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# S-duality and the prepotential in $\mathcal{N} = 2^*$ theories (I): the ADE algebras

M. Billó,<sup>a</sup> M. Frau,<sup>a</sup> F. Fucito,<sup>b</sup> A. Lerda<sup>c,a</sup> and J.F. Morales<sup>b</sup>

<sup>a</sup>*Dipartimento di Fisica, Università di Torino, and I.N.F.N. – sezione di Torino,  
Via P. Giuria 1, I-10125 Torino, Italy*

<sup>b</sup>*I.N.F.N. – sezione di Roma 2, and Dipartimento di Fisica, Università di Roma Tor Vergata,  
Via della Ricerca Scientifica, I-00133 Roma, Italy*

<sup>c</sup>*Dipartimento di Scienze e Innovazione Tecnologica, Università del Piemonte Orientale,  
and I.N.F.N. – Gruppo Collegato di Alessandria – sezione di Torino,  
Viale T. Michel 11, I-15121 Alessandria, Italy*

*E-mail:* [bill@to.infn.it](mailto:bill@to.infn.it), [frau@to.infn.it](mailto:frau@to.infn.it), [lerda@to.infn.it](mailto:lerda@to.infn.it),  
[fucito@roma2.infn.it](mailto:fucito@roma2.infn.it), [morales@roma2.infn.it](mailto:morales@roma2.infn.it)

**ABSTRACT:** The prepotential of  $\mathcal{N} = 2^*$  supersymmetric theories with unitary gauge groups in an  $\Omega$  background satisfies a modular anomaly equation that can be recursively solved order by order in an expansion for small mass. By requiring that S-duality acts on the prepotential as a Fourier transform we generalise this result to  $\mathcal{N} = 2^*$  theories with gauge algebras of the D and E type and show that their prepotentials can be written in terms of quasi-modular forms of  $SL(2, \mathbb{Z})$ . The results are checked against microscopic multi-instanton calculus based on localization for the A and D series and reproduce the known 1-instanton prepotential of the pure  $\mathcal{N} = 2$  theories for any gauge group of ADE type. Our results can also be used to obtain the multi-instanton terms in the exceptional theories for which the microscopic instanton calculus and the ADHM construction are not available.

**KEYWORDS:** Supersymmetric gauge theory, Duality in Gauge Field Theories, Solitons Monopoles and Instantons, Nonperturbative Effects

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**1 Introduction**

Gauge theories with extended supersymmetries show very remarkable behaviours. For example the maximally supersymmetric Yang-Mills theories in  $d = 4$ , briefly  $\mathcal{N} = 4$  SYM theories, are conjectured to enjoy S-duality invariance. S-duality is a strong/weak-coupling relation that exchanges electrically charged states with non-perturbative magnetically charged states [1]; over the years many tests of this conjecture have been carried out with success (see for instance [2, 3]). For theories with simply-laced gauge groups, S-duality maps the gauge group to itself, but when the gauge group is not simply laced, the gauge group of the S-dual theory is the GNO dual group [4].

Also  $\mathcal{N} = 2$  SYM theories are very interesting: even if they are less constrained than the  $\mathcal{N} = 4$  theories, it is still possible to study several of their properties in an exact way. Indeed their perturbative contribution is exhausted at the one-loop level and their non-perturbative behaviour is by now well understood, on the one hand, via the Seiberg-Witten [5, 6] description of their low energy effective theory and, on the other hand, via the direct computation of instanton effects by means of localization techniques [7–11]. A noticeable exception in this scenario is given by theories with exceptional gauge groups for which an ADHM construction of the instanton moduli space is still missing.<sup>1</sup>

Among the  $\mathcal{N} = 2$  models much attention has been devoted, in the recent years, to superconformal theories and to their mass deformations, which sit at the crossroad of many approaches to the non-perturbative description of quantum field theories and of their duality structures (see for example the collective review [18] and the references therein). This paper deals with the  $\mathcal{N} = 2^*$  SYM theories with simply laced gauge group  $G$  whose corresponding Lie algebra will be denoted by  $\mathfrak{g}$ . Beside the  $\mathcal{N} = 2$  gauge vector multiplet, these theories contain an adjoint hypermultiplet of mass  $m$  and represent a mass deformation of the  $\mathcal{N} = 4$  SYM theory. In an appropriate large- $m$  limit, the hypermultiplet decouples and the pure  $\mathcal{N} = 2$  SYM theory is retrieved. The  $\mathcal{N} = 2^*$  theory inherits from the  $\mathcal{N} = 4$  theory an interesting action of S-duality. In particular, S-duality acts non-trivially on the prepotential function  $F$  that encodes the low-energy effective dynamics on the Coulomb branch of moduli space. Upon expanding the prepotential in powers of the mass  $m$ , this action can be exploited to efficiently determine the non-perturbative expression of the prepotential. This is achieved by showing that the coefficients  $f_n$  of the mass expansion of  $F$  are (quasi)-modular functions of the gauge coupling  $\tau$  connected to each other by a recursion relation.

Such a recursion relation, which encodes the “modular anomaly” of the prepotential, was first pointed out for  $U(N)$  theories in [19] where it was derived from the Seiberg-Witten curve. The modular anomaly is related to the holomorphic anomaly of topological string amplitudes through local Calabi-Yau embeddings of the SW curves [20–23]. It has been studied also in presence of an  $\Omega$ -background [24–34], in the framework of the AGT correspondence [35–38] and in  $\mathcal{N} = 2$  conformal SQCD models with fundamental matter [30, 31, 39].

Here we review and streamline the derivation of the modular anomaly equation and the associated recursion relation directly from the S-duality requirement and for a generic simply-laced gauge group  $G$  (the non simply-laced groups will be discussed in a companion paper [40]). The modular anomaly equation leads to express the prepotential in terms of modular forms of  $\tau$  and of functions of the periods  $a$  which are written in terms of the root system of  $\mathfrak{g}$ , allowing for a unified treatment of all Lie algebras. In this way we can compute the prepotential also for the exceptional Lie algebras  $E_6$ ,  $E_7$  and  $E_8$  for which an ADHM construction of the instanton moduli space is not available.

This is the plan of the paper: in section 2 we discuss the behaviour of the  $\mathcal{N} = 2^*$  theories under S-duality and derive from it the modular anomaly equation satisfied by

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<sup>1</sup>See for example [12–17] for recent progresses on the description of instanton moduli spaces in theories with exceptional gauge groups.

the prepotential. In section 3 we exploit the recursion relation equivalent to the modular anomaly equation to compute exactly, *i.e.* to all orders in the instanton expansion, the first few coefficients in the mass expansion of the prepotential. The results are then generalised in section 4 to account for a non-trivial  $\Omega$ -background. In section 5 we will describe the direct microscopic computation of the instanton corrections for the algebras of type  $A_r$  and  $D_r$  using the equivariant localization methods. The purpose of this section is to clarify some subtle points of the multi-instanton calculus and to check successfully these microscopic results against the instanton expansion of the solutions of the modular anomaly equation derived in the previous section. Our conclusions are presented in section 6, while several technical material is confined in various appendices.

## 2 S-duality and modular anomaly

In this section we briefly review the structure of  $\mathcal{N} = 2^*$  theories with a gauge group  $G$  of ADE type and discuss the constraint that S-duality imposes on their prepotential.

### 2.1 The $SL(2, \mathbb{Z})$ -duality symmetry

The field content of these theories includes an  $\mathcal{N} = 2$  vector multiplet and a massive hypermultiplet, both transforming in the adjoint representation of  $G$ . The  $\mathcal{N} = 2$  gauge multiplet contains an adjoint complex scalar  $\varphi$ , whose vacuum expectation value can always be aligned along the Cartan directions and written in the diagonal form

$$\langle \varphi \rangle = a = \text{diag}(a_1, a_2, \dots, a_r), \tag{2.1}$$

with  $r$  denoting the rank of the gauge Lie algebra  $\mathfrak{g}$ . The parameters  $\{a_u\}$  span the Coulomb branch of the classical moduli space of the gauge theory. The low energy effective action on this branch is specified by a holomorphic function: the prepotential  $F(a)$ . Alternatively the gauge theory can be described in terms of the dual variables

$$a_u^D = \frac{1}{2\pi i} \frac{\partial F}{\partial a_u}. \tag{2.2}$$

In the following we will often write  $\partial_u$  for  $\frac{\partial}{\partial a_u}$ . We will also use a simplified vector notation, writing, for instance,  $a$  for the vector,  $\frac{\partial}{\partial a}$  for the gradient vector, and so on.

The effective coupling matrix, which also encodes the metric on the moduli space, is

$$\tau_{uv} = \partial_u a_v^D = \frac{1}{2\pi i} \partial_u \partial_v F. \tag{2.3}$$

The classical part of the prepotential reads simply

$$F^{\text{cl}} = \pi i \tau a^2 \tag{2.4}$$

where  $\tau$  is the complexified gauge coupling

$$\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2} \tag{2.5}$$

At this level, the dual periods and the effective coupling matrix are

$$a^D = \tau a, \quad \tau_{uv} = \tau \delta_{uv} . \tag{2.6}$$

In a Seiberg-Witten description of the theory,  $a$  and  $a^D$  describe the periods of the Seiberg-Witten differential along the  $2r$  cycles of the Riemann-surface defined by the Seiberg-Witten curve. The periods and dual periods can be assembled in a  $2r$ -dimensional vector  $(a^D, a)$  that transforms as a vector of the modular group  $\text{Sp}(4r, \mathbb{Z})$  of the Riemann surface. The two set of variables are suitable to describe the regimes of weak and strong coupling ( $g$  small and  $g$  large respectively) of the gauge theory. These two regimes are mapped to each other by S-duality which, as an element of  $\text{Sp}(4r, \mathbb{Z})$ , exchanges periods and dual periods and acts projectively on  $\tau$  by inverting it, namely

$$S(a) = a^D, \quad S(a^D) = -a, \quad S(\tau) = -\frac{1}{\tau} . \tag{2.7}$$

On the other hand, the T-duality acts as

$$T(a) = a, \quad T(a^D) = a^D + a, \quad T(\tau) = \tau + 1 . \tag{2.8}$$

S and T generate the  $\text{SL}(2, \mathbb{Z})$  modular group.

On the prepotential, the T-duality action is

$$T[F(a)] = F(a) + \pi i a^2 , \tag{2.9}$$

as one can see from the fact that only the classical part  $F^{\text{cl}}$  given in (2.4) transforms non-trivially under  $\tau \rightarrow \tau + 1$ . Indeed,  $\mathcal{N} = 2$  supersymmetry allows only for one-loop ( $\tau$ -independent) and instanton corrections (weighted by  $e^{2\pi i k \tau}$  with  $k$  integer) which are T-invariant.

The S-duality action is instead much less trivial since it maps the description of the theory in the variables  $a$  to its dual description in terms of  $a^D$ . Therefore  $S$  should map the prepotential  $F(a)$  to its Legendre transform:

$$S[F(a)] = \mathcal{L}[F](a^D) \tag{2.10}$$

where

$$\mathcal{L}[F](a^D) = F(a) - 2\pi i a \cdot a^D = F(a) - a \cdot \frac{\partial F}{\partial a} . \tag{2.11}$$

The classical part of the prepotential verifies immediately (2.10); in fact

$$S[F^{\text{cl}}] = -\frac{\pi i}{\tau} (a^D)^2 = \mathcal{L}[F^{\text{cl}}](a^D) . \tag{2.12}$$

The S-duality symmetry requirement (2.10) represents instead a highly non-trivial constraint on the *quantum* prepotential. As we will see, this constraint allows us to determine the exact form of the prepotential, order by order in the hypermultiplet mass, starting from very few microscopic data.

As mentioned in the introduction, this is known to happen for  $U(N)$  theories, where the prepotential satisfies a “modular anomaly” equation that has been discussed extensively in the literature [19]–[33]. In the following we derive the modular anomaly equation directly from the S-duality relation (2.10) and show that it holds for gauge theories with all gauge groups of ADE type.

