On Extension Dimension and [L]–Homotopy

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Abstract

There is a new approach in dimension theory, proposed by A. N. Dranishnikov and based on the concept of extension types of complexes. Following Dranishnikov, for a CW-complex L we introduce the definition of *extension type* $[L]$ of this complex. Further, for a space X we define the notion of *extension dimension* $\mathrm{e}-\dim$ of X, which generalizes both Lebesgue and cohomological dimensions.

An adequate homotopy and shape theories, which are specifically designed to work for at most $[L]$-dimensional spaces, have also been developed. Following A. Chigogidze, we present the concept of $[L]$*-homotopy.* This concept generalizes the concept of standard homotopy as well as of n–homotopy, introduced by R. H. Fox and studied by J.H.C. Whitehead. We also investigate the class of spaces which play a significant role in $[L]$-homotopy theory, namely, *absolute (neighborhood) extensors modulo a complex* (shortly A(N)E($[L]$)–spaces). Observe that A(N)E($[S^n]$)–spaces are precisely A(N)E(n)-spaces. The first result of the present thesis describes A(N)E($[L]$)-spaces in terms of local properties and provides an extension-dimensional version of Dugundji theorem.

Another result of the present work is related to the theory of continuous selections. The finite-dimensional selection theorem of E. Michael is very useful in geometric topology and is one of the central theorems in the theory of continuous selections of multivalued mappings. In the thesis we present the proof of an extension-dimensional version of the finite dimensional selection theorem. This version contains Michael's original finite dimensional theorem

as a special case.

The concept of $[L]$–homotopy naturally leads us to the definition of algebraic $[L]$–homotopy invariants, and, in particular, *[L]–homotopy groups.* We give a detailed description of $[L]$-homotopy groups introduced by Chigogidze.

The notion of *closed model category*, introduced by D. Quillen, gives an axiomatic approach to homotopy theory. It should be noted that while there exist several important examples of closed model category structures on the category of topological spaces **TOP**, the associated homotopies in all cases are very closely related to the ordinary homotopy. Based on the above mentioned $[L]$-homotopy groups we, in this thesis, provide the first examples of model category structures on **TOP** whose homotopies are substantially different from the ordinary one. Namely, we show that $[L]$-homotopy is indeed a homotopy in the sense of Quillen for each finite CW-complex L.

Observe that $[L]$-homotopy groups may differ from the usual homotopy groups even for polyhedra. The problem which arises in a natural way is to describe $[L]$-homotopy groups in terms of "usual" algebraic invariants of X and L (in particular, in terms of homotopy and homology groups). In the present work we compute the n-th $[L]$-homotopy group of S^n for a complex L whose extension type lies between extension types of S^n and S^{n+1}.

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Chapter 1

Introduction

Historically, one of the first approaches to the definition of dimension of a topological space gave rise to the notion of *Lebesgue dimension* $\dim X$. Recall that the inequality $\dim X \leq n$ means that any finite open covering of a space X has an open refinement of order $\leq n+1$. According to a well-known theorem of P. S. Alexandrov we can describe Lebesgue dimension in terms of extensions of mappings into the n–dimensional sphere S^n as follows. Let X be a normal space. Then $\dim X \leq n$ if and only if for any closed subspace A of X any continuous mapping $f\colon A \to S^n$ can be extended to a mapping of X to S^n.

The algebraic approach in dimension theory is presented by the notion of cohomological dimension. Recall that *cohomological dimension* $\dim_G X$ of a locally compact space X *with respect to an Abelian coefficient group* G is the largest integer n for which there exists a locally compact subspace A of X such that $H^n(A;G) \neq 0$. Here $H^k(Y;G)$ denotes the k-th cohomology group of the locally compact space Y (the reader can find the definition of these groups in [Bre]).

Cohomological dimension can also be characterized in terms of extensions of maps. Namely, $\dim_G X \leq n$ if and only if any mapping f of a closed subset A of X to $K(G,n)$ can be extended to a mapping of X to $K(G,n)$. Here $K(G,n)$ denotes *Eilenberg–MacLane complex*, i.e. CW–complex such that its n-th homotopy group is isomorphic to G and all other homotopy groups are trivial.

The notion of *extension dimension*, introduced by A.N. Dranishnikov, generalizes both the notions of Lebesgue and cohomological dimensions. Following [D2] and [D-Dy], we first define the notion of extension type of a CW–complex. For a topological space X and a CW–complex L we say that *L is an absolute extensor for X* (notation: $L \in \mathrm{AE}(X)$) if for any closed subspace $A \subset X$ any mapping of A to L can be extended to a mapping of X to L. Relation $L \in \mathrm{AE}(X)$ allows us to define a partial ordering on CW-complexes as follows: $L \leq M$ if and only if for a Polish space X property $L \in \mathrm{AE}(X)$ implies $M \in \mathrm{AE}(X)$. For a complex L we denote by $[L]$ the equivalence class of L with respect to this ordering and call this class *extension type of L*. If $L \in \mathrm{AE}(X)$ we say also that *extension dimension* of X is $\leq [L]$ (notation: $\mathrm{e}-\dim X \leq [L]$). Finally, extension dimension of space X is defined as follows: $\mathrm{e}-\dim X = \min\{[L] \mid L \in \mathrm{AE}(X)\}$. According to this definition and the above mentioned results, Lebesgue and cohomological dimensions can be described as follows:

$$\dim X \leq n \Leftrightarrow \mathrm{e}-\dim X \leq [S^n]$$

$$\dim_G X \leq n \Leftrightarrow \mathrm{e}-\dim X \leq [K(G,n)]$$

Observe that the class of all extension types has both maximal and minimal elements with respect to the partial ordering introduced above. The minimal element is the extension type of the 0-dimensional sphere S^0 (i.e. the two-point discrete space) and the maximal element is the extension type of the one-point space $\{\text{point}\}$ (or, equivalently, of any contractible complex). Observe also, that there is no complex L such that $[S^0] < [L] < [S^1]$. However, there exist complexes whose extension types are between $[S^1]$ and $[S^2]$ (see [Ch2] for details). Further, given two complexes $[K]$ and $[L]$ we have $\min\{[K],[L]\} = [K \vee L]$ where $L \vee K$ denotes the bouquet of complexes K and L. It is not hard to see that $[S^0] < [S^1] < \cdots < [S^n] < [S^{n+1}] < \cdots$ and $[\{\text{point}\}] = \sup\{[S^n] \mid n = 1, 2, \ldots\}$. Observe, however, that not every two complexes are comparable. For instance, it follows from [D-R] that the extension type $[\mathbb{R}P^2]$ of the projective plane is not comparable with $[S^n]$ for any $n \geq 2$.

Note that extension and Lebesgue dimensions coincide for locally compact polyhedra (see [Ch2, Proposition 2.7]), although they differ in general.

It should be pointed out that several important results of the classical dimension theory remain true for the extension dimension. For instance, the property of monotonicity and the countable sum theorem hold (see [D2, Ch]).

Further, adequate homotopy and shape theories are developed which are specifically designed to work for at most $[L]$-dimensional spaces (see [Ch2]). The following notion of $[L]$*–homotopy*, introduced by A. Chigogidze [Ch2], generalizes the notion of n–homotopy, proposed by R. H. Fox [F] and studied by J.H.C. Whitehead [Wh2]. For a finite complex L we say that two maps $f_0, f_1 \colon X \to Y$ are $[L]$*–homotopic* if for any mapping $h \colon Z \to X \times [0,1]$ of a space Z with $\mathrm{e-dim} Z \leq [L]$ there exists a map $H\colon Z \to Y$ extending the composition $(f_0 \oplus f_1)h|_{h^{-1}(X\times\{0,1\})}$. Observe that by letting $L = S^n$ in the above definition we obtain the notion of n–homotopy.

It is crucial to observe that $[L]$–homotopy provides the first example of a homotopy which substantially differs from the standard notion of homotopy.

An important role in the developing of $[L]$-homotopy theory belongs to the class of *absolute extensors modulo a complex* $[L]$. We say that a topological space Y is *an absolute (neighborhood) extensor in dimension* $[L]$ (notation: $Y \in \mathrm{A(N)E}([L])$) if for any space X with $\mathrm{e-dim} X \leq [L]$ any map of a closed subspace $A \subset X$ to L can be extended to a map of X (of some open neighborhood of A in X) to L.

Note that the notion of A(N)E($[L]$)–space generalizes that of A(N)E(n)–space. The following local characterization of A(N)E(n)–spaces is well-known (see Kuratowski [Ku]; for general form see Dugundji [Du]).

Theorem A. A metric space $Y \in \mathrm{ANE}(n)$ if and only if $Y \in LC^{n-1}$.

Recall that space Y is said to be *locally n–connected* (notation: $Y \in LC^n$) if for any point $y \in Y$ and each open neighborhood U of y there exists smaller neighborhood V of y such that any map of S^k $(k \leq n)$ to V is homotopic to a constant map in U.

Remark. The above characterization is not true in the infinite-dimensional case. There exists Borsuk's example [Bo] of a locally contractible and contractible space which is not an absolute extensor.

The first result of this thesis, presented in Chapter 3, is a Dugundji-type theorem for ANE($[L]$)–theory. Using the notion of $[L]$–homotopy we can define $[L]$–contractibility. A space Y is said to be *locally* $[L]$*–contractible* (notation: $Y \in LC^{[L]}$) if for any point $y \in Y$ and for each neighborhood U of y there exists smaller neighborhood V of y such that the inclusion map of V into U is $[L]$–homotopic (in U) to a constant map. Now our result can be stated as follows.

Theorem 3.3.2. Let L be a countable locally finite CW-complex such that $[L] \leq [S^n]$ for some n. Then for a Polish space Y property $Y \in LC^{[L]}$ implies $Y \in ANE([L])$.

Another result of the present work is related to the theory of continuous selections. Recall that a multivalued mapping $F\colon X \to Y$ is said to have a selection if there exists a single-valued mapping $f\colon X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$. The finite-dimensional selection theorem of E. Michael is very useful in geometric topology and it is one of the central theorems in the theory of continuous selections of multivalued mappings. Below we state this theorem in its original form (for the proof and necessary definitions see [M1]; generalization is obtained in [M]).

Theorem B. Let $F\colon X \to Y$ be a lower semi-continuous multivalued mapping of a paracompact space X to a complete metric space Y such that $\dim X \leq n$ and $F(x)$ is closed for any $x \in X$. If the collection $\{F(x) \mid x \in X\}$ is equi–LC^n and C^n, then F admits a continuous selection.

Next, we state a stronger selection theorem of Shchepin and Brodsky (the reader can find the proof and necessary definitions in [S-Br]).

Theorem C. Let $F\colon X \to Y$ be a multivalued mapping of a paracompact space X with $\dim X \leq n$ to a complete metric space Y. Suppose that F

admits a lower continuous, complete, fiberwise connected n–filtration. Then F has a continuous selection.

It is important to point out that the finite dimensional approximation theorem from [S-Br] was used in the proof of this selection theorem. This shows an interesting interference between selections and approximations of multivalued mappings.

The notion of filtration appears to be very useful in continuous selection theory (see [S-Br], [Br]). Recall that an increasing finite sequence of subspaces

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n \subset Z$$

is called a *filtration* of space Z of length n. A sequence of multivalued mappings $\{F_k \colon X \to Y\}_{k=0}^{n}$ is called a *filtration of multivalued mapping* F if $\{F_k(x)\}_{k=0}^{n}$ is a filtration of $F(x)$ for any $x \in X$. The notion of completeness for multivalued mapping is introduced by E. Michael [M]. A multivalued mapping $G \colon X \to Y$ is called *complete* if all sets $\{x\} \times G(x)$ are closed with respect to some G_δ-set $S \subset X \times Y$ containing the graph of this mapping.

Chapter 4 of the thesis is dedicated to the proof of an extension-dimensional version of the finite dimensional selection theorem. Of course, this version contains the original finite dimensional theorem as a special case. Note also that our proofs follow the ideas from the paper [S-Br].

Below we state our selection theorem, the proof of which will be presented in Chapter 4. The notions of $[L]$-continuous and fiberwise $[L]$-connected n-filtration are introduced in Section 2.6 of Chapter 2.

Theorem 4.3.5. Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Let X be a paracompact space of extension dimension $\mathrm{e-dim}X \leq [L]$.

Suppose that multivalued mapping $F\colon X \to Y$ into a complete metric space Y admits a lower $[L]$-continuous, complete, and fiberwise $[L]$-connected n-filtration $F_0 \subset F_1 \subset \cdots \subset F_n \subset F$. If $f\colon A \to Y$ is a continuous single-valued selection of F_0 over a closed subspace $A \subset X$, then there exists a continuous single-valued selection $\widetilde{f}\colon X \to Y$ of the mapping F such that $\widetilde{f}|_A = f$.

Further studies of $[L]$–homotopy lead to the definition of algebraic $[L]$–homotopy invariants, and, in particular, *[L]–homotopy groups.* In Chapter 5 we give detailed description of $[L]$-homotopy groups introduced by Chigogidze in [Ch2]. We also introduce the notion of an $[L]$-*complex* which generalizes the standard notion of a CW-complex.

The notion of *closed model category,* introduced by D. Quillen [Q], gives an axiomatic approach to homotopy theory. This notion turned out to be very useful for developing axiomatic homotopy theory in very general situations (see, for instance, [K], [K1], [H], [Bro]). We begin Chapter 5 outlining the basic definitions related to the concept of a model category. Recall that in order to describe a closed model category structure we need to distinguish three classes of morphisms, called *fibrations, cofibration* and *weak equivalences,* which satisfy certain axioms. Given a model category $\mathcal{C}$ one can build the "homotopy category" $\mathcal{H}o(\mathcal{C})$. This makes it possible to consider formal inverses of weak equivalences between objects of $\mathcal{C}$ if both domain and range are nice, namely, cofibrant and fibrant simultaneously. Thus, in the quotient category $\mathcal{H}o(\mathcal{C})$ weak equivalences become homotopy equivalences. This fundamental fact has a number of applications.

The axioms of model categories are verifiable in a wide variety [Dw-S], [Ba], [Ho] of situations. It should be noted, however, that these axioms are reminiscent of well-known properties of homotopies for topological spaces. Observe that the category **TOP** of topological spaces itself possesses a model category structure (see, for instance, [Dw-S]).

Theorem D. The category **TOP** of topological spaces admits a model category structure where a map f is a weak equivalence if f is a weak homotopy equivalence, i.e. for each $k = 0, 1, \ldots$ and each $x \in X$ the induced map $\pi_k(X, x) \to \pi_k(Y, f(x))$ is an isomorphism.

Later in Chapter 5 (Example 5.1.4(ii)) we describe another example of a model category structure on **TOP** whose weak equivalences are just standard homotopy equivalences [St]. Note that these two structures are essentially different.

Observe that CW-complexes are cofibrant and fibrant objects with respect to the model category structure described in Theorem D. Therefore by passing to the correspondent homotopy category and by a formal inversion of weak equivalences we see that weak equivalences between CW-complexes become invertible, i.e. coincide with the ordinary homotopy equivalences. This restates the following well known theorem of J.H.C.Whitehead [Wh, Wh1].

Theorem E. A map between CW-complexes (or, more generally, ANE-spaces)is a homotopy equivalence if and only if it induces isomorphisms of *all* homotopy groups.

Another classical approach in the homotopy theory is generated by the mentioned above notion of n-homotopy introduced in [F]. It should be pointed out that n-homotopies (and n-shapes [Ch1], [Ch4]) play a substantial role in the theory of Menger manifolds [Be], [Ch3] (see [Ch-F] for a discussion of categorical connections between n-homotopies and homotopies via the theories of manifolds modeled on Menger and Hilbert cubes respectively). The notion of n-homotopy lead to the concept of n-types introduced by Whitehead. Algebraic models for n-types (the so called cat^n-groups) were found in [L]. Next we state the theorem from [ED-HP] which describes another example of model category structures on **TOP**, closely related to the n-homotopies.

Theorem F. Let $n = 0, 1, \ldots$. The category **TOP** of topological spaces admits a model category structure where a map $f\colon X \to Y$ is a weak equivalence if it is a weak n-homotopy equivalence, i.e. for each $k = 0, 1, \ldots, n$ and each $x \in X$ the induced map $\pi_k(X, x) \to \pi_k(Y, f(x))$ is an isomorphism.

As before, we can consider the homotopy category $\mathcal{H}o_n(\mathbf{TOP})$. Note that $\mathcal{H}o_n(\mathbf{TOP})$ is a model category for the above mentioned notion of n-type. Further, the invertibility of weak equivalences in $\mathcal{H}o_n(\mathbf{TOP})$, in analogy with Theorem E, restates another well-known result of Whitehead [Wh].

Theorem G. A map between at most n-dimensional CW-complexes (or, more generally, at most n-dimensional LC^{n-1}-spaces) is an n-homotopy equivalence if and only if it induces isomorphisms of the k-th homotopy groups for each $k \leq n - 1$.

In the present thesis we describe a new example of a model category structure on **TOP**. Namely, in Chapter 5 we prove the following theorem.

Theorem 5.3.8. Let L be a finite CW-complex. The category **TOP** of topological spaces admits a model category structure where a map $f\colon X \to Y$

is a weak equivalence if it is a weak $[L]$-homotopy equivalence, i.e. for each $k = 0, 1, \dots, n$ and each $x \in X$ the induced map $\pi_k^{[L]}(X, x) \to \pi_k^{[L]}(Y, f(x))$ of $[L]$-homotopy groups is an isomorphism.

This theorem shows that $[L]$-homotopy is indeed a homotopy in Quillen's sense. It should be emphasized that the constructed model category is the first example of a model category on **TOP** which does not involve classical homotopies.

We will show also that the model categories introduced in the Theorem 5.3.8 for each finite complex L differs from the model category described in Theorem F even for $L = S^n$ (although in the latter case weak equivalences of these categories coincide).

Since weak equivalences become invertible in the corresponding homotopy category $\mathcal{H}o_{[L]}(\mathbf{TOP})$ we obtain the earlier announced result from [Ch2, Theorem 2.9].

Theorem 5.3.9. A map between $[L]$-complexes is an $[L]$-homotopy equivalence if and only if it induces isomorphisms of all $[L]$-homotopy groups.

This result also extends both theorems E (letting $L = \{\text{point}\}$) and G (letting $L = S^n$).

Our concluding result presents the first approach to a solution of the following natural problem: characterize $[L]$-homotopy groups in terms of "usual" algebraic invariants of X and L (in particular, in terms of homotopy and homology groups). Namely, we compute the n-th $[L]$-homotopy group of S^n for complex L whose extension type lies between extension types of S^n and S^{n+1}. The main result of Chapter 6 is as follows.

Theorem 6.3.4. Let L be a finite complex satisfying $[S^n] < [L] \leq [S^{n+1}]$ for some n. Then $\pi_n^{[L]}(S^n) = \mathbb{Z}$.

Chapter 2

Preliminary Information

The present chapter contains preliminary facts and definitions. The material is organized as follows. We begin by introducing our notations. Sections 2.2 and 2.3 are devoted to the notions of extension type and extension dimension. Further, in Section 2.4 we give the definition of an absolute (neighborhood) extensor modulo a complex and state the resolution theorem. We introduce the notion of $[L]$–homotopy in Section 2.5. Finally, Section 2.6 is devoted to multivalued mappings.

Throughout the thesis by a complex we always mean a countable and locally finite CW-complex. For the reader's convenience some facts and definitions mentioned in the Introduction are included in this chapter.

In the substantial part of this work we focus our attention on the metrizable separable spaces. However, in Chapter 4 we state and prove selection theorems for more general spaces, namely, for paracompact spaces.

2.1 Basic Definitions and Notations

In this section we recall definitions of basic notions and introduce our notations.

Let A be a subspace of a topological space X. We denote the closure, interior and boundary of A by $\mathrm{Cl}A$, $\mathrm{Int}A$ and $\mathrm{Bd}A$ respectively. The set of all open coverings of X is denoted by $\mathrm{cov}X$. If $u \in \mathrm{cov}X$ then $\cup u$ denotes the union of all elements of u. For a cover ω of space X and for a subset $A \subseteq X$ let $\mathrm{St}(A, \omega)$ denote the star of A with respect to ω. Identity mapping of space X is denoted by id_X.

As usual, for a given n, we denote the n-th homotopy group of X with the base point x by $\pi_n(X, x)$. If $f\colon (X, x) \to (Y, y)$ is a continuous mapping of pointed spaces then the induced mapping of homotopy groups is denoted by f_*.

Recall that the *Lebesgue dimension* $\mathrm{dim}X$ is equal to the smallest number n such that any open finite cover of X has an open subcover of order $\leq n+1$. If $X = \emptyset$ we let $\mathrm{dim}X = -1$.

A complete separable metrizable space is called a *Polish space.* Let X and Y be Polish spaces and $u \in \mathrm{cov}Y$. Then two mappings $f, g\colon X \to Y$ are said to be *u-close* if for each $x \in X$ there exists an element $U \in u$ such that $f(x)$ and $g(x)$ belong to U. By $C(X, Y)$ we denote the set of all continuous mappings from X to Y. If $f \in C(X, Y)$ and $u \in \mathrm{cov}Y$ then we let $\mathcal{B}(f, u) = \{g \in C(X, Y) \mid g \in C(X, Y) \text{ is } u\text{-close to f}\}$. The collection $\{\mathcal{B}(f, u) \mid f \in C(X, Y), u \in \mathrm{cov}Y\}$ form a topology on $C(X, Y)$. In what follows we assume that $C(X, Y)$ is endowed with this topology.

The space of all compact subspaces of a metric space X endowed with the Hausdorff metric is denoted by $\exp X$.

Recall that a closed subset A of space X is said to be a *Z-set* in X if the set $\{f \in C(X,X) \mid f(X) \cap A = \emptyset\}$ is dense in the space $C(X,X)$.

2.2 Extension Types

Following [D2] and [D-Dy], we will define the notion of *extension type of a complex* and state basic properties of extension types.

Definition 2.2.1. For spaces X and L, we say that L is an *absolute (neighborhood) extensor for* X (notation $L \in \mathrm{A(N)E}(X)$) if every map $f\colon A \to L$, defined on a closed subspace A of X, admits an extension $\tilde{f}\colon X \to L$ (respectively, $\tilde{f}\colon G \to L$) over X (respectively, over a neighborhood G of A in X).

As we mentioned in the Introduction, Lebesgue and cohomological dimensions can be described by means of this notion as follows:

$$[S^n] \in \mathrm{AE}(X) \Longleftrightarrow \dim X \leq n$$

and

$$K(G,n) \in \mathrm{AE}(X) \Longleftrightarrow \dim_G X \leq n.$$

Here $K(G,n)$ denotes *Eilenberg–MacLane complex*, i.e. CW–complex such that its n-th homotopy group is isomorphic to G and all other homotopy groups are trivial. Further, the above definition allows us to introduce the preorder relation $\leq$ onto the class of CW-complexes. Following Dranishnikov

[D2], we say that $L \leq K$ if for each Polish space X the condition $L \in \mathrm{AE}(X)$ implies the condition $K \in \mathrm{AE}(X)$. The preorder relation $\leq$ naturally generates the equivalence relation: $L \sim K$ if and only if $L \leq K$ and $K \leq L$. We denote by $[L]$ the equivalence class of L. These equivalence classes of complexes are called *extension types.* The above defined relation $\leq$ allows us to introduce a partial order in the class of extension types. This partial order will still be denoted by $\leq$.

