Piecewise constant subsolutions for the incompressible Euler and IPM equations

(Stückweise konstante Sublösungen für die inkompressiblen Euler und IPM Gleichungen)

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Contents

1 Introduction

1.1 Convex integration for the incompressible Euler equations

For more than 250 years, mathematicians study the incompressible Euler equations and there is still a substantial amount of ongoing work ahead. This set of nonlinear partial differential equations has been derived by Leonhard Euler and describes the motion of a velocity field v acting under the force of a pressure p. In this representation, viscosity effects are neglected. For some given initial data $v_0 : \mathbb{R}^n \to \mathbb{R}^n$ satisfying div $v_0 = 0$ and a positive time $T > 0$, the equations read

$$\partial_t v + \text{div } (v \otimes v) + \nabla p = 0 \tag{1.1}$$

$$\text{div } v = 0 \tag{1.2}$$

$$v(x, 0) = v_0(x) \tag{1.3}$$

in $\mathbb{R}^n \times [0, T]$. An extensive literature about classical solutions $(v, p) \in C^1(\mathbb{R}^n \times [0, T]; \mathbb{R}^n)$ has been established, for example we refer to [63] and [62] as good references about the topic. In two dimensions, global existence and uniqueness of solutions was proved. However for $n = 3$, the results mainly restrict to local well-posedness [58] and the blow-up criterion of Beale, Kato and Majda [7].

For this reason, and also motivated by mathematical physics like the theory of turbulence, one deals with weak solutions. We call $v \in L^2_{loc}(\mathbb{R}^n \times (0, T))$ a weak solution of (1.1)-(1.3) if

$$\int_0^T \int_{\mathbb{R}^n} v \ \partial_t \phi + (v \otimes v)\nabla\phi \ dxdt + \int_{\mathbb{R}^n} v_0(x)\phi(x, 0) \ dx = 0, \tag{1.4}$$

$$\int_{\mathbb{R}^n} v(x)\nabla\psi(x) \ dx = 0, \tag{1.5}$$

hold true for each $\phi \in C_c^\infty(\mathbb{R}^n \times [0, T))$ with div $\phi = 0$ and each $\psi \in C_c^\infty(\mathbb{R}^n)$.

Scheffer ([70]) and later on Shnirelman ([72]) were the first one observing the main drawback of weak solutions, namely its nonuniqueness. In particular, they could show the existence of nontrivial weak solutions to (1.1)-(1.3) with $v_0 = 0$, which have compact support in space und time. Clearly, such a solution does physically make no sense, since it violates the conservation of energy. Another striking result was given by Shnirelman in [73], where he constructed energy decreasing weak solutions, which belong to the energy space $L^\infty(0, \infty; L^2(\mathbb{R}^3))$. In [31], De Lellis and Szekelyhidi deduced the same results using the very different concept of convex integration. Furthermore the results could be recovered in any dimension and with bounded velocity and pressure.

The method of convex integration is based on Gromov's h-principle ([50]), which originally deals with nonuniqueness phenomena in differential geometry. A famous example is the celebrated Nash-Kuiper theorem, which shows a high flexibility of isometric emmbeddings of the sphere S^2 into $\mathbb{R}^3$ ([66], [55]). The proof relies on an iteration scheme, which in each step adds oscillations of high frequency on top of a suitable subsolution. Moreover, these oscillations are chosen to be one-dimensional in order to integrate them and thus leading to the name convex integration. Gromov generalized this technique to prove the h-principle in a wide class of geometric problems.

Surprisingly, it turned out that these ideas also apply to various problems in fluid dynamics. In [65], Müller and Sverak used the convex integration method combined with the general framework of Tartar to generate Lipschitz solutions of differential inclusions. The approach of Tartar addresses linear partial differential systems, which are coupled with nonlinear constraints. The essential part investigates plane-wave solutions in the phase space. I will explain this method in further details in Section 2.1. There exist many other approaches in the literature for the implementation of convex integration, such as in [27] or [53], [54], where Baire category arguments are used.

An application to nonlinear PDE was firstly given in [31] by De Lellis and Székelyhidi for the incompressible Euler equations. Following Tartar's framework, one starts to rewrite (1.1)-(1.3) as a system of linear PDE together with a nonlinear constraint. Motivated by the concept of Reynolds stress from the theory of turbulence, the notion of a subsolution for the incompressible Euler equations has been introduced, see also [32], [34], [78], [80]. Roughly speaking, we define a subsolution as a triple $(\overline{v}, \overline{u}, \overline{q})$ satisfying

$$\partial_t \overline{v} + \operatorname{div} \overline{u} + \nabla \overline{q} = 0 \tag{1.6}$$

$$\operatorname{div} \overline{v} = 0 \tag{1.7}$$

$$\overline{v} \otimes \overline{v} - \overline{u} \leq \overline{e} \cdot \mathrm{Id}, \tag{1.8}$$

where the energy density $\overline{e} \geq 0$ is a given function. In case of equality in (1.8), the velocity $\overline{v}$ is a weak solution of (1.1)-(1.2). Given a strict subsolution, that is, with strict inequality in (1.8), one can follow the idea of the proof of the Nash-Kuiper theorem and deduce the existence of infinitely many weak solutions (v, p) to (1.1)-(1.2) satisfying $\frac{1}{2}|v|^2 = \overline{e}$ for a.e. (x, t). This Subsolution Criterion of De Lellis and Szekelyhidi (see Theorem 2.11) leads to many interesting conclusions, such as an improvement of the results of Scheffer and Shnirelman (Theorem 2.13) or an global existence statement for weak solutions (Theorem 2.14). These topics will be discussed in more detail in Section 2.2. Beyond the incompressible Euler equations, the convex integration method also applies to a broad range of different nonlinear PDE, for example active scalar equations [74] or the compressible Euler equations [19], [18], [20].

Clearly, each classical C^1 solution of (1.1)-(1.3) satisfies

$$\frac{1}{2}\partial_t |v|^2 + \operatorname{div} \left(v \left(\frac{1}{2}|v|^2 + p \right) \right) = 0, \tag{1.9}$$

which can be checked by a straighforward calculation. This equation expresses the conservation of energy in the local form. An integration in space yields the global version

$$\frac{d}{dt} \int_{\mathbb{R}^n} \frac{|v(x,t)|^2}{2} dx = 0. \tag{1.10}$$

In the following we define the energy by

$$E(t) := \int_{\mathbb{R}^2} \frac{1}{2} |v(x,t)|^2 dx.$$

In accordance with energy inequalities for weak (Leray) solutions of the Navier-Stokes equations, in the vanishing viscosity limit we conclude several forms of energy inequalities for (1.1)-(1.3). Thus, a weak solution v of (1.1)-(1.3) satisfies the weak energy inequality if

$$E(t) \leq E(0) \text{ for all } t \geq 0. \tag{1.11}$$

and the strong energy inequality if

$$E(t) \leq E(s) \text{ for all } t \geq s \geq 0. \tag{1.12}$$

In situations, where $E(t)$ is not finite, it is more appropriate to use the local form, which was introduced by Duchon and Robert in [43]. Let $v \in L^3_{loc}(\mathbb{R}^n)$ be a weak solution of (1.1)-(1.3). Then, v satisfies the local energy inequality, provided that

$$\frac{1}{2}\partial_t |v|^2 + \text{div } \left(v \left(\frac{1}{2}|v|^2 + p \right) \right) \leq 0 \tag{1.13}$$

holds distributionally. Here, we require $v \in L^3_{loc}(\mathbb{R}^n)$, since then the equation for the pressure

$$\Delta p = \text{div div } (v \otimes v) \tag{1.14}$$

implies $p \in L^{\frac{3}{2}}_{loc}(\mathbb{R}^n)$ thanks to Calderon-Zygmund estimates. This yields $pv \in L^1_{loc}(\mathbb{R}^n)$, so that (1.13) is well-defined. Moreover, we say that v satisfies the (local) energy equality, if equality holds in (1.13) respectively (1.11).

A major consequence of the convex integration method is the nonuniqueness of weak solutions for (1.1)-(1.3) and the appearance of nonphysical solutions. In particular, it is possible to achieve full control on the energy, in the sense that for any positive continuous function $e : [0, \infty) \to \mathbb{R}$, there exist weak solutions (v, p) of (1.1)-(1.2) such that the corresponding energy $E(t)$ equals to e, see [31], [32]. This demonstrates a very wild behaviour of weak solutions. In this context, an interesting problem is the construction of Hölder continuous weak solutions $v \in C^{0,\theta}, \theta > 0$ via convex integration and the related conjecture of Onsager. The claim is that for each $\theta > \frac{1}{3}$, weak solutions conserve the energy, whereas for $0 < \theta < \frac{1}{3}$, there exist weak solutions for which the conservation of energy is not valid. Recently, this conjecture could be fully resolved in three dimensions, see [13], [30], [51].

To single out physically relevant solutions, a possible selection criteria is admissibility. We say that a weak solution is admissible, if (1.11) holds. In [59], p.153ff, Lions proposed another notion of a solution, so called dissipative solutions with the aim to obtain a form of weak-strong uniqueness in the following sense. Suppose that $v \in C([0,T]; L^2(\mathbb{R}^n))$ is a solution of (1.1)-(1.3) such that $(\nabla v + \nabla v^T) \in L^1([0,T]; L^\infty(\mathbb{R}^n))$. Then, each dissipative solution of (1.1)-(1.3) equals to v on $\mathbb{R}^n \times [0,T]$.

In fact, each admissible weak solution is a dissipative solution. Hence, despite the high flexibility of weak solutions, the condition of admissibility singles out the classical solution as long as it exists. This statement can even be refined to the weaker notion of admissible measure-valued solutions, see [12] and also [84].

Thus, admissibility serves as a useful selection principle, as it captures the classical solution and leads to uniqueness. Nevertheless, in the case that no classical solution exists, this statement fails. There exists a delicate class of velocity fields, so called wild initial data, which admit infinitely many admissible weak solutions. Obviously, such initial data can not be regular due to classical local existence of strong solutions and the weak-strong uniqueness. In [81], it was shown that the set of wild initial data is dense in the space of divergence-free L^2 vector fields, we also refer to Theorem 2.17 in Section 2.2. Recent results about density of Hölder continuous wild initial data can be found in [29], [30].

1.2 Vortex Sheet initial data

Appropriate candidates for wild initial data are vortex sheets. This type of velocities has a large importance in physics as it represents a well-established model for fluid interfaces, mixing layers or flows past an obstacle. In this thesis I will study the 2D case, although there are also some results in three dimensions available, see for example [15]. By a vortex sheet we mean a divergence-free velocity field $v_0 \in L^2_{loc}(\mathbb{R}^2)$ with corresponding vorticity $\omega_0 = \operatorname{curl} v_0$ being a finite Radon measure concentrated on a curve $\Gamma \subset \mathbb{R}^2$. Hence, v_0 describes the motion of a fluid, which moves with different tangential velocities along some interface Γ. Due to incompressibility, the normal component of v_0 is continuous in the whole $\mathbb{R}^2$, whereas the tangential component has a jump across Γ. In the two domains, which are separated by Γ, the flow is incompressible and irrotational.

Given some vortex sheet v_0, there are two different perspectives to describe the evolution of v_0 in time ([60],[85]). In the implicit approach we understand the interface as a solution of a PDE, in our case of the incompressible Euler equations, which carries the interface with it. Roughly speaking, we try to determine a (unique) weak solution of (1.1)-(1.3) with vortex sheet initial data v_0. In [35], Delort showed the remarkable result that for each initial vorticity $\omega_0 \in H^{-1}_{loc}(\mathbb{R}^2)$ of distinguished sign there exists a global weak solution. Later on, this result could be slightly improved to the case of mirror-symmetric flows in [61]. However, neither we know if these solutions are unique, nor we have any information about their structure, i.e. whether the vortex sheet structure remains for positive times.

The explicit approach characterizes the propagation of the interface via a time-dependent parametrization. In particular, we suppose that the initial vortex sheet remains a vortex sheet also for later times. To be more precise, let the interface $\Gamma(t)$ given as the graph of a function $x_2 = z(x_1, t)$ and let $\gamma(x_1, t)$ be the vorticity density given on this curve. Thus, the vorticity distribution is given by

$$\omega(x_1, x_2, t)dx_1dx_2 = \gamma(x_1, t)\delta(x_2 - z(x_1, t)).$$

From the Biot-Savart law we deduce for the corresponding velocity outside of the sheet $x_2 \neq z(x_1, t)$

$$v(x_1, x_2, t) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{(-x_2 + z(x_1'), x_1 - x_1')}{(x_1 - x_1')^2 + (x_2 - z(x_1'))^2}\gamma(x_1')dx_1'. \tag{1.15}$$

Especially, this velocity attains a discontinuity on Γ, therefore it is convenient to define the velocity on the sheet Γ as the average of the upper and lower limit. The Euler equations propose that the sheet is advected by the velocity generated by itself. Together with conservation of the vorticity along particle paths, we accomplish the evolution equations for the vortex sheet

$$\partial_t z(x_1,t) + v_1(x_1, z(x_1,t), t)\partial_x z(x_1,t) = v_2(x_1, z(x_1,t), t), \tag{1.16}$$
$$\partial_t \gamma(x_1,t) + \partial_{x_1}(\gamma(x_1,t) v_1(x_1, z(x_1,t), t)) = 0. \tag{1.17}$$

We refer to [63], p.191ff for a rigorous justification of the equivalence of weak solutions to (1.1)-(1.3) and solutions of (1.16)-(1.17). It is customary to take a Lagrangian parametrization of the sheet in terms of the circulation parameter α. This choice leads to the Birkhoff-Rott equation in complex variables

$$\partial_t \overline{z}(\alpha,t) = \frac{1}{2\pi i} \text{ PV} \int_{\mathbb{R}} \frac{1}{z(\alpha,t) - z(\alpha',t)} d\alpha'. \tag{1.18}$$

Here, $\overline{z}$ denotes the complex conjugate of z. Good derivations can also be found in [85], [6] or [62], p.359ff. We want to emphasize that the kernel in (1.18) is of degree -1, which leads to an ill-posed behaviour of the Birkhoff-Rott equation. The best available result one can expect is local well-posedness. Indeed, in [76] the authors used a version of the Cauchy-Kovalevskaya theorem to prove that for analytic initial data, (1.18) has a unique solution for short times. There are only some special examples of initial data close to the flat vortex sheet, for which global existence is known, see [41].

Additionally, in this paper Duchon and Robert showed the existence of a solution being initially analytic and which attains a singularity after finite time. Another example for such a breakdown of regularity has been established by Caflisch and Orellana in [14]. This ill-posed behaviour stems from the Kelvin-Helmholtz instability, a phenomenon, which was described for the first time by Helmholtz in 1868 and Kelvin in 1871. A mathematical explanation is given by the observation that solving the linearized equation of (1.18) for a small perturbation of the stationary flat solution $z(\alpha,t) = \alpha$ yields a solution with Fourier modes that grow exponentially in time, see [62], [63], p.200ff and [40], p.45ff. From the perspective of numerical experiments, this singularity appears in the tendency of the interface to roll up in spirals.

An interesting problem is the investigation of a regularity threshold for solutions of (1.18), which implies analyticity. The subtle examples from [41] and [14] show ill-posedness of the Birkhoff-Rott equation in $C^{1,\alpha}(\mathbb{R})$ for $\alpha > 0$ and in $H^n(\mathbb{R})$ for $n > \frac{3}{2}$. On the contrary, in [85], Wu verified well-posedness in $H^1_{loc}(\mathbb{R}^n)$. A way to overcome the bad behaviour of solutions beyond the formation of singularity is to define weak solutions for the Birkhoff-Rott equations. See for example [60], where the authors address the question of equivalence of such solutions with weak solutions to the Euler equations.

The simplest example is the flat vortex sheet, which is the periodic extension of

$$v_0(x) = \begin{cases} e_1, & x_n \in (-\pi, 0) \\ -e_1, & x_n \in (0, \pi) \end{cases}. \tag{1.19}$$

It turned out that this is indeed a wild initial data (see Theorem 2.15). In [77], Székelyhidi constructed a subsolution by a special ansatz, which reduces (1.6) to Burger's equation.

This subsolution is strict inside a zone around the sheet, which grows linearly in time. This resembles to the propagation of the singularity in the Kelvin Helmholtz instability and indicates that this instability is responsible for the wild behaviour. Because of the special symmetry, a similar construction is also possible for a rotational initial data ([5]). However, the initial data considered there is not curl-free outside of the interface and therefore not a vortex sheet. In Section 4.2 we give an extension of this result to a rotational vortex sheet initial data (Theorem 4.7).

Beyond these two examples, there are no other wild initial data known, such as more general vortex sheets, where the initial sheet is given by an arbitrary curve Γ. The goal of this thesis is to construct wild initial data attaining such a structure. We will mainly distinct the following situations.

1. In the first case, Γ can be expressed as the graph of a smooth function $z : \mathbb{R} \to \mathbb{R}$. This setting can be considered analogously on the periodic torus $\mathbb{T}^2$ for each smooth $z : [0, 1] \to [0, 1]$ with $z(0) = z(1)$.

2. Secondly, we investigate the case that Γ is a closed and smooth curve in $\mathbb{R}^2$.

After we recall in Section 2.2 how to apply the convex integration method to the incompressible Euler equations, it remains to find a strict subsolution $(\overline{v}, \overline{u}, \overline{q})$ together with a suitable energy density $\overline{e}$. As for the flat case, it will be essential to incorporate a growing zone U around the sheet, in which the subsolution is strict. Thanks to the subsolution criterion, we get infinitely many weak solutions in U, which emphasizes the idea that the turbulent behaviour is caused by the Kelvin-Helmholtz instability. This will give us not only an existence result for admissible weak solutions, but also captures information about the structure of these solutions as opposed to the existence result of Delort.

For this reason we start to characterize stationary vortex sheet solutions to (1.1)-(1.3) along curves $\Gamma \subset \mathbb{R}^2$ in Section 4.1. Such a result is only available if Γ is conformal, that is, if there exists a holomorphic function f in a neighbourhood of Γ such that $f(\Gamma) \subset \mathbb{R}$. Unfortunately, there seems to be no easy criterion for a curve to be conformal which is why we merely restrict to give a few concrete examples. For the construction of a subsolution, it will be substantial to introduce coordinates adapted to the geometry of the curve Γ. This is done in Section 4.2. The explicit construction of the subsolution and the main theorem follow in Section 4.3. We infer that a suitable class of vortex sheet initial data is wild.

Theorem 1.1. Let v_0 be a regular vortex sheet flow. Then there exists a time $T > 0$ and some $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$ and $0 < t < T$ there exist infinitely many weak solutions (v, p) of (1.1)-(1.3), which satisfy the local energy inequality (1.13). In particular, each regular vortex sheet flow v_0 is a wild initial data.

Regular vortex sheets have to be concentrated on smooth conformal curves and additionally have to satisfy suitable decay estimates at infinity, see Section 4.3 for the precise definition. Especially, these initial data include vortex sheets where the interface Γ coincides with ellipses.

The major step in the construction of the subsolution is to use a piecewise constant velocity $\overline{v}$. As opposed to a continuous subsolution inside the zone U, this has the major advantage to achieve strict inequality in (1.8) up to the boundary ∂U. The idea to use piecewise constant subsolutions (so called fan subsolutions) was already introduced in [18] for the compressible Euler equations in case of a flat interface. Since a piecewise constant subsolution is in general too restrictive, we merely choose one component of the subsolution to be piecewise constant, which leaves more freedom in choosing $\overline{u}, \overline{q}$.

Moreover, note that the construction of a strict subsolution in [77] is only valid on the torus $\mathbb{T}^2$. In order to obtain an analogous statement in $\mathbb{R}^2$, we have to work with admissibility in the sense of (1.13), as we did in Theorem 1.1. As a consequence, we slightly refine Theorem 2.15 and Theorem 2.16, since we establish the same result with admissibility in the local sense of (1.13).

Another main result of this thesis is devoted to the approximation of smooth subsolutions in Section 5. The motivation for this was to extend Theorem 2.15 for the flat vortex sheet to $\mathbb{R}^2$, that is, to conclude the existence of infinitely many weak solutions satisfying the local energy inequality. Observe that (1.13) reads in terms of the underlying subsolution

$$\partial_t \overline{e} + v \cdot \nabla \overline{q} \leq 0 \tag{1.20}$$

in the sense of distributions. This inequality becomes clearly easier in the case of a constant pressure $\overline{q}$. Thus, we propose that the local energy inequality can be established by approximating smooth subsolutions with piecewise constant ones and then to deduce (1.20) by an appropriate limit process.

The main Theorem can be stated as

Theorem 1.2. Let $\Omega \subset \mathbb{R}^2 \times (0, T)$ be an open and bounded set and Ω_t simply connected for all $t > 0$. Moreover, suppose that $(v, u, q) : \overline{\Omega} \to \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2} \times \mathbb{R}$ is a subsolution inside Ω such that $(v, u, q) \in C^1(\Omega) \cap C^0(\overline{\Omega})$. Then (v, u, q) can be approximated in $L^\infty(\Omega)$ by continuous and piecewise constant subsolutions $(\tilde{v}, \tilde{u}, \tilde{q})$.

There are two main ingredients for the proof. At first, we use the fact from [31] that subsolutions of the incompressible Euler equations can be presented in terms of a potential. Afterwards, we approximate this potential by a piecewise affine C^1 map in the spirit of Lemma 3.3 in [53].

However, it is not clear how to implement a limit process which proves (1.20). The problem is that in Theorem 1.2, we merely have an approximation in the L^∞-norm whereas in (1.20) the term $\nabla \overline{q}$ appears. Nevertheless, we think that Theorem 1.2 is a novelty, which may have useful applications in other problems. Furthermore, note that Theorem 1.1 already accomplishes the original goal, that is, to produce infinitely many weak solutions satisfying (1.13).

1.3 The IPM equations

Apart from the incompressible Euler equations, the second system of equations we want to investigate in greater detail are the incompressible porous media (IPM) equations given by

$$\partial_t \rho + \text{div } (\rho u) = 0 \tag{1.21}$$
$$\text{div } u = 0 \tag{1.22}$$
$$u + \nabla p = -(0, \rho) \tag{1.23}$$
$$\rho(x, 0) = \rho_0(x) \tag{1.24}$$

in $\mathbb{R}^2 \times [0, T]$. These equations model the dynamics of an incompressible fluid in a homogeneous and isotropic porous media. Although in physical applications one has to take boundary conditions into account (see for example [3]), we neglect these here for simplicity and consider the equations in $\mathbb{R}^2$. Note that the role of the velocity in the Euler equations is now filled by the density. Thus, instead of a vector-valued evolution equation, we are in the scalar case, which slightly lightens our calculations.

We essentially address with special initial value problems for (1.21)-(1.24), namely with initial densities of the form

$$\rho_0(x_1, x_2) = \begin{cases} \rho^+, & x_2 > z_0(x_1) \\ \rho^-, & x_2 < z_0(x_1) \end{cases} \tag{1.25}$$

where $\rho^+, \rho^- \in \mathbb{R}^+$ are constants and $z_0 : \mathbb{R} \to \mathbb{R}$ is a function. This can be viewed as an analogous version of vortex sheet initial data for the incompressible Euler equations. By $\Gamma := \{(s, z_0(s)) : s \in \mathbb{R}\}$ we denote the graph of z_0. We distinct the following cases: If $\rho^+ > \rho^-$, which means that the heavier fluid is on top, we speak of the unstable regime. The case $\rho^+ < \rho^-$ is called the stable regime. Taking the curl of (1.23), we can eliminate the pressure and obtain curl $u = -\partial_{x_1} \rho$. This motivates the definition of weak solutions in the following form.

Let $\rho_0 \in L^\infty(\mathbb{R}^2)$ and $T > 0$. We call $(\rho, u) \in L^\infty(\mathbb{R}^2 \times [0, T))$ a weak solution of (1.21)-(1.24) with initial data ρ_0 if

$$\int_0^T \int_{\mathbb{R}^2} \rho(\partial_t \phi + u \cdot \nabla \phi) dx dt = \int_{\mathbb{R}^2} \phi(x, 0) \rho_0(x) dx \quad \forall \phi \in C_c^\infty([0, T) \times \mathbb{R}^2)$$
$$\int_{\mathbb{R}^2} u \cdot \nabla \phi dx = 0 \quad \forall \phi \in C_c^\infty(\mathbb{R}^2)$$
$$\int_{\mathbb{R}^2} (u + (0, \rho)) \cdot \nabla^\perp \phi dx = 0 \quad \forall \phi \in C_c^\infty(\mathbb{R}^2).$$

Since for given $\rho(x, t)$ at a fixed time t, u is the solution of an elliptic problem by the Biot-Savart law, the equations (1.21)-(1.23) describe the evolution of the density in time. Assuming that $\rho(x, t)$ remains in the form (1.25) for positive times, we end up in a free boundary problem for the evolution of the sheet, which is commonly known as the Muskat problem. This equation corresponds to the Birkhoff-Rott equation in case of the Euler equations. If the sheet can be presented as a graph like above, the equation for the sheet $z(s, t)$ is given by

$$\partial_t z(s, t) = \frac{\rho^- - \rho^+}{2\pi} \int_{-\infty}^{\infty} (\partial_s z(s, t) - \partial_s z(\xi, t)) \frac{s - \xi}{(s - \xi)^2 + (z(s, t) - z(\xi, t))^2} d\xi, \tag{1.26}$$

see for example [25]. The same equation emerges for the description of the flow between closely spaced parallel sheets of glass, a so called Hele-Shaw cell. This can be taken as a simplified experimental model for the flow in a porous medium. The behaviour of the solution depends strongly on the sign of $\rho^- - \rho^+$. In the stable case, (1.26) is locally well-posed in $H^3(\mathbb{R})$, see [25] or [22] for an improved regularity. This is opposed to the more delicate unstable regime, which is ill-posed in $H^4(\mathbb{R})$ ([25]). In the flat case, where (1.25) is also a stationary solution, this can be verified by a linear stability analysis, which shows that initial disturbances grow exponentially in time. In particular, in the unstable case, there are no general existence results for (1.26) known.

Thus, it seems not suitable to study (1.21)-(1.24) in terms of a free boundary problem, at least in the unstable regime. In [67], Otto used a relaxational approach in the spirit of variational problems to reformulate the equations as a scalar conservation law. He proved existence of a unique entropy solution in the case of the flat initial data $z_0 \equiv 0$. In particular, this solution incorporates a growing zone around the sheet, in which a mixing behaviour can be observed. This coincides with the expected physical behaviour in the unstable case.

By means of this perspective, we call a weak solution admissible, if $\rho \in [\rho^-, \rho^+]$ (respectively $\rho \in [\rho^+, \rho^-]$ in the stable case). If we do not say anything else, from now on we assume without loss of generality $\rho^\pm = \pm 1$, see also Lemma 2.18 in Section 2.3.

The first observation of non-uniqueness of weak solutions was made in [24]. Since the constructed weak solutions start with $\rho_0 \equiv 0$ at time $t = 0$ and furthermore satisfy $|\rho| = 1$ almost everywhere for times $t > 0$, they are not admissible.

Later on, Székelyhidi was able to establish the existence of infinitely many admissible weak solutions of (1.21)-(1.24) in the unstable regime with flat initial data by the convex integration method (see [79]). In Section 2.3 we give a detailed analysis for the application of the convex integration technique to the IPM equations. This includes the definition of a subsolution for the IPM equations. We deduce an analogous result (Theorem 2.23) to the subsolution criterion and hence, as for the Euler equations, we will henceforth concentrate on the construction of subsolutions. Surprisingly, the weak solutions from [79] show the same behaviour of a mixing zone Ω_{mix} like the solution from Otto. In fact, the construction of the subsolution substantially based on a reduction to Burger's equation and this served as the original motivation to obtain an analogous result for the flat vortex sheet in case of the Euler equations in [77].

Recently, this result was generalized to arbitrary initial curves z_0 by Castro, Cordoba and Faraco, see [17]. The main theorem states that for each $z_0 \in H^5(\mathbb{R})$ there exist infinitely many admissible weak solutions to (1.21)-(1.24) with initial data (1.25) in the unstable case. In Section 2.3 we shortly demonstrate the major idea for the construction of the subsolution. Again, we can observe a mixing zone Ω_{mix} in the sense of the previous result for the flat interface. In particular, the method of convex integration implies that these weak solutions satisfy $|\rho(x, t)| = 1$ a.e., which means that the values of the density do not change in time. Hence, these weak solutions are admissible. Furthermore, inside of each ball $B \subset \Omega_{\text{mix}}$, the density takes both values ± 1. Because of this, these weak solutions are called mixing solutions.

In Section 3, I would like to propose an alternative proof of the result from [17]. The crucial point for the construction of the subsolution is the choice of a piecewise constant density. By means of the close analogies between the IPM and Euler equations we mentioned so far, this provides a basis for an analogous procedure in case of the incompressible Euler equations and consequently lead to Theorem 1.1.

Additionally, instead of solving a nonlinear equation for the interface $z(s,t)$ as in [17], it will be sufficient to choose the ansatz

$$z(s,t) = z_0(s) + a(s)t + b(s)\frac{1}{2}t^2.$$

Eventually, we obtain a slightly different regularity for the initial curve z_0.

Theorem 1.3.
Let $z_0(s) = \beta s + \overline{z}_0(s)$ with $\overline{z}_0 \in W^{4,1}(\mathbb{R}) \cap C^{4,\alpha}(\mathbb{R})$ for some $0 < \alpha < 1$ and $\beta \in \mathbb{R}$.
(i) In the unstable regime and for each $\beta \in \mathbb{R}$, there exists $T > 0$, such that for short times $0 < t < T$, there exist infinitely many admissible weak solutions to (1.21)-(1.24).
(ii) In the stable regime, suppose that $0 < \|\partial_s \overline{z}_0\|_{L^\infty(\mathbb{R})} < \sqrt{1+\beta^2} - 1$. Then there exists $T > 0$, such that for short times $0 < t < T$, there exist infinitely many admissible weak solutions to (1.21)-(1.24).

The second statement shows such a wild behaviour even in the stable case, unless the initial curve is horizontal. This somewhat surprising result was already noted in [17].

As opposed to the results of [79] and [17] for the unstable regime, where the propagation speed of the mixing zone Ω_{mix} is in the range $c \in (0,2)$, here the velocity merely satisfies $c \in (0,1)$. However, this is valid only for the simplest possible choice of a piecewise constant density. Imposing a piecewise constant approximation ρ_n of the continuous density in [17], the velocity indeed attains values arbitrarily close to 2 in the limit $n \to \infty$. Nevertheless, the time of existence T goes to zero as we reach the maximal speed $c = 2$, just like in [17]. This seems to indicate that there exist no weak solutions with the maximal propagation speed 2. On the contrary, in [79], we have a positive time of existence $T > \frac{1}{2}$ for all $c \in (0,2)$.

We will prove Theorem 1.3 in Section 3. We start to show how the ansatz of a piecewise constant density leads to a subsolution for (1.21)-(1.24) and thus proves Theorem 1.3. This will be done in Section 3.1, see Theorem 3.1. Afterwards, in Section 3.2, we study some well-known facts about the velocity, especially we give a detailed representation of u for piecewise constant densities in Theorem 3.8. Section 3.3 is devoted to the construction of the interface $z(s,t)$ by a power series ansatz and leads to Theorem 3.16, where we demonstrate a first rough existence result for curves $\overline{z}_0 \in W^{6,1}(\mathbb{R}) \cap C^{5,\alpha}(\mathbb{R})$. In Section 3.4, we improve the necessary regularity of the initial curve to $\overline{z}_0 \in W^{4,1}(\mathbb{R}) \cap C^{4,\alpha}(\mathbb{R})$, which bases on the special structure of the operators $T_F(G)$ similar to the Hilbert transform (Theorem 3.19). Finally we refine these results to more general piecewise constant densities in Section 3.5 and infer Theorem 3.22 therein.

1.4 Notations

Here, we want to introduce some notations we will use in the thesis. By $\mathbb{R}^n$ we denote the standard n-dimensional euclidean space with unit vectors $e_i, i = 1, \ldots, n$ and coordinates $x_1, \ldots, x_n$. Since we often work in the 2-dimensional case, we also write coordinates in the form (x, y). Especially, we impose so called flow coordinates $(\tilde{x}, \tilde{y})$ in Section 4.2. If $x = (x_1, x_2) \in \mathbb{R}^2$ is a vector, we write the perturbed vector as $x^\perp := (-x_2, x_1)$. Moreover, for $a, b \in \mathbb{R}^n$ we introduce the notation

$$a \otimes b = \begin{pmatrix} a_1 b_1 & \dots & a_n b_1 \\ \vdots & \vdots & \vdots \\ a_1 b_n & \dots & a_n b_n \end{pmatrix}.$$

The n-dimensional periodic torus is denoted by $\mathbb{T}^n$.

We propose the usual notations of $C^k(\mathbb{R}^n)$ for the k-times continuously differentiable functions and $C^{k,\alpha}(\mathbb{R}^n), \alpha \in (0, 1)$ for functions with Hölder continuous k-th derivative. The corresponding Hölder seminorm of some function $f \in C^{0,\alpha}(\mathbb{R}^n)$ is given by

$$[f]_{C^{0,\alpha}} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

For partial derivatives of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$ we take the equivalent representations $\partial_{x_i} f = \partial_i f = f_{x_i}$ and for higher derivatives we also use the notation $D^k f$. In Section 4.3 we will also use the notation

$$\|f\|_{\tilde{C}^k(\mathbb{R}^n)} := \sum_{j=0}^{k} \|\tilde{D}^j f\|_{L^\infty(\mathbb{R}^n)},$$

meaning that we sum the L^∞-norm of all derivatives with respect to flow coordinates up to order k. Moreover, by $L^q_t L^p_x$ we denote the space $L^q([0, T]; L^p(\mathbb{R}^n))$.

By supp(f) we mean the support of the function f. We say that f is of $o(1)$, if $\lim_{t\to 0} f(t) = 0$. In Section 3.4, we also have to work with the Hilbert transform of a function $f \in L^p(\mathbb{R}^n)$, denoted by $\mathcal{H}(f)$. In addition, we often have to consider functions with a singularity, for example in 0. Then we define the principal value integral by

$$\text{PV} \int_{\mathbb{R}^n} f(x)dx := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} f(x)dx.$$

The space of $m \times n$ matrices is written as $\mathbb{M}^{m\times n}$, the space of traceless $n \times n$ matrices as $\mathbb{M}^{n\times n}_0$ and the space of symmetric, traceless $n \times n$ matrices as $\mathcal{S}^{n\times n}_0$.

The convex hull of a set K is denoted by K^{co}. Furthermore we define the characteristic function of K by

$$\chi_K(x) := \begin{cases} 1 & , x \in K \\ 0 & , x \notin K \end{cases}.$$

When we consider the incompressible Euler equations, we usually denote weak solutions by (v, p) and the vorticity by $\omega := \text{curl } v = \partial_{x_1} v_2 - \partial_{x_2} v_1$. To make the difference clearer, we will use a bar for subsolutions, that is, we write $(\overline{v}, \overline{u}, \overline{q})$. On the contrary, for the IPM equations we denote the velocity component by u.

When dealing with vortex sheet flows respectively densities of Muskat type, by $\Gamma \subset \mathbb{R}^2$, we usually denote a smooth curve. We often assume that Γ can be presented as the graph of a function $z : \mathbb{R} \to \mathbb{R}$. The corresponding normal vector on Γ is denoted by n. Finally, when integrating a function f over a set containing Γ, we have to address the jump of f across Γ, denoted by $[f]$.

2 The Convex Integration method

As already mentioned, the technique of convex integration origins in the h-principle of Gromov ([50]), for which the Nash-Kuiper theorem is a typical example. It displays a paradox behaviour concerning isometric embeddings of the sphere S^2 into $\mathbb{R}^3$. In [66], [55] the authors established the existence of infinitely many isometric embeddings of class C^1, which can wrinkle S^2 into arbitrarily small regions. In contrast to this flexibility, we have rigidity in the class C^2, since the only C^2 isometric embedding of S^2 into $\mathbb{R}^3$ is the standard embedding modulo rigid motion. Our purpose is to isolate the key idea of the Nash-Kuiper theorem and transfer it into a more general context. In combination with the Tartar framework, this yields a powerful machinery producing a large set of solutions. Afterwards we investigate an application to the special cases of the incompressible Euler and IPM equations.

2.1 The Tartar framework

In this section we want to explain the method of convex integration in terms of the Tartar framework. For this, we mainly rely on the derivation of [78]. In addition, we refer to [53], [65], [54], [32] or [34] for good presentations.

At first we derive a general method which produces strongly convergent sequences from weakly convergent ones. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. We start with the following lemma about mollifications in $L^p(\Omega)$.

Lemma 2.1.
Let $\eta \in C_c^\infty(\mathbb{R}^n)$.If $u_k \rightharpoonup u$ in $L^p(\Omega)$ for some $p < \infty$, then

$$\eta * u_k \to \eta * u \quad \text{in } L^p(\Omega).$$

Proof. Write $f_k := \eta * u_k$ and $f := \eta * u$ which are well defined by setting $u_k = u = 0$ outside Ω. Let $x \in \mathbb{R}^n$. We have

$$\int_\Omega |\eta(x-y)|^{p'} dy \leq \|\eta\|_\infty^{p'} |\Omega| < \infty,$$

so the function $y \mapsto \eta(x-y)$ is in $L^{p'}(\Omega)$. We infer

$$f_k(x) = \int_\Omega \eta(x-y)u_k(y)dy \xrightarrow{k\to\infty} \int_\Omega \eta(x-y)u(y)dy = f(x),$$

which means that f_k converges pointwise to f. In particular, we see that $|f_k - f|^p$ converges pointwise to zero. Note that for each $x \in \mathbb{R}^n, k \in \mathbb{N}$

$$|f_k(x)|^p = \left| \int_{\mathbb{R}^n} \eta(x-y)u_k(y)dy \right|^p \leq \|\eta\|_{p'}^p \|u_k\|_p^p =: C < \infty$$

Thus, for any $x \in \mathbb{R}^n, k \in \mathbb{N}$

$$|f_k(x) - f(x)|^p \leq C_1(|f_k(x)|^p + |f(x)|^p) \leq C_1(C + |f(x)|^p) =: g(x)$$

and obviously, $g \in L^1(\Omega)$, since $f \in L^p(\Omega)$. Lebesgues convergence theorem finishs the proof. □

Let us shortly repeat some notions and standard results of Baire category theory. Assume Y is a complete metric space. We say that $E \subset Y$ is nowhere dense if $\overline{E}$ has empty interior. By a set of first category, we mean a countable union of nowhere dense sets. If a set is not of first category then we say it is of second category. Furthermore, the complement of a set of first category is called a residual set.

Theorem 2.2 (Baire's theorem)**.**
(i) Let Y be a complete metric space and $(V_k)_k$ a sequence of open and dense sets in Y. Then $\bigcap_{k=1}^{\infty} V_k$ is dense in Y.
(ii) Equivalently, each complete metric space Y is of second category.

Definition 2.3.
Let Y be a metric space. We say a function $J : Y \to \mathbb{R}$ is of class Baire 1 if it can be represented as a pointwise limit of continuous functions, that is, if there exist functions $J_n \in C(Y)$ such that $J_n(u) \to J(u)$ as $n \to \infty$ for all $u \in Y$.

Thanks to Baire's theorem we deduce

Proposition 2.4. Let Y be a complete metric space and $J : Y \to \mathbb{R}$ a Baire 1 function. Then the set of continuity points of J is dense in Y.

Proof. Define

$$E_{n,k} := \bigcap_{i,j \geq k} \left\{ u \in Y : |J_i(u) - J_j(u)| \leq \frac{1}{n} \right\}.$$

Let n, k and $i, j \geq k$ be given. Since J_i and J_j are continuous, it is obvious to see that $\{u \in Y : |J_i(u) - J_j(u)| \leq \frac{1}{n}\}$ is closed. Thus, $E_{n,k}$ is closed for each n, k. Let $u \in Y$ and $n \in \mathbb{N}$. Since $J_i(u) \xrightarrow{i \to \infty} J(u)$, $J_i(u)$ is a cauchy sequence. Hence, there exists $k \in \mathbb{N}$ such that $u \in E_{n,k}$. We deduce

$$Y = \bigcup_{k=1}^{\infty} E_{n,k} \quad \forall n \in \mathbb{N}.$$

Now define the open set

$$V_n := \bigcup_{k=1}^{\infty} \operatorname{int}(E_{n,k}),$$

To verify that V_n is dense, let $B \subset Y$ be an open set. Then $\overline{B}$ is closed and thus a complete metric space. From Theorem 2.2, (ii) we conclude that $\overline{B}$ is of second category. Now we write $\overline{B} = \bigcup_{k=1}^{\infty} (E_{n,k} \cap \overline{B})$.

If we suppose that $\operatorname{int}(E_{n,k} \cap \overline{B}) = \emptyset$ for each k, then we immediately see that $\overline{B}$ is a set of first category, a contradiction. Therefore we find some $k \in \mathbb{N}$ such that

$\text{int}(E_{n,k} \cap \overline{B}) \neq \emptyset$ or $(\text{int}E_{n,k} \cap B) \neq \emptyset$. But from this we infer $B \cap V_n \neq \emptyset$, so V_n is dense. Moreover, Theorem 2.2 tells us that

$$S := \bigcap_{n=1}^{\infty} V_n$$

is a dense set. If we show that S consists of continuity points of J, we are finished. Let $u \in S$ and $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{3}$. Then $u \in V_n$ and by definition of V_n we find $k \in \mathbb{N}$ and $\delta > 0$ such that $B_\delta(u) \subset E_{n,k}$. Thus for all $v \in B_\delta(u)$ and $i, j \geq k$ we get

$$|J_i(v) - J_j(v)| \leq \frac{1}{n} < \frac{\varepsilon}{3}.$$

Letting $i \to \infty$, we achieve for all $v \in B_\delta(u)$ and $j \geq k$

$$|J(v) - J_j(v)| < \frac{\varepsilon}{3}.$$

Because J_k is continuous, we additionally find a $\tilde{\delta} > 0$ such that $|J_k(u) - J_k(v)| < \frac{\varepsilon}{3}$ for all $v \in B_{\tilde{\delta}}(u)$. Defining $\delta_0 := \min(\delta, \tilde{\delta})$, we have

$$|J(u) - J(v)| \leq |J(u) - J_k(u)| + |J_k(u) - J_k(v)| + |J_k(v) - J(v)| < \varepsilon$$

for all $v \in B_{\delta_0}(u)$, so u is a continuity point of J. □

Let $X_0 \subset L^2(\Omega)$ be a bounded subset. By X we denote the closure of X_0 with respect to the weak L^2 topology. Since $X \subset L^2(\Omega)$ is bounded, the weak topology on X is metrizable. Hence, X is a compact metric space. Consider the functional $J : X \to \mathbb{R}$ defined by

$$J(u) = \int_\Omega |u(x)|^2 dx.$$

We see that J is not continuous, since $u_k \overset{L^2}{\rightharpoonup} u$ does not imply $J(u_k) \to J(u)$. However, we can approximate J pointwise by continuous maps.

Let $\rho \in C_c^\infty(\mathbb{R}^n)$ be a standard mollifier kernel, that is, $\rho \geq 0$ and $\int_{\mathbb{R}^n} \rho(x)dx = 1$ and define $\rho_\varepsilon(x) = \varepsilon^n \rho(\frac{x}{\varepsilon})$. For $\varepsilon > 0$ set

$$J_\varepsilon(u) = \int_\Omega |(\rho_\varepsilon * u)(x)|^2 dx.$$

Then, if $u_k \overset{L^2}{\rightharpoonup} u$, we conclude $J_\varepsilon(u_k) \to J_\varepsilon(u)$ as $k \to \infty$ from Lemma 2.1. Thanks to this, J_ε is continuous on X. Furthermore we have $J_\varepsilon(u) \overset{\varepsilon \to 0}{\longrightarrow} J(u)$ for each $u \in X$. This means that J is a Baire 1 function and Proposition 2.4 yields that

$$S := \{u \in X : u \text{ is a continuity point of } J\}$$

is dense in X. Observe that if $u \in S$, then

$$\forall u_k \in X \text{ with } u_k \overset{L^2}{\rightharpoonup} u \quad \text{we have} \quad u_k \overset{L^2}{\longrightarrow} u.$$

By means of this property, we call S the set of stable elements of X, meaning that these can not be weakly perturbed.

Proposition 2.5. Let X_0 be a bounded subset of $L^2(\Omega)$, denote by X its weak closure and let $I : X \to \mathbb{R}$ be a continuous functional with respect to the strong topology. Moreover assume

$$\forall u \in X_0 \ \exists (u_k)_k \subset X_0 \quad \text{with} \quad u_k \rightharpoonup u \ \text{ in } \ L^2(\Omega), \ I(u_k) \xrightarrow{k\to\infty} 0.$$

Then $S \subset \{u \in X : I(u) = 0\}$ and so $\{u \in X : I(u) = 0\}$ is dense.

Proof. Let $u \in S$. Then $u \in X = \overline{X_0}^{w,L^2}$ and thus we can find a sequence $(u_k)_k \subset X_0$ such that $u_k \xrightarrow{L^2} u$. Furthermore by assumption for each $k \in \mathbb{N}$ we find another sequence $(u_{k,j})_j \subset X_0$ satisfying

$$u_{k,j} \xrightharpoonup{j\to\infty} u_k \ \text{ in } \ L^2(\Omega), \ I(u_{k,j}) \xrightarrow{j\to\infty} 0.$$

We can choose a diagonal sequence $(\tilde{u}_k)_k \subset X_0$ such that $\tilde{u}_k \xrightharpoonup{L^2} u$ and $I(\tilde{u}_k) \to 0$ as $k \to \infty$. However, $u \in S$, so we conclude $\tilde{u}_k \xrightarrow{L^2} u$. Since I is continuous, this implies $I(\tilde{u}_k) \to I(u)$. Hence $I(u) = 0$. □

It is also possible to weaken the above approximation property and to require a perturbation property.

Proposition 2.6. Let $I : X \to \mathbb{R}_+$ be a continuous functional with respect to the strong topology. Suppose that

$$\begin{aligned} &\forall u \in X_0 \ \text{ with } \ I(u) > 0, \ \exists (u_k)_k \subset X_0 \ \text{ such that} \\ &u_k \rightharpoonup u \ \text{ in } \ L^2(\Omega) \ \text{ and } \ \liminf_{k\to\infty} \|u_k\|_2^2 \geq \|u\|_2^2 + \alpha, \end{aligned}$$

where $\alpha > 0$ depends only on $I(u) > 0$. Then $S \subset \{u \in X | I(u) = 0\}$ and in particular $\{u \in X | I(u) = 0\}$ is dense.

