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Exploring the total positivity of yields correlations

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We test the plausibility of the total positivity assumption of interest rates changes recently introduced in order to justify the presence of shift, slope and curvature for yield curves. To this aim, we introduce and discuss a test of total positivity of order k for covariance and correlation matrices. The explicit expressions of the test statistics are given for Gaussian samples and an extension to a distribution-free framework is made via a bootstrap method. After exploring with simulation the robustness of such tests, we show using real data how it is realistic to assume that correlation matrices of interest rates changes are totally positive of order two. Conclusions on total positivity of order three are more controversial.

Keywords: Bootstrap; Multiple tests; Schoenmakers & Coffey matrices; Partial correlations; Total positivity; Yield curve

JEL Classification: G12, C12

1. Introduction

The literature on correlation models of interest rates has paid a lot of attention to factor models where the movements of the yield curve are explained in terms of few unobservable latent variables (see e.g. Vasicek 1977, Ho *et al*. 1986, Hull and White 1990, Rebonato 2002). Empirical evidence (see e.g. Willner 1996, Golub and Tilman 1997, Longstaff *et al*. 1999) showed that changes in the shape of the yield curve are substantially imputable to three factors: the first one changing the interest rates of all maturities by almost identical amounts, the second one varying short-term interest rates in an opposite way with respect to long-term interest rates and the third one presenting the main effects on medium-term interest rates. Applying *principal component analysis* (PCA) to the yield curve expressed as a random vector in order to find these factors, it was noted (see Steeley 1990, Litterman and Scheinkman 1991) that the variability of the first three principal components explains most (more than 90%) of the total variability. The corresponding loadings vectors have been, respectively, termed as *shift*, *slope* and *curvature* (SSC from now on) because of the peculiar shape of their entries, characterized by a determined number of changes of sign and monotonicity.

Despite some criticism (see Lekkos 2000), the presence of SSC nowadays is considered (see e.g. Martellini *et al*. 2003) a basic feature that a correlation model for interest rates should enjoy jointly to some other 'standard' properties as positivity and decreasingness with respect to the difference in maturities of the correlation coefficients.

The works of Lord and Pelsser (Lord and Pelsser 2007) and Salinelli and Sgarra (Salinelli and Sgarra 2006, Salinelli and Sgarra 2007, Salinelli and Sgarra 2011) besides having introduced a formal definition of SSC (see also Forzani and Tolmasky 2003, Martellini *et al*. 2003) have faced the problem of justifying the presence of SSC in terms of some properties of the correlations of interest rates. In particular, these authors have formally justified the number of sign changes of SSC assuming the *total positivity* and/or *oscillatory* behavior up to order three (see Gantmacher and Krein 1961, Karlin 1968, Fallat and Johnson 2011) of the correlation matrix of rates. Roughly speaking, this corresponds to assuming the positivity of minors up to order three of the correlation matrix or of one of its appropriate powers. In the mentioned papers, the total positivity and the oscillatoriness assumptions have been related to the above-mentioned properties of correlations coefficients, and it has been proved that some of the most important models of interest rates satisfy them. Nevertheless, to our knowledge, no investigation on their empirical plausibility has been performed up to now.

In order to fill this gap, in this work, we test the significance of the total positivity hypothesis on a real data-set. To this aim, we introduce a statistical test for total positivity consisting in the simultaneous comparison of standardized minors of the sample correlation matrix with suitable critical values. The derivation of the test statistics and its distribution is based principally on the results appearing in Drton *et al*. (2008), where the first and second moments of minors of a covariance matrix are studied. Since this construction requires a Gaussianity assumption on data that is generally not verified by interest rates, we have

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also considered a distribution-free approach based on a bootstrap methodology. A simulation study has been conducted to assay the robustness of these test procedures under different assumptions on the distribution of the population, inspired by the statistical characteristics of the real data which have motivated the work. We apply our test to two different real data-sets in order to avoid possible effects due to the way in which the data are generated. The obtained results show how total positivity of order 2 occurs, whereas the conclusion on the assumption of total positivity of order 3 appears more complex. We briefly discuss the economic consequences of these results.

The outline of the paper is the following: in section 2, we recall the main definitions and results concerning the total positivity assumption on correlation matrices; next, in section 3, we introduce a total positivity test and discuss its robustness. In section 4, we apply our test to real data, obtaining our results on the total positivity assumptions and discussing them. Some possible directions for future research are sketched in section 5.

2. Yields correlation, PCA and total positivity

In this section, we briefly recall the notion of total positivity, how it has been related to some well-known properties of correlations between interest rates changes and how it was used in order to justify the presence of SSC. For details, we refer the reader to Lord and Pelsser (2007), Salinelli and Sgarra (2006), Salinelli and Sgarra (2007) and Salinelli and Sgarra (2011).

Let $\mathbf{X}$ be the random vector of standardized spot rates changes on a given maturity spectrum: for a discussion on this choice, we refer to Lardic *et al.* (2003). The first three principal components of $\mathbf{X}$ obtained by applying PCA (see e.g. Jolliffe 2004) explain more than 90% of the total variability of $\mathbf{X}$ and the corresponding loadings, the eigenvectors of the correlation matrix R of $\mathbf{X}$, have the particular shape illustrated in figure B3 below. Approximately, the shift has equal elements of the same sign with an humped shape, with elements first increasing and then decreasing; the slope has elements of opposite sign with similar magnitude at the opposite end of the maturity spectrum; the curvature has elements with the same signs at the opposite end of the maturity spectrum and of opposite sign in the middle.

As usual, we talk about the first three eigenvectors of a matrix, by meaning that they correspond to the first three eigenvalues assumed simple and sorted in descending order. To formally capture the behaviours described above, we recall that (see Gantmacher 1964) a vector $\mathbf{v} \in \mathbb{R}^n$ has a minimum $S^-(\mathbf{v})$ and a maximum $S^+(\mathbf{v})$ number of sign variations computed by discarding the zero elements or considering them both positive or negative, respectively. If $S^+(\mathbf{v}) = S^-(\mathbf{v})$, this common value is defined as the *number of sign variations* of $\mathbf{v}$. If $\Delta\mathbf{v} \in \mathbb{R}^{n-1}$ is defined by $(\Delta\mathbf{v})_i = v_{i+1} - v_i$ for $i = 1, \dots, n-1$, the following definition resumes the ones introduced in Salinelli and Sgarra (2006) and Salinelli and Sgarra (2011) in terms of changes of sign of the vectors $\mathbf{v}$ and $\Delta\mathbf{v}$. We will refer to correlation matrices, even if the definition also applies to covariance ones.

Definition 2.1 Let R be a $n \times n$, $n \geq 3$, correlation matrix having its first three eigenvalues simple, whose corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ have, by convention, Non-negative first element. We define:
$\mathbf{v}_1$ *weak shift* if $S^-(\mathbf{v}_1) = 0$, *shift* if it is weak shift and $S^-(\Delta\mathbf{v}_1) = 1$ where the first non-zero element of $\Delta\mathbf{v}_1$ is positive, *pure shift* if it is constant;
$\mathbf{v}_2$ *weak slope* if $S^-(\mathbf{v}_2) = 1$, *slope* if it is weak slope and $S^-(\Delta\mathbf{v}_2) = 0$;
$\mathbf{v}_3$ *weak curvature* if $S^-(\mathbf{v}_3) = 2$, *curvature* if it is weak curvature and $S^-(\Delta\mathbf{v}_3) = 1$.

The previous definition specifies the difference between two aspects of SSC: the weak form refers only to the number of the sign variations of the eigenvector elements (zero, one and two, respectively), the strict form requires information on the monotonicity changes too. Notice that the weak form of SSC coincides with the definition of level, slope and curvature given in Lord and Pelsser (2007).

In the empirical literature, both concrete cases of SSC and SSC in a weak form can be found. For example, in Martellini *et al.* (2003) (figures 3.16 and 3.17, p. 80), Golub and Tilman (1997) (Exhibit 5 p. 78), there are WSSC due to the presence of 'initial' and/or 'final' humps. Instead, in Lord and Pelsser (2007) (figures 1 and 2, p. 111), there are examples of SSC in a strict sense, whereas in Lardic *et al.* (2003), one finds both the cases. Anyway (see Martellini *et al.* 2003) 'every empirical investigation shows that the variance of the term structure of interest rates is explained to more than 90% using only the three first components' and '... these three factors have nice interpretations as being related, respectively, to parallel movement, slope oscillations and curvature of the term structure.'

The previous analysis makes clear how the presence of SSC, possibly in weak form, represents nowadays a standard property that a correlation model of interest rates has to present alongside the 'classic' requirements on correlations ρ_{ij}, namely:

(i) correlations are positive;
(ii) correlations decrease with respect to the difference in maturities:

$$1 \leq i < j < s \leq p \quad \Rightarrow \quad \rho_{ij} > \rho_{is} \tag{1}$$

$$1 \leq i < s < j \leq p \quad \Rightarrow \quad \rho_{ij} < \rho_{sj}\,. \tag{2}$$

We recall that in Salinelli and Sgarra (2011) it was showed that in a two-factor model, the presence of shift and slope implies properties (1) and (2).

An obvious but important question is whether the presence of SSC should be considered a property per se, or it is a consequence of (i) and (ii) and/or possibly of other properties not known yet.
Answering the question in the case of the first eigenvector of the correlation matrix $R = \left[\rho_{ij}\right]$ is easy: the presence of the weak shift, by (i), follows from the celebrated Perron–Frobenius Theorem. The existence of weak slope and weak curvature has instead been justified recurring to a mathematical assumption described in the following definition (see Karlin 1968, Fallat and Johnson 2011).

Definition 2.2 Given a $p \times p$ matrix A, the mth *compound matrix* of A, denoted by $A^{[m]}$, with $m \in \mathbb{N}$ and $m \leq p$, is the $\binom{p}{m}$-square matrix of the m minors of A. Then, for $k \in \{1, 2, \dots p\}$, the matrix A is called:

- *strictly totally positive of order* k, denoted by STP_k, if $A^{[m]}$ is positive for all $m = 1, \ldots, k$;
- *totally positive of order* k, denoted by TP_k, if $A^{[m]}$ is non-negative for all $m = 1, \ldots, k$;
- *oscillatory of order* k, denoted by OS_k, if A is TP_k and there exists a positive integer q such that A^q is STP_k.

It is possible to show (see Schriever 1982) that if A is a TP_k $(k \le p)$ $p \times p$ matrix, then A is OS_k if (i) $a_{ij} > 0$ for $|i - j| \le 1$, and its principal minors of order $\le k$ with consecutive lines are positive. These properties can be considered always satisfied by a correlation matrix of interest rates, hence it is sufficient to focus on the assumption of total positivity.
The importance of the matrices defined above is clarified by the following result (see Schriever 1982).

THEOREM 2.3 *Assume A is a $p \times p$ positive definite, symmetric matrix with eigenvalues $\lambda_1 \ge \lambda_2 \cdots \ge \lambda_p > 0$. If A is OS_k, then $\lambda_1 > \lambda_2 \cdots > \lambda_k > \lambda_{k+1} \ge \cdots \ge \lambda_p > 0$, i.e. at least the first k eigenvalues are simple and for $s \in \{1, \ldots, k\}$ the s-th eigenvector has exactly $s - 1$ changes of sign.*

By the previous theorem and the above considerations, we conclude that if R is TP_2, then its (first eigenvector is weak shift and its) second eigenvector is weak slope. If R is TP_3, then it has shift, slope and curvature in a weak form (the converse is not true, as illustrated in Salinelli and Sgarra (2011), example 6). Summing up, TP_2 and TP_3 are sufficient (but not necessary) conditions for the existence of SS and SSC in a weak form, respectively.

It is interesting to note that TP_2 correlation matrices whose elements are less than 1 satisfy the monotonicity properties (1) and (2) as showed in Salinelli and Sgarra (2006) (see theorem 16 and remark 17).

We conclude this section by illustrating some examples of correlation structures of yields satisfying some request of definition 2.2.