It is easy to see that the class of all extension types has both maximal and minimal elements with respect to this partial order. The minimal element is the extension type of the 0-dimensional sphere S^0 (i.e. the two-point discrete space) and the maximal element is the extension type of the one-point space $\{*\}$ (or, equivalently, of any contractible complex). Observe also, that $[S^0] = [L]$ if and only if L is not connected and $[S^1] \leq [L]$ if and only if L is connected. This implies that there is no complex L such that $[S^0] < [L] < [S^1]$. However, there exist complexes whose extension types are between $[S^1]$ and $[S^2]$, and in general, between $[S^n]$ and $[S^{n+1}]$ for $n \geq 1$ (see [Ch2] for details). Further, given two complexes $[K]$ and $[L]$ we have

$$\min\{[K],[L]\} = [K \vee L],$$

where $K \vee L$ denotes the bouquet of complexes K and L. It is not hard to see that

$$[S^0] < [S^1] < \cdots < [S^n] < [S^{n+1}] < \cdots$$

and

$$[\{*\}] = \sup\{[S^n] \mid n = 1,2,\ldots\}.$$

Observe, however, that not every two complexes are comparable. For instance, it follows from [D-R] that the extension type $[\mathbb{R}P^2]$ of the projective plane is not comparable with $[S^n]$ for any $n \geq 2$.

It is well-known that

$$[S^n] \leq [K(\mathbb{Z}, n)]$$

By the Fundamental theorem of cohomological dimension theory, obtained already in 1932 by Alexandrov [A] for metrizable compacta (see [A1] and [A2] for the general case) we have

$$\dim_{\mathbb{Z}} X = \dim X \text{ for any finite-dimensional compacum } X.$$

The following famous problem of cohomological dimension theory posed by Alexandrov remained unsolved for more than 50 years: *does the identity* $\dim_{\mathbb{Z}} X = \dim X$ *hold for an arbitrary metrizable compactum* X*?* The problem was solved by Dranishnikov [D1] who constructed an infinite-dimensional compactum having cohomological dimension equal to 3. Existence of this example implies that

$$[S^n] \neq [K(\mathbb{Z}, n)]$$

(see [Dy-Wa] for $n = 2$).

The Homotopy Extension Theorem implies the following trivial observation.

Proposition 2.2.2. *If* L *and* K *are homotopy equivalent complexes, then* $[L] = [K]$.

Observe however, that $[S^n \vee S^{n+1}] = [S^n]$. This shows that homotopy inequivalent complexes may have the same extension type.

2.3 Extension Dimension

In this section we introduce the notion of *extension dimension* of a topological space and state Dranishnikov's theorem, characterizing extension dimension [D].

Let L be a CW-complex. Following A. N. Dranishnikov (see [D2],[D-Dy]), we define *extension dimension* of a space X.

Definition 2.3.1. A space X is said to have *extension dimension* $\leq [L]$ (notation: $\mathrm{e}-\dim X \leq [L]$) if any mapping of its closed subspace $A \subset X$ into L admits an extension to the whole space X.

More precisely

$$\mathrm{e}-\dim(X) = \min\{[L]\colon L \in \mathrm{AE}(X)\}.$$

This definition implies that

$$\dim X \leq n \Longleftrightarrow \mathrm{e}-\dim X \leq [S^n]$$

and

$$\dim_{\mathbb{Z}} X \leq n \Longleftrightarrow \mathrm{e}-\dim X \leq [K(\mathbb{Z}, n)]\,.$$

Generally speaking, the extension dimension is not always defined (notice, however, that the inequality $\mathrm{e}-\dim X \leq [L]$ is meaningful for any space X). If we consider arbitrary (not necessarily countable) CW–complexes then this (in fact, set-theoretical) problem can be resolved (see [D2]). Observe also that extension dimension can be defined for wide classes of spaces. In view of this fact it is important to note that a spectral technique [Ch3] allows for reduction of all general questions to the case of Polish spaces [Ch].

It should be pointed out that several important results of the classical dimension theory remain true for the extension dimension. For instance, the following two propositions show that the property of monotonicity and the countable sum theorem hold (see [D2, Wa, Ch]).

Proposition 2.3.2. *If* $Y \subset X$ *then* $\mathrm{e}-\dim Y \leq \mathrm{e}-\dim X$.

Proposition 2.3.3. *If* $X = \cup\{X_k \mid k \in \omega\}$, *where* X_k *is a closed subspace of* X *such that* $\mathrm{e}-\dim X_k \leq [L]$ *then* $\mathrm{e}-\dim X \leq [L]$.

Further, extension dimension of a limit of an inverse sequence has the following property.

Proposition 2.3.4. [O, Ch] *Let* $X = \lim S$, *where* $S = \{X_k, p_k^{k+1}, \omega\}$ *be an inverse sequence of Polish spaces such that* $\mathrm{e}-\dim X_k \leq [L]$ *for each* $k \in \omega$. *Then* $\mathrm{e}-\dim X \leq [L]$.

Notice that extension and Lebesgue dimensions coincide for locally compact polyhedra (see [Ch2, Proposition 2.7]). The following statement shows that extension and Lebesgue dimensions differ in the general case.

Proposition 2.3.5. [Ch] *For a given complex* L *there exists a Polish space* X *such that* $\mathrm{e}-\dim X = [L]$. *If, in addition,* L *is finitely dominated, we may assume that* X *is compact.*

Moreover, if L is a finitely dominated complex then under additional set-theoretical assumptions there exists an example of a differentiable, countably compact, perfectly normal and hereditarily separable 4-manifold $M^{[L]}$ possessing an extraordinary dimensional property, namely, having extension dimension $\mathrm{e}-\dim M^{[L]} = [L]$ (see [Ch-F1]).

Extension dimension can be characterized in terms of cohomological dimension. The following theorem is a special case of Dranishnikov's characterization theorem.

Theorem 2.3.6. [D] *Let L be a simply-connected CW-complex and X be a finite-dimensional compactum. Then* $\mathrm{e} - \dim X \leq [L]$ *if and only if for any i we have* $\dim_{H_i(L)} X \leq i$.

Remark 2.3.7. The proof of Theorem 2.3.6 shows that for an arbitrary (not necessarily simply-connected) complex L and for a finite-dimensional compactum X the property $\mathrm{e} - \dim X \leq [L]$ implies $\dim_{H_i(L)} X \leq i$ for any i.

2.4 Absolute Extensors and [L]–Universal Spaces

In this section we introduce the notion of an *absolute extensor modulo a complex.* Following [Ch2] we give definitions of $[L]$*–soft* and *approximately* $[L]$*–soft* mappings and state some results related to these mappings. We also state an important theorem on existence and properties of $[L]$–universal spaces (resolution theorem).

Definition 2.4.1. [Ch2] A space X is said to be an absolute (neighborhood) extensor modulo $[L]$, or shortly A(N)E($[L]$)-space (notation: $X \in$ A(N)E($[L]$)) if $X \in$ A(N)E(Y) for each space Y with $\mathrm{e} - \dim Y \leq [L]$.

Obviously, $L \in$ AE($[L]$). The following proposition shows that under some restrictions on L and X we can assume in the above definition that Y is also a Polish space.

Proposition 2.4.2. [Br-Ch-Ka, Proposition A.3] *Let L be a countable locally finite CW-complex and X be a Polish space. If $X \in$ ANE$([L])$ for Polish spaces, then $X \in$ ANE$([L])$.*

Notice that the notion of absolute (neighborhood) extensor modulo $[L]$ generalizes two classical notions of absolute (neighborhood) extensor and absolute (neighborhood) extensor in dimension n. Indeed, A(N)E([point])-spaces are precisely A(N)E-spaces and A(N)E($[S^n]$)-spaces are precisely A(N)E(n)-spaces. Further, as in the classical case, open subspaces and retracts of A(N)E($[L]$)-spaces are A(N)E($[L]$)-spaces.

Following Chigogidze [Ch2], we introduce the notions of $[L]$-soft and $[L]$-invertible mappings.

Definition 2.4.3. A map $f\colon X \to Y$ is said to be $[L]$-*soft*, if for each Polish space B with $\mathrm{e}-\mathrm{dim} B \leq [L]$, for each closed subspace A of it, and for any two maps $g\colon A \to X$ and $h\colon B \to Y$ such that $f \circ g = h/A$, there exists a map $k\colon B \to X$ satisfying the conditions $k/A = g$ and $f \circ k = h$. If the above condition is satisfied in the cases when $A = \emptyset$ we say that f is $[L]$-*invertible.*

Clearly, every $[L]$-soft map is $[L]$-invertible. Observe that $[L]$-invertible mappings are surjective while $[L]$-soft maps are open [Ch, Remark 5.10]. The notions of absolute extensor modulo $[L]$ and $[L]$-soft mappings are related as follows (see [Ch2, Proposition 2.13]).

Proposition 2.4.4. *If a mapping $f\colon X \to Y$ between Polish spaces X and Y is $[L]$ soft then $X \in$ A(N)E($[L]$) if and only if $Y \in$ A(N)E($[L]$).*

In particular, $X \in$ AE($[L]$) if and only if the constant map $X \to$ point is $[L]$-soft. Notice also that all fibers of $[L]$–soft mapping are AE($[L]$)-spaces [Br-Ch-Ka].

Approximately $[L]$–soft mappings play an important role in the subsequent considerations.

Definition 2.4.5 (Definition 2.6, [Ch2]). A map $f\colon X \to Y$ is said to be *approximately* $[L]$*-soft*, if for each Polish space B with $\mathrm{e}-\dim(B) \leq [L]$, for each closed subset A of it, for an open cover $\mathcal{U} \in \mathrm{cov}(Y)$, and for any two maps $g\colon A \to X$ and $h\colon B \to Y$ such that $fg = h/A$, there is a map $k\colon B \to X$ satisfying the conditions $k/A = g$ and the composition fk is $\mathcal{U}$-close to h.

It is not hard to see that for an approximately $[L]$-soft mapping $f\colon X \to Y$ the property $Y \in$ A(N)E($[L]$) implies $X \in$ A(N)E($[L]$).

The following resolution theorem describes an important type of A(N)E($[L]$)–spaces.

Theorem 2.4.6 (Proposition 2.23, [Ch2]). *Let L be a finite complex and X be a locally finite CW-complex. Then there exist a locally compact metrizable space $\mu_X^{[L]}$ and an $[L]$-invertible and approximately $[L]$-soft proper map $f_X^{[L]}\colon \mu_X^{[L]} \to X$ satisfying the following conditions:*

(i) $\mu_X^{[L]} \in$ ANE($[L]$).

(ii) $\mathrm{e}-\dim(\mu_X^{[L]}) = [L]$.

(iii) *For any map $f\colon B \to \mu_X^{[L]}$, where B is a compact space with* $\mathrm{e}-\dim(B) \leq [L]$*, and for any open cover $\mathcal{U} \in \mathrm{cov}(\mu_X^{[L]})$ there is an embedding $g\colon B \to \nu_X^{[L]}$ which is $\mathcal{U}$-close to f and such that $f_X^{[L]} \circ g = f_X^{[L]} \circ f$.*

(iv) *If, in addition, X is a locally finite polyhedron and τ is its triangulation, then one can assume that for any $X_0 \subset X$ such that X_0 is a subpolyhedron of X with respect to τ, the inverse image $\left(f_X^{[L]}\right)^{-1}(X_0)$ is also a locally compact* ANE([L])*-space and the restriction* $f_X^{[L]} \mid \left(f_X^{[L]}\right)^{-1}(X_0)\colon \left(f_X^{[L]}\right)^{-1}(X_0) \to X_0$ *is also approximately* $[L]$*-soft.*

Proof of this statement, with necessary adjustments, follows Dranishnikov's construction [D-R] (see also the proof of [Ch2, Proposition 2.23]).

Following Chigogidze [Ch2], we define $[L]$-universal spaces.

Definition 2.4.7. For a given complex $[L]$ we say that a locally compact space X is strongly $[L]$–universal for compact spaces if for any locally compact space B of extension dimension $\mathrm{e}-\dim B \leq [L]$, its closed subspace A, any open cover $u \in \mathrm{cov}(X)$ and for any proper map $f\colon B \to X$, the restriction f/A of which is Z-embedding, there is a Z-embedding $g\colon B \to X$ which is u-close to f and such that $g/A = f/A$.

Observe that the notion of strongly $[L]$–universal spaces generalizes the notion of strongly n-universal spaces. Thus, strongly $[L]$-universal compacta play the role of Menger compacta and coincide with them in the case $L = S^n$. Letting $X = \{\text{point}\}$ in Theorem 2.4.6 we obtain the following theorem on existence of strongly $[L]$–universal spaces.

Theorem 2.4.8. [Ch2, Theorem 2.5] *Let L be a finite complex. Then there exists a compact metrizable space $\mu^{[L]}$ satisfying the following conditions:*

(i) $\mu^{[L]} \in \mathrm{AE}([L])$.

(ii) $\mathrm{e}-\dim(\mu^{[L]}) = [L]$.

(iii) $\mu^{[L]}$ *is strongly* $[L]$*-universal for compact spaces.*

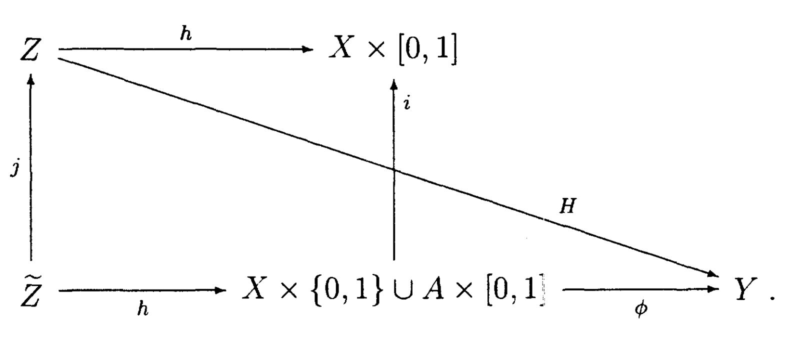
2.5 [L]-Homotopy

The notion of $[L]$*-homotopy*, introduced by A. Chigogidze, generalizes the notion of n–homotopy, proposed by R. H. Fox [F] and studied by J.H.C. Whitehead [Wh2]. We begin this section with the definition of $[L]$–homotopic maps. Throughout this section L stands for a finite CW-complex.

Definition 2.5.1. Let A be a subspace of a Polish space X and $f_0, f_1 \colon X \to Y$ be two maps such that $f_0(x) = f_1(x)$ for each $x \in A$. Then f_0 and f_1 are said to be $[L]$*-homotopic relative to* A (notation: $f_0 \overset{[L]}{\simeq} f_1 \operatorname{rel} A$) if for any map $h \colon Z \to X \times [0,1]$ where Z is a Polish space of extension dimension $\mathrm{e}-\dim Z \leq [L]$, there exists a map $H \colon Z \to Y$ such that

$$H(z) = \begin{cases} h(f_0(z)), & \text{if } z \in h^{-1}(X \times \{0\} \cup A \times [0,1]), \\ h(f_1(z)), & \text{if } z \in h^{-1}(X \times \{1\}). \end{cases}$$

The following diagram illustrates the situation:



Here $\widetilde{Z} = h^{-1}(X \times \{0,1\} \cup A \times [0,1])$ and

$$\phi(x) = \begin{cases} f_0(x), & \text{if } x \in X \times \{0\} \cup A \times [0,1], \\ f_1(x), & \text{if } x \in X \times \{1\}. \end{cases}$$

If $A = \emptyset$, then we say that f_0 and f_1 are $[L]$-homotopic (notation: $f_0 \overset{[L]}{\simeq} f_1$, see [Ch2, Definition 2.9]).

Observe that by letting $L = \{\text{point}\}$ in the above definition we obtain the notion of the usual homotopy and by letting $L = S^n$ we obtain the notion of the n–homotopy.

It should be noted that a number of basic properties of ordinary homotopies have their analogs for $[L]$-homotopies. Below we list some of them.

Example 2.5.2. (i) $[L]$–homotopic maps are $[K]$–homotopic for each K with $[K] \leq [L]$.

(ii) If X is AE($[L]$)-space, then id_X is $[L]$-homotopic to a constant map.

(iii) [Ch2, Proposition 2.25] Let L and K be connected complexes and $[K] < [L]$. Then the identity map id_K is not $[L]$–homotopic to a constant map.

We mentioned in Section 2.3 that the extension dimension of complexes coincides with the Lebesgue dimension. The following example shows that the concept of $[L]$–homotopy differs from the classical notions of homotopy or n-homotopy even for maps between complexes as follows.

Example 2.5.3. [Ch2] Let $n \geq 1$ and L be a connected complex such that $[S^n] < [L] < [S^{n+1}]$ (see [Ch2, Example 2.4(iii)]). Since $L \in \mathrm{AE}([L])$ by

Example 2.5.2 we have

$$\mathrm{id}_L \overset{[L]}{\simeq} \text{const, but } \mathrm{id}_L \overset{[S^{n+1}]}{\not\simeq} \text{const.}$$

Similarly,

$$\mathrm{id}_{[S^n]} \overset{[S^n]}{\simeq} \text{const, but } \mathrm{id}_{[S^n]}[S^n] \overset{[L]}{\not\simeq} \text{const.}$$

Further, absolute neighborhood extensors modulo $[L]$ introduced in Section 2.4 have an important property of $[L]$–homotopy extension. Here is an analog of the standard homotopy extension theorem [Ch2, Proposition 2.28].

Proposition 2.5.4. *Let L be a finitely dominated complex and X be a Polish $\mathrm{ANE}([L])$-space. Suppose that A is closed in a space B with $\mathrm{e}-\dim(B) \leq [L]$. If maps $f, g\colon A \to X$ are $[L]$-homotopic and f admits an extension $F\colon B \to X$, then g also admits an extension $G\colon B \to X$, and it may be assumed that $F \overset{[L]}{\simeq} G$.*

We also recall the following statement [Ch2, Proposition 2.26].

Proposition 2.5.5. *Let L be a finitely dominated complex and X be a Polish $\mathrm{ANE}([L])$-space X. Then there exists an open cover $\mathcal{U} \in \mathrm{cov}(X)$ such that any two $\mathcal{U}$-close maps of any space into X are $[L]$–homotopic.*

As the reader will see in Chapter 5, the class of approximately $[L]$-soft maps introduced in the previous section plays an important role in the $[L]$-homotopy theory. Between spaces of extension dimension not exceeding $[L]$, such maps provide basic examples of *$[L]$-homotopy equivalences.*

Definition 2.5.6. [Ch2, Definition 2.10] A map $f\colon X \to Y$ is said to be a $[L]$-homotopy equivalence if there exists a map $g\colon Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are $[L]$-homotopic to id_X and id_Y respectively.

Proposition 2.5.7. [Ch2] *Let $f\colon X \to Y$ be a map between $ANE([L])$-compacta and* $\mathrm{e-dim}Y \le [L]$*. If f is approximately $[L]$-soft then f is an $[L]$-homotopy equivalence.*

2.6 Multivalued Mappings

First part of this section contains preliminary definitions and notations related to the theory of selections and approximations of multivalued mappings. In particular, we recall an important notion of *filtered multivalued mapping.* In the second part we state the finite–dimensional approximation theorem from [Br-Ch-Ka]. Later we will apply this theorem in the proof of the finite dimensional selection theorem (see Chapter 4). Finally, we introduce the notion of *approximately $[L]$-invertible mapping* and state the theorem on $[L]$–invertibility of $UV^{[L]}$–filtered mapping (see [Br-Ch-Ka]) which will be applied also in Chapter 4.

Let $F\colon X \to Y$ be a multivalued mapping. As usual, the *graph* of a multivalued mapping F is the subset $\Gamma_F = \{(x,y) \in X \times Y\colon y \in F(x)\}$ of the product $X \times Y$.

The notion of *filtration* is very useful in continuous selection theory (see [S-Br], [Br]). Notice that in Chapter 4 we state our selection theorem in terms of filtrations of multivalued mappings.

Definition 2.6.1. An increasing finite sequence of subspaces

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n \subset Z$$

is called a *filtration* of space Z of length n. A sequence of multivalued mappings $\{F_k\colon X \to Y\}_{k=0}^{n}$ is called a *filtration of multivalued mapping* F if $\{F_k(x)\}_{k=0}^{n}$ is a filtration of $F(x)$ for any $x \in X$.

To construct a local selection we need our filtration of multivalued maps to be *complete* and *lower* $[L]$*-continuous.* The notion of completeness for multivalued mapping is introduced by E. Michael [M].

Definition 2.6.2. A multivalued mapping $G\colon X \to Y$ is called *complete* if all sets $\{x\} \times G(x)$ are closed with respect to some G_δ-set $S \subset X \times Y$ containing the graph of this mapping.

We say that a filtration of multivalued mappings $G_i\colon X \to Y$ is *complete* if every mapping G_i is complete.

In Section 4.1 we introduce the notion of a local property of multivalued mapping. To have a local property, multivalued mapping should have all fibers satisfying this local property, and, moreover, the fibers should satisfy this property uniformly. An important example of local property is local $[L]$-connectedness.

Definition 2.6.3. Let L be a CW-complex. A pair of spaces $V \subset U$ is said to be $[L]$*-connected* if for every paracompact space X of extension dimension $\mathrm{e-dim} X \leq [L]$ and for every closed subspace $A \subset X$ any mapping of A into V can be extended to a mapping of X into U.

The following proposition shows that under some restriction on L, U and V it suffices to check the property of $[L]$-connectedness for Polish spaces X.

Proposition 2.6.4. [Br-Ch-Ka, Corollary A.2] *Let L be a locally finite countable CW-complex and $V \subseteq U$ be a pair of Polish spaces. Then this pair is $[L]$-connected for paracompact spaces if and only if it is $[L]$-connected for Polish spaces.*

We call a multivalued mapping lower $[L]$-continuous if it is locally $[L]$-connected:

Definition 2.6.5. A multivalued mapping $F\colon X \to Y$ is called *$[L]$-continuous at a point* $(x, y) \in \Gamma_F$ of its graph if for any neighborhood Oy of the point $y \in Y$, there are a neighborhood $O'y$ of the point y and a neighborhood Ox of the point $x \in X$ such that for all $x' \in Ox$, the pair $F(x') \cap O'y \subset F(x') \cap Oy$ is $[L]$-connected.

A mapping which is $[L]$-continuous at all points of its graph is called *lower $[L]$-continuous.* We say that a filtration of multivalued mappings is *lower $[L]$-continuous* if every mapping of this filtration is lower $[L]$-continuous.

To construct a global selection we need our filtration of multivalued maps to be fiberwise $[L]$-connected.

Definition 2.6.6. A filtration of multivalued mappings $\{G_i\colon X \to Y\}_{i=0}^{n}$ is said to be *fiberwise $[L]$-connected* if for any point $x \in X$ and any $i < n$ the pair $G_i(x) \subset G_{i+1}(x)$ is $[L]$-connected.

For a subset $\mathcal{U}$ of the product $X \times Y$ we denote by $\mathcal{U}(x)$ the subset $\mathrm{pr}_Y(\mathcal{U} \cap \{x\} \times Y)$ of Y, where x is a point of X and pr_Y denotes the projection of $X \times Y$ onto Y. For a multivalued mapping $F\colon X \to Y$ we denote by $F^{\Gamma}(x)$ the subset $\{x\} \times F(x)$ of $X \times Y$. Recall that a multivalued mapping $F\colon X \to Y$ is said to be *upper semicontinuous* (shortly, u.s.c.) if its graph is closed in the product $X \times Y$. We say that multivalued mapping is *compact* if it is upper semicontinuous and compact-valued. A filtration consisting of compact multivalued mappings is called *compact.*

In what follows we will need the notion of *n-$UV^{[L]}$-filtered* multivalued mapping.

Definition 2.6.7. A pair of subspaces $K \subset K'$ of a space Z is called *$UV^{[L]}$-connected in Z* if any neighborhood U of K' contains a neighborhood V of K such that the pair $V \subset U$ is L-connected.

Definition 2.6.8. A filtration $\{F_i\colon X \to Y\}_{i=0}^{n}$ of u.s.c. maps is called *$UV^{[L]}$-connected n-filtration* if for any point $x \in X$ and any $i < n$ the pair $F_i(x) \subset F_{i+1}(x)$ is $UV^{[L]}$-connected in Y. We say that multivalued mapping F is *n-$UV^{[L]}$-filtered* if it contains an $UV^{[L]}$-connected n-filtration.