Proof. Let $u \in S$ and $I(u) > 0$. Choose a sequence $(u_k)_k \subset X_0$ such that $u_k \xrightharpoonup{L^2} u$. From $u \in S$ we deduce $u_k \to u$ strongly in $L^2(\Omega)$ and $I(u_k) \to I(u)$. Furthermore by assumption there exists $\alpha > 0$ such that for each $k \in \mathbb{N}$ we find a sequence $(u_{k,j})_j \subset X_0$ satisfying

$$u_{k,j} \xrightharpoonup{j\to\infty} u_k \ \text{ in } \ L^2(\Omega) \ \text{ and } \ \liminf_{k\to\infty} \|u_{k,j}\|_2^2 \geq \|u_k\|_2^2 + \alpha.$$

However, we can take a diagonal sequence $(\tilde{u}_k)_k \subset X_0$ such that $\tilde{u}_k \xrightharpoonup{L^2} u$ and $\liminf_{k\to\infty} \|\tilde{u}_k\|_2^2 \geq \|u\|_2^2 + \alpha$, which contradicts $u \in S$. □

Remark 2.7. We considered the L^2 case only for simplicity. Analogous results are also available in the L^p setting for $1 < p < \infty$.

The next aim is to apply these abstract results to the framework of compensated compactness, introduced by L.Tartar and R.DiPerna, see [82], [36]. For this, let $\mathcal{D} \subset \mathbb{R}^d$ be a bounded domain. We consider a general linear system of equations given by

$$\sum_{i=1}^{d} A_i \partial_i z = 0 \ \text{ in } \ \mathcal{D} \tag{2.1}$$

$$z(y) \in K \ \text{ for a.e. } \ y \in \mathcal{D}, \tag{2.2}$$

where $z : \mathcal{D} \to \mathbb{R}^N$ is the unknown, $A_i \in \mathbb{M}^{m\times N}$ are constant matrices and $K \subset \mathbb{R}^N$ is a given compact set. We want to construct many different solutions to (2.1)-(2.2). Therefore, we require the following assumptions.

(H1) The Wave Cone: There exists a closed cone $\Lambda \subset \mathbb{R}^N$ and a constant $C > 0$ such that for all $\hat{z} \in \Lambda$ there exists a sequence $z_k \in C_c^\infty(B_1(0); \mathbb{R}^N)$ such that

1. $\sum_{i=1}^d A_i \partial_i z_k = 0$ in $\mathcal{D}$ for each k,
2. $\mathrm{dist}(z_k, [-\hat{z}, \hat{z}]) \to 0$ uniformly as $k \to \infty$,
3. $z_k \rightharpoonup 0$ weakly in L^2 as $k \to \infty$,
4. $\int |z_k|^2 dy > C|\hat{z}|^2$.

(H2) The Λ-convex hull: There exists a bounded open set $U \subset \mathbb{R}^N$ with $U \cap K = \emptyset$ such that for all $z \in U$ with $\mathrm{dist}(z, K) \geq \alpha > 0$ there exists $\hat{z} \in \Lambda \cap S^{N-1}$ such that

$$z + t\hat{z} \in U \quad \text{for all} \quad |t| < \beta, \tag{2.3}$$

where $\beta = \beta(\alpha) > 0$.

(H3) Subsolutions: Let $X_0 \subset L^2(\mathcal{D})$ be a nonempty bounded set containing functions which are perturbable in an open subdomain $\mathcal{U} \subset \mathcal{D}$. By this we mean that each $z \in X_0$ is continuous on $\mathcal{U}$ with

$$z(y) \in U \quad \text{for each} \quad y \in \mathcal{U}. \tag{2.4}$$

Furthermore, if $z \in X_0$ and $w \in C_c(\mathcal{U})$ such that w solves (2.1) and $(z + w)(y) \in U$ for all $y \in \mathcal{U}$, then $z + w \in X_0$.

Similar to above, we denote the closure of X_0 with respect to the weak L^2 topology by X. We infer that X is a complete metric space, since the topology is metrizable on X, which is the consequence of the boundedness of X_0.

Theorem 2.8. Suppose that the conditions (H1)-(H3) are valid. Then, the set

$$\{z \in X : z(y) \in K \quad \text{for a.e.} \quad y \in \mathcal{U}\}$$

is residual in X.

In particular, we verified the existence of infinitely many weak solutions to (2.1)-(2.2). The proof relies on the following Lemma.

Lemma 2.9. There exists a continuous function $\Psi : \overline{U} \to [0, \infty)$ satisfying
(i) $\{\Psi = 0\} \subset K$,
(ii) For each $z \in X_0$ there exists a sequence $(z_k)_k \subset X_0$ such that $z_k \rightharpoonup z$ in L^2 and

$$\int_{\mathcal{U}} |z - z_k|^2 dy \geq \int_{\mathcal{U}} \Psi(z(y)) dy.$$

Proof. At first, thanks to (H1) and (H2), we conclude the following: There exists a continuous function $\Psi : \overline{U} \to [0, \infty)$ with $\{\Psi = 0\} \subset K$ such that for any $\overline{z} \in U$ there exists $V \subset\subset U$ and a sequence $(z_k)_k \subset C_c^\infty(B_1(0); \mathbb{R}^N)$ such that

(i) $\sum_{i=1}^d A_i \partial_i z_k = 0$ in $\mathcal{D}$ for each k,
(ii) $\overline{z} + z_k(y) \in V$ for all $y \in \mathcal{D}$,
(iii) $z_k \rightharpoonup 0$ weakly in L^2 as $k \to \infty$,
(iv) $\int |z_k|^2 dy > 2\Psi(\overline{z})$.

Indeed, for $\overline{z} \in U$ there exists some $\hat{z} \in \Lambda \cap S^{N-1}$ satisfying (2.3). Then, by (H1) we find a sequence $(z_k)_k \subset C_c^\infty(B_1(0); \mathbb{R}^N)$ satisfying (i) and (iii). Property (ii) follows from $\operatorname{dist}(z_k, [-\hat{z}, \hat{z}]) \to 0$ uniformly, (2.3) and the continuity of z_k. Choosing Ψ as a continuous function with roots in K and being bounded by $\frac{C}{2}$, where $C > 0$ is the constant from (H1), we accomplish (iv).

In the next step fix $y_0 \in \mathcal{U}$ and $r_0 > 0$. By applying the above statement to $\overline{z} = z(y_0)$ and thanks to a translation and rescaling of z_k via $y \mapsto r_0^{-1}(y - y_0)$, we get a sequence $(z_k)_k \subset C_c^\infty(B_{r_0}(y_0); \mathbb{R}^N)$ such that (i)-(iii) are valid and

(iv') $\int_{B_{r_0}(y_0)} |z_k|^2 dy > 2|B_{r_0}(y_0)|\Psi(z(y_0))$

holds. Furthermore, because of the continuity of z we can suppose that r_0 is so small that (ii) can be adjusted by

(ii') $z(y) + z_k(y) \in U$ for all y.

Since $y_0 \in \mathcal{U}$ was arbitrary, we can repeat this argument sufficiently often and obtain disjoint balls $B_i := B_{r_i}(y_i) \subset\subset \mathcal{U}$ for $i = 1, \ldots, n$ with associated sequences $(z_k^i)_k$ satisfying (i),(ii'),(iii) and (iv') for all i, k and the inequality

$$\int_{\mathcal{U}} \Psi(z(y))dy \leq 2 \sum_{i=1}^n |B_i| \Psi(z(y_i)).$$

Now we define

$$z_k := z + \sum_{i=1}^n z_k^i.$$

From (ii') we deduce $z_k(y) \in U$ for each $y \in \mathcal{U}$. Thus, (H3) implies that $z_k \in X_0$. In addition, we have $z_k \rightharpoonup z$ in L^2 and

$$\int_{\mathcal{U}} |z_k - z|^2 dy = \sum_{i=1}^n \int_{B_i} |z_k^i|^2 dy \geq 2 \sum_{i=1}^n |B_i| \Psi(z(y_i)) \geq \int_{\mathcal{U}} \Psi(z(y))dy,$$

which finishes the proof. □

The proof of Theorem 2.8 is now a direct consequence of Proposition 2.6 applied to $I(u) := \int_{\mathcal{U}} \Psi(z(y))dy$.

In general, a first candidate for the cone Λ in (H1) is the wave cone given by

$$\Lambda := \left\{ a \in \mathbb{R}^N : \left(\sum_{i=1}^d \xi_i A_i \right) a = 0 \text{ for some } \xi \in S^{d-1} \right\}. \tag{2.5}$$

We can characterize this set by observing that each $a \in \Lambda$ defines plane wave solutions to (2.1) through $z(x) = ah(x \cdot \xi)$ for each scalar function $h : \mathbb{R} \to \mathbb{R}$. However, in many situations, this set is too big and an appropriate modification is necessary.

A candidate for the set U in (H2) is the interior of K^{co}. This is due to the fact that whenever $U, K \subset \mathbb{R}^N$ are bounded and satisfy (H2), then we have $U \subset K^{co}$ (see [78] for a proof). Nevertheless, in most cases, choosing $U = K^{co}$ is still too large. Note that also for (H2) one has to take the cone Λ into account and thus one often defines U in the spirit of the Λ-convex hull K^Λ. To be precise, K^Λ is defined as the largest closed set, which can not be separated from K. A vector z does not belong to K^Λ, if there exists a Λ-convex function f such that $f \leq 0$ on K and $f(z) > 0$. Here, a function f is called Λ-convex if the map $s \mapsto f(z_0 + sz)$ is convex for each $z \in \Lambda$.

A typical example for the application of this framework are first order differential inclusions. In this case, Λ is given by the rank-one cone $\Lambda = \{z \in \mathbb{R}^{m\times n} : \text{rank } z \leq 1\}$ and accordingly U is closely related to the rank-one convex hull of K. We refer to [54] or [53] for more information about Λ-convex hulls.

We want to emphasize that the convex integration method applies to many partial differential equations in fluid dynamics, such as the Euler equations or active scalar equations([32], [18], [24] [74]). Next, we will show in detail, how this is established for the special cases of the incompressible Euler equations and the IPM equations.

2.2 Application to the incompressible Euler equations

To rewrite the Euler equations in the Tartar framework, it will simplify our computations to consider (1.1)-(1.3) in $\mathbb{T}^n$, since this is a bounded domain without boundary conditions. However, the convex integration method can be applied to the incompressible Euler equations also in $\mathbb{R}^n$, see [32]. Again, here we rely on the notes from [78], [80].

At first, it is essential to introduce a suitable definition of a subsolution, which is motivated by a macroscopic perspective and the notion of Reynold stress (see also [34] for a good derivation). Denote by $\overline{v}$ the macroscopically averaged velocity field. The precise definition of averaging is not important in these formal calculations. After such an averaging, (1.1)-(1.2) become

$$\partial_t \overline{v} + \text{div } (\overline{v} \otimes \overline{v} + R) + \nabla \overline{p} = 0 \tag{2.6}$$

$$\text{div } \overline{v} = 0, \tag{2.7}$$

where $R = \overline{v \otimes v} - \overline{v} \otimes \overline{v}$ is the Reynolds stress. The appearance of R is due to the fact that averaging does not commute with the nonlinearity. Following Tartar [82] and DiPerna [36], we separate the linear equations from the nonlinear constitutive relation. This means we impose a new variable $\overline{u}$ being the traceless part of $\overline{v} \otimes \overline{v} + R$. Thus, (2.6) becomes

$$\partial_t \overline{v} + \text{div } \overline{u} + \nabla \overline{p} = 0.$$

From $R = \overline{(v - \overline{v}) \otimes (v - \overline{v})}$, we deduce that R is a symmetric positive semidefinite matrix, i.e. $R \geq 0$. It is customary to rewrite this as

$$\overline{v} \otimes \overline{v} - \overline{u} \leq \frac{2}{n} \overline{e} \text{ Id},$$

with the macroscopic energy density $\overline{e} := \overline{\frac{1}{2}|v|^2}$ and the identity matrix Id. This motivates the following definition of subsolutions.

Definition 2.10. Let $\overline{e} \in L^1_{loc}(\mathbb{T}^n \times (0,T))$ such that $\overline{e} \geq 0$ holds. We call a triple $(\overline{v}, \overline{u}, \overline{q}) : \mathbb{T}^n \times (0,T) \to \mathbb{R}^n \times \mathcal{S}_0^{n\times n} \times \mathbb{R}$ a subsolution to the incompressible Euler equations with kinetic energy density $\overline{e}$, if
$\overline{v} \in L^2_{loc}(\mathbb{T}^n \times (0,T)), \overline{u} \in L^1_{loc}(\mathbb{T}^n \times (0,T))$ and $\overline{q}$ is a distribution,

$$\begin{cases} \partial_t \overline{v} + \operatorname{div} \overline{u} + \nabla \overline{q} = 0 \\ \operatorname{div} \overline{v} = 0. \end{cases} \quad \text{in the sense of distributions,} \tag{2.8}$$

and

$$\overline{v} \otimes \overline{v} - \overline{u} \leq \frac{2}{n} \overline{e} \ \mathrm{Id} \quad \text{a.e.} \tag{2.9}$$

Equivalently, we can express (2.9) as

$$e(\overline{v}, \overline{u}) \leq \overline{e} \quad \text{a.e.,} \tag{2.10}$$

where we define the generalized energy density

$$e(v,u) := \frac{n}{2} |v \otimes v - u|_\infty \quad \text{for } (v,u) \in \mathbb{R}^n \times \mathcal{S}_0^{n\times n}. \tag{2.11}$$

Here, by $|\cdot|_\infty$ we denote the operator norm of a matrix. For symmetric matrices this coincides with the largest eigenvalue. One can verify (see [32]) that $e : \mathbb{R}^n \times \mathcal{S}_0^{n\times n}$ is a convex function satisfying

$$\frac{1}{2}|v|^2 \leq e(v,u) \text{ with equality if and only if } u = v \otimes v - \frac{|v|^2}{n} \mathrm{Id}. \tag{2.12}$$

Obviously, the notion of a subsolution is a generalization of a weak solution, since we merely replace the nonlinearity $v \otimes v$ by a new variable $\overline{u}$ to obtain a linear system of equations. In addition, the inequality (2.9) arises and it captures how much the subsolution is away from being a weak solution. Taking the trace in (2.9), we get $\frac{1}{2}|\overline{v}|^2 \leq \overline{e}$ for a.e.(x,t). In case of having equality in this inequality, we infer from (2.12) that $\overline{u} = \overline{v} \otimes \overline{v} - \frac{|\overline{v}|^2}{n}$ Id. Then, $\overline{v}$ is actually a weak solution of the incompressible Euler equations with pressure $p = \overline{q} - \frac{1}{n}|\overline{v}|^2$. On the contrary, a nontrivial Reynolds stress $R > 0$ yields a strict inequality in (2.9) and the subsolution fails to be a weak solution. In this case we speak of a strict subsolution. The crucial idea of convex integration is that this strictness gives enough room to add high frequent oscillations to the subsolution. Roughly speaking, iterating this step until we reach equality in (2.9), leads to weak solutions. Since this can be done in a non unique way, many different solutions emerge from one common subsolution. The concrete result can be stated as

Theorem 2.11 (Subsolution Criterion)**.** Let $\overline{e} \in L^\infty(\mathbb{T}^n \times (0,T))$ and $(\overline{v}, \overline{u}, \overline{q})$ be a subsolution. Moreover, suppose there exists a subdomain $\mathcal{U} \subset \mathbb{T}^n \times (0,T)$ such that $(\overline{v}, \overline{u}, \overline{q})$ and $\overline{e}$ are continuous on $\mathcal{U}$ and

$$\begin{aligned} &e(\overline{v}, \overline{u}) < \overline{e} \ \text{ in } \mathcal{U} \\ &e(\overline{v}, \overline{u}) = \overline{e} \ \text{ a.e. in } \mathbb{T}^n \times (0,T) \setminus \mathcal{U}. \end{aligned} \tag{2.13}$$

Then there exist infinitely many weak solutions $v \in L^\infty_{loc}(\mathbb{T}^n \times (0,T))$ of the incompressible Euler equations satisfying

$$v = \overline{v} \text{ a.e. in } \mathbb{T}^n \times (0,T) \setminus \mathcal{U} \tag{2.14}$$

$$\frac{1}{2}|v|^2 = \overline{e} \text{ for a.e. } (x,t) \in \mathbb{T}^n \times (0,T) \tag{2.15}$$

$$p = \overline{q} - \frac{2}{n}\overline{e} \text{ for a.e. } (x,t) \in \mathbb{T}^n \times (0,T). \tag{2.16}$$

If in addition

$$\overline{v}(\cdot,t) \rightharpoonup v_0(\cdot) \text{ in } L^2_{loc}(\mathbb{T}^n) \text{ as } t \to 0, \tag{2.17}$$

then all the $v's$ so constructed solve the Cauchy problem (1.1)-(1.3).

Remark 2.12. (i) Condition (2.13) can be interpreted as saying that inside $\mathcal{U}$, where $(\overline{v}, \overline{u}, \overline{q})$ is not a solution, it should be a strict subsolution.
(ii) The same statement holds true on $\mathbb{R}^n$ instead of $\mathbb{T}^n$.
(iii) By a slight abuse of notation, in later applications in Theorem 2.15 and Section 4 we will denote the set $\mathcal{U}$, where the subsolution is strict, just by U. We will choose this set to be a growing zone around some sheet Γ. This is not the set U from condition (H2) of the Tartar framework.

A detailed proof can be found in [32]. However, we want to give some major ideas. Note that we can rewrite the Euler equations to fit in the general Tartar framework (2.1)-(2.2) of the previous section. With the notation from therein, we have

$$\begin{aligned} d &= n+1, \\ y &= (x,t), \\ \mathcal{D} &= \mathbb{T}^n \times (0,T), \\ z &= (\overline{v}, \overline{u}, \overline{q}), \\ \mathbb{R}^N &= \mathbb{R}^n \times \mathcal{S}_0^{n\times n} \times \mathbb{R}, \\ K &= \{(\overline{v}, \overline{u}, \overline{q}) \in \mathbb{R}^n \times \mathcal{S}_0^{n\times n} \times \mathbb{R} | e(\overline{v}, \overline{u}) = \overline{e}\}. \end{aligned}$$

Furthermore, (2.1) now reads (2.8). In the spirit of this framework, Theorem 2.11 follows from Theorem 2.8, provided that the conditions (H1)-(H3) are satisfied.

Regarding (H1), we start computing the wave cone. Suppose that $(\hat{v}, \hat{u}, \hat{q}) \in \Lambda$ as defined in (2.5). Then there exists $(\xi, c) \in \mathbb{R}^n \times \mathbb{R}$ such that

$$c\hat{v} + \xi\hat{u} + \xi\hat{q} = 0, \tag{2.18}$$

$$\xi \cdot \hat{v} = 0 \tag{2.19}$$

are satisfied. We can rewrite this as

$$P_{\hat{v}}\hat{u}P_{\hat{v}}\xi = -\hat{q}\xi,$$

where $P_{\hat v} = \mathrm{Id} - \frac{\hat v \otimes \hat v}{|\hat v|^2}$ denotes the projection onto the subspace orthogonal to $\hat v$. Thus, $(\hat v, \hat u, \hat q) \in \Lambda$ if $-\hat q$ is an eigenvalue of $P_{\hat v} \hat u P_{\hat v}$ with eigenvector ξ. Now for some given $\hat z = (\hat v, \hat u, \hat q) \in \Lambda$, we can choose the sequence

$$z_k(y) = \hat z \sin(ky \cdot (\xi, c)),$$

which satisfies the properties stated in (H1). However, z_k is not compactly supported. One possibility to overcome this problem is using a potential. Originally, this idea was introduced in [31], see also Lemma 5.1 in Section 5. In our situation, we set up the potential in such a way to get pressureless oscillations. This is substantial to ensure (2.16). More precisely, one can show that for each $\hat z \in \Lambda$, there exists a matrix-valued, constant coefficient, homogeneous linear differential operator of order l

$$A(\partial) : C_c^\infty(\mathbb{R}^{n+1}) \to C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^n \times \mathcal{S}_0^{n\times n} \times \mathbb{R})$$

and a space-time vector $\eta = (\xi, c) \in \mathbb{R}^{n+1}$ such that

$$A(\partial)\phi = (v, u, q) \text{ satisfies (2.8) for all } \phi \in C_c^\infty(\mathbb{R}^{n+1})$$
$$A(\partial)\phi(y) = \hat z \frac{d^l \psi}{ds^l}(y \cdot (\xi, c)) \quad \text{if} \quad \phi(y) = \psi(y \cdot (\xi, c)).$$

For the construction of $A(\partial)$ we refer to [32]. With such operators at hand, one can choose

$$z_k(y) = \frac{1}{k^l} A(\partial)(\psi(ky \cdot (\xi, c))\chi(y))$$

for some cut-off function $\chi \in C_c^\infty(\mathbb{R}^{n+1})$ and some suitable function ψ.

For (H2), at first observe that the set K is defined via the nonlinear constraint (2.9), which does not involve q. Moreover, we just showed that for each $\hat q$ we can find $(\hat v, \hat u)$ such that $(\hat v, \hat u, \hat q) \in \Lambda$. Hence, a natural candidate for the set U is

$$U = \{(\overline v, \overline u, \overline q) \in \mathbb{R}^n \times \mathcal{S}_0^{n\times n} \times \mathbb{R} | e(\overline v, \overline u) < \overline e\}.$$

Then, (H2) follows if we can verify that $U = \text{int } K^{co}$. For a proof of this we refer to [78].

Finally we define the space of subsolutions as

$$X_0 := \{(v, u, q) \text{ subsolution} | (v, u, q) = (\overline v, \overline u, \overline q) \text{ on } \mathcal{D} \setminus \mathcal{U},$$
$$(v, u, q) \text{ is continuous in } \mathcal{U}, e(v, u) < \overline e \text{ in } \mathcal{U}\}.$$

Here, $(\overline v, \overline u, \overline q)$ is the subsolution from the assumptions of Theorem 2.11. Then, obviously we have $X_0 \neq \emptyset$, since $(\overline v, \overline u, \overline q) \in X_0$. We infer that X_0 is a bounded subset of $L^2(\mathcal{D})$ and satisfies the conditions of (H3).

Now we want to present some important conclusions from the Subsolution Criterion, see also [34], [78], [32]. As a first result, we obtain in a very simple way a new proof of the famous nonuniqueness statement for weak solutions of the Euler equations due to Scheffer [70] and Shnirelman [72].

Theorem 2.13. For any dimension $n \geq 2$ there exist infinitely many compactly supported weak solutions to the incompressible Euler equations.

Proof. Set $(\overline{v}, \overline{u}, \overline{q}) \equiv 0$ and the energy

$$\overline{e} := \begin{cases} 1, & t \in (T_1, T_2) \\ 0, & \text{otherwise} \end{cases}$$

for $0 < T_1 < T_2 < T$. Moreover let $\mathcal{U} = \mathbb{T}^n \times (T_1, T_2)$. Then the statement follows from Theorem 2.11. □

A further consequence is the existence of global weak solutions for arbitrary initial data from [83].

Theorem 2.14. Let $v_0 \in L^2(\mathbb{T}^n)$ be a divergence-free vectorfield. Then there exist infinitely many global weak solutions of the Euler equations such that the energy

$$E(t) = \frac{1}{2} \int_{\mathbb{T}^n} |v(x,t)|^2 dx$$

is bounded. Furthermore, $E(t) \to 0$ as $t \to \infty$.

Proof. At first we construct a smooth subsolution with initial data v_0, i.e. a triple $(\overline{v}, \overline{u}, \overline{q})$ satisfying (2.22) and $\overline{v}_{|t=0} = v_0$. Afterwards we choose the energy function $\overline{e}$ in such a way that we obtain a strict subsolution, i.e. $e(\overline{v}, \overline{u}) < \overline{e}$ should be valid. For example we can take $\overline{e} := e(\overline{v}, \overline{u}) + \min(t, \frac{1}{t})$. To obtain weak solutions with bounded energy, note that the above Cauchy system for $(\overline{v}, \overline{u}, \overline{q})$ is underdetermined and admits many solutions. In particular, we can construct a solution with $\overline{v} \in L^\infty(0, T; L^2(\mathbb{T}^n))$ and $\overline{u} \in L^\infty(0, T; L^1(\mathbb{T}^n))$, see [83] for details. □

However, the energy in the previous theorem will not converge towards $\frac{1}{2}\|v_0\|^2_{L^2}$ as $t \to 0$. In general, there will occur instantaneous jumps. The energy can possibly even increase, which does physically make no sense. We already mentioned that the hope to get uniqueness from admissibility (1.11) turned out to be wrong. The next Theorem shows the existence of wild initial data, meaning those, which admit infinitely many admissible weak solutions. It is not that easy to establish the existence of wild initial data, since in addition to solving the Cauchy problem, the subsolution $(\overline{v}, \overline{u}, \overline{q})$ should satisfy a form of an energy inequality

$$\int_{\mathbb{T}^n} e(\overline{v}(x,t), \overline{u}(x,t)) dx < \frac{1}{2} \int_{\mathbb{T}^n} |v_0(x)|^2 dx.$$

If this inequality is true, we can choose some energy function $\overline{e}$ satisfying $e(\overline{v}, \overline{u}) < \overline{e}$ for a.e. $(x,t) \in \mathbb{T}^n \times (0, T)$ and

$$\int_{\mathbb{T}^n} \overline{e}(x,t) dx < \frac{1}{2} \int_{\mathbb{T}^n} |v_0(x)|^2 dx.$$

An application of the Subsolution Criterion yields infinitely many admissible weak solutions.

The most simple example, were this can be done, is the shear flow, see [77].

Theorem 2.15. There exist infinitely many admissible weak solutions with initial data (1.19).

Proof. As mentioned above, we only have to construct a suitable subsolution and apply Theorem 2.11 afterwards. We give a short sketch of the proof in two dimensions. At first, choose the ansatz

$$\overline{v} = (\alpha, 0), \overline{u} = \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix}, \overline{q} = \beta$$

with $\alpha = \alpha(x_2, t), \beta = \beta(x_2, t), \gamma = \gamma(x_2, t)$. Then (2.8) reads

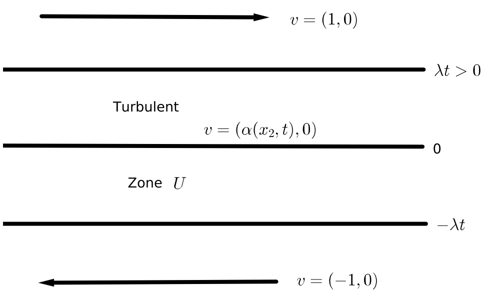
$$\partial_t \alpha + \partial_{x_2} \gamma = 0.$$

Set $\beta = \frac{1}{2}\alpha^2, \gamma = -\frac{\lambda}{2}(1 - \alpha^2), \lambda \in (0, 1)$. By this we can further reduce to Burger's equation

$$\partial_t \alpha + \frac{\lambda}{2}\alpha^2 = 0,$$

which has the unique rarefaction wave solution

$$\alpha(x_2, t) = \begin{cases} -1, & -\frac{1}{2} < x_2 < -\lambda t \\ \frac{x_2}{\lambda t}, & -\lambda t < x_2 < \lambda t \\ 1, & \lambda t < x_2 < \frac{1}{2} \end{cases}.$$



Moreover, we define the growing zone $U := \{|\alpha| < 1\} = \{(x,t) : |x_2| < \lambda t\}$ (this is the set $\mathcal{U}$ in Theorem 2.11). Then we see

$$e(\overline{v}, \overline{u}) = \frac{\alpha^2}{2} + \frac{\lambda}{2}(1 - \alpha^2) \begin{cases} = \frac{1}{2}, & \text{in } U^c \\ < \frac{1}{2}, & \text{in } U \end{cases}.$$

Thus, we can choose some continuous $\overline{e}$ with $e(\overline{v}, \overline{u}) < \overline{e} < \frac{1}{2}$ on U and $\overline{e} = \frac{1}{2}$ on U^c. □

Beyond the flat vortex sheet, one can show that a circular initial data is wild. By means of its symmetry, an analogous ansatz reduces (2.8) to Burger's equation as in the flat case.

Theorem 2.16. Let $0 < \rho < r_0 < R < \infty, \Omega = \{(r, \theta) \in \mathbb{R}^2 | \rho < r < R\}$ and consider the initial data

$$v_0(r, \theta) = \begin{cases} -\frac{1}{r^2}(\sin\theta, -\cos\theta), & \rho < r < r_0 \\ \frac{1}{r^2}(\sin\theta, -\cos\theta), & r_0 < r < R \end{cases}. \tag{2.20}$$

Then there exist infinitely many admissible weak solutions of the incompressible Euler equations with initial data (2.20).

We refer to [5] for a proof and want to note that this Theorem is also valid in n dimensions.

Apart from these two examples, there are no other explicite wild initial data known. Nevertheless, the set of wild initial data is indeed very large in the sense of the following Theorem.

Theorem 2.17. The set of wild initial data is dense in the space of divergence-free L^2 vector fields.

A good proof of this can be found in [78].

In conclusion, we have seen that the Subsolution Criterion leads to many important results. However, a lot of questions remain open. In Section 4, I will focus on constructing more concrete examples of wild initial data in the class of vortex sheets. The difficulty is to build suitable subsolutions for which we can apply Theorem 2.11. By means of Theorem 2.15 and Theorem 2.16, I try an analogous construction, in the sense that a growing turbulent zone U around the sheet arises in which the subsolution is strict. The main idea for this was developed while studying the anologous problem for the IPM equations.

2.3 Application to the IPM equations

Here, we want to apply the convex integration scheme to the IPM equations. For the following computations we refer to [79] and [17]. We start to separate (1.21)-(1.24) into a linear set of equations and a nonlinear constitutive relation. To keep calculations simple, for the moment we consider these equations on the two-dimensional periodic torus $\mathbb{T}^2$. However, we note that the same arguments can be transferred into $\mathbb{R}^2$ by the introduction of a suitable auxiliary space, see Remark 2.21, (ii). Beyond that, we impose $\tilde{u} := 2u + (0, \rho)$. Then, we write (1.21)-(1.23) in the form

$$\begin{aligned} \partial_t \rho + \operatorname{div} m &= 0, \\ \operatorname{div} (\tilde{u} - (0, \rho)) &= 0, \\ \operatorname{curl} (\tilde{u} + (0, \rho)) &= 0 \end{aligned} \tag{2.21}$$

together with the constitutive relations

$$m = \frac{1}{2}(\rho \tilde{u} - (0, 1)), \tag{2.22}$$

$$|\rho| = 1 \tag{2.23}$$

for a.e. $(x, t) \in \mathbb{T}^2 \times (0, T)$. Note that condition (2.23) is due to the assumption $\rho^{\pm} = \pm 1$ for the initial data (1.25). We always can reduce to this situation because of the following simple scaling law.

Lemma 2.18. Let $(\rho, u, p)(x, t)$ be a weak solution of (1.21)-(1.23) with initial density $\rho_0(x)$.

(i) For each $\lambda > 0$

$$\begin{aligned} \rho_\lambda(x,t) &:= \lambda\rho(x,\lambda t) \\ u_\lambda(x,t) &:= \lambda u(x,\lambda t) \\ p_\lambda(x,t) &:= \lambda p(x,\lambda t) \end{aligned}$$

is a weak solution of (1.21)-(1.23) with initial density $\rho_{\lambda,0}(x) = \lambda\rho_0(x)$.

(ii) For each constant $\overline{\rho} \in \mathbb{R}$, the triple $(\rho+\overline{\rho}, u, p - \overline{\rho}x_2)(x,t)$ is also a weak solution of (1.21)-(1.23) with initial density $\overline{\rho}_0(x) = \rho_0(x) + \overline{\rho}$.

The proof can be established by a straightforward computation. Thus, if (ρ, u, p) is a weak solution of (1.21)-(1.23) with initial data

$$\rho_0(x) = \begin{cases} 1, & x_2 > z_0(x_1) \\ -1, & x_2 < z_0(x_1) \end{cases},$$

then using Lemma 2.18,(i) for $\lambda = \frac{\rho^+ - \rho^-}{2}$ and afterwards statement (ii) with $\overline{\rho} = \frac{\rho^+ + \rho^-}{2}$, we obtain a weak solution with initial data (1.25).

With the notation from Section 2.1 we have

$$\begin{aligned} d &= 2+1, \\ y &= (x,t), \\ \mathcal{D} &= \mathbb{T}^2 \times (0,T), \\ z &= (\rho, \tilde{u}, m), \\ \mathbb{R}^N &= \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2, \\ K &= \left\{ (\rho, \tilde{u}, m) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 : |\rho| = 1, m = \frac{1}{2}\rho\tilde{u} \right\}. \end{aligned}$$

Furthermore, (2.1) becomes now (2.21). Especially, we infer that (2.22) is equivalent to $(\rho, \tilde{u}, m + \frac{1}{2}(0,1)) \in K$ for a.e. (x,t). However, in absence of boundary conditions, we can ignore the constant term $\frac{1}{2}(0,1)$. The next aim is to verify the conditions (H1)-(H3).

At first we determine the wave cone (2.5). Because of the special div-curl structure in (2.21), we deduce that

$$\Lambda = \{(\rho, \tilde{u}, m) : |\rho|^2 = |\tilde{u}|^2\}. \tag{2.24}$$

It turned out (see for example [24]), that this wave cone already satisfies the conditions of (H1).

Regarding (H2), we begin with the computation of the Λ-convex hull of K.

Proposition 2.19. We have

$$K^\Lambda = \left\{ (\rho, \tilde{u}, m) : |\rho| \le 1, \left| m - \frac{1}{2}\rho\tilde{u} \right| \le \frac{1}{2}(1-\rho^2) \right\}.$$

Proof. Thanks to the definition of K^Λ we conclude that whenever we have $z_1, z_2 \in K$ with $z_1 - z_2 \in \Lambda$, then $[z_1, z_2] \subset K^\Lambda$. Let $|\rho_0| \leq 1, \tilde{u}_0 \in \mathbb{R}^2$ and $e \in \mathbb{R}^2$ with $|e| = 1$. We set

$$\rho_1 = 1, \quad \tilde{u}_1 = \tilde{u}_0 + (1 - \rho_0)e \tag{2.25}$$
$$\rho_2 = -1, \quad \tilde{u}_2 = \tilde{u}_0 - (1 + \rho_0)e. \tag{2.26}$$

Thus, for each $m_1, m_2 \in \mathbb{R}^2$ we obviously have

$$(\rho_1 - \rho_2, \tilde{u}_1 - \tilde{u}_2, m_1 - m_2) \in \Lambda,$$

since $|\rho_1 - \rho_2| = 2 = 2|e| = |\tilde{u}_1 - \tilde{u}_2|$. In case of the concrete choice $m_1 = \frac{1}{2}\rho_1\tilde{u}_1$ and $m_2 = \frac{1}{2}\rho_2\tilde{u}_2$, we deduce that $(\rho_i, \tilde{u}_i, m_i) \in K$ for $i = 1, 2$. Hence, we observe that $\lambda(\rho_1, \tilde{u}_1, m_1) + (1 - \lambda)(\rho_2, \tilde{u}_2, m_2) \in K^\Lambda$ for all $\lambda \in [0, 1]$. Choosing $\lambda = \frac{1}{2}(1 + \rho_0)$, we infer $(\rho_0, \tilde{u}_0, m_0) \in K^\Lambda$, where

$$m_0 = \frac{1}{2}(1 + \rho_0)m_1 + \frac{1}{2}(1 - \rho_0)m_2 = \frac{1}{2}\rho_0\tilde{u}_0 + \frac{1}{2}(1 - \rho_0^2)e.$$

Thus, we established that each $(\rho, \tilde{u}, m)$ with

$$|\rho| \leq 1, \left| m - \frac{1}{2}\rho\tilde{u} \right| = \frac{1}{2}(1 - \rho^2)$$

is contained in K^Λ. Now, let $|\rho| \leq 1, \tilde{u} \in \mathbb{R}^2$ be arbitrary and $e \in \mathbb{R}^2, |e| = 1$. Then we set

$$\overline{m}_\pm := \frac{1}{2}\left(\rho\tilde{u} \pm (1 - \rho^2)e\right),$$

which clearly satisfies

$$\left| \overline{m}_\pm - \frac{1}{2}\rho\tilde{u} \right| = \frac{1}{2}(1 - \rho^2).$$

Since $(\rho, \tilde{u}, \overline{m}_\pm) \in K^\Lambda$ for $i = 1, 2$ and $(\rho, \tilde{u}, \overline{m}_+) - (\rho, \tilde{u}, \overline{m}_-) = (0, 0, (1 - \rho^2)e) \in \Lambda$, we conclude that each $(\rho, \tilde{u}, m)$ with

$$|\rho| \leq 1, \left| m - \frac{1}{2}\rho\tilde{u} \right| \leq \frac{1}{2}(1 - \rho^2)$$

is contained in K^Λ.

For the other inclusion, observe that

$$K \subset \left\{ z : g(z) \leq \frac{1}{2} \right\},$$

where we define $g(\rho, \tilde{u}, m) := f(\rho, \tilde{u}, m) + \frac{1}{4}(\rho^2 - |\tilde{u}|^2)$ and $f(\rho, \tilde{u}, m) := \left| m - \frac{1}{2}\rho\tilde{u} \right| + \frac{1}{4}(\rho^2 + |\tilde{u}|^2)$. Since f is a convex function and $\rho^2 - |\tilde{u}|^2$ is a Λ-convex function, also g is Λ-convex. Thus, $K^\Lambda \subset \left\{ z : g(z) \leq \frac{1}{2} \right\}$. $\square$

In the next step we need to find an appropriate open set U satisfying the conditions of (H2). Fortunately, the first natural candidate $U = \text{int } K^\Lambda$ or in our case equivalently

$$U = \left\{ (\rho, \tilde{u}, m) : |\rho| < 1, \left| m - \frac{1}{2}\rho\tilde{u} \right| < \frac{1}{2}(1-\rho^2) \right\}$$

already enjoys these properties. Indeed, we have

Lemma 2.20. There exists a constant $c > 0$ such that for each $(\rho, \tilde{u}, m) \in U$ we can find some $(\overline{\rho}, \overline{u}, \overline{m}) \in \Lambda \cap S^4$ with

$$(\rho, \tilde{u}, m) + s(\overline{\rho}, \overline{u}, \overline{m}) \in U$$

for all $|s| < c(1-\rho^2)$.

Proof. From the first part of the proof of Proposition 2.19 and (2.25),(2.26) we deduce that for any $(\rho, \tilde{u}, m)$ satisfying $|\rho| < 1$ and $|m - \frac{1}{2}\rho\tilde{u}| = \frac{1}{2}(1-\rho^2)$ there exists $(\overline{\rho}, \overline{u}, \overline{m}) \in \Lambda$ with $\overline{\rho} = 1$ such that

$$(\rho, \tilde{u}, m) + s(\overline{\rho}, \overline{u}, \overline{m}) \in K^\Lambda$$

for all $|s| < c_0(1-\rho^2)$ for some $c_0 > 0$. This follows since we constructed $(\rho_0, \tilde{u}_0, m_0)$ as a point on the segment $[(\rho_1, \tilde{u}_1, m_1), (\rho_2, \tilde{u}_2, m_2)]$. Together with a continuity argument, this yields the statement, provided that

$$\frac{1}{4}(1-\rho^2) < \left| m - \frac{1}{2}\rho\tilde{u} \right| < \frac{1}{2}(1-\rho^2).$$

For the case $\left| m - \frac{1}{2}\rho\tilde{u} \right| < \frac{1}{4}(1-\rho^2)$ away from ∂U, we simply choose $(\overline{\rho}, \overline{u}, \overline{m}) = (0, 0, \overline{m})$ with $|\overline{m}| = 1$ and $\overline{m}$ being parallel to $m - \frac{1}{2}\rho\tilde{u}$. □

Remark 2.21. (i) Note that the set K is not bounded. Because of this, the solutions constructed by the convex integration method will not be in L^∞, but merely in L^2. We can handle this problem by suitably modifying the sets K and K^Λ. Let $M > 1$ and define K_M as a compact subset of K as follows

$$K_M := \left\{ (\rho, \tilde{u}, m) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 : |\rho| = 1, m = \frac{1}{2}\rho\tilde{u}, |\tilde{u}| \leq M \right\}.$$

Although the computations become more technical, one can verify (see [79]) that K^Λ is given by the set of $(\rho, \tilde{u}, m)$, satisfying the following inequalities

$$|\rho| \leq 1, \tag{2.27}$$

$$\left| m - \frac{1}{2}\rho\tilde{u} \right| \leq \frac{1}{2}(1-\rho^2), \tag{2.28}$$

$$|\tilde{u}|^2 \leq M^2 - (1-\rho^2), \tag{2.29}$$

$$\left| m - \frac{1}{2}\tilde{u} \right| \leq \frac{M}{2}(1-\rho), \tag{2.30}$$

$$\left| m + \frac{1}{2}\tilde{u} \right| \leq \frac{M}{2}(1+\rho). \tag{2.31}$$

Analogously, one can show that $U_M := \text{int } (K_M)^\Lambda$, meaning the set defined with strict inequalities for (2.27)-(2.31), also satisfies (H2). By means of this extension we are able to accomplish bounded weak solutions.

(ii) To apply the foregoing results also in the unbounded set $\mathbb{R}^2$ instead of $\mathbb{T}^2$ we introduce the measures

$$d\mu = \frac{dx}{(1+|x|)^3} \text{ and } d\tilde{\mu} = d\mu dt.$$

Then we have $\mu(\mathbb{R}^2) < \infty$ and $L^\infty(\mathbb{R}^2) \subset L^2(d\mu)$ and thus we can work in the auxiliar space $L^2(d\tilde{\mu})$. As investigated in [17], the previous calculations for (H1), (H2) remain valid.

Because of this, from now on we will study (2.21) in $\mathbb{R}^2$. Moreover, we switch back to the velocity u instead of $\tilde{u}$. To deal with (H3), we first need to characterize a subsolution. The main idea is to construct subsolutions having a growing mixing zone Ω_{mix} around the sheet Γ, just as we had in case of the Euler equations.

Definition 2.22. Let $T > 0$. We call $(\rho, u, m) \in \left(L^\infty(\mathbb{R}^2 \times [0,T))\right)^3$ a subsolution of (1.21)-(1.24) if there exist open and simply connected domains $\Omega^\pm(t), \Omega_{\text{mix}}(t)$ satisfying $\overline{\Omega^+}(t) \cup \overline{\Omega^-}(t) \cup \Omega_{\text{mix}}(t) = \mathbb{R}^2$ for each $t \in (0,T)$ and

(i) $\rho(x,t) = \pm 1$ in $\Omega^\pm(t)$,

(ii)

$$\begin{cases} \partial_t \rho + \text{div } m &= 0 \\ \text{div } u &= 0 \\ \text{curl } u &= -\partial_{x_1}\rho \\ \rho(x,0) &= \rho_0(x) \end{cases} \tag{2.32}$$

holds in the sense of distributions in $\mathbb{R}^2 \times [0,T)$,

(iii) the inequality

$$\left| m - \rho u + \frac{1}{2}(0, 1-\rho^2) \right| \leq \frac{1}{2}\left(0, 1-\rho^2\right) \tag{2.33}$$

is satisfied with strict inequality in $\Omega_{\text{mix}}(t) \times (0,T)$ and equality in $\Omega^\pm(t) \times [0,T)$,

(iv) (ρ, u, m) is continuous in $\Omega_{\text{mix}}(t) \times (0,T)$.

Again, thanks to this definition we can cast (1.21)-(1.24) into the Tartar framework. Here, the inequality (iii) measures how far the subsolution is away from satisfying the nonlinear constraint. If we have equality in (2.33), then the subsolution is a weak solution of the IPM equations. In particular, this is satisfied for $|\rho| = 1$, that is, inside $\Omega^\pm(t)$. On the contrary, in $\Omega_{\text{mix}}(t)$ we have a strict inequality in (iii).

As already mentioned above, using the slightly sharper notion of an M-subsolution, additionally we need to require the inequalities (2.29)-(2.31) with strict inequality sign in $\Omega_{\text{mix}}(t)$ and equality in $\Omega^\pm(t)$. By means of the construction in Section 3, which yields a bounded velocity u (see Theorem 3.8), it is easy to satisfy these constraints for some $M > 1$ big enough. For simplicity, we will drop these additional constraints in the following.

Assume that some subsolution $(\overline{\rho}, \overline{u}, \overline{m}) \in \left(L^\infty(\mathbb{R}^2 \times [0,T))\right)^3$ is given. Then define the associated set

$$X_0 := \{(\rho, u, m) \in \left(L^\infty(\mathbb{R}^2 \times [0,T))\right)^3 : (\rho, u, m) \text{ is a subsolution and } (\rho, u, m) = (\overline{\rho}, \overline{u}, \overline{m}) \text{ a.e. in } \mathbb{R}^2 \setminus \overline{\Omega}_{mix}\}.$$

Obviously, this set is nonempty, since $(\overline{\rho}, \overline{u}, \overline{m}) \in X_0$. Observe that for each $(\rho, u, m) \in X_0$ we have $\|\rho\|_{L^2(d\tilde{\mu})}, \|u\|_{L^2(d\tilde{\mu})} < C$. Together with (2.33), we deduce that X_0 is a bounded subset of $L^2(d\tilde{\mu})$. Furthermore, one easily verifies that the perturbation property of (H3) is valid with these definitions. Thus, we infer

Theorem 2.23. Let X_0 be nonempty. Then the set of weak solutions to (1.21)-(1.24) is residual.

Remark 2.24. In [17], it is shown that the weak solutions (ρ, u) obtained via convex integration additionally satisfy the following mixing property. In $\Omega^\pm(t)$ we have $\rho \equiv \rho^\pm$ and inside the mixing zone $\Omega_{\text{mix}}(t)$ it holds $(\rho - \rho^+)(\rho - \rho^-) = 0$. Moreover, for each $B_r(x,t) \subset \bigcup_{0<t<T} \Omega_{\text{mix}}(t)$ we have

$$\int_{B_r(x,t)} (\rho - \rho^+) dx' dt' \int_{B_r(x,t)} (\rho - \rho^-) dx' dt' \neq 0.$$

This can be interpreted in the sense that ρ takes both values ρ^+ and ρ^- in each space-time ball inside the mixing zone, i.e. we have an infinite mixing. From this perspective, these weak solutions are called mixing solutions.

Hence, as for the Euler equations, the main difficulty is to find subsolutions in order to make the convex integration machinery work. In the following we want to show how the construction of subsolutions for Muskat type initial data of the form (1.25) has been established in the papers [79] and [17].

The approach for the horizontal interface $\Gamma \equiv 0$ in [79] resembles to the one in [77], namely using the x_1-invariance. Thus, we set $u \equiv 0,\ m = (0, m_2)$ and assume that ρ and m_2 only depend on x_2 and t. The ansatz $m_2 = -\alpha(1 - \rho^2)$ for some $\alpha \in (0,1)$ reduces (2.32) to Burger's equation just as for the Euler equations. We obtain the solution

$$\rho(x,t) = \begin{cases} 1 & , x_2 > 2\alpha t \\ \frac{x_2}{2\alpha t} & , |x_2| < 2\alpha t \\ -1 & , x_2 < -2\alpha t \end{cases}.$$

Furthermore, it is not hard to see that this subsolution satisfies (2.33) inside of the mixing zone $\Omega_{\text{mix}} = \{(x,t) : |x_2| < 2\alpha t\}$. Observe that the propagation speed $c = 2\alpha$ is in the range $(0, 2)$. In fact, the boarderline case $\alpha = 1$ gives the maximal possible propagation speed, in the sense that the mixing zone of each x_1-invariant subsolution is contained in $\{(x,t) : |x_2| < 2t\}$. Moreover, it turned out that in this case of maximal mixing, the subsolution coincides with the solution constructed by Otto in [67], [68]. Thus, there seems to be a selection criteria at least for subsolutions. We want to emphasize that the time of existence of this solution is independent of α.

