Diplomarbeit

On the time-analytic behavior of particle trajectories in an ideal and incompressible fluid flow

[revised version - December 5, 2017]

Universität Leipzig
Fakultät für Mathematik und Informatik
Mathematisches Institut

Verfasser:
Tobias Hertel

Betreuender Hochschullehrer:
Professor Dr. László Székelyhidi

Submission date: July 14, 2016

Abstract

This (Diplom-) thesis deals with the particle trajectories of an incompressible and ideal fluid flow in $n \geq 2$ dimensions. It presents a complete and detailed proof of the surprising fact that the trajectories of a smooth solution of the incompressible Euler equations are locally analytic in time. In following the approach of P. Serfati, a complex ordinary differential equation (ODE) is investigated which can be seen as a complex extension of a partial differential equation, which is solved by the trajectories. The right hand side of this ODE is in fact given by a singular integral operator which coincides with the pressure gradient along the trajectories. Eventually, we may apply the Cauchy-Lipschitz existence theorem involving holomorphic maps between complex Banach spaces in order to get a unique solution for the above mentioned ODE. This solution is real-analytic in time and coincides with the particle trajectories.

Declaration

I hereby declare that this thesis is my own work and effort and that it has not been submitted anywhere for any award. Where other sources of information have been used, they have been acknowledged.

Leipzig, July 14, 2016 Tobias Hertel

Contents

1. Introduction

The center of interest in this thesis is the behavior of a given particle in an incompressible and ideal fluid flow. More precisely, the focus lies on the particle trajectories related to a velocity field which solves the n-dimensional incompressible Euler equations which are given by

$$\begin{aligned} \partial_t u(x,t) + (u \cdot \nabla)\, u\,(x,t) &= -\nabla p(x,t) \\ \operatorname{div} u(x,t) &= 0 \\ u(x,0) &= u_0(x)\,. \end{aligned} \tag{E}$$

Here $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ and $p : \mathbb{R}^n \to \mathbb{R}$ denote the velocity and pressure respectively in each point $x \in \mathbb{R}^n$ at any time $t \in \mathbb{R}_+$. The particle trajectories i.e. the characteristic curves in the representation of Lagrange are solutions of the equation

$$\begin{aligned} &X_t(x,t) = u(X(x,t),t) \\ &X(x,0) = x\,. \end{aligned} \tag{U}$$

The aim of this thesis is to give a proof of the surprising fact that in spite of a low regularity of a solution to the n-dimensional Euler equations, the corresponding particle trajectories are locally analytic in time as long as the solution u exists. The result is surprising since one would expect that any solution u of (E) has finite time-regularity only, because the Euler equations contain only one time derivative. But along the trajectories $u(X(x,t),t)$ must be time-analytic too, because X is analytic in time and (U) holds.
After J.-Y. Chemin had proved the C^∞ time-regularity for the trajectories of a $C^{1,\alpha}$ initial velocity field u_0 (see [Che92] and also [Che98, p. 150] for $u_0 \in C^{m,\alpha}$, $m \geq 1$), P. Serfati showed that the particle trajectories which correspond to a solution of (E) with a $C^{m,\alpha}$ initial velocity field are even analytic in time, [Ser92; Ser95]. Although the proof was terse and left out many details, it was widely accepted to be mathematically sound. However, the claimed fact of time-analytic trajectories was again proved by P.Gamblin in 1993 and later by A. Shnirelman [Shn12] as well as by U. Frisch and V. Zheligovsky [FZ14] (in differing function spaces on the 3D torus). This thesis presents a complete and detailed proof of the time-analytic behavior of the particle trajectories corresponding to a $C^{m,\alpha}$-solution of (E) in the whole space of $n \geq 2$ dimensions. It is guided by the ideas and methods introduced by Philippe Serfati in [Ser95].

The structure of this thesis is as follows. A short notation overview is given in chapter two, whereas the proceedings and the main theorem are stated in chapter three. The fourth chapter serves to introduce the general theory concerning Banach-space valued holomorphy. In the fifth chapter we derive and investigate a complex ODE fulfilled by the particle trajectories and finally solve this equation in chapter six. Chapter seven gives the proof of the main theorem and some additional remarks.

2. Notation and Definitions

This chapter explains the notation of the most important concepts used throughout this document. A full list of symbols can be found on page 61.

Let in the following be $m \in \mathbb{N}_0,\ n \in \mathbb{N}$ and $\alpha \in (0,1)$.

- $\mathrm{L}^\infty\left(q(x), \mathbb{R}^n; \mathbb{C}\right)$ denotes the space of complex valued functions f which admit

$$\operatorname{ess\,sup}\ \|q{\cdot}f\|_{\mathbb{C}} < +\infty\,.$$

  $L^\infty(\mathbb{R}^n)^m := L^\infty(1, \mathbb{R}^n; \mathbb{C}^m)$, where $m \in \mathbb{N}$ (omitted if $m = 1$).

- $\boldsymbol{k}, \mu \in \mathbb{N}_0^n; \boldsymbol{r} \in \mathbb{R}_+^n$ usually denote multi-indices; resp. multi-radii for which we set the common notation: for $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$ set $|\boldsymbol{k}| := k_1 + \ldots k_n$, $\boldsymbol{k}! = k_1! \cdots k_n!$, $D_x^{\boldsymbol{k}} = \partial_{x_1}^{k_1} \partial_{x_2}^{k_2} \cdots \partial_{x_n}^{k_n}$, $a^{\boldsymbol{k}} := a_1^{k_1} \cdots a_n^{k_n}$ for $a \in \mathbb{C}^n$.

- $C^\alpha(\mathbb{R}^n) = C^\alpha(\mathbb{R}^n; \mathbb{C})$ is the space of (uniformly) α-Hölder continuous functions f which admit

$$\|f\|_\alpha := \|f\|_{C^\alpha(\mathbb{R}^n)} := \|f\|_\infty + [f]_\alpha := \sup_{x \in \mathbb{R}^n} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|^\alpha} < +\infty\,.$$

  Remark. For $f \in C^\alpha(\mathbb{R}^n)$ it is equivalent to say that f is continuous and $f \in L^\infty$ as well as $\frac{f(x) - f(x')}{\|x - x'\|^\alpha} \in L^\infty(\mathbb{R}_x^n \times \mathbb{R}_{x'}^n)$.

- $C^m(\mathbb{R}^n) = C^m(\mathbb{R}^n; \mathbb{C})$ denotes the space of functions $\mathbb{R}^n \to \mathbb{C}$ which are up to the m-th order continuously differentiable.