As we mentioned before, existence of a single-valued approximation of multivalued mapping is a key point in the construction of continuous selection.

Definition 2.6.9. We say that a multivalued mapping F admits *approximations* if every neighborhood of the graph of F contains the graph of a single-valued continuous mapping.

In order to state the approximation theorem we need the notion of *stable neighborhood.*

Definition 2.6.10. For a multivalued mapping $F\colon X \to Y$ an open neighborhood $U \subset X \times Y$ of a fiber $F^{\Gamma(x)}$ is said to be *F-stable with respect to* $x \in X$ if there exists an open neighborhood O_x of the point x and an open subset $V_x \subset Y$ such that $\Gamma_{F|_{O_x}} \subset O_x \times V_x \subset U$.

The neighborhood U of the graph is said to be *F-stable* if it is F-stable with respect to every point in X.

Theorem 2.6.11. [Br-Ch-Ka] *Let L be a CW-complex such that $[L] \leq [S^n]$ for some n. Let X be a paracompact space of extension dimension* $\mathrm{e}-\dim X \leq [L]$.

(1) *If $F\colon X \to Y$ is a multivalued mapping which admits $UV^{[L]}$-connected n-filtration, then any F-stable neighborhood of the graph Γ_F contains a graph of a single-valued continuous mapping of X to Y.*

(2)*Let $A \subset X$ be a closed subspace. If F admits $UV^{[L]}$-connected $(n+1)$-filtration $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{n+1}$, then for any F-stable neighborhood U of the graph Γ_F there exists F_0-stable neighborhood V of the graph $\Gamma_{F_0|_A}$ such that every single-valued continuous mapping $g\colon A \to Y$ with $\Gamma_g \subset V$ can be extended to a single-valued continuous mapping $f\colon X \to Y$ with $\Gamma_f \subset U$.*

We conclude this section by the following definition of *approximately $[L]$-invertible mapping* and the theorem on approximate $[L]$-invertibility.

Definition 2.6.12. [Br-Ch-Ka] A single-valued continuous surjective mapping $f\colon Y \to X$ of metric spaces is said to be approximately $[L]$-invertible if for any embedding of f into the projection $p\colon M \times N \to M$ of metric spaces where $M \in ANE([L])$ the following condition is satisfied:
for any neighborhood W of Y in $M \times N$ there exists open neighborhood U of

X in M such that for any mapping $g\colon Z \to U$ of paracompact space Z with $\mathrm{e}-\dim(Z) \leq [L]$ there exists a lifting $g'\colon Z \to W$ of g such that $pg' = g$.

Theorem 2.6.13. [Br-Ch-Ka] *Let L be a CW-complex such that $[L] \leq [S^n]$ for some n. Suppose that for a continuous single-valued surjective mapping of metric spaces f the multivalued mapping $F = f^{-1}$ admits a compact $UV^{[L]}$-connected n-filtration. Then f is approximately $[L]$-invertible.*

Chapter 3

$\mathrm{UV}^{[\mathrm{L}]}$–Compacta and ANE([L])–Spaces

We begin this chapter by presenting the concept of *simplices* and *skeleta* of an open covering, introduced in [Br-Ch-Ka]. This concept gives rise to a new approach allowing us to construct extensions of continuous mappings and continuous approximations. Further, we define the notion of $\mathrm{UV}^{[\mathrm{L}]}$–compactum and prove the invariance theorem, which shows that $UV^{[L]}$–property of a compact pair does not depend on embedding into ANE($[L]$)–spaces. We also establish some technical facts on lower $[L]$-continuity which will be used in the next chapter. Another result presented is an analog of the Dugundji theorem on local characterization of ANE([L])–spaces.

3.1 Preliminaries

One of the classical approaches to construction of continuous extensions or approximations possessing specific properties is as follows. One constructs a continuous mapping as the composition of the canonical mapping into the nerve of some covering and of a mapping of this nerve, defining the latter by induction on dimension of its skeleta. For instance, if we need to find an approximation of a UV^n-valued mapping and the domain space X has Lebesgue dimension n, then every point-image has trivial shape relative to X and relative to a nerve of some covering of X, which allows for construction of mapping from the nerve.

However, if extension dimension $\mathrm{e}-\dim X = [L]$ does not coincide with Lebesgue dimension of X, then $UV^{[L]}$-compactum does not have trivial shape relative to a nerve of fine covering of X, and the described technique is not applicable in this case. Therefore, we have to define the desired mapping directly. For some open fine covering Σ of X we consider the sets $\Sigma^{(k)} = \{x \in X \mid \mathrm{ord}_\Sigma x \leq k+1\}$ and construct a mapping extending it successively from $\Sigma^{(k)}$ to $\Sigma^{(k+1)}$. Here $\Sigma^{(k)}$ plays a role of "k-dimensional skeleton" of the cover Σ. For elements $s_0, s_1, \ldots, s_n \in \Sigma$ with non-empty intersection $\cap_{i=0}^n s_i$ we consider the set

$$[s_0, s_1, \ldots, s_n] = \bigcup_{i=0}^{n} s_i \setminus \bigcup_{i \neq 0,1,\ldots,n} s_i$$

as a closed "simplex" with vertices $s_0, \ldots, s_n$ and its "interior" $\langle s_0, s_1, \ldots, s_n\rangle = \cap_{i=0}^n s_i \cap \Sigma^{(n)}$. It is easy to check that the n-skeleton consists of n-simplices

$$\Sigma^{(n)} = \bigcup \{[s_{i_0}, s_{i_1}, \ldots, s_{i_n}] \mid \cap_{k=0}^n s_{i_k} \neq \emptyset\}$$

and that any "simplex" consists of its "boundary" and its "interior"

$$[s_0, s_1, \dots, s_n] = \bigcup_{m=0}^{n} [s_0, \dots, \widehat{s}_m, \dots, s_n] \cup \langle s_0, s_1, \dots, s_n \rangle.$$

Clearly, $\Sigma^{(k)}$ is closed in X and $\Sigma^{(n)} = X$ if the cover Σ has order $n+1$. The following property is important for our construction: the "interiors" of distinct k-dimensional "simplices" are mutually disjoint and

$$\Sigma^{(k)} = \bigcup \{ \langle s_{i_0}, s_{i_1}, \dots, s_{i_n} \rangle \mid \cap_{k=0}^{n} s_{i_k} \neq \emptyset \} \cup \Sigma^{(k-1)} \qquad (\dagger)$$

These notions of "skeleton" and "simplex" of a covering allow us to proceed in the usual way — by induction on "dimension" of "skeleta".

3.2 Lower $[L]$-Continuous Mappings and $\mathrm{UV}^{[\mathrm{L}]}$-Compacta

In this section we prove several important technical results about lower $[L]$-continuous mappings. In particular, these results allow us to show that $UV^{[L]}$-property of compactum does not depend on embedding of this compactum into ANE($[L]$)-space. Throughout this section we assume that $[L] \leq [S^n]$ for some n.

We begin with the definition of an $UV^{[L]}$-compactum.

Definition 3.2.1. A compact metric space K is called $UV^{[L]}$*-compactum* if the pair (K, K) is $UV^{[L]}$-connected in any ANR-space.

A multivalued mapping is called $UV^{[L]}$*-valued* if it takes any point to $UV^{[L]}$-compactum.

It should be pointed out that the notion of $UV^{[L]}$-compactum was earlier introduced in [Ch2, Definition 2.13]. The results of the present section (namely, invariance theorem) show, in particular, that in the case of complexes with the property $[L] \leq [S^n]$ these two definitions are equivalent. Another important fact to observe is the connection between approximately $[L]$-soft mappings and mappings all fibres of which are $UV^{[L]}$-compacta. Namely, approximately $[L]$-soft mappings between ANE($[L]$)-compacta are precisely $UV^{[L]}$-mappings [Ch2].

First we establish the following important property of a lower $[L]$-continuous multivalued mapping.

Theorem 3.2.2. *Let L be a CW-complex such that $[L] \leq [S^n]$ for some n. Suppose that $F\colon X \to Y$ is a lower $[L]$-continuous multivalued mapping of paracompact space X to metric space Y. Let K be a compact subspace of a fiber $F(x)$ for some point $x \in X$. Then for any $\varepsilon > 0$ there exist $\delta > 0$ and open neighborhood Ox of the point x such that for each $x' \in Ox$, for any paracompact space Z with $\mathrm{e-dim}X \leq [L]$, for each closed subspace A of Z and for any map $f\colon (A, Z) \to (O(K,\delta) \bigcap F(x'), O(K,\delta))$ there exists $g\colon Z \to F(x') \bigcap O(K,\varepsilon)$ such that $f|_A = g|_A$ and $\mathrm{dist}(f,g) < \varepsilon$.*

Proof. Consider arbitrary $\varepsilon > 0$. Applying Lemma 4.1.5 (proof of which the reader will find in Chapter 4, Section 4.1) choose sequence $\{\delta_{-1} < \delta_0 < \delta_1 < \cdots < \delta_n < \delta_{n+1} = \varepsilon\}$ of positive numbers and neighborhoods $\{O_i x\}_{i=0}^n$ of x such that for all $i = -1, 0, 1, \dots, n$ and for any points $x' \in O_i x$ and $y' \in F(x') \bigcap O(K, \delta_i)$ the pair $O(y', \delta_i) \bigcap F(x') \subset O(y', \delta_{i+1}/10) \bigcap F(x')$ is $[L]$-connected. Let $\{O(p_i, \delta_0/10) \mid i = 1, \dots, m\}$ be a finite covering of compactum K such that $p_i \in K$ for all i and choose δ such that $O(K,\delta) \subset$

$\bigcup_{i=1}^{m} O(p_i, \delta_0/10)$. Let $Ox = \bigcap_{i=1}^{n} O_i x$.

Fix $x' \in Ox$ and let $f\colon Z \to O(K,\delta)$ be a mapping such that $f(A) \subset F(x') \bigcap O(K,\delta)$ where Z has extension dimension $\mathrm{e-dim} Z \le [L]$. Let v be an open covering $\{V_p = f^{-1}O(p,\delta_0/10) \mid p = p_1,\dots,p_m\}$ of Z. Find an open locally finite covering Σ of Z such that closures of elements of Σ form strong star-refinement of v and order of Σ is $\le n+1$. For each $s \in \Sigma$ find $p(s) \in \{p_1,\dots,p_m\}$ such that $\mathrm{St}(s,\Sigma) \subset V_{p(s)} \in v$ and pick $y_s \in O(p(s),\delta_0/10) \bigcap F(x')$. Observe that $f(s) \subset O(p(s),\delta_0/10)$. Letting $g_{-1} = f|_A$ we shall inductively construct a sequence of mappings $\{g_k\colon \Sigma^{(k)} \cup A \to F(x')\}_{k=-1}^{n}$, such that g_k extends g_{k-1} and

$$g_k((\Sigma^{(k)} \cup A) \cap s) \subset O(y_s, \delta_{k+1}/2) \text{ for each } s \in \Sigma \qquad (*)$$

Since $\Sigma^{(n)} = Z$ and $\delta_{n+1} = \varepsilon$, the property $(*)$ implies that $g_n(Z) \subset O(K,\delta_0/10+\varepsilon/2) \subset O(K,\varepsilon)$. Moreover, g_n is ε-close to f, since for any $s \in \Sigma$ we have $\mathrm{dist}(f|_s, g_n|_s) < \mathrm{dist}(f|_s, p(s)) + \mathrm{dist}(p(s), y_s) + \mathrm{dist}(g_n|_s, y(s)) < \delta_0/10 + \delta_0/10 + \varepsilon/2 < \varepsilon$. Therefore, by letting $g = g_n$ we shall obtain desired mapping.

Suppose that g_k has been already constructed. It suffices to define g_{k+1} on the "interior" $\langle\sigma\rangle$ of each "simplex" $[\sigma] = [s_0, s_1, \dots, s_{k+1}]$. Let $[\sigma]' = [\sigma] \cap (\Sigma^{(k)} \cup A)$. By property $(*)$ of g_k we have $\mathrm{dist}(g_k([\sigma]'), y_{s_0}) < \delta_{k+1}/2 + \max_{i=1}^{k+1}\{\mathrm{dist}(y_{s_0}, y_{s_i})\}$. Further, since $f(s) \subset O(p(s),\delta_0/10)$ for any S and $s_0 \cap s_i \ne \varnothing$, we have $\mathrm{dist}(p(s_0), p(s_i)) < 2\delta_0/10$. Since $y_{s_i} \in O(p(s_i),\delta_0/10)$, we therefore obtain

$$\max_{i=1}^{k+1}\{\mathrm{dist}(y_{s_0}, y_{s_i})\} \le \mathrm{dist}(y_{s_0}, p(s_0)) + \mathrm{dist}(p(s_0), p(s_i)) + \mathrm{dist}(p(s_i), y_{s_i}) < \delta_0/10 + 2\delta_0/10 + \delta_0/10 = 2\delta_0/5.$$

Thus

$$g_k([\sigma]') \subset O(y_{s_0}, \delta_{k+1}/2 + 2\delta_0/5) \bigcap F(x') \subset O(y_{s_0}, \delta_{k+1} \bigcap F(x')$$

By the choice of Ox and δ_{k+2} the pair

$$O(y_{s_0}, \delta_{k+1}) \bigcap F(x') \subset O(y_{s_0}, \delta_{k+2}/10) \bigcap F(x')$$

is $[L]$-connected. Hence the map g_k can be extended to a map g_{k+1} such that $g_{k+1}([\sigma]) \subset O(y_{s_0}, \delta_{k+2}/10) \bigcap F(x')$. Let us check the property $(*)$. For any point $x \in (\Sigma^{(k)} \cup A) \cap s_i$ by the construction of g_{k+1} we have: $\operatorname{dist}(g_{k+1}(x), y_{s_i}) < \operatorname{dist}(g_{k+1}(x), y_{s_0}) + \operatorname{dist}(y_{s_0}, y_{s_i}) \leq \delta_{k+2}/10 + 2(\delta_0/5) < \delta_{k+2}/2$, as required. □

Corollary 3.2.3. *Let L be a CW-complex such that $[L] \leq [S^n]$ for some n. Let Y be a metric space, B be an* ANE$([L])$*-subspace of Y and K be a compact subspace of B. Then for any open neighborhood U of K in Y and for any $\varepsilon > 0$ there exists a neighborhood $V \subset O(K, \varepsilon)$ of K with the following property: for any paracompact space X with* $\mathrm{e-dim} X \leq [L]$, *any closed subspace A of X and for any map $f\colon X \to V$ with $f(A) \subset B$ there exists a map $g\colon X \to U \cap B$ such that g is ε-close to f and $g|_A = f|_A$.*

Next, we apply Theorem 3.2.2 to show that if a fibre of a lower $[L]$–continuous mapping contains a compact $UV^{[L]}$-pair, then the property of $UV^{[L]}$-connectedness is satisfied uniformly. Later in Chapter 4 we will make use of it to check that certain mappings are lower $[L]$–continuous.

Lemma 3.2.4. *Let L be a CW-complex such that $[L] \leq [S^n]$ for some n. Let $F\colon X \to Y$ be lower $[L]$-continuous multivalued mapping of topological space*

X to metric space Y. Suppose that a fiber $F(x)$ contains compact $UV^{[L]}$-pair $K \subset M$. Then for any neighborhood U of M in Y there exist neighborhoods V of K in Y and O_x of the point x in X such that for any point $x' \in O_x$ the pair $V \cap F(x') \subset U \cap F(x')$ is $[L]$-connected.

Proof. We can assume that Y is embedded into Banach space E. Then we may consider F as mapping into E. Consider $\varepsilon > 0$ and take a neighborhood $O(M, 3\varepsilon)$ of M in E. By Theorem 3.2.2 there exist $\delta < \varepsilon$ and a neighborhood O_x of the point x such that for any point $x' \in O_x$, for any space Z of extension dimension $\mathrm{e}-\dim Z \leq [L]$ and its closed subset $A \subset Z$, and for any mapping $\psi\colon (A, Z) \to (O(M,\delta) \cap F(x'), O(M,\delta))$ there exists a mapping $\psi'\colon Z \to F(x')$ such that $\psi'|_A = \psi|_A$ and $\mathrm{dist}(\psi, \psi') < \varepsilon$.

Homotopy Extension Theorem (see for example [Bo]), applied to the space E, allows us to find a number σ such that for any space Z, any closed subspace A of Z, and any two σ-close maps $f, g\colon A \to O(K, \sigma)$ such that f has an extension $f'\colon Z \to O(M, \delta)$, it follows that g also has an extension $g'\colon Z \to O(M, 2\delta)$ which is δ-close to f'. Since the pair $K \subset M$ in $F(x)$ has $UV^{[L]}$-property, we can find a number $\mu < \sigma$ such that the pair $O(K,\mu) \cap F(x) \subset O(K,\delta) \cap F(x)$ is $[L]$-connected. By Theorem 3.2.2 there exists $\nu < \mu$ such that for any space A of extension dimension $\mathrm{e}-\dim A \leq [L]$ and for any mapping $\varphi\colon A \to O(K,\nu)$ there is a mapping $\varphi'\colon A \to O(K,\mu) \cap F(x)$ with $\mathrm{dist}(\varphi, \varphi') < \mu$. Put $V = O(K, \nu)$.

Consider a point $x' \in O_x$, a space Z of extension dimension $\mathrm{e}-\dim Z \leq [L]$ and its closed subspace $A \subset Z$. Now any mapping $\varphi\colon A \to V \cap F(x')$ is μ-close to some mapping $\varphi'\colon A \to O(K,\mu) \cap F(x)$ which can be extended to a mapping $\widetilde{\varphi}'\colon Z \to O(M,\delta) \cap F(x)$. Since $\varphi|_A$ and $\varphi'|_A$ are σ-close maps into

$O(K,\sigma)$, φ can also be extended to a mapping $\psi\colon Z \to O(M, 2\delta)$ which is δ-close to $\widetilde{\varphi}'$. Finally, there is another extension $\psi'\colon Z \to O(M, 2\delta+\varepsilon) \cap F(x')$ of the mapping φ. Thus, the pair $V \cap F(x') \subset O(M, 3\varepsilon) \cap F(x')$ is $[L]$-connected. □

The following lemma shows that a compact pair has $UV^{[L]}$–property with respect to ANE($[L]$)–subspace if and only if it has this property with respect to ambient ANE($[L]$)–space.

Lemma 3.2.5. *Let L be a CW-complex such that $[L] \le [S^n]$ for some n. Consider spaces $K \subset M \subset Y \subset E$, where K and M are compacta, Y and E are metric* ANE($[L]$)*-spaces. Then $K \subset M$ is $UV^{[L]}$-pair in Y if and only if it is $UV^{[L]}$-pair in E.*

Proof. First suppose that $K \subset M$ is $UV^{[L]}$-pair in Y. Consider a multivalued mapping F of the unit interval $I = [0, 1]$ defined as follows: $F(0) = Y$ and $F(x) = E$ for any positive $x \in I$. It is easy to see that F is lower $[L]$-continuous. Now Lemma 3.2.4 implies the $UV^{[L]}$-property of the pair $K \subset M$ in E.

Now assume that $K \subset M$ is $UV^{[L]}$-pair in E. Let U be an open neighborhood of M in Y and consider an open neighborhood $O(M, 2\varepsilon)$ in E such that $O(M, 2\varepsilon) \cap Y \subset U$. Corollary 3.2.3 provides us with the positive number $\delta < \varepsilon$ such that for any space Z of extension dimension $\mathrm{e} - \dim Z \le [L]$ and its closed subset $A \subset Z$, and for any mapping $\psi\colon (A, Z) \to (O(K,\delta) \cap Y, O(K,\delta))$ there exists a mapping $\psi'\colon Z \to Y$ such that $\psi'|_A = \psi|_A$ and $\mathrm{dist}(\psi, \psi') < \varepsilon$. Applying the $UV^{[L]}$-property of the pair $K \subset M$ in E, we can find a neighborhood V' of K in E. Put $V = V' \cap Y$.

By the construction, any mapping $\varphi\colon A \to V$ of closed subset A of space Z of extension dimension $\mathrm{e-dim} Z \le [L]$ can be extended to a mapping $\psi\colon Z \to O(K, \delta)$. Finally, the choice of δ guarantees existence of an extension $\psi'\colon Z \to O(M, 2\varepsilon) \cap Y$ of the mapping φ. □

Now we are ready to prove invariance theorem.

Theorem 3.2.6. *Let L be a CW-complex such that $[L] \le [S^n]$ for some n. Suppose that a compact pair $K \subset M$ is $UV^{[L]}$-connected with respect to embedding in some Polish* ANE([L])*-space B. Then this pair is $UV^{[L]}$-connected with respect to any embedding in any Polish* ANE([L])*-space.*

Proof. By [Ch3, Theorem 2.3.17] we can find an embedding $i\colon M \to \mathbb{R}^\omega$ which can be extended to an embedding of any Polish space containing M.

If the pair $K \subset M$ is $UV^{[L]}$-connected in a Polish space B, we can extend i to an embedding of B in $\mathbb{R}^\omega$ and the pair $K \subset M$ is $UV^{[L]}$-connected in $\mathbb{R}^\omega$ by Lemma 3.2.5.

Let now Y be an arbitrary Polish ANE([L])-space, containing M. Extending i to an embedding of Y into $\mathbb{R}^\omega$, we obtain $UV^{[L]}$-connectedness of the pair $K \subset M$ in Y by Lemma 3.2.5. □

3.3 Local Characterization of ANE([L])-spaces

In this section we prove extension-dimensional version of Dugundji theorem. We show that in finite-dimensional case ANE([L])-spaces coincide with $LC^{[L]}$-*spaces.*

First we introduce two types of local property.

Definition 3.3.1. Let L be a CW-complex. A space X is said to be

(a) $[L]$-*contractible* (notation: $X \in LC^{[L]}$) if for any point $x \in X$ and each neighborhood U_x of x there exists a smaller neighborhood V_x such that the inclusion $V_x \hookrightarrow U_x$ is $[L]$-homotopic in U_x to a constant map.

(b) *locally absolute extensor modulo* $[L]$ (notation: $X \in LAE([L])$) if for any point $x \in X$ and each neighborhood U_x of x there exists a smaller neighborhood V_x such that the pair $V_x \subset U_x$ is $[L]$–connected.

Clearly, property (b) implies property (a). It is not hard to see that any ANE($[L]$)-space possesses both properties (a) and (b). The following theorem shows that in finite-dimensional case all these properties are equivalent for Polish spaces.

Theorem 3.3.2. *Let L be a locally finite countable CW-complex such that $[L] \leq [S^n]$ for some n. Then for a Polish space Y property $Y \in LC^{[L]}$ implies $Y \in$ ANE($[L]$).*

Proof. By virtue of Proposition 2.4.2, it suffices to check property $Y \in$ ANE($[L]$) for Polish spaces. Since any Polish space X with $\mathrm{e}-\dim X \leq [L]$ admits closed embedding into Polish $AE([L])$-space of extension dimension $\leq [L]$ [Ch], we may assume that $X \in AE([L])$.

Consider a closed subspace A of X. Let $f \colon A \to Y$ be a continuous mapping. By [Bo, Theorem 3.1.4] we can find an open covering ω of $X \backslash A$ with the following property: (i) for any point $a \in A$ and any its neighborhood O_a in X there exists a neighborhood V_a of a in X such that for all $W \in \omega$ if $W \cap V_a \neq \varnothing$ then $U \subset O_a$. Since $\dim(X \backslash A) \leq n$ (recall that $[L] \leq [S^n]$)

there exists an open refinement $u = \bigcup_{k=0}^{n} u_k$ of ω where u_k is a countable discrete system of open disjoint sets [Eng].