Example 2.4 A first model is the classical exponential one (see Rebonato 2002), where the correlation ρ_{ij} between maturities t_j and t_i is given by

$$\rho_{i,j} = \exp\left\{-\beta \left|t_j - t_i\right|\right\} \qquad \beta > 0. \tag{3}$$

It is evident that correlations satisfy properties (i) and (ii). A further characteristic is the homogeneity with respect to time: interest rates with the same maturity differences exhibit the same correlation. By setting $\rho = e^{-\beta}$, and identifying indices and maturities, one obtains the Toepliz correlation matrix (known in the numerical literature as the Kac-Murdok-Szegö matrix)

$$R = \begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{p-1} \\ \rho & 1 & \rho & \cdots & \rho^{p-2} \\ \rho^2 & \rho & 1 & \cdots & \rho^{p-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{p-1} & \rho^{p-2} & \rho^{p-3} & \cdots & 1 \end{bmatrix}. \tag{4}$$

In Salinelli and Sgarra (2006), it was proved the oscillatory property of R and in Salinelli and Sgarra (2007), some results on the existence of shift and slope were given.

A more general model also presenting time homogeneity but describing a correlation structure with very fast decay in maturity differences is

$$\rho_{ij} = \exp\left\{-\beta\, |i - j|^q\right\} \qquad \beta \in \mathbb{R}_+,\ q \in \mathbb{N}\setminus\{0\}. \tag{5}$$

In this case, the corresponding correlation matrix is TP, hence it has SSC in a weak sense.
The following model (see Rebonato 2002)

$$\rho_{i,j} = \exp\left\{-\beta \left|j^\gamma - i^\gamma\right|\right\} \qquad \beta \in \mathbb{R}_+,\ \gamma \in (0, 1). \tag{6}$$

has correlations satisfying (i) and (ii), but breaks the time-homogeneous behaviour. Furthermore, correlations increase descending on the diagonals. It represents a particular case of the so-called Schoenmakers–Coffey (see Schoenmakers and Coffey 2003) defined by

$$\rho_{i,j} = \frac{\min\left\{b_i, b_j\right\}}{\max\left\{b_i, b_j\right\}} \tag{7}$$

where the sequence $\{b_i\}$ is strictly positive, strictly increasing and log-concave, that is the sequence $\{b_i/b_{i+1}\}$ is strictly increasing. In Lord and Pelsser (2007), theorem 4 and corollary 4, p. 123, the oscillatoriness property of these matrices was stated.

Notice that model (4) represents a special case ($\{b_i/b_{i+1}\}$ is constant) of the Schoenmakers–Coffey structure, whereas (5) does not.

3. A testing procedure for total positivity

In this section, we address the problem of defining a method for detecting the total positivity of covariance or correlation matrices. Because a pure exploratory study, based only on the identification of the non-positive minors of the sample covariance or correlation matrices, would suffer of lack of generality since it is valid only for the observed data, an inferential approach for large samples has to be introduced in order to extend the results to the underlying population. Therefore, in what follows, we introduce and study a multiple testing procedure whose statistics are derived both under the assumption of Gaussian populations and in a distribution-free framework.

3.1. A Simes' test procedure

Consider a p-dimensional random vector $\mathbf{X} = \left(X_1, \ldots, X_p\right)^T$ with covariance matrix Σ, and denote by $\Sigma^{[m]}$, with $m \in \{1, \ldots, p\}$, its m-th compound matrix (see definition 2.2) whose entries are denoted by $\sigma_{ij}^{[m]}$ (clearly, $\Sigma_1 = \Sigma$). Since Σ is symmetric and positive semi-definite, its compound matrices $\Sigma^{[m]}$ are symmetric with non-negative main diagonal elements. Thanks to these aspects, testing total positivity of order k for Σ, with $k = 1, \ldots, p$, for a chosen significant level $\alpha \in (0, 1)$, is equivalent to testing the following multiple null hypothesis:

$$\mathcal{H} : \sigma_{i,j}^{[m]} \ge 0 \;\text{ for } 1 \le i < j \le \binom{p}{m}, \text{ with } m = 1, \ldots, k$$

against the alternative:

$$\mathcal{A} : \text{there exist a couple } (i, j) \text{ such that } \sigma_{i,j}^{[m]} < 0, \;\text{ for some } m.$$

In the following, we denote by $H_{ij}^{[m]}$ the marginal null hypothesis $\sigma_{ij}^{[m]} \ge 0$ for fixed i, j and m. In such a general framework, the number of tests that have to be performed to test TP_k is equal to $h_k = \frac{1}{2}\sum_{m=1}^{k}\binom{p}{m}\left(\binom{p}{m}-1\right)$.

Remark 1 When studying the yield curve, since correlations (and covariances) are strictly positive, it is useless to test TP_1. Consequently, the number of tests for the TP_k hypothesis, with $k = 2, \ldots, p$, reduces to: $h_k = \frac{1}{2}\sum_{m=2}^{k}\binom{p}{m}\left(\binom{p}{m}-1\right)$.

Let $\left\{\mathcal{X}_i = \left(X_{1,i}, \ldots, X_{p,i}\right)^T,\ i = 1, \ldots, n\right\}$ be a sample of i.i.d. observations from $\mathbf{X}$. Denote by $\widehat{\Sigma}$ the sample covariance matrix and $\widehat{\Sigma}^{[m]}$ its empirical m-th compound matrix with elements $\widehat{\sigma}_{ij}^{[m]}$, consistent estimators of $\sigma_{ij}^{[m]}$. We define the h -dimensional set of test statistics $\left\{Z_{i,j}^{[m]} : 1 \le i < j \le \binom{p}{m},\ m = 1, \ldots, k\right\}$, where

$$Z_{i,j}^{[m]} = \frac{\widehat{\sigma}_{i,j}^{[m]} - \sigma_{i,j}^{[m]}}{\sqrt{Var\left(\widehat{\sigma}_{i,j}^{[m]}\right)}} \tag{8}$$

is the standardized sample version of $\widehat{\sigma}_{i,j}^{[m]}$. If $\mathbf{X} \sim \mathcal{N}_p(0, \Sigma)$, the explicit expressions of the denominators in (8) can be obtained from the results appearing in Drton and Goia (2012) and Drton *et al.* (2008, section 5), where the first and second moments of the minors of a Wishart matrix are studied. For the sake of completeness, such expressions are recapped in the appendix 1: since in these formulas the minors of Σ appear explicitly, they have to be estimated. We use the correspondent minors of the empirical covariance matrix $\widehat{\Sigma}$.

Under the null hypothesis $H_{i,j}^{[m]}$ (i.e. when $\sigma_{i,j}^{[m]} = 0$), a direct application of the δ-method allows to conclude that the statistic $Z_{i,j}^{[m]}$ is asymptotically distributed as a standard Gaussian distribution (see the arguments in the proof of proposition 4 in Drton *et al.* (2007)), hence the corresponding p-value $P_{ij}^{[m]}$ can be calculated.

Since h_k increases rapidly with p and k, the use of the classical Bonferroni method, consisting in rejecting $\mathcal{H}$ if at least one of the p-values $P_{i,j}^{[m]}$ is less than α/h_k, could conduce to a too conservative test procedure for practical purposes: hence, we prefer to employ the Simes approach (see Simes 1986). Its decision rule is based on the ordered p-values $P_{i,j}^{[m]}(1) \le \cdots \le P_{i,j}^{[m]}(h_k)$, and it rejects $\mathcal{H}$ when $P_{i,j}^{[m]}(l) \le \alpha l/h_k$ for at least one l. Such method has the same critical values of the so-called 'false discovery rate controlling procedure' of Benjamini and Hochberg (see Benjamini and Hochberg 1995).

The test procedure illustrated above can be easily extended to correlation matrices R. Indeed, as pointed out in Drton *et al.* (2008, section 6), the ratio between a sample minor and its standard deviation is the same when one uses the sample covariance $\widehat{\Sigma}$ or sample correlation matrix $\widehat{R}$. Hence, denoting by $\rho_{ij}^{[m]}$ and $\widehat{\rho}_{ij}^{[m]}$ the entries of $R^{[m]}$ and $\widehat{R}^{[m]}$ (the m-th compounds of R and $\widehat{R}$ respectively), the test statistics is defined as:

$$Z_{i,j}^{[m]} = \frac{\widehat{\rho}_{ij}^{[m]} - \rho_{ij}^{[m]}}{\sqrt{Var\left(\widehat{\rho}_{ij}^{[m]}\right)}} \tag{9}$$

where the denominators are obtained by substituting $\widehat{\Sigma}$ with the empirical correlation matrix $\widehat{R}$ into the formulas in the appendix 1, and $\rho_{ij}^{[m]} = 0$ under the null hypothesis.

In order to complete the presentation of the test procedure, we observe that the expressions of the test statistics have been derived under the assumption of Gaussian population. In some empirical analysis this assumption is unreasonable. This is the case of the application which motivated our study (see the discussion in section 4). To extend the range of applicability of our test, one can use the bootstrapping methodology (see, for instance, Efron and Tibshirani 1986, Efron and Tibshirani 1993) to estimate the variances of empirical minors $\widehat{\sigma}_{i,j}^{[m]}$ (or $\widehat{\rho}_{ij}^{[m]}$). We consider the case of a covariance matrix Σ: the adaptation to correlation matrices is immediate. In brief, a large number B of bootstrap samples, obtained through the draws of n elements with replacement from the observed sample $\{\mathcal{X}_i,\ i = 1, \ldots, n\}$, is done. Hence, for each bootstrap sample, the empirical covariance matrix is estimated, and its minors $\widetilde{\sigma}_{i,j}^{[m]}(b)$ are computed. So $Var\left(\widehat{\sigma}_{i,j}^{[m]}\right)$ in (8) is estimated by:

$$\frac{1}{B-1}\sum_{b=1}^{B}\left(\widetilde{\sigma}_{i,j}^{[m]}(b) - \overline{\sigma}_{i,j}^{[m]}\right)^2$$

where $\overline{\sigma}_{ij}^{[m]} = \sum_{b=1}^{B}\widetilde{\sigma}_{ij}^{[m]}(b)/B$. Since the derived test statistics are, under the null hypothesis, asymptotically pivotal with Gaussian standard distribution, the p-values can be easily computed.

3.2. *Exploring finite sample properties of the test*

We conducted several simulation experiments to investigate the finite sample properties of the test for the total positivity of order two and three for correlation matrices by means of the estimation of the power for different population distributions, correlation structures and sample sizes. More in detail, we draw i.i.d. samples of size $n = 200$, 500 and 1000 from random vectors of dimension $p = 9$ (this choice is motivated by the size of the matrices involved in the application to the real data) having Gaussian and Student t with 5 and 10 degrees of freedom distributions, with various TP correlation matrices suitably perturbed in order to control the number of negative minors of order two and three ($\rho_{ij}^{[2]}$ and $\rho_{ij}^{[3]}$, in the above notation).

Since the study of yield correlation matrices is the main focus of this paper, to generate the correlation matrices, we adopted the ones of Schoenmakers–Coffey (7). In our experiments, correlation structures were defined according to the following models:

- power model: $b_i = i^{\delta}$, $0 < \delta < 1$,
- logarithmic model: $b_i = \log(i + \beta)$, $\beta > 1$,

where the coefficients δ and β and were chosen to reproduce some 'realistic' correlation structures of yield curves, namely presenting positive correlations that decrease with respect to the difference in maturities (see (1) and (2)). For each of the treated cases, the power has been estimated over 1000 Monte Carlo replications, as the proportion of times that the null hypothesis $\mathcal{H}$ was rejected using the Simes' test procedure at the nominal level $\alpha = 5\%$. Since the entries of Schoenmakers–Coffey matrices are strictly positive, we tested only TP_2 and TP_3 according to remark 1. As the two models do not lead to substantially different results, in the following, we illustrate in

Table 1. Number of negative minors of order two $\rho_{ij}^{[2]}$ and the smallest $\rho_{ij}^{[2]}$ ($\times 10^{-2}$) varying p_1.

p_1	Q	$\#\left\{\rho_{ij}^{[2]} < 0\right\}$ $(i < j)$	Power model $\varepsilon = 0.4$	$\varepsilon = 0.6$	$\varepsilon = 0.8$
1	1	28	−0.61	−0.92	−1.23
2	3	57	−0.71	−1.07	−1.43
3	6	71	−0.85	−1.27	−1.70
4	10	65	−1.04	−1.56	−2.08
5	15	45	−1.32	−1.99	−2.65
6	21	23	−1.79	−2.69	−3.59
7	28	7	−2.71	−4.07	−5.43

detail only the power case, deferring the logarithmic one to the appendix 2.