- $C_b(\mathbb{R}^n) = C_b(\mathbb{R}^n; \mathbb{C}); C_b^m(\mathbb{R}^n)$ is the Banach space of bounded, continuous functions $\mathbb{R}^n \to \mathbb{C}$; resp. the space of m-times continuously differentiable, norm-bounded functions, where the norm is given by

$$\|\,.\,\|_{C_b^m(\mathbb{R}^n)} = \sum_{0 \leq |\boldsymbol{k}| \leq m} \left\| D^{\boldsymbol{k}}(\,.\,) \right\|_{L^\infty}\,.$$

- $C^{m,\alpha}(\mathbb{R}^n) = C^{m,\alpha}(\mathbb{R}^n; \mathbb{C})$ is the subspace of $C_b^m(\mathbb{R}^n)$ whose functions admit α-Hölder continuous m-th order partial derivatives and are bounded in the appropriate norm. This norm (sometimes written as $\|\,.\,\|_{m,\alpha}$) is given by

$$\|\,.\,\|_{C^{m,\alpha}(\mathbb{R}^n)} = \sum_{0 \leq |\boldsymbol{k}| \leq m} \left\| D^{\boldsymbol{k}}(\,.\,) \right\|_{L^\infty} + \sum_{|\boldsymbol{k}| = m} \left[D^{\boldsymbol{k}}(\,.\,) \right]_\alpha\,.$$

Remark. This function space is for this proof in the center of interest and we would like to point out the following. It is equivalent to say, that $C^{m,\alpha}(\mathbb{R}^n)$ consists of up to the order m continuously differentiable functions f whose partial derivatives belong to $L^\infty(\mathbb{R}^n)$ and $\frac{D^{\boldsymbol{k}} f(x) - D^{\boldsymbol{k}} f(x')}{\|x-x'\|^\alpha} \in L^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_{x'})$ for $|\boldsymbol{k}| = m$.

- $H(O, E)$, for an open subset $O \subset \mathbb{C}^n$ and a complex Banach space E, denotes the space of holomorphic maps with values in E which are continuous on $\overline{O}$. See section 4.1 for further explications, especially for the case $E = C^{m,\alpha}(\mathbb{R}^n)$.
- JX denotes the Jacobi matrix of a vector field $X \in C^1(\mathbb{R}^n)^n$,i.e. $JX = DX$.
- $|JX| := \det(JX)$ is the Jacobi determinant of $X \in C^1(\mathbb{R}^n)^n$.
- $\varGamma$ denotes the fundamental solution of the Laplacian (i.e. the Newtonian potential), which is given by

$$\varGamma(x) = \begin{cases} C_n \frac{1}{\|x\|^{n-2}} & \text{, if } x \in \mathbb{R}^n \setminus \{0\} \text{ with } n \geq 3, \\ C_2 \log \|x\| & \text{, if } x \in \mathbb{R}^2 \setminus \{0\} \end{cases}$$
$$\varGamma(0) = 0,$$

where C_2, C_n are constants. Its derivatives are bounded by

$$|D^\mu \varGamma(x)| \leq C \frac{1}{\|x\|^{n-2+|\mu|}} \quad \text{, for } \mu \in \mathbb{N}_0^n \text{ and a constant } C > 0$$

(see for example [GT83, p. 17]).

- $\|\,.\,\|$ usually denotes the norm in $\mathbb{C}^n$, i.e. for $z \in \mathbb{C}^n$

$$\|z\| = \left(\sum_{i=1}^n z_i \bar{z}_i \right)^{1/2} .$$

3. Statement of the main theorem and challenges

We depart from a weak solution $u \in L^{\infty}_{loc}([0,T); C^{m,\alpha}(\mathbb{R}^n, \mathbb{R}^n))$ of the incompressible Euler equations in the whole space of $n \geq 2$ dimensions and a finite time interval $[0,T)$, $T > 0$. The vector field $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}^n$ denotes the velocity field of a fluid with a given initial velocity field $u(0) = u_0 \in C^{m,\alpha}(\mathbb{R}^n; \mathbb{R}^n)$, $m \geq 1$, $\alpha \in (0,1)$. If u solves (E), then the pressure p is in a later defined way uniquely associated to u by

$$-\Delta p = \mathrm{Tr}\,(Ju)^2 = \sum_{i,j=1}^{n} \partial_i \partial_j (u_i u_j)\,.$$

A solution to

$$\begin{aligned} X_t(x,t) &= u(X(x,t),t) \\ X(x,0) &= x, \end{aligned} \tag{U}$$

for $(x,t) \in \mathbb{R}^n \times [0,T)$, is a time-evolving vector field which assigns each point $x \in \mathbb{R}^n$ a curve parametrized by t, therefore the expression particle trajectories or characteristic curves is used. The unique existence of a solution to (U) is ensured by the Cauchy-Lipschitz existence theorem, if u is continuous and bounded in $(x,t) \in \mathbb{R}^n \times [0,T)$ as well as Lipschitz-continuous in $x \in \mathbb{R}^n$. Our main theorem then reads:

Theorem 3.1. *Let u solve the n-dimensional, incompressible Euler equations* (E) *with an initial velocity field $u_0 \in C^{m,\alpha}(\mathbb{R}^n; \mathbb{R}^n)$ ($m \geq 1$, $\alpha \in (0,1)$). Then the solution X of* (U) *is locally analytic in time as long as the solution exists.*

The proof, given in section 7.1, is guided by the following strategy. In section 5.2 it is shown that the solution of (U) also solves the partial differential equation

$$\begin{aligned} X_{tt}(x,t) &= -(\nabla p)(X(x,t),t) \quad , (x,t) \in \mathbb{R}^n \times [0,T) \\ (X, X_t)(x,0) &= (x, u_0(x))\,. \end{aligned}$$

The right hand side, namely the gradient of p, is given by a sum of singular integrals and coincides with $G = G(X, X_t)(x,t)$ for the solution X of (U), where $(x,t) \in \mathbb{R}^n \times [0,T)$. G is a well defined and bounded map $H(D_R, \Omega) \to H(D_R, C^{m,\alpha}(\mathbb{R}^n; \mathbb{C}^n))$, $R > 0$, where $\Omega \subset E$ is an open and bounded subset of the in section 6.1 in greater detail defined complex Banach space E. This operator is given by

$$\begin{aligned} G(X,Y)(x,t) = &\int_{\mathbb{R}^n} \nabla(a\Gamma)(X(x,t) - X(y,t))\,\mathrm{Tr}\left(JY \cdot \mathrm{cof}\,(JX)^{\top}\right)^2(y,t)|JX|(y,t)\,\mathrm{d}y \\ &+ \sum_{i,j} \int_{\mathbb{R}^n} \partial_i \partial_j \nabla((1-a)\Gamma)(X(x,t) - X(y,t))(Y_i Y_j)(y,t)|JX|(y,t)\,\mathrm{d}y \end{aligned}$$

for $X, Y : \mathbb{R}^n \times D_T \to \mathbb{C}^n$, where $D_T = \{z \in \mathbb{C} \mid |z| < T\}, T > 0$.

Eventually, in the proof of Theorem 6.1, the existence of $\hat{T} > 0$ is shown such that the complex ODE

$$\begin{aligned} \hat{X}''(\tau) &= G(\hat{X}, \hat{X}')(\tau) \quad , \tau \in D_{\hat{T}} \\ (\hat{X}, \hat{X}')(0) &= (Id, u_0), \end{aligned}$$

with $\hat{X}' = \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{X}$, admits a unique solution in

$$H\left(D_{\hat{T}}, \left\{f \in \mathrm{L}^\infty\left((1 + \|x\|)^{-1}, \mathbb{R}^n; \mathbb{C}^n\right) \mid JX \in C^{m-1,\alpha}(\mathbb{R}^n; \mathbb{C}^{n\times n})\right\}\right) .$$

This is the key ingredient for the proof of our main theorem, because this solution is real valued for $\tau \in (-\hat{T}, \hat{T})$ and coincides with the solution of (U) on $[0, \hat{T})$. The fulfilled PDEs from above also imply the following result.