It is interesting to note that the underlying subsolution $(\overline{\rho}, \overline{u}, \overline{m})$ already captures properties of the weak solutions, like the propagation speed of the mixing zone. Moreover, as already mentioned in [79], one can verify the existence of a sequence of weak solutions (ρ_k, v_k), such that $\rho_k \overset{*}{\rightharpoonup} \overline{\rho}$ in $L^\infty(\Omega_{\text{mix}})$. Hence, we can view $\overline{\rho}$ as a coarse-grained density.

In [17], the authors used a similar ansatz for a subsolution. Just like in the flat case, the density was chosen as a linear interpolation between $\rho^+ = 1$ and $\rho^- = -1$. Once the density is fixed, the velocity is defined as

$$u(x) := \mathcal{BS}(-\partial_{x_1}\rho) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}(-\partial_{y_1}\rho(y))dy.$$

To satisfy (2.32), it remains to find some m such that $\partial_t\rho + \operatorname{div} m = 0$ holds. Using special coordinates adapted to the sheet Γ and a suitable ansatz for m, one can solve this equation for m by a simple integration. Finally, to satisfy the strictness inequality (2.33), it turned out that the curve $z(s,t)$ has to solve an evolution equation in time given by

$$\begin{aligned} \partial_t z(s,t) &= \mathcal{M}u(s), \\ z(s,0) &= z_0(s), \end{aligned} \tag{2.34}$$

whith the operator

$$\mathcal{M}u(s) = -\frac{1}{2ct}\int_{-ct}^{ct}\frac{1}{\pi}\int_{-\infty}^{\infty}(\partial_s z(s) - \partial_s z(s'))\frac{1}{2ct}\int_{-ct}^{ct}\frac{(z_1(s)-z_1(s'))d\lambda'}{|z(s)-z(s')+(\lambda-\lambda')(0,1)|^2}ds'd\lambda.$$

Here, $c > 0$ is the propagation speed of the mixing zone Ω_{mix}. This operator resembles to the one in the Muskat problem (1.26). Solving this equation for z needs some effort, including pseudo-differential operator theory. Nevertheless, it could be established that for each initial data $z_0 \in H^5(\mathbb{R})$ and $0 < c < 2$, there exists a solution $z \in C([0,T]; H^4(\mathbb{R}))$, see Theorem 4.1 in [17]. In particular, this yields a subsolution in the sense of the above Definition 2.22.

In addition, there exist initial curves z_0, for which it is possible to find infinitely many admissible weak solutions even in the stable case. However, this is valid only for z_0 corresponding to straight interfaces except of the horizontal one. One can imagine that although we are in the stable case, due to gravity and the sloped interface, the heavier fluid starts to slip downwards, which causes a mixing of the fluids. This explains physically the emergence of the wild behaviour of weak solutions. The construction of a subsolution can be set up very similar to the flat case.

3 Piecewise constant density for the IPM equations

3.1 Subsolutions for the IPM equations

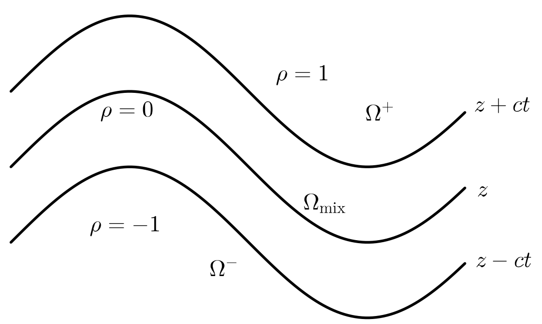
The most technical part in [17] is to establish an existence result for (2.34). In this section we want to show that it is indeed possible to construct a subsolution without solving such a nonlinear evolution equation. In view of Theorem 2.23, this yields a proof of Theorem 1.3. Throughout this section, we will assume that the initial curve z_0 can be split into $z_0(s) = \beta s + \overline{z}_0(s)$. Here, $\overline{z}_0$, can be viewed as a perturbation of the straight line βs. In the following, we start to consider the unstable regime.

The key idea is the observation that the choice of the density, adopted from the flat case, seems to be too restrictive in general. Instead, we allow a discontinuous density and set

$$\rho(x,t) = \begin{cases} 1 & , x \in \Omega^+(t) \\ 0 & , x \in \Omega_{\text{mix}}(t) \\ -1 & , x \in \Omega^-(t) \end{cases} , \tag{3.1}$$

where we define

$$\begin{aligned} \Omega^+(t) &= \{x \in \mathbb{R}^2 : x_2 > z(x_1,t) + ct\}, \\ \Omega_{\text{mix}}(t) &= \{x \in \mathbb{R}^2 : z(x_1,t) - ct < x_2 < z(x_1,t) + ct\}, \\ \Omega^-(t) &= \{x \in \mathbb{R}^2 : x_2 < z(x_1,t) - ct\}. \end{aligned} \tag{3.2}$$



Here, $c > 0$ is the propagation speed of $\Omega_{\text{mix}}(t)$. This is the simplest possible choice of a piecewise constant density. In Section 3.5 we will consider more general densities of this form. Note that we choose the mixing zone and the density in a symmetric way, in the sense that the two boundaries of $\Omega_{\text{mix}}(t)$ are given by the same curve z and propagate with equal speed c. This will lighten some calculations in the definition of the curve z in Section 3.3.

The choice (3.1) already determines the velocity by defining u via the Biot-Savart law $u = \mathcal{BS}(-\partial_{x_1}\rho)$, see also (3.18) and Theorem 3.8. In the next step, we set $m = \pm u$ in $\Omega^\pm(t)$, whereas inside $\Omega_{\text{mix}}(t)$ we define

$$m = \rho u - (1-\rho^2)\left(\gamma_1, \gamma_2 + \frac{1}{2}\right) = -\left(\gamma_1, \gamma_2 + \frac{1}{2}\right)$$

for some $\gamma : \Omega_{\text{mix}}(t) \to \mathbb{R}^2$. Then, (2.33) reads $|\gamma| < \frac{1}{2}$. The first equation in (2.32) becomes just div $\gamma = 0$, therefore we choose $\gamma = \nabla^\perp g$ for some function $g : \Omega_{\text{mix}}(t) \to \mathbb{R}$.

However, because of the discontinuity of ρ, we have to deal now with additional jump conditions on the boundary. We denote the two boundaries by

$$\Gamma^{\pm} = \{x \in \mathbb{R}^2 : x_2 = z(x_1, t) \pm ct\}$$

with corresponding normal vectors (pointing upwards)

$$n^{\pm} = \frac{(-\partial_s z, 1, \mp c - \partial_t z)}{\sqrt{1 + (\partial_s z)^2 + (\partial_t z \pm c)^2}}.$$

After multiplication with the normalizing factor, the jump conditions become

$$(\mp c - \partial_t z) \cdot (-1) + \left(\mp \left(\gamma_1, \gamma_2 + \frac{1}{2} \right) - u \right) \cdot (-\partial_s z, 1) = 0$$

or equivalently

$$\begin{aligned} \partial_t z - u \cdot (-\partial_s z, 1) + c - \frac{1}{2} &= \gamma \cdot (-\partial_s z, 1) \text{ on } \Gamma^+ \\ \partial_t z - u \cdot (-\partial_s z, 1) - c + \frac{1}{2} &= -\gamma \cdot (-\partial_s z, 1) \text{ on } \Gamma^-. \end{aligned}$$

By use of the following notations

$$\begin{aligned} \nu^{\pm}(s,t) &:= (u(s, z(s,t) \pm ct) \cdot (-\partial_s z(s,t), 1)), \\ \partial_\tau g &= \nabla g \cdot (1, \partial_s z), \end{aligned}$$

we infer the following conditions for the existence of a subsolution

$$|\nabla g| < \frac{1}{2} \text{ in } \Omega_{\text{mix}}(t) \tag{3.3}$$

$$\partial_t z - \nu^+ + c - \frac{1}{2} = \partial_\tau g \text{ on } \Gamma^+ \tag{3.4}$$

$$\partial_t z - \nu^- - c + \frac{1}{2} = -\partial_\tau g \text{ on } \Gamma^-. \tag{3.5}$$

In case of the stable initial data, we can argue similarly and conclude

$$|\nabla g| < \frac{1}{2} \text{ in } \Omega_{\text{mix}}(t) \tag{3.6}$$

$$\partial_t z - \nu^+ + c + \frac{1}{2} = -\partial_\tau g \text{ on } \Gamma^+ \tag{3.7}$$

$$\partial_t z - \nu^- - c - \frac{1}{2} = \partial_\tau g \text{ on } \Gamma^-. \tag{3.8}$$

The following theorem states a condition under which we can find a solution g of these systems.

Theorem 3.1. Let $z_0(s) = \beta s + \overline{z}_0(s)$ for some $\beta \in \mathbb{R}, \overline{z}_0 : \mathbb{R} \to \mathbb{R}$ and suppose that $z : \mathbb{R} \times [0,T) \to \mathbb{R}$ satisfies $z(\cdot,t) \in W^{1,\infty}(\mathbb{R})$ for each $t \in [0,T)$ and

$$\lim_{t\to 0} \frac{1}{t} \|2\partial_t z(\cdot,t) - \nu^+(\cdot,t) - \nu^-(\cdot,t)\|_{L^1(\mathbb{R})} = 0 \tag{3.9}$$

$$\lim_{t\to 0} \|\partial_t z(\cdot,t) - \nu^\pm(\cdot,t)\|_{L^\infty(\mathbb{R})} = 0 \tag{3.10}$$

$$z(s,0) = z_0(s) \quad \forall s \in \mathbb{R}. \tag{3.11}$$

(i) If $0 < c < 1$, then for each $t \in (0,T)$ with $T > 0$ sufficiently small, there exists a subsolution for (1.21)-(1.24) in the unstable regime.
(ii) If $\|\partial_s \overline{z}_0\|_{L^\infty(\mathbb{R})} < \sqrt{1+\beta^2} - 1$ and $c > 0$ is small enough, there exists $T > 0$ sufficiently small such that for each $t \in (0,T)$, there exists a subsolution for (1.21)-(1.24) in the stable regime.

Proof. (i) By means of the above derivation, each solution g of (3.3)-(3.5) gives rise to a subsolution. For the construction of such a function g, at first we introduce suitable new coordinates (s,λ) as in [17] by

$$(x_1, x_2) = (s, z(s,t) + \lambda),$$

with $s \in \mathbb{R}, \lambda \in [-ct, ct]$. For each C^1 function $f : \Omega_{\text{mix}}(t) \to \mathbb{R}$ we denote

$$\hat{f}(s,\lambda) := f(s, z(s,t) + \lambda).$$

Then we can compute the spacial derivatives in these new coordinates by using the chain rule

$$\hat{\nabla}\hat{f} = \begin{pmatrix} 1 & \partial_s z \\ 0 & 1 \end{pmatrix} \cdot \nabla f, \tag{3.12}$$

$$\nabla f = \begin{pmatrix} 1 & -\partial_s z \\ 0 & 1 \end{pmatrix} \cdot \hat{\nabla}\hat{f}, \tag{3.13}$$

where $\hat{\nabla} = (\partial_s, \partial_\lambda)$. From (3.12) we observe that $\partial_s \hat{g} = \partial_\tau \hat{g}$ and because of (3.4) and (3.5) we can prescribe g on the boundary $\Gamma^\pm$ by a simple integration

$$g(s, z(s,t) \pm ct) = \hat{g}(s, \pm ct) = \int_0^s \pm(\partial_t z - \nu^\pm) + \left(c - \frac{1}{2}\right) ds'.$$

The most appropriate choice for extending $\hat{g}$ to $\Omega_{\text{mix}}(t)$ is to take the linear interpolation. For $\lambda \in [-ct, ct]$ we define

$$\begin{aligned} g(s, z(s,t) + \lambda) = \hat{g}(s,\lambda) &= \frac{\lambda + ct}{2ct}(\hat{g}(s,ct)) - \frac{\lambda - ct}{2ct}(\hat{g}(s,-ct)) \\ &= s\left(c - \frac{1}{2}\right) + \frac{\lambda + ct}{2ct}\left(\int_0^s \partial_t z - \nu^+ ds'\right) + \frac{\lambda - ct}{2ct}\left(\int_0^s \partial_t z - \nu^- ds'\right). \end{aligned}$$

Thus, thanks to $0 < c < 1$ and (3.10), we estimate

$$|\partial_s \hat{g}| \le \left|c - \frac{1}{2}\right| + \|\partial_t z - \nu^+\|_{L^\infty(\mathbb{R})} + \|\partial_t z - \nu^-\|_{L^\infty(\mathbb{R})} < \frac{1}{2},$$

for small times t. In the λ-direction we calculate

$$\begin{aligned}|\partial_\lambda \hat{g}| &= \left|\frac{\hat{g}(s,ct)-\hat{g}(s,-ct)}{2ct}\right| \\ &\leq \frac{\|2\partial_t z(\cdot,t)-\nu^+(\cdot,t)-\nu^-(\cdot,t)\|_{L^1(\mathbb{R})}}{2ct} \to 0\end{aligned}$$

as $t\to 0$, because of (3.9). To verify (3.3), we deduce from (3.13)

$$|\nabla g| < |\partial_s \hat{g}| + |\partial_\lambda \hat{g}|(1+\|\partial_s z\|_{L^\infty(\mathbb{R})}) < \frac{1}{2}$$

for small times $t>0$.

(ii) Similarly to the unstable regime, we introduce new coordinates via

$$(x_1,x_2) = (s-\beta\lambda, z(s,t)+\lambda)$$

with $s\in\mathbb{R}, \lambda\in[-ct,ct]$ and for each C^1 function $f:\Omega_{\text{mix}}(t)\to\mathbb{R}$ we introduce

$$\hat{f}(s,\lambda) := f(s-\beta\lambda, z(s,t)+\lambda).$$

For the derivatives, we obtain

$$\hat{\nabla}\hat{f} = \begin{pmatrix}1 & \partial_s z\\ -\beta & 1\end{pmatrix}\cdot\nabla f, \tag{3.14}$$

$$\nabla f = \frac{1}{1+\beta\partial_s z}\begin{pmatrix}1 & -\partial_s z\\ \beta & 1\end{pmatrix}\cdot\hat{\nabla}\hat{f}. \tag{3.15}$$

Since we assumed $\|\partial_s \overline{z}_0\|_{L^\infty(\mathbb{R})}<\beta$, we also infer $\|\partial_s\overline{z}(\cdot,t)\|_{L^\infty(\mathbb{R})}<\beta$ for small times and hence $1+\beta\partial_s z = 1+\beta^2+\beta\partial_s\overline{z}>0$. In view of (3.7)-(3.8) we define

$$g(s,z(s,t)\pm ct) = \hat{g}(s,\pm ct) = \int_0^s \mp(\partial_t z-\nu^\pm) - \left(c+\frac{1}{2}\right)ds'$$

and for $\lambda\in[-ct,ct]$

$$\begin{aligned}g(s,z(s,t)+\lambda) = \hat{g}(s,\lambda) &= \frac{\lambda+ct}{2ct}(\hat{g}(s,ct)) - \frac{\lambda-ct}{2ct}(\hat{g}(s,-ct))\\ &= -s\left(c+\frac{1}{2}\right) - \frac{\lambda+ct}{2ct}\left(\int_0^s \partial_t z-\nu^+ ds'\right) - \frac{\lambda-ct}{2ct}\left(\int_0^s \partial_t z-\nu^- ds'\right).\end{aligned}$$

From (3.9), (3.10) we deduce

$$\left|\partial_s\hat{g} + \left(c+\frac{1}{2}\right)\right| \to 0 \text{ and } |\partial_\lambda\hat{g}|\to 0$$

as $t\to 0$. Together with (3.15) we get

$$|\nabla g| = \frac{\sqrt{1+\beta^2}}{1+\beta^2+\beta\partial_s\overline{z}}\left(c+\frac{1}{2}\right) + o(1)$$

as $t \to 0$ and observe that whenever

$$\frac{\sqrt{1+\beta^2}}{1+\beta^2+\beta\partial_s\overline{z}} < 1, \tag{3.16}$$

we can choose some small enough $c > 0$ such that for small times (3.6) is satisfied. A short calculation shows that (3.16) holds, provided that

$$\left|\frac{\beta\partial_s\overline{z}}{\sqrt{1+\beta^2}}\right| < \sqrt{1+\beta^2} - 1.$$

Thanks to the condition $\|\partial_s\overline{z}_0\|_{L^\infty(\mathbb{R})} < \sqrt{1+\beta^2} - 1$, this inequality is satisfied for small times. □

Remark 3.2.
(i) In particular, we observe that in the unstable regime $T \to 0$ as $c \to 1$, saying that the time of existence T goes to zero as we reach the boarder line velocity $c = 1$.
(ii) In the stable regime, we have to rule out the case of a horizontal line $\beta = 0$ just like in [17].

We deduce that to prove Theorem 1.3, it is enough to construct a curve z, which solves (3.9)-(3.11). Before that, we need to study in more detail the velocity u.

3.2 The velocity u

At first, we derive a concrete representation formula for the velocity u. We are only interested in the kinematics in the sense that for fixed time we are looking for a solution u of

$$\begin{cases} \operatorname{div} u &= 0 \\ \operatorname{curl} u &= -\partial_{x_1}\rho \end{cases} \quad \text{in } \mathbb{R}^2. \tag{3.17}$$

For this reason we drop the time variable in this section. Suppose for the moment that $\rho \in C^1_c(\mathbb{R}^2)$. Since $\operatorname{div} u = 0$, there exists some stream function ψ such that $u = \nabla^\perp\psi$. Thus, we obtain $-\Delta\psi = \partial_{x_1}\rho$ in $\mathbb{R}^2$, which leads to $\psi = \Phi_N * \partial_{x_1}\rho$, where by $\Phi_N(x) = -\frac{1}{2\pi}\log|x|$, we denote the Newton potential in two dimensions. Hence, a solution of (3.17) is given by

$$u(x) := \mathcal{BS}(-\partial_{x_1}\rho) := \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}(-\partial_{x_1}\rho)(y)dy. \tag{3.18}$$

Proposition 3.3. Let $\rho \in C^1_c(\mathbb{R}^2)$ and define u by (3.18). Then u solves (3.17) and furthermore u vanishes at infinity, $|u(x)| \to 0$ as $|x| \to \infty$.

Proof. At first, since ρ has compact support and by use of polar coordinates, we easily see that u is well-defined. Moreover, for large x we establish

$$|u(x)| \leq C\int_{\operatorname{supp}(\rho)}\frac{1}{|x-y|}dy \leq \frac{C}{\operatorname{dist}(x,\operatorname{supp}(\rho))} \to 0 \quad \text{as } |x| \to \infty.$$

By means of Proposition 2.17 from [62], p.74, we obtain

$$\partial_{x_1} u_1(x) = \frac{1}{2\pi} \text{ PV} \int_{\mathbb{R}^2} \partial_{x_1} \left(\frac{-x_2 + y_2}{|x-y|^2} \right) (-\partial_{x_1}\rho)(y) dy - \frac{(-\partial_{x_1}\rho)(x)}{2\pi} \int_{|x|=1} x_1 x_2 ds$$
$$\partial_{x_1} u_2(x) = \frac{1}{2\pi} \text{ PV} \int_{\mathbb{R}^2} \partial_{x_1} \left(\frac{x_1 - y_1}{|x-y|^2} \right) (-\partial_{x_1}\rho)(y) dy + \frac{(-\partial_{x_1}\rho)(x)}{2\pi} \int_{|x|=1} x_1^2 ds$$
$$\partial_{x_2} u_1(x) = \frac{1}{2\pi} \text{ PV} \int_{\mathbb{R}^2} \partial_{x_2} \left(\frac{-x_2 + y_2}{|x-y|^2} \right) (-\partial_{x_1}\rho)(y) dy - \frac{(-\partial_{x_1}\rho)(x)}{2\pi} \int_{|x|=1} x_2^2 ds$$
$$\partial_{x_2} u_2(x) = \frac{1}{2\pi} \text{ PV} \int_{\mathbb{R}^2} \partial_{x_2} \left(\frac{x_1 - y_1}{|x-y|^2} \right) (-\partial_{x_1}\rho)(y) dy + \frac{(-\partial_{x_1}\rho)(x)}{2\pi} \int_{|x|=1} x_1 x_2 ds$$

Beyond that, for $x \neq y$ we have

$$\partial_{x_1} \left(\frac{-x_2 + y_2}{|x-y|^2} \right) + \partial_{x_2} \left(\frac{x_1 - y_1}{|x-y|^2} \right) = 0,$$
$$\partial_{x_1} \left(\frac{x_1 - y_1}{|x-y|^2} \right) - \partial_{x_2} \left(\frac{-x_2 + y_2}{|x-y|^2} \right) = 0.$$

Moreover, we get

$$\int_{|x|=1} x_1 x_2 ds = 0 \quad \text{and} \quad \int_{|x|=1} x_1^2 ds = \int_{|x|=1} x_2^2 ds = \pi.$$

Thanks to the definition of the principal-value integral, we deduce (3.17). □

However, if the density is only in $L^\infty(\mathbb{R}^2)$ as in (1.25), the expression (3.18) is not well-defined. Nevertheless, we are able to modify (3.18) for such densities, in order to determine a solution u of (3.17) in the sense of distributions. To make sense of (3.18), one can argue as in [25] and view $\partial_{x_1}\rho$ as a delta distribution supported on Γ. Plugging this into (3.18), we infer

$$u(x) = \frac{\rho^+ - \rho^-}{2\pi} \text{ PV} \int_{\mathbb{R}} \frac{(z(\xi) - x_2, x_1 - \xi)}{(x_1 - \xi)^2 + (z(\xi) - x_2)^2} \partial_s z(\xi) d\xi. \tag{3.19}$$

Before we show that this expression defines a solution to (3.17), we want to address some useful lemmata.

Lemma 3.4. Let $a, b \in \mathbb{R}$ and $Z(s,\xi) = z(s-\xi) - z(s)$, where $z \in L^\infty(\mathbb{R})$. Then there exists a constant $C(a) > 0$ depending only on a such that

$$(i) \left| \text{ PV} \int_{|\xi|>1} \frac{\xi}{\xi^2 + (a\xi + b)^2} d\xi \right| \leq C(a). \tag{3.20}$$

$$(ii) \left| \text{ PV} \int_{|\xi|>1} \frac{\xi}{\xi^2 + (a\xi + b + Z(s,\xi))^2} d\xi \right| \leq C(a)(1 + \|z\|_{L^\infty(\mathbb{R})} + \|z\|^2_{L^\infty(\mathbb{R})}). \tag{3.21}$$

Proof. (i) Denote $\sigma = \frac{1}{1+a^2}$ and observe that by a change of coordinates $\xi \mapsto -\xi$ in the integral, we can assume without loss of generality that $ab > 0$. Then

$$\begin{aligned}
&\mathrm{PV}\int_{|\xi|>1} \frac{\xi}{\xi^2+(a\xi+b)^2}d\xi = \sigma \lim_{R\to\infty}\int_{1<|\xi|<R} \frac{\xi}{(\xi+ab\sigma)^2+b^2\sigma^2}d\xi \\
&= \frac{\sigma}{2}\lim_{R\to\infty}\int_{1<|\xi|<R} \frac{2(\xi+ab\sigma)}{(\xi+ab\sigma)^2+b^2\sigma^2}d\xi - \sigma \lim_{R\to\infty}\int_{1<|\xi|<R} \frac{ab\sigma}{(\xi+ab\sigma)^2+b^2\sigma^2}d\xi \\
&= \frac{\sigma}{2}\log\frac{(R+ab\sigma)^2+b^2\sigma^2}{(R-ab\sigma)^2+b^2\sigma^2} - \frac{\sigma}{2}\log\frac{(1+ab\sigma)^2+b^2\sigma^2}{(1-ab\sigma)^2+b^2\sigma^2} \\
&- \lim_{R\to\infty}\sigma a\left[\arctan\left(a+\frac{\xi}{b\sigma}\right)\right]_{-R}^{R} + \sigma a\left[\arctan\left(a+\frac{\xi}{b\sigma}\right)\right]_{-1}^{1} \\
&= -\frac{\sigma}{2}\log\frac{(1+ab\sigma)^2+b^2\sigma^2}{(1-ab\sigma)^2+b^2\sigma^2} - \mathrm{sign}(b)\sigma a\pi + \sigma a\left[\arctan\left(a+\frac{\xi}{b\sigma}\right)\right]_{-1}^{1}.
\end{aligned}$$

Furthermore we have the following inequality

$$\begin{aligned}
((1-ab\sigma)^2+b^2\sigma^2)(1+a^2)^2 &= (1+a^2-ab)^2+b^2 \\
&= 1+2a^2+a^4+a^2b^2+b^2-2ab-2a^3b \\
&= 1+a^2(a^2+b^2-2ab)+\left(2a^2-2ab+\frac{b^2}{2}\right)+\frac{b^2}{2} \\
&\geq 1+\frac{b^2}{2}.
\end{aligned}$$

Thanks to this and $ab > 0$, we deduce

$$\begin{aligned}
1 &\leq \frac{(1+ab\sigma)^2+b^2\sigma^2}{(1-ab\sigma)^2+b^2\sigma^2} = 1+\frac{4ab\sigma}{(1-ab\sigma)^2+b^2\sigma^2} \leq 1+\frac{4|ab|(1+a^2)^2}{1+\frac{b^2}{2}} \\
&\leq 1+2\sqrt{2}|a|(1+a^2)^2,
\end{aligned}$$

where we used $1+\frac{b^2}{2} \geq \sqrt{2}|b|$ in the last inequality. Finally, we conclude

$$\left|\mathrm{PV}\int_{|\xi|>1} \frac{\xi}{\xi^2+(a\xi+b)^2}d\xi\right| \leq \frac{\sigma}{2}\log\left(1+2\sqrt{2}|a|(1+a^2)^2\right)+2\sigma a\pi.$$

(ii) By taking the difference with the integral from (i), we compute

$$\begin{aligned}
&\left|\mathrm{PV}\int_{|\xi|>1} \frac{\xi}{\xi^2+(a\xi+b+Z)^2}d\xi\right| \leq \\
&\mathrm{PV}\int_{|\xi|>1} |\xi|\frac{Z^2+2|Z||a\xi+b|}{(\xi^2+(a\xi+b+Z)^2)(\xi^2+(a\xi+b)^2)}d\xi + \left|\mathrm{PV}\int_{|\xi|>1} \frac{\xi}{\xi^2+(a\xi+b)^2}d\xi\right| \\
&\leq \int_{|\xi|>1} \|z\|_{L^\infty(\mathbb{R})}^2\frac{1}{|\xi|^3}+\|z\|_{L^\infty(\mathbb{R})}\frac{1}{|\xi|^2}d\xi + C(a).
\end{aligned}$$

□

Lemma 3.5. Let $a, b \in \mathbb{R}, \alpha \in (0,1)$ and define $Z(s,\xi) = z(s-\xi) - z(s)$, for some $z \in W^{1,\infty}(\mathbb{R}) \cap C^{1,\alpha}(\mathbb{R})$. Then there exists a constant $C(a) > 0$ depending only on a such that

$$(i) \left| \operatorname{PV} \int_{|\xi|<1} \frac{\xi}{\xi^2 + (a\xi + b)^2} d\xi \right| \leq C(a). \tag{3.22}$$

$$(ii) \left| \operatorname{PV} \int_{|\xi|<1} \frac{\xi}{\xi^2 + (a\xi + b + Z(s,\xi))^2} d\xi \right| \leq C\big(a, \|\partial_s z\|_{L^\infty(\mathbb{R})}\big) + [\partial_s z]_{C^{0,\alpha}(\mathbb{R})} + [\partial_s z]^2_{C^{0,\alpha}(\mathbb{R})}. \tag{3.23}$$

Proof. (i) We can argue exactly as in the previous lemma to verify

$$\begin{aligned}
&\left| \operatorname{PV} \int_{|\xi|<1} \frac{\xi}{\xi^2 + (a\xi + b)^2} d\xi \right| = \left| \sigma \lim_{r \to 0} \int_{r<|\xi|<1} \frac{\xi}{(\xi + ab\sigma)^2 + b^2\sigma^2} d\xi \right| \\
&= \left| \frac{\sigma}{2} \log \frac{(1 + ab\sigma)^2 + b^2\sigma^2}{(1 - ab\sigma)^2 + b^2\sigma^2} - \sigma a \left[\arctan\left(a + \frac{\xi}{b\sigma} \right) \right]_{-1}^{1} \right| \\
&\leq \frac{\sigma}{2} \log(1 + 2\sqrt{2}|a|(1 + a^2)^2) + \sigma a \pi.
\end{aligned}$$

(ii) At first we impose $\tilde{a}(s) := a - \partial_s z(s)$ and $\tilde{Z}(s,\xi) := Z(s,\xi) + \xi \partial_s z(s)$. Moreover, if we write $Z(s,\xi) = z(s-\xi) - z(s) = -\xi \int_0^1 \partial_s z(s - \tau\xi) d\tau$, we have in the same way $\tilde{Z}(s,\xi) = -\xi \int_0^1 \partial_s z(s - \tau\xi) - \partial_s z(s) d\tau$. Hence,

$$\begin{aligned}
&\left| \operatorname{PV} \int_{|\xi|<1} \frac{\xi}{\xi^2 + (a\xi + b + Z)^2} d\xi \right| \\
&= \left| \operatorname{PV} \int_{|\xi|<1} \frac{\xi}{\xi^2 + (\tilde{a}\xi + b + \tilde{Z})^2} - \frac{\xi}{\xi^2 + (\tilde{a}\xi + b)^2} d\xi + \operatorname{PV} \int_{|\xi|<1} \frac{\xi}{\xi^2 + (\tilde{a}\xi + b)^2} d\xi \right| \\
&\leq \left| \operatorname{PV} \int_{|\xi|<1} \xi \frac{\tilde{Z}^2 + 2\tilde{Z}(\tilde{a}\xi + b)}{(\xi^2 + (\tilde{a}\xi + b + \tilde{Z})^2)(\xi^2 + (\tilde{a}\xi + b)^2)} d\xi \right| + \left| \operatorname{PV} \int_{|\xi|<1} \frac{\xi}{\xi^2 + (\tilde{a}\xi + b)^2} d\xi \right|.
\end{aligned}$$

The second summand can be bounded by $C(a, \|\partial_s z\|_{L^\infty(\mathbb{R})})$ using (i). For the first summand we use Young's inequality and the fact that $|\tilde{Z}| \leq [\partial_s z]_{C^{0,\alpha}(\mathbb{R})} \xi^{1+\alpha}$, since $z \in C^{1,\alpha}(\mathbb{R})$. Then we obtain

$$\begin{aligned}
&\left| \operatorname{PV} \int_{|\xi|<1} \frac{\xi}{\xi^2 + (a\xi + b + Z)^2} d\xi \right| \\
&\leq \left| \operatorname{PV} \int_{|\xi|<1} [\partial_s z]^2_{C^{0,\alpha}(\mathbb{R})} \xi^{2\alpha - 1} + [\partial_s z]_{C^{0,\alpha}(\mathbb{R})} \xi^{-1+\alpha} d\xi \right| + C(a, \|\partial_s z\|_{L^\infty(\mathbb{R})}).
\end{aligned}$$

□

Next, we want to introduce locally Hölder continuous functions.

Definition 3.6.
(i) We say that $f:\mathbb{R}\to\mathbb{R}$ is locally Hölder continuous with exponent $\alpha\in(0,1)$, if for each $x\in\mathbb{R}$ and $r>0$, there exists a constant $C=C(r)>0$ independent of x such that

$$\sup_{x\neq y, y\in B_r(x)} \frac{|f(x)-f(y)|}{|x-y|^\alpha} < C.$$

In this case we write $f\in C^{0,\alpha}_{loc}(\mathbb{R})$.

This definition amounts to a spacial uniform and local form of Hölder continuity. The goal is an extension of the Sobolev embedding theorem to unbounded domains.

Lemma 3.7.
(i) Suppose that $f\in L^1(\mathbb{R})\cap C^{0,\alpha}(\mathbb{R})$ for some $\alpha\in(0,1)$. Then $f\in L^\infty(\mathbb{R})$.
(ii) Let $1<p<\infty$. Then $L^1(\mathbb{R})\cap W^{1,p}(\mathbb{R})\subset C^{0,1-\frac{1}{p}}(\mathbb{R})$.

Proof. (i) Let $x\in\mathbb{R}$. Then we have for all $y\in B_1(x), y\neq x$ that $|f(x)-f(y)|\leq [f]_{C^{0,\alpha}(\mathbb{R})}|x-y|^\alpha\leq [f]_{C^{0,\alpha}(\mathbb{R})}$. Hence, $|f(x)|\leq|f(x)-f(y)|+|f(y)|\leq [f]_{C^{0,\alpha}(\mathbb{R})}+|f(y)|$. Taking the mean value over $B_1(x)$, we infer

$$|f(x)|\leq [f]_{C^{0,\alpha}(\mathbb{R})}+\frac{1}{2}\int_{B_1(x)}|f(y)|dy\leq [f]_{C^{0,\alpha}(\mathbb{R})}+\frac{1}{2}\|f\|_{L^1(\mathbb{R})}<\infty.$$

(ii) Let $f\in L^1(\mathbb{R})\cap W^{1,p}(\mathbb{R})$ and $x\in\mathbb{R}, r>0$. At first we show that $f\in C^{0,1-\frac{1}{p}}_{loc}(\mathbb{R})$. Therefore, we need to estimate the Hölder semi-norm of f on the bounded set $B_r(x)$ independently of x. Since $f\in W^{1,p}(\mathbb{R})$, this can be shown by use of Morrey's inequality just like in the proof of the Sobolev embedding on bounded sets, see for example [46], p.270. Since $B_r(x)$ was an arbitrary ball, the conclusion follows.

Exactly as in statement (i), also the local Hölder continuity yields $f\in L^\infty(\mathbb{R})$. We deduce the global Hölder continuity by

$$\begin{aligned}\sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^{1-\frac{1}{p}}} &\leq \sup_{x\neq y, y\in B_1(x)}\frac{|f(x)-f(y)|}{|x-y|^{1-\frac{1}{p}}}+\sup_{x\neq y,|x-y|>1}\frac{|f(x)-f(y)|}{|x-y|^{1-\frac{1}{p}}}\\ &\leq C+\frac{2\|f\|_{L^\infty(\mathbb{R})}}{1}.\end{aligned}$$

□

With these Lemmata at hand, we can show

Theorem 3.8. Let $z(s)=\beta s+\overline{z}(s)$ for some $\beta\in\mathbb{R}$ and let $\overline{z}\in C^{1,\alpha}(\mathbb{R})\cap W^{1,1}(\mathbb{R})$. Denote the corresponding graph by $\Gamma:=\{(s,z(s)):s\in\mathbb{R}\}$. Moreover, define the density $\rho:\mathbb{R}^2\to\mathbb{R}$ by

$$\rho(x_1,x_2)=\begin{cases}\rho^+, & x_2>z(x_1)\\ \rho^-, & x_2<z(x_1)\end{cases}, \tag{3.24}$$

with constants $\rho^+, \rho^- \in \mathbb{R}$. For $x \notin \Gamma$ we let $u(x)$ be defined as in (3.19). Then, the following conclusions hold.
(i) We have the equivalent representation

$$u(x) = \frac{\rho^+ - \rho^-}{2\pi} \text{ PV} \int_{\mathbb{R}} (1, \partial_s z(\xi)) \frac{x_1 - \xi}{(x_1 - \xi)^2 + (z(\xi) - x_2)^2} d\xi. \tag{3.25}$$

(ii) u is in $L^\infty(\mathbb{R}^2)$ and we have the estimate

$$\|u\|_{L^\infty} \leq C(\beta, \|\overline{z}\|_{L^\infty(\mathbb{R})}, \|\partial_s \overline{z}\|_{L^\infty(\mathbb{R})}, [\partial_s \overline{z}]_{C^{0,\alpha}(\mathbb{R})}). \tag{3.26}$$

(iii) u has well-defined boundary values on Γ

$$u(s, z(s)) = \frac{\rho^+ - \rho^-}{2\pi} \left(\text{PV} \int_{\mathbb{R}} \frac{(z(\xi) - z(s), s - \xi)}{(s - \xi)^2 + (z(\xi) - z(s))^2} \partial_s z(\xi) d\xi \pm \pi \partial_s z(s) \tau(s) \right), \tag{3.27}$$

where $\tau(s) = \frac{(1, \partial_s z(s))}{\sqrt{1 + (\partial_s z(s))^2}}$ is the unit tangential vector on Γ and the sign depends on the direction from which we approach to Γ. In particular, the normal component of the velocity on the sheet $\nu := u(s, z(s)) \cdot (-\partial_s z(s), 1)$ becomes

$$\nu = \frac{\rho^+ - \rho^-}{2\pi} \text{ PV} \int_{\mathbb{R}} (\partial_s z(\xi) - \partial_s z(s)) \frac{s - \xi}{(s - \xi)^2 + (z(\xi) - z(s))^2} d\xi, \tag{3.28}$$

and we observe that this is a continuous function in the whole $\mathbb{R}^2$. On the contrary, the tangential component of u has a jump across Γ.
(iv) u solves (3.17) in the sense of distributions.

Remark 3.9.
We can also investigate the more general case of N curves $z_j(s) = \beta s + \overline{z}_j(s)$ with $\overline{z}_j \in C^{1,\alpha}(\mathbb{R}) \cap W^{1,1}(\mathbb{R}), j = 1, \ldots, N$ and $z_j(s) > z_{j+1}(s)$ for all $s \in \mathbb{R}, j = 1, \ldots, N-1$. Then, if we define the density by

$$\rho(x_1, x_2) = \begin{cases} \rho_1 & , x_2 > z_1(x_1) \\ \rho_2 & , z_1(x_1) > x_2 > z_2(x_1) \\ \vdots & , \quad \vdots \\ \rho_{N+1} & , x_2 < z_N(x_1) \end{cases}$$

for $\rho_1, \rho_2, \ldots, \rho_{N+1} \in \mathbb{R}$, one easily obtains the analogous expressions by linearity

$$\begin{aligned} u(x) &= \sum_{j=1}^{N} \frac{\rho_j - \rho_{j+1}}{2\pi} \text{ PV} \int_{\mathbb{R}} \frac{(z_j(\xi) - x_2, x_1 - \xi)}{(x_1 - \xi)^2 + (z_j(\xi) - x_2)^2} \partial_s z_j(\xi) d\xi \\ &= \sum_{j=1}^{N} \frac{\rho_j - \rho_{j+1}}{2\pi} \text{ PV} \int_{\mathbb{R}} (1, \partial_s z_j(\xi)) \frac{x_1 - \xi}{(x_1 - \xi)^2 + (z_j(\xi) - x_2)^2} d\xi. \end{aligned}$$

Proof. At first we observe that $\overline{z} \in C^{1,\alpha}(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$ implies $\overline{z} \in W^{1,\infty}(\mathbb{R})$ by Lemma 3.7. The proof of (i) bases on the following trick, which was introduced in [25]. Thanks to the identity

$$\frac{1}{2}\partial_\xi \log\left(|(x_1,x_2)-(\xi,z(\xi))|^2\right) = -\frac{x_1-\xi}{|(x_1-\xi,x_2-z(\xi))|^2} - \frac{(x_2-z(\xi))\partial_x z(\xi)}{|(x_1-\xi,x_2-z(\xi))|^2}$$

and the fact that

$$\mathrm{PV}\int_{\mathbb{R}} \partial_\xi \log\left(|(x_1,x_2)-(\xi,z(\xi))|^2\right) d\xi = 0,$$

since for $x \notin \Gamma$ we have no singularity in the logarithm, one can easily rewrite (3.19) in the form (3.25).

(ii) This can be shown analogously to Lemma 4.5 in [17] and with help of the Lemmata 3.4, 3.5. We start with the first component and consider

$$\mathrm{PV}\int_{\mathbb{R}} \frac{x_1-\xi}{(x_1-\xi)^2+(z(\xi)-x_2)^2} d\xi.$$

Let $(x_1,x_2) \in \mathbb{R}^2$. Then we can express $x_2 = z(x_1)+\tau$ for some $\tau \in \mathbb{R}$. Moreover, we replace x_1 by s and impose $Z(s,\xi) := z(s-\xi)-z(s)$. Thus, the integral becomes

$$\begin{aligned} I(s,\tau) :=\ & \mathrm{PV}\int_{\mathbb{R}} \frac{s-\xi}{(s-\xi)^2+(Z(s,s-\xi)-\tau)^2} d\xi \\ =\ & \mathrm{PV}\int_{\mathbb{R}} \frac{\xi}{\xi^2+(Z(s,\xi)-\tau)^2} d\xi. \end{aligned}$$

We want to show that $I(\cdot,\tau)$ is in $L^\infty(\mathbb{R})$ uniformly in τ. For this purpose we split $\mathbb{R} = \{|\xi|<1\} \cup \{|\xi|>1\}$ and use Lemma 3.4 and Lemma 3.5. This gives immediately a bound on I depending only on β, $\|\overline{z}\|_{L^\infty(\mathbb{R})}$, $\|\partial_s \overline{z}\|_{L^\infty(\mathbb{R})}$ and $[\overline{z}]_{C^{0,\alpha}(\mathbb{R})}$, since $z(s) = \beta s + \overline{z}(s)$.

For the second component we have to bound the integral

$$\begin{aligned} J(s,\tau) :=\ & \mathrm{PV}\int_{\mathbb{R}} \partial_s z(\xi) \frac{s-\xi}{(s-\xi)^2+(Z(s,s-\xi)-\tau)^2} d\xi \\ =\ & \mathrm{PV}\int_{\mathbb{R}} \partial_s \overline{z}(s-\xi) \frac{\xi}{\xi^2+(Z(s,\xi)-\tau)^2} d\xi + \beta\ \mathrm{PV}\int_{\mathbb{R}} \frac{\xi}{\xi^2+(Z(s,\xi)-\tau)^2} d\xi. \end{aligned}$$

As before, we want to derive an estimate uniformly in τ. The second summand can be bounded as I and the first one is split into two terms

$$\begin{aligned} J_1 =\ & \mathrm{PV}\int_{|\xi|<1} \partial_s \overline{z}(s-\xi) \frac{\xi}{\xi^2+(Z(s,\xi)-\tau)^2} d\xi, \\ J_2 =\ & \mathrm{PV}\int_{|\xi|>1} \partial_s \overline{z}(s-\xi) \frac{\xi}{\xi^2+(Z(s,\xi)-\tau)^2} d\xi. \end{aligned}$$

Then we deduce by Lemma 3.5

$$
\begin{aligned}
|J_1| \leq & \left| \mathrm{PV} \int\limits_{|\xi|<1} (\partial_s \overline{z}(s-\xi) - \partial_s \overline{z}(s)) \frac{\xi}{\xi^2 + (Z(s,\xi)-\tau)^2} d\xi \right| \\
& + \left| \mathrm{PV} \int\limits_{|\xi|<1} \partial_s \overline{z}(s) \frac{\xi}{\xi^2 + (Z(s,\xi)-\tau)^2} d\xi \right| \\
& \leq C[\partial_s \overline{z}]_{C^{0,\alpha}(\mathbb{R})} \int\limits_{|\xi|<1} \frac{d\xi}{\xi^{1-\alpha}} + \|\partial_s \overline{z}\|_{L^\infty(\mathbb{R})} \Big(C(\beta, \|\partial_s z\|_{L^\infty(\mathbb{R})}) + [\partial_s z]_{C^{0,\alpha}(\mathbb{R})} + [\partial_s z]^2_{C^{0,\alpha}(\mathbb{R})} \Big).
\end{aligned}
$$

For J_2 we first note that $\partial_s \overline{z}(s-\xi) = -\partial_\xi \overline{z}(s-\xi)$ and then integrate by parts. This leads to

$$
\begin{aligned}
J_2 = & \overline{z}(s+1) \frac{-1}{1 + (Z(s,-1)-\tau)^2} - \overline{z}(s-1) \frac{1}{1 + (Z(s,1)-\tau)^2} \\
& + \int\limits_{|\xi|>1} \overline{z}(s-\xi) \frac{(Z(s,\xi)-\tau)^2 - \xi^2 + 2\xi(Z(s,\xi)-\tau)\partial_s z(s-\xi)}{(\xi^2 + (Z(s,\xi)-\tau)^2)^2} d\xi,
\end{aligned}
$$

which can be estimated by

$$
|J_2| \leq 2\|\overline{z}\|_{L^\infty(\mathbb{R})} + \|\overline{z}\|_{L^\infty(\mathbb{R})} \int\limits_{|\xi|>1} \frac{1 + \|\partial_s z\|_{L^\infty(\mathbb{R})}}{\xi^2} d\xi.
$$

Thus, we conclude (3.26).