## 2.2 The small mass expansion of the prepotential

When the mass  $m$  of the adjoint hypermultiplet vanishes, there are no quantum corrections to the prepotential since the supersymmetry is enhanced to  $\mathcal{N} = 4$ . When the mass is turned on, the supersymmetry is only  $\mathcal{N} = 2$  and the prepotential  $F$  is corrected. We write

$$F = F^{\text{cl}} + f = \pi i \tau a^2 + f, \tag{2.13}$$

where  $f$  is the quantum part of the prepotential. The dual periods and effective coupling  $\tau_{uv}$  also get quantum corrected and become non-trivial functions of  $\tau$ . As already mentioned,  $\mathcal{N} = 2$  supersymmetry allows perturbative corrections only at one-loop and non-perturbative corrections at all instanton numbers. Working in a mass expansion, we write the quantum prepotential as

$$f = f^{1\text{-loop}} + f^{\text{inst}} = \sum_{n=1}^{\infty} f_n \tag{2.14}$$

where

$$f_n = f_n^{1\text{-loop}} + f_n^{\text{inst}} \tag{2.15}$$

is proportional to  $m^{2n}$ .

The one-loop contribution to the prepotential has the form (see for instance [41])

$$f^{1\text{-loop}} = \frac{1}{4} \sum_{\alpha \in \Psi} \left[ -(\alpha \cdot a)^2 \log \left( \frac{\alpha \cdot a}{\Lambda} \right)^2 + (\alpha \cdot a + m)^2 \log \left( \frac{\alpha \cdot a + m}{\Lambda} \right)^2 \right] \tag{2.16}$$

where  $\Lambda$  is an arbitrary scale and  $\alpha$  is an element of the root system  $\Psi$  of the algebra  $\mathfrak{g}$ ;  $\alpha$  is an  $r$ -dimensional vector of components  $\alpha_u$ . The scalar product  $\alpha \cdot a$  represents the mass acquired by the complex  $W$ -boson associated to the root  $\alpha$  via the (super)-Higgs mechanism. Also the mass of the adjoint scalar along the root  $\alpha$  is shifted with respect to its original value  $m$  by the same amount. Expanding (2.16) in powers of  $m$ , all odd powers cancel upon summing over positive and negative roots, and we find

$$\begin{aligned} f^{1\text{-loop}} &= \frac{m^2}{4} \sum_{\alpha \in \Psi} \log \left( \frac{\alpha \cdot a}{\Lambda} \right)^2 - \sum_{n=2}^{\infty} \frac{m^{2n}}{4n(n-1)(2n-1)} C_{2n-2} \\ &= \frac{m^2}{4} \sum_{\alpha \in \Psi} \log \left( \frac{\alpha \cdot a}{\Lambda} \right)^2 - \frac{m^4}{24} C_2 - \frac{m^6}{120} C_4 - \frac{m^8}{336} C_6 - \dots \end{aligned} \tag{2.17}$$

where we defined

$$C_n = \sum_{\alpha \in \Psi} \frac{1}{(\alpha \cdot a)^n}. \tag{2.18}$$

The non-perturbative part of the prepotential receives contributions from the various instanton sectors, so  $f^{\text{inst}}$  is a series in the instanton weight

$$q = e^{-\frac{8\pi^2}{g^2} + i\theta} = e^{2\pi i \tau}. \tag{2.19}$$

The term of order  $q^k$  can be evaluated integrating over the moduli spaces of  $k$ -instantons by means of localization techniques when the gauge group  $G$  is one of the classical matrix groups. This excludes the exceptional groups  $E_{6,7,8}$ . We will review this computation in section 5. This procedure can in principle be carried out up to arbitrary order  $k$ ; in practice, however, it is computationally rather intense. It is important to observe that

$$f_1^{\text{inst}} = 0 \quad (2.20)$$

since instanton contributions start at order  $m^4$ . This can be seen by noticing that every mass insertion soaks two of the eight instanton fermionic zero modes of the  $\mathcal{N} = 4$  theory, so we need at least four powers of  $m$  to get a non-trivial result.

### 2.3 The modular anomaly equation

We now investigate the consequences of the S-duality relation (2.10) on the quantum prepotential  $f$ . First we observe that the prepotential has mass dimension 2, so on dimensional grounds all  $f_n$  with  $n \geq 2$  must be homogeneous functions of degree  $2 - 2n$  in  $a$ :

$$f_n(\lambda a) = \lambda^{2-2n} f_n(a) . \quad (2.21)$$

Moreover, they are non-trivial functions of  $\tau$  expressed as Fourier series in  $q$ . We therefore use the notation  $f_n(\tau, a)$  to express this fact.

Let us now compute first the two sides of the duality relation (2.10). The Legendre transform of  $F$  is

$$\mathcal{L}[F] = F - a \cdot \frac{\partial F}{\partial a} = -\pi i \tau a^2 + f - a \cdot \frac{\partial f}{\partial a} . \quad (2.22)$$

On the other hand, using (2.7), the S-transform of  $F$  is

$$S[F] = -\frac{\pi i}{\tau} (a^{\text{D}})^2 + f\left(-\frac{1}{\tau}, a^{\text{D}}\right) , \quad (2.23)$$

where, according to (2.2),

$$a^{\text{D}} = \frac{1}{2\pi i} \frac{\partial F}{\partial a} = \tau \left( a + \frac{1}{2\pi i \tau} \frac{\partial f}{\partial a} \right) . \quad (2.24)$$

Plugging (2.24) into (2.22), the S-duality relation  $S[F] = \mathcal{L}[F]$  can be written in the form

$$f\left(-\frac{1}{\tau}, a^{\text{D}}\right) = f(\tau, a) + \frac{1}{4\pi i \tau} \left( \frac{\partial f(\tau, a)}{\partial a} \right)^2 . \quad (2.25)$$

From (2.17) and (2.20) we notice that  $f_1$  is independent of  $\tau$  but dependent on  $\Lambda$ , so equation (2.25) at order  $m^2$  implies

$$f_1(\tau a, S(\Lambda)) = f_1(a, \Lambda) \quad (2.26)$$

where we have allowed an action of S-duality on the scale  $\Lambda$ . Using the explicit form of  $f_1$ , that is

$$f_1(a, \Lambda) = f_1^{1\text{-loop}}(a, \Lambda) = \frac{m^2}{4} \sum_{\alpha \in \Psi} \log \left( \frac{\alpha \cdot a}{\Lambda} \right)^2 , \quad (2.27)$$

we conclude that  $S(\Lambda) = \tau \Lambda$ .

At higher orders in the mass expansion, the differential equation (2.25) can be solved by taking  $f_n$ , for  $n \geq 2$ , to be an  $SL(2, \mathbb{Z})$  quasi-modular form of weight  $2n - 2$ . A basis of quasi-modular forms is given by the set of Eisenstein series  $\{E_2, E_4, E_6\}$ . More precisely  $E_4$  and  $E_6$  are true modular forms of weight 4 and 6 respectively, so under S-duality they transform as

$$E_4\left(-\frac{1}{\tau}\right) = \tau^4 E_4(\tau), \quad E_6\left(-\frac{1}{\tau}\right) = \tau^6 E_6(\tau). \quad (2.28)$$

The  $E_2$  series is instead a quasi-modular form of weight 2 because under S it gets shifted:

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2 \left( E_2(\tau) + \frac{6}{\pi i \tau} \right) \equiv \tau^2 (E_2(\tau) + \delta). \quad (2.29)$$

Here we introduced the notation  $\delta = \frac{6}{\pi i \tau}$  to avoid clutter in the subsequent formulæ.

We notice that all  $\delta$ -dependence should cancel in the duality relation since  $f$  is only a function of  $q$ , and that the quasi-modularity of  $f_n$  is due entirely to its dependence on  $E_2$ . Indicating explicitly this dependence, we have (for  $n \geq 2$ )

$$f_n\left(-\frac{1}{\tau}, a^D, E_2\left(-\frac{1}{\tau}\right)\right) = \tau^{2n-2} f_n(\tau, a^D, E_2 + \delta) = f_n\left(\tau, \frac{a^D}{\tau}, E_2 + \delta\right). \quad (2.30)$$

where in the last step we used the homogeneity property (2.21) of  $f_n$ . On the other hand we have

$$f_1(a^D, \tau\Lambda) = f_1\left(\frac{a^D}{\tau}, \Lambda\right). \quad (2.31)$$

Plugging (2.30) and (2.31) into the left hand side of (2.25), we find

$$\begin{aligned} f\left(-\frac{1}{\tau}, a^D, E_2\left(-\frac{1}{\tau}\right), \tau\Lambda\right) &= f\left(\tau, \frac{a^D}{\tau}, E_2 + \delta, \Lambda\right) \\ &= f\left(\tau, a + \frac{\delta}{12} \frac{\partial f}{\partial a}, E_2 + \delta, \Lambda\right) \\ &= f(\tau, a, E_2, \Lambda) + \delta \left[ \frac{\partial f}{\partial E_2} + \frac{1}{12} \left( \frac{\partial f}{\partial a} \right)^2 \right] (\tau, a, E_2, \Lambda) \\ &\quad + \frac{\delta^2}{2} \left[ \frac{\partial^2 f}{\partial E_2^2} + \frac{1}{144} \left( \frac{\partial f}{\partial a} \right)^2 \frac{\partial^2 f}{\partial a^2} + \frac{1}{6} \frac{\partial f}{\partial a} \cdot \frac{\partial^2 f}{\partial a \partial E_2} \right] (\tau, a, E_2, \Lambda) \\ &\quad + \dots \end{aligned} \quad (2.32)$$

where the dots stand for higher order terms in  $\delta$ . Comparing (2.32) with the right hand side of (2.25), one finds that at order  $\delta$  the following modular anomaly equation has to be satisfied

$$\frac{\partial f}{\partial E_2} + \frac{1}{24} \left( \frac{\partial f}{\partial a} \right)^2 = 0. \quad (2.33)$$

It is straightforward to check that the conditions obtained at higher orders in  $\delta$  follow from this equation. For instance, the term in  $\delta^2$  in (2.32) is easily shown to correspond to a further  $E_2$ -derivative of the modular anomaly equation (2.33); thus it vanishes, as requested by the comparison with the right hand side of (2.25).

Summarizing, the S-duality symmetry requirement (2.10) is satisfied if the coefficients  $f_n$  in the mass expansion of the quantum prepotential  $f$  are quasi-modular form of weight  $2n - 2$  and if  $f$  satisfies the modular anomaly equation (2.33).



### 3 The recursion relation

#### 3.1 The prepotential

Expanding the quantum prepotential  $f$  in mass powers according to (2.14), the requirement (2.33) turns into the relation

$$\frac{\partial f_n}{\partial E_2} = -\frac{1}{24} \sum_{m=1}^{n-1} \frac{\partial f_m}{\partial a} \cdot \frac{\partial f_{n-m}}{\partial a} \tag{3.1}$$

which allows to recursively determine the  $f_n$ 's in terms of the lower coefficients up to  $E_2$ -independent functions. The  $E_2$ -independent part can be fixed by using one-loop or lower- $k$  instanton data. Actually, to the order we will consider here, the one-loop data will be enough.