For each $U_i^0 \in u_0$ choose $a_i \in A$ such that $\operatorname{dist}(a_i, U_i^0) \leq \sup\{\operatorname{dist}(x, A) \mid x \in U_i^0\}$ and define a mapping f_0 on $W_0 = \cup\{U_i^0 \mid U_i^0 \in u_0\} \cup A$ as follows: $f_0|_A = f|_A$ and $f_0(U_i^0) = f(a_i)$. It is easily seen that f_0 is continuous.

By induction on $k = 1, \ldots, n$ we shall find neighborhoods W_k of A in $\bigcup_{j=0}^{k}\{U_i^j \mid U_i^j \in u_j\} \bigcup A$ and using f_{k-1} we shall extend f to $f_k \colon W_k \to Y$. Since u covers $X \backslash A$ the mapping f_n extends f to the neighborhood W_n of A in X.

Suppose that f_{k-1} has been already constructed. Since $Y \in LC^{[L]}$, for each $a \in A$ there exists a neighborhood O_a of a in X such that $f_{k-1}|_{O_a}$ is $[L]$-homotopic to a constant map in Y. Applying to O_a property (i) of u find neighborhood $V_a \subset O_a$. Put $\mathbf{V}_k = \bigcup\{V_a \mid a \in A\}$ and $W_k = \cup\{U_i^k \mid U_i^k \subset \mathbf{V}_k\} \cup W_{k-1}$. Observe that for all $U_i^k \in u_k$ we have: (ii) $f_{k-1}|_{U_i^k \cap W_{k-1}}$ is $[L]$-homotopic to a constant map in Y provided $U_i^k \subset \mathbf{V}_k$.

We shall define f_k as an extension of f_{k-1} from the set $W_{k-1} \backslash (\cup\{U_i^k \mid U_i^k \subset \mathbf{V}_k\})$. Since the system u_k is disjoint, we can define f_k independently on every $U_i^k \subset \mathbf{V}_k$. Consider an arbitrary $U_i^k \in u_k$ such that $U_i^k \subset \mathbf{V}_k$. If $W_{k-1} \backslash U_i^k$ is open in X, choose a point $a_i \in A$ such that $\operatorname{dist}(a_i, U_i^k) \leq \sup\{\operatorname{dist}(x, A) \mid x \in U_i^k\}$ and define $f_k(U_i^k) = f(a_i)$. Otherwise let G_i be an open neighborhood of $W_{k-1} \backslash U_i^k$ in $W_{k-1} \cup U_i^k$ such that $\overline{G_i} \cap (U_i^k \backslash W_{k-1}) = \varnothing$. Let $F_i = \overline{G_i} \cap U_i^k$.

Observe that $U_i^k \cap W_{k-1}$ is ANE($[L]$)-space as an open subspace of $AE([L])$-space X. Hence $\operatorname{Cone}(U_i^k \cap W_{k-1})$ is $AE([L])$-space and therefore inclusion of F_i into the base of the cone can be extended to a map of U_i^k into this cone. By (ii) there exists an extension of $f_{k-1}|_{F_i}$ to the set U_i^k. Let $f_k|_{U_i^k}$ be an extension of $f_{k-1}|_{F_i}$ such that $\operatorname{diam}(f_k(U_i^k)) < 2 \cdot \inf\{\operatorname{diam}(g(U_i^k)) \mid g \text{ extends } f_{k-1}|_{F_i}\}$.

Since u_k is discrete system it suffices to check continuity of f_k at every point $a \in A$. Fix $\varepsilon > 0$. Since $Y \in LC^{[L]}$ and f_{k-1} is continuous mapping there exists neighborhood O_a of a in X such that $f_{k-1}|_{O_a}$ is $[L]$-homotopic to a constant map in $\varepsilon/5$-neighborhood of $f(a)$. Applying property (i) of u to O_a find neighborhood V_a of a. Additionally, we may assume that $V_a = O(a, \delta)$ for some $\delta > 0$ such that $O(a, 3\delta) \subset O_a$. For all $U_i^k \in u_k$ such that $U_i^k \subset \mathbf{V}_k$ and $U_i^k \cap V_a \neq \varnothing$ we have $U_i^k \subset O_a$ by the choice of V_a. Therefore construction of $f_k|_{U_i^k}$ and choice of O_a imply $\operatorname{diam}(f_k(U_i^k)) < \frac{4}{5}\varepsilon$. If $W_{k-1} \backslash U_i^k$ is open in X then by the construction we have $f(U_i^k) = f(a_i)$ where $a_i \in O_a$. Hence $\operatorname{dist}(f(U_i^k), f(a)) < \varepsilon/5$ in this case. Otherwise $f_k|_{U_i^k}$ was obtained as an extension of f_{k-1} from nonempty set F_i and it follows that $\operatorname{dist}(f_k(U_i^k), f(a)) < \frac{4}{5}\varepsilon + \frac{1}{5}\varepsilon = \varepsilon$. Therefore $\operatorname{dist}(f_k(V_a), f(a)) < \varepsilon$ as required. □

Corollary 3.3.3. *Let L be a locally finite countable CW-complex such that $[L] \leq [S^n]$ for some n. The following properties are equivalent for any Polish space Y.*

(i) $Y \in LC^{[L]}$.

(ii) $Y \in LAE([L])$.

(iii) $Y \in \mathrm{ANE}([L])$.

Chapter 4

Selections Of Multivalued Mappings Of Finite-Dimensional Space

This chapter is devoted to the proof of extension-dimensional version of finite-dimensional selection theorem. In the first section of the present chapter we obtain various facts related to local properties of multivalued mappings. Further, in Section 4.2 we construct compact-valued upper semicontinuous selections. These results are applied in the proofs of selection theorems, which are contained in Section 4.3.

4.1 Local Properties of Multivalued Mappings

In this section we prove several technical statements on local properties of multivalued mappings. Notice that an important partial case of local property of multivalued mapping is $[L]$–continuity earlier introduced in Section 2.6. We follow definitions and notations from [Du-M].

Definition 4.1.1. An ordering α of subsets of a space Y is *proper* if the following conditions are satisfied:

(i) If $W\alpha V$, then $W \subset V$;

(ii) If $W \subset V$, and $V\alpha R$, then $W\alpha R$;

(iii) If $W\alpha V$, and $V \subset R$, then $W\alpha R$.

Given a proper ordering, we can introduce the corresponding local property and define the notion of lower α-continuity.

Definition 4.1.2. Let α be a proper ordering.

(a) A metric space Y is *locally of type* α if for each $y \in Y$ and any its neighborhood V there exists neighborhood W of y such that $W\alpha V$.

(b) A multivalued mapping $F\colon X \multimap Y$ of topological space X into metric space Y is said to be *lower α-continuous* if for any points $x \in X$ and $y \in F(x)$ and for any neighborhood V of y in Y there exist neighborhoods W of y in Y and U of x in X such that $(W \cap F(x'))\alpha(V \cap F(x'))$ provided $x' \in U$.

Below we list some important examples of local property and correspondent notions of lower α-continuity. Observe, in particular, that lower $[L]$-continuity is a partial case of lower α-continuity.

Example 4.1.3. (i) If $W\alpha V$ means that W is contractible in V, then locally of type α means locally contractible.

(ii) Suppose that $W\alpha V$ means that every continuous mapping of the n-sphere into W is homotopic to a constant mapping in V. Then locally of type α means precisely LC^n.

(iii) If in the previous example we consider the special case $n = -1$ then the property $W\alpha V$ means that V is non-empty. And lower α-continuity is lower semicontinuity.

(iv) If $W\alpha V$ means that the pair $W \subset V$ is $[L]$-connected, then locally of type α means local absolute extensor in dimension $[L]$, and lower α-continuity of multivalued mapping is precisely $[L]$-continuity.

The following lemma shows that the property of lower α-continuity is uniform in a certain sense.

Lemma 4.1.4. *Let $F\colon X \to Y$ be lower α-continuous multivalued mapping of topological space X to metric space Y. Consider a point $y \in F(x)$. Then for any $\varepsilon > 0$ there exist $\delta > 0$ and neighborhoods O_y of the point y in Y and O_x of the point x in X such that for any points $x' \in O_x$ and $y' \in F(x') \cap O_y$ we have $(O(y',\delta) \cap F(x'))\alpha(O(y',\varepsilon) \cap F(x'))$.*

Proof. Since the mapping F is lower α-continuous, we can find positive $\delta < \varepsilon/4$ and a neighborhood O_x of the point x such that

$$(O(y, 2\delta) \cap F(x'))\alpha(O(y, \varepsilon/2) \cap F(x'))$$

for every point $x' \in O_x$. Put $O_y = O(y,\delta)$. Then for every $x' \in O_x$ and every $y' \in F(x') \cap O_y$ we have inclusions $O(y',\delta) \subset O(y, 2\delta)$ and $O(y,\varepsilon/2) \subset O(y',\varepsilon)$. Therefore, $(O(y',\delta) \cap F(x'))\alpha(O(y',\varepsilon) \cap F(x'))$. □

The above lemma can be extended on the case of compact subset contained in a fibre of lower α-continuous mapping.

Lemma 4.1.5. *Let $F\colon X \to Y$ be lower α-continuous multivalued mapping of topological space X to metric space Y. Consider a compact subset K of the fiber $F(x)$. Then for any $\varepsilon > 0$ there exist $\delta > 0$ and neighborhoods OK of compactum K in Y and O_x of the point x in X such that for any points $x' \in O_x$ and $y' \in F(x') \cap OK$ we have $(O(y', \delta) \cap F(x'))\alpha(O(y', \varepsilon) \cap F(x'))$.*

Proof. For every point $y \in K$ find a number $\delta_y > 0$ and neighborhoods Oy of the point y and $O_y x$ of the point x provided by Lemma 4.1.4. Choose a finite subcovering $\{Oy_i\}_{i=1}^m$ of the cover $\{Oy\}_{y \in K}$ of compactum K and consider the corresponding numbers $\delta_1, \dots, \delta_m$ and neighborhoods $O_1 x, \dots, O_m x$ of the point x. Clearly, we can put

$$OK = \bigcup_{i=1}^m Oy_i, \qquad \delta = \min_{1 \le i \le m} \delta_i, \qquad Ox = \bigcap_{i=1}^m O_i x.$$

□

Now we further generalize Lemma 4.1.5 on the case of compact submapping. Later we will apply this generalization to construct coverings of graphs of multivalued mappings so that these coverings will have some specific properties.

Lemma 4.1.6. *Suppose that a lower α-continuous multivalued map $F\colon X \to Y$ of paracompact space X to metric space Y contains a compact submapping $H\colon X \to Y$. Then for any continuous positive function $\varepsilon\colon X \to \mathbb{R}$ there exist a continuous positive function $\delta\colon X \to \mathbb{R}$ and a neighborhood U of the* *graph* Γ_H *such that for any points* $x \in X$ *and* $y \in F(x) \cap U(x)$ *we have* $(O(y, \delta(x)) \cap F(x))\alpha(O(y, \varepsilon(x)) \cap F(x))$.

Proof. Applying Lemma 4.1.5, find for every point $x \in X$ a number $\sigma(x)$ and open neighborhoods Ox of the point x and $OH(x)$ of the compactum $H(x)$ such that $(O(y', \sigma(x)) \cap F(x'))\alpha(O(y', \varepsilon(x)/2) \cap F(x'))$ for any points $x' \in Ox$ and $y' \in F(x') \cap OH(x)$. It is easy to see that we may take neighborhood Ox to be so small that $H(Ox)$ is contained in $OH(x)$ and $\sup_{x' \in Ox}\varepsilon(x') < 2 \cdot \inf_{x' \in Ox}\varepsilon(x')$.

Find a locally finite cover $\omega = \{W_\lambda\}_{\lambda \in \Lambda}$ which refines the cover $\{Ox\}_{x \in X}$ and for every $\lambda \in \Lambda$ take a point x_λ such that W_λ is contained in Ox_λ. Let $\delta\colon X \to \mathbb{R}$ be a continuous positive function such that for every point $x \in X$ we have $\delta(x) \leq \min\{\sigma(x_\lambda) \mid x \in W_\lambda\}$. Put $U = \cup_{\lambda \in \Lambda} W_\lambda \times OH(x_\lambda)$. Observe that $H(W_\lambda)$ is contained in $OH(x_\lambda)$ and the sets W_λ cover X. This implies that U is a neighborhood of the graph Γ_H.

Consider an arbitrary point $\{x\} \times \{y\} \in U \cap \Gamma_F$. By the construction of U, there is a set W_λ containing x such that $\{x\} \times \{y\} \in W_\lambda \times OH(x_\lambda)$. Then $(O(y, \sigma(x_\lambda)) \cap F(x))\alpha(O(y, \varepsilon(x_\lambda)/2) \cap F(x))$. Therefore, since $\varepsilon(x) > \varepsilon(x_\lambda)/2$ and $\delta(x) \leq \sigma(x_\lambda)$, we have $(O(y, \delta(x)) \cap F(x))\alpha(O(y, \varepsilon(x)) \cap F(x))$. □

Further we are going to develop a certain technique for construction of lower α-continuous multivalued mappings. In order to do this we need to consider covers of the product $X \times Y$ of paracompact space X and metric space Y. It will be convenient to make use of "rectangular" covers. More precisely, we will consider covers of the form

$$\omega \times \varepsilon = \{W \times O(y, \varepsilon(x)) \mid x \in W \in \omega,\ x \in X\}$$

where ω is a covering of X and $\varepsilon\colon X \to \mathbb{R}$ is a continuous positive function.

Remark 4.1.7. Recall that a real-valued function $\varepsilon\colon X \to \mathbb{R}$ is said to be *locally positive* if for any point x there exists a neighborhood on which the infimum of the function is positive. If $\varepsilon(x)$ is an arbitrary locally positive function defined on a paracompact space, then we can find a positive continuous function which is less than this function, as follows. Consider a locally finite covering $\{W_\alpha\}$ of this paracompact space such that the function $\varepsilon(x)$ is greater than some positive number c_α on each element W_α of this covering. Construct a partition of the unity $\{\varphi_\alpha(x)\}$ subordinated to this covering. Letting $\varphi(x) = \sum_\alpha c_\alpha \cdot \varphi_\alpha(x)$ we obtain the desired continuous function.

Let $H\colon X \to Y$ be a compact multivalued mapping of paracompact space X to metric space Y. The following lemma shows that it is sufficient to consider only "rectangular" covers of this graph $\Gamma_H \subset X \times Y$ of the form $\omega \times \varepsilon$.

Lemma 4.1.8. *For any open cover γ of the graph $\Gamma_H \subset X \times Y$ of a compact multivalued mapping $H\colon X \to Y$ of paracompact space X to metric space Y there exist an open cover $\omega \in \operatorname{cov}X$ and a continuous positive function $\varepsilon\colon X \to \mathbb{R}$ such that the cover $\omega \times \varepsilon$ of the graph Γ_H refines γ.*

Proof. Take a point $x \in X$. For every point $\{x\} \times \{y\} \in \{x\} \times H(x)$ we fix its open neighborhood $O_y x \times Oy$ refining γ. Find a finite subcover $\{Oy_i\}_{i=1}^N$ of the cover $\{Oy\}_{y\in H(x)}$ of the compactum $H(x)$ and let $2\lambda(x)$ be its Lebesgue number. We put

$$Ox = (\bigcap_{i=1}^{N} O_{y_i}x) \cap \{x' \in X \mid H(x') \subset O(H(x), \lambda(x))\}$$

Then for any points $x' \in Ox$ and $y' \in H(x')$ the set $Ox \times O(y', \lambda(x))$ refines γ. Find an open locally finite cover $\omega \in \text{cov}X$ refining the cover $\{Ox\}_{x\in X}$. For every $W \in \omega$ fix an element Ox_W of the cover $\{Ox\}_{x\in X}$ such that $W \subset Ox_W$. Since the cover ω is locally finite, the function $\varepsilon'(x) = \min\limits_{x\in W\in\omega} \lambda(x_W)$ is locally positive. Let ε be any positive continuous function which is less than ε'. Then we define $\omega \times \varepsilon = \{W \times O(y, \varepsilon(x)) \mid x \in W \in \omega, y \in H(x) \subset Y\}$. $\square$

Consider a lower semicontinuous mapping $\Phi\colon X \to Y$. Suppose that Φ contains a compact submapping Ψ. We introduce the notion of starlike α-refinement, relative to a pair (Ψ, Φ), of coverings of the form $(\omega \times \varepsilon)$, where $\omega \in \text{cov}X$ and ε is a positive continuous function on X.

Definition 4.1.9. A covering $(\omega' \times \varepsilon')$ is called *starlike α-refined* into a covering $\omega \times \varepsilon$ relative to a pair (Ψ, Φ) if for any point $z \in \text{St}(\Gamma_\Psi, \omega' \times \varepsilon')$ there exists an element $W \times O(y, \varepsilon(x))$ of the cover $\omega \times \varepsilon$ containing the star $\text{St}(z, \omega' \times \varepsilon')$ and such that

$$(\text{St}(z, \omega' \times \varepsilon')(x') \cap \Phi(x'))\alpha(O(y, \varepsilon(x)) \cap \Phi(x'))$$

for any point $x' \in \text{pr}_X(\text{St}(z, \omega' \times \varepsilon'))$.

Further we will prove existence of starlike refinements. In order to do this we need the following lemma. We say that a positive continuous function $\delta\colon X \to \mathbb{R}$ vary within any element of a covering $\omega \in \text{cov}X$ less than by half if $\sup_{x\in W}\delta(x) < 2 \cdot \inf_{x\in W}\delta(x)$ for each $W \in \omega$.

Lemma 4.1.10. *Suppose that a positive continuous function $\delta\colon X \to \mathbb{R}$ vary within any element of the covering $\omega \in \text{cov}X$ less than by half. Then for any points $p_0 = \{x_0\} \times \{y_0\} \in X \times Y$ and $p = \{x\} \times \{y\} \in \text{St}(p_0, \omega \times \delta)$ the star $\text{St}(p_0, \omega \times \delta)$ is contained in the product $\text{St}(x_0, \omega) \times O(y, 16 \cdot \delta(x))$.*

Proof. For any point $x' \in \mathrm{St}(x_0, \omega)$ we have $\delta(x') \leq 2 \cdot \delta(x_0) \leq 4 \cdot \delta(x)$. Then the distance between points y_0 and y is less than $8 \cdot \delta(x)$. Clearly, every element of the cover $\omega \times \delta$ containing the point p_0 lies in the set $\mathrm{St}(x_0, \omega) \times O(y_0, 8 \cdot \delta(x))$. Therefore, the star $\mathrm{St}(p_0, \omega \times \delta)$ is contained in the product $\mathrm{St}(x_0, \omega) \times O(y, \mathrm{dist}(y, y_0) + 8 \cdot \delta(x))$. The lemma is proved. □

Now we prove existence of starlike refinements. We will apply this result in Section 4.3 in the procedure of obtaining a sequence of UV^L–filtered submappings fibres of which are uniformly bounded by decreasing to zero sequence of positive numbers. This in turn will guarantee a convergence of the sequence of multivalued mappings.

Lemma 4.1.11. *Suppose that a lower α-continuous multivalued mapping $F\colon X \to Y$ of paracompact space X to metric space Y contains a compact submapping $H\colon X \to Y$. Then for any continuous positive function $\varepsilon\colon X \to \mathbb{R}$ and any open cover $\omega \in covX$ there exist a continuous positive function $\delta\colon X \to \mathbb{R}$ and an open cover $\omega' \in covX$ such that the cover $\omega' \times \delta$ is starlike α-refined into a covering $\omega \times \varepsilon$ relative to a pair (H, F).*

Proof. Lemma 4.1.6 provides us with a neighborhood $\mathcal{U}$ of the graph Γ_H and continuous positive function $\sigma\colon X \to \mathbb{R}$ such that $16\sigma < \varepsilon$ and for any points $x \in X$ and $y \in F(x) \cap \mathcal{U}(x)$ we have $(O(y, 16\sigma(x)) \cap F(x)) \alpha (O(y, \varepsilon(x)) \cap F(x))$. According to Lemma 4.1.8 there exists a covering $\omega'' \times \nu$ of the graph Γ_H such that the star $\mathrm{St}(\Gamma_H, \omega'' \times \nu)$ is contained in $\mathcal{U}$. Define a continuous positive function $\delta\colon X \to \mathbb{R}$ letting $\delta(x) = \frac{1}{16} \min\{\sigma(x), \nu(x)\}$. Let a covering $\omega' \in covX$ be a starlike refinement of ω and ω''. We may also assume that the function ε vary within any element of the covering ω' less than by half.

This construction guarantees that for every point $p_0 = \{x_0\} \times \{y_0\} \in \mathrm{St}(\Gamma_H, \omega' \times \delta)$ the star $\mathrm{St}(p_0, \omega' \times \delta)$ is contained in $\mathcal{U}$. Indeed, the star $\mathrm{St}(x_0, \omega')$ is contained in some element V of the cover ω''. Consider a point $p = \{x\} \times \{y\} \in \Gamma_H \cap \mathrm{St}(p_0, \omega' \times \delta)$. By the construction of the cover $\omega'' \times \nu$ the set $V \times O(y, \nu(x))$ is contained in $\mathcal{U}$. By Lemma 4.1.10 the star $\mathrm{St}(p_0, \omega' \times \delta)$ is contained in $V \times O(y, 16\delta(x))$.

Consider an arbitrary point $x' \in \mathrm{St}(x_0, \omega')$ and suppose that the intersection of the set $\mathrm{St}(p_0, \omega' \times \delta)(x')$ with the fiber $F(x')$ is not empty and contains point y'. Then this intersection is contained in $O(y', 16\delta(x'))$. Since the point $\{x'\} \times \{y'\}$ lies in $\mathcal{U}$, then $(O(y', 16\sigma(x')) \cap F(x))\alpha(O(y', \varepsilon(x')) \cap F(x'))$. Fix an element W of the cover ω containing the star $\mathrm{St}(x_0, \omega')$. Clearly, the element $W \times O(y', \varepsilon(x'))$ of the cover $\omega \times \varepsilon$ contains the star $\mathrm{St}(p_0, \omega' \times \delta)$ (we apply Lemma 4.1.10) and the set $\{x'\} \times O(y', \varepsilon(x'))$. □

In what follows we call the set

$$\mathrm{st}(A, \omega) = \bigcup \{U \in \omega |\ A \subset U\}$$

the *small star* of a set A relative to a covering ω.

Given a compact multivalued mapping F and a complete lower α-continuous mapping Φ, the following lemma (actually, it is Lemma of Continuity of Star Trace from [S-Br]) allows for the construction of a complete and lower α-continuous submapping of the mapping F which is close to Φ.

Lemma 4.1.12. *Let ω be an open covering of a metric space Y, let $F\colon X \to Y$ be a compact multivalued mapping, and let $\Phi\colon X \to Y$ be a complete lower α-continuous mapping. Then the multivalued mapping G which assigns set $\Phi(x) \cap \mathrm{st}(F(x), \omega)$ to point $x \in X$ is complete and lower α-continuous.*

Proof. First, observe that the multivalued mapping G' which assigns small star $\mathrm{st}(F(x), \omega)$ to point $x \in X$ has open graph in the space $X \times Y$. Indeed, for a point $\{x\} \times \{y\} \in \Gamma_{G'}$ there is an element $W \in \omega$ containing the image $F(x)$. The upper semicontinuity of F implies that for some neighborhood $Ox \subset X$ of the point x the image $F(Ox)$ is contained in W. Hence the set $Ox \times W$ is an open neighborhood of the point $\{x\} \times \{y\}$ in the graph $\Gamma_{G'}$.

Finally, observe that the completeness and the lower α-continuity of mapping Φ imply these properties for the mapping $G = G' \cap \Phi$ by virtue of the openness of the graph $\Gamma_{G'}$. □

4.2 Compact-Valued Selections

This section is devoted to the construction of compact-valued upper semicontinuous selections for multivalued mappings. In particular, we prove Lemma 4.2.10 which allows us to find $UV^{[L]}$-connected subfiltration of lower $[L]$-continuous and $[L]$-connected filtration. This important result will be applied in the proof of selection theorems.

