences of $\sigma\mathcal{A}$. Especially, $\mathcal{A}$ is simple iff $\sigma\mathcal{A}$ is simple and $\mathcal{A}$ is subdirectly irreducible iff $\sigma\mathcal{A}$ is subdirectly irreducible. Now, in [22], the non-trivial result is shown that there exists a subdirectly irreducible HB-algebra which is not simple. So we get the non-trivial and interesting result that there exists a tense algebra which is subdirectly irreducible but not simple.

Proof. **HB** has the finite model property, by Theorem 25. So the finite model property of **Grz**.t follows from σ**HB** = **Grz**.t and the fact that σ preserves the finite model property, see Theorem 24. $\dashv$

Theorem 28 *All tense logics containing* **TLin** = **Grz**.$t \oplus$ **lin**$^G \oplus$ **lin**^{H} *have the finite model property. Here* **lin** $= \Box(\Box p \to q) \vee \Box(\Box q \to p)$.

Proof. Suppose that $\Lambda \supseteq$ **TLin** and that $\varphi \notin \Lambda$. Certainly there is a pointed descriptive frame $\langle \mathcal{G}, x\rangle = \langle\langle g, R, R^{-1}, A\rangle, x\rangle$ validating Λ such that $\mathcal{G} \models$ **Grz**.t and so that $\langle g, R\rangle$ is connected (i.e. $(\forall x, y)(xRy \vee yRx)$) such that there is a valuation V with $x \notin V(\varphi)$. Now call, for $a \in A$ and $S \in \{R, R^{-1}\}$, a point $y \in a$ *S-maximal in a* iff there does not exist a proper S-successor of x which is in a. The following fact is shown in [38].

Fact. (1) For all non-empty $a \in A$ and $S \in \{R, R^{-1}\}$ there exists a S-maximal $z \in a$. (2) If y is S-maximal in a for an $a \in A$ and an $S \in \{R, R^{-1}\}$, then $\{z : zRy \wedge yRz\} = \{y\}$, i.e., the cluster containing y consists only of y.

We take, for each subformula ψ of φ with $V(\psi) \neq \emptyset$ the S-maximal $z \in V(\psi)$ for $S \in \{R, R^{-1}\}$ and denote the set of all those z together with x by W. Consider the finite pointed frame $\langle \mathcal{H}, x\rangle := \langle\langle W, R \restriction W, R^{-1} \restriction W\rangle, x\rangle$ and define a valuation V' of this frame by putting $V'(p) = V(p) \cap W$, for all propositional variables p. Then it follows by induction from (1) of the fact above that $V'(\psi) = V(\psi) \cap W$, for all subformulas ψ of φ. Hence $x \notin V'(\varphi)$. On the other hand it is easily proved (by using (2) of the fact above) that there is a p-morphism f from $\mathcal{G}$ onto $\mathcal{H}$ with $f(x) = x$. It follows that the logic determined by $\langle \mathcal{H}, x\rangle$ contains the logic determined by $\langle \mathcal{G}, x\rangle$. Hence $\langle \mathcal{H}, x\rangle$ validates Λ and refutes φ. $\dashv$

It is of some interest to note that there are indeed a lot of logics containing **TLin** which are not normal. **TLin** $+_{\{G\}} Gp \leftrightarrow p$ is an example. So the situation is different from monomodal logic where it is known that there are no non-normal logics containing **S4.3** (cf. [34]).

Corollary 29 *All super-HB-logics containing* **LIN** *have the finite model property.*

Proof. By Theorem 28, all logics containing **TLin** have the finite model property. Now $\sigma\Lambda \supseteq$ **TLin**, for all $\Lambda \supseteq$ **LIN**. So all extensions of **LIN** have the finite model property since σ reflects the finite model property. $\dashv$

Theorem 30 *Suppose that* **HB** $\oplus\, \Gamma$ *has the finite model property,* $\Gamma \subseteq \mathcal{L}_{HB}$. *Then also the logics* **ML** $\oplus\, \Gamma$, **ML** $\oplus\, \Gamma \oplus \Box p \to p$, **ML** $\oplus\, \Gamma \oplus p \to \Diamond p$ *have the finite model property.*

Proof. Fix a logic Θ = **HB** $\oplus\, \Gamma$ with the finite model property. Then the logic $\sigma\Theta$ = **Grz**.$t \oplus \Gamma$ has the finite model property, by Theorem 24. By example 2.3 above we know that

$$\rho(\sigma(\Theta) \otimes \mathbf{K}_{\{\Box_1\Box_2\}}) = \mathbf{ML} \oplus \Gamma.$$

$$\rho(\sigma(\Theta) \otimes (\mathbf{K}_{\{\Box_1\}} \oplus \Box_1 p \to p) \otimes \mathbf{K}_{\{\Box_2\}}) = \mathbf{ML} \oplus \Gamma \oplus \Box p \to p.$$

$$\rho(\sigma(\Theta) \otimes \mathbf{K}_{\{\Box_1\}} \otimes (\mathbf{K}_{\{\Box_2\}} \oplus \Box_2 p \to p)) = \mathbf{ML} \oplus \Gamma \oplus p \to \Diamond p.$$