Corollary 3.1. *Local time-analyticity holds for u and ∇p along the trajectories as long as the solution exists.*

4. Banach space valued holomorphic maps

This chapter introduces the needed results for vector valued holomorphic maps. We will see that the theory of multi-dimensional holomorphy will extend directly to Banach space valued maps and so Cauchy's integral formula and the complex series expansion will also hold for these maps. A development of this theory set in an even more generalized frame of sequentially complete and locally convex vector spaces can be found for example in [Her89, ch. 1-3], from which certain proofs were adapted.

4.1. Cauchy's formula and Taylor series expansion

First we want to define holomorphic maps between subsets of $\mathbb{C}^n$ and complex Banach spaces in order to approach differentiability and analyticity of vector valued functions. Later, these properties will lead to interesting facts about operators between complex Banach spaces and will finally motivate assertions about the solvability of complex ordinary differential equations.

Definition 4.1 (holomorphic functions)**.** Let E be a complex Banach space and $O \subseteq \mathbb{C}^n$ open. Then $f : O \to E$ is said to be holomorphic, if it is continuous in O and if for any $z \in O$ there exists in E

$$\lim_{\substack{\tilde{z}\to 0\\ \tilde{z}\in\mathbb{C}\setminus\{0\}}} \frac{f(z+\tilde{z}e_i)-f(z)}{\tilde{z}} \quad , \forall i = 1,\dots,n\,.$$

Here, $(e_i)_{i=1}^n$ denotes the canonical base in $\mathbb{R}^n$. We then call the above limit the first order partial derivative in the variable z_i of f in z and write $\partial_{z_i} f(z)$. Higher partial derivatives will be denoted by $D^{\boldsymbol{k}} f := \partial_{z_1}^{k_1} \cdots \partial_{z_n}^{k_n} f$, where $\boldsymbol{k} \in \mathbb{N}_0^n$.
If $n = 1$, then we write the complex derivatives to the order $l \in \mathbb{N}_0$ of f as $D_z^l f$.

An equivalent definition of holomorphy is the following.

Definition 4.2. A continuous map $f : O \to E$ is holomorphic in O, if for any $z \in O$ there exists a $\mathbb{C}$-linear map $D_z f \in \mathcal{L}(\mathbb{C}^n, E)$ and a continuous map $r : \mathbb{C}^n \to E$ with $r(0) = 0$ such that

$$f(z+\tilde{z}) = f(z) + D_z f(\tilde{z}) + \|\tilde{z}\|\, r(\tilde{z}) \quad , \text{for any } \tilde{z} \in (O - z)\ .$$

Remark. We can even, in identifying $\mathbb{C}$ with $\mathbb{R}^2$, give another equivalent definition for holomorphy, here in the one dimensional case:
$f : O \to E$ is holomorphic in O, if f is continuously differentiable in any $(x, y) = z \in O \subseteq \mathbb{C} \simeq \mathbb{R}^2$ and there holds

$$f_x + \mathrm{i} f_y = 0$$

for all $(x, y) = z$ in O.

Notation. For an open and bounded subset $O \subset \mathbb{C}^n$ and a Banach space E, $H(O, E)$ will denote the set of all functions $\overline{O} \to E$ which are holomorphic in O and continuous on $\overline{O}$.

For a polydisc with multi-radius $\boldsymbol{r} = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$ and center 0, namely $D_{\boldsymbol{r}} := D_{\boldsymbol{r}}(0) = \{z \in \mathbb{C}^n \mid |z_i| < r_i \,, \forall i = 1, \ldots, n\}$, we use the abbreviation

$$E_{\boldsymbol{r}} := H(D_{\boldsymbol{r}}, E),$$

or equally in the one-dimensional case with $R > 0$

$$E_R := H(D_R, E)\,.$$

$E_{\boldsymbol{r}}$ is equipped with the norm

$$\|f\|_{E_{\boldsymbol{r}}} := \sup_{z \in D_{\boldsymbol{r}}} \|f(z)\|_E \,.$$

Many well known results for $\mathbb{C}$-valued holomorphic functions also hold in the infinite dimensional case as we will see in the following.

Theorem 4.1. *Let $f \in E_{\boldsymbol{r}}$ for a positive multi-radius $\boldsymbol{r} \in \mathbb{R}_+^n$ and a complex Banach space E. Then there holds Cauchy's integral formula for all $z \in D_{\boldsymbol{r}}$:*

$$f(z_1, \ldots, z_n) = \frac{1}{(2\pi \mathrm{i})^n} \int\limits_{|\xi_1| = r_1} \cdots \int\limits_{|\xi_n| = r_n} \frac{f(\xi_1, \ldots, \xi_n)}{(\xi_n - z_1) \cdots (\xi_n - z_n)} \,\mathrm{d}\xi_1 \ldots \mathrm{d}\xi_n \,. \tag{4.1}$$

Proof. Let first be $n = 1$ and f as above, then the integral

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{|\xi| = r} \frac{f(\xi)}{(\xi - z)} \,\mathrm{d}\xi$$

is well defined for all $z \in D_r$, since f is continuous on ∂D_r. For any $\varphi \in E'$ there holds

$$\begin{aligned} \varphi\left(\frac{1}{2\pi \mathrm{i}} \int_{|\xi| = r} \frac{f(\xi)}{(\xi - z)} \,\mathrm{d}\xi\right) &= \frac{1}{2\pi \mathrm{i}} \int_{|\xi| = r} \frac{\varphi\left(f(\xi)\right)}{(\xi - z)} \,\mathrm{d}\xi \\ &= \varphi\left(f(z)\right). \end{aligned}$$

The last equation holds by the usual Cauchy integral formula because $\varphi\left(f(.)\right)$ is a holomorphic function from D_r into $\mathbb{C}$ and continuous on $\overline{D_r}$. Since φ was arbitrarily chosen, we get Cauchy's integral formula by using a consequence of the Hahn-Banach theorem (see Lemma A.1 in Appendix A).
For $n \geq 2$ we have that f is holomorphic in every variable $z_i \in \mathbb{C}$, $\forall i = 1, \ldots, n$. Then (4.1) follows directly from the one dimensional case by iteration and the application of Fubini's theorem (see Theorem A.1). □

The multi-dimensional Cauchy's integral formula is an adequate tool to establish power series of Banach space valued maps. The following theorem expresses this fact in a more detailed way, initially for a complex disc around 0.

Theorem 4.2 (Taylor series expansion)**.** *For $f \in E_{\boldsymbol{r}}$ with a positive multi-radius $\boldsymbol{r}$ and a complex Banach space E, there holds the Taylor series expansion for any $z \in D_{\boldsymbol{r}}$, i.e.*

$$f(z) = f(0) + \sum_{|\boldsymbol{k}|>0} \frac{1}{\boldsymbol{k}!} D_0^{\boldsymbol{k}} f(0) z^{\boldsymbol{k}} . \tag{TE}$$

The coefficients are uniquely determined by the derivatives of f:

$$D_0^{\boldsymbol{k}} f(z) = \frac{\boldsymbol{k}!}{(2\pi \mathrm{i})^n} \int\limits_{|\xi|=\boldsymbol{r}} \frac{f(\xi)}{(\xi_1 - z_1)^{k_1+1} \dots (\xi_n - z_n)^{k_n+1}} \, \mathrm{d}\xi \tag{CF}$$

for any multi-index $\boldsymbol{k} \in \mathbb{N}_0^n$. The subscript 0 indicates the center of $D_{\boldsymbol{r}}$, while we integrate over $\partial D_{\boldsymbol{r}}$ and $|\xi| = \boldsymbol{r}$ means $|\xi_i| = r_i, \forall i = 1, \dots, n$ for $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$.