The following lemma establish an important property of approximately $[L]$-soft mappings related to $UV^{[L]}$-connectedness.

Lemma 4.2.1. *Let $f \colon X \to Y$ be a continuous single-valued mapping of compact metric spaces. Let $Y_1 \subset Y$ be a closed subset and X_1 be its inverse image $X_1 = f^{-1}(Y_1)$. If the mapping $f|_{X_1} \colon X_1 \to Y_1$ is approximately $[L]$-invertible and the pair $X_1 \subset X$ is $UV^{[L]}$-connected, then the pair $Y_1 \subset Y$ is also $UV^{[L]}$-connected.*

Proof. We can assume that f is a submapping of the projection $\pi \colon l_2 \times l_2 \to l_2$.

Let U be a neighborhood of the compact space Y in l_2. We need to find a neighborhood V of Y_1 such that the pair $V \subset U$ is $[L]$-connected.

Since the pair $X_1 \subset X$ is $UV^{[L]}$-connected we can find an open neighborhood W of X_1 such that the pair $W \subset \pi^{-1}(U)$ is $[L]$-connected. Since the mapping $f|_{X_1}$ is approximately $[L]$-invertible there exists a neighborhood V of Y_1 such that any mapping $g\colon Z \to V$ of space Z of extension dimension $\mathrm{e}-\dim Z \leq [L]$ admits a lifting map $\widetilde{g}\colon Z \to U$.

Further, if $g\colon A \to V$ is a mapping of closed subset $A \subset Z$ where $\mathrm{e}-\dim Z \leq [L]$, we consider a lifting map $\widetilde{g}\colon A \to W$ and extend it to a mapping $g'\colon Z \to \pi^{-1}(U)$. Finally, we can define an extension of g as $\pi \circ g'$. □

The following definition plays an important role in our consideration.

Definition 4.2.2. The *exponential of a pair* $\exp(A,B)$ is a subspace of $\exp B$ formed by compact sets $K \subset B$ containing A. We define the *$UV^{[L]}$-exponential of the pair* (A,B) as follows:

$$UV^{[L]}\text{-}\exp(A,B) = \{K \in \exp B \mid \text{ the pair } A \subset K \text{ is } UV^{[L]}\text{-connected}\}.$$

Now we prove a sequence of lemmas which finally allows for construction of compact $UV^{[L]}$-connected subfiltration. First we show that $UV^{[L]}$-exponential is a closed subset of the exponential of a pair.

Lemma 4.2.3. *For any pair (K,X) formed by a compact set K and a metric space X, the set* $\mathrm{UV}^{[L]}\text{-}\exp(K,X)$ *is closed in* $\exp(K,X)$.

Proof. Consider a sequence of compact sets $\{K_m\}_{m\geq 1}$ from the $\mathrm{UV}^{[L]}$-exponential of the pair (K,X) which is convergent with respect to the Hausdorff metric to a compact set K_0. Take a neighborhood U of K_0. There exists $m \geq 1$ such that $K_m \subset U$. By $UV^{[L]}$-connectedness of the pair $K \subset K_m$ we can find a neighborhood V of the compact set K such that the pair $V \subset U$ is $[L]$-connected. □

Definition 4.2.4. The *fiberwise exponential* of a multivalued map $F\colon X \to Y$ is the map $\exp F\colon X \to \exp Y$ which assigns $\exp F(x)$ to a point x.

Lemma 4.2.5. *The fiberwise exponential of a complete mapping is complete.*

Proof. Since the exponential of an open set is open and the exponential of an intersection coincides with the intersection of exponentials, the exponential of a G_δ-set is a G_δ-set. Since the exponential of a closed set is closed, the exponential of a fiber closed in a G_δ-set is closed in the exponential of a G_δ-set. □

Lemma 4.2.6. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Suppose that a metric space Z contains a compactum K and the pair $K \subset Z$ is $[L]$-connected. Then there exists a compactum $K' \subset Z$ containing K such that the pair $K \subset K'$ is $UV^{[L]}$-connected.*

Proof. Proposition 2.4.6 implies the existence of compactum X of extension dimension $\mathrm{e}-\dim X \leq [L]$ and a continuous mapping f of X onto K such that every fiber $f^{-1}(y)$ is $UV^{[L]}$–compactum. By Theorem 2.6.13, the mapping f is approximately $[L]$–invertible. By Theorem 2.4.8 there exists $AE([L])$-compactum $\mu^{[L]}$ containing X such that $\mathrm{e}-\dim\mu^{[L]} = [L]$. It is easy to see from Lemma 3.2.6 that the pair $X \subset \mu^{[L]}$ is $UV^{[L]}$-connected.

Since the pair $K \subset Z$ is $[L]$-connected, we can extend mapping f to a mapping $\widetilde{f}\colon \mu^{[L]} \to Z$. Put $K' = \widetilde{f}(\mu^{[L]})$. Then the pair $K \subset K'$ is $UV^{[L]}$-connected by Lemma 4.2.1. □

Definition 4.2.7. For a multivalued mapping $\Phi\colon X \to Y$ and its compact submapping Ψ we define the *fiberwise $UV^{[L]}$-exponential of the pair* $UV^{[L]}\text{-}\exp(\Psi, \Phi)\colon X \to \exp Y$ as a mapping assigning $UV^{[L]}\text{-}\exp(\Psi(x), \Phi(x))$ to a point $x \in X$.

Observe that fiberwise $UV^{[L]}$-exponential provides the basic tool for further construction of compact $UV^{[L]}$-connected subfiltration.

Next, we establish lower semicontinuity of fiberwise $UV^{[L]}$-exponential.

Lemma 4.2.8. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Suppose that a lower $[L]$-continuous mapping $\Phi\colon X \to Y$ of paracompact space X to metric space Y contains a compact submapping Ψ. Then the fiberwise* $\mathrm{UV}^{[L]}$*-exponential of the pair* $\mathrm{UV}^{[L]}\text{-}\exp(\Psi, \Phi)$ *is lower semicontinuous.*

Proof. Let $F = \mathrm{UV}^{[L]}\text{-}\exp(\Psi, \Phi)$. Consider a point $x \in X$ and a compact set $K \in F(x)$. Fix a positive number ε. By Lemma 3.2.4 there exist number $\delta < \varepsilon$ and neighborhood O'_x of the point x such that the pair $O(\Psi(x), \delta) \cap \Phi(x') \subset O(K, \varepsilon) \cap \Phi(x')$ is $[L]$-connected for any point $x' \in O'_x$. Since Φ is lower semicontinuous and K is compact, there exists a neighborhood O''_x of the point x such that $O(y, \varepsilon/2) \cap \Phi(x') \neq \emptyset$ for any points $y \in K$ and $x' \in O''_x$ (apply Lemma 4.1.5). Let O_x be a neighborhood of x such that $O_x \subset O'_x \cap O''_x$ and $\Psi(x') \subset O(\Psi(x), \delta)$ for every point $x' \in O_x$.

Consider any point $x' \in O_x$. According to Lemma 4.2.6 there exists a compactum $\widetilde{K} \subset \Phi(x') \cap O(K, \varepsilon)$ such that the pair $\Psi(x') \subset \widetilde{K}$ is $UV^{[L]}$-connected, and therefore $\widetilde{K} \in F(x')$. It remains to enlarge (if necessary) the compactum $\widetilde{K}$ to obtain a compactum K' with $\operatorname{dist}(\widetilde{K}, K') < \varepsilon$. By the choice of the neighborhood O''_x there is a finite set of points P in $\Phi(x')$ such that $\operatorname{dist}(K, P) < \varepsilon$. We put $K' = \widetilde{K} \cup P$. □

The following lemma describes the main step for the construction of compact $UV^{[L]}$-connected subfiltration.

Lemma 4.2.9. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Let $\Phi\colon X \to Y$ be a complete lower $[L]$-continuous mapping of a paracompact space into a complete metric space containing a compact submapping Ψ such that the pair $\Psi \subset \Phi$ is fiberwise $[L]$-connected. Then there exists a compact submapping Ψ' of the mapping Φ such that the pair $\Psi(x) \subset \Psi'(x)$ is $UV^{[L]}$-connected for any $x \in X$.*

Proof. Let $F = \mathrm{UV}^{[L]}\text{-}\exp(\Psi, \Phi)$. By Lemma 4.2.6, the mapping F has nonempty fibers. According to Lemma 4.2.8, F is lower semicontinuous. By Lemma 4.2.3, F is fiberwise closed in $\exp(\Psi, \Phi)$, and therefore, the completeness of this mapping follows from the completeness of the latter, which was established in Lemma 4.2.5. Further, by the compact-valued selection theorem from [S-Br], the mapping F admits a compact selection F'. Define a compact mapping $\Psi'\colon X \to Y$ by the equality $\Psi'(x) = \bigcup_{K \in F'(x)} K$. Since for any $K \in F'(x)$, the pair $\Psi(x) \subset K$ is $UV^{[L]}$-connected, then the pair $\Psi(x) \subset \Psi'(x)$ is also $UV^{[L]}$-connected. □

Next, we prove the main lemma of the present chapter.

Lemma 4.2.10. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Then any $[L]$-connected lower $[L]$-continuous increasing n-filtration*

$\Phi = \{\Phi_k\}$ *of complete mappings of a paracompact space to a complete metric space contains a compact* $\mathrm{UV}^{[L]}$*-connected* n*-subfiltration* $\Psi = \{\Psi_k\}$.

Proof. We construct filtration Ψ by induction so that Lemma 4.2.9 is applied on each step. We begin with the construction of a compact submapping $\Psi_0 \subset \Phi_0$. This can be done by virtue of the compact-valued selection theorem from [S-Br] (indeed, the initial term of the filtration Φ is lower semicontinuous). If compact $\mathrm{UV}^{[L]}$-connected filtration $\{\Psi_m\}_{m<k}$ has been constructed such that $\Psi_m \subset \Phi_m$ for $m < k$, then the pair $\Psi_{k-1} \subset \Phi_k$ satisfies the conditions of Lemma 4.2.9, and we use this lemma to complete the construction of the filtration. □

The following lemma is a generalization of Lemma 3.2.4.

Lemma 4.2.11. *Let* L *be a CW-complex such that* $[L] \leq [S^n]$ *for some* n. *Let* $F\colon X \to Y$ *be lower* $[L]$*-continuous multivalued mapping of paracompact space* X *to metric space* Y. *For a closed subset* $A \subset X$ *consider a compact submappings* $H \subset \widetilde{H}\colon A \to Y$ *of the mapping* $F|_A$. *If the pair* $H \subset \widetilde{H}$ *is fiberwise* $UV^{[L]}$*-connected, then for any neighborhood* $\mathcal{U}$ *of the graph* $\Gamma_{\widetilde{H}}$ *in the product* $X \times Y$ *there exists a neighborhood* $\mathcal{V}$ *of the graph* Γ_H *in the product* $X \times Y$ *such that the pair* $\mathcal{V}(x) \cap F(x) \subset \mathcal{U}(x) \cap F(x)$ *is* $[L]$*-connected for every* x *from some open neighborhood of the set* A.

Proof. Lemma 3.2.4 allows us to find for every point $x \in A$ an open neighborhood $O_x \subset X$ of the point x and an open neighborhood $V_x \subset Y$ of the set $H(x)$ such that the set $H(O_x \cap A)$ is contained in V_x and the pair $V_x \cap F(x') \subset \mathcal{U}(x') \cap F(x')$ is $[L]$-connected for every point $x' \in O_x$. Fix a closed neighborhood B of the set A such that $B \subset \cup_{x \in A} O_x$. Let $\Omega_1 = \{\omega_\lambda\}_{\lambda \in \Lambda}$ be a locally finite open (in B) cover of B refining the cover $\{O_x\}_{x\in A}$. For every $\lambda \in \Lambda$ we take a set $V_\lambda = V_x$ such that $\omega_\lambda \subset O_x$. Let $\Omega_2 \in \text{cov}B$ be a locally finite open cover starlike refining Ω_1. For $x \in \text{Int}B$ we define

$$\mathcal{V}(x) = \cap\{V_\lambda \mid \text{St}(x, \Omega_2) \subset \omega_\lambda\}.$$

Since the cover Ω_1 is locally finite, the set $\mathcal{V}(x)$ is an intersection of finitely many open sets, and, therefore, $\mathcal{V}(x)$ is open.

Since for every λ the pair $V_\lambda \cap F(x) \subset \mathcal{U}(x) \cap F(x)$ is $[L]$-connected, then the pair $\mathcal{V}(x) \cap F(x) \subset \mathcal{U}(x) \cap F(x)$ is $[L]$-connected. Since the cover Ω_2 is locally finite, then for every point $x \in \text{Int}B$ there is a neighborhood W_x such that for any point $x' \in W_x$ we have $\text{St}(x, \Omega_2) \subset \text{St}(x', \Omega_2)$. Therefore, for every $x' \in W_x$ we have $\mathcal{V}(x) \subset \mathcal{V}(x')$. Thus, the set $\mathcal{V}$ is open. □

Corollary 4.2.12. *Let L be a CW-complex such that $[L] \leq [S^n]$ for some n. Suppose that lower $[L]$-continuous multivalued mapping $F\colon X \to Y$ of paracompact space X to metric space Y contains a singlevalued continuous selection $f\colon A \to Y$ over the closed subset $A \subset X$. Then for any neighborhood $\mathcal{U}$ of the graph Γ_f in the product $X \times Y$ there exists a neighborhood $\mathcal{V}$ of the graph Γ_f in the product $X \times Y$ such that for every point $x \in \text{pr}_X\mathcal{V}$ the pair $\mathcal{V}(x) \cap F(x) \subset \mathcal{U}(x) \cap F(x)$ is $[L]$-connected.*

4.3 Selection Theorems

In this section we prove selection theorems. Let us present the sketch of our strategy. Given a lower $[L]$-continuous $[L]$-connected filtration we apply results of the previous section to construct compact $UV^{[L]}$-connected subfiltration. Next, we inductively construct a sequence of n-$UV^{[L]}$-filtered submappings so that diameters of their fibres decrease to zero uniformly and a decreasing sequence of neighborhoods of graphs of these submappings . Additionally, we will ensure that the sequence of fibres will be a Cauchy sequence. Further, since diameters of fibres decrease to zero, the limit mapping will be single-valued. Finally, our construction guarantees that the resulting mapping is a selection.

In what follows it is convenient to use the notion of a gauge of multivalued mapping.

Definition 4.3.1. The *gauge* of a multivalued mapping $F\colon X \to Y$ is defined as

$$\mathrm{cal}(F) = \sup\{\mathrm{diam}\, F(x) |\ x \in X\}.$$

The following lemma is a key point to the whole approach. It is important to observe that Approximation Theorem 2.6.11 is applied in the proof.

Lemma 4.3.2. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Let X be a paracompact space of extension dimension $\mathrm{e}-\mathrm{dim}X \leq [L]$. If a complete lower $[L]$-continuous mapping $\Phi\colon X \to Y$ into a complete metric space Y contains an n-$UV^{[L]}$-filtered compact submapping Ψ, then any neighborhood of the graph Γ_Ψ contains the graph of a compact n-$UV^{[L]}$-filtered submapping Ψ' of the mapping Φ whose gauge $\mathrm{cal}(\Psi')$ does not exceed any given ε.*

Proof. Let ε be an arbitrary positive number and $\mathcal{U}$ be an open neighborhood of the graph Γ_Ψ in the product $X \times Y$. Consider a covering $\omega_n \times \varepsilon_n$ of the graph Γ_Ψ such that the star $\mathrm{St}(\Gamma_\Psi, \omega_n \times \varepsilon_n)$ is contained in $\mathcal{U}$ (Lemma 4.1.8 is applied), while the function $\varepsilon_n(x)$ does not exceed $\varepsilon/3$.

Successively applying Lemma 4.1.11 to an $[L]$-continuous mapping Φ and for its compact submapping Ψ, we construct the coverings $\{\omega_k \times \varepsilon_k\}_{k=0}^{n-1}$ such that $\omega_k \times \varepsilon_k$ is starlike $[L]$-connectedly refined into $\omega_{k+1} \times \varepsilon_{k+1}$ for any $k < n$. Lemma 4.1.6 provides us with a neighborhood $\mathcal{V}$ of the graph Γ_Ψ in the product $X \times Y$ such that for any point $\{x\} \times \{y\} \in \mathcal{V}$ the star of this point relative to the covering $\omega_0 \times \varepsilon_0$ intersects the fiber $\{x\} \times \Phi(x)$.

By Theorem 2.6.11, there is a continuous singlevalued mapping $\psi \colon X \to Y$ whose graph is contained in $\mathcal{V}$. We define an $[L]$-connected n-filtration $\{G_m\}$ letting $G_m(x) = \Phi(x) \cap \mathrm{St}(\{x\} \times \psi(x), \omega_m \times \varepsilon_m)(x)$. Since the projection of the star $\mathrm{St}(\{x\} \times \psi(x), \omega_n \times \varepsilon_n)$ onto Y has the diameter less than ε, then cal$G_n < \varepsilon$. By Lemma 4.1.12 the filtration $\{G_m\}$ is complete and lower $[L]$-continuous. Finally, Lemma 4.2.10 allows us to find a compact $UV^{[L]}$-connected n-subfiltration $\Psi' = \{\Psi'_k\}$. □

The following theorem shows that complete lower $[L]$-continuous multivalued mapping which contains a $UV^{[L]}$-filtered compact submapping has a singlevalued selection. Moreover, the selection can be constructed so that its graph is contained in a given neighborhood of the graph of the submapping.

Theorem 4.3.3. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Let X be a paracompact space of extension dimension* $\mathrm{e-dim}X \leq [L]$. *If a complete lower $[L]$-continuous multivalued mapping Φ of X into a complete metric space Y contains an n-$UV^{[L]}$-filtered compact submapping Ψ, then Φ contains a singlevalued continuous selection s.*

Selection s can be chosen in such a way that the graph of this selection is contained in any given neighborhood $\mathcal{U}$ of the graph Γ_Ψ in the product $X \times Y$.

Proof. Let $\{\Psi_k\}_{k=0}^n$ be $UV^{[L]}$-filtration of Ψ. Denote Ψ_n by Ψ_n^0 and take an arbitrary neighborhood $\mathcal{U}_0$ of the graph Ψ_n^0. Consider a G_δ-subset $G \subset X \times Y$ such that all fibers of F are closed in G and fix open sets $G_i \subset X \times Y$ such that $G = \cap_{i=0}^\infty G_i$. By induction with the use of Lemma 4.3.2, we construct a sequence of n-$UV^{[L]}$-filtered mappings $\{\Psi_n^k\}_{k=1}^\infty$ and of open neighborhoods of graphs of these mappings $\{\mathcal{U}_k\}_{k=1}^\infty$ such that for any $k \geq 1$, the gauge $\mathrm{cal}\Psi_n^k$ does not exceed $\frac{1}{2^k}$, while the graph Ψ_n^k together with its neighborhood $\mathcal{U}_k$ is in $\mathcal{U}_{k-1} \cap G_{k-1}$. It is not difficult to choose the neighborhood $\mathcal{U}_k$ of the graph Ψ_n^k in such a way that the fibers $\mathcal{U}_k(x)$ have the diameter not more than $3 \cdot \mathrm{cal}\Psi_n^k = \frac{3}{2^k}$.

Then for any $m \geq k \geq 1$ and for any point $x \in X$, $\Psi_n^m(x) \subset O(\Psi_n^k(x), \frac{3}{2^k})$; this implies that $\{\Psi_n^k\}_{k=1}^\infty$ is a Cauchy sequence. Since Y is complete, there exists a limit s of this sequence. The mapping s is singlevalued by the condition $\mathrm{cal}\Psi_n^k < \frac{1}{2^k}$ and is upper semicontinuous (and, therefore, is continuous) by the upper semicontinuity of all the mappings Ψ_n^k. Clearly, for any $x \in X$ the point $s(x)$ lies in $G(x)$ and is a limit point of the set $F(x)$. Since $F(x)$ is closed in $G(x)$, then $s(x) \in F(x)$, i.e. s is a selection of the mapping F. □

Corollary 4.3.4. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Let X be a paracompact space of extension dimension $\mathrm{e-dim}X \leq [L]$. Let a complete lower $[L]$-continuous multivalued mapping Φ of X into a complete metric space Y contains an n-$UV^{[L]}$-filtered compact submapping Ψ which is singlevalued on some closed subset $A \subset X$. Then any neighborhood $\mathcal{U}$ of the graph Γ_Ψ in the product $X \times Y$ contains the graph of a singlevalued* *continuous selection s of the mapping Φ which coincides with $\Psi|_A$ on the set A.*

Proof. Apply Theorem 4.3.3 to the mapping F defined as follows:

$$F(x) = \begin{cases} \Psi(x), & \text{if } x \in A \\ \Phi(x), & \text{if } x \in X \setminus A. \end{cases}$$

□

Now we establish the main result of this chapter, namely, filtered selection theorem.

Theorem 4.3.5. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Let X be a paracompact space of extension dimension $\mathrm{e}-\dim X \leq [L]$. Suppose that multivalued mapping $F\colon X \to Y$ into a complete metric space Y admits a lower $[L]$-continuous, complete, and fiberwise $[L]$-connected n-filtration $F_0 \subset F_1 \subset \cdots \subset F_n \subset F$. If $f\colon A \to Y$ is a continuous singlevalued selection of F_0 over a closed subspace $A \subset X$, then there exists a continuous singlevalued selection $\widetilde{f}\colon X \to Y$ of the mapping F such that $\widetilde{f}|_A = f$.*

Proof. For every $i \leq n$ define a multivalued mapping $\Phi_i\colon X \to Y$ as follows:

$$\Phi_i(x) = \begin{cases} f(x), & \text{if } x \in A \\ F_i(x), & \text{if } x \in X \setminus A. \end{cases}$$

Then $\Phi_0 \subset \Phi_1 \subset \cdots \subset \Phi_n$ is lower $[L]$-continuous, complete, and fiberwise $[L]$-connected n-filtration. By Lemma 4.2.10 the mapping Φ_n contains a compact $\mathrm{UV}^{[L]}$-connected n-subfiltration. And application of Theorem 4.3.3 completes the proof. □

Our concluding version of selection theorem shows that the selection of complete lower $[L]$-continuous multivalued mapping over a closed subspace of the domain can be extended to a selection over some neighborhood of this subspace.

Theorem 4.3.6. *Let L be a finite CW-complex such that $[L] \leq [S^n]$ for some n. Let X be a paracompact space of extension dimension* $\mathrm{e}-\mathrm{dim}X \leq [L]$. *Let $F\colon X \to Y$ be a complete lower $[L]$-continuous multivalued mapping into a complete metric space. Suppose that $f\colon A \to Y$ is a continuous singlevalued selection of F over a closed subspace $A \subset X$. Then there exists a continuous extension of f to a selection of the mapping F over some neighborhood of the set A.*

Proof. Denote $\mathcal{U}_n = X \times Y$. Corollary 4.2.12 allows us to find open neighborhoods $\mathcal{U}_0 \subset \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_n$ of the graph Γ_f in $X \times Y$ such that for any $x \in \mathrm{pr}_X\mathcal{U}_0$ the pair $\mathcal{U}_i(x) \cap F(x) \subset \mathcal{U}_{i+1}(x) \cap F(x)$ is $[L]$-connected for every $i < n$. Consider a closed neighborhood OA of A which is contained in $\mathrm{pr}_X\mathcal{U}_0$. For every $i \leq n$ define a multivalued mapping $F_i\colon OA \to Y$ letting $F_i(x) = \mathcal{U}_i(x) \cap F(x)$. Then $F_0 \subset F_1 \subset \cdots \subset F_n = F|_{OA}$ is fiberwise $[L]$-connected n-filtration. The space OA, being a closed subset of X, is paracompact space of extension dimension $\leq [L]$. It is easy to verify that every mapping F_i is lower $[L]$-continuous and complete. Applying Theorem 4.3.5 we extend f to a selection of F over OA. □

Chapter 5

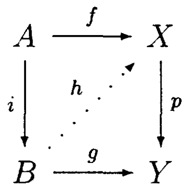
[L]–Homotopy Groups And Topological Model Categories

In this chapter we will show that for each finite complex L the category **TOP** of topological spaces possesses a model category structure in Quillen's sense, whose weak equivalences are weak $[L]$–homotopy equivalences. We begin by introducing the Quillen's concept of model category. We define the notion of $[L]$–complex which generalizes the notion of usual CW–complex. Next, we give detailed definitions related to $[L]$–homotopy groups introduced by Chigogidze in [Ch2]. Finally, we describe the corresponding closed model category structure on **TOP**. Throughout this chapter L denotes a finite complex.