3.2.1. Testing $\mathbf{TP_2}$. We first analyse the test for the TP_2 assumption. The Schoenmakers–Coffey correlation matrices were generated according to the power model with $\delta = 0.2$. Such matrices were perturbed by acting on the elements of the north-east (and, for symmetry, south-west) triangular submatrices as follows: let p_1 be an integer satisfying $1 \leq p_1 < p$; the perturbed terms $\widetilde{\rho}_{i,j} = \widetilde{\rho}_{j,i}$ were obtained from the original matrix by

$$\widetilde{\rho}_{i,j} = \widetilde{\rho}_{j,i} = \varepsilon\rho_{i,j-1} + (1-\varepsilon)\,\rho_{i,j}$$

where $i = 1, \ldots, p_1$, $j = p - p_1 + i, \ldots, p$, and $0 < \varepsilon < 1$. Hence, the number of perturbed terms in the north-west triangular submatrices is $Q = Q(p_1) = \sum_{j=1}^{p_1} j$. With a good choice of ε, such a technique produces correlation matrices having the monotonicity by row and column properties that are not TP_2. In the simulations, we used $p_1 = 1, \ldots, 7$, and $\varepsilon = 0.4$, 0.6 and 0.8.

Using the previous specifications, the number of strictly negative minors of order two $\rho_{ij}^{[2]}$ that have to be considered in the test (that is, the ones with $i < j$) increases with p_1 when $p_1 \leq 4$ and then decreases. On the other hand, the value of the smallest minor $\rho_{ij}^{[2]}$ decreases with Q, thus producing a more and more clear departure from the initial TP_2 situation. These behaviours can be appreciated by reading table 1 where the smallest minors $\rho_{ij}^{[2]}$ are reported for the various experimental conditions considered. All the perturbed matrices exhibit SSC.

The power comparisons when we use the test statistic (9) exploiting the expression of the variance derived explicitly under condition of Gaussianity are presented in figure 1. The panels display the estimated level and power against the number Q of perturbed terms: the case $Q = 0$, corresponding to the use of the not perturbed matrices, gives the estimation of the level of the test.

We can deduce from the graphs how the test produces good performances in the case of samples drawn from Gaussian populations or from a distribution which deviates slightly from Gaussianity (i.e. the Student t with 10 df): as expected, the estimated power is monotonic with respect to Q (this is coherent with the behaviour illustrated in table 1), increases with n and ε. In such cases, the test appears rather conservative, namely the estimated level is smaller than the nominal one: this fact is not surprising since we are dealing with multiple comparisons (see section 3.1).

When we work with a distribution having fat tails (i.e. the Student t with 5 df), the test appears very liberal: the estimated level is systematically greater than the nominal one and so the performances of the test are only apparently better than in the other two cases. This distortion is not even attenuated when the sample size n is large; therefore, the test does not appear robust to large deviations from Gaussianity.
To remedy the distort effects due to the use, when the population is not Gaussian, of an expression of the variance $Var\left(\widehat{\rho}_{i,j}^{[m]}\right)$ in (9) derived under Gaussianity assumptions, we estimate such variance by the bootstrap method proposed in section 3.1, using $B = 1000$ bootstrap samples. The behaviour of the estimated level and power in this case is shown in figure 2.

By inspecting the graphs, we note immediately that the bootstrap version of the test is robust to any deviation from Gaussianity of the population and that the performances are relatively good when n is large enough with respect to the magnitude of the perturbation. In particular, we observe that the estimated level (corresponding to $Q = 0$) is similar in all cases: the test appears rather uniformly conservative with respect to the population distributions, sample sizes and correlation structures. About the behaviour of the power with respect to the number of perturbed terms, the graphs display, as it is desirable, monotonicity. As claimed before, we use the asymptotic null distribution of the test statistic: notice that when the samples come from a Gaussian population, the test produces better results than in the other cases, but the differences decrease for large sample sizes, as it is reasonable to expect.

In conclusion, the Monte Carlo experiment suggests that the test for the TP_2 assumption based on the statistic derived for Gaussian population gives good performances for relatively large sample sizes, but it is not robust to deviation from Gaussianity. A robust version is provided when one uses the bootstrap approximation of the test statistic. The comments about the behaviour of the test when one varies the sample size n and the perturbation term ε remain valid for different choices of δ.

3.2.2. Testing $\mathbf{TP_3}$. To analyse the test for TP_3, we used the Schoenmakers–Coffey correlation matrices based on the power model with $\delta = 0.25$. To obtain correlation matrices that are TP_2 but not TP_3, we perturbed the square submatrices

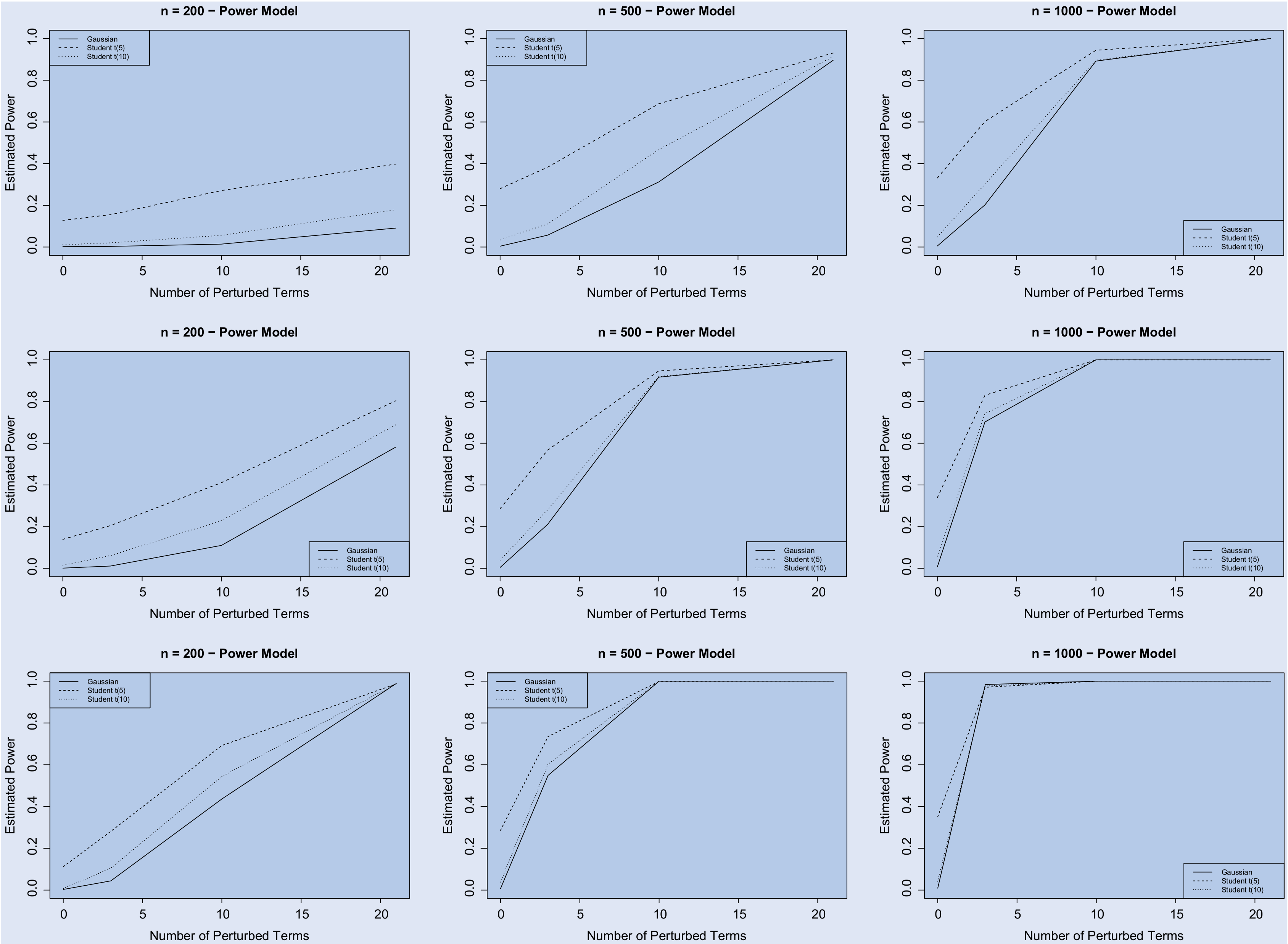

Figure 1. Estimated power of TP_2 test for Power Schoenmakers–Coffey correlation matrices with $n = 200$, 500 and 1000 and $\varepsilon = 0.4$, 0.6 and 0.8.

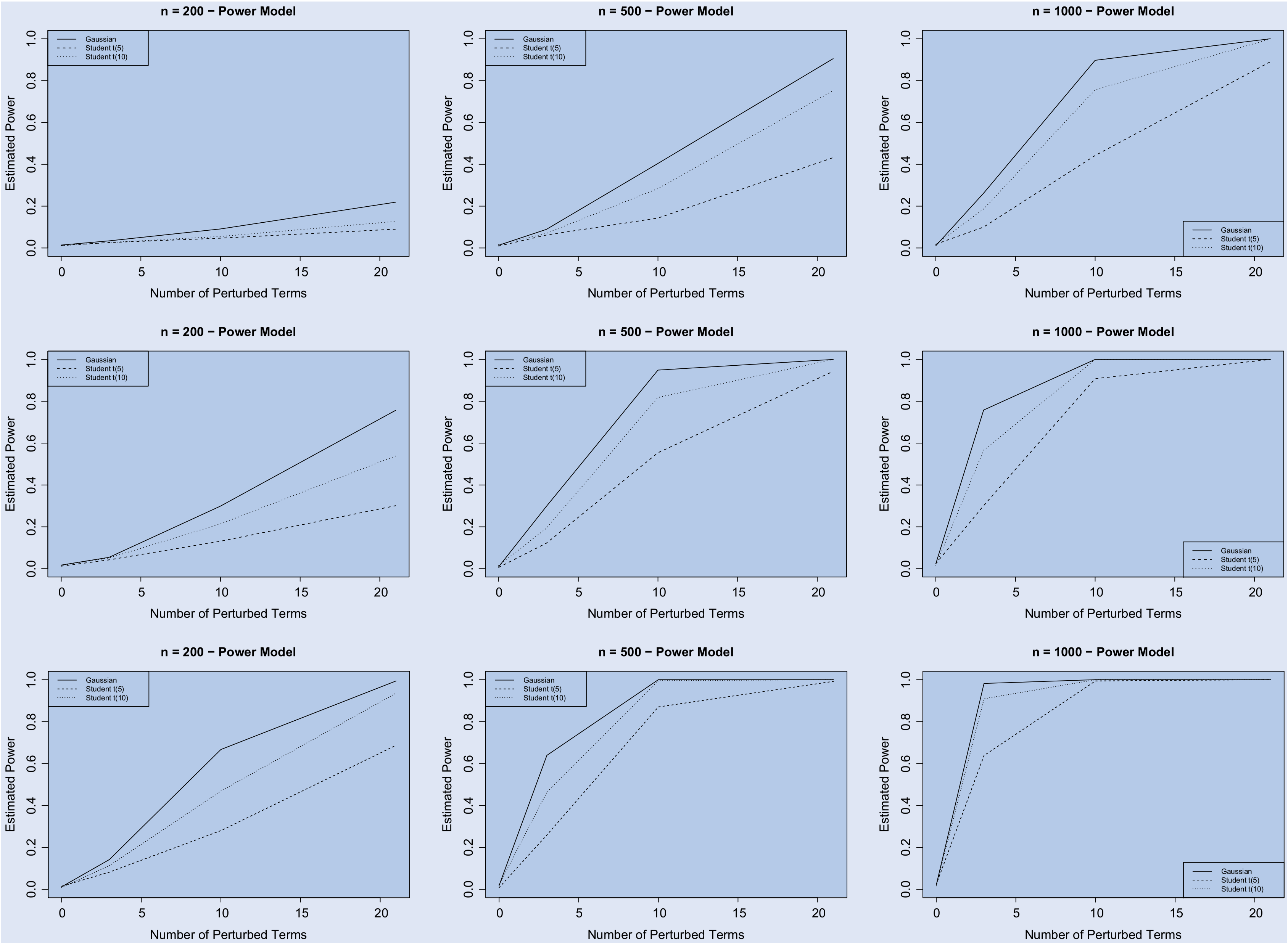

Figure 2. Estimated power of TP_2 test for Power Schoenmakers–Coffey correlation matrices with $n = 200$, 500 and 1000 and $\varepsilon = 0.4$, 0.6 and 0.8 with the standard error estimated by bootstrap.