Proof. We use the higher dimensional geometric series expansion

$$\frac{1}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} = \sum_{|\boldsymbol{k}|\geq 0} \frac{z_1^{k_1} \cdots z_n^{k_n}}{\xi_1^{k_1+1} \cdots \xi_n^{k_n+1}} =: \sum_{|\boldsymbol{k}|\geq 0} \frac{z^{\boldsymbol{k}}}{\xi^{\boldsymbol{k}+1}} ,$$

which holds for $|z_i| < |\xi_i| = r_i$, $\forall i = 1, \dots, n$. By substitution into Cauchy's integral formula we obtain for all $z \in D_{\boldsymbol{r}}$

$$\begin{aligned} f(z) = \frac{1}{(2\pi \mathrm{i})^n} \int\limits_{|\xi|=\boldsymbol{r}} \frac{f(\xi)}{(\xi_1 - z_1) \cdots (\xi_n - z_n)} \, \mathrm{d}\xi &= \frac{1}{(2\pi \mathrm{i})^n} \int\limits_{|\xi|=\boldsymbol{r}} f(\xi) \sum_{|\boldsymbol{k}|\geq 0} \frac{z^{\boldsymbol{k}}}{\xi^{\boldsymbol{k}+1}} \, \mathrm{d}\xi \\ &= \sum_{|\boldsymbol{k}|\geq 0} \frac{1}{(2\pi \mathrm{i})^n} \int\limits_{|\xi|=\boldsymbol{r}} \frac{f(\xi)}{\xi^{\boldsymbol{k}+1}} \, \mathrm{d}\xi \, z^{\boldsymbol{k}} . \end{aligned} \tag{4.2}$$

The change of integration and summation can be done, because the convergence of the geometric series is absolute and uniform for $\xi \in \partial D_{\boldsymbol{r}}$ and $z \in D_{\boldsymbol{r}}$.
Now f is expandable into a power series and it is therefore infinitely often differentiable. To show that the above series coincides with the Taylor expansion of f in $D_{\boldsymbol{r}}$, we need to prove (CF).
In the one dimensional case, the integral

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{|\xi|=r} \frac{f(\xi)}{(\xi - z)^{l+1}} \, \mathrm{d}\xi$$

is well defined for any $z \in D_r$, $r > 0$, and $l \in \mathbb{N}$, since f is continuous on ∂D_r. For any $\varphi \in E'$, where E' denotes the dual space of E, there holds

$$\varphi \left(\frac{l!}{2\pi \mathrm{i}} \int_{|\xi|=r} \frac{f(\xi)}{(\xi - z)^{l+1}} \, \mathrm{d}\xi \right) = \frac{l!}{2\pi \mathrm{i}} \int_{|\xi|=r} \frac{\varphi\left(f(\xi)\right)}{(\xi - z)^{l+1}} \, \mathrm{d}\xi$$

$$= D^l_z \varphi\left(f(z)\right).$$

Regarding the last equation, note that $\varphi\left(f(.)\right)$ is a holomorphic function from D_r into $\mathbb{C}$ and continuous on $\overline{D_r}$. After repeated application of

$$D^l_z\, \varphi\left(f(z)\right) = D^{l-1}_z \lim_{\tilde{z}\to z} \frac{\varphi\left(f(z)\right) - \varphi\left(f(\tilde{z})\right)}{z - \tilde{z}} = D^{l-1}_z \varphi\left(\lim_{\tilde{z}\to z} \frac{f(z) - f(\tilde{z})}{z - \tilde{z}}\right) = D^{l-1}_z \varphi\left(D_z f(z)\right)$$

we obtain

$$\varphi\left(\frac{l!}{2\pi \mathrm{i}} \int_{|\xi|=r} \frac{f(\xi)}{(\xi - z)^{l+1}}\, \mathrm{d}\xi\right) = \varphi\left(D^l_z\, f(z)\right).$$

Since φ was arbitrarily chosen, we get in using a consequence of the Hahn-Banach theorem (see Lemma A.1),

$$D^l_z f(z) = \frac{l!}{2\pi \mathrm{i}} \int\limits_{|\xi|=r} \frac{f(\xi)}{(\xi - z)^{l+1}}\, \mathrm{d}\xi \quad , \forall z \in D_r\,.$$

In order to indicate the center of integration, we write $D^l_z f(z) = D^l_0 f(z)$.
Now let $n \geq 2$, then f is by our hypothesis holomorphic in every variable $z_i \in \mathbb{C}$, $i = 1, \ldots, n$. Consequently, (CF) follows by iteration from the one dimensional case and Fubini's theorem. Ergo, the Taylor series expansion

$$f(z) = f(0) + \sum_{|\boldsymbol{k}|>0} \frac{1}{\boldsymbol{k}!} D^{\boldsymbol{k}}_0 f(0) z^{\boldsymbol{k}}$$

must hold for any $z \in D_{\boldsymbol{r}}$ as it corresponds to the series expansion (4.2). The uniqueness of the coefficients follows by identification of power series.

□

Remark. The center of the polydisc was chosen to be 0 for pure convenience. As the general case we get the following Corollary.

Corollary 4.1. *For $a \in \mathbb{C}^n$ and $\boldsymbol{r} \in \mathbb{R}^n_+$ let f be in $H(D_{\boldsymbol{r}}(a), E)$. Then for all $z \in D_{\boldsymbol{r}}(a)$ there holds*

$$f(z) = f(a) + \sum_{|\boldsymbol{k}|>0} \frac{1}{\boldsymbol{k}!} D^{\boldsymbol{k}}_a f(a)\, (z - a)^{\boldsymbol{k}}$$

where the coefficients are uniquely determined by

$$D^{\boldsymbol{k}}_a f(z) = \frac{\boldsymbol{k}!}{(2\pi \mathrm{i})^n} \int\limits_{|\xi - a| = \boldsymbol{r}} \frac{f(\xi)}{(\xi_1 - z_1)^{k_1+1} \ldots (\xi_n - z_n)^{k_n+1}}\, \mathrm{d}\xi$$

with $\boldsymbol{k} \in \mathbb{N}^n_0$ and for $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, $|\xi - a| = \boldsymbol{r}$ means $|\xi_i - a_i| = r_i\,, \forall i = 1, \ldots, n$.

4.2. Differentiation under the integral sign

Not only complex differentiability can be extended to vector valued functions. Integration and therefore parametric integrals are also well defined and enjoy the same properties as their real valued companions. The real case is for example considered in [Kön04, p. 282] whose proofs are just adapted to our needs.

Theorem 4.3. *Let $O \subseteq \mathbb{C}$ be open and E a complex Banach space. Let $f : \mathbb{R}^n \times O \to E$ be an integrable function with respect to the first variable which has the following properties:*

(i) $z \mapsto f(u, z)$ is continuous in O for a.e. $u \in \mathbb{R}^n$.

(ii) There exists $\Phi \in L^1(\mathbb{R}^n, \mathbb{R})$ such that for all $z \in O$ and a.e. $u \in \mathbb{R}^n$

$$\|f(u, z)\|_E \leq \Phi(u)\,.$$

Then

$$K(z) := \int_{\mathbb{R}^n} f(u, z)\,\mathrm{d}u$$

is continuous in O.