(iii) We rely on the proof in [15], which uses the Plemelj-formula. At first, consider (3.19) and take the limit $x \to x_0 \in \Gamma, x_0 = (\eta, z(\eta))$ for some $\eta \in \mathbb{R}$. Let $\varepsilon > 0$ and split

$$
\begin{aligned}
u(x) = v_\varepsilon(x) + w_\varepsilon(x) = & \frac{\rho^+ - \rho^-}{2\pi} \int\limits_{\eta-\varepsilon}^{\eta+\varepsilon} \frac{(z(\xi) - x_2, x_1 - \xi)}{(x_1-\xi)^2 + (z(\xi)-x_2)^2} \partial_s z(\xi) d\xi \\
& + \frac{\rho^+ - \rho^-}{2\pi} \int\limits_{\mathbb{R}\setminus B_\varepsilon(\eta)} \frac{(z(\xi) - x_2, x_1 - \xi)}{(x_1-\xi)^2 + (z(\xi)-x_2)^2} \partial_s z(\xi) d\xi.
\end{aligned}
$$

Obviously, by definition of the principal value integral,

$$
\lim_{\varepsilon\to 0} w_\varepsilon(x_0) = \frac{\rho^+ - \rho^-}{2\pi} \ \mathrm{PV} \int\limits_{\mathbb{R}} \frac{(z(\xi) - z(\eta), \eta - \xi)}{(\eta-\xi)^2 + (z(\eta) - z(\xi))^2} \cdot \partial_s z(\xi) d\xi
$$

and thus it remains to verify that $\lim\limits_{\varepsilon\to 0} \lim\limits_{x\to x_0} v_\varepsilon(x) = \pm \frac{\rho^+ - \rho^-}{2} \partial_s z(\eta) \tau_0$. By n_0 and τ_0 we denote the normal vector and the tangential vector to Γ in x_0 respectively. Furthermore, we write $G(x) = \frac{x^\perp}{|x|^2}$. Then we split

$$\begin{aligned}\frac{2\pi}{\rho^+ - \rho^-} v_\varepsilon(x) &= \int_{\eta-\varepsilon}^{\eta+\varepsilon} \partial_s z(\xi) G(x - (\xi, z(\xi))) d\xi = \int_{-\varepsilon}^{\varepsilon} \partial_s z(\eta) G(x - x_0 + \xi\tau_0) d\xi \\ &+ \int_{-\varepsilon}^{\varepsilon} (\partial_s z(\eta - \xi) - \partial_s z(\eta)) G(x - x_0 + \xi\tau_0) d\xi \\ &+ \int_{\eta-\varepsilon}^{\eta+\varepsilon} \partial_s z(\xi) G(x - (\xi, z(\xi))) d\xi - \int_{-\varepsilon}^{\varepsilon} \partial_s z(\eta - \xi) G(x - x_0 + \xi\tau_0) d\xi \\ &=: I_{1,\varepsilon}(x) + I_{2,\varepsilon}(x) + I_{3,\varepsilon}(x)\end{aligned}$$

For $I_{1,\varepsilon}(x)$ we first consider the case that $x \to x_0$ along the normal n_0, that is, $x = x_0 + \delta \cdot n_0$ with $\delta > 0$ or $\delta < 0$, depending on the site from which we approach to x_0. We deduce $|x - x_0 + \xi\tau_0|^2 = |\delta n_0 + \xi\tau_0|^2 = \delta^2 + \xi^2$. Since moreover,

$$\int_{-\varepsilon}^{\varepsilon} \frac{\xi}{\delta^2 + \xi^2} d\xi = 0,$$

we infer

$$I_{1,\varepsilon}(x) = 2\partial_s z(\eta) \int_0^{\varepsilon} \frac{\delta n_0^\perp}{\delta^2 + \xi^2} d\xi = 2\tau_0 \cdot \partial_x z(\eta) \arctan\left(\frac{\varepsilon}{\delta}\right).$$

Hence, we get

$$I_{1,\varepsilon}(x_0) = \lim_{\delta \to 0} 2\tau_0 \partial_s z(\eta) \arctan\left(\frac{\varepsilon}{\delta}\right) = \pm\pi \partial_s z(\eta) \cdot \tau_0$$

independently of ε. To conclude (3.27), we need to show $\lim_{\varepsilon \to 0} I_{i,\varepsilon}(x_0) = 0$ for $i = 2, 3$. Regarding $I_{2,\varepsilon}$, we recall $z \in C^{1,\alpha}(\mathbb{R})$ to estimate

$$\begin{aligned}|I_{2,\varepsilon}(x_0)| &= \left| \int_{-\varepsilon}^{\varepsilon} (\partial_s z(\eta - \xi) - \partial_s z(\eta)) G(\xi\tau_0) d\xi \right| \\ &\leq C \int_{-\varepsilon}^{\varepsilon} |\xi|^\alpha |\xi|^{-1} d\xi \leq C\varepsilon^\alpha \to 0 \quad \text{as } \varepsilon \to 0.\end{aligned}$$

We can argue similarly for $I_{3,\varepsilon}$ and use $z(\eta) = z(\eta - \xi) + \xi \partial_s z(\mu)$ for some $\mu \in (\eta - \xi, \eta)$

to establish

$$\begin{aligned}
|I_{3,\varepsilon}(x_0)| &= \left| \int_{-\varepsilon}^{\varepsilon} \partial_s z(\eta-\xi) \left(\frac{((\eta,z(\eta))-(\eta-\xi,z(\eta-\xi)))^\perp}{|\xi|^2+|z(\eta)-z(\eta-\xi)|^2} - \frac{\xi(-\partial_s z(\eta),1)}{\xi^2(1+\partial_s z(\eta)^2)} \right) d\xi \right| \\
&\le \int_{-\varepsilon}^{\varepsilon} |\partial_s z(\eta-\xi)| \left| \frac{(-\xi\partial_s z(\mu),\xi)}{|\xi|^2+|\xi\cdot\partial_s z(\mu)|^2} - \frac{(-\partial_s z(\eta),1)}{\xi(1+\partial_s z(\eta)^2)} \right| d\xi \\
&\le \int_{-\varepsilon}^{\varepsilon} \frac{C}{|\xi|} |\partial_s z(\eta)-\partial_s z(\mu)| \cdot \left(\frac{|(1+\partial_s z(\eta)\partial_s z(\mu), \partial_s z(\eta)+\partial_s z(\mu))|}{(1+\partial_s z(\eta)^2)(1+\partial_s z(\mu)^2)} \right) d\xi \\
&\le \int_{-\varepsilon}^{\varepsilon} \frac{C}{|\xi|^{1-\alpha}} d\xi \le C\varepsilon^\alpha \to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Thus, we proved (3.27), if we require that $x \to x_0$ along the normal direction. If $(x_n)_n$ is a sequence converging to x_0 along some arbitrary direction, the same limit is attained. Indeed, denote by $p_n \in \Gamma$ the projection of x_n onto Γ. Then the normal vector in p_n is given by $n(p_n) = \frac{x_n-p_n}{|x_n-p_n|}$. This yields

$$|u(x_n)-u(x_0)| \le |u(x_n)-u(p_n)| + |u(p_n)-u(x_0)| \to 0 \text{ as } n \to \infty,$$

where the first summand converges to zero, since $|x_n - p_n| \to 0$ as $n \to \infty$ and because of the previous result for the normal direction, which holds uniformly in x_0. The second term vanishes in the limit, since $z \in C^{1,\alpha}(\mathbb{R})$ and u is continuous.

(iv) It remains to prove that u solves (3.17) in the sense of distributions. For this, let $\phi \in C_c^\infty(\mathbb{R}^2)$ with $\text{supp}(\phi) \subset \{x_2 > z(x_1)\}$. Since the integrand of u in (3.19) has no singularity inside $\{x_2 > z(x_1)\}$, we can differentiate with respect to x directly under the integral. This can be established as follows. At first we decompose $\partial_s z(\xi) = \beta + \partial_s \overline{z}(\xi)$. Let $x \in \text{supp}(\phi)$. Then $|(x_1,x_2)-(\xi,z(\xi))| \ge m > 0$ for all $\xi \in \mathbb{R}$. Hence,

$$\left| \int_{\mathbb{R}} \partial_s \overline{z}(\xi) \frac{(z(\xi)-x_2, x_1-\xi)}{(x_1-\xi)^2+(z(\xi)-x_2)^2} d\xi \right| \le \frac{C}{m} \|\partial_s \overline{z}\|_{L^1(\mathbb{R})} < \infty$$

and similarly after differentiating the integrand with respect to $x_i, i = 1,2$

$$\left| \int_{\mathbb{R}} \partial_s \overline{z}(\xi) \partial_{x_i} \left(\frac{(z(\xi)-x_2, x_1-\xi)}{(x_1-\xi)^2+(z(\xi)-x_2)^2} \right) d\xi \right| \le \frac{C}{m^2} \|\partial_s \overline{z}\|_{L^1(\mathbb{R})} < \infty.$$

This justifies the differentiation under the integral for the summand involving $\partial_s \overline{z}(\xi)$. For the other summand we have to argue differently, since $\beta \notin L^1(\mathbb{R})$. In particular, we have to deal with a principal value integral. It is enough to investigate the x_1-derivative of the second component, since the differentiation with respect to x_2 and the argumentation for the first component are similarly. After a change of coordinates in (3.25), we need to show

$$\partial_{x_1} \left(\text{PV} \int_{\mathbb{R}} \beta \frac{\xi}{\xi^2+(z(x_1-\xi)-x_2)^2} d\xi \right) = \text{PV} \int_{\mathbb{R}} \beta \partial_{x_1} \left(\frac{\xi}{\xi^2+(z(x_1-\xi)-x_2)^2} \right) d\xi. \tag{3.29}$$

This amounts to justify the interchange of two limit processes, which follows if one of the limits is achieved uniformly with respect to the other one. In our case, it is enough to verify that for fixed $n \in \mathbb{N}$, the limit

$$\lim_{a\to\infty}\int_{-a}^{a} n\left(\frac{\xi}{\xi^2+(z(x_1+\frac{1}{n}-\xi)-x_2)^2}-\frac{\xi}{\xi^2+(z(x_1-\xi)-x_2)^2}\right)d\xi$$

is achieved uniformly in n. The integrand can be written as

$$\begin{aligned}
&n\xi\left(\frac{(z(x_1+\frac{1}{n}-\xi)-z(x_1-\xi))(2x_2+z(x_1+\frac{1}{n}-\xi)+z(x_1-\xi))}{(\xi^2+(z(x_1+\frac{1}{n}-\xi)-x_2)^2)(\xi^2+(z(x_1-\xi)-x_2)^2)}\right)\\
&=\frac{2x_2\xi(\beta+\partial_s\overline{z}(\eta))+\beta^2\xi(2x_1-2\xi+\frac{1}{n})+\xi\partial_s\overline{z}(\eta)(\overline{z}(x_1+\frac{1}{n}-\xi)+\overline{z}(x_1-\xi))}{(\xi^2+(z(x_1+\frac{1}{n}-\xi)-x_2)^2)(\xi^2+(z(x_1-\xi)-x_2)^2)}\\
&\le\frac{C_1}{|\xi|^2}+\frac{C_2}{|\xi|^3}\in L^1(\mathbb{R}\setminus(-1,1)),
\end{aligned}$$

where $\eta\in(x_1-\xi,x_1-\xi+\frac{1}{n})$ and $C_1,C_2>0$ are positive constants independent of n, since $\overline{z}\in W^{1,\infty}(\mathbb{R})$ and $x_1,x_2\in\operatorname{supp}(\phi)$, which is compact. This proves the uniform convergence of the principal value integral with respect to n and hence (3.29).

By means of this computation, we are allowed to differentiate under the integral. Now, thanks to

$$\operatorname{div}\left(\frac{(z(\xi)-x_2,x_1-\xi)}{(x_1-\xi)^2+(z(\xi)-x_2)^2}\right)=\operatorname{curl}\left(\frac{(z(\xi)-x_2,x_1-\xi)}{(x_1-\xi)^2+(z(\xi)-x_2)^2}\right)=0,$$

we easily deduce that div $u=$ curl $u=0$ in $\operatorname{supp}(\phi)$. Therefore we infer

$$\int_{\mathbb{R}^2}u\nabla\phi dx=\int_{\mathbb{R}^2}u\nabla^{\perp}\phi dx=0.$$

The same is true for $\operatorname{supp}(\phi)\subset\{x_2<z(x_1)\}$. Finally, we have to study the case that $\operatorname{supp}(\phi)\cap\Gamma\neq\emptyset$. Using (3.27), we infer

$$\int_{\mathbb{R}^2}u\nabla\phi dx=\int_{\Gamma}\phi[u]ndx=0,$$

where $[u]$ denotes the jump of u across Γ and n the normal vector on Γ. Here, the integral vanishes, since the jump of u is only in the tangential direction. Moreover, thanks to (iii), we get

$$\int_{\mathbb{R}^2}u\nabla^{\perp}\phi dx=\int_{\Gamma}\phi[u]\tau dx=(\rho^- -\rho^+)\int_{\mathbb{R}}\phi(s,z(s))\partial_s z(s)ds,$$

which concludes the proof of (3.17), since we have

$$-\partial_{x_1}\rho(x_1,x_2)=(\rho^+-\rho^-)\partial_{x_1}z(x_1)\delta(x_2-z(x_1)).$$

□

3.3 A power series ansatz

In this section we construct a solution z of (3.9)-(3.11). At first, we focus on showing (3.9) in the unstable regime. Using the special choice of the density (3.1) and Theorem 3.8, we infer for the normal component of the velocity on $\Gamma^{\pm}$

$$\begin{aligned}\nu^{\pm}(s,t) =& \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}} (\partial_s z(s-\xi,t)-\partial_s z(s,t))\frac{\xi}{\xi^2+(z(s-\xi,t)-z(s,t))^2}d\xi \\ &+\frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}} (\partial_s z(s-\xi,t)-\partial_s z(s,t))\frac{\xi}{\xi^2+(z(s-\xi,t)-z(s,t)\mp 2ct)^2}d\xi.\end{aligned} \tag{3.30}$$

We introduce the notation

$$Z_0(\xi,s) := z_0(s-\xi)-z_0(s). \tag{3.31}$$

In this sense, by a big letter we always denote the difference of the corresponding small letters, for example

$$Z(\xi,s,t) := z(s-\xi,t)-z(s,t)$$

for the sheet $z(s,t)$. Moreover, it will be useful to define

$$Z_0(\xi,s) = z_0(s-\xi)-z_0(s) = -\xi\int_0^1 \partial_s z_0(s-\xi\tau)d\tau =: \xi\psi_0(\xi,s) \tag{3.32}$$

and analogously

$$\partial_s Z_0(\xi,s) = \partial_s z_0(s-\xi)-\partial_s z_0(s) = -\xi\int_0^1 \partial_s^2 z_0(s-\xi\tau)d\tau =: \xi\psi_0'(\xi,s). \tag{3.33}$$

We use the same notation for Z,ψ_z,A,ψ_a and B,ψ_b later on. Additionally, since the initial curve is of the form $z_0(s)=\overline{z}_0(s)+\beta s$, we can write

$$Z_0(\xi,s) = \overline{Z}_0(\xi,s)-\beta\xi.$$

Similarly, we set $\overline{z}(s,t) := z(s,t)-\beta s$ and get $Z(\xi,s,t)=\overline{Z}(\xi,s,t)-\beta\xi$. We will adopt this notation to the general setting

$$\begin{aligned}\overline{F}(\xi,s,t) &= \overline{f}(s-\xi,t)-\overline{f}(s,t),\\ \overline{G}(\xi,s,t) &= \overline{g}(s-\xi,t)-\overline{g}(s,t),\\ F(\xi,s,t) &= f(s-\xi,t)-f(s,t)=\overline{F}(\xi,s,t)-\beta\xi,\\ G(\xi,s,t) &= g(s-\xi,t)-g(s,t)=\overline{G}(\xi,s,t)-\beta\xi.\end{aligned} \tag{3.34}$$

where $\beta\in\mathbb{R}$, $\overline{f},\overline{g}:\mathbb{R}\times[0,T)\to\mathbb{R}$ are arbitrary functions and we set similar as above $f(s,t)=\overline{f}(s,t)+\beta s,\;\; g(s,t)=\overline{g}(s,t)+\beta s.$

Furthermore, we introduce

$$D(s,t) := \nu^+(s,t) + \nu^-(s,t). \tag{3.35}$$

By means of the operators

$$T_F(G)(s,t) := \frac{1}{2\pi}\ \mathrm{PV}\int_{\mathbb{R}} \partial_s G(\xi,s,t)\frac{\xi}{\xi^2+F(\xi,s,t)^2}d\xi, \tag{3.36}$$

$$R_{F,c}(G)(s,t) := (T_{F+ct}(G) + T_{F-ct}(G))(s,t) \tag{3.37}$$

and in view of (3.30), we get the short hand notation $D = 2T_Z(Z) + R_{Z,2c}(Z)$.

In the following it will help to split up the operator $T_F(G)$ into Lebesgue integrals instead of a principal value integral. Therefore, we express $T_F(G)$ as

$$\begin{aligned}
T_F(G)(s,t) =& \frac{1}{2\pi}\int_{|\xi|<1} \frac{\partial_s\overline{G}(\xi,s,t)}{\xi}\frac{\xi^2}{\xi^2+F^2(\xi,s,t)}d\xi\\
&+\frac{1}{2\pi}\ \mathrm{PV}\int_{|\xi|>1}\partial_s\overline{g}(s-\xi,t)\frac{\xi}{\xi^2+F^2(\xi,s,t)}d\xi\\
&-\frac{\partial_s\overline{g}(s,t)}{2\pi}\ \mathrm{PV}\int_{|\xi|>1}\frac{\xi}{\xi^2+F^2(\xi,s,t)}d\xi\\
=&\frac{1}{2\pi}\int_{|\xi|<1} \frac{\partial_s\overline{G}(\xi,s,t)}{\xi}\frac{\xi^2}{\xi^2+F^2(\xi,s,t)}d\xi\\
&-\frac{1}{2\pi}\ \mathrm{PV}\int_{|\xi|>1}\partial_\xi\overline{g}(s-\xi,t)\frac{\xi}{\xi^2+F^2(\xi,s,t)}d\xi\\
&-\frac{\partial_s\overline{g}(s,t)}{2\pi}\ \mathrm{PV}\int_{|\xi|>1}\frac{\xi}{\xi^2+F^2(\xi,s,t)}-\frac{\xi}{(1+\beta^2)\xi^2}d\xi\\
=:& T_F^<(\overline{g})(s,t) + T_F^{>,1}(\overline{g})(s,t) + T_F^b(\overline{g})(s,t) + T_F^{>,2}(\overline{g})(s,t),
\end{aligned}$$

with

$$\begin{aligned}
T_F^<(\overline{g})(s,t) =& \frac{1}{2\pi}\int_{|\xi|<1}\frac{\partial_s\overline{g}(s-\xi,t)-\partial_s\overline{g}(s,t)}{\xi}\frac{\xi^2}{\xi^2+F^2(\xi,s,t)}d\xi\\
T_F^{>,1}(\overline{g})(s,t) =& \frac{1}{2\pi}\int_{|\xi|>1}\frac{\overline{g}(s-\xi,t)}{\xi^2}\frac{\xi^2(F(\xi,s,t)^2-\xi^2-2\xi F(\xi,s,t)\partial_\xi F(\xi,s,t))}{(\xi^2+F^2(\xi,s,t))^2}d\xi\\
T_F^b(\overline{g})(s,t) =& \frac{1}{2\pi}\left(\overline{g}(s+1,t)\frac{1}{1+F(-1,s,t)^2}+\overline{g}(s-1,t)\frac{1}{1+F(1,s,t)^2}\right)\\
T_F^{>,2}(\overline{g})(s,t) =& -\frac{\partial_s\overline{g}(s,t)}{2\pi(1+\beta^2)}\int_{|\xi|>1}\frac{1}{\xi^2}\frac{\xi(\overline{F}^2(\xi,s,t)-2\beta\xi\overline{F}(\xi,s,t))}{\xi^2+F^2(\xi,s,t)}d\xi.
\end{aligned}$$

Thus, D becomes

$$\begin{aligned}
D(s,t) &= 2T_Z(Z)(s,t) + R_{Z,2c}(Z)(s,t)\\
&= 2T_Z^<(\overline{z})(s,t) + 2T_Z^{>,1}(\overline{z})(s,t) + 2T_Z^b(\overline{z})(s,t) + 2T_Z^{>,2}(\overline{z})(s,t)\\
&\quad + T_{Z+2ct}^{>,1}(\overline{z})(s,t) + T_{Z+2ct}^b(\overline{z})(s,t) + T_{Z+2ct}^{>,2}(\overline{z})(s,t)\\
&\quad + T_{Z-2ct}^{>,1}(\overline{z})(s,t) + T_{Z-2ct}^b(\overline{z})(s,t) + T_{Z-2ct}^{>,2}(\overline{z})(s,t) + E_{Z,2c}(\overline{Z})(s,t),
\end{aligned} \tag{3.38}$$

where we set

$$E_{F,c}(G)(s,t) := \frac{1}{2\pi}\int_{|\xi|<1} \partial_s G(\xi,s,t)\frac{2\xi(\xi^2+F^2(\xi,s,t)+c^2t^2)}{(\xi^2+(F(\xi,s,t)+ct)^2)(\xi^2+(F(\xi,s,t)-ct)^2)}d\xi. \tag{3.39}$$

The definition of the operator $E_{F,c}(G)$ is due to a bad behaviour of the time derivatives of $T^{<}_{Z\pm 2ct}(\overline{z})$. To show (3.9) later on, we have to treat the term $E_{F,c}(G)$ separately to the other operators.

In the next Proposition, we study the regularity of a class of slightly more generalized versions of the above operators.

Proposition 3.10. Let $\overline{g} : \mathbb{R}\times[0,T) \to \mathbb{R}, n \in \mathbb{N}$ and suppose that Φ_1, Φ_2, Φ_3 are functions with bounded derivatives up to order n, $\Phi_1 \in W^{n,\infty}(\{|\xi|<1\}\times\mathbb{R}\times[0,T))$ and $\Phi_2, \Phi_3 \in W^{n,\infty}(\{|\xi|>1\}\times\mathbb{R}\times[0,T))$. We define

$$\begin{aligned}
H^1_{\Phi_1,\overline{g}}(s,t) &:= \frac{1}{2\pi}\int_{|\xi|<1}\frac{\overline{g}(s-\xi,t)-\overline{g}(s,t)}{\xi}\Phi_1(\xi,s,t)d\xi\\
H^2_{\Phi_2,\overline{g}}(s,t) &:= -\frac{\overline{g}(s,t)}{2\pi}\int_{|\xi|>1}\frac{1}{\xi^2}\Phi_2(\xi,s,t)d\xi\\
H^3_{\Phi_3,\overline{g}}(s,t) &:= \frac{1}{2\pi}\int_{|\xi|>1}\frac{\overline{g}(s-\xi,t)}{\xi^2}\Phi_3(\xi,s,t)d\xi.
\end{aligned}$$

(i) Assume that $\overline{g} \in W^{n+1,1}(\mathbb{R})$. Then for each $i = 1,2,3$ we have $H^i_{\Phi_i,\overline{g}}(\cdot,s,t) \in W^{n,1}(\mathbb{R})$ with

$$\|H^i_{\Phi_i,\overline{g}}\|_{W^{n,1}(\mathbb{R})} \le C(\|\Phi_i\|_{W^{n,\infty}}, \|\overline{g}\|_{W^{n+1,1}(\mathbb{R})}).$$

(ii) Assume that $\overline{g} \in W^{n,\infty}(\mathbb{R})\cap C^{n,\alpha}(\mathbb{R})$ for some $\alpha \in (0,1)$. Then for each $i = 1,2,3$ we have $H^i_{\Phi_i,\overline{g}}(\cdot,s,t) \in W^{n,\infty}(\mathbb{R})$ with

$$\|H^i_{\Phi_i,\overline{g}}\|_{W^{n,\infty}(\mathbb{R})} \le C(\|\Phi_i\|_{W^{n,\infty}}, \|\overline{g}\|_{W^{n,\infty}(\mathbb{R})}, [\partial^n_s\overline{g}]_{C^{0,\alpha}(\mathbb{R})}).$$

Proof. (i) We start with $H^1_{\Phi_1,\overline{g}}$ and the case $n=0$. Using Fubini's Theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}}\left|\frac{1}{2\pi}\int_{|\xi|<1}\frac{\overline{g}(s-\xi,t)-\overline{g}(s,t)}{\xi}\Phi_1(\xi,s,t)d\xi\right| ds\\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\left|\int_{|\xi|<1} -\int_0^1 \partial_s\overline{g}(s-\xi\tau)d\tau\Phi_1(\xi,s,t)d\xi\right| ds\\
&\le \frac{\|\Phi_1\|_{L^\infty}}{2\pi}\int_{|\xi|<1}\int_0^1 \|\partial_s\overline{g}\|_{L^1(\mathbb{R})}d\tau d\xi = \frac{\|\Phi_1\|_{L^\infty}}{\pi}\|\partial_s\overline{g}\|_{L^1(\mathbb{R})}.
\end{aligned}$$

For $n\ge 1$, we distinct the following cases. If the derivative in s falls onto $\overline{g}(s-\xi,t)-\overline{g}(s,t)$, then we can argue as for $n=0$ with $\overline{g}$ replaced by $\partial^n_s\overline{g}$. If the derivative falls onto Φ_1, we can use almost the same proof merely with $\|\Phi_1\|_{L^\infty}$ replaced by $\|\partial^n_s\Phi_1\|_{L^\infty}$.

The assertion for $H^2_{\Phi_2,\overline{g}}$ can be verified easily, since all derivatives of Φ_2 are bounded and thus

$$\left|\int_{|\xi|>1} \frac{1}{\xi^2}\partial_s^k \Phi_2(\xi,s,t)d\xi\right| < C\|D^k\Phi_2\|_{L^\infty} \text{ for all } k \in \mathbb{N}.$$

Eventually, $\overline{g} \in W^{n+1,1}(\mathbb{R})$ then implies $H^2_{\Phi_2,\overline{g}} \in W^{n,1}(\mathbb{R})$.

Finally, for $H^3_F(\overline{g})$ we have for each $k_1, k_2 \le n,\ k_1 + k_2 = k \le n$

$$\left|\int_{|\xi|>1} \frac{\partial_s^{k_1}\overline{g}(s-\xi,t)}{\xi^2}(\partial_s^{k_2}\Phi_3)(\xi,s,t)d\xi\right| \le C\|D^{k_2}\Phi_3\|_{L^\infty}\left(\partial_s^{k_1}\overline{g} * \frac{\chi_{|\xi|>1}}{|\xi|^2}\right).$$

From Young's inequality we conclude $H^3_{\Phi_3,\overline{g}} \in W^{n,1}(\mathbb{R})$.

(ii) This proof can be established essentially by the same steps as in (i). For $H^1_{\Phi_1,\overline{g}}$ and $n = 0$, we compute

$$\begin{aligned}&\left|\frac{1}{2\pi}\int_{|\xi|<1} \frac{\overline{g}(s-\xi,t)-\overline{g}(s,t)}{\xi}\Phi_1(\xi,s,t)d\xi\right| \\ &\le \frac{\|\Phi_1\|_{L^\infty}}{2\pi}\int_{|\xi|<1}[\overline{g}]_{C^{0,\alpha}(\mathbb{R})}\frac{1}{|\xi|^{1-\alpha}}d\xi \le C\|\Phi_1\|_{L^\infty}[\overline{g}]_{C^{0,\alpha}(\mathbb{R})}\end{aligned}$$

and similarly we deduce for $n \ge 1$. For $H^2_{\Phi_2,\overline{g}}$ and $H^3_{\Phi_3,\overline{g}}$ we argue as in (i) and use $\overline{g} \in W^{n,\infty}(\mathbb{R})$ instead of $\overline{g} \in W^{n+1,1}(\mathbb{R})$. □

Before we apply these results to the operator $T_F(G)$, we first state the following Lemma about derivatives of composited functions, see also [33], p.17.

Lemma 3.11. Let $\Omega \subset \mathbb{R}^N, \Phi : \Omega \to \mathbb{R}$ and $u : \mathbb{R}^n \to \Omega$ be smooth functions such that $\|(D^l\Phi)\circ u\|_{L^\infty(\mathbb{R}^n)}, \|D^l u\|_{L^\infty(\mathbb{R}^n)} < \infty$ for each $l \in \mathbb{N}$. Then for each $m \in \mathbb{N}\setminus\{0\}$, there exists a constant C depending on $N, n, m, \|D^l u\|_{L^\infty(\mathbb{R}^n)}, \|(D^l\Phi)\circ u\|_{L^\infty(\mathbb{R}^n)}$ for each $l = 1,\dots,m$ such that

$$\|D^m(\Phi\circ u)\|_{L^\infty(\mathbb{R}^n)} \le C. \tag{3.40}$$

Proof. The chain rule yields

$$D^m(\Phi\circ u) = \sum_{l=1}^m (D^l\Phi)\circ u \sum_\sigma C_{l,\sigma}(Du)^{\sigma_1}(D^2u)^{\sigma_2}\dots(D^mu)^{\sigma_m},$$

where $C_{l,\sigma} > 0$ are constants and the inner sum is taken over indices $\sigma = (\sigma_1,\dots,\sigma_m)$ satisfying

$$\sum_{j=1}^m \sigma_j = l, \quad \sum_{j=1}^m j\sigma_j = m.$$

Hence we conclude

$$\|D^m(\Phi\circ u)\|_{L^\infty(\mathbb{R}^n)} \le \sum_{l=1}^m \|(D^l\Phi)\circ u\|_{L^\infty(\mathbb{R}^n)}\sum_\sigma C_{l,\sigma}\|Du\|^{\sigma_1}_{L^\infty(\mathbb{R}^n)}\dots\|D^mu\|^{\sigma_m}_{L^\infty(\mathbb{R}^n)}.$$

This shows (3.40). □

In the following, we often have to apply this Lemma to rational functions.

Corollary 3.12. Let $k, n, m \geq 1$ and $\tilde{\Phi} : \mathbb{R}^{1+k} \to \mathbb{R}$ be a smooth rational function. Furthermore, let $F : \mathbb{R}^n \supset W \to V \subset \mathbb{R}^k$ be a vector field, which is in $W^{m,\infty}(W)$. Define $\Phi : W \to \mathbb{R}$ by $\Phi(x_1, \dots, x_n) := \tilde{\Phi}(x_1, F(x_1, \dots, x_n))$ and assume that Φ is bounded on W. Then, we also have $\Phi \in W^{m,\infty}(W)$.

Proof. By use of Lemma 3.11, the statement follows immediately, if we can show that $(D^l\tilde{\Phi}) \circ F \in L^\infty(W)$ for each $1 \leq l \leq m$. Obviously, it is enough to prove that $D^l\tilde{\Phi}$ is bounded on $W_1 \times V$, where $W_1 = \mathrm{Pr}_1(W)$ is the projection of W onto the first component. Thanks to the fact that $\tilde{\Phi}$ is a rational function, which is bounded on $W_1 \times V$, we conclude that also each derivative of $\tilde{\Phi}$ has to be bounded on $W_1 \times V$. Indeed, note that $D^l\tilde{\Phi}$ is again a rational function for each $l \geq 1$. If $D^l\tilde{\Phi}$ has a pole, than $\tilde{\Phi}$ would have the same pole, contradicting that $\tilde{\Phi}$ is bounded. Similarly, if there is a sequence $(x_j)_j \subset W_1 \times V$ with $|x_j| \to \infty$ and such that $|D^l\tilde{\Phi}(x_j)| \to \infty$, then again we deduce the contradiction $|\tilde{\Phi}(x_j)| \to \infty$. □

Proposition 3.13. Let F, G as in (3.34) and $n \in \mathbb{N}$.
(i) If $\overline{f} \in W^{n+1,\infty}(\mathbb{R})$ and $\overline{g} \in W^{n+2,1}(\mathbb{R})$, we have $T_F(G) \in W^{n,1}(\mathbb{R})$ with

$$\|T_F(G)\|_{W^{n,1}(\mathbb{R})} \leq C(\beta, \|\overline{f}\|_{W^{n+1,\infty}(\mathbb{R})}, \|\overline{g}\|_{W^{n+2,1}(\mathbb{R})}).$$

(ii) If $\overline{f} \in W^{n+1,\infty}(\mathbb{R})$ and $\overline{g} \in W^{n+1,\infty}(\mathbb{R}) \cap C^{n+1,\alpha}(\mathbb{R})$ for some $\alpha \in (0,1)$, we have $T_F(G) \in W^{n,\infty}(\mathbb{R})$ with

$$\|T_F(G)\|_{W^{n,\infty}(\mathbb{R})} \leq C(\beta, \|\overline{f}\|_{W^{n+1,\infty}(\mathbb{R})}, \|\overline{g}\|_{W^{n+1,\infty}(\mathbb{R})}, [\partial_s^{n+1}\overline{g}]_{C^{0,\alpha}(\mathbb{R})}).$$

Proof. Recall the decomposition

$$T_F(G)(s,t) = T_F^{<}(\overline{g})(s,t) + T_F^{>,1}(\overline{g})(s,t) + T_F^{b}(\overline{g})(s,t) + T_F^{>,2}(\overline{g})(s,t).$$

The statement for $T_F^b(\overline{g})$ is a direct consequence of the assumptions on $\overline{f}$ and $\overline{g}$.
For $T_F^{<}(\overline{g})(s,t), T_F^{>,1}(\overline{g})(s,t)$ and $T_F^{>,2}(\overline{g})(s,t)$ we observe

$$\begin{aligned}
T_F^{<}(\overline{g}) &= H^1_{\Phi_1, \partial_s \overline{g}} \text{ with } \Phi_1(\xi, s, t) = \frac{\xi^2}{\xi^2 + F^2(\xi, s, t)}, \\
T_F^{>,1}(\overline{g}) &= H^3_{\Phi_3, \overline{g}} \text{ with } \Phi_3(\xi, s, t) = \frac{\xi^2(F^2(\xi, s, t) - \xi^2 - 2\xi F(\xi, s, t)\partial_\xi F(\xi, s, t))}{(\xi^2 + F^2(\xi, s, t))^2}, \\
T_F^{>,2}(\overline{g}) &= H^2_{\Phi_2, \partial_s \overline{g}} \text{ with } \Phi_2(\xi, s, t) = \frac{\xi(\overline{F}^2(\xi, s, t) - 2\beta\xi\overline{F}(\xi, s, t))}{(1+\beta^2)(\xi^2 + F^2(\xi, s, t))}.
\end{aligned}$$

Thus, the statement follows, if we can show that the $\Phi_i, i = 1, 2, 3$ satisfy the conditions of Proposition 3.10, namely that $\Phi_i \in W^{n,\infty}$. For this purpose we rely on Corollary 3.12. We have $\Phi_1(\xi, s, t) = \tilde{\Phi}_1(\xi, \cdot) \circ \overline{F}(\xi, s, t)$, with

$$\tilde{\Phi}_1(x_1, x_2) := \frac{x_1^2}{x_1^2 + (x_2 - \beta x_1)^2}.$$

It remains to verify that Φ_1 is bounded, which is clear, since $|\tilde{\Phi}_1| \leq 1$ on $\mathbb{R}^2$.

Similarly, we impose $\Phi_3(\xi, s, t) = \tilde{\Phi}_3(\xi, \cdot, \cdot) \circ (\overline{F}(\xi, s, t), \partial_\xi \overline{F}(\xi, s, t))$, where

$$\tilde{\Phi}_3(x_1, x_2, x_3) := \frac{(x_2 - \beta x_1)^2 - x_1^2 - 2x_1(x_2 - \beta x_1)(x_3 - \beta)}{(x_1^2 + (x_2 - \beta x_1)^2)^2}.$$

Let $M_1 > 0$ be a constant such that $\|\partial_\xi \overline{F}\|_{L^\infty} < M_1$. Then, for each $|x_1| > 1, x_2 \in \mathbb{R}$ and $x_3 \in [-M_1, M_1]$ we estimate

$$|\tilde{\Phi}_3(x_1, x_2, x_3)| \leq \frac{1 + M_1 + \beta}{x_1^2 + x_2^2} \leq 1 + M_1 + \beta,$$

which shows that Φ_3 is bounded.

Finally, we write $\Phi_2(\xi, s, t) = \tilde{\Phi}_2(\xi, \cdot) \circ \overline{F}(\xi, s, t)$, with

$$\tilde{\Phi}_2(x_1, x_2) := \frac{x_1(x_2^2 - 2\beta x_1 x_2)}{(1 + \beta^2)(x_1^2 + (x_2 - \beta x_1)^2)}.$$

Here, by $M_2 > 0$ we denote a constant satisfying $\|\overline{F}\|_{L^\infty} \leq M_2$. Therefore, on the set $\{|\xi| > 1\} \times [-M_2, M_2]$, $\tilde{\Phi}_2$ can be bounded by

$$|\tilde{\Phi}_2(x_1, x_2)| \leq \frac{M_2^2}{(1 + \beta^2)|x_1|} + \frac{2\beta M_2}{(1 + \beta^2)}.$$

This concludes the proof. □

To construct z satisfying (3.9), one could try to find the exact solution of

$$\partial_t z(s, t) = \frac{1}{2} D(s, t). \tag{3.41}$$

This ansatz would substantially correspond to the one in [17]. Especially, we need to establish uniform estimates for integral operators. If we compare with the results for the Muskat problem in [25] and [22], we notice that local well-posedness was proven only in the horizontal and stable case, i.e. for $\beta = 0$ and $\partial_t z = -T_Z(Z)$. However, this is exactly the case we singled out in Theorem 1.3. Furthermore, due to the nonlinear character, it is not clear, whether one could extend these results to (3.41).

Fortunately, to satisfy (3.9), it is not necessary to solve the equation exactly, since an estimate of $o(t)$ is sufficient. As opposed to [17], we propose a formal power series ansatz up to second order for the construction of $z(s, t)$. This motivates the ansatz

$$z(s, t) = z_0(s) + a(s)t + \frac{1}{2} b(s) t^2. \tag{3.42}$$

Obviously, the natural choice for $a(s)$ in the sense of a power series ansatz is

$$\begin{aligned} a(s) &= \frac{1}{2} D(s, 0) = T_{Z_0}(Z_0) + \frac{1}{2} R_{Z_0, 2c}(Z_0) = 2T_{Z_0}(\overline{Z}_0) \\ &= \frac{1}{\pi} \operatorname{PV} \int_{\mathbb{R}} (\partial_s \overline{z}_0(s - \xi) - \partial_s \overline{z}_0(s)) \frac{\xi}{\xi^2 + (z_0(s - \xi) - z_0(s))^2} d\xi. \end{aligned} \tag{3.43}$$

In particular, $a(s)$ only depends on the initial curve.

For the function b we want to choose $b(s) = \frac{1}{2}\partial_t D(s,0)$. Suppose for the moment that the curve z is already fixed by $z(s,t) = z_0(s) + a(s)t + \frac{1}{2}\tilde{b}(s)t^2$ for some function $\tilde{b}(s) \in W^{1,\infty}(\mathbb{R})$. Furthermore, in the spirit of (3.31), we set

$$A(\xi,s) := a(s-\xi) - a(s), \tag{3.44}$$

$$\tilde{B}(\xi,s) := \tilde{b}(s-\xi) - \tilde{b}(s). \tag{3.45}$$

To derive b, we explicitely compute

$$\begin{aligned} &\frac{T_Z(Z)(s,t) - T_Z(Z)(s,0)}{t} = \\ &\frac{1}{2\pi t}\left(\mathrm{PV}\int_{\mathbb{R}} \left(\partial_s Z_0 + t\partial_s A + \frac{t^2}{2}\partial_s \tilde{B}\right) \frac{\xi}{\xi^2 + (Z_0 + tA + \frac{t^2}{2}\tilde{B})^2} - \partial_s Z_0 \frac{\xi}{\xi^2 + Z_0^2} d\xi \right) \\ &= \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}} \left(\partial_s A + \frac{t}{2}\partial_s \tilde{B}\right) \frac{\xi}{\xi^2 + (Z_0 + tA + \frac{t^2}{2}\tilde{B})^2} \\ &\quad - \partial_s Z_0 \frac{\xi(2Z_0 A + tZ_0\tilde{B} + tA^2 + t^2 A\tilde{B} + \frac{t^3}{4}\tilde{B}^2)}{(\xi^2 + Z_0^2)(\xi^2 + (Z_0 + tA + \frac{t^2}{2}\tilde{B})^2)} d\xi. \end{aligned}$$

Thanks to this, we infer

$$\lim_{t\to 0} \frac{T_Z(Z)(s,t) - T_Z(Z)(s,0)}{t} = \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}} \partial_s A \frac{\xi}{\xi^2 + Z_0^2} - \partial_s Z_0 \frac{2\xi Z_0 A}{(\xi^2 + Z_0^2)^2} d\xi.$$

In the same way we observe

$$\begin{aligned} &\frac{R_{Z,2c}(Z)(s,t) - R_{Z,2c}(Z)(s,0)}{2t} = \frac{1}{4\pi t}\,\mathrm{PV}\int_{\mathbb{R}} \left(\partial_s Z_0 + t\partial_s A + \frac{t^2}{2}\partial_s \tilde{B}\right) \cdot \\ &\left(\frac{\xi}{\xi^2 + (Z_0 + tA + 2ct + \frac{t^2}{2}\tilde{B})^2} + \frac{\xi}{\xi^2 + (Z_0 + tA - 2ct + \frac{t^2}{2}\tilde{B})^2} \right) - 2\partial_s Z_0 \frac{\xi}{\xi^2 + Z_0^2} d\xi \\ &= \frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}} \frac{2\xi(\xi^2 + 4c^2t^2 + (Z_0 + tA + \frac{t^2}{2}\tilde{B})^2)\left(\partial_s A + \frac{t}{2}\partial_s \tilde{B}\right)}{(\xi^2 + (Z_0 + tA + 2ct + \frac{t^2}{2}\tilde{B})^2)(\xi^2 + (Z_0 + tA - 2ct + \frac{t^2}{2}\tilde{B})^2)} \\ &\quad - \partial_s Z_0 \frac{\xi((tA + 2ct + \frac{t^2}{2}\tilde{B})^2 + 2Z_0(tA + 2ct + \frac{t^2}{2}\tilde{B}))}{t(\xi^2 + Z_0^2)(\xi^2 + (Z_0 + tA + 2ct + \frac{t^2}{2}\tilde{B})^2)} \\ &\quad - \partial_s Z_0 \frac{\xi((tA - 2ct + \frac{t^2}{2}\tilde{B})^2 + 2Z_0(tA - 2ct + \frac{t^2}{2}\tilde{B}))}{t(\xi^2 + Z_0^2)(\xi^2 + (Z_0 + tA - 2ct + \frac{t^2}{2}\tilde{B})^2)}\, d\xi \end{aligned}$$

After a change of coordinates $\xi \mapsto \frac{\xi}{t}$ and Lebesgue's convergence theorem we deduce

$$\begin{aligned} &\lim_{t\to 0} \frac{R_{Z,2c}(Z)(s,t) - R_{Z,2c}(Z)(s,0)}{2t} = \frac{1}{2\pi}\,\mathrm{PV}\int_{\mathbb{R}} \partial_s A \frac{\xi}{\xi^2 + Z_0^2} - \partial_s Z_0 \frac{2\xi Z_0 A}{(\xi^2 + Z_0^2)^2} d\xi \\ &- \frac{1}{4\pi}\,\mathrm{PV}\int_{\mathbb{R}} \partial_s^2 z_0(s) \frac{4c^2 + 4c\xi\partial_s z_0(s)}{(1 + \partial_s z_0(s)^2)(\xi^2 + (\partial_s z_0(s)\xi + 2c)^2)} \\ &\qquad + \partial_s^2 z_0(s) \frac{4c^2 - 4c\xi\partial_s z_0(s)}{(1 + \partial_s z_0(s)^2)(\xi^2 + (\partial_s z_0(s)\xi - 2c)^2)} d\xi. \end{aligned}$$

In conclusion, we established

$$b(s) := \frac{1}{2}\partial_t D(s,0) = 2T_{Z_0}(A)(s) - 2\tilde{T}_{Z_0}(Z_0)(s) + P_{2c}(s), \tag{3.46}$$

where we define

$$(\tilde{T}_F(G))(s,t) := \frac{1}{2\pi}\ \mathrm{PV}\int_{\mathbb{R}} \partial_s G(\xi,s,t)\frac{2\xi Z_0(\xi,s)A(\xi,s)}{(\xi^2+F(\xi,s,t)^2)^2}d\xi \tag{3.47}$$

and

$$P_c(s) := \frac{-1}{4\pi}\frac{\partial_s^2 z_0(s)}{1+\partial_s z_0(s)^2}\ \mathrm{PV}\int_{\mathbb{R}} \frac{2c^2(\xi^2(1-\partial_s z_0(s)^2)+c^2)}{(\xi^2+(\partial_s z_0(s)\xi+c)^2)(\xi^2+(\partial_s z_0(s)\xi-c)^2)}d\xi. \tag{3.48}$$

In particular, we see that $\frac{1}{2}\partial_t D(s,0)$ does not depend on the concrete choice of $\tilde{b}$, but only on z_0, a and the propagation speed c. In this sense, $b(s)$ is well-defined.

In this definition of b we see the reason for choosing the same propagation speed c for the two boundaries $\Gamma^{\pm}$. Otherwise there would emerge terms of the form

$$\frac{\partial_s^2 z_0(s)}{1+\partial_s z_0(s)^2}\ \mathrm{PV}\int_{\mathbb{R}} \frac{4c\xi\partial_s z_0(s)}{\xi^2+(\partial_s z_0(s)\xi \pm 2c)^2}d\xi,$$

in which the integrand is of order $|\xi|^{-1}$. Although this term is still well-defined thanks to Lemma 3.4, it takes some more effort to verify this for the derivatives with respect to s, which we need to do in the following Lemma. However, in (3.46) these terms cancel each other in such a way that the remaining term P_{2c} has an integrand of order $|\xi|^{-2}$. This lightens the computations on $\{|\xi|>1\}$. For the same reason, we find more easily a majorant for this term before applying Lebesgue's convergence theorem in the derivation of b.

After we fixed a, b and z via (3.43), (3.46) and (3.42), we investigate regularity properties of a and b.

Lemma 3.14. Let $n \in \mathbb{N}, n \geq 5$ and $\overline{z}_0 \in W^{n,1}(\mathbb{R}) \cap C^{n-1,\alpha}(\mathbb{R})$ for some $\alpha \in (0,1)$. Then
(i) $a \in W^{n-2,1}(\mathbb{R}) \cap W^{n-2,\infty}(\mathbb{R}) \cap C^{n-3,\alpha}(\mathbb{R})$.
(ii) $b \in W^{n-4,1}(\mathbb{R}) \cap W^{n-4,\infty}(\mathbb{R}) \cap C^{n-5,\alpha}(\mathbb{R})$.

Proof. At first, Lemma 3.7 yields $\overline{z}_0 \in W^{n-1,\infty}(\mathbb{R})$.
(i) Since we know that $a = 2T_{Z_0}(\overline{Z}_0)$, from Proposition 3.13 we immediately obtain $a \in W^{n-2,1}(\mathbb{R}) \cap W^{n-2,\infty}(\mathbb{R})$. Thus, we deduce $a \in W^{n-2,p}(\mathbb{R})$ for each $1 \leq p \leq \infty$ and thanks to Lemma 3.7 we infer $a \in C^{n-3,\alpha}(\mathbb{R})$.