5.1 Model Categories

In this section we present Quillen's concept of model category. We provide a number of examples of model category structures on **TOP**. In order to describe these structures we recall the notions of Serre, Hurewicz and n-fibrations and definitions of various types of homotopy equivalences.

In what follows it is convenient to use the following notions of lifting properties. We say that a map $i\colon A \to B$ has the *left lifting property* (LLP) with respect to a map $p\colon X \to Y$ and p has the *right lifting property* (RLP) with respect to i if, for every commutative square diagram of unbroken arrows



there exists a lift (the broken arrow) $h\colon B \to X$ such that $h \circ i = f$ and $p \circ h = g$.

Further we will use the following notations. Let I^k denote the k-dimensional unit cube, $\mathrm{Bd}I^k$ denote its boundary and V^{k-1} — the union of all faces of I^k except for $I^{k-1} \times \{1\}$, i.e. $V^{k-1} = \mathrm{Cl}(\mathrm{Bd}I^k - I^{k-1} \times \{1\})$.

Next we give definitions of *Serre, Hurewicz and n-fibrations* in terms of right lifting property.

Definition 5.1.1. Let $p\colon E \to B$ be a mapping possessing the RLP with respect to each mapping of a family $\mathcal{F}$. Then p is called a

(a) *Serre fibration* if $\mathcal{F} = \{i\colon D^n \to D^n \times I \mid n \geq 0\}$ where D^n denotes the standard n-disk and i is the natural embedding for each n.

(b) *Hurewicz fibration* if $\mathcal{F} = \{i\colon X \to X \times I\}$ where X is arbitrary topological space and i is the natural embedding.

(c) *n-fibration* (see [ED-HP]) if $\mathcal{F} = \{i\colon V^{k-1} \to I^k \mid k = 1, 2 \ldots, n+1\} \cup \{i\colon V^{n+1} \to \mathrm{Bd}I^{n+2}\}$, where i is the natural inclusion for each k.

Let us recall definitions of *homotopy equivalence, weak homotopy equivalence and n-homotopy equivalence.*

Definition 5.1.2. Let $f\colon X \to Y$ be a mapping of topological spaces. Then f is called a

(a) *homotopy equivalence*, if there exists a mapping $g\colon Y \to X$ such that compositions $g \circ f$ and $f \circ g$ are homotopic to identity mappings id_X and id_Y of X and Y respectively.

(b) *weak homotopy equivalence* if for each basepoint $x \in X$ the induced map $f_*\colon \pi_n(X, x) \to \pi_n(Y, f(x))$ is a bijection of pointed sets for $n = 0$ and an isomorphism of groups for $n \geq 1$.

(c) *weak n-homotopy equivalence* if for each basepoint $x \in X$ the induced map $f_*\colon \pi_k(X, x) \to \pi_k(Y, f(x))$ is a bijection of pointed sets for $k = 0$ and an isomorphism of groups for $k = 1, 2, \ldots, n$.

The following *Quillen's concept of the model category* plays the main role throughout the present chapter.

Definition 5.1.3. A model category is a category $\mathcal{C}$ with three distinguished classes of maps:

(i) weak equivalences,

(ii) fibrations,

(iii) cofibrations,

each of which is closed under compositions and contains all identity maps. A map which is both a fibration (respectively, cofibration) and a weak equivalence is called an *acyclic* fibration (respectively, acyclic cofibration). The following axioms must be satisfied.

(MC1) Finite limits and colimits exist in $\mathcal{C}$.

(MC2) If f and g are maps in $\mathcal{C}$ such that gf is defined and if two of the three maps f, g, gf are weak equivalences, then so is the third.

(MC3) If f is a retract of g and g is a fibration, cofibration or a weak equivalence, then so is f.

(MC4) Given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

of unbroken arrows, a lift (the broken arrow) exists in either of the following two situations: (i) i is a cofibration and p is an acyclic fibration, or (ii) i is an acyclic cofibration and p is a fibration.

(MC5) Any map f can be factored in two ways: (i) $f = pi$, where i is a cofibration and p is an acyclic fibration, and (ii) $f = qj$, where j is an acyclic cofibration and q is a fibration.

Below we list important examples of model categories.

Example 5.1.4. (i) The category **TOP** of topological spaces with Quillen's structure [Q]. Fibrations of these model category are precisely Serre fibrations and weak equivalences coincide with the weak homotopy equivalences. A map is a cofibration if it has the left lifting property with respect to any acyclic fibration.

(ii) The category of topological spaces **TOP** with Strøm structure [St]. The class of fibrations coincides with the class of Hurewicz fibrations. A map is said to be a weak equivalence if it is a homotopy equivalence. A map $j\colon A \to X$ is a cofibration if it is a closed map and it has LLP with respect to the mapping $d_0\colon Y^I \to Y$ where Y is arbitrary topological space, Y^I denotes path space of Y and $d_0(\alpha) = \alpha(0)$ for each $\alpha \in Y^I$.

(iii) The category of topological spaces **TOP** with the structure of n-equivalences, n-fibrations and n-cofibrations (see [ED-HP]). Fix $n \geq 0$. A map is called a fibration if it is n-fibration. A map is called a weak equivalence if it is weak n-homotopy equivalence. A map is a cofibration if it has the LLP with respect to any acyclic fibration.

If $\mathcal{C}$ is a model category, then it has an initial object and a terminal object. Namely, the initial object is given by the colimit and the terminal object is given by the limit of the empty diagram. An object in $\mathcal{C}$ is called *cofibrant* if the map from the initial object to it is a cofibration. Similarly an object in $\mathcal{C}$ is called *fibrant* if the map from it to the terminal object is a fibration. By $\mathcal{C}_{cf}$ we denote the full subcategory of $\mathcal{C}$ consisting of objects which are simultaneously cofibrant and fibrant. Observe that in the model category structure introduced in Example 5.1.4(i) CW–complexes are cofibrant and fibrant simultaneously.

The following important observation is well-known ([Dw-S, Lemma 4.24], [Ho, Proposition 1.2.8]).

Proposition 5.1.5. *Suppose $\mathcal{C}$ is a model category. Then a map of $\mathcal{C}_{cf}$ is a weak equivalence if and only if it is a homotopy equivalence.*

If we apply this proposition to the model category introduced in Example 5.1.4(i), we arrive to the classical theorem of J.H.C.Whitehead [Wh, Wh1].

Theorem 5.1.6. *A map between CW–complexes (or, more generally,* ANE*–spaces) is a homotopy equivalence if and only if it induces isomorphisms of all homotopy groups.*

5.2 $[L]$–Complexes and $[L]$–Homotopy Groups

The class of CW–complexes plays an important role in developing of the classical homotopy theory. In this section we introduce the notion of $[L]$–complexes which is designed for the needs of $[L]$–homotopy theory. Secondly, we present a detailed description of $[L]$–homotopy groups introduced by Chigogidze [Ch2].

5.2.1 $[L]$–Complexes

We begin with the following technical proposition which will be extensively used further. In particular, we will make use of it to show that certain maps are $[L]$–homotopy equivalences.

Proposition 5.2.1. *Let $p\colon X \to Y$ be an approximately $[L]$-soft map between Polish spaces. Let also $f_1, f_2\colon A \to X$ be two maps, defined on a Polish space A, such that*

(a) $f_1(a_0) = f_2(a_0)$ *for some point* $a_0 \in A$;

(b) $p \circ f_0 \overset{[L]}{\simeq} p \circ f_1 \operatorname{rel} a_0$.

Then $f_0 \overset{[L]}{\simeq} f_1 \operatorname{rel} a_0$.

Proof. Consider the map $\phi\colon A \times \{0,1\} \cup \{a_0\} \times I \to Y$, defined by letting

$$\phi(a,t) = \begin{cases} p(f_t(a)), \text{ if } a \in A \text{ and } t = 0,1; \\ p(f_0(a_0)), \text{ if } a = a_0 \text{ and } t \in I. \end{cases}$$

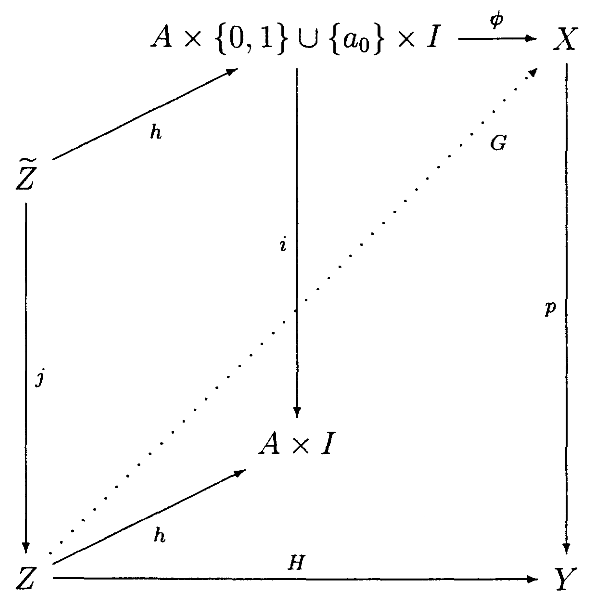
Let also $h\colon Z \to A \times I$ be an $[L]$-invertible map such that $\mathrm{e}-\dim Z \leq [L]$ and $\widetilde{Z} = h^{-1}(A \times \{0,1\} \cup \{a_0\} \times I)$. By (b), there exists a map $H\colon Z \to Y$ which extends the composition $\phi \circ h|\widetilde{Z}\colon \widetilde{Z} \to Y$.

Next consider the map $\varphi\colon A \times \{0,1\} \cup \{a_0\} \times I \to X$

$$\varphi(a,t) = \begin{cases} f_t(a), \text{ if } a \in A \text{ and } t = 0,1; \\ f_0(a_0), \text{ if } a = a_0 \text{ and } t \in [0,1]. \end{cases}$$

which, according to (a), is well defined.

It is easy to see that the following diagram of unbroken arrows commutes (here i and j denote the corresponding inclusion maps).



In particular, $p \circ \varphi \circ h|\widetilde{Z} = \phi \circ h|\widetilde{Z} = H|\widetilde{Z}$. Finally, since p is approximately $[L]$-soft and since $\mathrm{e}-\dim Z \leq \{L\}$ it follows that there exists a map $G \colon Z \to X$ (the broken arrow in the above diagram) such that $G|\widetilde{Z} = \varphi \circ h|\widetilde{Z}$ (notice also that above diagram is not commutative if G is included in it, however, it can be made approximately commutative). This proves that $f_0 \overset{[L]}{\simeq} f_1 \operatorname{rel} a_0$.

□

In order to define $[L]$-homotopy groups we need to introduce the notions of $[L]$-spheres and $[L]$-disks. For each $n \geq 0$ we consider $[L]$-dimensional compact AE($[L]$)–spaces $D^n_{[L]}$ which admit ($[L]$-invertible) approximately $[L]$-soft maps onto n-dimensional disk D^n (see Proposition 2.4.6). Note that Proposition 2.4.6(iv) allows us to consider pairs $(S^n_{[L]}, D^{n+1}_{[L]})$ as approximately $[L]$-soft preimages of the standard pairs (S^n, D^{n+1}) consisting of the $(n+1)$-dimensional disk and its boundary $\partial D^{n+1} = S^n$, $n \geq 0$. We say that $D^n_{[L]}$

is an n-$[L]$*-disk*. Further, by n-$[L]$*-spheres* (which are denoted by $S^n_{[L]}$) we mean compacta which admit approximately $[L]$-soft maps onto n-dimensional sphere S^n . It is important to note that generally speaking for any given n there are many n-$[L]$-disks and n-$[L]$-spheres. Observe, however, that being AE($[L]$)-compacta, all n-$[L]$-disks are $[L]$-homotopy equivalent to the one point spaces. As for $[L]$-spheres, Proposition 5.2.1 shows that all n-$[L]$-spheres are also $[L]$-homotopy equivalent. As above, generally speaking $[L]$-homotopy equivalence between two $[L]$-spheres is not uniquely defined. Nevertheless, it is possible to distinguish a certain class of $[L]$-homotopy equivalences between $[L]$-spheres. Namely, further we will define the notion of $[L]$-homotopy equivalence of canonical type, which, in turn, will allow for a correct definition of $[L]$-homotopy groups.

Next, we define *[L]-polyhedra* and *[L]-complexes* (see [Ch2, Section 2.6.2]).

Definition 5.2.2. We say that a space X of extension dimension $\mathrm{e-dim}X \leq [L]$ is a (finite) $[L]$-polyhedron if it admits a proper approximately $[L]$-soft map $f\colon X \to Y$ onto a locally finite polyhedron Y such that f possesses property (iv) of Proposition 2.4.6 for some (finite) triangulation τ of Y. By $[L]$-complexes we understand spaces that are $[L]$-homotopy equivalent to $[L]$-polyhedra.

The above definition guarantees that any $[L]$-polyhedron can be constructed by attaching "cells" (i.e. $[L]$-disks) along their "boundaries" (i.e. $[L]$-spheres) by means of inclusion maps. This procedure is similar to the standard procedure of building of classical CW-complexes. Therefore a number of properties of $[L]$-polyhedra can be obtained by easy modifications of standard constructions and proofs (see, for instance, [Fr-Pi], [Sw]).

Remark 5.2.3. Let X be an $[L]$-polyhedron. According to the above definition there exists a corresponding approximately $[L]$-soft mapping $f\colon X \to Y$ of X onto a locally finite polyhedron Y. Let $Y^{(n)}$ denote the n-skeleton of Y. Then for any $y \in Y^{(0)}$ we have $f^{-1}(y) \in AE([L])$. Put $X^{(n)} = f^{-1}(Y^{(n)})$. It is natural to refer to $X^{(n)}$ as *n-$[L]$-skeleton* of X. Clearly, $X^{(n)}$ is closed in X for any n. Since Y is locally finite for any point $x \in X$ there exists an open in X neighborhood Ox such that $Ox \subset X^{(n)}$ for some n. This implies that a subset F of X is closed if and only if $F \cap X^{(n)}$ is closed for any n. Therefore a topology of $[L]$-polyhedra resembles a standard CW-topology of complexes.

Clearly, standard polyhedra are obtained by letting $L = \{\text{point}\}$ (or $L = S^n$ for n-dimensional polyhedra). However, class of $[S^n]$-polyhedra is obviously wider than class of n-dimensional polyhdera.

5.2.2 $[L]$-Homotopy Groups

Our next step is to introduce an important algebraic $[L]$-homotopy invariant — *$[L]$-homotopy groups*. Note that $[L]$-spheres $S^n_{[L]}$ will play the role of the unit n-spheres S^n.

Let (X, x_0) be a pointed space. Fix $n \geq 1$ and a point $s \in S^n$. Let $S^n_{[L]}$ be a n-$[L]$-sphere, i.e. a compactum of extension dimension at most $[L]$ which admits an approximately $[L]$-soft mapping onto S^n. Let $f\colon S^n_{[L]} \to S^n$ be the corresponding approximately $[L]$-soft mapping. Fix a point $s_{[L]} \in f^{-1}(s)$.

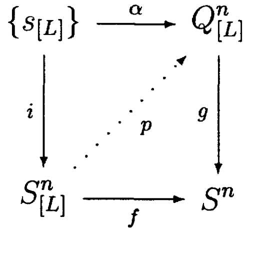
Consider the set $\pi^{[L]}_n(X, x_0) = \left[(S^n_{[L]}, s_{[L]}), (X, x_0)\right]_{[L]}$ of relative $[L]$-homotopy classes of maps of pointed spaces. In what follows we will describe a group structure on this set. As we mentioned above, the choice of n-$[L]$-

sphere is not unique. Therefore we have to show that the set $\pi_n^{[L]}(X, x_0)$ does not depend on the choice of $[L]$–sphere. In order to check this fact we will introduce the notion of $[L]$-homotopy equivalence of canonical type.

Consider another n-$[L]$–sphere $Q^n_{[L]}$ and let $g\colon Q^n_{[L]} \to S^n$ be the corresponding approximately $[L]$–soft mapping. Choose a point $q_{[L]} \in g^{-1}(s)$ and let $\widetilde{\pi}_n^{[L]}(X, x_0) = \left[(Q^n_{[L]}, q_{[L]}), (X, x_0)\right]_{[L]}$.

Since S^n is an ANE-compactum there exists an open cover $\mathcal{U} \in \operatorname{cov}(S^n)$ such that any two $\mathcal{U}$-close maps (defined on any compactum) are homotopic as maps into S^n (we use Proposition 2.5.5 with $L = \{*\}$).

Compacta $S^n_{[L]}$ and $Q^n_{[L]}$ can be included in the following commutative diagram (consisting of unbroken arrows)



where $\alpha(s_{[L]}) = q_{[L]}$.

Since g is approximately $[L]$–soft, there exists a mapping $p\colon S^n_{[L]} \to Q^n_{[L]}$ (the broken arrow in the above diagram) such that $p(s_{[L]}) = q_{[L]}$ (i.e. $p \circ i = \alpha$) and $g \circ p$ is $\mathcal{U}$-close to f (and therefore the above diagram becomes approximately commutative if p is included in it). Similarly, there exists a mapping $q\colon Q^n_{[L]} \to S^n_{[L]}$ such that $q(q_{[L]}) = s_{[L]}$ and $f \circ q$ is $\mathcal{U}$-close to g. Choice of the cover $\mathcal{U}$ guarantees that $g \circ p \simeq f$ and $f \circ q \simeq g$. We may assume without loss of generality that these are homotopies relative to the given points $s_{[L]}$ and $q_{[L]}$.

Next note that

$$g \circ p \circ q \simeq f \circ q \operatorname{rel} q_{[L]} \simeq g \operatorname{rel} q_{[L]}$$

and

$$f \circ q \circ p \simeq g \circ p \operatorname{rel} s_{[L]} \simeq f \operatorname{rel} s_{[L]}.$$

By virtue of Proposition 5.2.1, $q \circ p \overset{[L]}{\simeq} \mathrm{id}_{Q^n_{[L]}} \operatorname{rel} q_{[L]}$ and $p \circ q \overset{[L]}{\simeq} \mathrm{id}_{S^n_{[L]}} \operatorname{rel} s_{[L]}$. We call mappings constructed as described above *$[L]$-homotopy equivalences of canonical type.* It is important to observe that any two $[L]$–homotopy equivalences of canonical type are $[L]$–homotopic.

Now we use of mappings p and q to define a mapping $\phi\colon \pi_n^{[L]}(X, x_0) \to \widetilde{\pi}_n^{[L]}(X, x_0)$. Consider an element $\alpha \in \pi_n^{[L]}(X, x_0)$ and let $a\colon (S^n_{[L]}, s_{[L]}) \to (X, x_0)$ be its representative. We let $\phi(\alpha) = \beta$, where β is a relative $[L]$–homotopy class of the composition $a \circ q\colon (Q^n_{[L]}, q_{[L]}) \to (X, x_0)$. Similarly, we define a mapping $\psi\colon \widetilde{\pi}_n^{[L]}(X, x_0) \to \pi_n^{[L]}(X, x_0)$ using the mapping p. It is easy to check that ϕ and ψ are well-defined and are inverses to each other. Hence both mappings ϕ and ψ provide a bijection between the sets $\pi_n^{[L]}(X, x_0)$ and $\widetilde{\pi}_n^{[L]}(X, x_0)$. Therefore, the set $\pi_n^{[L]}(X, x_0)$ does not depend on the choice of n-$[L]$–sphere. The construction of $[L]$-homotopy equivalences of canonical type guarantees that bijections ϕ and ψ do not depend on the choice of $[L]$–homotopy equivalences of canonical type.

Let $f\colon (X, x_0) \to (Y, y_0)$ be a mapping and $[f]_{[L]} \in [(X, x_0), (Y, y_0)]_{[L]}$ be its relative $[L]$–homotopy class. Then a natural map $\pi_n^{[L]}([f]_{[L]})\colon \pi_n^{[L]}(X, x_0) \to \pi_n^{[L]}(Y, y_0)$ can be defined in a standard way.

Next, we introduce a group structure on $\pi_n^{[L]}(X, x_0)$ such that natural maps $\pi_n^{[L]}([f]_{[L]})$ are well-defined homomorphisms. It should be emphasized that this structure will be defined so that bijections ϕ and ψ, generated by $[L]$–homotopy equivalences of canonical type, are group isomorphisms. Hence $[L]$-homotopy groups will be well-defined.

Next we define operation on the set $\pi_n^{[L]}(X, x_0)$. We pick two elements α and β of the set $\pi_n^{[L]}(X, x_0)$ and their representatives $a, b\colon (S^n_{[L]}, s_{[L]}) \to (X, x_0)$. Let S^n_+ and S^n_- denote the upper and lower hemispheres, respectively, and E denote an equator of S^n containing the point s. Let $f\colon S^n_{[L]} \to S^n$ be approximately $[L]$–soft mapping. Let $h\colon S^n \to S^n \vee S^n$ be the homotopy comultiplication defining the standard H-cogroup structure (see, for instance, [Sw, Definition 2.16]) of the sphere S^n. Let also $f_-\colon S^n_{[L]} \to S^n$ and $f_+\colon S^n_{[L]} \to S^n$ be two copies of the map f.

Given mappings a and b, we need to construct new mapping $a * b$ of a pointed n-$[L]$–sphere to (X, x_0) whose $[L]$-homotopy class will represent the product of α and β. Since f_+ is approximately $[L]$-soft there exists a mapping $\widetilde{a}\colon f^{-1}(S^n_+) \to S^n_{[L]}$ such that $\widetilde{a}(f^{-1}(E)) = s_{[L]}$ and the composition $f_+ \circ \widetilde{a}$ is $\mathcal{U}$–close to the composition $h \circ f|_{f^{-1}(S^n_+)}$. Similarly, there exists a mapping $\widetilde{b}\colon f^{-1}(S^n_-) \to S^n_{[L]}$ such that $\widetilde{b}(f^{-1}(E)) = s_{[L]}$ and the composition $f_- \circ \widetilde{b}$ is $\mathcal{U}$–close to the composition $h \circ f|_{f^{-1}(S^n_-)}$.

Now we can define the composition $a * b = (\widetilde{a} \cup \widetilde{b}) \circ (a \vee b)\colon (S^n_{[L]}, E') \to (X, x_0)$ by letting

$$(a * b)(x) = \begin{cases} a(\widetilde{a}(x)), & \text{if } x \in f^{-1}(S^n_+) \\ b(\widetilde{b}(x)), & \text{if } x \in f^{-1}(S^n_-). \end{cases}$$

It is not hard to verify that the relative $[L]$–homotopy class $[a * b]_{[L]}$ of this composition does not depend on the choice of representatives a and b (and mappings $\widetilde{a}$ and $\widetilde{b}$). This allows us to define the product of α and β by letting $\alpha * \beta = [a * b]_{[L]}$.

Next, the identity element in $\pi_n^{[L]}(X, x_0)$ is given by $\epsilon = [e]_{[L]}$ where e is a constant mapping which sends $S^n_{[L]}$ to the point x_0. It is routine to check that ϵ is indeed the identity with respect to the introduced above multiplication.