Proof. We prove sequential continuity. Let $\{z_n\}_{n\geq 1} \subset O$ be a sequence which converges to $z \in O$. Since f is continuous, there holds $f(u, z_n) \to f(u, z)$ as $n \to \infty$ for a.e. $u \in \mathbb{R}^n$. We set

$$\varphi_n(u) := f(u, z_n)\ .$$

Then $\varphi_n(\,.\,)$ converges a.e. point wise against $f(\,.\,, z)$ for all $z \in O$ and is bounded a.e. by $\Phi \in L^1(\mathbb{R}^n, \mathbb{R})$. Hence, Lebesgue's theorem of dominated convergence (see Theorem A.2) yields

$$\lim_{n\to\infty} K(z_n) = \lim_{n\to\infty} \int_{\mathbb{R}^n} \varphi_n(u)\,\mathrm{d}u = \int_{\mathbb{R}^n} \lim_{n\to\infty} \varphi_n(u)\,\mathrm{d}u = \int_{\mathbb{R}^n} f(u, z)\,\mathrm{d}u = K(z)\,.$$

□

Theorem 4.4. *Let $O \subseteq \mathbb{C}$ be open, E a complex Banach space and $f : \mathbb{R}^n \times O \to E$ an integrable function with respect to the first variable which has the following properties:*

(i) $z \mapsto f(u, z)$ is holomorphic in O for a.e. $u \in \mathbb{R}^n$.

(ii) There exists $\Phi \in L^1(\mathbb{R}^n, \mathbb{R})$ such that for all $z \in O$ and a.e. $u \in \mathbb{R}^n$

$$\|f(u, z)\|_E \leq \Phi(u)\,.$$

Then

$$K(z) := \int\limits_{\mathbb{R}^n} f(u,z)\,\mathrm{d}u$$

is holomorphic in O and its derivative is given by

$$K'(z) = \int\limits_{\mathbb{R}^n} \partial_z f(u,z)\,\mathrm{d}u \quad ,\forall z \in O\,.$$

Proof. To show that K is holomorphic, we show equivalently, in identifying $\mathbb{C}$ with $\mathbb{R}^2$ [$z = x + \mathrm{i}y =: (x,y)$], that K is continuously differentiable in $O \subseteq \mathbb{C} \simeq \mathbb{R}^2$ and

$$K_x + \mathrm{i}K_y = 0\,. \tag{CR}$$

$f(u,.)$ is holomorphic in O for a.e. $u \in \mathbb{R}^n$ and therefore continuously differentiable and $f_x + \mathrm{i}f_y = 0$ in O. We define for any $u \in \mathbb{R}^n$ and $h \in \mathbb{R}$ sufficiently small

$$\varphi_h(u) := \frac{f(u,x+h,y) - f(u,x,y)}{h}$$

and get almost everywhere in $\mathbb{R}^n$

$$\lim_{h\to 0} \varphi_h(u) = \partial_x f(u,x,y)\,.$$

Since f is continuously differentiable, it inherits local Lipschitz continuity which yields

$$\|\varphi_h(u)\|_E \le \|\partial_x f(u,x,y)\|_E \qquad \text{for a.e.}\, u \in \mathbb{R}^n\,.$$

In addition we have for $a \in O$ and $r > 0$, such that $\overline{D_r(a)} \subset O$, and for all $z \in D_{r/2}(a)$:

$$\begin{aligned}\|\partial_x f(u,x,y)\|_E = \|\partial_z f(u,z)\|_E &= \left\| \frac{1}{2\pi\mathrm{i}} \int\limits_{|\xi|=r} \frac{f(u,\xi)}{(\xi-z)^2}\mathrm{d}\xi \right\|_E \\ &\le r \cdot \max_{|\xi|=r} \left\| \frac{f(u,\xi)}{(\xi-z)^2} \right\|_E \le \tfrac{4}{r} \cdot \max_{|\xi|=r} \|f(u,\xi)\|_E \le \tfrac{4}{r}\,\Phi(u)\ .\end{aligned}$$

Hence, for a.e. $u \in \mathbb{R}^n$ one obtains

$$\|\varphi_h(u)\|_E \le \tfrac{4}{r}\,\Phi(u)\,.$$

Lebesgue's theorem of dominated convergence (see Theorem A.2) now assures the existence of

$$\lim_{h\to 0} \int\limits_{\mathbb{R}^n} \varphi_h(u)\,\mathrm{d}u = \lim_{h\to 0} \int\limits_{\mathbb{R}^n} \frac{f(u,x+h,y) - f(u,x,y)}{h}\,\mathrm{d}u = \lim_{h\to 0} \frac{K(x+h,y) - K(x,y)}{h}$$

with the limits

$$\int_{\mathbb{R}^n} \lim_{h\to 0} \varphi_h(u)\,\mathrm{d}u = \int_{\mathbb{R}^n} \partial_x f(u,x,y)\,\mathrm{d}u = \partial_x K(x,y)\,\mathrm{d}u\,.$$

The same argumentation for y and Theorem 4.3 yield the continuous differentiability of K and indeed we get (CR):

$$\begin{aligned} K_x(x,y) + \mathrm{i}K_y(x,y) &= \int_{\mathbb{R}^n} \partial_x f(u,x,y)\,\mathrm{d}u + \mathrm{i}\int_{\mathbb{R}^n} \partial_y f(u,x,y)\,\mathrm{d}u \\ &= \int_{\mathbb{R}^n} \partial_x f(u,x,y) + \mathrm{i}\,\partial_y f(u,x,y)\,\mathrm{d}u = 0 \end{aligned}$$

for all $(x,y) = z \in D_{2/r}(a)$. This proves the proposition, since a was arbitrarily chosen. $\square$

From the foregoing proof one obtains the following result for real differentiability.

Corollary 4.2. *Let $W \subseteq \mathbb{R}^n$ be an open subset, E a Banach space and $f : \mathbb{R}^n \times W \to E$ an integrable function with respect to the first variable which has the following properties:*

(i) $x \mapsto f(u,x)$ is an element of $C^1(W;E)$ for a.e. $u \in \mathbb{R}^n$.

(ii) There exists $\Phi \in L^1(\mathbb{R}^n, \mathbb{R})$ such that for all $x \in W$ and a.e. $u \in \mathbb{R}^n$

$$\|\partial_i f(u,x)\|_E \leq \Phi(u) \quad ,\forall i = 1,\ldots,n\,.$$

Then

$$K(x) := \int_{\mathbb{R}^n} f(u,x)\,\mathrm{d}u$$

is an element of $C^1(W;E)$ and its derivative is given by

$$\nabla K(x) = \int_{\mathbb{R}^n} \nabla_x f(u,x)\,\mathrm{d}u \quad ,\forall x \in W\,.$$

4.3. Analyticity on Banach spaces

In this section we want to discuss the analytic behavior of maps between two complex Banach spaces E and F. These operators must have a certain holomorphic property which will be outlined in the following theorem. In fact, we will show that these operators are developable in power series whose coefficients are restrictions of multi-linear maps on the diagonal of a cross product of E. See [Her89, ch. 2 and 3] or [PT87, p. 133] for the proceeding result.