(ii) It is enough to verify $b \in W^{n-4,1}(\mathbb{R}) \cap W^{n-4,\infty}(\mathbb{R})$, since then Lemma 3.7 implies $b \in C^{n-5,\alpha}(\mathbb{R})$. Recall that $b = 2T_{Z_0}(A) - 2\tilde{T}_{Z_0}(Z_0) + P_{2c}$. The statement for $T_{Z_0}(A)$ is a consequence of Proposition 3.13 and (i). To deal with $\tilde{T}_{Z_0}(Z_0)$, we decompose

$$\begin{aligned}
\tilde{T}_{Z_0}(Z_0) &= \tilde{T}_{Z_0}(\overline{Z}_0) = \tilde{T}^{<}_{Z_0}(\overline{z}_0) + \tilde{T}^{>,1}_{Z_0}(\overline{z}_0) + \tilde{T}^{>,2}_{Z_0}(\overline{z}_0),\\
\tilde{T}^{<}_{Z_0}(\overline{z}_0)(s) &:= \frac{1}{2\pi}\int_{|\xi|<1} \partial_s \overline{Z}_0(\xi,s)\frac{2\xi Z_0(\xi,s)A(\xi,s)}{(\xi^2+Z_0(\xi,s)^2)^2}d\xi,\\
&= \frac{1}{2\pi}\int_{|\xi|<1} \frac{\partial_s\overline{z}_0(s-\xi)-\partial_s\overline{z}_0(s)}{\xi}\frac{2\xi^4\psi_0(\xi,s)\psi_a(\xi,s)}{(\xi^2+Z_0(\xi,s)^2)^2}d\xi,\\
\tilde{T}^{>,1}_{Z_0}(\overline{z}_0)(s) &:= \frac{1}{2\pi}\int_{|\xi|>1} \frac{\partial_s\overline{z}_0(s-\xi)}{\xi^2}\frac{2\xi^3 Z_0(\xi,s)A(\xi,s)}{(\xi^2+Z_0(\xi,s)^2)^2}d\xi,\\
\tilde{T}^{>,2}_{Z_0}(\overline{z}_0)(s) &:= -\frac{\partial_s\overline{z}_0(s)}{2\pi}\int_{|\xi|>1} \frac{1}{\xi^2}\frac{2\xi^3 Z_0(\xi,s)A(\xi,s)}{(\xi^2+Z_0(\xi,s)^2)^2}d\xi,
\end{aligned}$$

where we use definition (3.32). Adapting the notation from Proposition 3.10, we can write $\tilde{T}^{<}_{Z_0}(\overline{z}_0) = H^1_{\Phi_1}(\partial_s\overline{z}_0), \tilde{T}^{>,1}_{Z_0}(\overline{z}_0)(s) = H^3_{\Phi_3}(\partial_s\overline{z}_0), \tilde{T}^{>,2}_{Z_0}(\overline{z}_0)(s) = H^2_{\Phi_2}(\partial_s\overline{z}_0)$ with

$$\begin{aligned}
\Phi_1(\xi,s,t) &= \frac{2\xi^4\psi_0(\xi,s)\psi_a(\xi,s)}{(\xi^2+Z_0(\xi,s)^2)^2}\\
\Phi_2(\xi,s,t) = \Phi_3(\xi,s,t) &= \frac{2\xi^3 Z_0(\xi,s)A(\xi,s)}{(\xi^2+Z_0(\xi,s)^2)^2}.
\end{aligned}$$

Hence, if Φ_1, Φ_2 have bounded derivatives up to order $n-4$, then Proposition 3.10 yields $\tilde{T}_{Z_0}(\overline{Z}_0) \in W^{n-4,1}(\mathbb{R}) \cap W^{n-4,\infty}(\mathbb{R})$, since the assumption on $\overline{z}_0$ especially gives $\overline{z}_0 \in W^{n-2,1}(\mathbb{R}) \cap C^{n-2,\alpha}(\mathbb{R})$.

For Φ_1, we first note that thanks to $\overline{z}_0 \in W^{n-1,\infty}(\mathbb{R})$, we deduce $\psi_0(\xi,\cdot) \in W^{n-2,\infty}(\mathbb{R})$ for fixed $|\xi| \in \mathbb{R}$. Similarly, from (i) we conclude $\psi_a \in W^{n-3,\infty}(\mathbb{R}^2)$. We can express $\Phi_1(\xi,s,t) = \tilde{\Phi}_1(\xi,\cdot,\cdot,\cdot) \circ (\psi_0(\xi,s), \psi_a(\xi,s), \overline{Z}_0(\xi,s))$, where

$$\tilde{\Phi}_1(x_1,x_2,x_3,x_4) := \frac{2x_1^4 x_2 x_3}{(x_1^2+(x_4-\beta x_1)^2)^2}.$$

Because of Corollary 3.12, it is sufficient to establish an estimate for $\tilde{\Phi}_1$ on the set $[-1,1]\times[-\|\psi_0\|_{L^\infty(\mathbb{R}^2)}, \|\psi_0\|_{L^\infty(\mathbb{R}^2)}]\times[-\|\psi_a\|_{L^\infty(\mathbb{R}^2)}, \|\psi_a\|_{L^\infty(\mathbb{R}^2)}]\times[-\|\overline{z}_0\|_{L^\infty(\mathbb{R})}, \|\overline{z}_0\|_{L^\infty(\mathbb{R})}]$. Indeed, here we obtain the bound

$$|\tilde{\Phi}_1| \leq 2\|\psi_0\|_{L^\infty(\mathbb{R}^2)}\|\psi_a\|_{L^\infty(\mathbb{R}^2)}.$$

For Φ_2, Φ_3 we argue analogously and write $\Phi_2(\xi,s,t) = \tilde{\Phi}_2(\xi,\cdot,\cdot) \circ (\overline{Z}_0(\xi,s), A(\xi,s))$ with

$$\tilde{\Phi}_2(x_1,x_2,x_3) := \frac{2x_1^3(x_2-\beta x_1)x_3}{(x_1^2+(x_2-\beta x_1)^2)^2}.$$

Then, on $\{\mathbb{R}\setminus(-1,1)\}\times[-\|\overline{z}_0\|_{L^\infty(\mathbb{R})}, \|\overline{z}_0\|_{L^\infty(\mathbb{R})}]\times[-\|a\|_{L^\infty(\mathbb{R})}, \|a\|_{L^\infty(\mathbb{R})}]$, we estimate

$$|\tilde{\Phi}_2| \leq \|a\|_{L^\infty(\mathbb{R})}.$$

From Corollary 3.12, we accomplish the statement.

Finally, we investigate the term

$$P_{2c}(s) = \frac{-1}{4\pi} \frac{\partial_s^2 z_0(s)}{1 + \partial_s z_0(s)^2} \, \mathrm{PV} \int_{\mathbb{R}} \frac{8c^2(\xi^2(1 - \partial_s z_0(s)^2) + 4c^2)}{(\xi^2 + (\partial_s z_0(s)\xi + 2c)^2)(\xi^2 + (\partial_s z_0(s)\xi - 2c)^2)} d\xi.$$

By means of $\overline{z}_0 \in W^{n-2,1}(\mathbb{R}) \cap W^{n-2,\infty}(\mathbb{R})$, we obtain $P_{2c} \in W^{n-4,1}(\mathbb{R}) \cap W^{n-4,\infty}(\mathbb{R})$ provided that

$$\left\| \partial_s^k \left(\mathrm{PV} \int_{\mathbb{R}} \frac{8c^2(\xi^2(1 - \partial_s z_0(s)^2) + 4c^2)}{(\xi^2 + (\partial_s z_0(s)\xi + 2c)^2)(\xi^2 + (\partial_s z_0(s)\xi - 2c)^2)} d\xi \right) \right\|_{L^\infty(\mathbb{R})} < \infty$$

for each $0 \leq k \leq n-4$. For this purpose, we split up the domain of integration into $\mathbb{R} = \{|\xi| < 1\} \cup \{|\xi| > 1\}$. On $\{|\xi| < 1\}$, we can estimate the integrand by a constant $C > 0$, since there is no singularity in the integrand. This holds true for each derivative $0 \leq k \leq n-4$. On $\{|\xi| > 1\}$ and for $k = 0$, the integrand can be bounded by $C|\xi|^{-2}$ and therefore the integral is finite. For higher derivatives, we rewrite the integrand as $|\xi|^{-2}\Phi$, where Φ is a composition of a bounded rational function $\tilde{\Phi}$ with $\partial_s z_0$. An application of Corollary 3.12 finishes the proof. □

Now we demonstrate that this choice of z indeed gives rise to a subsolution. For the next calculations we often have to deal with terms involving the time variable t. Therefore the following Lemma will be helpful.

Lemma 3.15. Let $z(\cdot, t) \in W^{1,\infty}(\mathbb{R})$ for each $t \in [0, T]$. Then there exists a positive constant $C > 0$ such that

$$\frac{1}{\xi^2 + (Z(s,\xi,t) \pm t)^2} \leq \frac{C}{\xi^2 + t^2} \tag{3.49}$$

for all $s, \xi \in \mathbb{R}$ and each $0 < t \leq T$. In particular, we can estimate

$$\frac{1}{\xi^2 + (Z(s,\xi,t) \pm t)^2} \leq \frac{C}{\xi^2} \quad \text{or} \quad \frac{1}{\xi^2 + (Z(s,\xi,t) \pm t)^2} \leq \frac{C}{t^2},$$

i.e. by space or time variable respectively.

Proof. We express $Z(s,\xi,t) = \psi(s,\xi,t)\xi$, where $\psi(s,\xi,t) = -\int_0^1 \partial_s z(s - \tau\xi, t) d\tau$ is a bounded function. To establish (3.49), it is enough to prove

$$\xi^2 + (Z(s,\xi,t) \pm t)^2 \geq Ct^2 + \frac{1}{2}\xi^2$$

for some $C > 0$. Equivalently, this reads

$$\frac{1}{2}y^2 + (y\psi \pm 1)^2 \geq C,$$

where we set $y := \frac{\xi}{t}$. Obviously, this is satisfied for some $C > 0$, since the left hand side can never become zero. □

By means of the foregoing preparations, we can state a first existence result for a curve z satisfying (3.9)-(3.11), which in turn implies the existence of a subsolution.

Theorem 3.16. Define the curve $z(s,t)$ by (3.42), with $z_0(s) = \beta s + \overline{z}_0(s)$, $\beta \in \mathbb{R}$, define $a(s)$ by (3.43) and $b(s)$ by (3.46). Furthermore, suppose that for some $\alpha \in (0,1)$ we have

$$\overline{z}_0 \in W^{6,1}(\mathbb{R}) \cap C^{5,\alpha}(\mathbb{R}). \tag{3.50}$$

Then, $z(\cdot,t) \in C^{1,\alpha}(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ satisfies (3.9)-(3.11) on $[0,T)$ for some $T > 0$.

Remark 3.17. In the stable regime and for $\beta \in \mathbb{R} \setminus \{0\}$, the sign of $\nu^{\pm}$ changes and consequently also the sign of $D(s,t), a(s)$ and $b(s)$. Of course, this does not affect the estimates here and therefore the same statement remains true for the stable regime.

Proof. At first, from Lemma 3.14 we conclude $a \in W^{4,1}(\mathbb{R}) \cap W^{4,\infty}(\mathbb{R}) \cap C^{3,\alpha}(\mathbb{R})$ and $b \in W^{2,1}(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R}) \cap C^{1,\alpha}(\mathbb{R})$. In particular, we get $z \in C^{1,\alpha}(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$.

Obviously, (3.11) is satisfied. We start to verify (3.9), that is,

$$\frac{1}{t}\|2\partial_t z(s,t) - D(s,t)\|_{L^1(\mathbb{R})} \to 0 \text{ as } t \to 0.$$

By the choice of a and b, this is equivalent to

$$\left\| \frac{D(s,t) - D(s,0)}{t} - \partial_t D(s,0) \right\|_{L^1(\mathbb{R})} \to 0 \text{ as } t \to 0. \tag{3.51}$$

Observe that

$$\begin{aligned} \frac{D(s,t) - D(s,0)}{t} - \partial_t D(s,0) &= \int_0^1 \partial_t D(s,\tau t)d\tau - \partial_t D(s,0) \\ &= t \int_0^1 \tau \int_0^1 \partial_t^2 D(s,\tau'\tau t)d\tau' d\tau. \end{aligned}$$

Hence, to deduce (3.51), we need to show

$$\|\partial_t^2 D\|_{L_t^\infty L_s^1} < \infty. \tag{3.52}$$

Using the representation of D from (3.38), it is enough to establish (3.52) for each of the operators separately. We start with $T_Z^<(\overline{z})$. Recall that

$$T_Z^<(\overline{z})(s,t) = \frac{1}{2\pi} \int_{|\xi|<1} \frac{\partial_s \overline{z}(s-\xi,t) - \partial_s \overline{z}(s,t)}{\xi} \frac{\xi^2}{\xi^2 + Z^2(\xi,s,t)} d\xi$$

with $z(s,t) = z_0(s) + ta(s) + \frac{t^2}{2}b(s)$ and $\overline{z}(s,t) = \overline{z}_0(s) + ta(s) + \frac{t^2}{2}b(s)$. If both time derivatives fall onto the first factor $(\partial_s \overline{Z}(\xi,s,t))\xi^{-1}$, we obtain $T_Z^<(b)$. Exactly as in Proposition 3.13, we infer $\|T_Z^<(b)\|_{L_t^\infty L_s^1} < \infty$, since we know that $\partial_s^2 b \in L^1(\mathbb{R})$. For later purposes in Section 3.4 we want to emphasize that this is the only term, where we need this strong regularity assumption on b.

Suppose that one time derivative falls on each factor of the integrand. In this case, we estimate

$$\begin{aligned}&\left|-\frac{1}{2\pi}\int_{|\xi|<1}\frac{\partial_s A(\xi,s)+t\partial_s B(\xi,s)}{\xi}\frac{2\xi^2 Z(\xi,s,t)(A(\xi,s)+tB(\xi,s))}{(\xi^2+Z^2(\xi,s,t))^2}d\xi\right|\\&\le\frac{[\partial_s a]_{C^{0,\alpha}(\mathbb{R})}+T[\partial_s b]_{C^{0,\alpha}(\mathbb{R})}}{\pi}\int_{|\xi|<1}\frac{|\psi_z(\xi,s,t)(\psi_a(\xi,s)+T\psi_b(\xi,s))|}{|\xi|^{1-\alpha}}d\xi.\end{aligned}$$

This term is in L^1_s uniformly in t, since $\partial_s z\in L^\infty$ and $a,b\in W^{1,1}(\mathbb{R})$. Finally, if both time derivatives fall onto $\frac{\xi^2}{\xi^2+Z^2(\xi,s,t)}$, we can argue analogously. Thus, we conclude (3.52) for $T_Z^<(\overline{z})$.

Except of $E_{Z,2c}(\overline{Z})$, all remaining operators in (3.38) can be handled similarly. On one hand, for $k=0,1,2$ we have $\partial_t^k\overline{z}(s,t)\in W^{1,1}(\mathbb{R})$, from which we deduce the L^1_s-bound. On the other hand, for the other factor $\Phi(\xi,s,t)$ in the integral, we know $\partial_t^k\Phi\in L^\infty$ uniformly in t for $k=0,1,2$, which follows from Corollary 3.12.

Finally, we have to investigate $E_{Z,2c}(\overline{Z})$. However, the above argumentation does not work here, since $\partial_t^2 E_{Z,2c}(\overline{Z})$ is not well-defined for positive times. Due to an appropriate cancellation, this term behaves well only at time $t=0$. Thus, it is enough to verify (3.51) for $E_{Z,2c}(\overline{Z})$, as we will establish now explicitely. At first we compute in detail the term $\partial_t E_{Z,2c}(\overline{Z})(s,0)$, which resembles the computations for the definition of b. Therefore we decompose

$$\frac{1}{t}(E_{Z,2c}(\overline{Z})(s,t)-E_{Z,2c}(\overline{Z})(s,0))=I_1(s,t)+I_2(s,t)+o(1)$$

as $t\to 0$, where we set

$$I_1(s,t):=\frac{1}{2\pi t}\int_{|\xi|<1}\partial_s\overline{Z}_0\xi\left(\frac{1}{\xi^2+(Z+2ct)^2}+\frac{1}{\xi^2+(Z-2ct)^2}-\frac{2}{\xi^2+Z_0^2}\right)d\xi,\quad(3.53)$$

$$I_2(s,t):=\frac{1}{2\pi}\int_{|\xi|<1}\partial_s A\frac{2\xi(\xi^2+Z^2+4c^2t^2)}{(\xi^2+(Z+2ct)^2)(\xi^2+(Z-2ct)^2)}d\xi.\quad(3.54)$$

Moreover, we split I_1 into

$$\begin{aligned}I_1&=\frac{1}{2\pi t}\int_{|\xi|<1}\psi_0'(\xi,s)\xi^2\left(\frac{4c^2t^2+8ctZ+Z_0^2-Z^2}{(\xi^2+Z_0^2)(\xi^2+(Z+2ct)^2)}+\frac{4c^2t^2-8ctZ+Z_0^2-Z^2}{(\xi^2+Z_0^2)(\xi^2+(Z-2ct)^2)}\right)d\xi\\&=\frac{1}{2\pi t}\int_{|\xi|<1}\frac{\psi_0'(\xi,s)}{1+\psi_0^2(\xi,s)}\left(\frac{4c^2t^2+8ct\xi\psi_z(\xi,s,t)}{\xi^2+(\xi\psi_z(\xi,s,t)+2ct)^2}+\frac{4c^2t^2-8ct\xi\psi_z(\xi,s,t)}{\xi^2+(\xi\psi_z(\xi,s,t)-2ct)^2}\right)d\xi\\&-\frac{1}{2\pi}\int_{|\xi|<1}\frac{\psi_0'(\xi,s)}{1+\psi_0^2(\xi,s)}\left(\frac{2\xi^2\psi_0(\xi,s)\psi_a(\xi,s)}{\xi^2+(\xi\psi_z(\xi,s,t)+2ct)^2}+\frac{2\xi^2\psi_0(\xi,s)\psi_a(\xi,s)}{\xi^2+(\xi\psi_z(\xi,s,t)-2ct)^2}\right)d\xi+o(1)\\&=\frac{1}{2\pi}\int_{|\xi|<\frac{1}{t}}\frac{\psi_0'(t\xi,s)}{1+\psi_0^2(t\xi,s)}\left(\frac{4c^2+8c\xi\psi_z(t\xi,s,t)}{\xi^2+(\xi\psi_z(t\xi,s,t)+2c)^2}+\frac{4c^2-8c\xi\psi_z(t\xi,s,t)}{\xi^2+(\xi\psi_z(t\xi,s,t)-2c)^2}\right)d\xi\\&-\frac{1}{2\pi}\int_{|\xi|<1}\frac{\psi_0'(\xi,s)}{1+\psi_0^2(\xi,s)}\left(\frac{2\xi^2\psi_0(\xi,s)\psi_a(\xi,s)}{\xi^2+(\xi\psi_z(\xi,s,t)+2ct)^2}+\frac{2\xi^2\psi_0(\xi,s)\psi_a(\xi,s)}{\xi^2+(\xi\psi_z(\xi,s,t)-2ct)^2}\right)d\xi+o(1)\\&=:I_{11}(s,t)+I_{12}(s,t)+o(1).\end{aligned}$$

Note that

$$\begin{aligned}\psi_z(t\xi,s,t) =& -\int_0^1 \partial_s z(s-t\xi\tau,t)d\tau\\ =& -\frac{1}{t}\int_0^t \partial_s z_0(s-\xi\tau)+t\partial_s a(s-\xi\tau)+\frac{1}{2}t^2\partial_s b(s-\xi\tau)d\tau \to -\partial_s z_0(s)\end{aligned}$$

as $t\to 0$, because of $\overline{z}_0\in C^{5,\alpha}(\mathbb{R})$. Hence, in the limit $t\to 0$ we infer

$$\begin{aligned}\partial_t E_{Z,2c}(\overline{Z})(s,0) =& I_{11}(s,0)+I_{12}(s,0)+I_2(s,0)\\ =& \frac{1}{2\pi}\int_{\mathbb{R}} \frac{-\partial_s^2 z_0(s)}{1+\partial_s z_0(s)^2}\frac{8c^2(\xi^2+\xi^2\partial_s z_0(s)^2+4c^2)-64c^2\xi^2\partial_s z_0(s)^2}{(\xi^2+(-\xi\partial_s z_0(s)+2c)^2)(\xi^2+(-\xi\partial_s z_0(s)-2c)^2)}d\xi\\ &-\frac{1}{2\pi}\int_{|\xi|<1} \frac{2\psi_0(\xi,s)\psi_a(\xi,s)\psi_0'(\xi,s)}{(1+\psi_0^2(\xi,s))^2}-\partial_s A(\xi,s)\frac{2\xi}{\xi^2+Z_0^2(\xi,s)}d\xi.\end{aligned}$$

For (3.9), it remains to show

$$\left\|\frac{E_{Z,2c}(\overline{Z})(s,t)-E_{Z,2c}(\overline{Z})(s,0)}{t}-\partial_t E_{Z,2c}(\overline{Z})(s,0)\right\|_{L^1(\mathbb{R})}\to 0 \text{ as } t\to 0. \tag{3.55}$$

We start with I_2 and compute

$$\begin{aligned}&I_2(s,t)-I_2(s,0)=\\ &\frac{1}{2\pi}\int_{|\xi|<1} 2\xi\partial_s A\cdot\left(\frac{(\xi^2+Z^2+4c^2t^2)(\xi^2+Z_0^2)-(\xi^2+(Z+2ct)^2)(\xi^2+(Z-2ct)^2)}{(\xi^2+(Z+2ct)^2)(\xi^2+(Z-2ct)^2)(\xi^2+Z_0^2)}\right)d\xi\\ &=\frac{1}{2\pi}\int_{|\xi|<1} 2\xi\partial_s A\cdot\\ &\frac{16c^2t^2Z^2-4c^2t^2(\xi^2+Z^2+4c^2t^2)-(\xi^2+Z^2+4c^2t^2)(2Z_0(tA+\frac{t^2}{2}B)+(tA+\frac{t^2}{2}B)^2)}{(\xi^2+(Z+2ct)^2)(\xi^2+(Z-2ct)^2)(\xi^2+Z_0^2)}d\xi.\end{aligned}$$

We can pull out a factor $\sqrt{t}$, which ensures the limit zero as $t\to 0$. The remaining integrand can be estimated with help of Lemma 3.15 and $\partial_s z\in L^\infty(\mathbb{R})$ by $C|\xi|^{-\frac{1}{2}}$, which is integrable on $\{|\xi|<1\}$. Thanks to $\partial_s A=\xi\psi_a'$ and $a\in W^{2,1}(\mathbb{R})$, we accomplish the L^1-bound in s.

The next term to investigate is

$$\begin{aligned}&I_{12}(s,t)-I_{12}(s,0)=-\frac{1}{2\pi}\int_{|\xi|<1}\frac{2\psi_0(\xi,s)\psi_a(\xi,s)\psi_0'(\xi,s)}{1+\psi_0^2(\xi,s)}\cdot\\ &\left(\frac{\xi^2}{\xi^2+(\xi\psi_z(\xi,s,t)+2ct)^2}-\frac{1}{1+\psi_0^2(\xi,s)}+\frac{\xi^2}{\xi^2+(\xi\psi_z(\xi,s,t)-2ct)^2}-\frac{1}{1+\psi_0^2(\xi,s)}\right)d\xi\\ &=-\frac{1}{2\pi}\int_{|\xi|<1}\frac{2\psi_0(\xi,s)\psi_a(\xi,s)\psi_0'(\xi,s)}{1+\psi_0^2(\xi,s)}\cdot\\ &\left(\frac{\xi^2(\psi_0^2-\psi_z^2)-4ct\xi\psi_z-4c^2t^2}{(\xi^2+(\xi\psi_z+2ct)^2)(1+\psi_0^2)}+\frac{\xi^2(\psi_0^2-\psi_z^2)+4ct\xi\psi_z-4c^2t^2}{(\xi^2+(\xi\psi_z-2ct)^2)(1+\psi_0^2)}\right)d\xi.\end{aligned}$$

Using

$$\begin{aligned}\psi_z(\xi,s,t)-\psi_0(\xi,s,t) &= -\int_0^1 \partial_s z(s-\xi\tau,t)-\partial_s z_0(s-\xi\tau)d\tau \\ &= -\int_0^1 t\partial_s a(s-\xi\tau)+\frac{t^2}{2}\partial_s b(s-\xi\tau)d\tau,\end{aligned}$$

we observe that the argumentation resembles to the one for I_2. Indeed, after pulling out a factor $\sqrt{t}$, we can estimate the integrand by $C|\xi|^{-\frac{1}{2}}$. The L^1-bound in s is a consequence of $\partial_s^2 z_0 \in L^1(\mathbb{R})$ coming from the factor $\psi_0(\xi,s)$ in front of the integrand.

Finally we study I_{11}. Since

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathbb{R}}\left|\int_{|\xi|>\frac{1}{t}}\frac{-\partial_s^2 z_0(s)}{1+\partial_s z_0(s)^2}\frac{8c^2(\xi^2+\xi^2\partial_s z_0(s)^2+4c^2)-64c^2\xi^2\partial_s z_0(s)^2}{(\xi^2+(-\xi\partial_s z_0(s)+2c)^2)(\xi^2+(-\xi\partial_s z_0(s)-2c)^2)}d\xi\right| ds \\ &\le C\|\partial_s^2 z_0\|_{L^1(\mathbb{R})}\int_{|\xi|>\frac{1}{t}}\frac{1}{|\xi|^2}d\xi \to 0 \text{ as } t\to 0,\end{aligned}$$

it is sufficient to consider $I_{11}(s,0)$ on $\{|\xi|<\frac{1}{t}\}$. For this purpose, we rewrite the difference $I_{11}(s,t)-I_{11}(s,0)=H_1(s,t)+H_2(s,t)$, where

$$\begin{aligned}H_1(s,t) &= \frac{1}{2\pi}\int_{|\xi|<\frac{1}{t}}\frac{\psi_0'(t\xi,s)}{1+\psi_0^2(t\xi,s)}\left(\frac{8c^2(\xi^2+\xi^2\psi_z^2+4c^2)-64c^2\xi^2\psi_z^2}{(\xi^2+(\xi\psi_z+2c)^2)(\xi^2+(\xi\psi_z-2c)^2)}\right. \\ &\quad\left.-\frac{8c^2(\xi^2+\xi^2\partial_s z_0(s)^2+4c^2)-64c^2\xi^2\partial_s z_0(s)^2}{(\xi^2+(-\xi\partial_s z_0(s)+2c)^2)(\xi^2+(-\xi\partial_s z_0(s)-2c)^2)}\right), \\ H_2(s,t) &= \frac{1}{2\pi}\int_{|\xi|<\frac{1}{t}}K_1(\xi,s,t)K_2(\xi,s,t)d\xi, \\ K_1(\xi,s) &= \frac{8c^2(\xi^2+\xi^2\partial_s z_0(s)^2+4c^2)-64c^2\xi^2\partial_s z_0(s)^2}{(\xi^2+(-\xi\partial_s z_0(s)+2c)^2)(\xi^2+(-\xi\partial_s z_0(s)-2c)^2)}, \\ K_2(\xi,s,t) &= \frac{\psi_0'(t\xi,s)}{1+\psi_0^2(t\xi,s)}-\frac{-\partial_s^2 z_0(s)}{1+\partial_s z_0^2(s)} \\ &= \frac{(-\int_0^1\partial_s^2 z_0(s-\tau\xi t)-\partial_s^2 z_0(s)d\tau)(1+\partial_s z_0^2(s))+\partial_s^2 z_0(s)(\psi_0^2(s)-\partial_s z_0^2(s))}{(1+\psi_0^2)(1+\partial_s z_0^2(s))}.\end{aligned}$$

Because of

$$|\psi_0^2(s)-\partial_s z_0^2(s)| \le 2|\partial_s z_0(s)||\xi t|\int_0^1\tau\int_0^1|\partial_s^2 z_0(s-\tau\tau'\xi t)|d\tau' d\tau$$

and

$$-\int_0^1\partial_s^2 z_0(s-\tau\xi t)-\partial_s^2 z_0(s)d\tau = \xi t\int_0^1\tau\int_0^1\partial_s^3 z_0(s-\tau\tau'\xi t)d\tau d\tau',$$

we can pull out a factor ξt in K_2. Moreover, we can estimate $|K_1| \leq C(\|\partial_s z_0\|^2_{L^\infty(\mathbb{R})})$ on $\{|\xi| < 1\}$ and $|K_1| \leq \frac{C(\|\partial_s z_0\|^2_{L^\infty(\mathbb{R})})}{|\xi|^2}$ on $\{1 < |\xi| < \frac{1}{t}\}$. Hence, we deduce

$$\begin{aligned}\|H_2(\cdot,t)\|_{L^1_s(\mathbb{R})} \leq Ct \int_{|\xi|<1} \|\partial_s^3 z_0\|_{L^1(\mathbb{R})} + \|\partial_s z_0\|_{L^1(\mathbb{R})} d\xi \\ + C\int_{1<|\xi|<\frac{1}{t}} \frac{|\xi t|}{|\xi|^2}(\|\partial_s^3 z_0\|_{L^1(\mathbb{R})} + \|\partial_s z_0\|_{L^1(\mathbb{R})}) d\xi \\ \leq Ct(\|\partial_s^3 z_0\|_{L^1(\mathbb{R})} + \|\partial_s z_0\|_{L^1(\mathbb{R})}) + C\sqrt{t}(\|\partial_s^3 z_0\|_{L^1(\mathbb{R})} + \|\partial_s z_0\|_{L^1(\mathbb{R})}),\end{aligned}$$

where we used that $\sqrt{|\xi t|} \leq 1$ on $\{1 < |\xi| < \frac{1}{t}\}$. In particular, we observe that $\|H_2(\cdot,t)\|_{L^1_s(\mathbb{R})}$ tends to zero as $t \to 0$. For H_1, we first compute

$$\begin{aligned}L :=& \frac{1}{(\xi^2 + (\xi\psi_z + 2c)^2)(\xi^2 + (\xi\psi_z - 2c)^2)} \\ &- \frac{1}{(\xi^2 + (-\xi\partial_s z_0(s) + 2c)^2)(\xi^2 + (-\xi\partial_s z_0(s) - 2c)^2)} \\ =&\frac{2\xi^4(\partial_s z_0^2(s) - \psi_z^2) + (\xi^2\partial_s z_0^2(s) - 4c^2)^2 - (\xi^2\psi_z^2 - 4c^2)^2}{(\xi^2 + (\xi\psi_z + 2c)^2)(\xi^2 + (\xi\psi_z - 2c)^2)(\xi^2 + (-\xi\partial_s z_0(s) + 2c)^2)(\xi^2 + (-\xi\partial_s z_0(s) - 2c)^2)} \\ =&\frac{\xi^2(\partial_s z_0^2(s) - \psi_z^2)(2\xi^2 - 8c^2 + \xi^2(\partial_s z_0^2(s) + \psi_z^2))}{(\xi^2 + (\xi\psi_z + 2c)^2)(\xi^2 + (\xi\psi_z - 2c)^2)(\xi^2 + (-\xi\partial_s z_0(s) + 2c)^2)(\xi^2 + (-\xi\partial_s z_0(s) - 2c)^2)}.\end{aligned}$$

Thus, we get

$$\begin{aligned}H_1(s,t) =&\frac{1}{2\pi}\int_{|\xi|<\frac{1}{t}} \frac{\psi_0'}{1+\psi_0^2}\Bigg(L(8c^2(\xi^2+4c^2) - 56c^2\xi^2\psi_z^2) \\ &- \frac{56c^2\xi^2(\psi_z^2 - \partial_s z_0^2(s))}{(\xi^2 + (-\xi\partial_s z_0(s) + 2c)^2)(\xi^2 + (-\xi\partial_s z_0(s) - 2c)^2)}\Bigg) d\xi.\end{aligned}$$

Note that there is no singularity in H_1 and moreover

$$|\partial_s z_0^2(s) - \psi_z^2| \leq 2\|\partial_s z\|_{L^\infty(\mathbb{R})}\left(|\xi t|\|\partial_s^2 z_0\|_{L^\infty(\mathbb{R})} + t\|\partial_s a\|_{L^\infty(\mathbb{R})} + \frac{t^2}{2}\|\partial_s b\|_{L^\infty(\mathbb{R})}\right).$$

Now we can argue similarly as for H_2. Because of the factor ψ_0' in the integrand of H_1, we estimate the L^1-norm by $\|\partial_s^2 z_0\|_{L^1(\mathbb{R})}$, which yields

$$\|H_1(\cdot,t)\|_{L^1_s(\mathbb{R})} \leq Ct\|\partial_s^2 z_0\|_{L^1(\mathbb{R})}\int_{|\xi|<1} 1 d\xi + C\sqrt{t}\|\partial_s^2 z_0\|_{L^1(\mathbb{R})}\int_{1<|\xi|<\frac{1}{t}} \frac{1}{|\xi|^{\frac{3}{2}}} d\xi \to 0$$

as $t \to 0$. This completes (3.51) and thus also (3.9).

In the last step, it remains to address with (3.10). Let $\varepsilon \in (0,\alpha)$. First note that in the same way we accomplished (3.9), we infer an L^∞-bound

$$\|2\partial_t z(\cdot,t) - D(\cdot,t)\|_{L^\infty_s(\mathbb{R})} \leq Ct \leq Ct^{\alpha-\varepsilon} \quad \forall\, t \in [0,T).$$

For the one term where we used $\partial_s^2 b \in L^1(\mathbb{R})$, also $b \in C^{1,\alpha}(\mathbb{R})$ is sufficient to attain the L^∞-bound. By means of this, we do not need any further conditions on the second derivatives of b. Hence, (3.10) follows, if we can proof

$$\|\nu^+(\cdot,t) - \nu^-(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq Ct^{\alpha-\varepsilon} \quad \forall\, t \in [0,T).$$

We have

$$\begin{aligned}\nu^+(s,t) - \nu^-(s,t) &= \frac{1}{2\pi}\ \mathrm{PV}\int\limits_{\mathbb{R}} \xi \partial_s Z \left(\frac{1}{\xi^2 + (Z-2ct)^2} - \frac{1}{\xi^2+(Z+2ct)^2}\right) d\xi \\ &= \frac{t^{\alpha-\varepsilon}}{2\pi}\int\limits_{\mathbb{R}} \frac{4c\xi Z \partial_s \overline{Z} t^{1-\alpha+\varepsilon}}{(\xi^2+(Z-2ct)^2)(\xi^2+(Z+2ct)^2)} d\xi.\end{aligned}$$

Thanks to $\partial_s \overline{Z} \in L^\infty(\mathbb{R})$, on $\{|\xi| > 1\}$ the integrand can be estimated by $C|\xi|^{-2}$, which yields the L^∞-bound. On $\{|\xi| < 1\}$, we apply Lemma 3.15 and $\overline{z}_0, a, b \in C^{1,\alpha}(\mathbb{R})$ to estimate

$$\begin{aligned}&\nu^+(s,t) - \nu^-(s,t) = \frac{t^{\alpha-\varepsilon}}{2\pi}\int\limits_{|\xi|<1} \frac{4c\xi Z \partial_s \overline{Z} t^{1-\alpha+\varepsilon}}{(\xi^2+(Z-2ct)^2)(\xi^2+(Z+2ct)^2)} d\xi \\ &\leq C\frac{t^{\alpha-\varepsilon}}{2\pi}([\partial_s \overline{z}_0]_{C^{0,\alpha}(\mathbb{R})} + [\partial_s a]_{C^{0,\alpha}(\mathbb{R})} + [\partial_s b]_{C^{0,\alpha}(\mathbb{R})}) \int\limits_{|\xi|<1} \frac{|\xi|^{2+\alpha} t^{1-\alpha+\varepsilon} d\xi}{(\xi^2+(Z-2ct)^2)(\xi^2+(Z+2ct)^2)} \\ &\leq Ct^{\alpha-\varepsilon}([\partial_s \overline{z}_0]_{C^{0,\alpha}(\mathbb{R})} + [\partial_s a]_{C^{0,\alpha}(\mathbb{R})} + [\partial_s b]_{C^{0,\alpha}(\mathbb{R})}) \int\limits_{|\xi|<1} \frac{1}{|\xi|^{1-\varepsilon}} d\xi.\end{aligned}$$

Since $\varepsilon > 0$, the last integral is finite. This finishes the proof of (3.10). □

3.4 Regularity

The proof of Theorem 3.16 mainly based on Proposition 3.13, which can be refined by the observation that the operators $T_F(G)$ resemble the Hilbert transform $\mathcal{H}(g)$. By means of this, we demonstrate that instead of $\overline{z}_0 \in W^{6,1}(\mathbb{R}) \cap C^{5,\alpha}(\mathbb{R})$, the condition $\overline{z}_0 \in W^{4,1}(\mathbb{R}) \cap C^{4,\alpha}(\mathbb{R})$ is sufficient to prove Theorem 1.3.

For some function $f \in L^p(\mathbb{R})$ and $1 < p < \infty$, the Hilbert transform of f is defined as

$$\mathcal{H}(f)(s) := \ \mathrm{PV}\int_{\mathbb{R}} \frac{f(s-\xi)}{\pi\xi} d\xi.$$

It is well-known (see for example [75]) that $\mathcal{H} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is a linear and bounded operator for each $1 < p < \infty$. Furthermore we have the identity

$$\mathcal{H}(\mathcal{H}(f)) = -f. \tag{3.56}$$

For the computations in this section, we will often rely on the following lemma.

Lemma 3.18.
(i) Let $f \in L^1(\mathbb{R})$. Then $h^>(s) := \text{PV} \int_{|\xi|>1} \frac{\partial_s f(s-\xi)}{\xi} d\xi \in L^1(\mathbb{R})$.
(ii) For each $f \in W^{2,1}(\mathbb{R})$ we have $(\mathcal{H}(\partial_s f))(s) \in L^1(\mathbb{R})$.

Proof. (i) An integration by parts yields

$$\begin{aligned} h^>(s) &= -\text{PV} \int_{|\xi|>1} \frac{\partial_\xi f(s-\xi)}{\xi} d\xi = f(s-1) + f(s+1) - \text{PV} \int_{|\xi|>1} \frac{f(s-\xi)}{\xi^2} d\xi \\ &= f(s-1) + f(s+1) - \left(f * \frac{\chi_{|\xi|>1}}{\xi^2} \right). \end{aligned}$$

Using Young's inequality and $f \in L^1(\mathbb{R})$ gives the assertion.
(ii) We split $\mathcal{H}(\partial_s f)(s) = \pi(h^<(s) + h^>(s))$, where $h^>$ is defined as in (i) and

$$h^<(s) = \text{PV} \int_{|\xi|<1} \frac{(\partial_s f)(s-\xi)}{\xi} d\xi = \text{PV} \int_{|\xi|<1} \frac{(\partial_s f)(s-\xi) - (\partial_s f)(s)}{\xi} d\xi.$$

Since $f \in W^{2,1}(\mathbb{R})$, we conclude from Proposition 3.10 that $h^< \in L^1(\mathbb{R})$. Together with (i) this finishes the proof. ☐

Theorem 3.19. Define the curve $z(s,t)$ by (3.42), with $z_0(s) = \beta s + \overline{z}_0(s)$, $\beta \in \mathbb{R}$, define $a(s)$ by (3.43) and $b(s)$ by (3.46). Furthermore, suppose that for some $\alpha \in (0,1)$ we have

$$\overline{z}_0 \in W^{4,1}(\mathbb{R}) \cap C^{4,\alpha}(\mathbb{R}), \tag{3.57}$$
$$a \in W^{2,1}(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R}) \cap C^{1,\alpha}(\mathbb{R}), \tag{3.58}$$
$$b \in W^{1,1}(\mathbb{R}) \cap C^{1,\alpha}(\mathbb{R}). \tag{3.59}$$

Then, $z(\cdot,t) \in C^{1,\alpha}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ solves the equations (3.9)-(3.11).

Remark 3.20. Lemma 3.21 verifies that (3.58) and (3.59) already follow from (3.57).

Proof. At first, from Lemma 3.7, we infer $\overline{z}_0 \in W^{4,\infty}(\mathbb{R})$ and $b \in W^{1,\infty}(\mathbb{R})$.

Step 1: We can adopt the same proof as in Theorem 3.16 excluded one term, for which the conditions (3.57)-(3.59) are not sufficient, namely $T_Z^<(b)$. We need to show that $\|T_Z^<(b)\|_{L_t^\infty L_s^1} < \infty$, however this time without using $b \in W^{2,1}(\mathbb{R})$. Recall the definition $b = 2T_{Z_0}(A) - 2\tilde{T}_{Z_0}(Z_0) + P_{2c}$. In Lemma 3.14 we already established that (3.57) implies $-2\tilde{T}_{Z_0}(Z_0) + P_{2c} \in W^{2,1}(\mathbb{R})$. Thus, we merely have to consider $b_1(s) := T_{Z_0}(A)(s)$. Accordingly, we write $B_1(\xi,s) = b_1(s-\xi) - b_1(s)$. It remains to prove an L_s^1-bound uniformly in t for

$$T_Z^<(b_1)(s) = \frac{1}{2\pi} \int\limits_{|\xi|<1} (\partial_s b_1(s-\xi) - \partial_s b_1(s)) \left(\frac{\xi}{\xi^2 + Z^2(\xi,s)} \right) d\xi.$$

For this purpose, we use the fact that this integral is almost of the form $\mathcal{H}(\partial_s b_1)$. For a reduction to the Hilbert transform, it will be more convenient to prove an L^1-bound for

$$T_Z(B_1)(s) = \frac{1}{2\pi} \text{PV} \int\limits_{\mathbb{R}} \partial_s B_1(\xi,s) \left(\frac{\xi}{\xi^2 + Z^2(\xi,s)} \right) d\xi$$

instead of $T_Z^<(b_1)$. This is equivalent, since $T_Z(B_1) = T_Z^<(b_1)+T_Z^{>,1}(b_1)+T_Z^{>,2}(b_1)+T_Z^b(b_1)$ and the L^1-bound for $T_Z^{>,1}(b_1), T_Z^{>,2}(b_1), T_Z^b(b_1)$ follows already from $b_1 \in W^{1,1}(\mathbb{R})$ as in Proposition 3.10.

Step 2: The important observation is the fact that not only $T_Z(B_1)$ has a similar structure like $\mathcal{H}(\partial_s b_1)$, but also $b_1 = T_{Z_0}(A)$ is almost of the form $\mathcal{H}(\partial_s a)$. Thus, we have to investigate a term like $\mathcal{H}(\mathcal{H}(\partial_s^2 a)) = -\partial_s^2 a$, which admits a better bound. To be precise, we start by rewriting

$$\begin{aligned} T_Z(B_1)(s) &= \frac{1}{2\pi}\left(\text{PV}\int_{\mathbb{R}} \left(\frac{\xi}{\xi^2+Z^2} - \frac{\xi}{(1+\beta^2)\xi^2}\right)\partial_s B_1 d\xi + \text{PV}\int_{\mathbb{R}} \frac{\partial_s B_1}{(1+\beta^2)\xi} d\xi \right) \\ &= \frac{1}{2\pi}\,\text{PV}\int_{\mathbb{R}} \frac{\xi\partial_s B_1(\beta^2\xi^2 - Z^2)}{(1+\beta^2)\xi^2(\xi^2+Z^2)} d\xi + \frac{1}{2(1+\beta^2)}\mathcal{H}(\partial_s b_1) := l_1(s) + l_2(s), \end{aligned}$$

which means we decompose $T_Z(B_1)$ into a Hilbert transform term l_2 and an error term l_1.

We begin with the estimate of l_1. Similar as in $\tilde{T}_{Z_0}(Z_0)$ in Lemma 3.14, we decompose $l_1 = l_1^< + l_1^{>,1} + l_1^{>,2}$. From Proposition 3.10 we deduce that $l_1^{>,1}, l_1^{>,2} \in L^1(\mathbb{R})$, since $b_1 \in W^{1,1}(\mathbb{R})$ and the corresponding functions Φ_2, Φ_3 are bounded. This is a consequence of the estimate $|\beta^2\xi^2 - Z^2| \le C(1+|\xi|)$. For $l_1^<$ we have to argue slightly different, since we want to avoid using $\partial_s^2 b \in L^1(\mathbb{R})$. Instead we estimate

$$\|l_1\|_{L^1(\mathbb{R})} \le C\int_{\mathbb{R}} [\partial_s b_1]_{C^{0,\alpha}(\mathbb{R})}\,\text{PV}\int_{|\xi|<1} \frac{|\beta||\xi|(|\overline{Z}_0| + |A| + |B|) + |\overline{Z}_0|^2 + |A|^2 + |B|^2}{|\xi|^{3-\alpha}} d\xi ds.$$

For the first summand, we apply (3.32) and Fubini's Theorem to verify

$$\begin{aligned} &C|\beta|\int_{\mathbb{R}} [\partial_s b_1]_{C^{0,\alpha}(\mathbb{R})}\,\text{PV}\int_{|\xi|<1} \frac{|\overline{Z}_0| + |A| + |B|}{|\xi|^{2-\alpha}} d\xi ds \\ &\le C|\beta|[\partial_s b_1]_{C^{0,\alpha}(\mathbb{R})}\int_{|\xi|<1} \frac{1}{|\xi|^{1-\alpha}}\int_0^1 \|\partial_s \overline{z}_0\|_{L^1(\mathbb{R})} + \|\partial_s a\|_{L^1(\mathbb{R})} + \|\partial_s b\|_{L^1(\mathbb{R})} d\tau d\xi \\ &\le C|\beta|[\partial_s b_1]_{C^{0,\alpha}(\mathbb{R})}(\|\partial_s \overline{z}_0\|_{L^1(\mathbb{R})} + \|\partial_s a\|_{L^1(\mathbb{R})} + \|\partial_s b\|_{L^1(\mathbb{R})}). \end{aligned}$$

On the other hand, we estimate the quadratic terms by

$$C\int_{\mathbb{R}} [\partial_s b_1]_{C^{0,\alpha}(\mathbb{R})}\text{PV}\int_{|\xi|<1} \frac{|\overline{Z}_0|^2}{|\xi|^{3-\alpha}} d\xi ds \le C[\partial_s b_1]_{C^{0,\alpha}(\mathbb{R})}\|\partial_s \overline{z}_0\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}} \text{PV}\int_{|\xi|<1} \frac{|\overline{Z}_0|}{|\xi|^{2-\alpha}} d\xi ds,$$

since $\overline{z}_0 \in W^{1,\infty}(\mathbb{R})$. Then we argue as for the linear summands before. The same holds true for the terms involving A and B. We emphasize that all estimates in Step 2 hold uniformly in t.

Step 3: Since l_2 is time-independent, we only need to show an L^1-bound. We repeat the decomposition from Step 2 once more to get a term of the form $\mathcal{H}(\mathcal{H}(\partial_s^2 a))$, for which we can use (3.56). Thus, we split $l_2 = l_{21} + l_{22}$, where

$$l_{21} = \frac{\mathcal{H}(\partial_s b_1 - \partial_s \tilde{b}_1)}{2(1+\beta^2)},$$

$$l_{22} = \frac{\mathcal{H}(\partial_s \tilde{b}_1)}{2(1+\beta^2)}$$

and

$$\tilde{b}_1(s) := \frac{1}{2\pi} \operatorname{PV} \int_{\mathbb{R}} \partial_s A \frac{\xi}{\xi^2(1+\partial_s z_0(s)^2)} d\xi = \frac{1}{2(1+\partial_s z_0(s)^2)} \mathcal{H}(\partial_s a)(s).$$

At first we study the error term l_{21} and neglect for simplicity the prefactor $(2(1+\beta^2))^{-1}$. The above definitions yield

$$b_1(s) - \tilde{b}_1(s) = \frac{1}{2\pi} \operatorname{PV} \int_{\mathbb{R}} \partial_s A \frac{\xi(\xi^2 \partial_s z_0(s)^2 - Z_0^2)}{\xi^2(\xi^2 + Z_0^2)(1+\partial_s z_0(s)^2)} d\xi.$$

If we can show that $b_1 - \tilde{b}_1 \in W^{2,1}(\mathbb{R})$, we deduce that $l_{21} \in L^1(\mathbb{R})$ from Lemma 3.18.