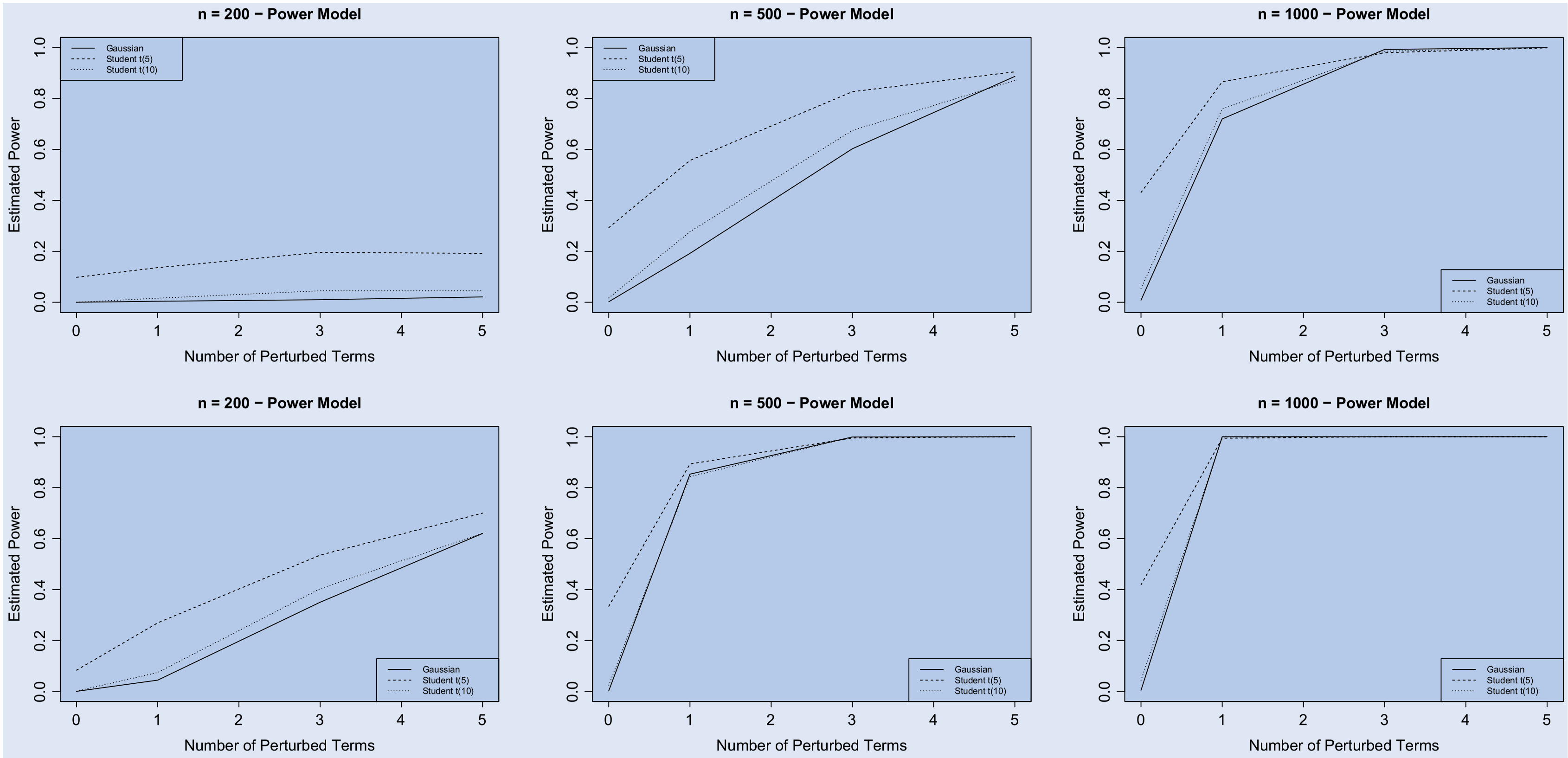

Figure 3. Estimated power of TP_3 test for Power Schoenmakers–Coffey correlation matrices with $n = 200$, 500 and 1000 and $\varepsilon = 0.97$ and 0.95, using the variance computed for Gaussian population.

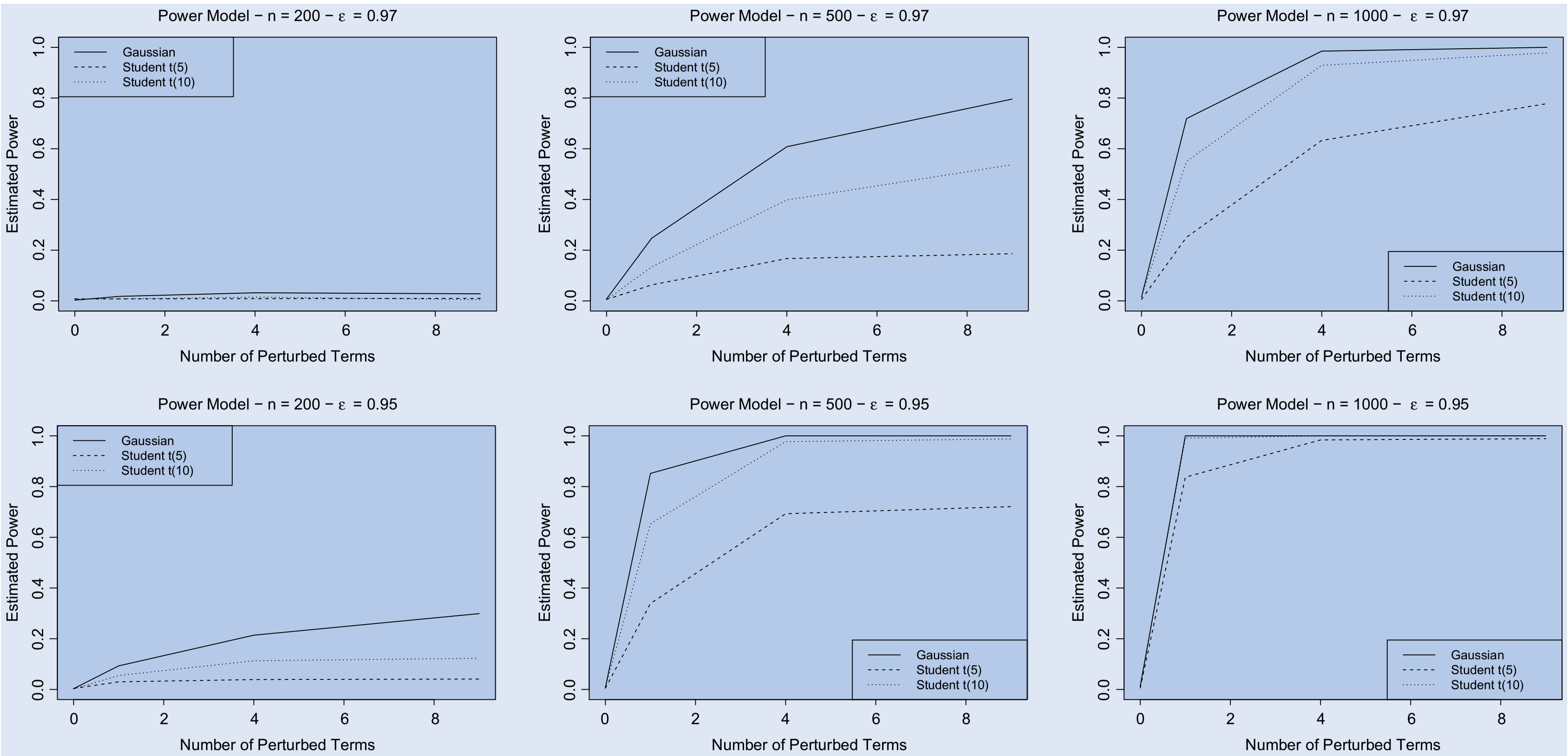

Figure 4. Estimated power of TP_3 test for Power Schoenmakers–Coffey correlation matrices with $n = 200$, 500 and 1000 and $\varepsilon = 0.97$ and 0.95, using the bootstrap estimated variance.

Table 2. Augmented Dickey–Fuller and Phillips–Perron test statistics computed on the BFV dataset.

Maturities	1Y	2Y	3Y	4Y	5Y	7Y	8Y	9Y	10Y
Augmented Dickey–Fuller test statistics									
France	−9.62	−10.68	−11.05	−11.27	−11.40	−11.41	−11.29	−11.18	−11.26
Germany	−9.57	−10.58	−10.87	−11.15	−11.26	−11.25	−11.22	−11.15	−11.29
Italy	−9.63	−10.86	−11.11	−11.44	−11.50	−11.24	−11.43	−11.33	−11.28
UK	−9.23	−10.34	−10.90	−11.05	−11.10	−10.99	−10.97	−11.19	−11.44
Phillips–Perron test statistics									
France	−1249.1	−1247.6	−1251.5	−1265.9	−1260.5	−1251.8	−1253.4	−1251.4	−1239.7
Germany	−1213.6	−1250.4	−1250.0	−1246.4	−1267.4	−1251.5	−1235.3	−1220.1	−1240.9
Italy	−1233.2	−1240.0	−1218.0	−1195.4	−1184.2	−1177.8	−1142.3	−1114.5	−1119.9
UK	−1137.8	−1103.8	−1110.0	−1137.7	−1172.7	−1224.2	−1231.6	−1228.7	−1231.4

in the north-east (and, for symmetry, south-west) corner of the original TP matrices, by multiplying each of their elements by a suitable ε, with $0 < \varepsilon < 1$:

$$\widetilde{\rho}_{i,j} = \widetilde{\rho}_{j,i} = \varepsilon \rho_{i,j} \tag{10}$$

where $i = 1, \ldots, p_1$, $j = p - p_1 + 1, \ldots, p$, with p_1 integer such that $1 \leq p_1 < p$. Since it is rather complex to obtain correlation matrices which are TP_2 but not TP_3, we limit our simulations study only to few cases and we use $p_1 = 1, 2, 3$ and $\varepsilon = 0.95$ and 0.97 (corresponding to a large and small deviation from TP_3 structure, respectively).

Differently from the one introduced in section 3.2.1, perturbation (10) produces very different effects on the compound matrices, while preserving SSC. More in detail, if one considers $\varepsilon = 0.95$ and 0.97, when p_1 increases from 1 to 3, the number of negative minors of order three $\rho_{ij}^{[3]}$ (with $i < j$) becomes 378, 580 and then 279, and the value of the smallest of them decreases for increasing p_1. The behaviours of the estimated power, obtained when we use the test statistic with variance derived under Gaussianity assumptions, are plotted against the number of perturbed terms (i.e. p_1^2) in figure 3.

As in the TP_2 case, the non-robustness of the test in case of substantial deviations from Gaussianity emerges. When we use the bootstrap estimation of the variance, we obtain the trends for estimated powers shown in figure 4, from which emerges the goodness of this approach.

4. Analysis of the case study

In this section, we perform a numerical study in order to support the effectiveness of the total positivity assumption of the yield correlation matrices by the empirical evidence. We consider two different data-sets in order to verify that our results are independent of the methodology used to generate the data since it is known that it may affect the presence of SSC (see Lekkos 2000, Alexander and Lvov 2003, Lardic *et al*. 2003, Lord and Pelsser 2007).

The first data-set used in the analysis consists of Bloomberg Fair Value (BFV) curves for sovereign bonds in four European countries (France, Germany, Italy and the UK) from 3rd January 2004 to 29th December 2008, for a total of 1305 daily yield curves. These last are derived over 15 maturities ranging from 3 months to 30 years. More in detail, the maturity spectra are: short-term (3 and 6 months), medium-term (from 1 year to 5 years and from 7 to 10 years), and long-term (15, 20, 25 and 30 years) spectra. The data are zero-coupon curves generated by a proprietary optimization model mainly based on the use of piece-wise linear functions and bootstrapping; for more details see Lee (2007).

The second data-set consists in Euro Swap data in the same period and for the same maturities of the previous case. In such a market, swaps are quoted on a daily basis, and therefore no interpolation is in principle is required: no effect due to interpolation should act. As the results illustrated below are substantially the same in the two cases, we will show explicitly only those for the first data-set.