Definition 4.3 (Local Lipschitz-continuity)**.** A function K which maps a Banach space E into a Banach space F is called locally Lipschitz-continuous, if for any $x \in E$ there exists a neighborhood $V \subset E$ of x and a constant $C > 0$ such that for any $y, \tilde{y} \in V$ there holds

$$\|K(y) - K(\tilde{y})\|_F \leq C \|y - \tilde{y}\|_E .$$

Theorem 4.5. *Let E and F be complex Banach spaces, $\Omega \subseteq E$ open and $K : \Omega \to F$ a bounded map. Furthermore, let $K \circ h \in F_{\boldsymbol{r}}$ for any $h \in \Omega_{\boldsymbol{r}}$ with $\boldsymbol{r} \in \mathbb{R}^n_+$. Then the following propositions hold for fixed $a \in \Omega$:*

1. *For $x \in E, 0 < \|x\|_E < R$ with $0 < R < \operatorname{dist}(a, \partial\Omega)$ (if there exists a boundary), there holds*
$$K(a + x) = K(a) + \sum_{l \geq 1} \hat{D}^l_a K(x)$$
and the $\hat{D}^l_a K$ are uniquely determined by
$$\hat{D}^l_a K(x) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|\xi|=\varepsilon} \frac{K(a + \xi x)}{\xi^{l+1}} \,\mathrm{d}\xi$$
with $0 < \varepsilon \leq \frac{R}{\|x\|}$ arbitrarily chosen.

2. *The $\hat{D}^l_a K$ can be identified with k-linear symmetric maps K^l_a which are extendable to $E^l = E \times \cdots \times E$ (written l-times). They take the form*
$$K^l_a(x_1, \ldots, x_l) = \frac{1}{(2\pi \mathrm{i})^l} \int\limits_{|\xi_1|=\varepsilon} \cdots \int\limits_{|\xi_l|=\varepsilon} \frac{K(a + \xi_1 x_1 + \ldots + \xi_l x_l)}{\xi_1^2 \cdots \xi_l^2} \,\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_l$$
for $(x_1, \ldots x_l) \in E^l$ and ε sufficiently small.

3. *For any $x \in E$, $0 < \|x\|_E < R$ there holds*
$$K(a + x) = K(a) + \sum_{l \geq 1} K^l_a(\underbrace{x, \ldots, x}_{l-times})$$

and, for any $0 < R' < R$,

$$\sum_{l \geq 1} \sup_{x \in B_{R'}} \|K_a^l(x, \dots, x)\|_F \leq \sup_{y \in B_R} \|K(a+y)\|_F \cdot \tfrac{R'}{R-R'} < +\infty \,.$$

4. *The operator* K *is locally Lipschitz-continuous in* Ω.

Proof. (1.) Fix $a \in \Omega$, let $x \in B_R^* := B_R(0) \backslash \{0\}$ and $\| \,.\, \| := \| \,.\, \|_E$, for a fixed $0 < R < \operatorname{dist}(a, \partial\Omega)$ and set

$$\varrho := \frac{R}{\|x\|} \,.$$

Then, $\varrho > 1$ and $\zeta \mapsto a + \zeta x$ is a member of $\Omega_\varrho = H(D_\varrho, \Omega)$. Therefore, $\zeta \mapsto K(a + \zeta x)$ is an element of $F_\varrho = H(D_\varrho, F)$ by hypothesis. Hence, Theorem 4.1 yields that

$$K(a + \zeta x) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|\xi| = \varrho} \frac{K(a + \xi x)}{(\xi - \zeta)} \,\mathrm{d}\xi$$

for all $\zeta \in D_\varrho$, and we obtain from Theorem 4.2 that

$$K(a + \zeta x) = K(a) + \sum_{l \geq 1} \frac{1}{l!} D_0^l K(a + \tau x)|_{\tau = 0} \cdot \zeta^l$$

with uniquely determined

$$D_0^l K(a + \tau x)|_{\tau=0} = \frac{l!}{2\pi \mathrm{i}} \int\limits_{|\xi| = \varrho} \frac{K(a + \xi x)}{\xi^{l+1}} \,\mathrm{d}\xi \,.$$

Clearly, by choosing $\zeta = 1$, we get

$$K(a + x) = K(a) + \sum_{l \geq 1} \hat{D}_a^l K(x).$$

(2.) Let $a \in \Omega$ be fixed and $x \in B_R^*$.
In considering the case $l = 1$ we get directly for any $x \in B_R^*$

$$\hat{D}_a^1 K(x) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|\xi| = \frac{R}{\|x\|}} \frac{K(a + \xi x)}{\xi^2} \,\mathrm{d}\xi = K_a^1(x) \,.$$

Then, (TE) in Theorem 4.2 yields, for $z_1, z_2 \in D_{1/2R}$ and $x_1, x_2 \in B_R^*$, that

$$z_1 x_1 + z_2 x_2 \in B_R$$

and so

$$\begin{aligned} K(a+z_1x_1+z_2x_2) &= K(a) + \sum_{|\boldsymbol{l}|\geq 1} \tfrac{1}{\boldsymbol{l}!} z^{\boldsymbol{l}} D_{\boldsymbol{0}}^{\boldsymbol{l}} K(a+\tau x_1+\eta x_2)\big|_{\tau,\eta=0} \\ &= K(a) + z_1 \cdot D_0^{(1,0)} K(a+\tau x_1+\eta x_2)\Big|_{\tau,\eta=0} \\ &\quad + z_2 \cdot D_0^{(0,1)} K(a+\tau x_1+\eta x_2)\Big|_{\tau,\eta=0} \\ &\quad + \sum_{|\boldsymbol{l}|\geq 2} \tfrac{1}{\boldsymbol{l}!} z^{\boldsymbol{l}} D_0^{\boldsymbol{l}} K(a+\tau x_1+\eta x_2)\big|_{\tau,\eta=0} \ . \end{aligned}$$

Also, in considering Cauchy's integral formula, with $0<\varepsilon\leq \min\left\{\frac{R}{2\|x_1\|},\frac{R}{2\|x_2\|}\right\}$, we obtain

$$\begin{aligned} D_0^{(1,0)} K(a+\tau x_1+\eta x_2)\Big|_{\tau,\eta=0} &= \frac{1}{(2\pi\mathrm{i})^2} \int\limits_{|\xi|=\varepsilon}\int\limits_{|\zeta|=\varepsilon} \frac{K(a+\xi x_1+\zeta x_2)}{\xi^2\zeta}\,\mathrm{d}\zeta\mathrm{d}\xi \\ &= \frac{1}{2\pi\mathrm{i}} \int\limits_{|\xi|=\varepsilon} \frac{K(a+\xi x_1+\zeta x_2)|_{\zeta=0}}{\xi^2}\,\mathrm{d}\xi \\ &= K_a^1(x_1) \end{aligned}$$

and equally

$$D_0^{(0,1)} K(a+\tau x_1+\eta x_2)\Big|_{\tau,\eta=0} = K_a^1(x_2)\ .$$

Hence,

$$\begin{aligned} K(a+z_1x_1+z_2x_2) = K(a) &+ z_1K_a^1(x_1) + z_2K_a^1(x_2) \\ &+ \sum_{|\boldsymbol{l}|\geq 2} \tfrac{1}{\boldsymbol{l}!} z^{\boldsymbol{l}} D_{\boldsymbol{0}}^{\boldsymbol{l}} K(a+\tau x_1+\eta x_2)\big|_{\tau,\eta=0} \ . \end{aligned}$$

On the other hand, we have for $z_1=z_2=:z$, that

$$\begin{aligned} K(a+z(x_1+x_2)) &= K(a) + z\cdot \hat{D}_a^1 K(x_1+x_2) + \sum_{l\geq 2} z^l \hat{D}_a^l K(x_1+x_2) \\ &= K(a) + z\cdot K_a^1(x_1+x_2) \quad + \sum_{l\geq 2} z^l \hat{D}_a^l K(x_1+x_2)\,. \end{aligned}$$

Thus, we obtain the additivity of K_a^1 by identification of power series.