For this purpose, we have to make some effort in using only L^∞-estimates instead of L^1-estimates for a because of (3.58). Because of this, we sometimes have to argue slightly different as in Proposition 3.10. We start to derive an L^1-bound for $b_1 - \tilde{b}_1$. Therefore, we decompose $b_1 - \tilde{b}_1 = (b_1 - \tilde{b}_1)^< + (b_1 - \tilde{b}_1)^>$ with

$$\begin{aligned}
(b_1 - \tilde{b}_1)^< &= \frac{1}{2\pi} \operatorname{PV} \int_{|\xi|<1} \partial_s A \frac{\xi(\xi^2 \partial_s z_0(s)^2 - Z_0^2)}{\xi^2(\xi^2 + Z_0^2)(1+\partial_s z_0(s)^2)} d\xi \\
(b_1 - \tilde{b}_1)^> &= \frac{1}{2\pi} \operatorname{PV} \int_{|\xi|>1} \partial_s A \frac{\xi(\xi^2 \partial_s z_0(s)^2 - Z_0^2)}{\xi^2(\xi^2 + Z_0^2)(1+\partial_s z_0(s)^2)} d\xi.
\end{aligned}$$

For $(b_1 - \tilde{b}_1)^>$, we observe

$$\xi^2 \partial_s z_0(s)^2 - Z_0^2 = \xi^2(\partial_s \overline{z}_0^2(s) + 2\beta \partial_s \overline{z}_0(s)) - \overline{Z}_0^2 + 2\beta\xi \overline{Z}_0 \tag{3.60}$$

and thus obtain

$$(b_1 - \tilde{b}_1)^> = \frac{(\partial_s \overline{z}_0^2(s) + 2\beta \partial_s \overline{z}_0(s))}{1+\partial_s z_0^2} \left(T_{Z_0}^{>,1}(a) + T_{Z_0}^{>,2}(a) + T_{Z_0}^{b}(a) \right) + H^2_{\Phi,\overline{z}_0} + H^3_{\Phi,\overline{z}_0},$$

where Φ is a bounded function given by

$$\Phi = \frac{\xi \partial_s A (2\beta\xi - \overline{Z}_0)}{(\xi^2 + Z_0^2)(1+\partial_s z_0(s)^2)}.$$

Thanks to Proposition 3.10,(i), Corollary 3.13,(ii) and $\overline{z}_0, a \in W^{1,\infty}(\mathbb{R}), \overline{z}_0 \in W^{1,1}(\mathbb{R})$ we infer $(b_1 - \tilde{b}_1)^> \in L^1(\mathbb{R})$.

Now we address with $(b_1 - \tilde{b}_1)^<$. We emphasize that taking a difference in this error term creates an extra factor ξ. Thus, we can overcome to use that $\partial_s^2 a \in L^1(\mathbb{R})$. Instead, thanks to

$$\begin{aligned}
|\xi^2 \partial_s z_0(s)^2 - Z_0^2| &= |\xi \partial_s z_0(s) - Z_0| \cdot |\xi \partial_s z_0(s) + Z_0| \\
&= |z_0(s-\xi) - z_0(s) - \xi \partial_s z_0(s)| \cdot |z_0(s-\xi) - z_0(s) + \xi \partial_s z_0(s)| \\
&\leq 2|\xi|^2 \|\partial_s z_0\|_{L^\infty(\mathbb{R})} \cdot \left| \int_0^1 \partial_s z_0(s) - \partial_s z_0(s-\tau\xi) d\tau \right| \\
&= 2|\xi|^3 \|\partial_s z_0\|_{L^\infty(\mathbb{R})} \cdot \left| \int_0^1 \tau \int_0^1 \partial_s^2 z_0(s-\tau'\tau\xi) d\tau' d\tau \right|
\end{aligned} \tag{3.61}$$

and $a \in W^{1,\infty}(\mathbb{R}), \partial_s z_0 \in L^\infty(\mathbb{R}), \partial_s^2 \overline{z}_0 \in L^1(\mathbb{R})$, we establish

$$\begin{aligned}\|(b_1 - \tilde{b}_1)^<\|_{L^1(\mathbb{R})} &\leq C\|\partial_s a\|_{L^\infty(\mathbb{R})}\|\partial_s z_0\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}}\int_{|\xi|<1} \left|\int_0^1 \tau \int_0^1 \partial_s^2 z_0(s-\tau'\tau\xi)d\tau' d\tau\right| d\xi ds \\ &\leq C\|\partial_s a\|_{L^\infty(\mathbb{R})}\|\partial_s z_0\|_{L^\infty(\mathbb{R})} \int_{|\xi|<1} \left|\int_0^1 \tau \int_0^1 \|\partial_s^2 z_0\|_{L^1(\mathbb{R})} d\tau' d\tau\right| d\xi < \infty.\end{aligned}$$

More generally, we obtain in the same way $(b_1 - \tilde{b}_1) \in W^{2,1}(\mathbb{R})$, since we know $\overline{z}_0 \in W^{4,1}(\mathbb{R}) \cap W^{4,\infty}(\mathbb{R})$ and $a \in W^{3,\infty}(\mathbb{R})$.

Step 4: Finally, we need to show an L^1-bound for the term l_{22}. Without the factor $(2(1+\partial_s z_0(s)^2))^{-1}$ in $\tilde{b}_1$, we could directly use (3.56). However, we have to decompose l_{22} once more. For simplicity, we drop the prefactor $(2(1+\beta^2))^{-1}$ and split up

$$l_{22} = \mathcal{H}(\partial_s \tilde{b}_1)(s) = \text{ PV}\int_{|\xi|<1} \frac{(\partial_s \tilde{b}_1)(s-\xi)}{\pi\xi} + \text{ PV}\int_{|\xi|>1} \frac{(\partial_s \tilde{b}_1)(s-\xi)}{\pi\xi} d\xi =: k_1 + k_2.$$

We immediately get $k_2 \in L^1(\mathbb{R})$ from Lemma 3.18 and $\tilde{b}_1 \in L^1(\mathbb{R})$, which is the consequence of $b_1 \in L^1(\mathbb{R})$ and $b_1 - \tilde{b}_1 \in L^1(\mathbb{R})$. For k_1, we explicitely compute the derivative of $\tilde{b}_1$ and split up once more $k_1 = k_{11} + k_{12}$, where

$$\begin{aligned} k_{11} &= \text{ PV} \int\limits_{|\xi|<1} \frac{(\mathcal{H}(\partial_s^2 a))(s-\xi)}{2\pi\xi(1+\partial_s z_0(s-\xi)^2)} d\xi, \\ k_{12} &= \text{ PV} \int\limits_{|\xi|<1} \frac{\mathcal{H}(\partial_s a)(s-\xi)\partial_s \left(\frac{1}{1+(\partial_s z_0(s-\xi))^2}\right)}{2\pi\xi} d\xi. \end{aligned}$$

Then we have

$$\begin{aligned} k_{11}(s) &= \int_{|\xi|<1} \frac{\mathcal{H}(\partial_s^2 a)(s-\xi)}{2\pi\xi} \frac{\partial_s z_0(s-\xi)^2 - \partial_s z_0(s)^2}{(1+\partial_s z_0(s-\xi)^2)(1+\partial_s z_0(s)^2)} d\xi \\ &\quad + \frac{1}{2(1+\partial_s z_0(s)^2)} \text{ PV} \int\limits_{|\xi|<1} \frac{(\mathcal{H}(\partial_s^2 a))(s-\xi)}{\pi\xi} d\xi \\ &= \int_{|\xi|<1} \frac{\mathcal{H}(\partial_s^2 a)(s-\xi)}{2\pi\xi} \frac{\partial_s z_0(s-\xi)^2 - \partial_s z_0(s)^2}{(1+\partial_s z_0(s-\xi)^2)(1+\partial_s z_0(s)^2)} d\xi \\ &\quad + \frac{1}{2(1+\partial_s z_0(s)^2)} \mathcal{H}(\mathcal{H}(\partial_s^2 a)) - \frac{1}{2(1+\partial_s z_0(s)^2)} \text{ PV} \int\limits_{|\xi|>1} \frac{(\mathcal{H}(\partial_s^2 a))(s-\xi)}{\pi\xi} d\xi \\ &=: k_{111} + k_{112} + k_{113}. \end{aligned}$$

We start with k_{113} and notice that $\mathcal{H}(\partial_s^2 a) = \partial_s \mathcal{H}(\partial_s a)$. By means of Lemma 3.18,(i), we observe that $k_{113} \in L^1(\mathbb{R})$ if $\mathcal{H}(\partial_s a) \in L^1(\mathbb{R})$. However, since $a \in W^{2,1}(\mathbb{R})$, this follows again from Lemma 3.18,(ii).

In the next step, $a \in W^{2,1}(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$ implies $a \in W^{2,p}(\mathbb{R})$ for each $1 < p < \infty$. Hence, because of (3.56), we infer

$$k_{112}(s) = \frac{-\partial_s^2 a(s)}{2(1+\partial_s z_0(s)^2)},$$

which is clearly in $L^1(\mathbb{R})$. For the last term we estimate

$$\begin{aligned}\int_{\mathbb{R}} |k_{111}(s)|ds &\leq C\|\partial_s z_0\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \int_{|\xi|<1} |\mathcal{H}(\partial_s^2 a)(s-\xi)| \int_0^1 |\partial_s^2 \overline{z}_0(s-\tau\xi)| d\tau d\xi ds \\ &= C\|\partial_s z_0\|_{L^\infty(\mathbb{R})} \int_{|\xi|<1} \int_0^1 \int_{\mathbb{R}} |\mathcal{H}(\partial_s^2 a)(s-\xi)||\partial_s^2 \overline{z}_0(s-\tau\xi)| ds d\tau d\xi \\ &\leq C\|\partial_s z_0\|_{L^\infty(\mathbb{R})} \int_{|\xi|<1} \int_0^1 \int_{\mathbb{R}} |\mathcal{H}(\partial_s^2 a)(s-\xi)|^2 + |\partial_s^2 \overline{z}_0(s-\tau\xi)|^2 ds d\tau d\xi \\ &\leq C\|\partial_s z_0\|_{L^\infty(\mathbb{R})} \left(\|\mathcal{H}(\partial_s^2 a)\|_{L^2(\mathbb{R})}^2 + \|\partial_s^2 \overline{z}_0\|_{L^2(\mathbb{R})}^2 \right),\end{aligned}$$

where we used Young's inequality in the third step. Note that we have $\overline{z}_0 \in W^{2,2}(\mathbb{R})$ and $\mathcal{H}(\partial_s^2 a) \in L^2(\mathbb{R})$, which follows from $a \in W^{2,2}(\mathbb{R})$.

The term k_{12} can be estimated as follows. Denote $\psi(s) := \mathcal{H}(\partial_s a)(s)\partial_s \left(\frac{1}{1+(\partial_s z_0(s))^2}\right)$. Then

$$k_{12} = \frac{1}{2\pi} \operatorname{PV} \int_{|\xi|<1} \frac{\psi(s-\xi)}{\xi} d\xi = \frac{1}{2\pi} \operatorname{PV} \int_{|\xi|<1} \frac{\psi(s-\xi)-\psi(s)}{\xi} d\xi.$$

Thanks to Proposition 3.10, we conclude $k_{12} \in L^1(\mathbb{R})$, provided that $\partial_s \psi \in L^1(\mathbb{R})$. This can be easily computed

$$\begin{aligned}|\partial_s \psi(s)| &= \left| (\partial_s \mathcal{H}(\partial_s a))(s) \frac{-2\partial_s z_0(s)\partial_s^2 z_0(s)}{(1+\partial_s z_0(s-\xi)^2)^2} + \mathcal{H}(\partial_s a)(s)\partial_s^2 \left(\frac{1}{1+(\partial_s z_0(s))^2}\right) \right| \\ &\leq \frac{1}{2}(\mathcal{H}(\partial_s^2 a)(s))^2 + \frac{2(\partial_s z_0)^2(s)(\partial_s^2 z_0)^2(s)}{(1+(\partial_s z_0(s))^2)^4} + \left| \mathcal{H}(\partial_s a)(s)\partial_s^2 \left(\frac{1}{1+(\partial_s z_0(s))^2}\right) \right|,\end{aligned}$$

by an application of Young's inequality. The first two summands are in $L^1(\mathbb{R})$, because of $\mathcal{H}(\partial_s^2 a) \in L^2(\mathbb{R})$ and $\overline{z}_0 \in W^{2,\infty}(\mathbb{R}) \cap W^{2,1}(\mathbb{R})$ respectively. As already mentioned for k_{113}, we know that $\mathcal{H}(\partial_s a) \in L^1(\mathbb{R})$ and consequently we are done provided that $\partial_s^2 \left(\frac{1}{1+(\partial_s z_0(s))^2}\right) \in L^\infty(\mathbb{R})$. However, by a straightforward computation, this follows immediately from $\overline{z}_0 \in W^{3,\infty}(\mathbb{R})$. Hence, we established that $l_{22} \in L^1(\mathbb{R})$, which finishes the proof. □

We want to emphasize that instead of using the rough estimates of Corollary 3.13, in the sense that we require $b \in W^{2,1}(\mathbb{R})$ and $a \in W^{4,1}(\mathbb{R})$, we improved now by one derivative and deduce that (3.58) and (3.59) are sufficient. Finally, we verify that (3.58) and (3.59) are already a consequence of (3.57).

Lemma 3.21. Suppose that for some $\alpha \in (0,1)$, the condition (3.57) holds. Then (3.58) and (3.59) are satisfied.

Proof. First of all, in view of Lemma 3.7, we deduce that $\overline{z}_0 \in W^{4,\infty}(\mathbb{R})$ and thus also $\overline{z}_0 \in W^{4,p}(\mathbb{R})$ for each $1 < p < \infty$. Thanks to Lemma 3.7, it is enough to establish $a \in W^{2,1}(\mathbb{R}) \cap W^{2,p}(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R})$ and $b \in W^{1,1}(\mathbb{R}) \cap W^{2,p}(\mathbb{R})$ for some $1 < p < \infty$.

To prove $a \in W^{2,1}(\mathbb{R}) \cap W^{2,p}(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R})$ we refer to Proposition 3.13 and (3.57).

Thus, we turn to $b = 2T_{Z_0}(A) - 2\tilde{T}_{Z_0}(Z_0) + P_{2c}$. As stated at the beginning of the proof of Theorem 3.19, from (3.57) and Lemma 3.14 we conclude in the same way that $\tilde{T}_{Z_0}(Z_0) + P_{2c} \in W^{2,1}(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$. Hence, it remains to estimate $b_1 = T_{Z_0}(A)$.

Similarly as in the proof of Theorem 3.19, we rewrite in terms of the Hilbert transform

$$\begin{aligned} b_1(s) &= \frac{1}{2\pi} \operatorname{PV} \int_{\mathbb{R}} \frac{\xi \partial_s A}{\xi^2 + Z_0^2} - \frac{\xi \partial_s A}{\xi^2(1 + \partial_s z_0(s)^2)} d\xi + \frac{1}{2\pi} \operatorname{PV} \int_{\mathbb{R}} \frac{\xi \partial_s A}{\xi^2(1 + \partial_s z_0(s)^2)} d\xi \\ &= \frac{1}{2\pi} \operatorname{PV} \int_{\mathbb{R}} \frac{\xi \partial_s A(\xi^2 \partial_s z_0(s)^2 - Z_0^2)}{\xi^2(\xi^2 + Z_0^2)(1 + \partial_s z_0(s)^2)} d\xi + \frac{1}{2(1 + \partial_s z_0(s)^2)} \mathcal{H}(\partial_s a) \\ &:= b_{11}(s) + b_{12}(s). \end{aligned}$$

Observe that b_{11} coincides with $b_1 - \tilde{b}_1$ in Step 3 of the proof of Theorem 3.19 and in the same way from (3.57) and (3.58) we infer $b_{11} \in W^{2,1}(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R})$. In particular, this yields $b_{11} \in W^{1,1}(\mathbb{R}) \cap W^{2,p}(\mathbb{R})$.

Now we address with b_{12}. Since the prefactor can be estimated by 1, it is enough to consider $\mathcal{H}(\partial_s a)$. We will do the same trick as for b_{11} once more and split into

$$\mathcal{H}(\partial_s a) = \mathcal{H}(\partial_s a - \partial_s \tilde{a}) + \mathcal{H}(\partial_s \tilde{a}),$$

where we set

$$\tilde{a}(s) := \frac{1}{\pi} \operatorname{PV} \int_{\mathbb{R}} \frac{\xi \partial_s Z_0}{\xi^2(1 + \partial_s z_0(s)^2)} d\xi = \frac{1}{(1 + \partial_s z_0(s)^2)} \mathcal{H}(\partial_s \overline{z}_0).$$

Here, note that $\partial_s Z_0 = \partial_s \overline{Z}_0$. We need to establish $\mathcal{H}(\partial_s a - \partial_s \tilde{a}) \in W^{1,1}(\mathbb{R}) \cap W^{2,p}(\mathbb{R})$. For this purpose, we compute

$$a(s) - \tilde{a}(s) = \frac{1}{\pi} \operatorname{PV} \int_{\mathbb{R}} \frac{\partial_s \overline{Z}_0}{\xi} \frac{\xi^2 \partial_s z_0(s)^2 - Z_0^2}{(\xi^2 + Z_0^2)(1 + \partial_s z_0(s)^2)} d\xi,$$

which is the same as $2b_{11}$ with $\partial_s A$ replaced by $\partial_s \overline{Z}_0$. We claim that from (3.57), we obtain

$$a - \tilde{a} \in W^{3,q}(\mathbb{R}) \quad \forall\, 1 \leq q \leq \infty. \tag{3.62}$$

This can be seen as follows. As for $b_1 - \tilde{b}_1$ in Step 3 of the proof of Theorem 3.19, we decompose $a - \tilde{a} = (a - \tilde{a})^< + (a - \tilde{a})^>$. Then, by (3.60) we have analogously

$$(a - \tilde{a})^> = \frac{(\partial_s \overline{z}_0^2(s) + 2\beta \partial_s \overline{z}_0(s))}{1 + \partial_s z_0^2} \left(T_{Z_0}^{>,1}(\overline{z}_0) + T_{Z_0}^{>,2}(\overline{z}_0) + T_{Z_0}^{b}(\overline{z}_0) \right) + H_{\Phi,\overline{z}_0}^2 + H_{\Phi,\overline{z}_0}^3,$$

where

$$\Phi = \frac{\xi \partial_s \overline{Z}_0 (2\beta\xi - \overline{Z}_0)}{(\xi^2 + Z_0^2)(1 + \partial_s z_0(s)^2)}.$$

Thanks to Corollary 3.12 and (3.57), we infer $\Phi \in W^{3,\infty}(\mathbb{R})$. Then Proposition 3.10,(i), Proposition 3.13,(ii) and (3.57) yield $(a-\tilde{a})^{>} \in W^{3,1}(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R})$. The other term equals $(a-\tilde{a})^{<} = 2H^1_{\Phi_1,\partial_s \overline{z}_0}$, where

$$\Phi_1(\xi,s) = \frac{\xi^2 \partial_s z_0(s)^2 - Z_0^2}{(\xi^2+Z_0^2)(1+\partial_s z_0(s)^2)}.$$

Again, $\Phi_1 \in W^{3,\infty}(\mathbb{R})$ and Proposition 3.10 imply that $(a-\tilde{a})^{<} \in W^{2,1}(\mathbb{R}) \cap W^{3,\infty}(\mathbb{R})$. Beyond that, to show $(a-\tilde{a})^{<} \in W^{3,1}(\mathbb{R})$, we need an L^1-bound for

$$\mathrm{PV}\int_{\mathbb{R}} \frac{\partial_s^4 \overline{Z}_0}{\xi} \frac{\xi^2 \partial_s z_0(s)^2 - Z_0^2}{(\xi^2+Z_0^2)(1+\partial_s z_0(s)^2)} d\xi.$$

This follows by using (3.61) as we did for $b_1 - \tilde{b}_1$. Hence, this gives (3.62).

From Lemma 3.18 we infer $\mathcal{H}(\partial_s(a-\tilde{a})) \in L^1(\mathbb{R})$, since $a-\tilde{a} \in W^{2,1}(\mathbb{R})$ and in the same way also $\mathcal{H}(\partial_s(a-\tilde{a})) \in W^{1,1}(\mathbb{R})$, because of (3.62). For the $W^{2,p}(\mathbb{R})$-bound with $p>1$, we have to use the fact that for $p \in (1,\infty)$ the Hilbert transform is a linear and bounded operator $\mathcal{H} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$. Therefore, we observe that $\mathcal{H}(\partial_s(a-\tilde{a})) \in W^{2,p}(\mathbb{R})$ if $a-\tilde{a} \in W^{3,p}(\mathbb{R})$. This follows again from (3.62).

Finally, it remains to verify $\mathcal{H}(\partial_s \tilde{a}) \in W^{1,1}(\mathbb{R}) \cap W^{2,p}(\mathbb{R})$ for some $p>1$. The statement $\mathcal{H}(\partial_s \tilde{a}) \in W^{2,p}(\mathbb{R})$ follows from $\tilde{a} \in W^{3,p}(\mathbb{R})$, which is a consequence of the definition of $\tilde{a}$, the boundedness of the Hilbert transform in $L^p(\mathbb{R})$ and (3.57).

From (3.58),(3.62) and Lemma 3.18 we get $\mathcal{H}(\partial_s \tilde{a}) \in L^1(\mathbb{R})$ and we are left with showing $\mathcal{H}(\partial_s^2 \tilde{a}) \in L^1(\mathbb{R})$. Analogously to the proof of Theorem 3.19, we split up into

$$\mathcal{H}(\partial_s^2 \tilde{a})(s) = \mathrm{PV}\int_{|\xi|<1} \frac{(\partial_s^2 \tilde{a})(s-\xi)}{\pi\xi} + \mathrm{PV}\int_{|\xi|>1} \frac{(\partial_s^2 \tilde{a})(s-\xi)}{\pi\xi} d\xi =: m_1 + m_2.$$

$m_2 \in L^1(\mathbb{R})$ can be established by Lemma 3.18 and $\partial_s \tilde{a} \in L^1(\mathbb{R})$, which is the consequence of (3.58) and (3.62). For m_1, we explicitely compute $\partial_s^2 \tilde{a}$ and thus split up once more $m_1 = m_{11} + m_{12} + m_{13}$, where

$$m_{11} = \mathrm{PV}\int_{|\xi|<1} \frac{(\mathcal{H}(\partial_s^3 z_0))(s-\xi)}{(1+\partial_s z_0(s-\xi)^2)\pi\xi} d\xi,$$

$$m_{12} = 2\,\mathrm{PV}\int_{|\xi|<1} \frac{\mathcal{H}(\partial_s^2 z_0)(s-\xi)\partial_s\left(\frac{1}{1+(\partial_s z_0)^2}\right)(s-\xi)}{\pi\xi} d\xi$$

$$m_{13} = \mathrm{PV}\int_{|\xi|<1} \frac{\mathcal{H}(\partial_s z_0)(s-\xi)\partial_s^2\left(\frac{1}{1+(\partial_s z_0)^2}\right)(s-\xi)}{\pi\xi} d\xi.$$

Replacing a by $\partial_s z_0$, we deduce that $m_{11}, m_{12} \in L^1(\mathbb{R})$ just as we did for k_{11}, k_{12} in Theorem 3.19 and because of (3.57). The term m_{13} can be handled analogously as m_{12}, that is, the L^1-bound follows under the condition that $\partial_s^2\left(\frac{1}{1+(\partial_s z_0(s))^2}\right) \in L^2(\mathbb{R})$ and $\partial_s^3\left(\frac{1}{1+(\partial_s z_0(s))^2}\right) \in L^\infty(\mathbb{R})$. This merely requires a simple calculation and (3.57). □

By means of these results, the condition $\overline{z}_0 \in W^{4,1}(\mathbb{R}) \cap C^{4,\alpha}(\mathbb{R})$ suffices to apply Theorem 3.19. This finishes the proof of Theorem 1.3.

3.5 Approximation of a continuous density

The main drawback of the subsolution we have constructed here compared to the one from [17] appears in the propagation speed of the mixing zone $\Omega_{\text{mix}}(t)$ in the unstable regime. Whereas in the latter, one can choose $c \in (0,2)$, here we merely reach values in $(0,1)$. To overcome this difference, we propose to approximate the linear density function from [17] by piecewise constant densities. In the spirit of the previous sections, we now demonstrate how to construct a subsolution for such densities. Beyond, we establish that the velocity indeed admits values arbitrary close to 2.

Let $n \in \mathbb{N}$. Then define

$$\begin{aligned}
\Omega^0(t) :=& \{x \in \mathbb{R}^2 : -c_1 t < x_2 - z_1(x_1,t) < c_1 t\} \\
\Omega^{\pm i}(t) :=& \{x \in \mathbb{R}^2 : \; c_i t < \pm(x_2 - z_i(x_1,t)) \quad \text{and} \quad \pm(x_2 - z_{i+1}(x_1,t)) < c_{i+1} t\} \\
& \forall\, i = 1,\dots,n-1 \\
\Omega^{\pm}(t) :=& \{x \in \mathbb{R}^2 : \pm(x_2 - z_n(x_1,t)) > c_n t\},
\end{aligned}$$

where $0 < c_1 < c_2 < \ldots < c_n$ are the velocities of the boundaries $\Gamma^{\pm i}$, $i = 1,\dots,n$, which are given by

$$\Gamma^{\pm i} := \{x \in \mathbb{R}^2 : x_2 = z_i(x_1,t) \pm c_i t\} \quad \forall\, i = 1,\dots,n.$$

The associated normal vectors on $\Gamma^{\pm i}$ (pointing upwards) become

$$n^{\pm i} = \frac{(-\partial_s z_i, 1, \mp c_i - \partial_t z_i)}{\sqrt{1 + (\partial_s z_i)^2 + (\partial_t z_i \pm c_i)^2}}, \quad i = 1,\dots,n. \tag{3.63}$$

Furthermore, the curves $z_i(x_1,t)$ are still to be determined for each $i = 1,\dots,n$, which will be done at the end of this section. In particular, we will choose

$$z_i(s,t) = z_0(s) + a(s)t + \frac{1}{2} b_i(s) t^2.$$

In view of $c_i < c_{i+1}$, this definition guarantees that the sets $\Omega^{\pm i}(t)$ are well-defined in the sense that the curve $\Gamma^{+(i+1)}$ lies above Γ^{+i} for small times. The mixing zone is defined as $\Omega_{\text{mix}}(t) := \Omega^0(t) \cup \bigcup_{i=1}^{n-1} \Omega^i(t) \cup \Omega^{-i}(t)$. Analogously to (3.1) we impose the density

$$\rho(x,t) := \begin{cases} 1 & , x \in \Omega^+(t) \\ \pm\frac{i}{n} & , x \in \Omega^{\pm i}(t), \quad i = 0,\dots,n-1\ . \\ -1 & , x \in \Omega^-(t) \end{cases} \tag{3.64}$$

In particular, we have a constant density jump of $\frac{1}{n}$ across each boundary $\Gamma^{\pm i}$. The reason for choosing the domains and the density in a symmetric way around the initial curve $\Gamma = \{(s, z_0(s))\}$ is the same as before in the case $n = 1$, namely, to cancel delicate terms which would arise for non-symmetric curves in the definition of the coefficients $b_i^{\pm}(s)$ of the curve z_i. Moreover, note that the subsolution we construct here is not continuous inside the mixing zone. Nevertheless, this does not cause problems, since we apply the convex integration method in each zone $\Omega^{\pm i}$ separately.

Once we have defined the density as above in (3.64), we obtain an expression for the velocity u from Theorem 3.8 and the remark afterwards. Therefore, the normal component $\nu^{\pm i}(s,t) := u(s, z_i(s,t) \pm c_i t) \cdot (-\partial_s z_i(s,t), 1)$ on $\Gamma^{\pm i}$ becomes

$$\nu^{\pm i}(s,t) = \frac{1}{2\pi n} \sum_{j=-n, j\neq 0}^{n} \mathrm{PV} \int_{\mathbb{R}} \partial_s Z_{ji}(\xi, s, t) \frac{\xi}{\xi^2 + (Z_{ji}(\xi,s,t) \pm (c_j - c_i)t)^2} d\xi, \tag{3.65}$$

with $Z_{ji}(\xi, s, t) := z_j(s-\xi, t) - z_i(s,t)$ and the convention $c_{-j} = -c_j, z_{-j} = z_j$ for $j = 1, \dots, n$.

Next, we define $m = \pm u$ in $\Omega^{\pm}(t)$, whereas for $i = 0, \dots, n-1$ we set

$$m = \pm \frac{i}{n} u - \left(1 - \frac{i^2}{n^2}\right) \left(\gamma_1^{\pm i}, \gamma_2^{\pm i} + \frac{1}{2}\right) \text{ in } \Omega^{\pm i}(t). \tag{3.66}$$

In order to get a subsolution, the functions $\gamma^{\pm i}$ have to satisfy div $\gamma^{\pm i} = 0$ and $|\gamma^{\pm i}| < \frac{1}{2}$ in $\Omega^{\pm i}(t)$. Therefore, we choose $\gamma^{\pm i} = (\nabla^\perp g^{\pm i})$, where $g^{\pm i} : \Omega^{\pm i}(t) \to \mathbb{R}, i = 0, \dots, n-1$ are functions still to be determined. Similarly as in Section 3.1, we have to consider jump conditions on each $\Gamma^{\pm i}$, namely

$$|\nabla g^{\pm i}| < \frac{1}{2} \text{ in } \Omega^{\pm i}(t), \ \forall \ i = 0, \dots, n-1 \tag{3.67}$$

$$\begin{aligned} \partial_t z_i - \nu^{\pm i} \pm c_i \mp \frac{2i-1}{2n} &= \pm \left(\frac{n^2 - (i-1)^2}{n}\right) \partial_\tau^i g^{\pm(i-1)} \\ &\mp \left(\frac{n^2 - i^2}{n}\right) \partial_\tau^i g^{\pm i} \text{ on } \Gamma^{\pm i}, \ \forall \ i = 1, \dots, n. \end{aligned} \tag{3.68}$$

Here, we define $\partial_\tau^j g^{\pm i} := \nabla g^{\pm i} \cdot (1, \partial_s z_j)$. The following refined version of Theorem 3.1 addresses the existence of a solution $g^{\pm i}$ of (3.67)-(3.68).

Theorem 3.22. Assume that

$$0 < c_i < \frac{2i-1}{n} \tag{3.69}$$

for $i = 1, \dots, n$ and let $z_0(s) = \beta s + \overline{z}_0(s)$ for some $\beta \in \mathbb{R}$ and $\overline{z}_0 : \mathbb{R} \to \mathbb{R}$. Moreover, we suppose that for each $i = 1, \dots, n$ the curves $z_i : \mathbb{R} \times [0, T) \to \mathbb{R}$ satisfy

$$\lim_{t\to 0} \frac{1}{t} \|2\partial_t z_i(\cdot, t) - \nu^{+i}(\cdot, t) - \nu^{-i}(\cdot, t)\|_{L^1(\mathbb{R})} = 0 \tag{3.70}$$

$$\lim_{t\to 0} \|\partial_t z_i(\cdot, t) - \nu^{\pm i}(\cdot, t)\|_{L^\infty(\mathbb{R})} = 0 \tag{3.71}$$

$$z_i(s, 0) = z_0(s) \quad \forall \ s \in \mathbb{R}. \tag{3.72}$$

Then for each $t \in (0, T)$ with $T = T(n) > 0$ sufficiently small, there exists a subsolution for (1.21)-(1.24) in the unstable regime.

Proof. By means of the above derivation, each solution $g^{\pm i}$ of (3.67)-(3.68) leads to a subsolution. As in Theorem 3.1, in each $\Omega^{\pm i}(t)$, we introduce new coordinates (s, λ) by

$$(x_1, x_2) = (s, z_{i+1}(s,t) + \lambda),$$

with $s \in \mathbb{R}, \lambda \in (c_i t + z_i(s,t) - z_{i+1}(s,t), c_{i+1}t)$ respectively $\lambda \in (-c_{i+1}t, -c_i t + z_i(s,t) - z_{i+1}(s,t))$ and denote

$$\hat{g}^{\pm i}(s,\lambda) := g^{\pm i}(s, z_{i+1}(s,t) + \lambda).$$

We start with defining $g^{\pm(n-1)} : \Omega^{\pm(n-1)}(t) \to \mathbb{R}$ such that (3.67) and (3.68) for $i = n$ are satisfied. It will be appropriate to choose the boundary value for $g^{\pm(n-1)}$ in the whole of $\Omega^{\pm(n-1)}(t)$. Thus, according to (3.68), we set

$$g^{\pm(n-1)}(s, z_n(s) + \lambda) = \hat{g}^{\pm(n-1)}(s,\lambda) = \int_0^s \frac{\pm n}{2n-1}(\partial_t z_n - \nu^{\pm n}) + \left(\frac{nc_n}{2n-1} - \frac{1}{2}\right) ds'$$

independently of λ. From this definition we infer

$$|\partial_\tau^n \hat{g}^{\pm(n-1)}| = |\partial_s \hat{g}^{\pm(n-1)}| \leq \left|\frac{n}{2n-1}c_n - \frac{1}{2}\right| + \|\partial_t z_n - \nu^{\pm n}\|_{L^\infty(\mathbb{R})} < \frac{1}{2},$$

for small times t, thanks to $0 < c_n < \frac{2n-1}{n}$ and (3.71). Because of $|\partial_\lambda g^{\pm(n-1)}| = 0$, we accomplish (3.67) for $i = n-1$ analogously as in Theorem 3.1.

For the general case, we argue by induction. At first, we claim that by constructing the functions $g^{\pm i}$ independently of λ as above, the boundary condition (3.68) reads

$$\sum_{j=i+1}^{n} (\partial_t z_j - \nu^{\pm j} \pm c_j) \mp \frac{n^2 - i^2}{2n} = \pm \left(\frac{n^2 - i^2}{n}\right) \partial_\tau^{i+1} g^{\pm i} \text{ on } \Gamma^{\pm(i+1)} \tag{3.73}$$

for each $i = 0, \ldots, n-1$. We already verified the case $i = n-1$. Suppose that (3.73) is true for some i. We need to show this equation for $i-1$. Since we constructed the function $g^{\pm i}$ in $\Omega^{\pm i}(t)$ independent of λ, we have $\partial_{x_2} g^{\pm i} = \partial_\lambda \hat{g}^{\pm i} = 0$ and therefore $\partial_s \hat{g}^{\pm i} = \partial_{x_1} g^{\pm i}$. In particular, this implies

$$\partial_\tau^i g^{\pm i} = \partial_s \hat{g}^{\pm i} = \pm \sum_{j=i+1}^{n} \frac{n}{n^2 - i^2}(\partial_t z_j - \nu^{\pm j} \pm c_j) - \frac{1}{2}.$$

Hence, we can plug this into (3.68) and deduce on $\Gamma^{\pm i}$

$$\begin{aligned}(\partial_t z_i - \nu^{\pm i}) \pm \left(c_i - \frac{2i-1}{2n}\right) &= \pm\, n \left(1 - \frac{(i-1)^2}{n^2}\right) \partial_\tau^i g^{\pm(i-1)} \\ &\quad - \sum_{j=i+1}^{n} (\partial_t z_j - \nu^{\pm j} \pm c_j) \pm \frac{n^2 - i^2}{2n}\end{aligned}$$

which is equivalent to

$$\sum_{j=i}^{n} (\partial_t z_j - \nu^{\pm j} \pm c_j) \mp \frac{n^2 - (i-1)^2}{2n} = \pm \left(\frac{n^2 - (i-1)^2}{n}\right) \partial_\tau^i g^{\pm(i-1)},$$

that is, (3.73) for $i-1$.

We use (3.73) now for the construction of $g^{\pm i}$ in $\Omega^{\pm i}(t)$ independent of λ as

$$g^{\pm i}(s, z_{i+1}(s)+\lambda) = \int_0^s \frac{\pm n}{n^2-i^2}\left(\sum_{j=i+1}^n (\partial_t z_j - \nu^{\pm j} \pm c_j)\right) - \frac{1}{2} ds'$$

for each $i = 1, \ldots, n-1$. From this choice and (3.71) we deduce for small times

$$|\partial_\tau^{i+1} \hat{g}^{\pm i}| \leq \left| \frac{n}{n^2-i^2}\left(\sum_{j=i+1}^n c_j\right) - \frac{1}{2}\right| + \frac{n}{n^2-i^2}\sum_{j=i+1}^n \|\partial_t z_j - \nu^{\pm j}\|_{L^\infty(\mathbb{R})} < \frac{1}{2},$$

provided that additionally

$$0 < \frac{n}{n^2-i^2}\left(\sum_{j=i+1}^n c_j\right) < 1.$$

This follows from (3.69), since we know $0 < c_j < \frac{2j-1}{n}$ for $j = i+1, \ldots, n$ and thus conclude

$$\frac{n}{n^2-i^2}\sum_{j=i+1}^n \frac{2j-1}{n} = \frac{n}{n^2-i^2}\left(\frac{2}{n}\left(\frac{n(n+1)-i(i+1)}{2}\right) - \frac{n-i}{n}\right) = 1.$$

Thanks to $\partial_\lambda \hat{g}^{\pm i} = 0$ we infer (3.67).

Finally, it remains to investigate the construction of g^0 in $\Omega^0(t)$. We emphasize that in this case, we have two boundary conditions (3.73) for $i = 0$ given by

$$\sum_{j=1}^n (\partial_t z_j - \nu^{\pm j} \pm c_j) \mp \frac{n}{2} = \pm n \partial_\tau^1 g^0 \quad \text{on } \Gamma^{\pm 1}.$$

Hence, it will be convenient to define g^0 as in Theorem 3.1 by

$$\begin{aligned} g^0(s, z_1(s)+\lambda) = {} & \hat{g}^0(s,\lambda) = \frac{\lambda + c_1 t}{2c_1 t}(\hat{g}^0(s, c_1 t)) - \frac{\lambda - c_1 t}{2c_1 t}(\hat{g}^0(s, -c_1 t)) \\ = {} & s\left(\frac{1}{n}\sum_{j=1}^n c_j - \frac{1}{2}\right) + \frac{\lambda + c_1 t}{2c_1 t}\left(\int_0^s \frac{1}{n}\sum_{j=1}^n \partial_t z_j - \nu^{+j} ds'\right) \\ & + \frac{\lambda - c_1 t}{2c_1 t}\left(\int_0^s \frac{1}{n}\sum_{j=1}^n \partial_t z_j - \nu^{-j} ds'\right). \end{aligned}$$

Together with (3.71), this leads to

$$|\partial_\tau^1 \hat{g}^0| = |\partial_s \hat{g}^0| \leq \left|\frac{1}{n}\sum_{j=1}^n c_j - \frac{1}{2}\right| + \sum_{j=1}^n \|\partial_t z_j - \nu^{+j}\|_{L^\infty(\mathbb{R})} + \|\partial_t z_j - \nu^{-j}\|_{L^\infty(\mathbb{R})} < \frac{1}{2},$$

for small times t, since by (3.69) we observe $0 < \frac{1}{n}\sum_{j=1}^{n} c_j < 1$. For the λ-direction we calculate

$$|\partial_\lambda \hat{g}^0| = \left|\frac{\hat{g}^0(s,c_1t) - \hat{g}^0(s,-c_1t)}{2c_1t}\right| \leq \sum_{j=1}^{n} \frac{\|2\partial_t z_j(\cdot,t) - \nu^{+j}(\cdot,t) - \nu^{-j}(\cdot,t)\|_{L^1(\mathbb{R})}}{2c_1tn} \to 0$$

as $t \to 0$, because of (3.70). This establishes (3.67) and the proof. □

Hence, in order to obtain a subsolution, we need to construct curves $z_i, i = 1,\dots,n$ satisfying (3.70)-(3.72). This can be done analogously as in Section 3.3. Indeed, we introduce for $i = 1,\dots,n$ the operators

$$D_i(s,t) := \nu^{+i}(s,t) + \nu^{-i}(s,t) = \frac{1}{n}\left(2T_{Z_i}(Z_i) + \sum_{j\in\{-n,\dots,n\}\setminus\{0,i\}} R_{Z_{ji},c_j-c_i}(Z_{ji})\right).$$

For each curve z_i we choose a power series ansatz $z_i(s,t) = z_0(s) + a_i(s)t + \frac{1}{2}b_i(s)t^2$ and as in (3.43), (3.46) define

$$\begin{aligned} a_i(s) &= \frac{1}{2}D_i(s,0) = 2T_{Z_0}(\overline{Z}_0)(s), \\ b_i(s) &= \frac{1}{2}\partial_t D_i(s,0) = 2T_{Z_0}(A) - 2\tilde{T}_{Z_0}(Z_0)(s) + \frac{1}{n}\sum_{j\in\{-n,\dots,n\}\setminus\{0,i\}} P_{(c_j-c_i)}(s), \end{aligned}$$

Here, for the derivation of b_i, we argue as in Section 3.3 and with $\tilde{B}(\xi,s)$ replaced by $\tilde{B}_{ji}(\xi,s) = \tilde{b}_j(s-\xi) - \tilde{b}_i(s)$. As opposed to $\tilde{B}(\xi,s)$, we can not pull out a factor ξ from $\tilde{B}_{ji}(\xi,s)$. Nevertheless, this does not cause any problems, because we only need an extra factor ξ on the set $\{|\xi| < 1\}$. Since there is always an additional factor t in front of $\tilde{B}_{ji}(\xi,s)$, we can pull one $\sqrt{t}$ out of the integral and see that the remaining integrand is in $L^1(\{|\xi| < 1\})$. This is the same argument as for the integrals I_2, I_{12} in the proof of Theorem 3.16. Therefore, all terms including $\tilde{B}_{ji}(\xi,s)$ vanish in the limit $t \to 0$.

Moreover, it is clear that Lemma 3.14 remains valid for b_i, since the estimates for P_c do not depend on the concrete values of c. We still have to convince ourselves that the curves z_i satisfy (3.70)-(3.72). For this purpose, we take the same proof as for Theorem 3.16 and need to show that

$$\|\partial_t^2 D_i\|_{L_t^\infty L_s^1} < \infty, \tag{3.74}$$

for which we split

$$\begin{aligned} D_i(s,t) = &\frac{2}{n}\left(T_{Z_i}^{<}(\overline{z}_i)(s,t) + T_{Z_i}^{>,1}(\overline{z}_i)(s,t) + T_{Z_i}^{b}(\overline{z}_i)(s,t) + T_{Z_i}^{>,2}(\overline{z}_i)(s,t)\right) \\ &+ \frac{1}{n}\sum_{j\in\{-n,\dots,n\}\setminus\{0,i\}}\Bigg(T^{>,1}_{Z_{ji}+(c_j-c_i)t}(\overline{z}_j)(s,t) + T^{b}_{Z_{ji}+(c_j-c_i)t}(\overline{z}_j)(s,t) \\ &+ T^{>,2}_{Z_{ji}+(c_j-c_i)t}(\overline{z}_i)(s,t) + T^{>,1}_{Z_{ji}-(c_j-c_i)t}(\overline{z}_j)(s,t) + T^{b}_{Z_{ji}-(c_j-c_i)t}(\overline{z}_j)(s,t) \\ &+ T^{>,2}_{Z_{ji}-(c_j-c_i)t}(\overline{z}_i)(s,t) + E_{Z_{ji},c_j-c_i}(\overline{Z}_{ji})(s,t)\Bigg). \end{aligned}$$

Again, we show (3.74) for each summand separately. For the operators $T^{<}_{Z_i}(\overline{z}_i), T^{>,1}_{Z_i}(\overline{z}_i)$, $T^{b}_{Z_i}(\overline{z}_i)$ and $T^{>,2}_{Z_i}(\overline{z}_i)$, it is obvious to see that we can adopt the same proof as in Theorem 3.16. The same is true for each other summand of D_i except of $E_{Z_{ji},c_j-c_i}(\overline{Z}_{ji})$, since the main difference is that Z is replaced by $Z_{ji} = Z_0 + tA + \frac{1}{2}t^2 B_{ji}$. As stated above, this merely restricts us in not getting an additional factor ξ. However, on $\{|\xi| > 1\}$ we do not need this property at all, since instead we use the L^∞-bound of Z_{ji}. It remains to deal with

$$E_{Z_{ji},c_j-c_i}(\overline{Z}_{ji})(s,t) =$$
$$\frac{1}{2\pi}\int_{|\xi|<1} \frac{2\xi(\xi^2 + Z_{ji}^2(\xi,s,t) + (c_j-c_i)^2t^2)(\partial_s\overline{z}_j(s-\xi,t) - \partial_s\overline{z}_i(s,t))}{(\xi^2 + (Z_{ji}(\xi,s,t) + (c_j-c_i)t)^2)(\xi^2 + (Z_{ji}(\xi,s,t) - (c_j-c_i)t)^2)} d\xi.$$

For the derivation of $\partial_t E_{Z_{ji},c_j-c_i}(\overline{Z}_{ji})(s,0)$, we decompose

$$\frac{1}{t}(E_{Z_{ji},c_j-c_i}(\overline{Z}_{ji})(s,t) - E_{Z_{ji},c_j-c_i}(\overline{Z}_{ji})(s,0)) = I_{1,ji} + I_{2,ji} + I_{3,ji},$$

where we set

$$I_{3,ji}(s,t) = \frac{\sqrt{t}}{2\pi}\int_{|\xi|<1} \frac{2\xi\sqrt{t}(\xi^2 + Z_{ji}^2(\xi,s,t) + (c_j-c_i)^2t^2)\partial_s B_{ji}(\xi,s,t)}{(\xi^2 + (Z_{ji}(\xi,s,t) + (c_j-c_i)t)^2)(\xi^2 + (Z_{ji}(\xi,s,t) - (c_j-c_i)t)^2)} d\xi$$

and $I_{1,ji}, I_{2,ji}$ are defined as in (3.53), (3.54) with Z and $2c$ replaced by Z_{ji} and $c_j - c_i$. By the same argument as above, we deduce that the integrand of $I_{3,ji}$ can be estimated against $C|\xi|^{-\frac{1}{2}}$. Hence, this term vanishes as $t \to 0$. Just like in Theorem 3.16 we split $I_{1,ji} = I_{11,ji} + I_{12,ji}$, this time with $\xi(\psi_0(\xi,s) + t\psi_a(\xi,s)) + \frac{1}{2}t^2 B_{ji}(\xi,s,t)$ instead of $\xi\psi_z(\xi,s,t)$. Because of the prefactor t^2 in front of B_{ji}, this term vanishes in the limit $t \to 0$ and we obtain the same limit for $\partial_t E_{Z_{ji},c_j-c_i}(\overline{Z}_{ji})(s,0)$ as in Theorem 3.16, except of the velocity $c_j - c_i$ instead of $2c$. To show the corresponding limit to (3.55), we can adapt the same proof. The only modification we have to take care of, is to replace $t^2 B$ by $t^2 B_{ji}$ inside of the terms Z or $\xi\psi_z$. However, thanks to the prefactor t^2, the proof remains valid by the same arguments. This is also true for verifying (3.71) analogous to (3.10).

In Section 3.4, we improved the regularity of the term $b_1 = T_{Z_0}(A)$, which can be done the same way for the curves z_i, since there is no dependence of the index i in b_1. Thus, Theorem 3.19 remains here valid too.

This means that for large $n \gg 1$, we established the existence of a subsolution containing a piecewise constant density (3.64), which approximates the density ρ from [17]. Furthermore, the propagation speed $c_n = 2 - \frac{1}{n}$ of $\partial\Omega_{\mathrm{mix}} = \Gamma^{+n} \cup \Gamma^{-n}$ becomes arbitrarily close to 2 in the limit $n \to \infty$. However, in contrary to the flat case in [79], the closer c_n becomes to 2, the shorter will be the time of existence T.