In order to finish our construction, given an element $\alpha \in \pi_n^{[L]}(X, x_0)$ we need to define its inverse $\alpha^{-1} \in \pi_n^{[L]}(X, x_0)$ with respect to the operation $*$. Let $a\colon (S^n_{[L]}, s_{[L]}) \to (X, x_0)$ be a representative of α. Let $g\colon S^n \to S^n$ be the mapping such that $g(x_1, \dots, x_{n+1}) = (x_1, \dots, -x_{n+1})$, which fixes the equator E. Since f is approximately $[L]$–soft there exists a mapping $\widetilde{g}\colon S^n_{[L]} \to S^n_{[L]}$ such that $\widetilde{g}(s_{[L]}) = s_{[L]}$ and composition $f \circ \widetilde{g}$ is $\mathcal{U}$–close to the composition $g \circ f$. The $[L]$-homotopy class of the composition $\widetilde{a} = a \circ \widetilde{g}\colon (S^n_{[L]}, s_{[L]}) \to (X, x_0)$ does not depend on the choice of representative a and of the mapping $\widetilde{g}$. And we let $\alpha^{-1} = [a \circ \widetilde{g}]_{[L]}$. Again, routine verification shows that $\alpha^{-1}\alpha = \alpha\alpha^{-1} = \epsilon \in \pi_n^{[L]}(X, x_0)$.

It is necessary to point out that $[L]$-homotopy groups may differ from usual homotopy groups even for polyhedra. We conclude this section by two examples.

Example 5.2.4. (i) If $[L] = [\{\text{point}\}]$, then the k-th $[\{\text{point}\}]$-sphere is (homotopically) the same as the usual sphere S^k and the notion of $[L]$-homotopy coincides with the notion of usual homotopy. Therefore the system of $[\{\text{point}\}]$-homotopy groups of any space X coincides with the system of usual homotopy groups $\pi_1(X), \pi_2(X), \dots \pi_n(X), \dots$.

(ii) If $[L] = [S^n]$, then the k-th $[S^n]$-sphere $S^k_{[S^n]}$ is ($[S^n]$-homotopically) the same as the usual sphere S^k for $k \leq n-1$ and as a point for $k \geq n$. Hence the system of $[S^n]$-homotopy groups of a space X is as follows

$$\pi_1(X), \pi_2(X), \ldots \pi_{n-1}(X), 0, 0, \ldots, 0, \ldots$$

5.3 Model Category Structure on TOP Generated By a Finite Complex

In this section we prove the main result of the present chapter which states that the category **TOP** admits a model category structure whose weak equivalences are *weak* $[L]$*-homotopy equivalences*. First, we introduce the required classes of morphisms.

Definition 5.3.1. A map $f\colon X \to Y$ of spaces is called a *weak* $[L]$*-homotopy equivalence* if for each basepoint $x \in X$ the map $f_*\colon \pi_n^{[L]}(X, x) \to \pi_n^{[L]}(Y, f(x))$ is a bijection of pointed sets for $n = 0$ and an isomorphism of groups for $n \geq 1$.

Definition 5.3.2. Let $f\colon X \to Y$ be a map in **TOP**. We say that f is

$(\mathrm{i})_L$ a weak equivalence if it is a weak $[L]$-homotopy equivalence,

$(\mathrm{ii})_L$ a fibration if it has the RLP with respect to inclusions of finite $[L]$-polyhedra $A \subset B$ such that both A and B are AE($[L]$)–spaces.

$(\mathrm{iii})_L$ a cofibration if it has the LLP with respect to acyclic fibrations.

Below we shall use the following fact, proof of which is trivial.

Proposition 5.3.3. *Let $f\colon S^n_{[L]} \to X$ be a mapping of n-$[L]$-sphere to a topological space X. If f is $[L]$-homotopic to a constant map then for any embedding of $S^n_{[L]}$ to a space Z of extension dimension $\leq [L]$ the mapping f can be extended to a mapping of Z to X. Conversely, suppose that $S^n_{[L]}$ is a subspace of space Z of extension dimension $\leq [L]$ such that $Z \in \mathrm{AE}([L])$ and f admits an extension over Z. Then f is $[L]$-homotopic to a constant map.*

In order to check that classes of morphisms introduced in Definitions 5.3.1 and 5.3.2 satisfy the axioms of closed model category, we need to characterize acyclic $[L]$–fibrations in terms of RLP. First we prove the following important lemma.

Lemma 5.3.4. *Let $p\colon X \to Y$ be a $[L]$-fibration. If p is acyclic then for each $n = 0, 1, \ldots$ there exist pairs of $[L]$-polyhedra $S^n_{[L]} \subset D^{n+1}_{[L]}$ where $D^{n+1}_{[L]}$ is an $(n+1)$-$[L]$-disk and $S^n_{[L]}$ is a corresponding n-$[L]$-sphere, such that p has the RLP with respect to these pairs.*

Proof. In what follows the standard $(n+1)$–disk centered at the origin O (of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$) and n–sphere of radius r are denoted by D^{n+1}_r and $S^n_r = \partial D^{n+1}_r$, respectively. Consider an approximately $[L]$–soft mapping $f\colon D \to D^{n+1}_1 = D^{n+1}$ of a compactum D having extension dimension $\leq [L]$ onto D^{n+1}. Put $S^n_{[L]} = f^{-1}(S^n_1)$, $S^n_{[L]}(0) = f^{-1}(S^n_{1/2})$, $C^{n+1}_{[L]} = f^{-1}(D^{n+1}\backslash \mathrm{Int}(D^{n+1}_{1/2}))$ and $D_0 = f^{-1}(D^{n+1}_{1/2})$. Let τ denote a triangulation of D^{n+1} such that $S^n_{1/2}$, S^n_1, $D^{n+1}_{1/2}$ and $D^{n+1}\backslash \mathrm{Int}(D^{n+1}_{1/2})$ are subpolyhedra of D^{n+1} with respect to τ. Proposition 2.4.6 allows us to assume that restrictions of f on preimages of subcomplexes of D^{n+1} with respect to τ are also approximately $[L]$-soft.

This implies, in particular, that $D_0 \in \mathrm{AE}([L])$, $S^n_{[L]} \in \mathrm{ANE}([L])$ and that $C^{n+1}_{[L]}$ is an $(n+1)$-$[L]$-cylinder, i.e. ANE([L])–compactum of extension dimension $\leq [L]$ admitting an approximately $[L]$-soft mapping $f\colon C^{n+1}_{[L]} \to S^n \times I$, where $I = [0,1]$ denotes unit interval.

Consider a quotient space $D^{n+1}_{[L]} = D/D_0$ and let $\phi\colon D \to D/D_0$ be the corresponding quotient mapping. Since $D \in \mathrm{AE}([L])$ and $D_0 \in \mathrm{AE}([L])$ we conclude that $D^{n+1}_{[L]} \in \mathrm{AE}([L])$. Put $s_{[L]} = \phi(D_0)$. Define a mapping $g\colon D^{n+1}_{[L]} \to D^{n+1}$ as follows. Put $g(s_{[L]}) = O$ and for each point $x \in D^{n+1}_{[L]}$, distinct from $s_{[L]}$, let $g(x) = h \circ f(x)$, where $h\colon D^{n+1} \to D^{n+1}$ is a mapping which collapses $D^{n+1}_{1/2}$ to the point O. Let also τ' denote a triangulation of D^{n+1} obtained from τ by means of collapsing D^{n+1} to a point (and enlarging of the resulting cellular structure). It is easy to check that the thus constructed space $D^{n+1}_{[L]}$ is a $AE([L])$–space. Further, it is not hard to verify that the mapping g is approximately $[L]$–soft and restriction of g on a preimage of any subcomplex of D^{n+1} with respect to τ' is also approximately $[L]$–soft. This implies, in particular, that $D^{n+1}_{[L]}$ is $[L]$–polyhedron and $[L]$–disk. Further in the proof we shall refer to the compacta constructed above simply as $[L]$–disk, corresponding $[L]$–sphere and $[L]$–cylinder, and denote them by $D^{n+1}_{[L]}$, $S^n_{[L]}$ and $C^{n+1}_{[L]}$, respectively. Note also that $D^{n+1}_{[L]}$ is homeomorphic to the quotient space $C^{n+1}_{[L]}/S^n_{[L]}(0)$.

Let us check that the constructed pair $S^n_{[L]} \subset D^{n+1}_{[L]}$ satisfies the condition of the lemma. Consider two mappings $f\colon D^{n+1}_{[L]} \to Y$ and $G\colon S^n_{[L]} \to X$ such that $p \circ G = f|_{S^n_{[L]}}$. Since p is acyclic and $f|_{S^n_{[L]}}$ represents trivial element of the group $\pi^{[L]}_n(Y)$, there exists a mapping $\overline{G}\colon D^{n+1}_{[L]} \to X$ extending G.

Construct a compactum $\widehat{S^{n+1}_{[L]}} = (D^{n+1}_{[L]})_+ \cup (D^{n+1}_{[L]})_-$ by gluing of two copies of $D^{n+1}_{[L]}$ along $S^n_{[L]}$. Clearly, $\widehat{S^{n+1}_{[L]}}$ admits an approximately $[L]$–soft mapping onto S^{n+1} and hence represents an $(n+1)$-$[L]$–sphere. Fix the point $s_{[L]} \in (D^{n+1}_{[L]})_-$ (see the construction of $D^{n+1}_{[L]}$ above). Define a mapping $h\colon (\widehat{S^{n+1}_{[L]}}, s_{[L]}) \to (Y, h(s_{[L]}))$ as follows:

$$h(x) = \begin{cases} f(x), & \text{if } x \in (D^{n+1}_{[L]})_+ \\ p \circ \overline{G}(x), & \text{if } x \in (D^{n+1}_{[L]})_- \end{cases}$$

Put $y_0 = h(s_{[L]})$. Notice that h represents an element α of the group $\pi^{[L]}_{n+1}(Y, y_0)$. Since p is acyclic, there exists a mapping $H\colon (\widehat{S^{n+1}_{[L]}}, s_{[L]}) \to (X, x_0)$ where $x_0 \in p^{-1}(y_0)$ such that composition $p \circ H\colon (\widehat{S^{n+1}_{[L]}}, s_{[L]}) \to (Y, y_0)$ represents an element of the group $\pi^{[L]}_{n+1}(Y, y_0)$ which is $[L]$–homotopically inverse to α.

Let $C^{n+1}_{[L]}$ be an n-$[L]$–cylinder corresponding to $D^{n+1}_{[L]}$. Consider the space $\widehat{D^{n+1}_{[L]}} = D^{n+1}_{[L]} \cup C^{n+1}_{[L]}$ obtained by gluing of $D^{n+1}_{[L]}$ and $C^{n+1}_{[L]}$ along $S^n_{[L]}$ and the space $\widetilde{S^{n+1}_{[L]}} = (\widehat{D^{n+1}_{[L]}})_+ \cup (\widehat{D^{n+1}_{[L]}})_-$ obtained by gluing of two copies of $\widehat{D^{n+1}_{[L]}}$ along $S^n_{[L]}(0)$. The corresponding quotient mapping is denoted by ϕ. It is easy to see that compactum $\widetilde{S^{n+1}_{[L]}}$ represents a $(n+1)$-$[L]$–sphere. Let

$$\widetilde{D^{n+1}_{[L]}} = (C^{n+1}_{[L]})_+ \cup_\phi (\widehat{D^{n+1}_{[L]}})_- \subset \widetilde{S^{n+1}_{[L]}}$$

Clearly, $\widetilde{D^{n+1}_{[L]}}$ is an $(n+1)$-$[L]$–disk and $[L]$–polyhedron. Define a mapping

$$k\colon (\widetilde{S^{n+1}_{[L]}}, S^n_{[L]}(0)) \to (Y, y_0)$$

as follows:

$$k(x) = \begin{cases} h(x), & \text{if } x \in (\widehat{D^{n+1}_{[L]}})_+ \backslash S^n_{[L]}(0) \\ p \circ H(x), & \text{if } x \in (\widehat{D^{n+1}_{[L]}})_- \backslash S^n_{[L]}(0) \\ y_0, & \text{if } x \in S^n_{[L]}(0) \end{cases}$$

Define a mapping $K\colon (\widetilde{D^{n+1}_{[L]}}) \to X$ such that $k|_{\widetilde{D^{n+1}_{[L]}}} = p \circ K$ letting

$$K(x) = \begin{cases} \overline{G}(x), & \text{if } x \in (C^{n+1}_{[L]})_+ \setminus S^n_{[L]}(0) \\ H(x), & \text{if } x \in (\widehat{D^{n+1}_{[L]}})_- \setminus S^n_{[L]}(0) \\ x_0, & \text{if } x \in S^n_{[L]}(0) \end{cases}$$

Clearly the mapping k represents the product of α and α^{-1} in the group $\pi^{[L]}_{n+1}(Y, y_0)$ and hence is $[L]$–homotopically trivial.

Let $\mu^{[L]}$ be a strongly $[L]$–universal AE($[L]$)–compactum of extension dimension $[L]$ provided by Proposition 2.4.6. According to (iv) of this proposition we may assume that $\mu^{[L]}$ is a $[L]$-polyhedron. Consider an embedding $i\colon \widetilde{S^{n+1}_{[L]}} \hookrightarrow \mu^{[L]}$. By Proposition 5.3.3, there exists an extension $\overline{k}\colon \mu^{[L]} \to Y$ of k. Applying the RLP of p to the pair $\widetilde{D^{n+1}_{[L]}} \subset \mu^{[L]}$ we obtain a lifting $\overline{K}\colon Z \to X$ of $\overline{k}$ which is an extension of K. The mapping $F = \overline{K}|_{(D^{n+1}_{[L]})_+}\colon (D^{n+1}_{[L]})_+ \to X$ provides a desired extension of G. $\square$

Now we are ready to obtain desired characterization.

Theorem 5.3.5. *An $[L]$-fibration $p\colon X \to Y$ is acyclic iff it has the RLP with respect to any pair $S^n_{[L]} \subset D^{n+1}_{[L]}$ where $D^{n+1}_{[L]}$ is an $(n+1)$-$[L]$-disk and $S^n_{[L]}$ is a corresponding n-$[L]$-sphere, $n = 0, 1, \ldots$.*

Proof. We begin by proving that condition of the lemma is necessary. Let $S^n_{[L]} \subset D^{n+1}_{[L]}$ be a pair of $[L]$-polyhedra ($(n+1)$-$[L]$-disk and corresponding n-$[L]$-sphere) provided by Lemma 5.3.4. Let also τ denote the correspondent triangulation of D^{n+1} and $f\colon D^{n+1}_{[L]} \to D^{n+1}$ be the correspondent approximately $[L]$-soft mapping. Consider $[L]$-dimensional AE($[L]$)–space $\mathbb{D}^{n+1}_{[L]}$ which satisfies conditions (i)-(iv) of Proposition 2.4.6 and the corresponding

$[L]$-invertible and approximately $[L]$-soft mapping $f^{[L]}\colon \mathbb{D}^{n+1}_{[L]} \to D^{n+1}$. We assume that condition (iv) of Proposition 2.4.6 is satisfied with respect to the same triangulation τ of D^{n+1}. Put $\mathbb{S}^n_{[L]} = (f^{[L]})^{-1}(S^n)$. Then $\mathbb{S}^n_{[L]} \in ANE([L])$ and the restriction $f^{[L]}|_{\mathbb{S}^n_{[L]}}$ is also approximately $[L]$-soft.

The further proof consists of two steps. First we apply the RLP of p with respect to the pair $S^n_{[L]} \subset D^{n+1}_{[L]}$ to verify that p has RLP with respect to the pair $(\mathbb{S}^n_{[L]}, \mathbb{D}^{n+1}_{[L]})$. Then we use this fact to show that p has RLP with respect to arbitrary pair of n-$[L]$–sphere and $(n+1)$-$[L]$–disk.

Since the mapping $f^{[L]}$ is $[L]$-invertible there exists a mapping $\tilde{f}$ such that $f^{[L]} \circ \tilde{f} = f$. Next, Proposition 2.4.6(iii) provides us with an embedding $g\colon D^{n+1}_{[L]} \to \mathbb{D}^{n+1}_{[L]}$ such that

$$f^{[L]} \circ g = f^{[L]} \circ \tilde{f} = f.$$

It is easy to check that

$$g(S^n_{[L]}) \subset \mathbb{S}^n_{[L]} \text{ and } g(D^{n+1}_{[L]})\backslash S^n_{[L]} \subset \mathbb{D}^{n+1}_{[L]}\backslash \mathbb{S}^n_{[L]}.$$

In what follows we identify compacta $D^{n+1}_{[L]}$ and $S^n_{[L]}$ with their images $g(D^{n+1}_{[L]})$ and $g(S^n_{[L]})$. Consider the space $\widetilde{D^{n+1}_{[L]}} = \mathbb{S}^n_{[L]} \cup D^{n+1}_{[L]}$. It is not hard to see that $\widetilde{D^{n+1}_{[L]}} \in \mathrm{AE}([L])$. Moreover, $\widetilde{D^{n+1}_{[L]}}$ is $(n+1)$-$[L]$–disk and $[L]$-polyhedron. Let now $F\colon \mathbb{D}^{n+1}_{[L]} \to Y$ and $G\colon \mathbb{S}^n_{[L]} \to X$ be mappings such that $F|_{\mathbb{S}^n_{[L]}} = p \circ G$. We have to find a lifting $\overline{G}$ of F extending G. Since p has RLP with respect to the pair $S^n_{[L]} \subset D^{n+1}_{[L]}$ there exists an extension $\widetilde{G}\colon \widetilde{D^{n+1}_{[L]}} \to X$ of G such that $F|_{\widetilde{D^{n+1}_{[L]}}} = p \circ \widetilde{G}$. Now the right lifting property of $[L]$-fibration applied to the pair $\widetilde{D^{n+1}_{[L]}} \subset \mathbb{D}^{n+1}_{[L]}$ provides us with the desired lifting $\overline{G}$.

Let now $(S^n_{[L]}, D^{n+1}_{[L]})$ denote an arbitrary pair of $(n+1)$-$[L]$–disk and n-$[L]$–sphere. The corresponding approximately $[L]$-soft mapping of $D^{n+1}_{[L]}$ onto

D^{n+1} is denoted still by f. As above, we can find an embedding $D^{n+1}_{[L]} \hookrightarrow \mathbb{D}^{n+1}_{[L]}$ such that $S^n_{[L]} \hookrightarrow \mathbb{S}^n_{[L]}$ and $D^{n+1}_{[L]} \backslash S^n_{[L]} \hookrightarrow \mathbb{D}^{n+1}_{[L]} \backslash \mathbb{S}^n_{[L]}$. Since both mappings $f^{[L]}|_{\mathbb{S}^n_{[L]}}$ and $f|_{S^n_{[L]}}$ are approximately $[L]$–soft there exists a retraction $r\colon \mathbb{S}^n_{[L]} \to S^n_{[L]}$ (which is $[L]$-homotopy equivalence). Since $D^{n+1}_{[L]} \in \mathrm{AE}([L])$ this retraction can be extended to a retraction $R\colon \mathbb{D}^{n+1}_{[L]} \to D^{n+1}_{[L]}$. Consider two mappings $F\colon D^{n+1}_{[L]} \to Y$ and $G\colon S^n_{[L]} \to X$ such that $F|_{S^n_{[L]}} = p \circ G$. Since, as shown above, the mapping p possesses RLP with respect to the pair $\mathbb{S}^n_{[L]} \subset \mathbb{D}^{n+1}_{[L]}$ the mapping $F \circ R\colon \mathbb{D}^{n+1}_{[L]} \to Y$ has a lifting $\overline{G}\colon \mathbb{D}^{n+1}_{[L]} \to X$ which extends the mapping $G \circ r\colon \mathbb{S}^n_{[L]} \to X$. Clearly, the restriction $\overline{G}|_{D^n_{[L]}} \to X$ is a lifting of F extending G, as required. Proof of the necessity is completed.

Now we show that the condition of the Theorem is also sufficient. Consider a fibration $p\colon X \to Y$ which possesses the RLP with respect to any pair $S^n_{[L]} \subset D^{n+1}_{[L]}$ where $D^{n+1}_{[L]}$ is an $(n+1)$-$[L]$–disk and $S^n_{[L]}$ is a corresponding n-$[L]$–sphere, $n = 0, 1, \ldots$. We need to show that p is acyclic. Fix $n = 0, 1, \ldots$ and a point $x_0 \in X$. Let $y_0 = p(x_0)$. Pick an arbitrary element α of the group $\pi^{[L]}_n(Y, y_0)$. Let $\widetilde{S^n_{[L]}} = D^n_{[L]}/S^{n-1}_{[L]}$ be a quotient space, where $D^n_{[L]}$ is a n-$[L]$–disk. Let ϕ be the correspondent quotient mapping and put $s_{[L]} = \phi(S^{n-1}_{[L]})$. Obviously, compactum $\widetilde{S^n_{[L]}}$ admits an approximately $[L]$–soft mapping onto S^n. Therefore element α can be represented by means of a mapping $\widetilde{f}\colon \widetilde{S^n_{[L]}} \to Y$ such that $\widetilde{f}(s_{[L]}) = y_0$. The mapping $\widetilde{f}$ allows us to define mapping $f\colon D^{n+1}_{[L]} \to Y$ as follows.

$$f(x) = \begin{cases} \widetilde{f}(x), & \text{if } x \in D^n_{[L]} \backslash S^{n-1}_{[L]} \\ x_0, & \text{if } x \in S^{n-1}_{[L]} \end{cases}$$

Let $G\colon S^{n-1}_{[L]} \to x_0$ be a constant mapping. Applying RLP of p to the pair $S^{n-1}_{[L]} \subset D^n_{[L]}$ we can find a lifting $F\colon D^n_{[L]} \to X$ of f extending G. Now we define mapping $\widetilde{F}\colon \widetilde{S^n_{[L]}} \to X$ letting

$$\widetilde{F}(x) = \begin{cases} F(x), & \text{if } x \in \widetilde{S^n_{[L]}} \backslash s_{[L]} \\ x_0, & \text{if } x = s_{[L]} \end{cases}$$

Clearly, $\widetilde{f} = p \circ \widetilde{F}$. This shows that p induces an epimorphism of n^{th}-$[L]$–homotopy groups.

Consider now a mapping $F\colon S^n_{[L]} \to X$ where $S^n_{[L]}$ coincides with the subset $S^n_{[L]}$ of $(n+1)$-$[L]$–disk $D^{n+1}_{[L]}$, such that a composition $p \circ F\colon S^n_{[L]} \to Y$ is $[L]$–homotopically trivial. This implies existence of an extension $g\colon D^{n+1}_{[L]} \to Y$ of $p \circ F$. Applying RLP of p to the pair $S^n_{[L]} \subset D^{n+1}_{[L]}$ we can find a lifting $G\colon D^{n+1}_{[L]} \to X$ of g extending F. By Proposition 5.3.3 the mapping F is also $[L]$–homotopically trivial. This shows that p induces a monomorphism of n^{th}-$[L]$–homotopy groups.

Therefore fibration p is acyclic. □

The just proved theorem can be applied, in particular, as follows.