Let us start by determining  $f_2$  which, being a quasi-modular form of weight 2, can only be proportional to  $E_2$ . For  $n = 2$  the recursion relation (3.1) simply reads

$$\frac{\partial f_2}{\partial E_2} = -\frac{1}{24} \frac{\partial f_1}{\partial a} \cdot \frac{\partial f_1}{\partial a} = -\frac{m^4}{96} \sum_{\alpha, \beta \in \Psi} \frac{\alpha \cdot \beta}{(\alpha \cdot a)(\beta \cdot a)}, \tag{3.2}$$

where in the second step we have used the expression (2.26) for  $f_1$ . The sum over the roots  $\alpha, \beta \in \Psi$  can be rewritten as

$$\sum_{\alpha, \beta \in \Psi} \frac{\alpha \cdot \beta}{(\alpha \cdot a)(\beta \cdot a)} = 4 \sum_{\alpha \in \Psi} \frac{1}{(\alpha \cdot a)^2} + \sum_{\alpha \neq \pm \beta \in \Psi} \frac{\alpha \cdot \beta}{(\alpha \cdot a)(\beta \cdot a)}. \tag{3.3}$$

The first term corresponds to the cases  $\alpha = \pm \beta$  and comes with an overall factor of 4 since for any ADE Lie algebra all roots have length square 2:  $\alpha \cdot \alpha = 2$  (see appendix A for details on the root system of the ADE algebras). In the second term of (3.3), for any  $\beta \neq \alpha$  we have either  $\alpha \cdot \beta = \pm 1$  or  $\alpha \cdot \beta = 0$ , and so both  $\beta$  and  $-\beta$  give the same contribution. Therefore, we can limit ourselves to sum over the roots  $\beta \in \Psi(\alpha)$  where

$$\Psi(\alpha) = \{\beta \in \Psi : \alpha \cdot \beta = 1\}, \tag{3.4}$$

and get

$$\sum_{\alpha, \beta \in \Psi} \frac{\alpha \cdot \beta}{(\alpha \cdot a)(\beta \cdot a)} = 4 \sum_{\alpha \in \Psi} \frac{1}{(\alpha \cdot a)^2} + 2 \sum_{\alpha \in \Psi} \sum_{\beta \in \Psi(\alpha)} \frac{1}{(\alpha \cdot \phi)(\beta \cdot \phi)}. \tag{3.5}$$

The first term is proportional to  $C_2$  as one can see from (2.18), while the second term suggests to introduce a more general sum over the root lattice, namely

$$C_{n; m_1 m_2 \dots m_\ell} = \sum_{\alpha \in \Psi} \sum_{\beta_1 \neq \beta_2 \neq \dots \beta_\ell \in \Psi(\alpha)} \frac{1}{(\alpha \cdot a)^n (\beta_1 \cdot a)^{m_1} (\beta_2 \cdot a)^{m_2} \dots (\beta_\ell \cdot a)^{m_\ell}}. \tag{3.6}$$

As we will show in the following, these sums will be useful to express all higher prepotential coefficients in a very compact way. The properties of these sums are discussed in appendix D where in particular we show that  $C_{1;1}$  is identically vanishing. We therefore have

$$\sum_{\alpha, \beta \in \Psi} \frac{\alpha \cdot \beta}{(\alpha \cdot a)(\beta \cdot a)} = 4 C_2 + 2 C_{1;1} = 4 C_2 \tag{3.7}$$

Using this in (3.2) and integrating with respect to  $E_2$ , we finally obtain

$$f_2 = -\frac{m^4}{24} E_2 C_2 = -\frac{m^4}{24} (1 - 24q - 72q^2 - 96q^3 \dots) C_2, \quad (3.8)$$

where in the second step we inserted the Fourier expansion of  $E_2$ . We observe that the  $q^0$ -term matches the  $m^4$  contribution in the one-loop result in (2.17). The higher order terms in the  $q$ -expansion are a prediction for the instanton corrections to  $f_2$ . As we will see in section 5, these predictions can be tested and verified for the first few instanton numbers in various gauge groups of the A and D series using localization methods. For the exceptional groups  $E_{6,7,8}$ , instead, these are truly predictions since the multi-instanton calculus is not available in these cases.

We now consider the next mass order. For  $f_3$ , from (3.1), we have

$$\frac{\partial f_3}{\partial E_2} = -\frac{1}{12} \frac{\partial f_1}{\partial a} \cdot \frac{\partial f_2}{\partial a} = -\frac{m^6}{288} E_2 \sum_{\alpha, \beta \in \Psi} \frac{\alpha \cdot \beta}{(\alpha \cdot a)^3 (\beta \cdot a)}, \quad (3.9)$$

where we have used the explicit expressions of  $f_1$  and  $f_2$  given in (2.26) and (3.8) to do the second step. Manipulating the root sums as before and using the identity (D.13), we can rewrite (3.9) as

$$\frac{\partial f_3}{\partial E_2} = -\frac{m^6}{72} E_2 \left( C_4 + \frac{1}{4} C_{2;11} \right). \quad (3.10)$$

Integrating with respect to  $E_2$ , we find

$$f_3 = -\frac{m^6}{144} E_2^2 \left( C_4 + \frac{1}{4} C_{2;11} \right) + x E_4, \quad (3.11)$$

where we have taken into account that the “integration constant” must have modular weight 4 and thus must be proportional to  $E_4$ . The coefficient  $x$  must be chosen in such a way that in the perturbative limit, where  $E_2$  and  $E_4$  become 1, one recovers the  $m^6$  term in the one-loop result (2.17). This requires that

$$x = -m^6 \left( \frac{C_4}{720} - \frac{C_{2;11}}{576} \right). \quad (3.12)$$

Plugging this back into (3.11), we finally obtain

$$f_3 = -\frac{m^6}{720} (5E_2^2 + E_4) C_4 - \frac{m^6}{288} (E_2^2 - E_4) \times \frac{1}{2} C_{2;11}. \quad (3.13)$$

Expanding the Eisenstein series in powers of  $q$  we find

$$f_3 = -\frac{m^6}{120} C_4 + q \frac{m^6}{2} C_{2;1,1} + q^2 m^6 (-6C_4 + 3C_{2;11}) + q^3 m^6 (-32C_4 + 6C_{2;11}) + \dots \quad (3.14)$$

from which we can explicitly read the multi-instanton corrections.

Using the recursion relation and the comparison with the perturbative expression, we have determined also the terms of order  $m^8$  and  $m^{10}$  in the prepotential. We now collect

all our results up to  $f_5$ :

$$f_1 = \frac{m^2}{4} \sum_{\alpha \in \Psi} \log \left( \frac{\alpha \cdot a}{\Lambda} \right)^2, \quad (3.15a)$$

$$f_2 = -\frac{m^4}{24} E_2 C_2, \quad (3.15b)$$

$$f_3 = -\frac{m^6}{720} (5E_2^2 + E_4) C_4 - \frac{m^6}{288} (E_2^2 - E_4) \times \frac{1}{2} C_{2;11}, \quad (3.15c)$$

$$\begin{aligned} f_4 = & -\frac{m^8}{90720} (175E_2^3 + 84E_2E_4 + 11E_6) C_6 \\ & + \frac{m^8}{8640} (5E_2^3 - 3E_2E_4 - 2E_6) \left( C_{4;2} + \frac{1}{12} C_{3;3} \right) \\ & - \frac{m^8}{1728} (E_2^3 - 3E_2E_4 + 2E_6) \times \frac{1}{24} C_{2;1111}, \end{aligned} \quad (3.15d)$$

$$\begin{aligned} f_5 = & -\frac{m^{10}}{362880} (245E_2^4 + 196E_2^2E_4 + 44E_2E_6 + 19E_4^2) C_8 \\ & + \frac{m^{10}}{145152} (35E_2^4 - 7E_2^2E_4 - 18E_2E_6 - 10E_4^2) \left( C_{6;2} - \frac{13}{45} C_{3;3} \right) \\ & + \frac{m^{10}}{82944} (E_2^2 - E_4)^2 \left( \frac{5}{12} C_{4;4} - 3C_{4;22} - C_{3;32} - C_{4;31} \right) \\ & - \frac{m^{10}}{6912} (E_2^4 - 6E_2^2E_4 + 8E_6E_2 - 3E_4^2) \times \frac{1}{720} C_{2;111111}. \end{aligned} \quad (3.15e)$$

If we were to proceed to the next order, *i.e.* to  $f_6$ , after having determined all terms containing  $E_2$ , we would still have to fix a purely modular term of order 12. Since there are two independent modular forms of weight 12, namely  $E_6^2$  and  $E_4^3$ , we could no longer fix the coefficient of these two forms by comparison to the one-loop result only; we would also need to know the one-instanton result, if available. Having done this, however, all the subsequent instanton corrections would be predicted. The covariance of the prepotential under S-duality, implemented through the recursion relation, is a symmetry requirement: it is not sufficient by itself to determine the dynamics, and, in particular, it does not eliminate the need to evaluate explicitly the non-perturbative corrections. Still, it minimizes the number of such computations.

### 3.2 1-instanton terms

Let us consider the 1-instanton terms in the prepotential. Substituting the  $q$ -expansion of the Eisenstein series into (3.15) one can see that the only terms which contribute at order  $q$  are those proportional to  $C_{2;11\dots}$ , whose coefficients follow an obvious pattern:

$$\begin{aligned} F_{k=1} &= m^4 C_2 + \frac{m^6}{2!} C_{2;11} + \frac{m^8}{4!} C_{2;1111} + \frac{m^{10}}{6!} C_{2;111111} + \dots \\ &= \sum_{\alpha \in \Psi} \frac{m^4}{(\alpha \cdot a)^2} \sum_{\ell=0} \frac{m^{2\ell}}{\ell!} \sum_{\beta_1 \neq \beta_2 \neq \dots \beta_\ell \in \Psi(\alpha)} \frac{1}{(\beta_1 \cdot a) \cdots (\beta_\ell \cdot a)} \end{aligned} \quad (3.16)$$

where in the second line we have used the explicit definition of the sums  $C_{2;11\dots}$ . This pattern extends to all orders in  $m$ , and we can rewrite the above expression as<sup>2</sup>

$$F_{k=1} = \sum_{\alpha \in \Psi} \frac{m^4}{(\alpha \cdot a)^2} \prod_{\beta \in \Psi(\alpha)} \left( 1 + \frac{m}{\beta \cdot a} \right). \quad (3.17)$$

We now show that, in the decoupling limit in which the  $\mathcal{N} = 2^*$  theory reduces to the pure  $\mathcal{N} = 2$  SYM, the above result agrees with the explicit computations that are present in the literature [14–17]. In the decoupling limit the mass  $m$  is sent to infinity and the instanton weight  $q$  to zero, keeping constant the dynamically generated scale  $\widehat{\Lambda}$  defined as

$$\widehat{\Lambda}^{2h^\vee} = m^{2h^\vee} q. \quad (3.18)$$

Here  $2h^\vee$  is the one-loop  $\beta$ -function coefficient of the pure  $\mathcal{N} = 2$  theory, expressed in terms of the dual Coxeter number of the Lie algebra  $\mathfrak{g}$ . For the single-laced algebras these numbers are given by

	$A_r$	$D_r$	$E_6$	$E_7$	$E_8$
$h^\vee$	$r$	$2r - 2$	$12$	$18$	$30$

Since the number of roots  $\beta$  in the set  $\Psi(\alpha)$  is  $2h^\vee - 4$ , the highest mass power in (3.17) is exactly  $m^{2h^\vee}$ , and so it is consistent to take the decoupling limit, in which all terms proportional to non-maximal powers of  $m$  vanish. Doing this, we remain with

$$q F_{k=1} \rightarrow \widehat{\Lambda}^{2h^\vee} \sum_{\alpha \in \Psi} \frac{1}{(\alpha \cdot a)^2} \prod_{\beta \in \Psi(\alpha)} \frac{1}{\beta \cdot a}. \quad (3.19)$$

This expression has been derived in [15] following a completely different approach.<sup>3</sup> Our result in (3.17) generalizes this to the  $\mathcal{N} = 2^*$  case.

## 4 The recursion relation in the $\Omega$ -background

The general features described in the previous section hold also when the  $\mathcal{N} = 2^*$  theories are formulated in an  $\Omega$ -background [9]. In fact we are going to show that for a generic gauge group of the ADE series the  $\Omega$ -deformed prepotential satisfies a generalized recursion relation, thus extending the analysis of [24–32] for the  $SU(2)$  theory and of [33] where the  $SU(N)$  theories were considered.

We parametrize the  $\Omega$ -background by  $\epsilon_1$  and  $\epsilon_2$  and, for later convenience, introduce the following combinations

$$\epsilon = \epsilon_1 + \epsilon_2, \quad h = \sqrt{\epsilon_1 \epsilon_2}. \quad (4.1)$$

<sup>2</sup>Note that the terms with odd powers of  $m$  that we obtain expanding the product in the right hand side vanish identically, as discussed in appendix D.

<sup>3</sup>In [15] also the non-simply laced groups are considered; in the companion paper [40] we will show that also in these cases our treatment reproduces, in the decoupling limit, their expression.

The deformed prepotential can still be written as in (2.13)–(2.15), but both the one-loop and the instanton parts receive corrections in  $\epsilon$  and  $h$ . In particular, the one-loop term becomes [9, 28, 30]

$$f^{1\text{-loop}} = h^2 \sum_{\alpha \in \Psi} \left[ \log \Gamma_2(\alpha \cdot \phi) - \log \Gamma_2(\alpha \cdot \phi + m + \epsilon) \right] \quad (4.2)$$

where  $\Gamma_2$  is the Barnes double  $\Gamma$ -function (see appendix C). By expanding for small values of  $m$ ,  $\epsilon$  and  $h$ , we obtain

$$f_1^{1\text{-loop}} = \frac{M^2}{4} \sum_{\alpha \in \Psi} \log \left( \frac{\alpha \cdot \phi}{\Lambda} \right)^2, \quad (4.3a)$$

$$f_2^{1\text{-loop}} = -\frac{M^2(M^2 + h^2)}{24} C_2, \quad (4.3b)$$

$$f_3^{1\text{-loop}} = -\frac{M^2(M^2 + h^2)(2M^2 + 3h^2 - \epsilon^2)}{240} C_4, \quad (4.3c)$$

$$f_4^{1\text{-loop}} = -\frac{M^2(M^2 + h^2)(3M^4 + 10h^4 + 2\epsilon^4 + 11h^2M^2 - 4\epsilon^2M^2 - 10h^2\epsilon^2)}{1008} C_6, \quad (4.3d)$$

where we have defined

$$M^2 \equiv m^2 - \frac{\epsilon^2}{4}. \quad (4.4)$$

As in the undeformed case, also here  $f_1$  does not receive instantonic corrections, so we have

$$f_1 = f_1^{1\text{-loop}}, \quad (4.5)$$

while all other  $f_n$ 's with  $n \geq 2$  have contributions at any order in the instanton expansion.