For what concerns the methodological choices on which we based the data processing, we referred to the large literature about practical aspects of PCA on yields curves, in particular to the conclusions of Lardic *et al*. (2003), summarized in the following. First, we use daily data which guarantee more accurate results than monthly ones. Second, we work on the rate changes instead of the levels in order to have the stationarity of the marginal (univariate) time series. To confirm this, we applied to the available rate changes the classical Augmented Dickey–Fuller and Phillips–Perron tests for the null hypothesis of the presence of a unit root against the alternative of stationarity in time series (the values of the test statistics are collected in table 2): both tests lead us to systematically reject the null hypothesis of a unit root at the nominal level 5% (the analysed series p-value is about 0.01 for each maturity). Finally, we compute the correlation matrix from the data of the medium-term maturity spectrum: as claimed in the cited paper, this choice allows to obtain the maximum variance explained by the first three factors and prevent the influence of the more volatile short-term rates.

4.1. Analysis of the whole period

We begin our empirical study considering the entire set of observations, from 2004 to 2008. The implementation of the PCA on yield curves produces, as was to be expected, some loadings having the typical patterns described in section 2: looking at the plots in figure 5, we can recognize the shift, the slope and the curvature factors, respectively. Moreover, these three factors explain approximately the 99% of the total variance for all the countries as we can deduce by table 3.

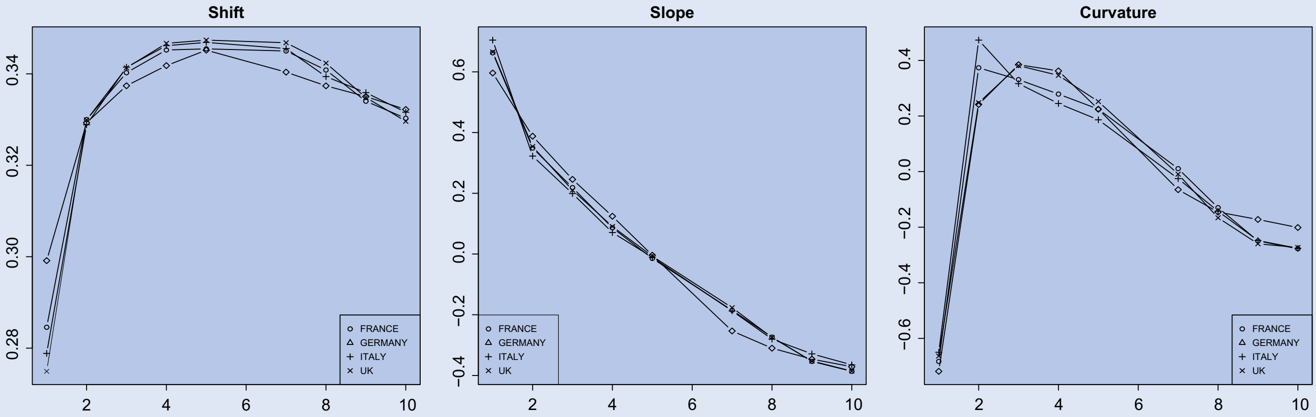

Figure 5. Sensitivities with respect to the first three factors for France, Germany, Italy the and UK.

Table 3. Cumulative percentage of variance explained by the first three factors (BFV data).

Country	Factor 1(%)	Factor 2(%)	Factor 3(%)
France	90.9	97.3	99.3
Germany	90.6	97.0	99.1
Italy	88.6	96.4	98.5
UK	91.8	97.9	99.3

Table 4. Multivariate skewness and kurtosis indices on the BFV dataset.

Country	$b_{1,9}$	$b^{\star}_{1,9}$	$b_{2,9}$	$b^{\star}_{2,9}$
France	21.55	4690.4	552.83	582.77
Germany	40.68	8853.9	509.31	526.89
Italy	90.40	19677.6	648.23	705.28
UK	32.99	7180.6	403.18	390.61

Table 5. The ordered first small six p-values for TP_2 test and the corresponding reference values $\alpha l/360$, $l = 1, 2, \ldots, 6$.

Ordered p-values						
Reference values	0.00008	0.00016	0.00024	0.00032	0.00040	0.00048
France	0.307	0.352	0.359	0.369	0.489	0.500
Germany	0.007	0.020	0.046	0.093	0.113	0.161
Italy	0.155	0.273	0.422	0.494	0.511	0.542
UK	0.608	0.660	0.819	0.850	0.860	0.861

Having found the characteristic shapes for the first three loadings, we focus on verifying the compatibility between our first data-set and the total positivity assumptions.

Before applying the test procedure, we carry out an analysis to verify if the Gaussianity of the data could be assumed: this is very important in order to decide if is reasonable to use the explicit formulas of the test statistics or their bootstrap version. To this aim, we compute the Mardia's multivariate skewness and kurtosis indices $b_{1,p}$ and $b_{2,p}$ (here $p = 9$), and we perform the goodness-of-tests based on these statistics (see Mardia 1970). The results from the data are reported in table 4: they say us that the assumption of Gaussianity cannot be accepted because $b_{1,9}$ and the difference $b_{2,9} - 99$ are widely significant (the p-values are very close to zero in all the cases). These results lead us to prefer the bootstrap approach.

We test total positivity of order $m = 2$ and $m = 3$ for the 9×9 empirical correlation matrices at the level $\alpha = 5\%$. The variances of the empirical minors are evaluated through the bootstrap procedure illustrated in section 3 with $B = 2000$ replications. Because of the strict positivity of the correlation matrices, we only consider the minors of order two and three in our test (except for the principal minors).

First, we consider the case $m = 2$: for all the countries, we accept the null hypothesis of total positivity. Indeed all the estimated ordered p-values are considerably larger than the reference values of the test $\alpha l/h_2$, with $l = 1, 2, \ldots, h_2$, $h_2 = 630$, as one can appreciate reading table 5, where the first six ordered p-values and the corresponding reference values are reported.

Combining this result with the fact that all the terms of the empirical correlation matrices of yield curves are significantly greater than zero, in the light of definition 2.2, we can conclude that the empirical evidence supports the assumption of oscillatory of order 2 of the correlation matrices.

For $m = 3$ and all treated countries, the test procedure leads to rejecting the null hypothesis of total positivity of order 3 at the level 5%. This result is supported, in all the cases, by the number of minors that are significantly negative: 47 for France, 252 for Germany, 57 for Italy and 157 for the UK, over 4116 tested minors. Figure 6 visualizes p-value levels in the 3-rd compound matrices for France and the UK: the cells show, by varying the intensity in a scale of grey, the p-value of the corresponding minors, ordered in the lexicographical way. White areas identify regions where assigned p-values are close to zero corresponding to significantly negative minors. For symmetry reasons, only the upper triangular part has been visualized.

It seems that smallest values are concentrated in some region of the compound matrices, that is, they can be related to terms having maturities in a specific part of the spectrum. As a consequence, a specific reduction of the analysed spectrum by deleting some maturities on the edge could lead to the acceptance the TP_3 hypothesis. Then, we repeated the test considering correlation matrices of size $p \times p$, with $p = 4, 5, 6$, that refer to spectra segments from 1 to 4 year, from 1 to 5 year and 1 to 7 year, respectively. The number of significantly negative minors for all cases, reported in table 6, shows that, in general, only a drastic reduction of the maturity spectrum can allow to accept the TP_3 assumption. This conclusion holds true also by taking similar windows of maturities shifted ahead. However, when the TP_3 assumption is rejected, the percentage of significantly negative minors over tested ones is confined between 1 and 6%, similarly to what happens by considering the whole spectrum.

To conclude this section, we sum up the main results obtained on Euro Swap data. As claimed before, the analysis conducted lead to substantially similar conclusions to the ones obtained on spot rates: also in this case, the Gaussianity of data has to be rejected and the application of the bootstrap version of our test leads to the acceptance of the TP_2 hypothesis, whereas for the TP_3 assumption, the test procedure is in favour of the alternative hypothesis (with 115 minors significantly negative over 4116). Also for Swap data, only a drastic contraction of the maturities spectrum could lead to accept TP_3.

4.2. Analysis of sub-periods

To complete the analysis, we investigate what happens when we use samples defined over sub-periods instead of the whole temporal window. To do this, we employ ‘rolling-samples’ with partial overlaps over time. More in detail, let $r_t = (r_{1,t}, \ldots, r_{9,t})^T$ be the observed rate changes at time t, we consider k samples of n consecutive observations each one having index $t = (j-1)\,\Delta + 1, \ldots, (j-1)\,\Delta + n$, with $j = 1, \ldots, k$ and where Δ is the number of overlapped observations $(1 \le \Delta \le n)$. Since the simulations in section 3.2 show that the TP_2 and TP_3 tests work well only for relatively large sample sizes, in our study, we used $n = 500$ to guarantee reliable results and $\Delta = 200$ to avoid too a wide overlap: thus, we obtained $k = 5$ sub-samples from the original one, having indices $t = 1, \ldots, 500$, the first one, $t = 201, \ldots, 700$, the second one and so on.

Table 6. Number of significant negative minors of order 3 varying the maturity spectrum.

Maturity spectrum	1Y–4Y	1–5Y	1Y–7Y
# tested minors	21	90	295
France	0	1	7
Germany	0	3	27
Italy	0	0	0
UK	0	1	7

For all the sub-periods, we performed the Gaussianity tests based on Mardia’s multivariate skewness and kurtosis indices (see statistics in table 7) that allow to reject the hypothesis of Gaussianity. Then, we applied the bootstrap version of the tests for TP_2 and TP_3 hypothesis: if the hypothesis TP_2 can be accepted for all the sub-samples, the TP_3 is rejected for some sub-periods but can be accepted for others (see table 7 where the number of significant negative minors of order 3 for each country and each period is reported).

The same analysis has been repeated for Swap data. Also in this case, Gaussianity has to be rejected and results of the tests for TP_2 and TP_3 confirm the above conclusions: TP_2 can be accepted for all the sub-periods, whereas TP_3 is not found only in sub-sample 2 (that is, when indices are: 201–700).

4.3. Discussion

The above study leads us to conclude that totally positive of order 2 correlations between spot (or swap) rates should be considered a standard hypothesis to require to any model of the yield curve. A first consequence is the indirect confirmation of the presence of the slope in a weak sense. A further consequence of our result is the possibility to give an indirect answer to the following conjecture presented in Lord and Pelsser (2007): a quasi-correlation matrix R with strictly positive entries displays shift and slope, if it satisfies (1), (2) and $\rho_{i,i+j} \le \rho_{i+1,i+j+1}$, i.e. the correlations increase when we move from north-west to south-east. Indeed, this conjecture was introduced in an attempt to find a condition that, referring only to the classical properties of the correlations between rates, would guarantee the presence of shift and slope (in a weak sense). Our empirical analysis shows instead that TP_2 is not an ‘ad hoc’ theoretical assumption but rather a fact, thus making it unnecessary to look for an answer to the conjecture. However, the conjecture could still be relevant if it is interpreted in the sense that it might make it easier to analyse the properties of matrices by focusing on the three mentioned properties, but dropping the requirement that the matrix we are dealing with is a proper correlation matrix.