Corollary 5.3.6. *Every $[L]$-polyhedron is cofibrant and fibrant object in the category* **TOP** *with weak equivalences, fibrations and cofibrations, defined as in Definition 5.3.2.*

Proof. Definition of $[L]$–fibration implies that constant mapping of every topological space is a fibration. Let us show that every $[L]$–polyhedron X is cofibrant. According to the definition we need to check that for any acyclic fibration $p\colon A \to B$ and for any mapping $f\colon X \to B$ there exists a lifting

$F\colon X \to A$. Recall that any $[L]$–polyhedron can be obtained by attaching n-$[L]$-cells. This allows us to construct a lifting F by induction. If $g\colon X \to Y$ is an approximately $[L]$-soft mapping corresponding to X (see Definition 5.2.2), then for each $y \in Y^{(0)}$ we have $g^{-1}(y) \in \mathrm{AE}([L])$. We begin inductive construction applying RLP of p to the pairs $x \in g^{-1}(y)$ for each $y \in Y^{(0)}$ (the point x is arbitrary). Inductive steps are performed by successive use of Theorem 5.3.5. The resulting map F is continuous by virtue of the fact that C is a closed subspace of X if and only if $C \cap X^{(n)}$ is closed for each n (see Remark 5.2.3). □

Axiom (MC5) of model category holds by the following proposition.

Lemma 5.3.7. *Every map $p\colon X \to Y$ in* **TOP** *can be factored in either of two ways:*

(a) $f = p_\infty i_\infty$, where $i_\infty\colon X \to X'$ is a cofibration and $p_\infty\colon X' \to Y$ is an acyclic $[L]$-fibration.

(b) $f = q_\infty j_\infty$, where $j_\infty\colon X \to X'$ is a weak $[L]$-homotopy equivalence which has the LLP with respect to $[L]$-fibrations, and $q_\infty\colon X' \to Y$ is a $[L]$-fibration.

Proof. **(a)** Consider the set $\mathcal{F}$ of all inclusions $\{f_t\colon A_t \hookrightarrow B_t; t \in T\}$ where B_t is an n-$[L]$ disk and A_t is a corresponding $(n-1)$-$[L]$-sphere, $n = 0, 1, \ldots,$ or A_t and B_t are finite $[L]$-polyhedra such that both A_t and B_t are $\mathrm{AE}([L])$–spaces. For each $t \in T$ let $S(t)$ be the set containing all pairs of maps (g, h) such that the following diagram

$$\begin{array}{ccc} A_t & \xrightarrow{g} & X \\ {\scriptstyle f_t}\downarrow & & \downarrow{\scriptstyle p} \\ B_t & \xrightarrow{h} & Y \end{array}$$

commutes. By gluing a copy of B_t to X along A_t for every commutative diagram of the above form we obtain the Gluing Construction $G^1(\mathcal{F}, p)$. Let $i_1 \colon X \to G^1(\mathcal{F}, p)$ denote the natural embedding. The universal property of colimits guarantees existence of a map $p_1 \colon G^1(\mathcal{F}, p) \to Y$ such that $p_1 i_1 = p$. Inductively repeating the procedure, we arrive to the following infinite commutative diagram

$$\begin{array}{ccccccccccc} X & \xrightarrow{i_1} & G^1(\mathcal{F},p) & \xrightarrow{i_2} & G^2(\mathcal{F},p) & \xrightarrow{i_3} & \cdots & \xrightarrow{i_k} & G^k(\mathcal{F},p) & \longrightarrow & \cdots \\ {\scriptstyle p}\downarrow & & {\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow & & \downarrow & & {\scriptstyle p_k}\downarrow & & \\ Y & \xrightarrow{\text{id}} & Y & \xrightarrow{\text{id}} & Y & \xrightarrow{\text{id}} & \cdots & \xrightarrow{\text{id}} & Y & \longrightarrow & \cdots, \end{array}$$

Next consider the colimit $G^\infty(\mathcal{F}, p)$ of the upper row (i.e. the Infinite Gluing Construction) in the above diagram. There exist natural maps $i_\infty \colon X \to G^\infty(\mathcal{F}, p)$ and $p_\infty \colon G^\infty(\mathcal{F}, p) \to Y$ such that $p = p_\infty i_\infty$.

The small object argument (see [Dw-S, Proposition 7.17]) and the proof of [Dw-S, Lemma 8.12]) implies that the map $p_\infty \colon G^\infty(\mathcal{F}, p) \to Y$ has the RLP with respect to each of the maps in the family $\mathcal{F}$. According to definition and by Theorem 5.3.5, we therefore conclude that p_∞ is an acyclic $[L]$-fibration.

Since every map $f_t \in \mathcal{F}$ is either an inclusion of finite $[L]$–polyhedra which are AE($[L]$)–spaces or an inclusion of $(n-1)^{th}$-$[L]$–sphere into n^{th}-$[L]$–disk for some n, it follows from Definition 5.3.2(ii)$_L$ and Theorem 5.3.5 that f_t has the LLP with respect to acyclic $[L]$-fibrations. The colimit universality property of the Infinite Gluing Construction implies that the map i_∞ also has the LLP with respect to all acyclic $[L]$-fibrations and therefore is a $[L]$-cofibration according to Definition 5.3.2(iii)$_L$.

(b) Now we consider the set $\mathcal{F}$ of all inclusions $\{f_t \colon A_t \hookrightarrow B_t; t \in T\}$ of finite $[L]$-polyhedra such that both A_t and B_t are AE($[L]$)–spaces. For each $t \in T$ let $S(t)$ denote the set containing all pairs of maps (g, h) such that the following diagram

$$\begin{array}{ccc} A_t & \xrightarrow{g} & X \\ {\scriptstyle f_t}\downarrow & & \downarrow{\scriptstyle q} \\ B_t & \xrightarrow{h} & Y \end{array}$$

commutes. As in the part (b), we obtain the Infinite Gluing Construction $G^\infty(\mathcal{F}, q)$ and natural maps $j_\infty \colon X \to G^\infty(\mathcal{F}, q)$ and $q_\infty \colon G^\infty(\mathcal{F}, q) \to Y$ such that $q = q_\infty j_\infty$.

It follows from the small object argument (see [Dw-S, Proposition 7.17]) and the proof of [Dw-S, Lemma 8.12]) that the map $q_\infty \colon G^\infty(\mathcal{F}, q) \to Y$ has the RLP with respect to each of the maps in the family $\mathcal{F}$. This, according to definition, means that q_∞ is a $[L]$-fibration.

Observe that every map $f_t \in \mathcal{F}$ is an inclusion of finite $[L]$-polyhedra which are AE($[L]$)–spaces, and therefore f_t has the LLP with respect to $[L]$-fibrations (Definition 5.3.2(ii)$_L$). It is easy to see that j_∞ is a weak $[L]$-homotopy equivalence. The colimit universality property of the Infinite Gluing Construction implies that the map j_∞ also has the LLP with respect to $[L]$-fibrations. □

Now we are ready to establish the main result.

Theorem 5.3.8. *Let L be a finite complex. Then the category* **TOP** *with weak equivalences, fibrations and cofibrations, defined as in Definition 5.3.2, is a model category.*

Proof. Axioms (MC1)–(MC2) are trivially satisfied. Since the notions of $[L]$–fibrations and $[L]$–cofibrations are defined by lifting properties, the classes of $[L]$–fibrations and $[L]$–cofibrations are closed by retracts. Further, a retract of an isomorphism is an isomorphism. These facts imply Axiom (MC3). Axiom (MC5) holds according to Lemma 5.3.7.

It remains to verify Axiom (MC4). We need only to check that acyclic $[L]$–cofibrations have the LLP with respect to all $[L]$–fibrations ($[L]$–cofibrations have the property, required in (MC4), by definition). Consider a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

of unbroken arrows, where i is an acyclic $[L]$–cofibration and p is a $[L]$–fibration. We need to show existence of h. By Lemma 5.3.7 we have the following commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{j_\infty} & A' \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle q_\infty} \\ B & \xrightarrow{\mathrm{id}} & B \end{array}$$

where j_∞ is a weak $[L]$–homotopy equivalence having the LLP with respect to all $[L]$–fibrations and q_∞ is an $[L]$–fibration.

Observe that i and j_∞ are weak $[L]$–homotopy equivalences. Therefore axiom (MC2) implies that q_∞ is a weak $[L]$–homotopy equivalence. Therefore q_∞ is an acyclic $[L]$–fibration. Since i is a $[L]$–cofibration, there exists a lifting $h'\colon B \to A'$ of q_∞. Since j_∞ has the LLP with respect to $[L]$–fibrations, there exists a lifting $h''\colon A' \to X$ of a composition $g \circ q_\infty\colon A' \to Y$. We let $h = h'' \circ h'$. It is easily seen that h has the required property. □

Theorem 5.3.8 and Corollary 5.3.6 imply the following important statement, which generalizes classical theorems of Whitehead.

Theorem 5.3.9. *A map between $[L]$–complexes is an $[L]$–homotopy equivalence if and only if it induces isomorphisms of all $[L]$–homotopy groups.*

We conclude this chapter showing that the notion of $[L]$–fibration for $L = S^n$ differs from the usual notion of n–fibration. Thus, even if L coincides with the standard n-dimensional sphere S^n, the model category structure in theorem 5.3.8 differs from the one described in Example 5.1.4(iii) (note, however, that these two structures have identical weak equivalences). The question whether the model category structure generated on **TOP** by $L = \{\text{point}\}$ coincides with the one described in Example 5.1.4(i) is open (although classes of weak equivalences are identical and the notion of $[L]$-homotopy coincides with the notion of usual homotopy in this case).

Now we construct an example which shows that the notions of n-fibration and S^n-fibration differ. Let $n \geq 0$ and let X and Y be copies of the $(n +$

1)-dimensional universal Menger compactum μ^{n+1}. Let $p\colon X \to Y$ be the Dranishnikov's resolution constructed in [D3] (see also [Ch3, §4.2]). The map p is polyhedrally $(n+1)$–soft. Hence it is an acyclic $(n+1)$–fibration. Recall that p is n-soft, but not $(n+1)$–soft. The latter fact implies existence of at most $(n+1)$-dimensional compactum B_0, a closed embedding $i_0\colon A_0 \hookrightarrow B_0$ and of two maps $\alpha\colon A_0 \to X$ and $\beta\colon B_0 \to Y$ such that there is no lifting of β, extending α.

Let A and B denote two additional copies of the Menger compactum μ^{n+1} and let $i\colon A \to B$ be a Z–embedding. Since i is a Z-embedding, we can find embedding $j_0\colon A_0 \to A$ and $j\colon B_0 \to B$ such that $j \circ i_0 = i \circ j_0$ and $j(B_0 \setminus i_0(A_0)) \subseteq B \setminus i(A)$. Since X is an AE$(n+1)$–compactum and dim$A = n+1$, there exists a map $g\colon A \to X$ such that $\alpha = g \circ i_0$. Define map $h\colon i(A) \cup j(B_0) \to Y$ letting

$$h(b) = \begin{cases} p(g(i^{-1}(b))), & \text{if } b \in i(A), \\ \beta(j^{-1}(b)), & \text{if } b \in j(B_0) \end{cases}$$

Since $Y \in$ AE$(n+1)$ and dim$B = n+1$, there exists a map $f\colon B \to Y$ such that $f|\,(i(A) \cup j(B_0)) = h$. Note that $p \circ g = f \circ i$. Observe that the map f does not have lifting extending g (otherwise this would imply existence of lifting of β extending α). Hence p is not a $[S^{n+1}]$–fibration.

Chapter 6

[L]-Homotopy Groups: Some Computations

Perhaps the most natural problem related to $[L]$-homotopy groups is the problem of computation. More specifically, the problem can be stated as follows: describe $[L]$-homotopy groups of a space X in terms of "usual" algebraic invariants of X and L (in particular, in terms of homotopy and homology groups). The first step towards this goal is apparently computation of n-th $[L]$-homotopy group of S^n for complex L whose extension type lies between extension types of S^n and S^{n+1}. In the present chapter we perform this step. The main result of this chapter is published in [Ka].

6.1 Cohomological Properties

In this section we investigate some cohomological properties of complex L satisfying the condition $[L] \leq S^n$.

Proposition 6.1.1. *Let L be a finite complex such that $[L] \leq [S^{n+1}]$ and n is the minimal integer with this property. Then for any $q \leq n$ group $H_q(L)$ is torsion.*

Proof. Suppose there exists $q \leq n$ such that $H^q(L) \cong \mathbb{Z} \oplus G$. To get a contradiction we will show that $[L] \leq [S^q]$. Consider a compactum X such that $L \in AE(X)$. Observe that X is finite-dimensional since $[L] \leq [S^{n+1}]$ by our assumption. Put $H = H_q(L)$. By Theorem 2.3.6 and Remark 2.3.7 we have $\mathrm{c-dim}_H X \leq q$. Hence for any closed subspace A of X and for any integer $i \geq 1$ we have $H^{q+i}(X, A; H) = \{0\}$. By universal coefficients formula

$$H^{q+i}(X, A; H) \cong (H^{q+i}(X, A) \otimes H) \oplus \mathrm{Tor}(H^{q+i+1}(X, A), H)$$

Hence $H^{q+i}(X, A) \otimes H = \{0\}$. On the other hand we have $H^{q+i}(X, A) \otimes H = H^{q+i}(X, A) \otimes (\mathbb{Z} \oplus G) = H^{q+i}(X, A) \oplus (H^{q+i}(X, A) \otimes G)$. Therefore $H^{q+i}(X, A) = 0$ for all $i \geq 1$ and hence $\mathrm{c-dim} X \leq q$. Since X is finite-dimensional, the last inequality implies $\dim X \leq q$ and hence $S^q \in AE(X)$. □

Using universal coefficients formula connecting homology and cohomology and taking into account that homology and cohomology groups of L are finitely generated, we obtain the following corollary.

Corollary 6.1.2. *Let L be a finite complex such that $[L] \leq [S^{n+1}]$ and n is the minimal integer with this property. Then for any $q \leq n$ group $H^q(L)$ is torsion.*

Proposition 6.1.3. *Let L be a finite complex such that $[L] \leq [S^{n+1}]$ and n is the minimal integer with this property. Then there exist arbitrary large integers p such that $H^q(L; \mathbb{Z}_p)$ is trivial for any $q \leq n$.*

Proof. Corollary 6.1.2 implies that $H^q(L) \cong \bigoplus_{i=1}^{k_q} \mathbb{Z}_{m_i(q)}$ for any $q \leq n$. Additionally, let $\mathrm{Tor}H^{n+1}(L) = \bigoplus_{i=1}^{k_{n+1}} \mathbb{Z}_{m_i(n+1)}$

Consider any integer p such that $(p, m_i(q)) = 1$ for all $q \leq n+1$ and $i = 1, \dots, k_q$. Universal coefficients formula implies that $H^q(L; \mathbb{Z}_p)$ is trivial for every $q \leq n$. □

6.2 Properties of [L]-Homotopy Groups

In this section we establish some of the properties of $[L]$-homotopy groups.

Remark 6.2.1. Let L be a finite complex satisfying double inequality $[S^n] < [L] \leq [S^{n+1}]$ for some fixed n. Then by virtue of Proposition 2.5.7 compactum $S^n_{[L]}$ is $[L]$-homotopically equivalent to S^n. Therefore for any space X group $\pi_n^{[L]}(X)$ is isomorphic to $\pi_n(X)/N([L])$ where $N([L])$ denotes the relation of $[L]$-homotopy equivalence between elements of $\pi_n(S^n)$.

This remark implies the following proposition.

Proposition 6.2.2. *Let L be a finite complex satisfying double inequality $[S^n] < [L] \leq [S^{n+1}]$. Then for the group $\pi_n^{[L]}(S^n)$ there are three variants: $\pi_n^{[L]}(S^n) \cong \mathbb{Z}$, $\pi_n^{[L]}(S^n) \cong \mathbb{Z}_m$ for some integer m or this group is trivial.*

Let us characterize the hypothetical equality $\pi_n^{[L]}(S^n) \cong \mathbb{Z}_m$ in terms of extensions of maps.

Proposition 6.2.3. *Let L be a finite complex satisfying double inequality $[S^n] < [L] \leq [S^{n+1}]$. If $\pi_n^{[L]}(S^n) \cong \mathbb{Z}_m$ then for any space X such that* $\mathrm{e-dim}X \leq [L]$, *for any closed subspace A of X and for any mapping $f\colon A \to$*

S^n *there exists an extension* $\overline{h}\colon X \to S^n$ *of the composition* $h = z_m f$*, where* $z_m\colon S^n \to S^n$ *is a mapping of degree* m*.*

Proof. Suppose that $\pi_n^{[L]}(S^n) \cong \mathbb{Z}_m$. Then Remark 6.2.1 implies that the map $z_m\colon S^n \to S^n$ is $[L]$-homotopic to a constant map. Let us show that $h = z_m f\colon A \to S^n$ is also $[L]$-homotopic to a constant map. This fact combined with $[L]$–homotopy extension property (see Proposition 2.5.4) will prove our statement.

Consider a space Z such that $\mathrm{e-dim} Z \le [L]$ and a mapping $H\colon Z \to A \times I$, where $I = [0,1]$ denotes the unit interval. Pick a point $s \in S^n$. Consider mappings $f_i\colon A \times \{i\} \to S^n$, $i = 0,1$ where $f_0 = z_m f$ and $f_1 \equiv s$ is a constant map. Define $F\colon A \times I \to S^n \times I$ as follows: $F(a,t) = (f(a),t)$ for each $a \in A$ and $t \in I$. Let $f_0' \equiv z_m$ and $f_1' \equiv s$ be mappings such that $f_i'\colon S^n \times \{i\} \to S^n$, $i = 0,1$.

Consider a composition $G = FH\colon Z \to S^n \times I$. By our assumption f_0' is $[L]$-homotopic to f_1'. Therefore a mapping $g\colon G^{-1}(S^n \times \{0\} \bigcup S^n \times \{1\}) \to S^n$, defined as $g|_{G^{-1}(S^n \times \{i\})} = f_i' G$ for $i = 0,1$, can be extended to a mapping of Z to S^n. On the other hand $G^{-1}(S^n \times \{i\}) = H^{-1}(A \times \{i\})$ and $g|_{G^{-1}(S^n \times \{i\})} = f_i' f H|_{H^{-1}(A \times \{i\})} = f_i H|_{H^{-1}(A \times \{i\})}$ for $i = 0,1$. This remark completes the proof. □

6.3 Computation of $\pi_n^{[L]}(S^n)$

In this section we prove the main result of the present chapter, namely, we show that $\pi_n^{[L]}(S^n) \cong \mathbb{Z}$. In order to establish this fact we need to construct a compactum with special properties. We will apply technique used in the proof of Theorem 2.4 in [D-R].

Recall the following definition [D-R].

Definition 6.3.1. Inverse sequence $S = \{X_i, p_i^{i+1} \colon i \in \omega\}$ consisting of metrizable compacta is said to be *L-resolvable* if for any i, for any closed subspace A of X_i and any map $f \colon A \to L$ there exists $k \geq i$ such that the composition $fp_i^k \colon (p_i^k)^{-1}A \to L$ can be extended to a mapping of X_k to L.

The following lemma expresses an important property of $[L]$-resolvable inverse sequences.

Lemma 6.3.2. [D-R] *Suppose that L is a countable complex and X is a compactum such that $X = limS$ where $S = \{(X_i, \lambda_i), q_i^{i+1} \colon i \in \omega\}$ is an L-resolvable inverse system of compact polyhedra X_i with triangulations λ_i such that* $\mathrm{mesh}\{\lambda_i\} \to 0$. *Then* $\mathrm{e-dim}X \leq [L]$.

Further we shall need the following simple proposition.

Proposition 6.3.3. *Let X be a metrizable compactum, A be a closed subspace of X. Then for any mapping $f \colon A \to S^n$ existence of extension $\overline{f} \colon X \to S^n$ of f implies that $\delta^*_{X,A}(f^*(\zeta)) = 0$ in group $H^{n+1}(X, A; \mathbb{Z}_k)$ for any integer k, where ζ is a generator of $H^n(S^n, \mathbb{Z}_k)$.*

Proof. Let $\overline{f}$ be an extension of f. Commutativity of the following diagram proves the statement.

$$\begin{array}{ccc} H^n(A;\mathbb{Z}_k) & \xrightarrow{\delta^*_{X,A}} & H^{n+1}(X,A;\mathbb{Z}_k) \\ \uparrow{\scriptstyle f^*} & & \uparrow{\scriptstyle \overline{f}^*} \\ H^n(S^n;\mathbb{Z}_k) & \xrightarrow{\delta^*_{S^n,S^n}} & H^{n+1}(S^n,S^n;\mathbb{Z}_k) = \{0\} \end{array}$$

□

We are ready now to prove the main theorem of the present chapter.

Theorem 6.3.4. *Let L be a finite complex satisfying $[S^n] < [L] \leq [S^{n+1}]$ for some n. Then $\pi_n^{[L]}(S^n) \cong \mathbb{Z}$.*

Proof. Suppose the opposite, i.e. $\pi_n^{[L]}(S^n) \cong \mathbb{Z}_m$ (we use Proposition 6.2.2; the same arguments can be used to show that $\pi_n^{[L]}(S^n)$ is non-trivial). Slightly modifying construction of compactum used in [D-R, Theorem 2.4], obtain an inverse sequence $S = \{(X_i, \tau_i), p_i^{i+1} \colon i \in \omega\}$ consisting of compact polyhedra with fixed triangulations $\{\tau_i\}$ such that $\mathrm{mesh}\tau_i \to 0$, S is $[L]$-resolvable, $X_0 = D^{n+1}$ where D^{n+1} is $(n+1)$-dimensional disk and for any $x \in X_i$ we have $(p_i^{i+1})^{-1}x \simeq L$ or $*$.

Let $X = \lim S$. Observe that $\mathrm{e-dim}(X) \leq [L]$ by Lemma 6.3.2. Let $p_0 \colon X \to D^{n+1}$ be a limit projection. Consider $p \geq m+1$ which satisfies Proposition 6.1.3. By Vietoris-Begle theorem [Sp] and by our choice of p, for every index i and any $X_i' \subset X_i$ a homomorphism $(p_i^{i+1})^* \colon H^k(X_i'; \mathbb{Z}_p) \to H^k((p_i^{i+1})^{-1}X_i'; \mathbb{Z}_p)$ is an isomorphism for $k \leq n$ and a monomorphism for $k = n+1$. Therefore for each $D' \subset X_0 = D^{n+1}$ a homomorphism $p_0^* \colon H^k(D'; \mathbb{Z}_p) \to H^k((p_0)^{-1}D'; \mathbb{Z}_p)$ is an isomorphism for $k \leq n$ and a monomorphism for $k = n+1$. In particular, $H^n(X; \mathbb{Z}_p) = \{0\}$ since $X_0 = D^{n+1}$.

Put $A = (p_0)^{-1}S^n$. Let $\zeta \in H^n(S^n; \mathbb{Z}_p) \cong \mathbb{Z}_p$ be a generator. Since $p_0^* \colon H^n(S^n; \mathbb{Z}_p) \to H^n(A; \mathbb{Z}_p)$ is an isomorphism, $p_0^*(\zeta)$ is a generator of $H^n(A, \mathbb{Z}_p) \cong \mathbb{Z}_p$. In particular, $p_0^*(\zeta)$ is an element of order p. From exact sequence of the pair (X, A)

$$\ldots \to H^n(X; \mathbb{Z}_p) = \{0\} \xrightarrow{i_{X,A}} H^n(A; \mathbb{Z}_p) \xrightarrow{\delta^*_{X,A}} H^{n+1}(X, A; \mathbb{Z}_p) \to \ldots$$

we see that $\delta^*_{X,A}$ is a monomorphism and hence $\delta^*_{X,A}(p_0^*(\zeta)) \in H^{n+1}(X, A; \mathbb{Z}_p)$

is an element of order p.

Consider now the composition $h = z_m p_0$. By our assumption and by Proposition 6.2.3 this map can be extended to a mapping of X to S^n. This fact combined with Proposition 6.3.3 implies that $\delta^*_{X,A}(h^*(\zeta)) = 0$ in $H^{n+1}(X, A; \mathbb{Z}_p)$. On the other hand, $\delta^*_{X,A}(h^*(\zeta)) = m\delta^*_{X,A}(p_0^*(\zeta))$. Contradiction. Therefore $\pi_n^{[L]}(S^n) \cong \mathbb{Z}$. $\square$

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