The exact  $q$ -dependence of the deformed  $f_n$ 's can be determined by requiring that the prepotential transforms properly under S-duality. In an  $\Omega$ -background this means that S-duality acts on the prepotential as a Fourier transform [29–32], namely

$$\exp \left( -\frac{S[F](a^D)}{h^2} \right) = \left( \frac{i\tau}{h^2} \right)^{r/2} \int d^r x \exp \left( \frac{2\pi i a^D \cdot x - F(x)}{h^2} \right) \quad (4.6)$$

where  $r$  is the rank of the gauge group. This interpretation of S-duality is fully consistent with the interpretation of  $a$  and  $a^D$  as canonical conjugate variables, on which S acts as a canonical transformation, and of

$$Z = \exp \left( -\frac{F}{h^2} \right) \quad (4.7)$$

as a wave function in a quantization of this phase space, with  $h^2 = \epsilon_1 \epsilon_2$  playing the rôle of the Planck constant [21–23].

If we compute the Fourier transform (4.6) in the saddle point approximation for  $h \rightarrow 0$  and denote by  $a$  the solution of the saddle point equation, that is

$$2\pi i a^D - \partial_x F(x) \Big|_{x=a} = 0, \quad (4.8)$$

then the leading contribution to the integral in (4.6) is

$$\exp\left(-\frac{S[F](a^D)}{h^2}\right) = \exp\left(-\frac{F(a) - a \cdot \partial_a F(a)}{h^2} - \frac{1}{2} \text{tr} \log\left(\delta_{uv} + \frac{1}{2\pi i \tau} \partial_u \partial_v f\right)\right) + \dots \quad (4.9)$$

where the  $\text{tr} \log$  part comes from the Gaussian integration around the saddle point and the ellipses stand for subleading terms in  $h$ . The dominant contribution for  $h \rightarrow 0$  reproduces the Legendre transform of the prepotential as expected, but there are corrections for finite  $h$ . Indeed we have

$$\begin{aligned} S[F] &= \mathcal{L}[F] + \frac{h^2}{2} \text{tr} \log\left(\delta_{uv} + \frac{1}{2\pi i \tau} \partial_u \partial_v f\right) + \dots \\ &= \mathcal{L}[F] + \delta \left(\frac{h^2}{24} \Delta f\right) + \mathcal{O}(\delta^2) + \dots \end{aligned} \quad (4.10)$$

where we have used  $\delta = \frac{6}{\pi i \tau}$ , as before, and defined  $\Delta = \sum_u \partial_u^2$ .

Repeating the same steps described in section 2 (see also sections 3 and 4 of [31] for more details), one can show that (4.10) leads to the following recursion relation for the prepotential coefficients  $f_n$ 's:

$$\frac{\partial f_n}{\partial E_2} = -\frac{1}{24} \sum_{m=1}^{n-1} \frac{\partial f_m}{\partial a} \cdot \frac{\partial f_{n-m}}{\partial a} + \frac{h^2}{24} \Delta f_{n-1} . \quad (4.11)$$

The recursive computation of the  $f_n$ 's can then proceed along the lines we have discussed in the undeformed theory. At level two we find

$$\frac{\partial f_2}{\partial E_2} = -\frac{1}{24} \frac{\partial f_1}{\partial a} \cdot \frac{\partial f_1}{\partial a} + \frac{h^2}{24} \Delta f_1 = -\frac{1}{24} M^2 (M^2 + h^2) C_2 \quad (4.12)$$

where we have used

$$\Delta f_1 = -\frac{M^2}{2} \sum_{\alpha \in \Psi} \frac{(\alpha \cdot \alpha)}{(\alpha \cdot \phi)^2} = -M^2 C_2 . \quad (4.13)$$

Integrating with respect to  $E_2$  we get

$$f_2 = -\frac{1}{24} M^2 (M^2 + h^2) E_2 C_2 . \quad (4.14)$$

It is immediate to see that in the perturbative limit, when  $E_2$  reduces to 1, this correctly reproduces (4.3b).

Using this result we can write the differential equation constraining  $f_3$ , namely

$$\begin{aligned} \frac{\partial f_3}{\partial E_2} &= -\frac{1}{12} \frac{\partial f_1}{\partial a} \cdot \frac{\partial f_2}{\partial a} + \frac{h^2}{24} \Delta f_2 \\ &= -\frac{1}{144} M^2 (M^2 + h^2) ((2M^2 + 3h^2) C_4 + M^2 C_{3;1}) E_2 . \end{aligned} \quad (4.15)$$

Integrating with respect to  $E_2$  and fixing the dependence on  $E_4$  in such a way to reproduce the perturbative result (4.3c), we get

$$\begin{aligned} f_3 &= -\frac{1}{288} M^2 (M^2 + h^2) \left[ \frac{1}{5} ((2M^2 + 3h^2)(5E_2^2 + E_4) - 6\epsilon^2 E_4) C_4 \right. \\ &\quad \left. + \frac{1}{2} M^2 (E_2^2 - E_4) C_{2;11} \right] \end{aligned} \quad (4.16)$$

where we have used the identity  $C_{3;1} = C_{2;11}$  proven in appendix D.

In a similar way we can determine  $f_4$ . The result we get is

$$\begin{aligned}
f_4 = & -\frac{1}{1728}M^2(M^2 + h^2) \left\{ \left[ \frac{2}{105}(2M^2 + 3h^2)(2M^2 + 5h^2)(35E_2^3 + 21E_2E_4 + 4E_6) \right. \right. \\
& + \frac{2}{21}M^2(M^2 + h^2)(7E_2^3 - E_6) - \frac{12}{35}\epsilon^2(2M^2 + 5h^2)(7E_2E_4 + 3E_6) + \frac{24}{7}\epsilon^4E_6 \left. \right] C_6 \\
& - \left[ \frac{1}{10}M^2(2M^2 + 3h^2)(5E_2^3 - 3E_2E_4 - 2E_6) - \frac{6}{5}\epsilon^2M^2(E_2E_4 - E_6) \right] C_{4;2} \\
& - \left[ \frac{1}{60}M^2(M^2 + 4h^2)(5E_2^3 - 3E_2E_4 - 2E_6) - \frac{3}{5}\epsilon^2M^2(E_2E_4 - E_6) \right] C_{3;3} \\
& \left. + \frac{1}{24}M^4(E_2^3 - 3E_2E_4 + 2E_6) C_{2;1111} \right\} . \tag{4.17}
\end{aligned}$$

This procedure can be carried out for the next orders in the mass expansion, but the results become lengthy and we see no reason to explicitly report them.

#### 4.1 1-instanton terms

While the exact expressions of the  $f_n$ 's are rather involved, their 1-instanton part is quite simple and it is possible to write a compact expression which generalizes the one we have derived in (3.17) for the underformed theory. Indeed, inserting the  $q$ -expansions of the Eisenstein series in (4.14), (4.16) and (4.17) one can see that only few terms contribute at order  $q$ :

$$\begin{aligned}
f_2|_{k=1} &= M^2(M^2 + h^2) C_2, \\
f_3|_{k=1} &= M^2(M^2 + h^2) \left[ \epsilon^2 C_4 + \frac{1}{2}M^2 C_{2;11} \right], \\
f_4|_{k=1} &= M^2(M^2 + h^2) \left[ \epsilon^4 C_6 + \frac{1}{2}\epsilon^2 M^2 C_{4;11} + \frac{1}{24}M^2(M^2 - \epsilon^2) C_{2;1111} \right]
\end{aligned} \tag{4.18}$$

where in the last equation we have used the first identity given in (D.17). Actually this pattern extends also to higher  $f_n$ 's, as we have verified by computing the 1-instanton prepotential using localization techniques described in the next section. Altogether we find

$$\begin{aligned}
F_{k=1} &= \sum_{n=2} f_n|_{k=1} \\
&= M^2(M^2 + h^2) \left[ (C_2 + \epsilon^2 C_4 + \epsilon^4 C_6 + \epsilon^6 C_8 \dots) \right. \\
&\quad + \frac{1}{2}M^2(C_{2;11} + \epsilon^2 C_{4;11} + \epsilon^4 C_{6;11} + \dots) \\
&\quad + \frac{1}{24}M^2(M^2 - \epsilon^2)(C_{2;1111} + \epsilon^2 C_{4;1111} + \dots) \\
&\quad \left. + \frac{1}{720}M^2(M^4 - 3M^2\epsilon^2 + 3\epsilon^4)(C_{2;111111} + \dots) + \dots \right] . \tag{4.19}
\end{aligned}$$

This pattern suggests to introduce the following notation

$$\begin{aligned}
 g_{2n} &= \frac{1}{(2n)!} \left( C_{2;\underbrace{11\dots 1}_{2n}} + \epsilon^2 C_{4;\underbrace{11\dots 1}_{2n}} + \epsilon^4 C_{6;\underbrace{11\dots 1}_{2n}} + \dots \right) \\
 &= \frac{1}{(2n)!} \sum_{\alpha \in \Psi} \sum_{\beta_1 \neq \dots \beta_{2n} \in \Psi(\alpha)} \frac{1}{(\alpha \cdot a)(\alpha \cdot a + \epsilon)(\beta_1 \cdot a) \dots (\beta_{2n} \cdot a)},
 \end{aligned} \tag{4.20}$$

so that (4.19) becomes

$$F_{k=1} = M^2(M^2 + h^2) \left[ g_0 + M^2 g_2 + M^2(M^2 - \epsilon^2) g_4 + M^2(M^4 - 3M^2\epsilon^2 + 3\epsilon^4) g_6 + \dots \right]. \tag{4.21}$$

Notice that in the sums  $g_{2n}$  all odd powers in the  $\epsilon$ -expansion of the second line of (4.20) vanish upon summing over the roots and that, for a given algebra  $\mathfrak{g}$ , the highest non-vanishing expression of this kind is  $g_{2h^\vee - 4}$ , since the order of  $\Psi(\alpha)$  is  $2h^\vee - 4$ . It is interesting to observe that the  $g$ 's can be expressed in terms of a generating function

$$G(x) = \sum_{n=0}^{2h^\vee - 4} g_n x^n = \sum_{\alpha \in \Psi} \frac{1}{(\alpha \cdot a)(\alpha \cdot a + \epsilon)} \prod_{\beta \in \Psi(\alpha)} \left( 1 + \frac{x}{\beta \cdot a} \right) \tag{4.22}$$

where

$$g_n = \frac{1}{n!} \left. \frac{\partial^n G(x)}{\partial x^n} \right|_{x=0}. \tag{4.23}$$

It is also possible to recognize a pattern in the mass- and  $\epsilon$ -dependent expressions that multiply the  $g_n$ 's in eq. (4.21). Writing the latter in the form

$$F_{k=1} = M^2(M^2 + h^2) \sum_{n=0}^{2h^\vee - 4} g_n \epsilon^n H_n \left( \frac{M^2}{\epsilon^2} \right), \tag{4.24}$$

one can see that the polynomials  $H_n$  are connected to the Euler polynomials  $\mathcal{E}_n$  according to

$$H_n \left( \frac{M^2}{\epsilon^2} \right) = \frac{1}{2} \left[ \mathcal{E}_n \left( \frac{1}{2} + \frac{m}{\epsilon} \right) + \mathcal{E}_n \left( \frac{1}{2} - \frac{m}{\epsilon} \right) \right] \tag{4.25}$$

(recall that  $M^2 = m^2 - \frac{\epsilon^2}{4}$ ). In turn, the Euler polynomials are defined by

$$\frac{2e^{zt}}{e^t + 1} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{E}_n(z) t^n. \tag{4.26}$$