We observe furthermore that, for $x \in B_R^*$,

$$K_a^1(x) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|\xi|=\varepsilon} \frac{K(a+\xi x)}{\xi^{l+1}} \,\mathrm{d}\xi$$

is independent of ε as long as $0 < \varepsilon \leq \frac{R}{\|x\|}$. Consequently, K_a^1 can be extended, first to $E^* = E \setminus \{0\}$ and then, in setting $K_a^1(0) = 0$, to E.
Let $\alpha \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, then for any $x \in B_R^*$ there holds

$$K_a^1(\alpha x) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|\xi|=\varepsilon} \frac{K(a+\alpha\xi x)}{\xi^2} \,\mathrm{d}\xi \underset{(\xi'=\alpha\xi)}{=} \frac{1}{2\pi \mathrm{i}} \int\limits_{|\xi'|=\varepsilon'} \alpha \frac{K(a+\xi' x)}{\xi'^2} \,\mathrm{d}\xi' = \alpha K_a^1(x)$$

with $0 < \varepsilon < \frac{R}{|\alpha|\|x\|}$ and $\varepsilon' = |\alpha|\varepsilon$.

We showed that the first derivative of a map which satisfies the conditions stated in this theorem is actually a linear map. We will proceed by induction on $l \in \mathbb{N}$ to show that the K_a^l are l-linear maps. We assume, for fixed $l \in \mathbb{N}$, that the map K_a^l, given by

$$K_a^l(x_1, \ldots, x_l) = \frac{1}{(2\pi \mathrm{i})^l} \int\limits_{|\xi_1|=\varepsilon} \cdots \int\limits_{|\xi_l|=\varepsilon} \frac{K(a+\xi_1 x_1 + \ldots + \xi_l x_l)}{\xi_1^2 \cdots \xi_l^2} \,\mathrm{d}\xi_1 \ldots \mathrm{d}\xi_l,$$

$$0 < \varepsilon \leq \min\left\{ \frac{R}{l\,\|x_i\|} \;\mid\; i = 1, \ldots, l \right\},$$

is an element of the Banach space $\mathcal{L}^l(E;F)$. We remark that $a \mapsto K_a^l(\,.\,)$ is a well defined and uniformly continuous map $\Omega \to \mathcal{L}^l(E;F)$. Consider the function

$$\zeta \mapsto \varphi \circ K_{a+\zeta x_0}^l(x_1, \ldots, x_l) = \frac{1}{(2\pi \mathrm{i})^l} \int\limits_{|\xi_1|=\varepsilon} \cdots \int\limits_{|\xi_l|=\varepsilon} \frac{\varphi \circ K(a+\zeta x_0 + \xi_1 x_1 + \ldots + \xi_l x_l)}{\xi_1^2 \cdots \xi_l^2} \,\mathrm{d}\xi_1 \ldots \mathrm{d}\xi_l$$

with $x_0, \ldots, x_l \in E$ and $|\zeta| \leq \varepsilon'$ for $0 < \varepsilon' \leq \min\left\{ \frac{R}{(l+1)\|x_i\|} \mid i = 0, \ldots, l \right\}$ and $\varphi \in F'$ arbitrary. The above formula is holomorphic in ζ by Theorem 4.4 and the arbitrary choice of φ implies

$$K_{a+\,.\,x_0}^l(x_1, \ldots, x_l) \in H(D_{\varepsilon'}, F) \qquad , \text{ for } (x_1, \ldots, x_l) \in E^l\,.$$

Hence, for $x_0, x_1, \ldots, x_l \in E^*$ we get with the same argumentation as before that

$$\hat{D}_a^1 K_a^l(x_1, \ldots, x_l)(x_0) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|\xi_0|=\varepsilon} \cdots \int\limits_{|\xi_l|=\varepsilon} \frac{K(a+\xi_0 x_0 + \xi_1 x_1 + \ldots + \xi_l x_l)}{\xi_0^2 \cdots \xi_l^2} \,\mathrm{d}\xi_0 \ldots \mathrm{d}\xi_l$$

is linear in x_0, where $0 < \varepsilon \leq \min\left\{ \frac{R}{(l+1)\|x_i\|} \mid i = 0, \ldots, l \right\}$. Thus, we have, in setting $K_a^l(0) = 0$, that $\hat{D}_a^1 K_a^l \in \mathcal{L}(E; \mathcal{L}^l(E;F))$, whereas $\mathcal{L}(E; \mathcal{L}^l(E;F))$ is isomorphic to $\mathcal{L}^{l+1}(E;F)$. Consequently, we may write

$$\hat{D}^1_a K^l_a(x_1,\ldots,x_l)(x_0) = K^{l+1}_a(x_0,x_1,\ldots,x_l),$$

which is $l+1$-linear and symmetric by Fubini's theorem. The l-linearity follows now for all $l \in \mathbb{N}$ by the induction principle. We can again extend the K^l_a to E^l since the above integral is independent of ε as long as it is sufficiently small. Moreover, the K^l_a are bounded and therefore, as l-linear maps, continuous on E^l.
Finally, we restrict K^l_a to the diagonal of $B_R(0)^l$ and set $x \coloneqq x_1 = \ldots = x_l$, then

$$\begin{aligned} K^l_a(\underbrace{x,\ldots,x}_{l-times}) &= \partial_{z_1}\cdots\partial_{z_l} K(a+z_1x+\ldots+z_lx)\Big|_{z_1,\ldots z_l=0} \\ &= \frac{\mathrm{d}^l}{\mathrm{d}z^l} K(a+zx)\Big|_{z=0} \\ &= \hat{D}^l_a K(x)\,. \end{aligned}$$

(3.) The previous calculations imply directly that

$$K(a+x) = K(a) + \sum_{l\geq 1} K^l_a(\underbrace{x,\ldots,x}_{l-times}) \qquad ,\forall x \in B_R\,.$$

Let $0 < R' < R < \operatorname{dist}(a,\partial\Omega)$ and $M \coloneqq \sup_{x\in B_R} \|K(a+x)\|_F$, we then obtain for any $l \in \mathbb{N}$:

$$\begin{aligned} \sup_{x\in B_{R'}} \|K^l_a(x,\ldots,x)\|_F &= \sup_{x\in B_{R'}} \left\| \frac{1}{2\pi\mathrm{i}} \int\limits_{|\xi|=\frac{R}{\|x\|}} \frac{K(a+\xi x)}{\xi^{l+1}}\,\mathrm{d}\xi \right\|_F \\ &\leq \sup_{x\in B_{R'}} \frac{1}{2\pi} \int\limits_{|\xi|=\frac{R}{\|x\|}} \frac{M}{|\xi|^{l+1}}\,\mathrm{d}\xi \\ &= \sup_{x\in B_{R'}} \left(\tfrac{\|x\|}{R}\right)^l M \\ &= \left(\tfrac{R'}{R}\right)^l M\,. \end{aligned}$$