As for the concrete meaning of the TP_2 hypothesis, following Lord and Pelsser, since correlations of interest rates changes are positive, the TP_2 condition can be expressed as

$$\frac{\rho_{jl} - \rho_{jk}}{\rho_{jl}} \ge \frac{\rho_{il} - \rho_{ik}}{\rho_{il}} \qquad \forall i \le j \text{ and } k \le l$$

equivalent to

$$\frac{\rho_{jl} - \rho_{jk}}{k - l} \cdot \frac{l}{\rho_{jl}} \ge \frac{\rho_{il} - \rho_{ik}}{k - l}\,\frac{l}{\rho_{il}}.$$

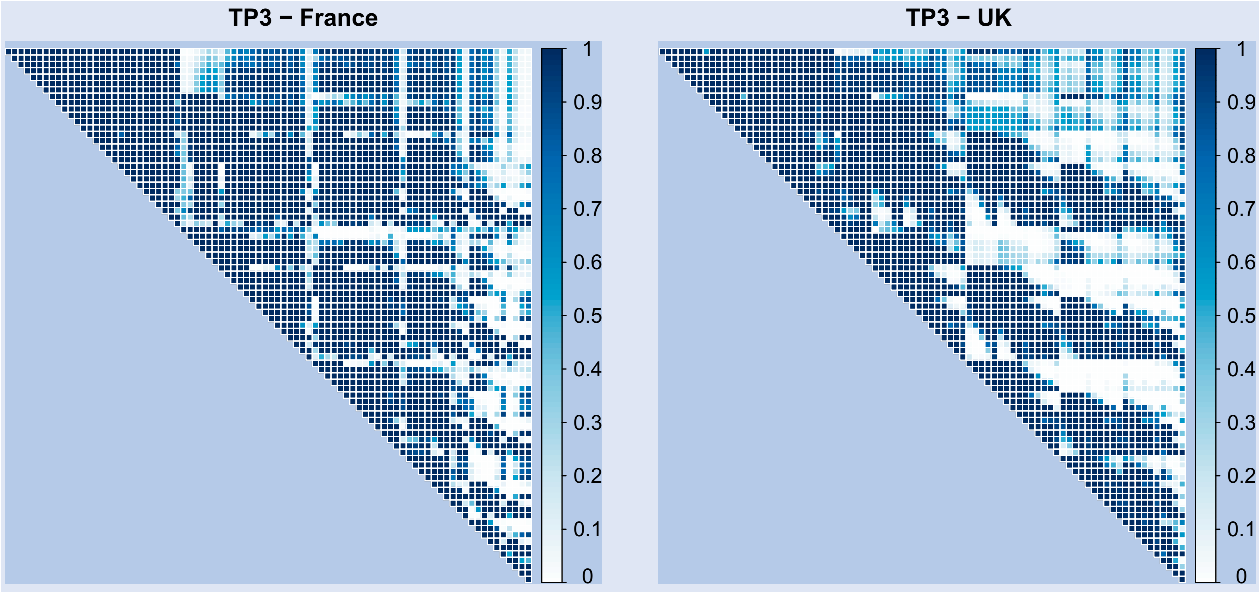

Figure 6. Evaluated p-values for TP_3 test in the 3rd compound matrices for France and the UK.

Table 7. Main results for sub-periods: multivariate skewness and kurtosis indices, and number of significantly negative minors of order 3 ($\lessdot 0$ stays for 'significantly less than zero').

Indices		Sample 1 1–500	Sample 2 201–700	Sample 3 401–900	Sample 4 601–1100	Sample 5 801–1300
France	$b_{1,9}$	30.4	18.8	30.7	45.4	25.9
	$b_{2,9}$	265.4	237.6	290.8	326.7	370.3
	$\#\left\{\widehat{\sigma}_{ij}^{(3)} \lessdot 0\right\}$	19	101	174	18	0
Germany	$b_{1,9}$	29.8	22.3	27.6	70.2	39.3
	$b_{2,9}$	279.3	272.0	256.6	366.6	322.4
	$\#\left\{\widehat{\sigma}_{ij}^{(3)} \lessdot 0\right\}$	139	64	163	172	72
Italy	$b_{1,9}$	170.3	68.7	49.8	51.1	56.4
	$b_{2,9}$	471.5	376.2	400.7	346.4	371.7
	$\#\left\{\widehat{\sigma}_{ij}^{(3)} \lessdot 0\right\}$	299	257	142	0	0
UK	$b_{1,9}$	19.2	31.4	27.1	119.5	53.2
	$b_{2,9}$	259.5	304.7	297.7	448.5	337.0
	$\#\left\{\widehat{\sigma}_{ij}^{(3)} \lessdot 0\right\}$	295	189	232	12	0

This means that the partial (discrete) elasticity of the correlation starting from l up to k decreases when tenors increase, representing a condition on the slopes of the correlation curves. A further consideration is possible in terms of partial correlation coefficients of the first order. We recall that, given three square integrable random variables X_1, X_2 and X_3, the first order partial coefficient of X_1 and X_2 given X_3 is defined by

$$\rho_{X_1X_2.X_3} = \frac{\rho_{X_1X_2} - \rho_{X_1X_3} \cdot \rho_{X_2X_3}}{\sqrt{\left(1-\rho^2_{X_1X_3}\right)\left(1-\rho^2_{X_2X_3}\right)}}.$$

The coefficients $\rho_{X_1X_3.X_2}$ and $\rho_{X_2X_3.X_1}$ are defined in the same way. It is possible to show (see theorem A1 in the appendix 3) that when the random variables X_i represent the change in interest rates corresponding to different maturities, properties (1) and (2) give the positivity of $\rho_{X_1X_2.X_3}$ when the maturity associated to X_3 is the shortest or the longest, while the TP_2 property gives the non-positivity of $\rho_{X_1X_2.X_3}$ when the maturity associated to X_3 is intermediate between those associated with X_1 and X_2.

About the TP_3 hypothesis, the provided results seem to suggest that when one considers some short periods of observations, the estimated correlation matrices could be compatible with the TP_3 assumption, while this does not happen when one takes data over a relatively long time interval. Hence, one can conjecture that the daily rate changes are drawn from a mixture of distributions having correlation structures compatible with TP_3 structure which, however, changes over time. Thus, taking a large set of data that refers to a long period, the resultant estimated correlation matrix does not preserve the TP_3 characteristics which, however, can be found in sub-samples related to shorter time windows. We observe that, however, an interpretation of the TP_3 in the same spirit showed for TP_2 is not available to our knowledge. This fact surely can represent an obstacle to financially interpreting any empirical evidence. In the case where additional empirical evidence would lead to reject the TP_3 hypothesis, its replacement with an (weaker) alternative assumption does not seem available today in the theoretical literature. In our opinion, only a supplement of theoretical investigation could give some new insights. These aspects need a deep study which goes beyond the aim of this paper. A last consideration: the robustness of our results on the TP_2 property suggests that this latter should be possessed by every 'good' model of interest rates. Evidently, this reasoning could not be applied to the TP_3 one.

5. Conclusions

In this paper, we have explored the empirical plausibility of the TP_2 and TP_3 assumptions for correlation matrices of interest rates, introduced recently from a theoretical standpoint to justify some spectral properties of these matrices. We have proposed a total positivity test for covariance and/or correlation matrices of random vectors. We have shown how to extend via bootstrap the original result holding for Gaussian population to a distribution-free framework, also performing a brief robustness analysis via simulation. Using our test on real data, we concluded that the TP_2 hypothesis has to be systematically accepted, while for the TP_3 assumption, the results signal a more complex situation which seems to depend on the one hand, on the analysed spectra segment and, on the other, on the temporal window in which the data lie. This opens the way to a deeper future investigation of the TP_3 assumption.

The possible extensions of our work are several. On one side, it would be interesting to look for further empirical evidence, possibly based on different and larger data-sets, of our results. Moreover, it seems to be interesting to perform a complete theoretical study of the properties of our TP test both in the Gaussian case and for more general situations, as suggested by the simulations presented. A deeper mathematical investigation devoted to optimize the number of minors involved in the TP test might reduce or eliminate dependencies between the marginal tests. Finally, it could be useful to extend our TP test to the functional statistic framework where the empirical problem treated here finds its natural analogue.

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Disclosure statement

No potential conflict of interest was reported by the authors.

References

Alexander, C. and Lvov, D., Statistical properties of forward LIBOR rates. Working Paper, University of Reading, 2003.

Benjamini, Y. and Hochberg, Y., Controlling the false discovery rate: A practical and powerful approach to multiple testing. *J. Roy. Stat. Soc.: Ser. B*, 1995, **57**, 289–300.

Drton, M. and Goia, A., Correction: Moments of minors of Wishart matrices. *Ann. Stat.*, 2012, **40**, 1283–1284.

Drton, M., Massam, H. and Olkin, I., Moments of minors of Wishart matrices. *Ann. Stat.*, 2008, **36**, 2261–2283.

Drton, M., Sturmfels, B. and Sullivan, S., Algebraic factor analysis: Tetrads, pentads and beyond. *Prob. Theory Rel. Fields*, 2007, **138**, 463–493.

Efron, B. and Tibshirani, R., Bootstrap methods for standard errors, confidence intervals, and other measures of statistical accuracy. *Stat. Sci.*, 1986, **1**, 54–77.

Efron, B. and Tibshirani, R., *An Introduction to the Bootstrap*, 1993 (Chapman and Hall: New York).

Fallat, S.M. and Johnson, C.R., *Totally Nonnegative Matrices*, 2011 (Princeton University Press: Princeton, NJ).

Forzani, L. and Tolmasky, C., A family of models explaining the level-slope-curvature effect. *Int. J. Theor. Appl. Finance*, 2003, **6**(3), 239–255.

Gantmacher, F.R., *The Theory of Matrices*, 1964 (Chelsea Publishing Company: New York).

Gantmacher, F.R. and Krein, M.G., *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*, 1961 (Department of Commerce: Washington).

Golub, B.W. and Tilman, L.M., Measuring yield curve risk using principal components analysis, value at risk, and key rate durations. *J. Portfolio Manage.*, 1997, **23**(4), 72–94.

Ho, T. and Lee, S., Term structure movements and pricing interest rate contingent claims. *J. Finance*, 1986, **41**, 1011–1029.

Hull, J. and White, A., Pricing interest-rate derivative securities. *Rev. Financ. Stud.*, 1990, **3**, 573–592.

Jolliffe, I.T., *Principal Component Analysis*, 2004 (Springer-Verlag: Berlin).

Karlin, S., *Total Positivity*. Vol. 1, 1968 (Stanford University Press: Stanford, CA).

Lardic, S., Priaulet, P. and Priaulet, S., PCA of the yield curve dynamics: Questions of methodologies. *J. Bond Trading Manage.*, 2003, **1**, 327–349.

Lee, M., Bloomberg fair value market curves. Bloomberg L.P. presented at International Bond Market Conference 2007, Taipei, 2007.

Lekkos, I., A critique of factor analysis of interest rates. *J. Derivatives*, 2000, **8**(1), 72–83.

Litterman, R. and Scheinkman, J., Common factors affecting bond returns. *J. Fixed Inc.*, 1991, **1**, 54–61.

Longstaff, F., Santa-Clara, P. and Schwartz, S., The relative valuation of caps and swaptions: Theory and empirical evidence. UCLA Working Paper, 1999.

Lord, R. and Pelsser, A., Level, slope and curvature: Art or artefact? *Appl. Math. Finance*, 2007, **14**(2), 105–130.

Mardia, K.V., Measures of multivariate skewnees and kurtosis with applications. *Biometrika*, 1970, **57**(3), 519–530.

Martellini, L., Priaulet, P. and Priaulet, S., *Fixed-income Securities: Valuation, Risk Management and Portfolio Strategies*, 2003 (Wiley: Chichester).

Rebonato, R., *Modern Pricing of Interest-rate Derivatives*, 2002 (Princeton University Press: Princeton, NJ).

Salinelli, E. and Sgarra, C., Correlation matrices of yields and total positivity. *Linear Algebra Appl.*, 2006, **418**(1–2), 682–692.

Salinelli, E. and Sgarra, C., Shift, slope and curvature for a class of yields correlation matrices. *Linear Algebra Appl.*, 2007, **426**(1–2), 650–666.

Salinelli, E. and Sgarra, C., Some results on correlation matrices for interest rates. *Acta Appl. Math.*, 2011, **115**, 291–318.

Schoenmakers, J.G.M. and Coffey, B., Systematic generation of parametric correlation structures for the LIBOR market model. *Int. J. Theor. Appl. Finance*, 2003, **6**, 507–519.

Schriever, B.F., *Ordering Properties in Correspondence Analysis*, report SW80, 1982 (Mathematisch Centrum: Amsterdam).

Simes, R.J., An improved Bonferroni procedure for multiple tests of significance. *Biometrika*, 1986, **73**, 751–754.

Steeley, J.M., Modelling the dynamics of the term structure of interest rates. *Econ. Soc. Rev.*, 1990, **21**(4), 337–361.
Vasicek, O., An equilibrium characterization of the term structure. *J. Financ. Econ.*, 1977, **5**, 177–188.
Willner, R., A new tool for portfolio managers: Level, slope, and curvature durations. *J. Fixed Inc.*, 1996, **6**(1), 48–59.

Appendix 1. Moments of minors of Wishart matrices

Consider a p-dimensional random vector $\mathbf{X} \sim \mathcal{N}_p(0, \Sigma)$, with $p \geq 2m$ ($m \geq 1$ integer), with positive definite covariance matrix Σ. Let $\mathcal{X} = (X_{ij}) \in \mathbb{R}^{p\times n}$ be a matrix whose columns are independent random vectors distributed according to $\mathcal{N}_p(0, \Sigma)$. Then $S = \mathcal{X}\mathcal{X}^T$ is distributed according to the Wishart distribution with scale parameter Σ and n degrees of freedom.