With this definition one can easily check that all  $H_{2n+1}$  are vanishing, while the  $H_{2n}$  reproduce the expressions appearing in (4.21). Inserting (4.25) and (4.23) into (4.24), we then obtain

$$\begin{aligned}
 F_{k=1} &= \frac{1}{2} \left( m^2 - \frac{\epsilon^2}{4} \right) \left( m^2 - \frac{\epsilon^2}{4} + h^2 \right) \\
 &\quad \times \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \mathcal{E}_n \left( \frac{1}{2} + \frac{m}{\epsilon} \right) + \mathcal{E}_n \left( \frac{1}{2} - \frac{m}{\epsilon} \right) \right] \left( \epsilon \frac{\partial}{\partial x} \right)^n G(x) \Big|_{x=0}.
 \end{aligned} \tag{4.27}$$



Since  $G(x)$  is a polynomial of order  $2h^\vee - 4$ , all terms with  $n > 2h^\vee - 4$  in the sum vanish and (4.27) is simply another way to write (4.24). However, this allows us to use the property (4.26) of the Euler polynomials in order to write

$$\begin{aligned}
 F_{k=1} &= \left(m^2 - \frac{\epsilon^2}{4}\right) \left(m^2 - \frac{\epsilon^2}{4} + h^2\right) \left( \frac{e^{(\frac{\epsilon}{2}+m)\partial_x}}{e^{\epsilon\partial_x} + 1} + \frac{e^{(\frac{\epsilon}{2}-m)\partial_x}}{e^{\epsilon\partial_x} + 1} \right) G(x) \Big|_{x=0} \\
 &= \left(m^2 - \frac{\epsilon^2}{4}\right) \left(m^2 - \frac{\epsilon^2}{4} + h^2\right) \sum_{n=0}^{2h^\vee-4} \left(-\epsilon \frac{\partial}{\partial x}\right)^n \left[ G(x + \frac{\epsilon}{2} + m) + G(x + \frac{\epsilon}{2} - m) \right] \Big|_{x=0}
 \end{aligned}
 \tag{4.28}$$

where in the second line we truncated the expansion of the geometric series since, as we stressed above,  $G(x)$  is a polynomial of order  $2h^\vee - 4$ . It is not difficult to check that in the limit  $\epsilon \rightarrow 0$  we recover the 1-instanton expression given in (3.17) and that keeping the  $\epsilon$  but decoupling the matter hypermultiplet we recover the same formula obtained in [15] for the pure  $\mathcal{N} = 2$  theories from the coherent states of the W-algebras.

## 5 Multi-instanton calculations

In this section, we test the results for the  $\mathcal{N} = 2^*$  prepotential obtained from the modular recursion equation against a direct microscopic computation of the first instanton corrections based on equivariant localization techniques [7–11] (see also [42] for further details). To do so we first recall a few basic facts about the instanton moduli space and the multi instanton calculus starting from the gauge theories with unitary groups.

### 5.1 Multi-instantons for the $U(N)$ gauge theory

The moduli space of  $k$ -instantons in the  $\mathcal{N} = 2^*$  theory with gauge group  $U(N)$  can be built from the open strings connecting a stack of  $k$  D(-1) and  $N$  D3-branes in Type IIB string theory. The gauge theory prepotential can be viewed as the free energy of the statistical system describing the lowest modes of the open strings with at least one end-point on the D(-1) branes that account for the the instanton moduli [43–47].

The partition function  $Z_k$  of the system can be computed using localization methods. To achieve full localization all symmetries of the system have to be broken. The gauge symmetries on the world volumes of the D3 and D(-1) branes can be broken by distributing them along a transverse complex plane  $\mathbb{C}$ . We label their positions in this plane by  $a_u$  (with  $u = 1, \dots, N$ ) and  $\chi_i$  (with  $i = 1, \dots, k$ ), respectively. The  $SO(4) \times SO(4)$  Lorentz symmetry of the spacetime transverse to this plane can be broken by turning on an  $\Omega$ -background with parameters  $\epsilon_1, \epsilon_2, \epsilon_3$  and  $\epsilon_4$ . The first two parameters,  $\epsilon_1$  and  $\epsilon_2$ , describe a gravitational background, while  $\epsilon_3$  and  $\epsilon_4$  are related to the mass of the adjoint hypermultiplet.

In table 1 we list all moduli for given  $k$  and  $N$ , together with their transformation properties with respect to the various symmetry groups. In the first column we have grouped the moduli into  $Q$ -pairs of the supersymmetric charge  $Q$  used for localization and labeled by their  $SO(4) \times SO(4)$  quantum numbers with spinor indices  $\alpha, \dot{\alpha}, a, \dot{a}$ , all taking two values. The neutral bosonic moduli include the eight instanton positions transverse to

$(\phi, \psi)$	$(-1)^{F_\phi}$	$U(N) \times U(k)$	$\lambda_\phi$
$(B_{\alpha\dot{\alpha}}, M_{\alpha\dot{\alpha}})$	+	$(\mathbf{1}, \square \bar{\square})$	$\chi_{ij} + \epsilon_1, \chi_{ij} + \epsilon_2$
$(B_{a\dot{a}}, M_{\dot{a}a})$	+	$(\mathbf{1}, \square \bar{\square})$	$\chi_{ij} + \epsilon_3, \chi_{ij} + \epsilon_4$
$(N_{(\dot{\alpha}\dot{\beta})}, D_{(\dot{\alpha}\dot{\beta})})$	-	$(\mathbf{1}, \square \bar{\square})$	$\sqrt{\chi_{ij}}, \chi_{ij} + \epsilon_1 + \epsilon_2$
$(\bar{\chi}, N)$	+	$(\mathbf{1}, \square \bar{\square})$	$\sqrt{\chi_{ij}}$
$(N_{\alpha a}, D_{\alpha a})$	-	$(\mathbf{1}, \square \bar{\square})$	$\chi_{ij} + \epsilon_1 + \epsilon_3, \chi_{ij} + \epsilon_1 + \epsilon_4$
$(w_{\dot{\alpha}}, \mu_{\dot{a}})$	+	$(\bar{\square}, \square)$	$\chi_i - a_u + \frac{\epsilon_1 + \epsilon_2}{2}$
$(\bar{w}_{\dot{\alpha}}, \bar{\mu}_{\dot{a}})$	+	$(\square, \bar{\square})$	$-\chi_i + a_u + \frac{\epsilon_1 + \epsilon_2}{2}$
$(h_a, \mu_a)$	-	$(\bar{\square}, \square)$	$\chi_i - a_u + \frac{\epsilon_3 - \epsilon_4}{2}$
$(\bar{h}_a, \bar{\mu}_a)$	-	$(\square, \bar{\square})$	$-\chi_i + a_u + \frac{\epsilon_3 - \epsilon_4}{2}$

**Table 1.** Instanton moduli for the  $U(N)$  gauge theory. The columns display, respectively, the moduli in a ADHM-like notation, their statistics, their transformation properties with respect to the gauge and instanton symmetry groups and their  $Q^2$ -eigenvalues  $\lambda_\phi$ . The notation  $\chi_{ij}$  stands for  $\chi_i - \chi_j$ .

$\mathbb{C}$ , denoted by  $B_{\alpha\dot{\alpha}}$  and  $B_{a\dot{a}}$ , and the positions along  $\mathbb{C}$ , denoted by  $\chi$  and  $\bar{\chi}$ . The charged bosonic moduli  $w_{\dot{\alpha}}$  and  $\bar{w}_{\dot{\alpha}}$  describe the size and the orientation of the instantons, while the auxiliary fields  $D_{\dot{\alpha}\dot{\beta}}$ ,  $D_{\alpha a}$  and  $h_a$  take care of the generalised ADHM constraints. The field  $\chi$  can be viewed as the  $U(k)$  gauge parameter and thus it should be integrated out in order to achieve  $U(k)$ -invariance.

The  $k$ -instanton partition function is given by the complex superdeterminant of  $Q^2$ , which can be computed from the data reported in the last column of the above table. The result is

$$Z_k = \oint \prod_{i=1}^k \frac{d\chi_i}{2\pi i} \Delta(0) \prod_{\phi} \lambda_{\phi}^{(-1)^{F_{\phi}+1}} = \oint \prod_{i=1}^k \frac{d\chi_i}{2\pi i} z_k^{\text{gauge}} z_k^{\text{matter}} \quad (5.1)$$

where  $\Delta(0) = \prod_{i \neq j} \chi_{ij}$  is the Vandermonde determinant and

$$z_k^{\text{gauge}} = \frac{(-1)^k}{k!} \left( \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^k \frac{\Delta(0) \Delta(\epsilon_1 + \epsilon_2)}{\Delta(\epsilon_1) \Delta(\epsilon_2)} \prod_{i=1}^k \frac{1}{P(\chi_i + \frac{\epsilon_1 + \epsilon_2}{2}) P(\chi_i - \frac{\epsilon_1 + \epsilon_2}{2})}, \quad (5.2a)$$

$$z_k^{\text{matter}} = \left( \frac{(\epsilon_1 + \epsilon_3)(\epsilon_1 + \epsilon_4)}{\epsilon_3 \epsilon_4} \right)^k \frac{\Delta(\epsilon_1 + \epsilon_3) \Delta(\epsilon_1 + \epsilon_4)}{\Delta(\epsilon_3) \Delta(\epsilon_4)} \prod_{i=1}^k P(\chi_i + \frac{\epsilon_3 - \epsilon_4}{2}) P(\chi_i - \frac{\epsilon_3 - \epsilon_4}{2}) \quad (5.2b)$$

with

$$P(x) = \prod_{u=1}^N (x - a_u), \quad \Delta(x) = \prod_{i < j} (x^2 - \chi_{ij}^2). \quad (5.3)$$

The integrals in (5.1) are computed by closing the contours in the upper-half complex  $\chi_i$ -planes after giving  $\epsilon_1, \epsilon_2, \epsilon_3$  and  $\epsilon_4$  an imaginary part with the following prescription

$$\text{Im}(\epsilon_4) \gg \text{Im}(\epsilon_3) \gg \text{Im}(\epsilon_2) \gg \text{Im}(\epsilon_1) > 0. \quad (5.4)$$

This choice allows us to unambiguously compute all integrals in (5.1) and obtain the instanton partition of the  $U(N)$  theory

$$Z_{\text{inst}} = 1 + \sum_{k=1} q^k Z_k \quad (5.5)$$

where  $q = e^{2\pi i \tau}$ . At the end of the calculations we have to set

$$\epsilon_3 = m - \frac{\epsilon_1 + \epsilon_2}{2}, \quad \epsilon_4 = -m - \frac{\epsilon_1 + \epsilon_2}{2} \quad (5.6)$$

in order to express the result in terms of the hypermultiplet mass  $m$  in the normalization of the previous sections. The prepotential is then given by

$$F_{\text{inst}} = -\epsilon_1 \epsilon_2 \log Z_{\text{inst}} = \sum_{k=1} q^k F_k ; \quad (5.7)$$

by taking the limit  $\epsilon_1, \epsilon_2 \rightarrow 0$  one finally recovers the prepotential of the undeformed gauge theory.

**1-instanton terms.** At  $k = 1$  there is one integral to compute; it is very easy to see that the poles of (5.1) are located at

$$\chi_1 = a_u + \frac{\epsilon_1 + \epsilon_2}{2}. \quad (5.8)$$

Evaluating the residues, using (5.6) and summing over  $u$  we find

$$F_{k=1} = -\epsilon_1 \epsilon_2 Z_1 = - \left( m^2 - \frac{(\epsilon_1 - \epsilon_2)^2}{4} \right) \sum_{u=1}^N \prod_{v \neq u} \frac{(a_{uv} + \frac{\epsilon_1 + \epsilon_2}{2})^2 - m^2}{a_{uv}(a_{uv} + \epsilon_1 + \epsilon_2)} \quad (5.9)$$

where  $a_{uv} = a_u - a_v$ . For example for  $U(2)$  we have

$$F_{k=1} \Big|_{U(2)} = (M^2 + h^2) \left[ -2 + \frac{M^2}{a_{12}(a_{12} + \epsilon)} + \frac{M^2}{a_{21}(a_{21} + \epsilon)} \right] \quad (5.10)$$

where  $M^2$  and  $\epsilon$  are defined in (4.4) and (4.1). Notice that the terms proportional to  $M^2$  in the square brackets precisely reconstruct the sum  $g_0$  defined in (4.20). To get the prepotential for the  $SU(2)$  theory we simply have to set  $a_1 = -a_2 = a$  in the above expression; in this way we get

$$F_{k=1} \Big|_{SU(2)} = \frac{2(M^2 + h^2)(M^2 + \epsilon^2 - 4a^2)}{4a^2 - \epsilon^2}. \quad (5.11)$$

For unitary groups of higher rank, the expanded expression of the 1-instanton prepotential is more cumbersome; however it is possible to check that (5.9) can be written as<sup>4</sup>

$$F_{k=1} = M^2(M^2 + h^2) \left[ g_0 + M^2 g_2 + M^2(M^2 - \epsilon^2) g_4 + M^2(M^4 - 3M^2\epsilon^2 + 3\epsilon^4) g_6 + \dots \right] \quad (5.12)$$

---

<sup>4</sup>We discard all  $a$ -independent terms.

in agreement with (4.21). The equality between (5.9) and (5.12) (up to  $a$ -independent terms) is not immediate to see, but nevertheless it is true. In the undeformed theory, *i.e.*  $\epsilon, h \rightarrow 0$  and  $M^2 \rightarrow m^2$  the above results further simplify and reduce to those obtained long ago in [48] using a completely different approach.