Certainly the logics $\mathbf{K}_{\{\Box\}}$ and $\mathbf{K}_{\{\Box\}} \oplus \Box p \to p$ have the finite model property. Thus, for all the logics Λ of the theorem there always is a fusion $\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3$ such that Λ_i, $1 \leq i \leq 3$, has the finite model property and so that $\rho(\Lambda_1 \otimes \Lambda_2 \otimes \Lambda_3) = \Lambda$. Now the finite model property of Λ follows from the fact that the finite model property is preserved under ρ and under forming fusion (cf. [24], [14]). $\dashv$

It is interesting to notice that we also obtain results on the finite model property of modal logics based on intermediate logics. Indeed, if $\Lambda = \mathbf{IntK}_{\Box\Diamond} \oplus \Gamma$, $\Gamma \subseteq \mathcal{L}_{\mathbf{Int}}$ is complete, then $\Lambda^B = \mathbf{ML} \oplus \Gamma$ is a conservative extension of Λ and obviously Λ has the finite model property whenever Λ^B has the finite model property. Thus, e.g., $\mathbf{IntK}_{\Box\Diamond} \oplus \Gamma$ has the finite model property whenever Γ is disjunction-free.

Finally we note some consequences concerning the structure of the lattice of normal super-HB-logics. Since σ restricted to the lattices of normal super-HB-logics is an isomorphism onto the lattice of normal tense logics containing $\mathbf{Grz}.t$ every result on the lattice of normal tense logics has a straightforward translation for the lattice of normal super-HB-logics. One of the most important concepts in the study of lattices of logics is the notion of a splitting (cf. e.g. [4], [32], [23], [39]). Recall that a finite rooted frame $\mathcal{G}$ (or, equivalently, a finite subdirectly irreducible algebra $\mathcal{A}$) is said to *split* a lattice D of logics iff there exists a smallest logic $\Lambda \in D$ such that $\mathcal{G} \not\models \Lambda$. In this case Λ is called the *splitting-companion* of $\mathcal{G}$. Spittings are basic for studying intermediate logics since all finite rooted $\mathbf{Int}$-frames split the lattice of intermediate logics (cf. e.g. [4], [32]). Splittings of lattices of modal logics based on intermediate logics are studied in [39]. Now the situation for normal super-HB-logics is clarified to some extent by the following result.

Theorem 31 *A finite and rooted HB-frame splits the lattice of normal super-HB-logics iff it coincides with* $\boxed{\bullet}$ *or with* $\boxed{\bullet \rightarrow \bullet}$ *. The splitting companion of* $\boxed{\bullet}$ *is the inconsistent logic and the splitting companion of* $\boxed{\bullet \rightarrow \bullet}$ *is the logic determined by* $\boxed{\bullet}$ *. Thus the lattice of normal super-HB-logics contains precisely one coatom*[16]*, namely the logic determined by* $\boxed{\bullet}$ *and it contains precisely one logic of codimension 2, namely the logic determined by* $\boxed{\bullet \rightarrow \bullet}$*. It contains, however, infinitely many logics of codimension 3.*

Proof. All this is shown in [23] for the lattice of normal tense logics containing $\mathbf{Grz}.t$. Now apply σ. $\dashv$

[16] An element d of a lattice D has codimension n, $n \in \omega$, iff n is the length of the maximal strict $\leq$-chain from d to $\top$. Elements of codimension 1 are called coatoms.

References

[1] R. Balbes & Ph. Dwinger. Distributive Lattices. University of Missouri Press, 1974

[2] J. van Benthem. The logic of time. Reidel, Dordrecht, 1983

[3] J. van Benthem. Temporal Logic, in Gabbay, Hogger, and Robinson (editors), Handbook of Logic in Artificial Intelligence and Logic Programming, Volume 4, pages 241 - 350, 1995

[4] W. Blok. Varieties of interior algebras, Dissertation, University of Amsterdam, 1976

[5] W. Blok & P. Köhler. Algebraic Semantics for quasi-classical modal logics, Journal of Symbolic Logic 48: 941 - 964, 1983

[6] Basic Tense logic. In D. Gabbay & F. Guenthner, editors, Handbook of Philosophical Logic, volume 2, pages 89 - 133, 1984

[7] M. Bosic & K. Došen. Models for normal intuitionistic modal logics. Studia Logica, 43: 217 - 245, 1984

[8] A. V. Chagrov & M. V. Zakharyaschev. Modal companions of intermediate propositional logics, Studia Logica 51: 49 - 82, 1992

[9] A. V. Chagrov & M. V. Zakharyaschev. Modal and superintuitionistic logics. Oxford University Press, 1996

[10] M. Dummett & E. Lemmon. Modal Logics between S4 and S5, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik 5: 250 - 264, 1959

[11] L. Esakia. On varieties of Grzegorczyk algebras, in Studies in non-classical logics and set theory, pages 257 - 287, Moscow, Nauka, 1979

[12] K. Fine. Logics containing K4, Part I, Journal of Symbolic Logic 39: 229 - 237, 1974

[13] K. Fine. Logics containing K4, Part II, Journal of Symbolic Logic 50: 619 - 651, 1985

[14] K. Fine & G. Schurz. Transfer Theorems for stratified modal logics, in Proceedings of the Arthur Prior Memorial Conference, Christchurch, New Zealand, 1991.