In Drton *et al.* (2008) the first and second moments of the minors of S are studied. The minors are specified by two subsets $I, J \subseteq [p] = \{1, \ldots, p\}$ of equal cardinality $|I| = |J| = m$. We denote with

$$\left\{ \begin{matrix} p \\ m \end{matrix} \right\} = \{I \subseteq [p] : |I| = m\}, \qquad m \in [p]$$

the set constituted by all the subsets of $[p]$ of size m.

In Drton and Goia (2012) and Drton *et al.* (2008), the authors prove the following results (we adopt the shorthand notation $S_{I\times I} = S_{II}$ and $S_{I\times J} = S_{IJ}$):

- Let $I, J \in \left\{ \begin{smallmatrix} p \\ m \end{smallmatrix} \right\}$ and $n \geq m$, then
$$\mathbb{E}\left[S^{[m]}\right] = \frac{n!}{(n-m)!}\Sigma^{[m]}.$$
- If $I \in \left\{ \begin{smallmatrix} p \\ m \end{smallmatrix} \right\}$, then
$$Var\left[\det(S_{II})\right] = \frac{n!}{(n-m)!}\left\{\frac{(n+2)!}{(n+2-m)!} - \frac{n!}{(n-m)!}\right\} \times \det(\Sigma_{II})^2.$$
- Let $I, J \in \left\{ \begin{smallmatrix} p \\ m \end{smallmatrix} \right\}$ be two disjoint subsets. Then the off-diagonal minor $\det(S_{IJ})$ has variance
$$\begin{aligned} &Var\left[\det(S_{IJ})\right] \\ &= \frac{n!}{(n-m)!}\det(\Sigma_{IJ})^2\left\{\frac{(n+2)!}{(n+2-m)!} - \frac{n!}{(n-m)!}\right\} \\ &\quad + \frac{n!}{(n-m)!}\det\left(\Sigma_{IJ\times IJ}\right) \\ &\quad \times \left[\sum_{k=0}^{m-1}(m-k)!\frac{(n+2)!}{(n+2-k)!}(-1)^k \right. \\ &\quad \left. \mathrm{tr}\left\{\left(\Sigma_{IJ}\Sigma^{IJ}\right)^{[k]}\right\}\right] \end{aligned}$$
where $\Sigma_{IJ\times IJ}$ is the $(I\cup J)\times(I\cup J)$-submatrix of Σ and Σ^{IJ} denotes the $I\times J$-submatrix of the inverse of $\Sigma_{IJ\times IJ}$.
- Let $I, J \in \left\{ \begin{smallmatrix} p \\ m \end{smallmatrix} \right\}$ have intersection $C := I \cap J$ of cardinality $|C| = c$. Define $\overline{I} = I \setminus C$, $\overline{J} = J \setminus C$ and $\overline{IJ} = \overline{I} \cup \overline{J}$. Then the minor $\det(S_{IJ})$ has variance
$$\begin{aligned} &Var\left[\det(S_{IJ})\right] \\ &= \det(\Sigma_{IJ})^2\frac{n!}{(n-m)!}\left[\frac{(n+2)!}{(n+2-m)!} - \frac{n!}{(n-m)!}\right] \\ &\quad + \det(\Sigma_{CC})^2\det(\bar{\Sigma}_{\bar{I}\bar{J}\times\bar{I}\bar{J}})\frac{(n+2)!}{(n+2-c)!}\cdot\frac{n!}{(n-m)!} \\ &\quad \times \left[\sum_{k=0}^{m-c-1}\frac{(m-c-k)!(n+2-c)!}{(n+2-c-k)!}(-1)^k \right. \\ &\quad \left. \mathrm{tr}\left\{\left(\bar{\Sigma}_{\bar{I}\bar{J}}\bar{\Sigma}^{\bar{I}\bar{J}}\right)^{[k]}\right\}\right], \end{aligned}$$
where $\overline{\Sigma} = \Sigma_{([r]\setminus C)\times([r]\setminus C)} - \Sigma_{([r]\setminus C)\times C}\Sigma^{-1}_{C\times C}\Sigma_{C\times([r]\setminus C)}$.

Appendix 2. Simulation study: the logarithmic model

In the following, we collect the results of the simulation study conducted over the same experimental conditions of section 3.2, when the correlation structures are defined according to the logarithmic model $b_i = \log(i+\beta)$, $\beta > 1$, with $\beta = 5$.

We consider first the test for TP_2 assumption. The smallest minors $\rho^{[2]}_{ij}$ for the various experimental conditions considered are showed in table A1.

The power comparisons, when we use the test statistic (9) exploiting the expression of the variance derived explicitly under condition of Gaussianity, are presented in figure B1.

Comparing these plots with those of section 3.2 (see figure 1), it emerges that the behaviour of the test looks moderately related to the model which defines the correlation structure: the increase of the power with Q in the so-called 'power model' case is slightly slower than in the 'logarithmic' one, and that occurs because to perturb the 'power model' generates a more rapid departure from the TP_2 structure (see table A1). The behaviour of the estimated level and power in the bootstrap case is shown in figure B2.

For what concerns the TP_3 case, we observe that when one perturbs the matrices generated according to the 'logarithmic model', an increase of p_1 does not produce a substantial departure from the initial TP_3 condition: if from one hand, the number of considered negative minors $\rho^{[3]}_{ij}$ exhibits a behaviour similar to the 'power case', on the other, their values tend to have an average decreasing with p_1 but, at same time, their dispersion becomes smaller (and the minimum values increase). Figures B3 and B4 illustrate the power results in both the Gaussian and bootstrap cases. By a comparison with figures 3 and 4 in section 3.2, one can observe the greater sensitivity to the behaviour of the minors of order three in the 'logarithmic' case compared to the 'exponential' one.

Appendix 3. TP_2 and partial correlation

THEOREM A1 *The first order partial correlation coefficients*

$$\rho_{i_1 i_2 . i_3} = \frac{\rho_{i_1 i_2} - \rho_{i_1 i_3}\cdot\rho_{i_2 i_3}}{\sqrt{\left(1-\rho^2_{i_1 i_3}\right)\left(1-\rho^2_{i_2 i_3}\right)}}$$

of a TP_2 correlation matrix $R = [\rho_{ij}]_{i,j=1,\ldots,p}$ with $\rho_{ij} \in (0,1)$ for $i \neq j$, are positive if $i_1 < i_2 < i_3$ or $i_3 < i_1 < i_2$, negative for $i_1 < i_3 < i_2$.

Proof Since R is TP_2 and $\rho_{ij} \in (0,1)$, then (1) and (2) hold true. If $i_1 < i_2 < i_3$, then (1) implies $\rho_{i_1 i_2} - \rho_{i_1 i_3} > 0$ and since $\rho_{i_2 i_3} \in (0,1)$, we obtain $\rho_{i_1 i_2 . i_3} > 0$. The same conclusion is true when $i_3 < i_1 < i_2$ because (2) gives $\rho_{i_1 i_2} - \rho_{i_2 i_3} > 0$ and $\rho_{i_1 i_3} \in (0,1)$. Finally, if $i_1 < i_3 < i_2$, then (1) and (2) give, respectively, $\rho_{i_1 i_2} - \rho_{i_1 i_3} < 0$ and $\rho_{i_1 i_2} - \rho_{i_2 i_3} < 0$, hence the sign of $\rho_{i_1 i_2 . i_3}$ is not predictable. However, since in this case,

$$\rho_{i_1 i_2} - \rho_{i_1 i_3}\cdot\rho_{i_2 i_3} = -\det\begin{pmatrix} \rho_{i_1 i_3} & \rho_{i_1 i_2} \\ 1 & \rho_{i_2 i_3} \end{pmatrix}$$

the TP_2 assumption gives $\rho_{i_1 i_2 . i_3} \leq 0$.
Notice that this last inequality would be strict if R were STP_2. □

Table A1. Number of negative minors of order two $\rho_{ij}^{[2]}$ and the smallest $\rho_{ij}^{[2]}$ $(\times 10^{-2})$ varying p_1.

p_1	Q	$\#\left\{\rho_{ij}^{[2]} < 0\right\}$ $(i < j)$	Logarithmic model $\varepsilon = 0.4$	$\varepsilon = 0.6$	$\varepsilon = 0.8$
1	1	28	−0.79	−1.18	−1.57
2	3	57	−0.90	−1.35	−1.80
3	6	71	−1.05	−1.57	−2.09
4	10	65	−1.24	−1.86	−2.47
5	15	45	−1.49	−2.24	−2.99
6	21	23	−1.85	−2.77	−3.70
7	28	7	−2.37	−3.55	−4.73

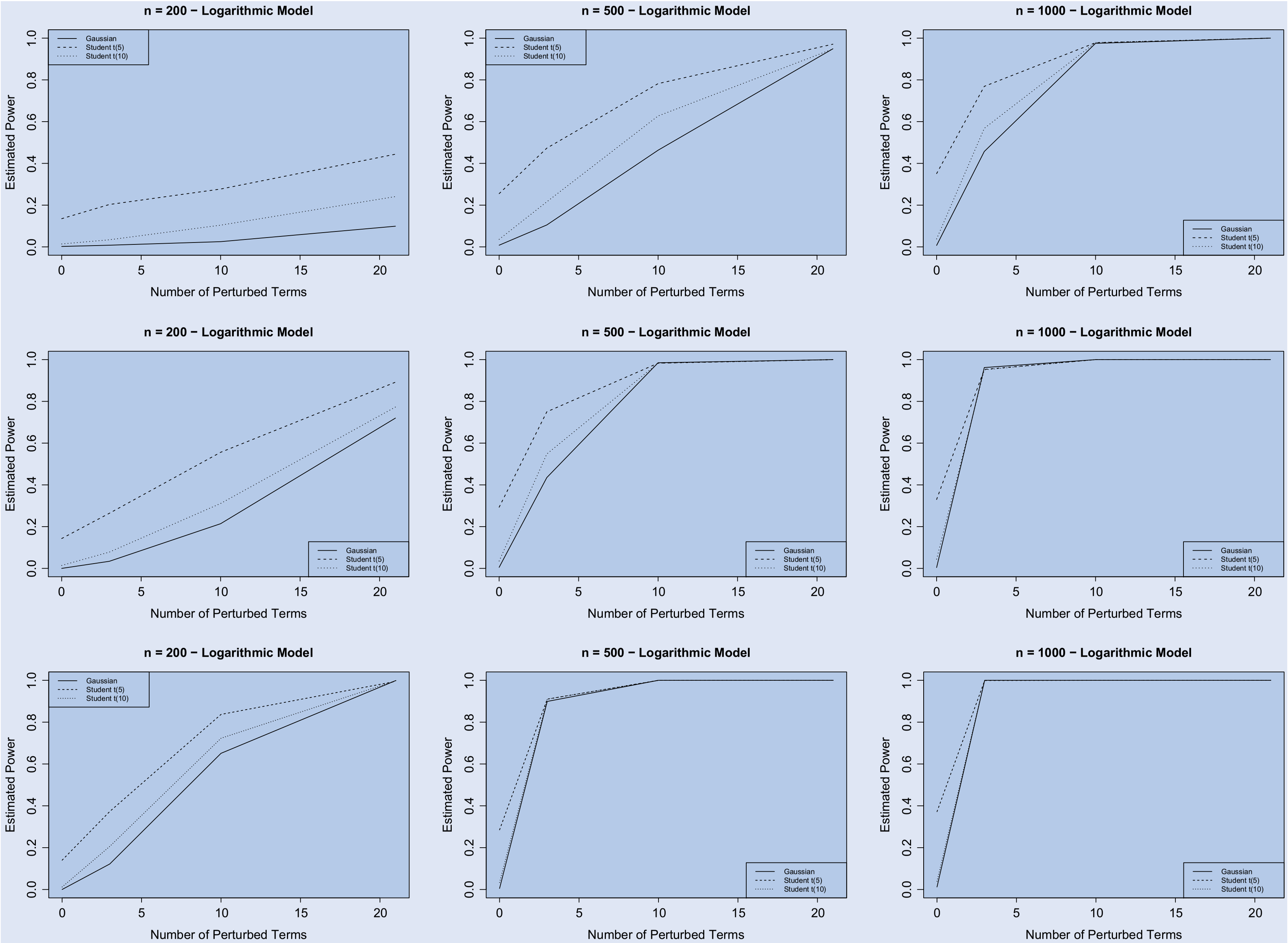

Figure B1. Estimated power of TP_2 test for logarithmic Schoenmakers–Coffey correlation matrices with $n = 200$, 500 and 1000 and $\varepsilon = 0.4$, 0.6 and 0.8.