**2-instanton terms.** At  $k = 2$  there are two integrals in (5.1) to compute. The procedure we have described above is straightforward to implement and with the prescription (5.4) no ambiguities arise. To avoid long formulas we write some explicit 2-instanton terms only in the  $\epsilon, h \rightarrow 0$  limit. For example in the undeformed U(2) theory, we get

$$\begin{aligned}
 F_{k=2} \Big|_{U(2)} &= -3m^2 + 6m^4 \frac{1}{a_{12}^2} - 12m^6 \frac{1}{a_{12}^4} + 5m^8 \frac{1}{a_{12}^6} + \dots \\
 &= -3m^2 + 3m^4 C_2 - 6m^6 C_4 + \frac{5m^8}{2} C_6 + \dots
 \end{aligned}
 \tag{5.13}$$

where in the second line we have used the sums  $C_n$ 's defined in (2.18) and the dots stand for subleading terms in the mass expansion.

Likewise for U(3) we find

$$\begin{aligned}
 F_{k=2} \Big|_{U(3)} &= -\frac{9m^2}{2} + 6m^4 \left( \frac{1}{a_{12}^2} + \frac{1}{a_{13}^2} + \frac{1}{a_{23}^2} \right) \\
 &\quad - 12m^6 \left( \frac{1}{a_{12}^4} + \frac{1}{a_{13}^4} + \frac{1}{a_{23}^4} + \frac{a_1^2 + a_2^2 + a_3^2 - a_1 a_2 - a_1 a_3 - a_2 a_3}{a_{12}^2 a_{13}^2 a_{23}^2} \right) + \dots \\
 &= -\frac{9m^2}{2} + 3m^4 C_2 - 6m^6 C_4 + 3m^6 C_{2;11} + \dots
 \end{aligned}
 \tag{5.14}$$

We have explicitly checked up to U(5) that the same pattern occurs, namely that the 2-instanton prepotential is (up to  $a$ -independent terms)

$$F_{k=2} = 3m^4 C_2 - 6m^6 C_4 + 3m^6 C_{2;11} + \frac{5m^8}{2} C_6 + 6m^8 C_{4;2} + \frac{m^8}{2} C_{3;3} + \frac{m^8}{2} C_{2;1111} + \dots
 \tag{5.15}$$

This result is in total agreement with the 2-instanton prepotential that can be obtained from (3.15) by expanding the Eisenstein series; moreover it clearly shows the advantage of using the root lattice sums  $C_{n;m_1\dots}$  that allow us to write a single expression valid for all U( $N$ )'s groups.

Finally we mention that for the unitary groups it is possible to push the calculations to higher instanton numbers as we have shown in [33].

## 5.2 Multi-instantons for the SO(2N) gauge theory

The moduli space of the SO(2N) gauge theory is obtained from that of the U(2N) theory by using the projector  $(1 + \Omega I)/2$  where  $\Omega$  is the orientifold operator that changes the orientation of the open strings and  $I$  reflects the moduli carrying an index  $\dot{\alpha}$ , *i.e.* transforming in the fundamental representation of the SU(2)<sub>L</sub> factor of the spacetime Lorentz group [49]. As a result, the symmetry of the brane system reduces to SO(2N)  $\times$  Sp(2k). The instanton moduli and their transformation properties are listed in table 2 which uses the same notation as table 1.

$(\phi, \psi)$	$(-1)^{F_\phi}$	$\text{SO}(2N) \times \text{Sp}(2k)$	$\lambda_\phi$
$(B_{\alpha\dot{\alpha}}, M_{\alpha\dot{\alpha}})$	+	$(\mathbf{1}, \square)$	$\chi_{ij} + \epsilon_1, \chi_{ij} + \epsilon_2$
$(B_{a\dot{a}}, M_{a\dot{a}})$	+	$(\mathbf{1}, \square)$	$\chi_{ij} + \epsilon_3, \chi_{ij} + \epsilon_4$
$(N_{(\dot{a}\dot{b})}, D_{(\dot{a}\dot{b})})$	-	$(\mathbf{1}, \square)$	$\sqrt{\chi_{ij}}, \chi_{ij} + \epsilon_1 + \epsilon_2$
$(\bar{\chi}, N)$	+	$(\mathbf{1}, \square)$	$\sqrt{\chi_{ij}}$
$(N_{\alpha\alpha}, D_{\alpha\alpha})$	-	$(\mathbf{1}, \square)$	$\chi_{ij} + \epsilon_1 + \epsilon_3, \chi_{ij} + \epsilon_1 + \epsilon_4$
$(w_{\dot{\alpha}}, \mu_{\dot{\alpha}})$	+	$(\square, \square)$	$\chi_i - a_u + \frac{\epsilon_1 + \epsilon_2}{2}$
$(h_a, \mu_a)$	-	$(\square, \square)$	$\chi_i - a_u + \frac{\epsilon_3 - \epsilon_4}{2}$

**Table 2.** Instanton moduli for the  $\text{SO}(2N)$  gauge theory. The columns display the moduli, their statistics, their transformation properties with respect to the gauge and instanton symmetry groups and their  $Q^2$ -eigenvalues  $\lambda_\phi$ .

Collecting the eigenvalues  $\lambda_\phi$  for all moduli, we find that the  $k$ -instanton partition function is

$$Z_k = \oint \prod_{i=1}^k \frac{d\chi_i}{2\pi i} z_k^{\text{gauge}} z_k^{\text{matter}} \quad (5.16)$$

where

$$z_k^{\text{gauge}} = \frac{(-1)^k}{2^k k!} \left( \frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2} \right)^k \frac{\Delta(0) \Delta(\epsilon_1 + \epsilon_2)}{\Delta(\epsilon_1) \Delta(\epsilon_2)} \prod_{i=1}^k \frac{4\chi_i^2 (4\chi_i^2 - (\epsilon_1 + \epsilon_2)^2)}{P(\chi_i + \frac{\epsilon_1 + \epsilon_2}{2}) P(\chi_i - \frac{\epsilon_1 + \epsilon_2}{2})}, \quad (5.17a)$$

$$z_k^{\text{matter}} = \left( \frac{(\epsilon_1 + \epsilon_3)(\epsilon_1 + \epsilon_4)}{\epsilon_3 \epsilon_4} \right)^k \frac{\Delta(\epsilon_1 + \epsilon_3) \Delta(\epsilon_1 + \epsilon_4)}{\Delta(\epsilon_3) \Delta(\epsilon_4)} \prod_{i=1}^k \frac{P(\chi_i + \frac{\epsilon_3 - \epsilon_4}{2}) P(\chi_i - \frac{\epsilon_3 - \epsilon_4}{2})}{(4\chi_i^2 - \epsilon_3^2)(4\chi_i^2 - \epsilon_4^2)} \quad (5.17b)$$

with

$$P(x) = \prod_{u=1}^N (x^2 - a_u^2), \quad \Delta(x) = \prod_{i < j}^k ((\chi_i - \chi_j)^2 - x^2) ((\chi_i + \chi_j)^2 - x^2). \quad (5.18)$$

Once again the integrals in (5.16) are computed by closing the contours in the upper-half complex  $\chi_i$ -planes with the prescription (5.4). It is important to stress that unlike in the  $U(N)$  theory, the integral (5.16) receives non-trivial contributions also from poles located at  $\chi_i = \epsilon_3$ ,  $\chi_i = \epsilon_4$ ,  $\chi_{ij} = \epsilon_3$  and  $\chi_{ij} = \epsilon_4$ . The contributions of the corresponding residues are crucial to find an expression which is polynomial in the hypermultiplet mass as one expects on general grounds. Only at the very end of the computation one should use the identification (5.6) in order to write the final results in terms of the vacuum expectation values  $a_u$  and the mass  $m$  in the normalization used in the previous sections.

**1-instanton terms.** For  $k = 1$  the poles of (5.16) are located at

$$\chi_1 = \left\{ \pm a_u + \frac{\epsilon_1 + \epsilon_2}{2}, \frac{\epsilon_3}{2}, \frac{\epsilon_4}{2} \right\} \quad \text{for } u = 1, \dots, N \quad (5.19)$$

The  $k = 1$  prepotential can then be written as

$$F_{k=1} = -\epsilon_1 \epsilon_2 Z_1 = \sum_{u=1}^N f_{+a_u + \frac{\epsilon_1 + \epsilon_2}{2}} + \sum_{u=1}^N f_{-a_u + \frac{\epsilon_1 + \epsilon_2}{2}} + f_{\frac{\epsilon_3}{2}} + f_{\frac{\epsilon_4}{2}} \quad (5.20)$$

with

$$\begin{aligned} f_{\pm a_u + \frac{\epsilon_1 + \epsilon_2}{2}} &= -(\epsilon_1 + \epsilon_3)(\epsilon_1 + \epsilon_4) \frac{(\pm 2a_u + \epsilon_1 + \epsilon_2)(\pm a_u + \epsilon_1 + \epsilon_2)}{(\pm 2a_u + \epsilon_1 + \epsilon_2 - \epsilon_3)(\pm 2a_u + \epsilon_1 + \epsilon_2 - \epsilon_4)} \\ &\quad \times \prod_{v \neq u} \frac{((\pm a_u - \epsilon_3)^2 - a_v^2)((\pm a_u - \epsilon_4)^2 - a_v^2)}{(a_u^2 - a_v^2)((\pm a_u + \epsilon_1 + \epsilon_2)^2 - a_v^2)}, \\ f_{\frac{\epsilon_3}{2}} &= -\frac{(\epsilon_1 + \epsilon_3)(\epsilon_1 + \epsilon_4)(\epsilon_3 - \epsilon_1 - \epsilon_2)}{8(\epsilon_3 - \epsilon_4)} \prod_{u=1}^N \frac{(2\epsilon_3 - \epsilon_4)^2 - a_u^2}{(\epsilon_3 - \epsilon)^2 - a_u^2}, \\ f_{\frac{\epsilon_4}{2}} &= -\frac{(\epsilon_1 + \epsilon_3)(\epsilon_1 + \epsilon_4)(\epsilon_4 - \epsilon_1 - \epsilon_2)}{8(\epsilon_4 - \epsilon_3)} \prod_{u=1}^N \frac{(2\epsilon_4 - \epsilon_3)^2 - a_u^2}{(\epsilon_4 - \epsilon)^2 - a_u^2}. \end{aligned}$$

For example for  $\text{SO}(4)$  these formulas lead to

$$\begin{aligned} F_{k=1} \Big|_{\text{SO}(4)} &= (M^2 + h^2) \left[ -\frac{17}{8} + \frac{M^2}{(a_1 + a_2)(a_1 + a_2 + \epsilon)} + \frac{M^2}{(-a_1 - a_2)(-a_1 - a_2 + \epsilon)} \right. \\ &\quad \left. + \frac{M^2}{(a_1 - a_2)(a_1 - a_2 + \epsilon)} + \frac{M^2}{(-a_1 + a_2)(-a_1 + a_2 + \epsilon)} \right] \quad (5.21) \end{aligned}$$

where we have used (5.6), (4.4) and (4.1). Inside the square brackets the terms proportional to  $M^2$  precisely reconstruct the sum  $g_0$  defined in (4.20) so that this result is in perfect agreement with (4.21). We also notice that (5.21) is related to the  $\text{SU}(2)$  prepotential (5.11). Indeed, upon comparison, we have

$$F_{k=1} \Big|_{\text{SO}(4)}(a_1, a_2) = F_{k=1} \Big|_{\text{SU}(2)}(a_L) + F_{k=1} \Big|_{\text{SU}(2)}(a_R) + \frac{15}{8}(M^2 + h^2) \quad (5.22)$$

where

$$a_1 = a_L + a_R, \quad a_2 = a_L - a_R, \quad (5.23)$$

so the two prepotentials match up to an  $a$ -independent function as they should, since  $\text{SO}(4) \sim \text{SU}(2) \times \text{SU}(2)$ .