[15] G. Fischer Servi. Semantics for a class of Intuitionistic Modal Calculi, in M. L. Dalla Chiara (ed.), Italian Studies in the Philosophy of Science, Reidel, Dordrecht, 59 - 72, 1980

[16] G. Fischer Servi. Axiomatizations for some Intuitionistic Modal Logics, Rend. Sem. Mat. Univers. Polit. 42: 179 - 194, 1984

[17] J. Font. Modality and Possibility in Some Intuitionistic Modal Logics, Notre Dame Journal of Formal Logic 27: 533 - 546, 1986

[18] K. Gödel. Eine Interpretation des intuitionistischen Aussagenkalküls, Ergebnisse eines mathematischen Kolloquiums 6: 39 - 40, 1933

[19] R. Goldblatt. Metamathematics of Modal Logic, Reports on Mathematical Logic 6: 41 - 78, 1976, 7: 21 - 52, 1976

[20] R. Goldblatt. Logics of Time and Computation, Number 7 in CSLI Lecture Notes, CSLI, 1987

[21] A. Grzegorczyk. A philosophically plausible formal interpretation of intuitionistic logic, Indag. Math. 26: 596 - 601

[22] R. Köhler. A subdirectly irreducible double Heyting algebra which is not simple, Algebra Universalis 10: 189 - 194, 1980

[23] M. Kracht. Even more on the lattice of tense logics, Arch. Math. Logic 31: 243 - 257, 1992

[24] M. Kracht & F. Wolter. Properties of independently axiomatizable bimodal logics, Journal of Symbolic Logic 56: 1469 - 1485, 1991

[25] S. Kripke. A semantical analysis of intuitionistic logic I, in J. Crossley & M. Dummett (eds.) Formal systems and Recursive Functions, North-Holland, Amsterdam, 92 - 129, 1965

[26] M. Makkai & G. Reyes. Completeness results for intuitionistic and modal logics in a categorical setting. Annals of Pure ans Applied Logic 72: 25 - 101, 1995

[27] L. Maksimova & V. Rybakov. Lattices of modal logics, Algebra and Logic 13: 105 - 122, 1974

[28] H. Ono. On some intuitionistic modal logics. Publ. Kyoto University 13, 687 - 722, 1977.

[29] C. Rauszer. Semi-Boolean algebras and their applications to intuitionistic logic with dual operators. Fund. Math. 83: 219 - 249, 1974

[30] C. Rauszer. A formalization of propositional calculus of H-B logic, Studia Logica 33, 1974

[31] C. Rauszer. An algebraic and Kripke-style approach to a certain extension of intuitionistic logic, Dissertationes Mathematicae, vol. CLXVII, Warszawa 1980

[32] W. Rautenberg. Klassische und nichtklassische Aussagenlogik, Wiesbaden, 1979

[33] K. Segerberg. An Essay in Classical Modal Logic, Uppsala 1971

[34] K. Segerberg. That every extension of **S4.3** is normal, in S. Kanger (editor), Proceedings of the Third Scandinavian Logic Symposium, pages 194 - 196, Amsterdam, 1976

[35] S. K. Thomason. Semantic analysis of tense logics, Journal of Symbolic Logic 37: 150 - 158, 1972

[36] A. Troelstra & D. van Dalen. Constructivism in Mathematics, Vol I, North-Holland, Amsterdam 1988

[37] R. Wojcicki. Theory of Logical Calculi, Dordrecht 1988

[38] F. Wolter. The finite model property in tense logic, Journal of Symbolic Logic 60: 757 - 774, 1995

[39] F. Wolter. Superintuitionistic companions of classical modal logics, to appear in Studia Logica, 1996

[40] F. Wolter. Completeness and decidability of tense logics closely related to logics containing **K4**, to appear in Journal of Symbolic Logic, 1996

[41] F. Wolter & M. Zakharyaschev. Intuitionistic Modal Logics as Fragments of Classical Bimodal Logics, in Logic at Work, Essays in honour of H. Rasiowa, forthcoming

[42] F. Wolter & M. Zakharyaschev. On the relation between intuitionistic and classical modal logics, to appear in Algebra and Logic, 1996

[43] M. Zakharyaschev. Canonical Formulas for K4, Part II, to appear in Journal of Symbolic Logic, 1996