4 Wild initial data for the incompressible Euler equations

The observation that already a quite simple ansatz of a zero density inside the mixing zone yields a subsolution for the IPM equations gives hope that a similar approach also works in case of the incompressible Euler equations. We will show now that this is indeed the case.

4.1 Stationary vortex sheet solutions in the plane

In this section we want to investigate vortex sheet flows, which are stationary solutions to the incompressible Euler equations. From now on we restrict to the two-dimensional case and use coordinates $(x, y) \in \mathbb{R}^2$. Suppose that a smooth curve $\Gamma \subset \mathbb{R}^2$ and a vortex sheet strength $\gamma : \mathbb{R} \to \mathbb{R}$ are given. Then we can define a vortex sheet flow by use of the Biot-Savart kernel as in (1.15). However, it turns out that for the following computations, such vortex sheet flows are not a suitable choice. Beyond that, note that the flat vortex sheet flow (1.19) is obviously not of the form (1.15), since by definition of the Biot-Savart kernel, such flows vanish at infinity. Therefore we propose another approach, namely the use of holomorphic functions, we also refer to [1], p.120ff.

We start by considering a holomorphic function $f : V \to \mathbb{C}, f = \phi + i\psi$ and $V \subset \mathbb{C}$ being an open set. Because of the Cauchy-Riemann equations, we deduce that the vector field $w = (w_1, w_2) : V \to \mathbb{R}^2$, defined by

$$w := \nabla\phi = -\nabla^\perp\psi,$$

is incompressible and curl-free. Writing the Euler equations in the equivalent form

$$\begin{aligned} \partial_t v + (\text{curl } v) \cdot v^\perp + \nabla \left(p + \frac{1}{2}|v|^2 \right) &= 0, \\ \text{div } v &= 0, \end{aligned}$$

we observe that $v = w, p = C - \frac{1}{2}|w|^2$ for a constant $C \in \mathbb{R}$ defines a stationary solution. Hence, each holomorphic function f induces a stationary and curl-free solution of the incompressible Euler equations in V. We would like to emphasize that the streamlines of such flows coincide with the level sets of ψ, i.e. $\{\psi = C : C \in \mathbb{R}\}$.

However, we are interested in vortex sheet initial data, in which case the vorticity is a measure concentrated on a curve Γ. For this reason, we have to produce a tangential jump across Γ, which can be achieved as follows.

For simplicity, let $f : \mathbb{C} \to \mathbb{C}$ and assume for the moment that the zero set of the imaginary part of f, i.e. $\{\psi = 0\}$, coincides with the graph of a smooth function $z : \mathbb{R} \to \mathbb{R}$. Then we set $\Gamma := \{\psi = 0\}$ and obviously, Γ separates $\mathbb{R}^2 \cong \mathbb{C}$ into two connected domains A and B above and below Γ. As derived above, the function f induces the velocity field

$$w_f = (w_{f,1}, w_{f,2}) = \nabla\phi = -\nabla^\perp\psi,$$

which together with $p_f := C - \frac{1}{2}|w_f|^2$ yields a stationary solution of the Euler equations in $\mathbb{R}^2$. Then we define the vortex sheet flow

$$\tilde{w}_f(x) := \begin{cases} w_f(x) & , x \in A \\ -w_f(x) & , x \in B \end{cases} \tag{4.1}$$

in order to impose a tangential jump across Γ. Moreover, we set the corresponding pressure $\tilde{p}_f = C - \frac{1}{2}|\tilde{w}_f|^2$, which is continuous on the whole $\mathbb{R}^2$ in contrary to $\tilde{w}_f$.

Definition 4.1. Let $V \subset \mathbb{R}^2$ be an open set. We call a set $\Gamma \subset V$ conformal, if there exists a holomorphic function $f : V \to \mathbb{C}, f = \phi + i\psi$ such that Γ coincides with a connected component of $\{\psi = 0\}$, i.e. $f(\Gamma) \subset \mathbb{R}$.

Theorem 4.2. Let $\Gamma \subset \mathbb{R}^2$ be a smooth and conformal curve, which can be presented as the graph of a function. Then there exists a stationary vortex sheet solution (v, p) of the Euler equations. The vorticity $\omega = \operatorname{curl} v$ is a measure concentrated on Γ. Furthermore, v is tangential to Γ, i.e. $v \cdot n = 0$ on Γ, where n denotes the normal vector on Γ.

Proof. As derived above, we set

$$(v, p) := (\tilde{w}_f, \tilde{p}_f).$$

Then, by construction, v is a vortex sheet with vorticity concentrated on Γ and obviously v is tangential to Γ, since

$$v \cdot n = \pm \nabla^\perp \psi \cdot \nabla \psi = 0.$$

It remains to establish that (v, p) is a stationary solution to the Euler equations. Obviously, v remains a curl-free solution in the regions A and B. But we still have to make sure that the tangential jump across Γ causes no problems. Let $\varphi \in C_c^\infty(\mathbb{R}^2)$. The incompressibility condition reads

$$\int_{\mathbb{R}^2} \nabla\varphi \cdot v \; dxdy = \int_\Gamma \varphi[v] \cdot n \; dx = 0,$$

where by $[v]$ we denote the jump of v across Γ. Similarly, we infer

$$\begin{aligned} \int_{\mathbb{R}^2} (v \otimes v)\nabla\varphi + \nabla\varphi \cdot p \; dxdy &= \int_\Gamma \varphi\,(v \otimes v) \cdot n + \varphi[p] \cdot n \; dx \\ &= \int_\Gamma \varphi\,(v \cdot n)\,v + \varphi[p] \cdot n \; dx = 0, \end{aligned}$$

since the jump of p vanishes due to continuity. □

Remark 4.3. (i) The same result can be established on the Torus $\mathbb{T}^2$. Indeed, starting with a holomorphic function $f : \mathbb{T}^2 \to \mathbb{T}^2$ and identifying $\mathbb{T}^2$ with $[0,1]^2$, we can continue f periodically to a holomorphic function in $\mathbb{R}^2$. As before, this f induces a stationary and periodic vortex sheet flow $\tilde{w}_f$ tangential to $\Gamma = \{\psi = 0\}$.

(ii) In the same way, Theorem 4.2 also holds true in the case that Γ is a closed, smooth and conformal curve in $\mathbb{R}^2$.

We want to state a few concrete examples of conformal sets and corresponding stationary vortex sheet solutions.

Example 4.4. (i) Consider the identity $f(z) = z$, which induces the constant velocity field $w_f = (1, 0)$ and the curve $\Gamma = \{y = 0\}$. The streamlines correspond to straight lines parallel to the x-axis. Eventually, this yields the flat vortex sheet

$$\tilde{w}_F(x, y) = \begin{cases} (1, 0) & , y > 0 \\ -(1, 0) & , y < 0 \end{cases}. \tag{4.2}$$

(ii) The function $f(z) = -i \log(z)$ attains circles around the origin as streamlines and induces the velocity field $w_f = \frac{1}{x^2+y^2}(-y, x) = \frac{1}{r} e_\theta$. In particular, we obtain the curve $\Gamma = \{r = 1\}$ as the unit circle. As a consequence, this yields the circular vortex sheet flow

$$\tilde{w}_f(r, \theta) = \begin{cases} \frac{1}{r} e_\theta & , r > 1 \\ -\frac{1}{r} e_\theta & , 0 < r < 1 \end{cases}. \tag{4.3}$$

(iii) Taking $f(z) = z^2$, we deduce $w_f(x, y) = (2x, -2y)$. This velocity field is tangential to $\Gamma = \{(x, 0) : x \in \mathbb{R}\} \cup \{(0, y) : y \in \mathbb{R}\}$, which is not a smooth curve, because of the stagnation point in $(0, 0)$. In this point, the velocity w_f vanishes. Nonetheless, as in Theorem 4.2, this flow induces a stationary vortex sheet solution in $\mathbb{R}^2$ by

$$\tilde{w}_f(x, y) := \begin{cases} w_f(x, y), & xy > 0 \\ -w_f(x, y), & xy < 0 \end{cases}.$$

The question arises, which curves $\Gamma \subset \mathbb{R}^2$ are conformal. This leads to the problem of characterizing level sets of harmonic functions, which seems to be quite difficult. One can not expect that for a given smooth curve Γ, there exists a harmonic function vanishing on Γ. Results in the literature restrict to topological properties ([52]) or curvature bounds for the level sets ([16]). The only concrete examples I could find are the following from [47].

- Parabolas: By translation and rescaling one can consider without loss of generality the example $\Gamma = \{y = x^2\}$. Then, it is shown that the imaginary part of the holomorphic function $f(z) = \cosh(\pi\sqrt{4z - 1})$ vanishes on Γ.

- Hyperbolas: In the case that the angle α between the asymptotes of a hyperbola is a rational multiple of π, say for example $\alpha = \frac{m\pi}{n}$, we can choose the function $f(z) = \cos(2n(\arcsin(z)))$. If the angle is a irrational multiple of π, there does not exist a harmonic function vanishing on such a hyperbola.

- Ellipses are conformal, which can be established by taking $f(z) = F(\arcsin(z))$, for a suitable elliptic function F. Note that f necessarily has to have a singularity, otherwise the level set can not equal to a closed curve.

We want to emphasize that there is another possibility to see that ellipses are conformal, namely the method of conformal transformation. In our special situation we make use of the Joukowski transformation $F(z) = z + \frac{c^2}{z}$ for some $0 \leq c \leq a$. This function maps ellipses of the form

$$\frac{x^2}{(a+\frac{c^2}{a})^2} + \frac{y^2}{(a-\frac{c^2}{a})^2} = 1$$

onto circles with radius a. If we compose the holomorphic function $-i\log(z)$ with the Joukowski transformation, we obtain the holomorphic function $-i\log(F(z))$ on $\mathbb{C} \setminus \{0\}$. Then we see that the streamlines of this function correspond to ellipses.

The same function F is also used to model a flow behind an obstacle. For example, the choice $c = 1$ yields $F(z) = z + \frac{1}{z} = \Phi + i\Psi$ and the velocity field

$$w_F(x,y) = \nabla\Phi = (1,0) - \frac{1}{(x^2+y^2)^2}(x^2-y^2, 2xy).$$

The associated zero set is given by $\Gamma = \{\Psi = 0\} = \{x^2+y^2 = 1\} \cup \{y = 0\}$. This set is not smooth, since stagnation points emerge in $(\pm 1, 0)$. Nevertheless, we can still define some vortex sheet flow for each $(x,y) \in \mathbb{R}^2 \setminus \{x^2+y^2 \leq 1\}$ by

$$\tilde{w}_F(x,y) := \begin{cases} w_F(x,y), & y > 0 \\ -w_F(x,y), & y < 0 \end{cases}.$$

Inside the domain $\mathbb{R}^2 \setminus \{x^2+y^2 \leq 1\}$, this is a stationary solution of the Euler equations, which is tangential to Γ. This example describes the flow along a circular obstacle placed in $\{x^2+y^2 \leq 1\}$. By a suitable transformation, F can be used to characterize flows along obstacles of arbitrary shape. Thanks to this, the Joukowski transformation has a substantial relevance in engineering sciences. Note that $w_F \to (1,0)$ as $|(x,y)| \to \infty$, i.e. w_F converges to the flat flow at infinity.

Thus, we have shown at least some explicite examples of conformal sets Γ for which we have a stationary vortex sheet solution. Unfortunately, we do not have a formula, which characterizes how $\tilde{w}_f$ depends on the curve Γ. This is in contrast to vortex sheet initial data given by the Biot-Savart law (1.15). Although we have a concrete representation formula here, it is not clear, whether one can produce stationary vortex sheet solutions of this form at all. The problem arises in choosing the vorticity strength in such a way that v is tangential to Γ. Furthermore, we want to emphasize the different behaviour at infinity of these two types of vortex sheet data. Whereas velocities of the form (1.15) tend to zero as $|x| \to \infty$, stationary vortex sheets of the form (4.1) do not necessarily vanish at infinity, which can be seen for instance at (4.2).

4.2 Flow coordinates

Suppose that some stationary vortex sheet solution $v_0 := \tilde{w}_f$ is given by (4.1) and that v_0 is tangential to a smooth conformal curve Γ, for example an ellipse. Adapting the notation from the previous section, the underlying holomorphic potential $f = \phi + i\psi$ induces the flow $w_f = \nabla\phi = \nabla^{\perp}\psi$. From now on we drop the f and just write w or $\tilde{w}$. The sheet is given by $\Gamma = \{\psi = 0\}$.

Recall from Section 2, that we want to construct a subsolution $(\overline{v}, \overline{u}, \overline{q})$ being strict inside of a growing zone U around the sheet Γ. In the spirit of Theorem 1.19, we choose

$$U = \{(x, y, t) : |\psi(x, y)| < \lambda t,\ 0 < t < T\} \quad \text{for some } \lambda, T > 0. \tag{4.4}$$

On the other hand, outside of U we want to stay with the strong and curl-free solution $(\tilde{w}, C - \frac{1}{2}|\tilde{w}|^2)$ of the Euler equations.

For this purpose, at first we should introduce suitable new coordinates $(\tilde{x}, \tilde{y})$ adapted to the geometry of the curve Γ. We call them flow coordinates.

For the stationary solution of the flat vortex sheet, we notice that the streamlines coincide with horizontal lines $\{y = \text{const}\}$. Therefore a common choice is to set

$$\tilde{y} := \psi(x, y). \tag{4.5}$$

Of course, in this case the most obvious choice for $\tilde{x}$ is to define

$$\tilde{x} := \phi(x, y). \tag{4.6}$$

Note that these coordinates are orthogonal to each other

$$e_{\tilde{x}} = \frac{w}{|w|} = \frac{1}{\sqrt{w_1^2 + w_2^2}}(w_1, w_2), \quad e_{\tilde{y}} = \frac{w^\perp}{|w|} = \frac{1}{\sqrt{w_1^2 + w_2^2}}(-w_2, w_1).$$

The derivatives in these new coordinates can be computed as follows.

Lemma 4.5. For each C^1 function h we have

$$\nabla h = \begin{pmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{pmatrix} \cdot \tilde{\nabla} h, \tag{4.7}$$

$$\tilde{\nabla} h = \frac{1}{|w|^2} \begin{pmatrix} w_1 & w_2 \\ -w_2 & w_1 \end{pmatrix} \cdot \nabla h. \tag{4.8}$$

Proof. The first identity follows simply by using the chain rule and the second one by computing the inverse matrix. □

For the construction of a suitable subsolution inside of U, at first we will determine the velocity $\overline{v}$ in these new coordinates. If we want to get an analogous result to the flat case, $\overline{v}$ should be directed parallel to the stationary flow, i.e. moving in $\tilde{x}$-direction. Let $\alpha = \alpha(\tilde{y}, t)$ be a function independent of $\tilde{x}$. Then we infer that

$$\overline{v} = \alpha(\tilde{y}, t) w, \tag{4.9}$$

is indeed a divergence-free vector field, since

$$\operatorname{div} \overline{v} = \nabla\alpha \cdot w + \alpha \cdot \operatorname{div} w = (w_1 \cdot \alpha_{\tilde{x}} - w_2 \alpha_{\tilde{y}}, w_2 \cdot \alpha_{\tilde{x}} + w_1 \alpha_{\tilde{y}}) \cdot w = 0.$$

Furthermore, we compute

$$\overline{v} \otimes \overline{v} = \alpha^2 \begin{pmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 \\ w_2 & -w_1 \end{pmatrix} \begin{pmatrix} \alpha^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 & w_2 \\ w_2 & -w_1 \end{pmatrix}.$$

Thanks to this, it is appropriate to impose the ansatz

$$\begin{aligned}\overline{u} &= \begin{pmatrix} w_1 & w_2 \\ w_2 & -w_1 \end{pmatrix} \begin{pmatrix} \beta & -\gamma \\ -\gamma & -\beta \end{pmatrix} \begin{pmatrix} w_1 & w_2 \\ w_2 & -w_1 \end{pmatrix} \\ &= \begin{pmatrix} \beta(w_1^2 - w_2^2) - 2w_1w_2\gamma & 2w_1w_2\beta + \gamma(w_1^2 - w_2^2) \\ 2w_1w_2\beta + \gamma(w_1^2 - w_2^2) & -\beta(w_1^2 - w_2^2) + 2w_1w_2\gamma \end{pmatrix}\end{aligned} \tag{4.10}$$

where $\beta = \beta(\tilde{x}, \tilde{y}, t)$ and $\gamma = \gamma(\tilde{x}, \tilde{y}, t)$. Moreover, let $\overline{q} = \overline{q}(\tilde{x}, \tilde{y}, t)$. Since ϕ, ψ are harmonic, we have $\partial_x w_1 + \partial_y w_2 = 0$ and $\partial_y w_1 = \partial_x w_2$. This leads to

$$\partial_x(w_1^2 - w_2^2) + 2\partial_y(w_1w_2) = 0 \quad \text{and} \quad 2\partial_x(w_1w_2) - \partial_y(w_1^2 - w_2^2) = 0. \tag{4.11}$$

Using (4.11), we deduce

$$\begin{aligned}\operatorname{div} \overline{u} &= (w_1^2 - w_2^2)(w_1\beta_{\tilde{x}} - w_2\beta_{\tilde{y}} + w_2\gamma_{\tilde{x}} + w_1\gamma_{\tilde{y}})e_x \\ &\quad + 2w_1w_2(w_1\beta_{\tilde{y}} - w_1\gamma_{\tilde{x}} + w_2\gamma_{\tilde{y}} + w_2\beta_{\tilde{x}})e_x \\ &\quad + (w_1^2 - w_2^2)(w_1\gamma_{\tilde{x}} - w_2\gamma_{\tilde{y}} - w_2\beta_{\tilde{x}} - w_1\beta_{\tilde{y}})e_y \\ &\quad + 2w_1w_2(w_1\beta_{\tilde{x}} - w_2\beta_{\tilde{y}} + w_2\gamma_{\tilde{x}} + w_1\gamma_{\tilde{y}})e_y, \\ \partial_t\overline{v} &= \partial_t\alpha(w_1e_x + w_2e_y), \\ \nabla\overline{q} &= (w_1\overline{q}_{\tilde{x}} - w_2\overline{q}_{\tilde{y}})e_x + (w_2\overline{q}_{\tilde{x}} + w_1\overline{q}_{\tilde{y}})e_x.\end{aligned}$$

Therefore, the equation $\partial_t\overline{v} + \operatorname{div}\overline{u} + \nabla\overline{q} = 0$ reads

$$0 = w_1\partial_t\alpha + w_1|w|^2\beta_{\tilde{x}} + w_2|w|^2\beta_{\tilde{y}} - w_2|w|^2\gamma_{\tilde{x}} + w_1|w|^2\gamma_{\tilde{y}} + w_1\overline{q}_{\tilde{x}} - w_2\overline{q}_{\tilde{y}} \tag{4.12}$$
$$0 = w_2\partial_t\alpha + w_2|w|^2\beta_{\tilde{x}} - w_1|w|^2\beta_{\tilde{y}} + w_1|w|^2\gamma_{\tilde{x}} + w_2|w|^2\gamma_{\tilde{y}} + w_2\overline{q}_{\tilde{x}} + w_1\overline{q}_{\tilde{y}}. \tag{4.13}$$

We evaluate (4.12)w_1 + (4.13)w_2 and (4.12)w_2 - (4.13)w_1 and afterwards divide by $|w|^2$ to arrive at the equivalent system

$$0 = \partial_t\alpha + |w|^2\beta_{\tilde{x}} + |w|^2\gamma_{\tilde{y}} + \overline{q}_{\tilde{x}} \tag{4.14}$$
$$0 = |w|^2\beta_{\tilde{y}} - |w|^2\gamma_{\tilde{x}} - \overline{q}_{\tilde{y}}, \tag{4.15}$$

which can be written as

$$\partial_t \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + |w|^2 \tilde{\operatorname{div}} \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix} + \tilde{\nabla}\overline{q} = 0. \tag{4.16}$$

We observe that in flow coordinates, we have almost the same equation for being a subsolution, except of the extra factor $|w|^2$ in front of the divergence-term. This term captures the geometry of the curve Γ. Note that the stationary solution outside of U corresponds to $\alpha \equiv \pm 1, \beta \equiv \overline{q} \equiv \frac{1}{2}$ and $\gamma \equiv 0$.

For the special choice $\beta = \beta(\tilde{y}, t), \gamma = \gamma(\tilde{y}, t), \overline{q} = \overline{q}(\tilde{y}, t)$, (4.14) and (4.15) decouple and reduce to

$$0 = \partial_t\alpha + |w|^2\gamma_{\tilde{y}} \tag{4.17}$$
$$0 = |w|^2\beta_{\tilde{y}} - \overline{q}_{\tilde{y}}. \tag{4.18}$$

Assuming that the term $|w|^2$ is independent of $\tilde{x}$, we could solve (4.18) simply by defining

$$\overline{q} = \int_0^{\tilde{y}} |w|^2 \beta_{\tilde{y}} d\tilde{y}'$$

and (4.17) could be solved analogously to the flat vortex sheet case reducing to a first order PDE similar to Burger's equation.

Hence, the question arises for which curves Γ we have the property that $|w|^2$ is independent of $\tilde{x}$. This is the content of the following lemma.

Lemma 4.6. The term $|w|^2$ is independent of $\tilde{x}$ if and only if the curve Γ is given by a straight line $\Gamma = \{ax + by = C\}$ or by a circle $\Gamma = \{x^2 + y^2 = r^2\}$.

Proof. Suppose Γ with $\partial_{\tilde{x}}|w|^2 = 0$ is given. Then note that since $|w|^2 = |\nabla\phi|^2$ is independent of $\tilde{x}$, this term is constant along the streamlines $\{\psi = C\}$. Therefore the level sets of $|\nabla\phi|^2$ coincide with the streamlines and as a consequence we deduce that $\nabla\psi$ and $\nabla(|\nabla\phi|^2)$ are parallel to each other. Thus, the equation $\nabla\phi \cdot \nabla\psi = 0$ leads to

$$\nabla\phi \cdot \nabla(|\nabla\phi|^2) = 0,$$

or equivalently

$$0 = 2\phi_x\phi_y\phi_{xy} + \phi_{xx}(\phi_x^2 - \phi_y^2). \tag{4.19}$$

In the next step consider an arbitrary level set of ϕ, for example $\{\phi = 1\}$ and denote this curve by $\gamma(s) = (\gamma_1(s), \gamma_2(s))$. Then $\phi(\gamma(s)) = 1 \quad \forall s \in \mathbb{R}$. Differentiating gives

$$0 = \nabla\phi(\gamma(s)) \cdot \dot{\gamma}(s). \tag{4.20}$$

We differentiate once more and obtain

$$0 = \phi_{xx}\dot{\gamma_1}^2 + \phi_{xy}\dot{\gamma_1}\dot{\gamma_2} + \phi_x\ddot{\gamma_1} + \phi_{yy}\dot{\gamma_2}^2 + \phi_{xy}\dot{\gamma_1}\dot{\gamma_2} + \phi_y\ddot{\gamma_2} \tag{4.21}$$

Now, from (4.20) we conclude $\nabla\phi = c\dot{\gamma}^\perp$ for some function $c = c(s)$. Plugging this into (4.19), we obtain

$$0 = c^2(-2\phi_{xy}\dot{\gamma_1}\dot{\gamma_2} + \phi_{xx}(\dot{\gamma_2}^2 - \dot{\gamma_1}^2)).$$

Finally, this equation together with $\Delta\phi = 0$ and (4.21) leads to

$$0 = \nabla\phi \cdot \ddot{\gamma} = c\dot{\gamma}^\perp \cdot \ddot{\gamma}.$$

Using the representation formula for the curvature of a planar curve

$$\kappa(s) = \frac{-\dot{\gamma(s)} \cdot \ddot{\gamma(s)}^\perp}{|\dot{\gamma(s)}|^3}$$

we get $\kappa(s) = 0$. Thus, $\gamma(s)$ has to be a straight line. Since obviously the level sets of ϕ are not allowed to cross each other, there are only two possibilities:
In the first case, the level sets of ϕ are parallel lines, which means that the streamlines are also parallel lines. Secondly, the level sets of ϕ can be lines going through the origin $(0, 0)$. Note that in this case the function $f = \phi + i\psi$ has a singularity in $(0, 0)$. Then, the streamlines coincide with circles around the origin. □

So there are only these two special curves, where the symmetry leads to such a simple structure. It is worth to investigate the solvability of (4.17) and (4.18) in these cases.

Without loss of generality, in case of a straight line, we reduce to the horizontal one $\{y = 0\}$. For this flat vortex sheet we have $|w|^2 = 1$, so (4.18) is solved by setting $\beta = \overline{q}$. For (4.17), we define $\gamma = -\frac{\lambda}{2}(1 - \alpha^2)$ and so reduce to Burger's equation

$$\partial_t \alpha + \frac{\lambda}{2}\partial_y.\alpha^2 = 0$$

The solution is obtained just like in [77].

For the case of the rotational initial data, we first study the paper [5] and denote the sets $\Omega = \{x \in \mathbb{R}^2 : \rho < |x| < R\}$ for $\rho, R > 0$ and $\Omega_T := \Omega \times [0, T]$. In [5], the authors address the initial data

$$\hat{v}_0(r) = \begin{cases} -\frac{1}{r^2} e_\theta & \rho < r < r_0 \\ \frac{1}{r^2} e_\theta & r_0 < r < R \end{cases}, \tag{4.22}$$

which is not curl-free and thus can not be inferred from a holomorphic potential. Because of this, we can not define flow coordinates $(\tilde{x}, \tilde{y})$ and therefore we are working in polar coordinates (r, θ). In this description, one can reduce the equation $\partial_t v + \operatorname{div} u + \nabla q = 0$ to the system

$$0 = \partial_t \hat{\alpha} + \partial_r \hat{\gamma} + \frac{2}{r}\hat{\gamma} \tag{4.23}$$

$$0 = \partial_r \hat{\beta} + \frac{2}{r}\hat{\beta} + \partial_r \hat{q}. \tag{4.24}$$

(4.24) is solved by setting $\hat{\beta} = -\frac{1}{2}\hat{\alpha}^2$ and $\hat{q}$ as the stationary pressure

$$\hat{q} = \frac{1}{2}\hat{\alpha}^2 + \int_\rho^r \frac{\hat{\alpha}^2}{s} ds. \tag{4.25}$$

So we only concentrate on (4.23) now. Defining

$$\hat{\alpha} := \frac{1}{r^2}\hat{f}(r,t) \quad \text{and} \quad \hat{\gamma} := -\frac{\lambda}{2r^2}(1 - \hat{f}^2),$$

for some $\hat{f} : (0, \infty) \times (0, T) \to \mathbb{R}$, (4.23) turned into Burger's equation

$$\partial_t \hat{f} + \frac{\lambda}{2}\partial_r(\hat{f}^2) = 0, \quad \hat{f}(r, 0) = \begin{cases} -1 & \rho < r < r_0 \\ 1 & r_0 < r < R \end{cases}.$$

Similar to the flat case, there exists a solution $\hat{f}$ of this equation and thus the existence of a strict subsolution with initial data (4.22) was shown.

Beyond that, a similar construction is possible for the initial datum

$$\tilde{v}_0(r) = \begin{cases} -\frac{1}{r} e_\theta & r < r_0 \\ \frac{1}{r} e_\theta & r > r_0 \end{cases}. \tag{4.26}$$

As already stated in Example 4.4,(ii), this velocity field arises from the holomorphic function $f(z) = -i\log(z)$ and yields flow coordinates $\tilde{x} = \theta, \tilde{y} = -\log(r)$.

If we take the same ansatz as before, we have to solve Burger's equation for $\hat{f}$ with initial datum

$$\hat{f}(r,0) = \begin{cases} -r & \rho < r < r_0 \\ r & r_0 < r < R \end{cases}.$$

However, the corresponding solution will not remain stationary outside the region U. In particular we observe that $\hat{\gamma} = -\frac{\lambda}{2r^2}(1-\hat{f}^2) \neq 0$ in U^c and then $e(v,u) \neq \frac{1}{2}|\tilde{v}_0|^2$. Therefore one should search for another ansatz. Now we impose new functions denoted by $\tilde{\cdot}$ instead of $\hat{\cdot}$. The choice for $\tilde{\beta} = -\frac{1}{2}\tilde{\alpha}^2$ and $\tilde{q}$ by (4.25) will be the same. However, for $\tilde{\gamma}$ we choose

$$\tilde{\gamma} := -\frac{\lambda}{2}\left(\frac{1}{r^2} - \tilde{\alpha}^2\right).$$

Plugging this ansatz into (4.23), we arrive at

$$0 = \partial_t \tilde{\alpha} + \frac{\lambda}{2r^2}\partial_r(r^2\tilde{\alpha}^2).$$

The definition $\tilde{f}(r,t) := r\tilde{\alpha}(r,t)$ leads to the first order PDE

$$\partial_t \tilde{f} + \frac{\lambda}{2r}\partial_r(\tilde{f}^2) = 0, \quad \tilde{f}(r,0) = \begin{cases} -1 & \rho < r < r_0 \\ 1 & r_0 < r < R \end{cases},$$

which is not Burger's equation. Nevertheless, we can still solve this equation by use of the method of characteristics.

We denote $x(s) = (x_1(s), x_2(s)) = (r,t), z(s) = \tilde{f}(x(s)), p(s) = (\partial_t \tilde{f}(x(s)), \partial_r \tilde{f}(x(s)))$. Thus, the characteristic equations become

$$x'(s) = \left(\frac{\lambda}{x_1(s)} z(s), 1\right) \quad \text{and} \quad z'(s) = \left(\frac{\lambda}{x_1(s)} z(s), 1\right) p(s) = 0.$$

We deduce $z(s) \equiv z(0), x_2(s) \equiv s$ and $x_1'(s) = \frac{\lambda z(0)}{x_1(s)}$. Solving the ODE for x_1 easily gives $x_1(s) = \sqrt{2\lambda z(0)s + \overline{r}^2}$, for an initial point $(\overline{r}, 0)$. Using the initial data $z(0) = \tilde{f}(r,0)$, we accomplish

$$\begin{aligned} \tilde{f}(r,t) = z(s) &= \begin{cases} -1, & \rho < \sqrt{r^2 + 2\lambda t} < r_0 \\ 1, & r_0 < \sqrt{r^2 - 2\lambda t} < R \end{cases} \\ &= \begin{cases} -1, & \rho < r < \sqrt{r_0^2 - 2\lambda t} \\ 1, & \sqrt{r_0^2 + 2\lambda t} < r < R \end{cases}. \end{aligned}$$

Observe that $\tilde{f}$ is still not defined in the range $\sqrt{r_0^2 - 2\lambda t} < r < \sqrt{r_0^2 + 2\lambda t}$. Thus we try to interpolate and propose the ansatz $\tilde{f}(r,t) = \frac{r^n}{t^m}$. Plugging into the PDE for $\tilde{f}$ and comparing the powers yields $n = 2, m = 1$. The continuity of $\tilde{f}$ then implies

$$\tilde{f}(r,t) = \begin{cases} -1, & \sqrt{\rho^2 - 2\lambda t} < r < \sqrt{r_0^2 - 2\lambda t} \\ \frac{r^2 - r_0^2}{2\lambda t}, & \sqrt{r_0^2 - 2\lambda t} < r < \sqrt{r_0^2 + 2\lambda t} \\ 1, & \sqrt{r_0^2 + 2\lambda t} < r < \sqrt{R^2 + 2\lambda t} \end{cases}. \tag{4.27}$$

Indeed, one can check easily that $\frac{r^2-r_0^2}{2\lambda t}$ is a solution of the PDE for $\tilde{f}$. Thus, as opposed to the flat vortex sheet, the domain U now does not grow linearly in time, but with order $\sim \sqrt{t}$. Furthermore, for the generalized energy we get

$$e(\overline{v},\overline{u}) = \frac{1}{2r^2}(1-(1-\lambda)(1-\tilde{f}^2)),$$

from which we see $e(\overline{v},\overline{u}) < \frac{1}{2}|\tilde{v}_0|^2$ if and only if $|\tilde{f}| < 1$, that is inside U. Hence we established the following result.

Theorem 4.7. For each $\lambda, \varepsilon > 0$ there exists a subsolution $(\overline{v},\overline{u},\overline{q})$ in Ω_T with respect to the energy density $\overline{e} = \frac{1}{2r^2}(1-\varepsilon(1-\lambda)(1-\tilde{f}^2))$ and initial datum (4.26). Here, $\tilde{f}$ is given by (4.27). This subsolution is strict inside the zone

$$U = \left\{ x \in \mathbb{R}^2 : \sqrt{r_0^2 - 2\lambda t} < |x| < \sqrt{r_0^2 + 2\lambda t} \right\},$$

which means that we have

$$\begin{aligned} e(\overline{v},\overline{u}) &< \overline{e} \quad \text{in } U, \\ e(\overline{v},\overline{u}) &= \overline{e} \quad \text{in } \Omega_T \setminus U. \end{aligned}$$

Remark 4.8. Actually, we can show that this subsolution also satisfies (4.23), (4.24) written in flow coordinates, that is, it solves (4.17), (4.18). Indeed, recall that we have $\tilde{x} = \theta, \tilde{y} = -\log(r)$ and $w = \frac{1}{r}(\sin(\theta), \cos(\theta))$. In particular, we obtain $|w|^2 = r^{-2} = e^{2\tilde{y}}$. We choose the ansatz $\alpha = r\tilde{\alpha}$ and $\beta = -r^2\tilde{\beta}, \gamma = -r^2\tilde{\gamma}$ and compute

$$\begin{aligned} 0 &= \partial_t\tilde{\alpha} + \frac{\lambda}{2r^2}\partial_r(r^2\tilde{\alpha}^2) = \partial_t\tilde{\alpha} - \frac{\lambda}{2}e^{3\tilde{y}}\partial_{\tilde{y}}(e^{-2\tilde{y}}\tilde{\alpha}^2) \\ &= \partial_t\tilde{\alpha} + \frac{\lambda}{2}e^{3\tilde{y}}\partial_{\tilde{y}}(1 - e^{-2\tilde{y}}\tilde{\alpha}^2) = e^{-\tilde{y}}\partial_t\tilde{\alpha} + e^{2\tilde{y}}\partial_{\tilde{y}}(-\frac{\lambda}{2}(e^{2\tilde{y}} - \tilde{\alpha}^2)\cdot(-e^{-2\tilde{y}})) \\ &= \partial_t\alpha + (u^2+v^2)\partial_{\tilde{y}}(\gamma), \end{aligned}$$

which is (4.17). In the same way we infer that (4.18) and (4.24) coincide

$$\begin{aligned} &\partial_{\tilde{y}}\overline{q} = e^{2\tilde{y}}\beta_{\tilde{y}} \\ \Leftrightarrow\ &\partial_r\overline{q} = r^{-2}\partial_r\beta = r^{-2}\partial_r\left(\frac{r^2}{2}\tilde{\alpha}^2\right) = \partial_r\left(\frac{\tilde{\alpha}^2}{2}\right) + \frac{\tilde{\alpha}^2}{r}. \end{aligned}$$

An integration in r gives the stationary pressure (4.25).

In conclusion, we have seen that in the two special cases of the flat vortex sheet and the circle, the subsolution equation can be reduced to a first order PDE, which allows us to construct a strict subsolution inside of a growing domain U. Since in general $|w|^2$ is not independent of $\tilde{x}$, we can not reduce to such a simple first order PDE and consequently need a different ansatz for the construction.

Recall that the stationary vortex sheet solution w is induced by a holomorphic map $f : \mathbb{C} \to \mathbb{C}$ and that $f = \phi + i\psi, (\phi,\psi) = (\tilde{x},\tilde{y})$. Thanks to this, we hope that f does not only map Γ onto $\{y=0\}$, but in the same way f pulls back the subsolution from the flat case. Thus, denote by (v_*, u_*, q_*) the strict subsolution belonging to the flat vortex sheet as constructed in [77]. Then define for $(x,y,t) \in U$

$$\overline{v}(x,y,t) := Df^T(x,y)v_*(f(x,y),t) \tag{4.28}$$
$$\overline{u}(x,y,t) := Df^T(x,y)u_*(f(x,y),t)Df(x,y) \tag{4.29}$$
$$\overline{q}(x,y,t) := |\nabla f_2(x,y)|^2 q_*(f(x,y),t). \tag{4.30}$$

Since f is holomorphic, one easily verifies that

$$\text{cof } Df^T = Df^T = \begin{pmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{pmatrix}.$$

By a straightforward computation we conclude that $\overline{v}$ is divergence-free. To establish that $(\overline{v}, \overline{u}, \overline{q})$ is a subsolution, we evaluate $E_f := \partial_t \overline{v} + \text{div } \overline{u} + \nabla \overline{q}$ and obtain

$$E_f = 2D^2 f_2(x,y)\nabla f_2(x,y) \cdot q_*(f(x,y),t) + (1 - |\nabla f_2|^2) Df^T \partial_t v_*(f(x,y),t) \neq 0.$$

Unless $f_2 \equiv y$ like in the flat case, this error term E_f does not vanish. Hence, a simple transformation of the flat subsolution (v_*, u_*, q_*) is not an appropriate choice. By means of this computation, it seems that an ansatz for the subsolution in the spirit of Burger's equation does not fit for a general curve.

A slightly weaker ansatz consists in transforming merely the velocity by (4.28), that is, inside U we set

$$\overline{v}(x,y,t) = \frac{\psi(x,y)}{\lambda t} w(x,y,t) \tag{4.31}$$

in analogy to the flat vortex sheet case. As stated above, this velocity is divergence-free. It remains to determine $\overline{u} = \begin{pmatrix} \beta & \gamma \\ \gamma & -\beta \end{pmatrix}$ and $\overline{q}$ such that the subsolution equation

$$\partial_t \overline{v} + \text{div } \overline{u} + \nabla \overline{q} = 0 \tag{4.32}$$

is valid in U and such that this is a strict subsolution, which means there exists an energy function $\overline{e}$ with

$$e(\overline{v}, \overline{u}) < \overline{e}, \tag{4.33}$$
$$\partial_t \overline{e} + \text{div } (\overline{v} \cdot \overline{q}) \leq 0. \tag{4.34}$$

The second inequality will be necessary to obtain weak solutions, which satisfy the local energy inequality. Moreover, we need to impose boundary conditions on ∂U

$$\beta = \frac{w_1^2 - w_2^2}{2}, \tag{4.35}$$
$$\gamma = w_1 w_2, \tag{4.36}$$
$$\overline{q} = C(t). \tag{4.37}$$

We can view (4.32) together with (4.35)-(4.36) as an overdetermined div-curl system for (β, γ) and we have to choose the pressure $\overline{q}$ in such a way that this system is solvable. Furthermore, the equations (4.33)-(4.34) have to hold true. This seems to be a delicate problem and we were not able to find a suitable ansatz for a solution. Since the only assumption we made so far is the choice (4.28), this indicates that already this ansatz is too strong. We believe that a velocity, which behaves like a solution of Burger's equation, only fits for the simple flat case due to the special symmetry. This is also demonstrated by our computations for the rotational initial data. Thus, it is reasonable to assume that for general curves, formula (4.28) is no appropriate ansatz. Instead, we will choose from now on an arbitrary velocity, in particular we allow discontinuities on ∂U.

4.3 Piecewise constant velocity

The goal of this section is the explicit construction of a strict subsolution leading to a proof of Theorem 1.1. We start with the following definition.

Definition 4.9. Let $T > 0$, Γ be a conformal set and v_0 be the corresponding stationary vortex sheet solution of the form (4.1). Moreover, by w we denote the harmonic vector field and by $(\tilde{x}, \tilde{y})$ the flow coordinates. Then we call v_0 a regular vortex sheet flow, if there exists a constant $m > 0$ such that

$$0 < m < |w|^2(\tilde{x}, \tilde{y}) \text{ for all } \tilde{x} \in \mathbb{R}, |\tilde{y}| < T \tag{4.38}$$

$$\nabla w(\cdot, \tilde{y}) \in L^\infty(\mathbb{R}) \text{ for all } |\tilde{y}| < T \tag{4.39}$$

$$\left(\frac{w_1}{|w|^2}\right)(\cdot, \tilde{y}), \left(\frac{w_2}{|w|^2}\right)(\cdot, \tilde{y}) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \text{ for all } |\tilde{y}| < T. \tag{4.40}$$

Note that for each smooth and conformal curve Γ, condition (4.38) is satisfied, since w is always tangential to Γ. Hence, stagnation points, where $w = 0$, can only arise at discontinuities of Γ. For example, this can be observed for the flow at a right angle from Example 4.4, (iii). Whereas also (4.39) seems to be a natural condition, it is not obvious whether (4.40) is valid in the general case.

On $\mathbb{R}^2$, we notice that (4.40) is not satisfied for the flat vortex sheet. On the other hand, for any other smooth and conformal curve Γ, which can be represented as the graph of a function, w is harmonic and non-constant. Hence, we conclude that $|w(x)| \to \infty$ as $|x| \to \infty$ by the Liouville Theorem. In light of this, condition (4.40) seems reasonable at least for sufficiently fast increasing $|w|$. Nevertheless, we can not give any concrete examples of such regular vortex sheet flows.

However, if Γ is a closed, smooth and conformal curve in $\mathbb{R}^2$, we have $\tilde{x} \in [0, 2\pi]$. Then $w(\cdot, \tilde{y})$ is a smooth and bounded velocity field on the compact interval $[0, 2\pi]$ and (4.40) easily follows from (4.38). Thus we obtain regular vortex sheet flows for such Γ. In particular, this is the case for ellipses. The same argument applies to smooth, periodic and conformal curves on the torus $\mathbb{T}^2$.

To start the construction of a subsolution, assume that some regular vortex sheet flow v_0 is given and recall the definition of U by (4.4). We denote the two boundaries of U by

$\Gamma^\pm = \{\tilde{y} = \pm\lambda t\}$. Then the outward normal vectors on $\Gamma^\pm$ pointing into U^c are given by

$$n^\pm = \frac{(\mp w_2, \pm w_1, -\lambda)}{\sqrt{|w|^2 + \lambda^2}}.$$

Outside of U, we choose the subsolution to coincide with the classical stationary solution (v_0, p). The delicate part is to set up a strict subsolution inside of U in order to apply Theorem 2.11. At first, motivated from the choice of a piecewise constant density for the IPM equations in Section 3, we choose

$$\overline{v} = 0. \tag{4.41}$$

It remains to define $\overline{u} = \begin{pmatrix} \overline{\beta} & \overline{\gamma} \\ \overline{\gamma} & -\overline{\beta} \end{pmatrix}$ and $\overline{q}$ such that the two equations $\operatorname{div} \overline{u} + \nabla \overline{q} = 0$ are satisfied. These equations become easier if we decouple them in $\overline{\beta}$ and $\overline{\gamma}$. For this reason we introduce

$$k_1 := \overline{q} - \left(\overline{\beta} - \frac{w_1^2 - w_2^2}{2}\right) \tag{4.42}$$

$$k_2 := -(\overline{\gamma} - w_1 w_2) \tag{4.43}$$

$$k_3 := \overline{q} + \left(\overline{\beta} - \frac{w_1^2 - w_2^2}{2}\right). \tag{4.44}$$

Then it is easy to check that because of $\operatorname{div} w = \operatorname{curl} w = 0$, we obtain

$$\partial_x \overline{\beta} + \partial_y \overline{\gamma} + \partial_x \overline{q} = \partial_x \left(k_3 + \frac{w_1^2 - w_2^2}{2}\right) + \partial_y (w_1 w_2 - k_2) = \operatorname{curl}\,(k_2, k_3)$$

$$\partial_x \overline{\gamma} - \partial_y \overline{\beta} + \partial_y \overline{q} = \partial_y \left(k_1 - \frac{w_1^2 - w_2^2}{2}\right) + \partial_x (w_1 w_2 - k_2) = -\operatorname{curl}\,(k_1, k_2).$$

Thus we arrive at

$$\operatorname{curl}\,(k_1, k_2) = \operatorname{curl}\,(k_2, k_3) = 0 \text{ in } U. \tag{4.45}$$

In addition we have to find an energy density $\overline{e}$ such that $e(\overline{v}, \overline{u}) < \overline{e}$ in U. In view of (4.41) this becomes $\overline{\beta}^2 + \overline{\gamma}^2 < \overline{e}^2$. Using that

$$\overline{\beta} = \frac{k_3 - k_1 + w_1^2 - w_2^2}{2},$$

$$\overline{\gamma} = w_1 w_2 - k_2,$$

$$\overline{q} = \frac{k_1 + k_3}{2},$$

this corresponds to

$$4\overline{e}^2 > (k_3 - k_1 + w_1^2 - w_2^2)^2 + (2k_2 - 2w_1 w_2)^2.$$

Hence, the strictness condition reads

$$\left|(k_3 - k_1, 2k_2) + (w_1^2 - w_2^2, -2w_1w_2)\right| < 2\overline{e} \text{ in } U. \tag{4.46}$$

Furthermore, to obtain weak solutions, which satisfy the local energy inequality, we have to impose (4.34) in U. Thanks to (4.41) we infer

$$\partial_t \overline{e} \leq 0 \text{ in } U. \tag{4.47}$$

We still need to study the jump conditions on $\Gamma^{\pm}$, which arise because of the discontinuity of $\overline{v}$. We start with (4.47). Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\varphi \geq 0$. Then we need to verify

$$\int_{\mathbb{R}^2} \partial_t \varphi \; \overline{e} + \nabla\varphi \cdot \overline{v} \; \overline{q} \; dx \geq 0.$$

We integrate by parts in U and U^c separately and deduce

$$\begin{aligned} 0 \leq &- \int_U \varphi \; \partial_t \overline{e} \; dx - \int_{U^c} \varphi \; \partial_t \left(\frac{|w|^2}{2}\right) + \varphi \text{ div } (Cw) dx \\ &+ \int_{\partial U} \varphi \left(\overline{e} - \frac{|w|^2}{2}\right)(-\lambda) \pm \varphi(0 \mp Cw) \cdot w^\perp dx. \end{aligned}$$

In view of (4.47), this is valid if

$$\overline{e} \leq \frac{|w|^2}{2} \text{ on } \Gamma^{\pm}. \tag{4.48}$$

Finally, we deal with the jump condition of (4.45), namely

$$(-\lambda)(0 \mp 1)w \mp \begin{pmatrix} \overline{\beta} - \frac{w_1^2 - w_2^2}{2} & \overline{\gamma} - w_1w_2 \\ \overline{\gamma} - w_1w_2 & -\overline{\beta} + \frac{w_1^2 - w_2^2}{2} \end{pmatrix} \cdot (w_2, -w_1)^T \mp (\overline{q} - 0)(w_2, -w_1) = 0 \text{ on } \Gamma^{\pm}$$

which is equivalent to

$$0 = \begin{pmatrix} k_2 & k_3 \\ -k_1 & -k_2 \end{pmatrix} \cdot (w_1, w_2)^T - \lambda w \text{ on } \Gamma^{\pm}. \tag{4.49}$$

This means that whenever we find a triple (k_1, k_2, k_3) and an energy function $\overline{e}$ satisfying (4.45)-(4.49), we get a strict subsolution with respect to $\overline{e}$ in U. After these considerations, we establish the following result, being the main ingredient for the proof of Theorem 1.1.