The explicit expressions of the prepotential for groups of higher rank quickly become rather involved; nevertheless we have checked up to  $\text{SO}(12)$  that the 1-instanton result (5.20) can be written as

$$F_{k=1} = M^2(M^2 + h^2) \left[ g_0 + \frac{M^2}{2} g_2 + \frac{M^2(M^2 - \epsilon^2)}{24} g_4 + \frac{M^2(M^4 - 3M^2\epsilon^2 + 3\epsilon^4)}{720} g_6 + \dots \right] \quad (5.24)$$

This is the same expression we found for the  $U(N)$  theories (see (5.12)) and is in perfect agreement with what follows from solving the recursion relation as discussed in section 4. Furthermore, in the limit  $\epsilon, h \rightarrow 0$  we exactly recover the results obtained in [48] using a very different approach.

**2-instanton terms.** At  $k = 2$  one has to compute two integrals. Again, to avoid long formulas we only write an example in the  $\epsilon, h \rightarrow 0$  limit for the purpose of illustration. For  $SO(4)$ , up to  $a$ -independent terms we find

$$\begin{aligned}
 F_{k=2} \Big|_{SO(4)} &= 12m^4 \frac{a_1^2 + a_2^2}{(a_1^2 - a_2^2)^2} - 24m^6 \frac{a_1^4 + 6a_1^2 a_2^2 + a_2^4}{(a_1^2 - a_2^2)^4} \\
 &\quad + 10m^8 \frac{a_1^6 + 15a_1^4 a_2^2 + 15a_1^2 a_2^4 + a_2^6}{(a_1^2 - a_2^2)^6} \tag{5.25} \\
 &= 3m^4 C_2 - 6m^6 C_4 + 3m^6 C_{2;11} + \frac{5m^8}{2} C_6 + 6m^8 C_{4;2} + \frac{m^8}{2} C_{3;3} + \frac{m^8}{2} C_{2;1111}
 \end{aligned}$$

where the last line follows upon using the sums (3.6) over the lattice root of  $SO(2N)$ . Formally, this is the same expression found for the unitary theories and agrees with the results obtained in section 3 from the recursion relations.

We have verified that this agreement persist up for higher rank groups up to  $SO(12)$ . This fact puts our findings on a very solid ground.

## 6 Conclusions

In this paper we have shown that S-duality in  $\mathcal{N} = 2^*$  gauge theories with simply-laced gauge groups requires that the quantum prepotential satisfies a modular anomaly equation which in turn allows to recursively determine the prepotential itself. It is very satisfactory that these conditions can be expressed in a unified form involving sums over the roots of the gauge algebra without resorting to the specific details of the algebra itself. This is the key to extend our results to the case of exceptional algebras, where the lack of an ADHM construction of the instanton moduli space does not allow the application of the traditional methods of investigation. The differential equation coming from the anomaly, irrespective of the gauge algebra, needs an external output to fix all the terms in the prepotential. Given that  $f_n$  is a modular form of weight  $2n - 2$ , in solving the recursion relation (3.1) we can add to  $f_n$  all monomials in the Eisenstein series which have weight  $2n - 2$  but do not contain  $E_2$ . The coefficients in front of these terms are determined by comparing with the perturbative expansion and when this is not enough by resorting to microscopic instanton computations. Given that no microscopic instanton computations exist for the exceptional gauge groups this could seem a problem. Luckily enough, the results for  $f_n$  given in terms of sums over the roots of the algebra are universal and thus should hold for the exceptional algebras as well. We believe our results be a very solid conjecture which we have successfully tested for the lowest instanton number with the result for pure  $\mathcal{N} = 2$  theory existing in literature [13, 15] and provide an elegant generalisation to the  $\mathcal{N} = 2^*$  case, as well as precise predictions for higher instanton corrections.

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## A Notations and conventions for the root systems

In this appendix we list our conventions for the root systems of the simply-laced algebras. We consider both the classical algebras  $A_r = \mathfrak{su}(r+1)$  and  $D_r = \mathfrak{so}(2r)$ , and the exceptional ones  $E_6$ ,  $E_7$  and  $E_8$ .

We denote by  $\Psi$  the set of all roots  $\alpha$  and by  $\Psi(\alpha)$  the set

$$\Psi(\alpha) = \{\beta \in \Psi : \alpha \cdot \beta = 1\} . \tag{A.1}$$

The order of this set is

$$\text{ord}(\Psi(\alpha)) = 2h^\vee - 4, \tag{A.2}$$

where  $h^\vee$  is the dual Coxeter number of the algebra.

To write the roots of the different ADE algebras we use the standard orthonormal basis in  $\mathbb{R}^r$ :  $\{\mathbf{e}_i; 1 \leq i \leq r\}$  and our conventions are such that, for every ADE algebra  $(\alpha \cdot \alpha) = 2$ .

**The roots of  $A_r$ .** The set  $\Psi$  of the roots of  $A_r$  is

$$\Psi = \{ \pm (\mathbf{e}_i - \mathbf{e}_j); 1 \leq i < j \leq r \} . \tag{A.3}$$

It is easy to see that

$$\text{ord}(\Psi) = r(r-1) \quad \text{and} \quad \text{ord}(\Psi(\alpha)) = 2r - 4, \tag{A.4}$$

since the dual Coxeter number for  $A_r$  is  $h^\vee = r$ .

**The roots of  $D_r$ .** For  $D_r$  the roots are given by

$$\Psi = \{ \pm \mathbf{e}_i \pm \mathbf{e}_j; 1 \leq i < j \leq r \}, \tag{A.5}$$

with all possible signs. It is easy to see that

$$\text{ord}(\Psi) = 2r(r-1) \quad \text{and} \quad \text{ord}(\Psi(\alpha)) = 4r - 8, \tag{A.6}$$

since the dual Coxeter number for  $D_r$  is  $h^\vee = 2r - 2$ .

**The roots of  $E_6$ .**  $E_6$  has 72 roots given by

$$\Psi = \{ \pm \mathbf{e}_i \pm \mathbf{e}_j; 1 \leq i < j \leq 5 \} \cup \left\{ \frac{1}{2} (\pm \mathbf{e}_1 \pm \dots \pm \mathbf{e}_5 \pm \sqrt{3} \mathbf{e}_6) \right\}, \tag{A.7}$$

where the elements of the second set must have an even number of minus signs. In this case

$$\text{ord}(\Psi(\alpha)) = 20, \tag{A.8}$$

since  $h^\vee = 12$ .



**The roots of  $E_7$ .** The 126 roots of  $E_7$  are

$$\Psi = \{ \pm \mathbf{e}_i \pm \mathbf{e}_j ; 1 \leq i < j \leq 6 \} \cup \{ \pm \sqrt{2} \mathbf{e}_7 \} \cup \{ \frac{1}{2} ( \pm \mathbf{e}_1 \pm \dots \pm \mathbf{e}_6 \pm \sqrt{2} \mathbf{e}_7 ) \}, \quad (\text{A.9})$$

where the elements of the third set must have an odd (even) number of minus signs in the  $(\mathbf{e}_1, \dots, \mathbf{e}_6)$  components if the  $\mathbf{e}_7$  is positive (negative). Moreover we have

$$\text{ord}(\Psi(\alpha)) = 32, \quad (\text{A.10})$$

since in this case  $h^\vee = 18$ .

**The roots of  $E_8$ .**  $E_8$  has 240 roots given by

$$\Psi = \{ \pm \mathbf{e}_i \pm \mathbf{e}_j ; 1 \leq i < j \leq 8 \} \cup \{ \frac{1}{2} ( \pm \mathbf{e}_1 \pm \dots \pm \mathbf{e}_7 \pm \mathbf{e}_8 ) \}, \quad (\text{A.11})$$

where the element of the second set must have an even number of minus signs. It is easy to see that that

$$\text{ord}(\Psi(\alpha)) = 56, \quad (\text{A.12})$$

since in this case  $h^\vee = 30$ .

## B Eisenstein series and their modular properties

The Eisenstein series  $E_{2n}$  are holomorphic functions of  $\tau$  defined as

$$E_{2n} = \frac{1}{2\zeta(2n)} \sum_{m,n \in \mathbb{Z}^2 \setminus \{0,0\}} \frac{1}{(m + n\tau)^{2n}}. \quad (\text{B.1})$$

For  $n > 1$ , they are modular forms of degree  $2n$ : under an  $\text{SL}(2, \mathbb{Z})$  transformation

$$\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d} \quad \text{with } a, b, c, d \in \mathbb{Z} \quad \text{and } ad - bc = 1 \quad (\text{B.2})$$

one has

$$E_{2n}(\tau') = (c\tau + d)^{2n} E_{2n}(\tau). \quad (\text{B.3})$$

For  $n = 1$ , the  $E_2$  series is instead quasi-modular:

$$E_2(\tau') = (c\tau + d)^2 E_2(\tau) + \frac{6}{i\pi} c(c\tau + d). \quad (\text{B.4})$$

All the modular forms of degree  $2n > 6$  can be expressed in terms of  $E_4$  and  $E_6$ ; the quasi-modular forms instead can be expressed as polynomials in  $E_2$ ,  $E_4$  and  $E_6$ .

The Eisenstein series admit a Fourier expansion in terms of  $q = e^{2\pi i\tau}$  of the form

$$E_{2n} = 1 + \frac{2}{\zeta(1-2n)} \sum_{k=1}^{\infty} \sigma_{2n-1}(k) q^k, \quad (\text{B.5})$$

where  $\sigma_p(k)$  is the sum of the  $p$ -th powers of the divisors of  $k$ . In particular, this amounts to

$$\begin{aligned} E_2 &= 1 - 24 \sum_{k=1}^{\infty} \sigma_1(k)q^k = 1 - 24q - 72q^2 - 96q^3 + \dots, \\ E_4 &= 1 + 240 \sum_{k=1}^{\infty} \sigma_3(k)q^k = 1 + 240q + 2160q^2 + 6720q^3 + \dots, \\ E_6 &= 1 - 504 \sum_{k=1}^{\infty} \sigma_5(k)q^k = 1 - 504q - 16632q^2 - 122976q^3 + \dots. \end{aligned} \tag{B.6}$$

Using these expansions it is easy to see that a generic quasi-modular function of weight 4

$$M_4(q) = \alpha E_2^2 + \beta E_4 = (\alpha + \beta) + 48(5\beta - \alpha)q + \dots \tag{B.7}$$

has no perturbative contribution if  $\alpha = -\beta$ , like for instance  $E_2^2 - E_4$ , and it has no 1-instanton contribution if  $\alpha = 5\beta$ , like for instance  $5E_2^2 + E_4$ . These are precisely the combinations that appear in the prepotential coefficient  $f_3$  (see (3.13)). Analogously, for a weight 6 quasi-modular function

$$M_6(q) = \alpha E_2^3 + \beta E_2 E_4 + \gamma E_6 = (\alpha + \beta + \gamma) + 72(-\alpha + 3\beta - 7\gamma)q + \dots \tag{B.8}$$

the perturbative and 1-instanton contributions vanishes respectively for  $\alpha = -\beta - \gamma$  and  $\alpha = 3\beta - 7\gamma$ . This is the case for the combinations that appear in the prepotential coefficient  $f_4$  in (3.15).