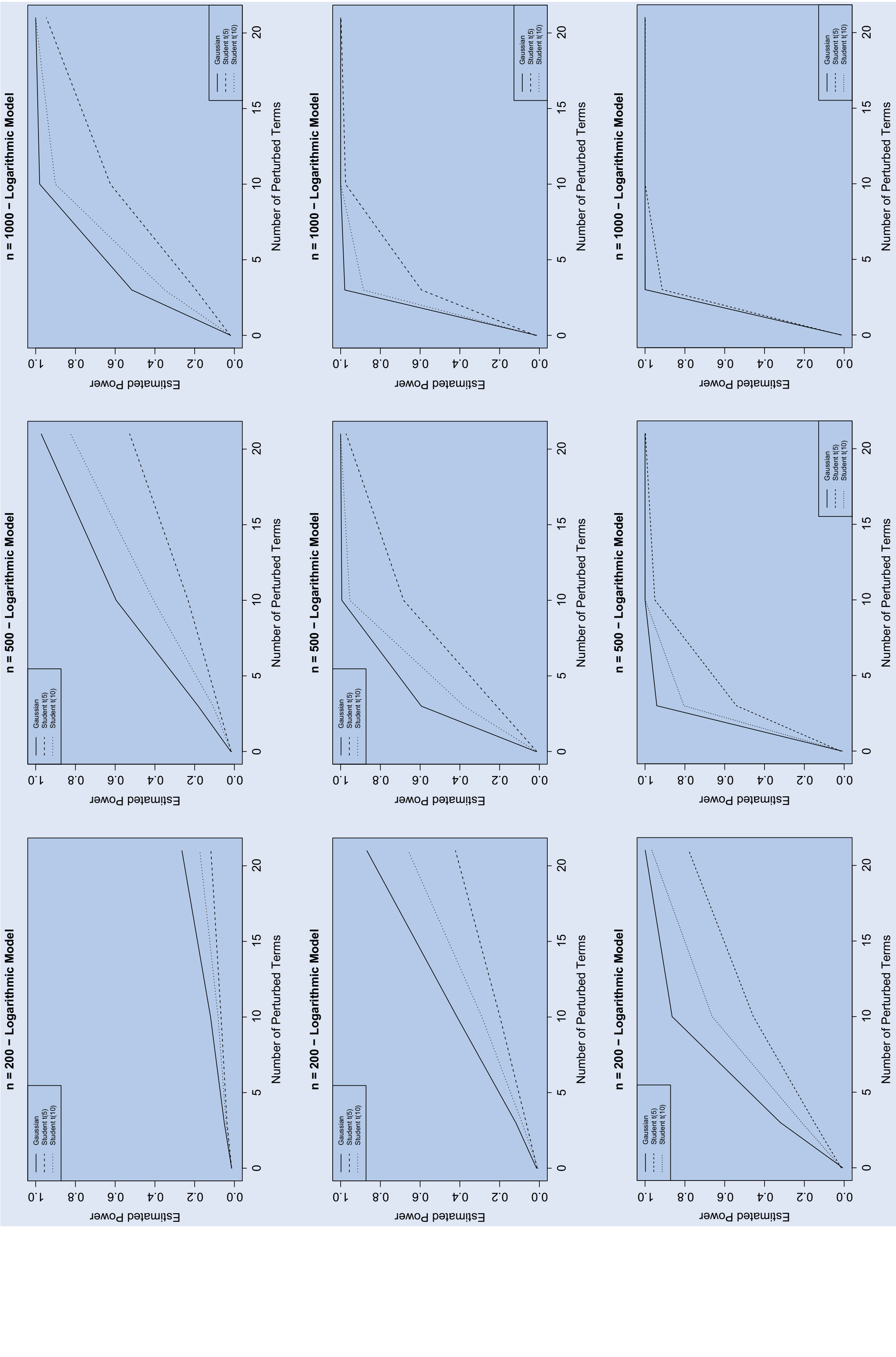

Figure B2. Estimated power of TP_2 test for logarithmic Schoenmakers–Coffey correlation matrices with $n = 200$, 500 and 1000 and $\varepsilon = 0.4$, 0.6 and 0.8 with the standard error estimated by bootstrap.

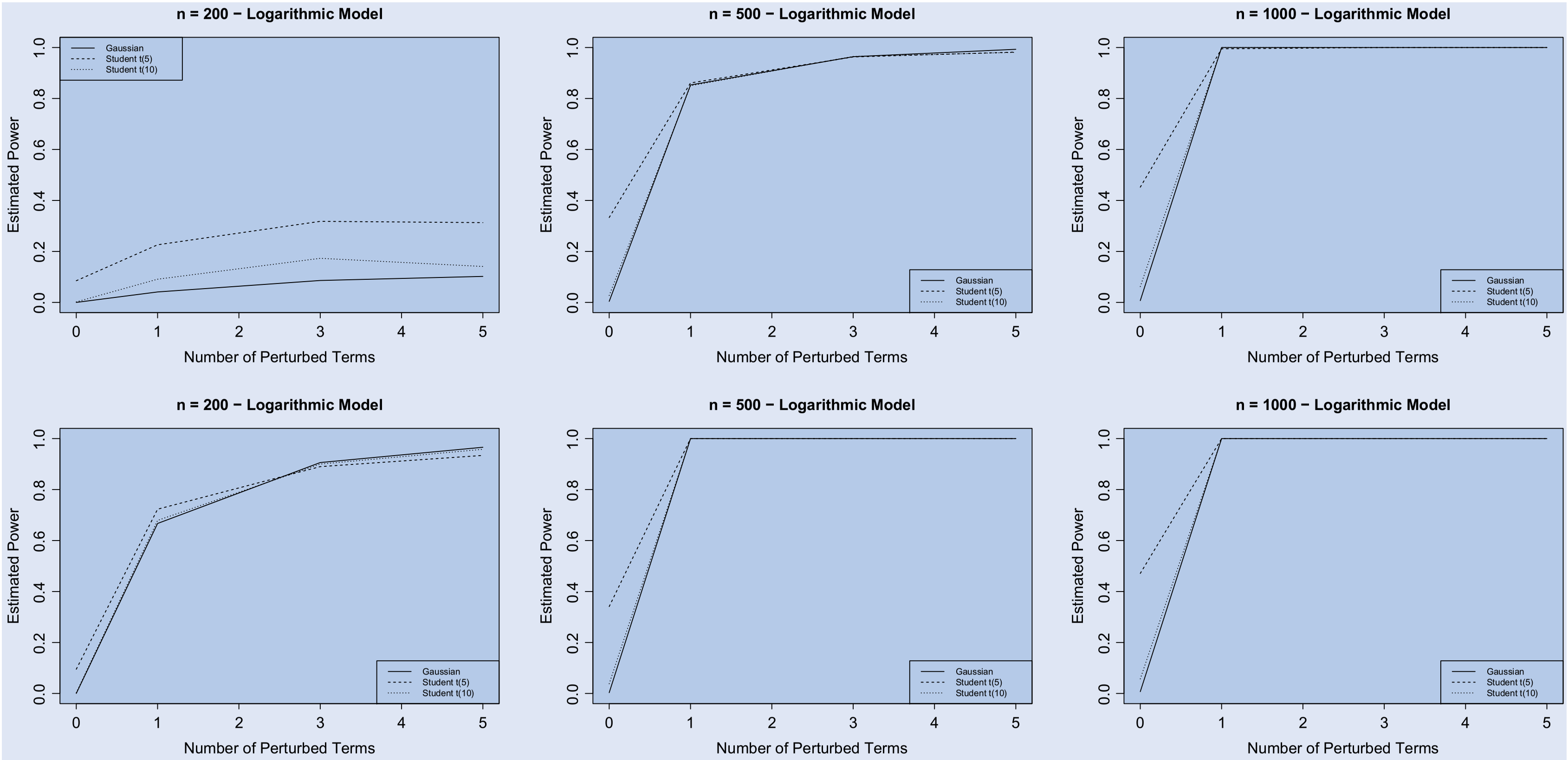

Figure B3. Estimated power of TP_3 test for Logarithmic Schoenmakers–Coffey correlation matrices with $n = 200, 500$ and 1000 and $\varepsilon = 0.97$ and 0.95 using the variance computed for Gaussian population.

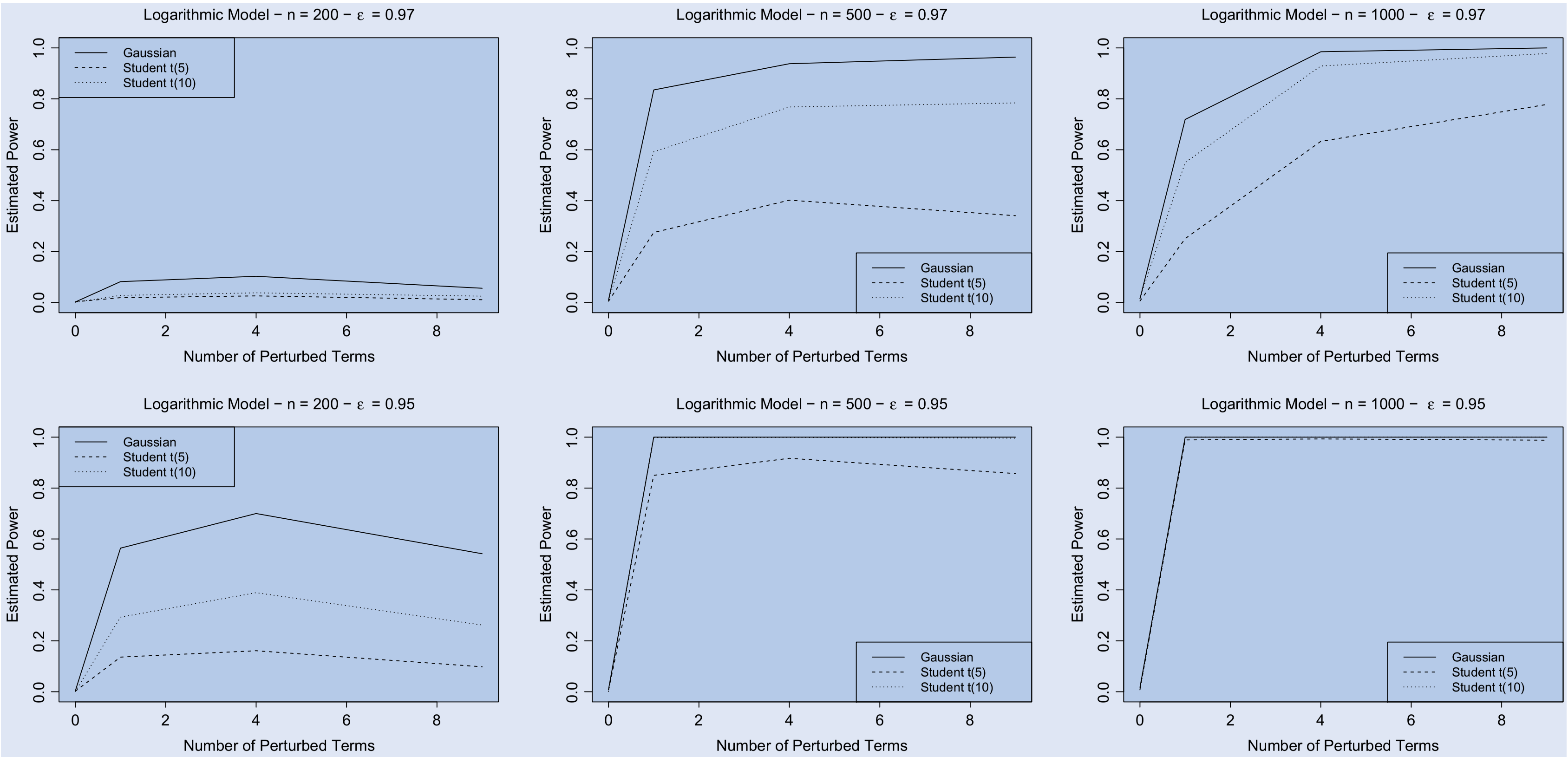

Figure B4. Estimated power of TP_3 test for logarithmic Schoenmakers–Coffey correlation matrices with $n = 200$, 500 and 1000 and $\varepsilon = 0.97$ and 0.95, using the bootstrap estimated variance.