Proposition 4.10. Let v_0 be a regular vortex sheet flow. Then there exists a time $T > 0$ and some $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$ and $0 < t < T$ there exists a strict subsolution $(\overline{v}, \overline{u}, \overline{q})$ with respect to some energy function $\overline{e}$ in U satisfying (4.45)-(4.49). In particular we have $\overline{v} = 0$ in U.

Proof. Obviously, each function $g : \overline{U} \to \mathbb{R}$, $g \in C^2(\overline{U})$ defines a solution (k_1, k_2, k_3) to (4.45) via the identification

$$D^2 g = \begin{pmatrix} k_1 & k_2 \\ k_2 & k_3 \end{pmatrix}.$$

Hence, we are left with constructing functions $g \in C^2(\overline{U})$ and $\overline{e} \in C(\overline{U})$ such that (4.46)-(4.49) are satisfied.

At first we claim that each function $g \in C^2(\overline{U})$ with

$$g(\tilde{x}, \pm\lambda t) = \lambda \int_0^{\tilde{x}} \frac{1}{|w|^2(\tilde{x}', \pm\lambda t)} \left(-w_1(\tilde{x}', \pm\lambda t) \int_0^{\tilde{x}'} \left(\frac{w_2}{|w|^2}\right)(\tilde{x}'', \pm\lambda t) d\tilde{x}'' \right. \\ \left. + w_2(\tilde{x}', \pm\lambda t) \int_0^{\tilde{x}'} \left(\frac{w_1}{|w|^2}\right)(\tilde{x}'', \pm\lambda t) d\tilde{x}'' \right) d\tilde{x}' \ \text{ on } \Gamma^{\pm} \tag{4.50}$$

and

$$\partial_{\tilde{y}} g(\tilde{x}, \pm\lambda t) = \frac{\lambda}{|w|^2(\tilde{x}, \pm\lambda t)} \left(w_2(\tilde{x}, \pm\lambda t) \int_0^{\tilde{x}} \left(\frac{w_2}{|w|^2}\right)(\tilde{x}', \pm\lambda t) d\tilde{x}' \right. \\ \left. + w_1(\tilde{x}, \pm\lambda t) \int_0^{\tilde{x}} \left(\frac{w_1}{|w|^2}\right)(\tilde{x}', \pm\lambda t) d\tilde{x}' \right) \ \text{ on } \Gamma^{\pm} \tag{4.51}$$

already satisfies condition (4.49). Indeed, since $g \in C^2(\overline{U})$ and because of (4.50), we deduce

$$\partial_{\tilde{x}} g(\tilde{x}, \pm\lambda t) = \frac{\lambda}{|w|^2(\tilde{x}, \pm\lambda t)} \left(-w_1(\tilde{x}, \pm\lambda t) \int_0^{\tilde{x}} \left(\frac{w_2}{|w|^2}\right)(\tilde{x}', \pm\lambda t) d\tilde{x}' \right. \\ \left. + w_2(\tilde{x}, \pm\lambda t) \int_0^{\tilde{x}} \left(\frac{w_1}{|w|^2}\right)(\tilde{x}', \pm\lambda t) d\tilde{x}' \right) \ \text{ on } \Gamma^{\pm}. \tag{4.52}$$

Inside of U we can use Lemma 4.5, which together with the continuity of the derivatives of g and (4.51), (4.52) yields

$$\begin{aligned} \partial_y g(\tilde{x}, \pm\lambda t) &= \lim_{\tilde{y} \to \pm\lambda t} \partial_y g(\tilde{x}, \tilde{y}) = \lim_{\tilde{y} \to \pm\lambda t} (w_2 \partial_{\tilde{x}} g + w_1 \partial_{\tilde{y}} g)(\tilde{x}, \tilde{y}) \\ &= w_2(\tilde{x}, \pm\lambda t) \lim_{\tilde{y} \to \pm\lambda t} \partial_{\tilde{x}} g(\tilde{x}, \tilde{y}) + w_1(\tilde{x}, \pm\lambda t) \lim_{\tilde{y} \to \pm\lambda t} \partial_{\tilde{y}} g(\tilde{x}, \tilde{y}) \\ &= \lambda \int_0^{\tilde{x}} \left(\frac{w_1}{|w|^2}\right)(\tilde{x}', \pm\lambda t) d\tilde{x}' \ \text{ on } \Gamma^{\pm} \end{aligned}$$

and analogously

$$\partial_x g(\tilde{x}, \pm\lambda t) = -\lambda \int_0^{\tilde{x}} \left(\frac{w_2}{|w|^2}\right)(\tilde{x}', \pm\lambda t) d\tilde{x}' \ \text{ on } \Gamma^{\pm}.$$

Thus, we conclude

$$|w|^2(\partial_{\tilde{x}} \partial_y g, -\partial_{\tilde{x}} \partial_x g) = \lambda w \ \text{ on } \Gamma^{\pm} \tag{4.53}$$

and as before due to Lemma 4.5 and the continuity of the second derivatives of g, we establish

$$\lambda w = (\nabla(\partial_y g) \cdot w, -\nabla(\partial_x g) \cdot w) \ \text{ on } \Gamma^{\pm},$$

which is equivalent to (4.49). Note that the expressions (4.50)-(4.52) are well-defined because of (4.38)-(4.40).

Similarly, we can write (4.46) in the form

$$\left|P(g) + (w_1^2 - w_2^2, -2w_1w_2)\right| < 2\overline{e} \text{ in } U, \tag{4.54}$$

where

$$\begin{aligned} P(g) :=\Big(& 2\partial_y w_2 \partial_{\tilde{x}} g + 2\partial_x w_2 \partial_{\tilde{y}} g + (w_1^2 - w_2^2)(\partial_{\tilde{y}}^2 g - \partial_{\tilde{x}}^2 g) + 4w_1w_2\partial_{\tilde{x}\tilde{y}}^2 g, \\ & 2\partial_x w_2 \partial_{\tilde{x}} g + 2\partial_y w_2 \partial_{\tilde{y}} g - 2w_1w_2(\partial_{\tilde{y}}^2 g - \partial_{\tilde{x}}^2 g) + 2(w_1^2 - w_2^2)\partial_{\tilde{x}\tilde{y}}^2 g\Big). \end{aligned} \tag{4.55}$$

We construct g and $\overline{e}$ in three steps.

Step 1: Construction of a function $\tilde{g} \in C^2(\overline{U})$ satisfying (4.49) and

$$|P(\tilde{g})| < \frac{7}{8}|w|^2 \text{ on } \Gamma^{\pm}. \tag{4.56}$$

We define $\tilde{g}$ in $\overline{U}$ as follows. In order to satisfy (4.49), we prescribe $\tilde{g}$ and $\partial_{\tilde{y}}\tilde{g}$ on $\Gamma^{\pm}$ by (4.50) and (4.51). Furthermore we require $\partial_{\tilde{y}}^2\tilde{g} = \partial_{\tilde{x}}^2\tilde{g}$ on $\Gamma^{\pm}$, where the right hand side is a given function, i.e. the $\tilde{x}$-derivative of (4.52). Since we do not prescribe any conditions on $\tilde{g}$ inside U, it is always possible to find some $\tilde{g} \in C^2(\overline{U})$ satisfying these boundary conditions. Moreover, since the boundary values are smooth functions, we can choose $\tilde{g}$ also being smooth, which means $\|\tilde{D}^k\tilde{g}\|_{L^\infty(\overline{U})} < \infty$ for each $k \in \mathbb{N}$ and this also includes time derivatives. Next, we investigate inequality (4.56). Thanks to the choice of $\partial_{\tilde{y}}^2\tilde{g}$ on $\Gamma^{\pm}$, we have to verify

$$|(\partial_y w_2 \partial_{\tilde{x}}\tilde{g} + \partial_x w_2 \partial_{\tilde{y}}\tilde{g} + 2w_1w_2\partial_{\tilde{x}\tilde{y}}^2\tilde{g}, \partial_x w_2 \partial_{\tilde{x}}\tilde{g} + \partial_y w_2 \partial_{\tilde{y}}\tilde{g} + (w_1^2 - w_2^2)\partial_{\tilde{x}\tilde{y}}^2\tilde{g})| < \frac{7|w|^2}{16},$$

which holds true if

$$|\nabla w_2|(|\partial_{\tilde{x}}\tilde{g}| + |\partial_{\tilde{y}}\tilde{g}|) + |w|^2|\partial_{\tilde{x}\tilde{y}}^2\tilde{g}| < \frac{7|w|^2}{16} \text{ on } \Gamma^{\pm}. \tag{4.57}$$

This is ensured, provided we have

$$|\partial_{\tilde{x}\tilde{y}}^2\tilde{g}| < \frac{7}{32} \text{ on } \Gamma^{\pm}, \tag{4.58}$$

$$|\nabla w_2|(|\partial_{\tilde{x}}\tilde{g}| + |\partial_{\tilde{y}}\tilde{g}|) < \frac{7m}{32} \text{ on } \Gamma^{\pm}. \tag{4.59}$$

To show these inequalities, we take the $\tilde{x}$-derivative of (4.51) and get

$$\begin{aligned} |\partial_{\tilde{x}\tilde{y}}^2\tilde{g}| &= \left|\partial_{\tilde{x}}\left(\frac{\lambda}{|w|^2}\right)\left(w_2\int_0^{\tilde{x}}\frac{w_2}{|w|^2}d\tilde{x}' + w_1\int_0^{\tilde{x}}\frac{w_1}{|w|^2}d\tilde{x}'\right)\right. \\ &\quad \left. + \frac{\lambda}{|w|^2}\left((|w|^2) + \partial_{\tilde{x}}w_2\int_0^{\tilde{x}}\frac{w_2}{|w|^2}d\tilde{x}' + \partial_{\tilde{x}}w_1\int_0^{\tilde{x}}\frac{w_1}{|w|^2}d\tilde{x}'\right)\right| \\ &= \lambda\left|\partial_{\tilde{x}}\left(\frac{w_2}{|w|^2}\right)\int_0^{\tilde{x}}\frac{w_2}{|w|^2}d\tilde{x}' + \partial_{\tilde{x}}\left(\frac{w_1}{|w|^2}\right)\int_0^{\tilde{x}}\frac{w_1}{|w|^2}d\tilde{x}' + 1\right| \\ &\le C\lambda < \frac{7}{32}, \end{aligned}$$

for some λ small enough and for a positive constant $C > 0$ depending on $m, \|\nabla w\|_{L^\infty(\mathbb{R})}$ and $\|\frac{w_i}{|w|^2}\|_{L^1(\mathbb{R})}, i = 1, 2$. This proves (4.58). Similarly, the equations (4.52) and (4.51) together with (4.40) imply (4.59) for small enough λ.

We emphasize that the strict inequality on ∂U in (4.57) is the essential ingredient for the construction of the strict subsolution and due to the choice of a piecewise constant velocity component $\overline{v}$. On the contrary, for the continuous velocity from the flat case, this argument does not work.

Step 2: Construction of g
We define $g := \tilde{g} - \frac{1}{2}\tilde{y}^2$. Then, this g still satisfies (4.49), since this condition only involves $\tilde{x}$-derivatives of g, see (4.53). Moreover, for small times we deduce

$$|P(g) + (w_1^2 - w_2^2, -2w_1w_2)| = |P(\tilde{g}) - 2\tilde{y}\nabla w_2| < \frac{7}{8}|w|^2 \text{ on } \Gamma^\pm, \tag{4.60}$$

because of (4.56) and since

$$|2\tilde{y}\nabla w_2| \leq 2\lambda t \|\nabla w_2\|_{L^\infty(\mathbb{R})}$$

is arbitrary small for $t \ll 1$.

For later purposes, it will be useful to compute

$$\overline{q} = \frac{1}{2}(\partial_x^2 g + \partial_y^2 g) = \frac{|w|^2}{2}\left(\partial_{\tilde{x}}^2 g + \partial_{\tilde{y}}^2 g\right)$$

and by means of that

$$\begin{aligned}|\nabla\overline{q}| &\leq |w||\nabla w|\|g\|_{\tilde{C}^2(\overline{U})} + \frac{|w|^3}{2}\|g\|_{\tilde{C}^3(\overline{U})} \leq (|w||\nabla w| + |w|^3)(1 + \|\tilde{g}\|_{\tilde{C}^2(\overline{U})} + \|\tilde{g}\|_{\tilde{C}^3(\overline{U})}) \\ &\leq C_1|w|^3,\end{aligned}$$

where the constant C_1 depends only on $m, \|\nabla w\|_{L^\infty(\mathbb{R})}$ and $\|\tilde{g}\|_{\tilde{C}^3(\overline{U})}$.

Step 3: Construction of $\overline{e}$
At first, in $\overline{U}$ we set

$$2\tilde{e} := \frac{1}{16}|w|^2 + |P(g) + (w_1^2 - w_2^2, -2w_1w_2)| = \frac{1}{16}|w|^2 + |P(\tilde{g}) - 2\tilde{y}\nabla w_2|,$$

Then, obviously this $\tilde{e}$ satisfies (4.46) or equivalently (4.54) and also $2\tilde{e} < |w|^2$ on $\Gamma^\pm$, because of (4.60). To deal with (4.47), we estimate

$$|\partial_t\tilde{e}| \leq \frac{1}{2}\partial_t(|P(\tilde{g})|) \leq |\nabla w_2|\partial_t(|\tilde{\nabla}\tilde{g}|) + |w|^2\partial_t(|\partial_{\tilde{x}}^2\tilde{g}| + |\partial_{\tilde{y}}^2\tilde{g}| + |\partial_{\tilde{x}\tilde{y}}^2\tilde{g}|) \leq C_2|w|^2,$$

where $C_2 > 0$ depends only on $m, \|\nabla w\|_{L^\infty(\mathbb{R})}$ and $\|\tilde{g}\|_{\tilde{C}^3(\overline{U})}$. Beyond that, we infer

$$|2\tilde{e}| \leq \frac{1}{16}|w|^2 + 2\|\nabla w\|_{L^\infty(\mathbb{R})}\|\tilde{g}\|_{\tilde{C}^1(\overline{U})} + |w|^2\|\tilde{g}\|_{\tilde{C}^2(\overline{U})} + 2\lambda t\|\nabla w\|_{L^\infty(\mathbb{R})} \leq C_3|w|^2,$$

for some constant C_3 depending only on m, $\|\nabla w\|_{L^\infty(\mathbb{R})}$ and $\|\tilde{g}\|_{\tilde{C}^2(\overline{U})}$. Finally we set

$$\overline{e} := \tilde{e} - C_2 n|w|^2 t$$

with some constant $n > 1$, which we choose large enough to ensure

$$\partial_t \overline{e} + \sqrt{2\tilde{e}}|\nabla \overline{q}| \leq 0 \text{ in } U. \tag{4.61}$$

Indeed, we compute

$$\begin{aligned}\partial_t \overline{e} + \sqrt{2\tilde{e}}|\nabla \overline{q}| &= \partial_t \tilde{e} - C_2 n|w|^2 + \sqrt{2\tilde{e}}|\nabla \overline{q}| \\ &\leq (1-n)C_2|w|^2 + \sqrt{2\tilde{e}}|\nabla \overline{q}| \\ &\leq (1-n)C_2|w|^2 + \sqrt{C_3}C_1|w|^4 \leq 0\end{aligned}$$

for n large enough. In particular, (4.47) holds and since $\tilde{e}$ satisfies (4.46) and (4.48), also $\overline{e}$ does so for small times. This completes the construction of g and $\overline{e}$ satisfying (4.46)-(4.49) and thus we established the existence of a strict subsolution. □

Remark 4.11. Although the flat vortex sheet is not a regular vortex sheet flow, the same result holds true. Indeed, since $w = (1, 0)$ and $(\tilde{x}, \tilde{y}) = (x, y)$, we can construct the strict subsolution simply by defining $g(x, y) = \lambda xy - \frac{1}{2}y^2$ with $0 < \lambda < \frac{1}{4}$. In this case we get

$$\left|P(g) + (w_1^2 - w_2^2, -2w_1 w_2)\right| = 2\lambda.$$

Then, with $2\overline{e} := \frac{1}{4} + 2\lambda - t$, we conclude that (4.45)-(4.49) are satisfied for small times.

The proof of Theorem 1.1 is now an easy consequence.

Proof. Once we have constructed a strict subsolution by Proposition 4.10, we can apply Theorem 2.11 to obtain infinitely many weak solutions (v, p) with initial data v_0 satisfying $\frac{1}{2}|v|^2 = \overline{e}$ and $p = \overline{q} - \frac{1}{2}|v|^2$. It remains to verify that these weak solutions satisfy the local energy inequality

$$\partial_t \overline{e} + \text{div } (v\overline{q}) \leq 0$$

distributionally in $\mathbb{R}^2$. Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\varphi \geq 0$. We have to show

$$\int_{\mathbb{R}^2} \partial_t \varphi \ \overline{e} + \nabla\varphi \cdot v \ \overline{q} \ dx \geq 0.$$

For this, we integrate by parts in U and U^c separately and get

$$\begin{aligned}0 \leq &- \int_U \varphi \ \partial_t \overline{e} + \varphi \ v \cdot \nabla \overline{q} \ dx - \int_{U^c} \varphi \ \partial_t \left(\frac{|w|^2}{2}\right) \pm \varphi \text{ div } (wC) dx \\ &+ \int_{\Gamma^+} \varphi \left(\overline{e} - \frac{|w|^2}{2}\right)(-\lambda) + \varphi(\overline{q}v - Cw) \cdot w^\perp dx \\ &+ \int_{\Gamma^-} \varphi \left(\overline{e} - \frac{|w|^2}{2}\right)(-\lambda) - \varphi(\overline{q}v + Cw) \cdot w^\perp dx.\end{aligned}$$

Clearly, the integral over U^c vanishes, since w is a stationary solution here. Thanks to (4.61), it remains to show that the boundary value integrals are positive. This follows from (4.48) and the fact that $v \cdot w^\perp = w \cdot w^\perp = 0$ on $\Gamma^\pm$ since v is divergence-free. □

5 Approximation of smooth subsolutions by piecewise constant subsolutions

This last section is devoted to the proof of Theorem 1.2. At first we recall a result from [31], which states that the Euler equations can be rewritten as a differential inclusion. For this purpose, let $\Omega \subset \mathbb{R}^2 \times (0,T)$ be an open set in space-time and consider a subsolution $(v,u,q) : \Omega \to \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2} \times \mathbb{R}$. Then we introduce

$$U := \begin{pmatrix} u_{11}+q & u_{12} & v_1 \\ u_{12} & -u_{11}+q & v_2 \\ v_1 & v_2 & 0 \end{pmatrix}. \tag{5.1}$$

Hence, it is immediate to see that the equations

$$\begin{cases} \partial_t v + \text{div } u + \nabla q = 0 \\ \text{div } v = 0 \end{cases}$$

are equivalent to

$$\text{div}_{(x,t)} U = 0.$$

Here, we set $\text{div}_{(x,t)} f = \partial_{x_1} f_1 + \partial_{x_2} f_2 + \partial_t f_3$ for each vector field $f : \mathbb{R}^2 \times (0,T) \to \mathbb{R}^3$. By means of this compact notation, we now establish the existence of a potential for subsolutions. In the following, we identify the coordinate x_3 with the time t.

Lemma 5.1 (Potential for subsolutions)**.**
Consider a C^k subsolution $(v,u,q) : \Omega \to \mathbb{R}^2 \times \mathcal{S}_0^{2\times 2} \times \mathbb{R}, k \geq 1$, on some open set $\Omega \subset \mathbb{R}^2 \times (0,T)$ with the property that $\Omega_{x_3} = \{(y_1,y_2,y_3) \in \Omega : y_3 = x_3\}$ is simply connected for each positive $x_3 > 0$. Then there exists a potential $w \in C^{k+1}(\Omega;\mathbb{R}^3)$ with div $w = 0$ such that

$$U = \begin{pmatrix} \partial_2 w_1 & \frac{1}{2}\partial_2 w_2 - \frac{1}{2}\partial_1 w_1 & \frac{1}{2}\partial_2 w_3 \\ \frac{1}{2}\partial_2 w_2 - \frac{1}{2}\partial_1 w_1 & -\partial_1 w_2 & -\frac{1}{2}\partial_1 w_3 \\ \frac{1}{2}\partial_2 w_3 & -\frac{1}{2}\partial_1 w_3 & 0 \end{pmatrix}. \tag{5.2}$$

On the contrary, each divergence-free potential $w \in C^{k+1}(\Omega;\mathbb{R}^3)$ gives rise to a subsolution via (5.2) and (5.1).

Proof. Notice that $\text{div}_{(x,t)} U = 0$ means that the divergence of each row is zero. Applied to the third row of U we get $\partial_1 v_1 + \partial_2 v_2 = 0$. Since Ω_t is simply connected for each $t > 0$, we conclude the existence of $w_3 : \Omega \to \mathbb{R}$ such that

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\frac{1}{2}\nabla^\perp w_3 = \begin{pmatrix} \frac{1}{2}\partial_2 w_3 \\ -\frac{1}{2}\partial_1 w_3 \end{pmatrix}.$$

From $v \in C^k(\Omega)$ we infer $w_3 \in C^{k+1}(\Omega)$. Hence,

$$U = \begin{pmatrix} u_{11}+q & u_{12} & \frac{1}{2}\partial_2 w_3 \\ u_{12} & -u_{11}+q & -\frac{1}{2}\partial_1 w_3 \\ \frac{1}{2}\partial_2 w_3 & -\frac{1}{2}\partial_1 w_3 & 0 \end{pmatrix}.$$

Evaluating $\operatorname{div}_{(x,t)} U = 0$ for the first two rows gives

$$\partial_1(u_{11}+q)+\partial_2 u_{12}+\frac{1}{2}\partial_3\partial_2 w_3 = 0 \text{ and } \partial_1 u_{12}+\partial_2(-u_{11}+q)-\frac{1}{2}\partial_3\partial_1 w_3 = 0,$$

which is equivalent to

$$\partial_1(u_{11}+q)+\partial_2\left(u_{12}+\frac{1}{2}\partial_3 w_3\right) = 0 \text{ and } \partial_1\left(u_{12}-\frac{1}{2}\partial_3 w_3\right)+\partial_2(-u_{11}+q) = 0.$$

As before we deduce the existence of $w_1, w_2 : \Omega \to \mathbb{R}$ such that

$$\begin{pmatrix} u_{11}+q \\ u_{12}+\frac{1}{2}\partial_3 w_3 \end{pmatrix} = -\nabla^\perp w_1 = \begin{pmatrix} \partial_2 w_1 \\ -\partial_1 w_1 \end{pmatrix},$$
$$\begin{pmatrix} u_{12}-\frac{1}{2}\partial_3 w_3 \\ -u_{11}+q \end{pmatrix} = -\nabla^\perp w_2 = \begin{pmatrix} \partial_2 w_2 \\ -\partial_1 w_2 \end{pmatrix}.$$

Note that since $\partial_3 w_3 \in C^k(\Omega)$, the left sides in both equalities are in $C^k(\Omega)$ and thus we infer $w_1, w_2 \in C^{k+1}(\Omega)$. We immediately obtain $u_{11}+q = \partial_2 w_1$ and $-u_{11}+q = -\partial_1 w_2$. Adding respectively subtracting the remaining two equations yields

$$u_{12} = \frac{1}{2}\partial_2 w_2 - \frac{1}{2}\partial_1 w_1 \quad \text{and} \quad \partial_3 w_3 = -\partial_1 w_1 - \partial_2 w_2.$$

The last equation means that $\operatorname{div}(w_1, w_2, w_3) = 0$ and we observe that U satisfies (5.2). The other direction can be verified by a straightforward computation following the same lines. □

The main ingredient for our approximation result is a variant of Lemma 3.3 from [53]. It states that each $f \in C^1$ with a symmetric gradient (i.e. being curl-free) can be approximated uniformly in the C^1-norm by piecewise affine functions $g \in C^1$ also having a symmetric gradient. The property of g to be C^1 and piecewise affine seems to be delicate at a first look. Nonetheless, such functions can indeed be constructed using an affine decomposition which is not locally finite. The goal is to apply such a result to the potentials of a subsolution from Lemma 5.1. For this purpose, we give a slightly different version of such an approximation, where we address functions with a trace-free gradient (i.e. being divergence-free).

Proposition 5.2.
Let $\Omega \subset \mathbb{R}^3$ be open and bounded, $f \in C^1(\overline{\Omega};\mathbb{R}^3) \cap C^2(\Omega;\mathbb{R}^3)$ with $Df \in \mathbb{M}_0^{3\times 3}$ everywhere in $\overline{\Omega}$. Then, for any lowersemicontinuous function $\varepsilon : \Omega \to (0,\infty)$ there exists a piecewise affine function $g \in C^1(\overline{\Omega})$ with

$$\begin{aligned} &(i) && Dg \in \mathbb{M}_0^{3\times 3}, \\ &(ii) && |f(x)-g(x)| + |Df(x)-Dg(x)| < \varepsilon(x) \quad \forall\, x \in \Omega, \\ &(iii) && f \equiv g \text{ and } Df \equiv Dg \text{ on } \partial\Omega. \end{aligned}$$

Proof. Since the statement is a local one, we can additionally assume that $\varepsilon(x) \equiv \varepsilon > 0$ is constant. Indeed, suppose we can construct some g satisfying $(i)-(iii)$ for constant $\varepsilon > 0$. Let ε be any lowersemicontinuous function and consider some ball $B_r(x) \subset \Omega$. Then we find some piecewise affine $g_{r,x}$ satisfying $(i), (iii)$ and (ii) for $\varepsilon_{r,x} := \min\{\varepsilon(y) : y \in \overline{B_r(x)}\}$. Thanks to an exhaustion argument for Ω, we obtain a Cauchy sequence in C^1, which converges to some $g \in C^1(\overline{\Omega})$ satisfying $(i) - (iii)$.

Furthermore, note that it is enough to prove the following statement: For each $\varepsilon > 0$, there exist $\tilde{f} \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ and an open set G such that

$$\begin{aligned}
&\text{(a)} \quad \overline{G} \subset \Omega,\ |\partial G| = 0,\ |G| > \delta \cdot |\Omega|,\ \text{ for some } 0 < \delta < 1,\\
&\text{(b)} \quad D\tilde{f} \in \mathbb{M}_0^{3\times 3} \ \text{ everywhere in } \Omega,\\
&\text{(c)} \quad \tilde{f}_{|G} \ \text{ is locally affine},\\
&\text{(d)} \quad \|f - \tilde{f}\|_{L^\infty(\Omega)} + \|Df - D\tilde{f}\|_{L^\infty(\Omega)} < \varepsilon,\\
&\text{(e)} \quad f \equiv \tilde{f} \ \text{ and } \ Df \equiv D\tilde{f} \ \text{ on } \partial\Omega.
\end{aligned}$$

Whenever this result holds, we will say that f has an (ε, δ) affine approximation on Ω. The general statement can be derived from this as follows. Take $\delta > 0$ arbitrary. Then construct an $\left(\frac{\varepsilon}{2}, \delta\right)$ affine approximation of f on Ω, which we denote by $\tilde{f}_1$, together with the corresponding set G_1. Set $V_1 := G_1$. In the second step we choose an $\left(\frac{\varepsilon}{4}, \delta\right)$ affine approximation of $\tilde{f}_1$ on $\Omega \setminus \overline{V}_1$. By this we obtain some $\tilde{f}_2 \in C^1(\overline{\Omega \setminus \overline{V}_1})$ and an open set $G_2 \subset \Omega \setminus \overline{V}_1$. Set $V_2 := G_1 \cup G_2$. Note that by defining $\tilde{f}_2 = \tilde{f}_1$ on $\overline{V}_1$, we have $\tilde{f}_2 \in C^1(\overline{\Omega})$ because of property (e). We proceed iteratively in this way, which means that in the n-th step we construct an $(\frac{\varepsilon}{2^n}, \delta)$ affine approximation of $\tilde{f}_{n-1}$ on $\Omega \setminus \overline{V}_{n-1}$. We extend $\tilde{f}_n$ to Ω by $\tilde{f}_n := \tilde{f}_{n-1}$ on $\overline{V}_{n-1}$ and define $V_n := V_{n-1} \cup G_n$. This procedure yields a sequence of functions $(\tilde{f}_n)_n \subset C^1(\overline{\Omega}) \cap C^2(\Omega)$ and open sets $V_n \subset V_{n+1} \subset \ldots \subset \Omega$ such that

$$\begin{aligned}
&\text{(a')} \quad \overline{V}_n \subset \Omega,\ |\partial V_n| = 0 \ \text{ and } \ |\Omega \setminus V_n| < (1-\delta)^n |\Omega|,\\
&\text{(b')} \quad D\tilde{f}_n \in \mathbb{M}_0^{3\times 3} \ \text{ everywhere in } \Omega,\\
&\text{(c')} \quad \tilde{f}_{n+l} \equiv \tilde{f}_n \ \text{ on } V_n \ \text{ for each } l \in \mathbb{N}, l > 0 \ \text{ and is locally affine there},\\
&\text{(d')} \quad \|f - \tilde{f}_n\|_{L^\infty(\Omega)} + \|Df - D\tilde{f}_n\|_{L^\infty(\Omega)} < \varepsilon,\\
&\text{(e')} \quad f \equiv \tilde{f}_n \ \text{ and } \ Df \equiv D\tilde{f}_n \ \text{ on } \partial\Omega.
\end{aligned}$$

Due to properties (a'),(c') and (d'), the sequence $(\tilde{f}_n)_n$ is Cauchy in the C^1-norm. We conclude that $g = \lim_{n\to\infty} \tilde{f}_n \in C^1(\overline{\Omega})$ satisfies $Dg \in \mathbb{M}_0^{3\times 3}$ as well as the right approximation properties $(ii), (iii)$ of f. Moreover, g is piecewise affine, since g is locally affine on $\bigcup_{n\in\mathbb{N}} V_n$, which is a set of full measure.

Hence, it remains to show the basic construction, that is, constructing $\tilde{f}$ and G satisfying (a)-(e). At first, we choose a smooth function $\psi : \mathbb{R}^3 \to [0,1]$ with

$$\psi(x) = \begin{cases} 0, & |x| < \frac{3}{4} \\ 1, & |x| > \frac{4}{5} \end{cases}$$

and set $c_1 := \|D^2\psi\|_\infty + \|D\psi\|_\infty > 1$. In the next step we determine finitely many disjoint closed balls $B_i = B_{r_i}(x_i) \subset \Omega, i = 1, \ldots, n$ such that

$$\left|\bigcup_{i=1}^{n} B_i\right| > \frac{64}{27}\delta|\Omega|. \tag{5.3}$$

To this end, let $y \in \Omega$ and $0 < r < \frac{1}{2}\text{dist}(y, \partial\Omega)$. Since $f \in C^2(\Omega)$, we define

$$M(r) := \tilde{C}r \sup_{z \in B_{2r}(y)} |D^2 f(z)| < \infty,$$

where $\tilde{C}$ is a constant, which will be specified later. Obviously, $M(r) \to 0$ as $r \to 0$, thus for each $y \in \Omega$ there exist some $r = r(y) \in (0, 1)$ such that

$$\tilde{C}r\|D^2 f\|_{L^\infty(B_{2r}(y))} < \frac{\varepsilon}{4}. \tag{5.4}$$

Let $A \subset\subset \Omega$ with $|A| > N_B \frac{64}{27}\delta|\Omega|$, the constant N_B will be specified below. Define the family of balls $\mathcal{B} = \{B_r(y) : y \in \Omega\}$ with r like in (5.4). Then, for each $a \in A$ there exists some $r > 0$ such that $B_r(a) \in \mathcal{B}$, i.e. $\mathcal{B}$ is a Besicovitch cover of A. Thanks to Besicovitch's covering theorem (see for example [44], p.30 ff.), we infer the existence of a constant N_B, depending only on the dimension 3, and N_B pairwise disjoint families $\mathcal{B}_1, \ldots, \mathcal{B}_{N_B} \subset \mathcal{B}$ which cover the set A, that is,

$$A \subset \bigcup_{j=1}^{N_B} \bigcup_{B_i \in \mathcal{B}_j} B_i.$$

Hence, at least one $\mathcal{B}_j$ consists out of disjoint closed balls $B_i = B_{r_i}(x_i)$ satisfying

$$\bigcup_{B_i \in \mathcal{B}_j} |B_i| > \frac{64}{27}\delta|\Omega|$$

and such that for each (x_i, r_i) (5.4) is valid. In particular, there exists a finite subset of disjoint balls $B_i \in \mathcal{B}_j, i = 1, \ldots, n$ satisfying (5.3). Beyond that, we have the inequality

$$\text{osc}_{B_i} Df \le \sup_{x \neq y} |x - y| \frac{|Df(x) - Df(y)|}{|x - y|} \le r_i \|D^2 f\|_{L^\infty(B_i)}. \tag{5.5}$$

In the next step set $A_i := Df(x_i) \in \mathbb{M}_0^{3\times 3}$ for each $i = 1, \ldots, n$. Obviously, this implies div $(f(x) - A_i \cdot x) = 0$. Therefore, we investigate the following problem on B_i

$$\begin{cases} \text{curl } F_i(x) = h_i(x) & \text{in } B_i \\ \text{div } F_i(x) = 0 & \text{in } B_i \end{cases}, \tag{5.6}$$

where we set $h_i(x) := f(x) - A_i x - f(x_i) + A_i x_i$. We set up a solution of (5.6) as follows. First choose some smooth cut-off function $\chi_i \in C_c^\infty(\mathbb{R}^3)$ such that

$$\chi_i(x) = \begin{cases} 1 & , x \in B_i = B_{r_i}(x_i) \\ 0 & , x \in \mathbb{R}^3 \setminus B_{2r_i}(x_i) \end{cases}.$$

Note $\|\chi_i\|_{C^k(\mathbb{R}^3)} \leq C r_i^{-k}$ for $k = 1, 2$. Set $\tilde{h}_i(x) := \chi_i(x) \cdot h_i(x)$. Then $\tilde{h}_i \in C_c^2(\mathbb{R}^3)$. By $u_i : \mathbb{R}^3 \to \mathbb{R}^3$ we denote the solution of the problem

$$\begin{cases} \text{curl } u_i(x) = \tilde{h}_i(x) & \text{in } \mathbb{R}^3 \\ \text{div } u_i(x) = 0 & \text{in } \mathbb{R}^3 \\ |u_i(x)| \to 0 & \text{as } |x| \to \infty \end{cases}. \tag{5.7}$$

Since $\tilde{h}_i$ has compact support, the solution is given by

$$u_i(x) = \int_{\mathbb{R}^3} \phi(x-y) \cdot \text{curl } \tilde{h}_i(y) dy,$$

where $\phi(x) = \frac{1}{4\pi|x|}$ is the Newton potential in three dimensions.

Finally we define $F_i : B_i \to \mathbb{R}^3$ by

$$F_i(x) := u_i(x) - u_i(x_i + r_i e_3).$$

One easily checks that F_i solves (5.6). Set $\tilde{F}_i(x) = \psi(\frac{x-x_i}{r_i}) F_i(x)$.

Obviously, $F_i(x_i + r_i e_3) = 0$, which implies for any $x \in B_i$

$$|F_i(x)| = |F_i(x) - F_i(x_i + r_i e_3)| \leq |x - (x_i + r_i e_3)| \|DF_i\|_{L^\infty(B_i)} \leq 2r_i \|DF_i\|_{L^\infty(B_i)}.$$

Thus, we get $\|F_i\|_{L^\infty(B_i)} \leq 2r_i \|DF_i\|_{L^\infty(B_i)}$. Moreover, thanks to curl $F_i(x_i) = 0$, we obtain analogously

$$\|h_i\|_{L^\infty(B_i)} = \|\text{curl } F_i\|_{L^\infty(B_i)} \leq r_i \|Df - A_i\|_{L^\infty(B_i)}.$$

Due to these computations, we deduce

$$\begin{aligned} \|D(\text{curl } \tilde{F}_i)\|_{L^\infty(B_i)} \leq & \frac{c_1}{r_i^2} \|F_i\|_{L^\infty(B_i)} + \frac{c_1}{r_i} \Big(\|DF_i\|_{L^\infty(B_i)} + \|\text{curl } F_i\|_{L^\infty(B_i)} \Big) \\ & + \|D(\text{curl } F_i)\|_{L^\infty(B_i)} \\ \leq & \frac{3c_1}{r_i} \|DF_i\|_{L^\infty(B_i)} + 2c_1 \|Df - A_i\|_{L^\infty(B_i)}. \end{aligned} \tag{5.8}$$

To estimate the L^∞-norm of DF_i, we use the definition of u_i. Let $x \in B_i$. Then

$$u_i(x) = \int_{\mathbb{R}^3} \phi(y) \cdot \text{curl } \tilde{h}_i(x-y) dy = \int_{B_{2r_i}(x-x_i)} \phi(y) \cdot \text{curl } \tilde{h}_i(x-y) dy$$

and we infer

$$\begin{aligned} |u_i(x)| &\leq \|D\tilde{h}_i\|_{L^\infty(B_{2r_i}(x_i))} \int_{B_{3r_i}(0)} |\phi(y)| dy \\ &\leq C r_i^2 \left(\|Df - A_i\|_{L^\infty(B_{2r_i}(x_i))} + \frac{C}{r_i} \|h_i\|_{L^\infty(B_{2r_i}(x_i))} \right) \\ &\leq C r_i^2 \|Df - A_i\|_{L^\infty(B_{2r_i}(x_i))} \leq C r_i^3 \|D^2 f\|_{L^\infty(B_{2r_i}(x_i))}. \end{aligned}$$

To deal with the C^1-norm of u_i, we argue as in [46], p.23, 24. Let $j \in \{1,2,3\}$. Then

$$\frac{\partial u_i}{\partial x_j}(x) = \int_{B_{2r_i}(x-x_i)} \phi(y) \cdot \left(\frac{\partial}{\partial x_j} \left(\text{curl } \tilde{h}_i \right) \right) (x-y) dy.$$

Analogously as for $|u_i|$ we estimate

$$\begin{aligned}
\left| \frac{\partial u_i}{\partial x_j}(x) \right| &\leq \|D^2 \tilde{h}_i\|_{L^\infty(B_{2r_i}(x_i))} \int\limits_{B_{3r_i}(0)} |\phi(y)| dy \\
&\leq C r_i^2 \left(\|D^2 f\|_{L^\infty(B_{2r_i}(x_i))} + \frac{1}{r_i} \|Df - A_i\|_{L^\infty(B_{2r_i}(x_i))} + \frac{1}{r_i^2} \|h_i\|_{L^\infty(B_{2r_i}(x_i))} \right) \\
&\leq C r_i^2 \|D^2 f\|_{L^\infty(B_{2r_i}(x_i))}.
\end{aligned}$$

This yields

$$\|DF_i\|_{L^\infty(B_i)} = \|Du_i\|_{L^\infty(B_i)} \leq C r_i^2 \|D^2 f\|_{L^\infty(B_{2r_i}(x_i))}$$

and thanks to (5.8), (5.4) we observe

$$\begin{aligned}
\|D(\text{curl } \tilde{F}_i)\|_{L^\infty(B_i)} &\leq 3Cc_1 r_i \|D^2 f\|_{L^\infty(B_{2r_i}(x_i))} + 2c_1 r_i \|D^2 f\|_{L^\infty(B_{2r_i}(x_i))} \\
&\leq (2+3C) c_1 r_i \|D^2 f\|_{L^\infty(B_{2r_i}(x_i))} < \frac{\varepsilon}{4}.
\end{aligned}$$

Note that the constant $\tilde{C}$ from (5.4) is chosen as $(2+3C)c_1$, which is independent of r_i.

Finally, we define $\tilde{f}_i := \text{curl } \tilde{F}_i + A_i x + f(x_i) - A_i x_i \in C^2(B_i)$. Then, for $x \in B_i$ we have

$$\tilde{f}_i(x) = \begin{cases} A_i x + f(x_i) - A_i x_i, & |x - x_i| < \frac{3r_i}{4} \\ f(x), & |x - x_i| > \frac{4r_i}{5} \end{cases}.$$

In particular, $\tilde{f}_i$ is affine for $|x - x_i| < \frac{3r_i}{4}$. Furthermore,

$$\|D\tilde{f}_i - A_i\|_{L^\infty(B_i)} = \|D(\text{curl } \tilde{F}_i)\|_{L^\infty(B_i)} \leq \frac{\varepsilon}{4}$$

and this implies together with (5.4) and (5.5) that $\|D\tilde{f}_i - Df\|_{L^\infty(B_i)} \leq \frac{\varepsilon}{2}$. Using that $r_i < 1$ and $\tilde{f}_i(x_i) - f(x_i) = 0$, we conclude

$$\|\tilde{f}_i - f\|_{L^\infty(B_i)} \leq r_i \|D\tilde{f}_i - Df\|_{L^\infty(B_i)} < \frac{\varepsilon}{2}.$$

Hence, defining

$$G = \bigcup_{i=1}^n B_{\frac{3}{4}r_i} \quad \text{and} \quad \tilde{f}(x) = \begin{cases} \tilde{f}_i(x), & x \in B_i \\ f(x), & \text{else} \end{cases},$$

we observe that the properties (a)-(e) are satisfied. This completes the proof. □

The proof of Theorem 1.2 is now a consequence of Lemma 5.1 and Proposition 5.2.

Proof. Because of Lemma 5.1 there exists a divergence-free potential $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ for (v, u, q). Let $\varepsilon > 0$. Thanks to Proposition 5.2, w can be approximated by some piecewise affine and divergence-free $g \in C^1(\overline{\Omega})$. Hence, Lemma 5.1 yields a piecewise constant and continuous subsolution

$$(\tilde{v}, \tilde{u}, \tilde{q}) = \left(\begin{pmatrix} \frac{1}{2}\partial_2 g_3 \\ -\frac{1}{2}\partial_1 g_3 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}(\partial_2 g_1 + \partial_1 g_2) & \frac{1}{2}(\partial_2 g_2 - \partial_1 g_1) \\ \frac{1}{2}(\partial_2 g_2 - \partial_1 g_1) & -\frac{1}{2}(\partial_2 g_1 + \partial_1 g_2) \end{pmatrix}, \frac{1}{2}(\partial_2 g_1 - \partial_1 g_2) \right)$$

in the distributional sense. Moreover, due to $\|Dw - Dg\|_{L^\infty(\Omega)} \leq \varepsilon$, we conclude that $\|(v, u, q) - (\tilde{v}, \tilde{u}, \tilde{q})\|_{L^\infty(\Omega)} \leq \varepsilon$. In particular, the functions coincide on the boundary, that is, we have $(v, u, q) \equiv (\tilde{v}, \tilde{u}, \tilde{q})$ on $\partial\Omega$. □

Remark 5.3. To keep calculations simple, we studied only the spacial two-dimensional case. Beyond that, one can analogously derive a version of Proposition 5.2 in dimension $n \geq 2$. Furthermore, Lemma 3.4 from [31] states that each skew-symmetric tensor field $E_{ij}^{kl} \in C^\infty(\mathbb{R}^{n+1})$ gives rise to a subsolution. However, it is not clear, whether each subsolution can be represented in such a way and how to apply Proposition 5.2 afterwards to such a potential. From this perspective, an extension of Theorem 1.2 to higher dimensions needs more subtle arguments.

6 Open problems

- The main part of this thesis addresses the construction of a strict subsolution having piecewise constant components. Therefore one could try to extend this method to other equations in fluid dynamics for which the convex integration method is implemented. The goal is to characterize initial data showing a corresponding wild behaviour. A natural candidate to investigate are active scalar equations

$$\begin{cases} \partial_t \theta + v \cdot \nabla_x \theta = 0, \\ \operatorname{div} v = 0, \\ v = T[\theta], \end{cases}$$

  where T is a suitable integral operator. Choosing T as the Biot-Savart kernel, we obtain the IPM equations. Another well-known example are the surface quasi geostrophic equations, where $T = \nabla^{\perp}(-\Delta)^{-\frac{1}{2}}$.

  Additionally, a first result in the sense of Theorem 1.1 for the compressible Euler equations was already derived in [19], however this is merely done for the simple flat case.

- Characterize in more detail the set of regular vortex sheet flows and give more examples apart from ellipses. Therefore one should determine some geometric conditions or regularity assumptions on a curve Γ, which ensure that it is conformal and such that the corresponding velocity field w satisfies the conditions (4.38)-(4.40). Perhaps, there is a way to resign on (4.38), which could yield regular vortex sheet flows along non-smooth curves. It would be useful to gain insights in the behaviour of the turbulent zone of a strict subsolution locally around stagnation points, such as for flows passing an obstacle.

- An application of Theorem 1.1 excludes vortex sheet initial data of the form (1.15). To construct strict subsolutions for such initial data, a suitable ansatz should take the Birkhoff-Rott equation (1.18) into account. Since these flows are not a stationary solution of the Euler equations, we propose to use time-dependent flow coordinates, which do not arise in the easier case of stationary initial data. Furthermore, leaving the notion of conformal curves, which bases on holomorphic functions, could help to accomplish analogous results in dimensions greater than two.

- In the way we imposed the notion of admissibility for the IPM equations, it is not clear if this notion leads to a corresponding statement of weak-strong uniqueness as for the Euler equations. Thus, it would be interesting to prove such a result or if necessary to introduce a suitable adjusted definition of admissibility.

- A further task is to determine suitable selection criteria for weak solutions. By means of the presence of wild initial data we infer that admissibility is not sufficient for uniqueness. Thus, we search for different conditions, such as the maximal dissipation rate. In [77], Székelyhidi analysed the flat vortex sheet and inferred that any weak solution, which arises from the subsolution he constructed, has a maximal possible energy dissipation of

$$\frac{dE}{dt} = -\frac{1}{6}.$$

The question arises, if we can determine similar maximal dissipation rates for general conformal curves $\Gamma \subset \mathbb{R}^2$ of regular vortex sheets and if there are weak solutions, which indeed attain these rates.

In case of the IPM equations, the maximal propagation speed of the turbulent zone, we can reach, is given by 2. Surprisingly, this is exactly the expansion rate of the solution constructed by Otto in [67]. One should look for a rigorous statement that connects these two solutions, which have been derived completely different from each other.

- Due to the Kelvin-Helmholtz instability and the ill-posed character of the Birkhoff-Rott equation, various types of generalized models for the Euler equations have been proposed in the literature to deal with vortex sheet initial data. In [42], Duchon and Robert introduced the foliated Euler system, which describes a two-phase flow. This system indeed admits continuous solutions for some vortex sheet initial data and additionally yields a linearly well-posed problem for the description of the Rayleigh-Taylor instability, i.e. where a heavier fluid is superposed over a lighter one with a horizontal interface. In fact, solutions of the foliated euler system are merely special measure-valued solutions. Furthermore, Brenier introduced the homogenized vortex sheet equations in [9], which can be viewed as a further generalization of the foliated Euler flow from Duchon and Robert. It would be worth to look for possible connections with strict subsolutions, in the sense that these generalized models give insights from a broader perspective.

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