The  $E_2$  series is related to the Dedekind  $\eta$ -function

$$\eta(q) = q^{1/24} \prod_{k=1}^{\infty} (1 - q^k). \tag{B.9}$$

Indeed, taking the logarithm of this definition we get

$$\log \left( \frac{\eta(q)}{q^{1/24}} \right) = \sum_{r=1}^{\infty} \log(1 - q^r) = - \sum_{k=1}^{\infty} \frac{\sigma_1(k)}{k} q^k. \tag{B.10}$$

If we apply now to this relation the derivative operator  $q \frac{d}{dq}$  we get

$$q \frac{d}{dq} \log \left( \frac{\eta}{q^{1/24}} \right) = - \sum_{k=1}^{\infty} \sigma_1(k) q^k = \frac{E_2 - 1}{24}. \tag{B.11}$$

Applying repeatedly the operator  $q \frac{d}{dq}$  to this last expression we also find

$$\left( q \frac{d}{dq} \right)^{n-1} (E_2 - 1) = -24 \sum_{k=1}^{\infty} k^{n-1} \sigma_1(k) q^k. \tag{B.12}$$

Finally, one has

$$\begin{aligned} q \frac{d}{dq} E_2 &= \frac{1}{12} (E_2^2 - E_4), \\ q \frac{d}{dq} E_4 &= \frac{1}{3} (E_2 E_4 - E_6), \\ q \frac{d}{dq} E_6 &= \frac{1}{2} (E_2 E_6 - E_4^2). \end{aligned} \tag{B.13}$$

### C The $\Gamma_2$ -function

The Barnes double  $\Gamma$ -function is defined as

$$\begin{aligned} \log \Gamma_2(x|\epsilon_1, \epsilon_2) &= \frac{d}{ds} \left( \frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{dt}{t} \frac{t^s e^{-xt}}{(1 - e^{-\epsilon_1 t})(1 - e^{-\epsilon_2 t})} \right) \Big|_{s=0} \\ &= \log \left( \frac{x}{\Lambda} \right)^2 \left( -\frac{1}{4} b_0 x^2 + \frac{1}{2} b_1 x - \frac{b_2}{4} \right) + \left( \frac{3}{4} b_0 x^2 - b_1 x \right) + \sum_{n=3}^\infty \frac{b_n x^{2-n}}{n(n-1)(n-2)} \end{aligned} \quad (\text{C.1})$$

where the coefficients  $b_n$  are given by

$$\frac{1}{(1 - e^{-\epsilon_1 t})(1 - e^{-\epsilon_2 t})} = \sum_{n=0}^\infty \frac{b_n}{n!} t^{n-2}. \quad (\text{C.2})$$

The first few of them are

$$b_0 = \frac{1}{\epsilon_1 \epsilon_2} = \frac{1}{h^2}, \quad b_1 = \frac{\epsilon_1 + \epsilon_2}{2\epsilon_1 \epsilon_2} = \frac{\epsilon}{h^2}, \quad b_2 = \frac{\epsilon_1^2 + 3\epsilon_1 \epsilon_2 + \epsilon_2^2}{6\epsilon_1 \epsilon_2} = \frac{4\epsilon^2 + h^2}{6h^2}. \quad (\text{C.3})$$

### D Useful formulas for the root lattice sums

Let us first recall the definition of the lattice sums  $C_{n;m_1 m_2 \dots}$  introduced in (3.6) which we rewrite here for convenience:

$$C_{n;m_1 m_2 \dots m_\ell} = \sum_{\alpha \in \Psi} \sum_{\beta_1 \neq \beta_2 \neq \dots \beta_\ell \in \Psi(\alpha)} \frac{1}{(\alpha \cdot a)^n (\beta_1 \cdot a)^{m_1} (\beta_2 \cdot a)^{m_2} \dots (\beta_\ell \cdot a)^{m_\ell}} \quad (\text{D.1})$$

where  $\Psi$  is the root lattice and  $\Psi(\alpha)$  is the set:

$$\Psi(\alpha) = \{ \beta \in \Psi : \alpha \cdot \beta = 1 \}. \quad (\text{D.2})$$

Actually, not all these sums are independent of each other, since there exist various algebraic identities among them which we are going to discuss.

First of all, from the definition (D.1) it is straightforward to see that

$$C_n = 0 \quad \text{for } n \text{ odd}, \quad (\text{D.3})$$

and, more generally

$$C_{n;m_1 m_2 \dots m_\ell} = 0 \quad \text{for } (n + \sum_\ell m_\ell) \text{ odd}. \quad (\text{D.4})$$

Many different sets of relations among the  $C$ 's can be proved by dividing the sums in (D.1) into sums over closed orbits of the Weyl reflection group and exploiting the properties of the partial sums. As a first example of this strategy, let us consider the set formed by a given couple of roots  $(\alpha, \beta)$  with  $\beta \in \Psi(\alpha)$ , together with its images under mutual reflection:

$$\{ (\alpha, \beta), (-\alpha, \beta - \alpha), (\alpha - \beta, -\beta) \}. \quad (\text{D.5})$$

This set forms a closed orbit of the Weyl reflections group, up to irrelevant overall signs. It is straightforward to see that

$$\frac{1}{(\alpha \cdot a)(\beta \cdot a)} + \frac{1}{(\alpha \cdot a)((\alpha - \beta) \cdot a)} - \frac{1}{((\alpha - \beta) \cdot a)(\beta \cdot a)} = 0. \quad (\text{D.6})$$

From this equation we can immediately prove that

$$C_{1;1} = \sum_{\alpha \in \Psi} \sum_{\beta \in \Psi(\alpha)} \frac{1}{(\alpha \cdot a)(\beta \cdot a)} = 0 . \tag{D.7}$$

In fact, the sum in  $C_{1;1}$  contains (twice) all the images of any couple under Weyl reflections. The expression in the above sum vanishes identically already when summed over the components of the set in (D.5).

The implication of (D.6) are however far more reaching. In fact, multiplying it by  $1/(\alpha \cdot a)^2$  and then summing over  $\alpha \in \Psi$  and  $\beta \in \Psi(\alpha)$ , we easily find

$$2 C_{3;1} = \widehat{C}_{2;11} , \tag{D.8}$$

with

$$\widehat{C}_{2;11} = \sum_{\alpha \in \Psi} \sum_{\beta \in \Psi(\alpha)} \frac{1}{(\alpha \cdot a)^2(\beta \cdot a)((\alpha - \beta) \cdot a)} . \tag{D.9}$$

Likewise, multiplying (D.6) by  $1/(\alpha \cdot a)(\beta \cdot a)$  and then summing over the roots, we find

$$C_{2;2} = -2 \widehat{C}_{2;11} . \tag{D.10}$$

These two relations together imply

$$C_{2;2} = -4 C_{3;1} , \tag{D.11}$$

but, using again the strategy of summing over the closed orbits of the Weyl group, it is easy to check that

$$\widehat{C}_{2;11} = C_{2;11} , \tag{D.12}$$

thus obtaining the identity

$$2 C_{3;1} = C_{2;11} . \tag{D.13}$$

This is the identity used to write the expression (3.11) for  $f_3$  given in the main text.

This method can be easily generalized to derive many other identities. In fact, multiplying (D.6) by

$$\frac{1}{(\alpha \cdot a)^{n-1}(\beta \cdot a)^{m-1}} \tag{D.14}$$

and summing over the roots, one gets

$$C_{n;m} = -\widehat{C}_{n;m-1p} - \widehat{C}_{n-1;mp} . \tag{D.15}$$

This relation, together with the symmetry properties of the different  $C$ 's, can be recursively used to link together all lattice sums with two indices at a given level. For example, at level 6 we have

$$4C_{5;1} + 2C_{4;2} + C_{3;3} = 0 . \tag{D.16}$$

All identities among the root lattice sums, presumably, can be derived following the strategy we have outlined; however, as the level increases, the algebra becomes more involved and thus it is often more convenient to check the identities in a numerical way.

Here we list some other identities among the level 6 sums that are needed in order to write the expression for  $f_4$  as in eq. (3.15d), namely

$$\begin{aligned} C_{4;11} &= -C_{4;2} - \frac{1}{2}C_{3;3} + \frac{1}{12}C_{2;1111}, & C_{3;21} &= -\frac{1}{2}C_{3;3} - \frac{1}{4}C_{2;1111}, \\ C_{2;31} &= \frac{1}{2}C_{3;3}, & C_{3;111} &= \frac{1}{2}C_{2;1111}, & C_{2;211} &= -\frac{2}{3}C_{2;1111}. \end{aligned} \quad (\text{D.17})$$

## E An observation on the prepotential

It is a fact that in all prepotential coefficients  $f_n$  the combinations of Eisenstein series appearing in front of the sum  $C_{2;11\dots 1}/\ell!$  have a linear term in  $q$  whose coefficient is exactly 1. This comes about through the following mechanism. From (3.15b) we see that the coefficient of  $C_2$  in  $f_2$  is (for simplicity we omit now the overall powers of  $m$  which are easily reinstated)

$$-\frac{E_2}{24} = -\frac{1}{24} + q + \dots, \quad (\text{E.1})$$

so it enjoys this property. The coefficient of  $C_{2;11}/2$  in  $f_3$  (see (3.15c)), of  $C_{2;1111}/24$  in  $f_4$  (see (3.15d)), and so on, turn out to be obtained from (E.1) by repeated logarithmic  $q$ -derivatives; indeed using (B.13) one has

$$-\frac{E_2}{24} \xrightarrow{q \frac{d}{dq}} -\frac{E_2^2 - E_4}{288} \xrightarrow{q \frac{d}{dq}} -\frac{E_2^3 - 3E_2E_4 + 2E_6}{1728} \xrightarrow{q \frac{d}{dq}} \dots \quad (\text{E.2})$$

so they again contain exactly  $q$  in their Fourier expansion. Due to this structure, the entire part of the prepotential which contains the expressions  $C_{2;11\dots 1}/\ell!$ , and not just its 1-instanton component, can be written in a compact way, analogously to (3.17). Reinstating the mass prefactors and introducing the operator

$$\mathcal{D} = m^2 q \frac{d}{dq} = \frac{m^2}{2\pi i} \frac{d}{d\tau} \quad (\text{E.3})$$

such terms are

$$\begin{aligned} &\left( C_2 + \frac{1}{2!}C_{2;11}\mathcal{D} + \frac{1}{4!}C_{2;1111}\mathcal{D}^2 + \frac{1}{6!}C_{2;111111}\mathcal{D}^3 + \dots \right) \left( -\frac{E_2}{24} \right) \\ &= \sum_{\alpha \in \Psi} \frac{1}{(\alpha \cdot a)^2} \prod_{\beta \in \Psi(\alpha)} \left( 1 + \frac{\mathcal{D}^{\frac{1}{2}}}{\beta \cdot a} \right) \left( -\frac{E_2}{24} \right). \end{aligned} \quad (\text{E.4})$$

Notice that only integer powers of  $\mathcal{D}$  remain in the expansion of the product above, due to the properties of the root systems.

Actually, all terms in the prepotential can be grouped into series of terms connected by the action of the operator  $\mathcal{D}$ ; writing only the terms up to  $f_4$  for simplicity, we may write

$$\begin{aligned} f_2 &= \mathcal{A}_{(1)} C_2, \\ f_3 &= \mathcal{A}_{(2)} C_4 + \mathcal{D}\mathcal{A}_{(1)} \frac{1}{2}C_{2;1,1}, \\ f_4 &= \mathcal{A}_{(3)} C_6 - \mathcal{D}\mathcal{A}_{(2)} \frac{1}{2} \left( C_{4;2} + \frac{1}{12}C_{3;3} \right) + \mathcal{D}^2\mathcal{A}_{(1)} \frac{1}{24}C_{2;1,1,1,1}, \end{aligned} \quad (\text{E.5})$$

where

$$\begin{aligned}
 \mathcal{A}_{(1)} &= -\frac{m^4}{24} E_2, \\
 \mathcal{A}_{(2)} &= -\frac{m^6}{720} (5E_2^2 + E_4), \\
 \mathcal{A}_{(3)} &= -\frac{m^8}{90720} (175E_2^3 + 84E_2E_4 + 11E_6).
 \end{aligned}
 \tag{E.6}$$

Thus, at each order  $m^{2n}$ , a new modular form  $\mathcal{A}_{(n)}$  appears in front of the structure  $C_n$ . For  $n > 2$ , the term of order  $q$  in this form vanishes; this agrees with the fact that in the microscopic computation of the 1-instanton corrections the structure  $C_n$  cannot appear. All the rest of  $f_n$  is organized in structures proportional to multiple  $\mathcal{D}$  derivatives of the lower  $\mathcal{A}_{(k)}$  coefficients, which have therefore neither perturbative terms (since the  $q$ -derivative kills the constant term in  $\mathcal{A}_{(k)}$ ) nor 1-instanton corrections, except for the structures  $C_{2;11\dots}$ , proportional to derivatives of  $\mathcal{A}_{(1)}$ , that we considered above.

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