Thus, there holds

$$\sum_{l\geq 1} \sup_{x\in B_{R'}} \|K^l_a(x,\ldots,x)\|_F \leq \sum_{l\geq 1} \left(\tfrac{R'}{R}\right)^l M = M\left(\tfrac{R}{R-R'} - 1\right) = M\tfrac{R'}{R-R'}\,.$$

(4.) For $x, \tilde{x} \in B_{R'/2}$, with $0 < R' < R < \operatorname{dist}(a,\partial\Omega)$ as before, we have $x - \tilde{x} \in B_{R'}$. Then the previous estimates of K^l_a and the l-linearity yield

$$\begin{aligned}\|K(a+x)-K(a+\tilde{x})\|_F &\leq \sum_{l\geq 1} \|K_a^l(x-\tilde{x},\dots,x-\tilde{x})\|_F \\ &\leq \|x-\tilde{x}\| \sum_{l\geq 1} \frac{\|x-\tilde{x}\|^{l-1}}{R^l} M \leq L\,\|x-\tilde{x}\|\end{aligned}$$

with $L = M\frac{1}{R-R'}$. Hence local Lipschitz-continuity of K follows, since $a \in \Omega$ was arbitrarily chosen. $\square$

4.4. The Cauchy-Lipschitz existence theorem

Let us now switch to one of the key theorems in this approach. It makes under rather low requirements an assertion about existence and uniqueness of a solution to a complex ODE.

Theorem 4.6 (Cauchy-Lipschitz existence Theorem)**.** *Let E be a complex Banach space, $z_0 \in \mathbb{C}, y_0 \in E$ and $G \subseteq \mathbb{C} \times E$ open and connected with $(z_0, y_0) \in G$ and $\overline{D_R(z_0)} \times \overline{B_r(y_0)} \subseteq G$ for some $R, r > 0$. Let also K be a continuous and bounded map from G into E which is locally Lipschitz-continuous in the second variable and set*

$$S := \sup_{(z,y)\in G} \|K(z,y)\|_E$$

as well as

$$\varrho := \min\{R, \frac{r}{S}\}.$$

If $z \mapsto K(z, y(z))$ is holomorphic in $D_\varrho(z_0)$ for any $y \in H(D_\varrho(z_0), B_r(y_0))$, then there exists a unique solution of

$$\begin{aligned} y'(z) &= K(z, y(z)) \\ y(z_0) &= y_0 \end{aligned} \tag{DE}$$

for all $z \in D_\varrho(z_0)$.

Remark. In the above theorem E can also be taken as a finite product of complex Banach spaces.

Proof. By the assumptions of holomorphy made on K we can rewrite (DE) into an equivalent integral equation, namely for all $z \in D_\varrho(z_0)$

$$y(z) = y_0 + \int_{z_0}^{z} K(w, y(w))\,\mathrm{d}w \tag{IE}$$

and set

$$(Ty)(z) := y_0 + \int_{z_0}^{z} K(w, y(w))\,\mathrm{d}w, \quad \forall\, z \in D_\varrho(z_0),$$

where the above integrals are taken over any continuous path in $D_\varrho(z_0)$ connecting z_0 and z. Then, T is well defined on $N := \{u \in H(D_\varrho(z_0), E) \mid \sup_{z\in D_\varrho(z_0)} \|u(z) - y_0\|_E \le r\}$.

We now show that T is a contractive map from N into N. To prove the latter statement, let $y \in N$ and $z \in \overline{D_\varrho(z_0)}$, then there holds

$$\begin{aligned} \|(Ty)(z) - y_0\|_E \le \left\| \int_{z_0}^{z} K(w, y(w))\,\mathrm{d}w \right\|_E &\le \left| \int_{z_0}^{z} \|K(w, y(w))\|_E\,\mathrm{d}w \right| \\ &\le S\,|z - z_0| \le S\,\varrho \le r, \end{aligned}$$

hence $Ty \in N$.

To show that T is a contraction, let us remark that since $\overline{B_r(y_0)}$ is compact in E and K locally Lipschitz-continuous in the second variable, there exists a constant $L > 0$ such that

$$\|K(z,y) - K(z,\tilde{y})\|_E \leq L\,\|y - \tilde{y}\|_E, \quad \forall\, y, \tilde{y} \in \overline{B_r(y_0)}.$$

We introduce an equivalent norm on $H(D_r(z_0), E)$:

$$|\!|\!| u |\!|\!| \coloneqq \sup_{z \in D_r(z_0)} \{\, \|u(z)\|_E \cdot \mathrm{e}^{(-2L|z-z_0|)} \,\}$$

Then there holds for all $y, \tilde{y} \in N$:

$$\begin{aligned}
\|Ty(z) - T\tilde{y}(z)\|_E &\leq \left| \int_{z_0}^{z} \|K(w,y(w)) - K(w,\tilde{y}(w))\|_E \,\mathrm{d}w \right| \\
&\leq \left| L \cdot \int_{z_0}^{z} \|y(w) - \tilde{y}(w)\|_E \cdot \mathrm{e}^{-2L|w-z_0|} \cdot \mathrm{e}^{2L|w-z_0|} \,\mathrm{d}w \right| \\
&\leq L \cdot |\!|\!| y - \tilde{y} |\!|\!| \cdot \left| \int_{z_0}^{z} \mathrm{e}^{2L|w-z_0|} \,\mathrm{d}w \right| \\
&\leq L \cdot |\!|\!| y - \tilde{y} |\!|\!| \cdot \tfrac{1}{2L} (\mathrm{e}^{2L|z-z_0|} - 1) \\
&\leq \tfrac{1}{2} |\!|\!| y - \tilde{y} |\!|\!| \cdot \mathrm{e}^{2L|z-z_0|},
\end{aligned}$$

which is equivalent to

$$|\!|\!| Ty - T\tilde{y} |\!|\!| \leq \tfrac{1}{2} |\!|\!| y - \tilde{y} |\!|\!| \,.$$

Now T being a contraction on a closed and connected subset of a Banach space fulfills the propositions of Banach's fixed point theorem (see Theorem A.3). Hence, (IE) admits a unique solution in N which implies a unique solution of (DE) in N.

$\square$

Remark. In fact, the requirements for the last theorem are satisfied, if $K : \Omega \to E$ is a bounded map for $\Omega \subseteq E$ open, connected and $K(\Omega_R) \subseteq E_R$ for $R > 0$, which are precisely the conditions of Theorem 4.5.

5. Derivation of a second order PDE

5.1. Representation of the pressure-gradient

We recall the Euler equations for an ideal and incompressible fluid flow in $n \geq 2$ dimensions:

$$\begin{aligned} \partial_t u(x,t) + (u \cdot \nabla) u(x,t) &= -\nabla p(x,t) && \text{(Ea)} \\ \operatorname{div} u(x,t) &= 0 \qquad (E) && \text{(Eb)} \\ u(x,0) &= u_0(x), && \text{(Ec)} \end{aligned}$$

where u and p denote the velocity and pressure respectively in a given fluid. In this approach, the investigation of particle trajectories in a fluid flow requires a solution of the Euler system (E). Let therefore $u \in L^\infty_{loc}([0,T), C^{m,\alpha}(\mathbb{R}^n;\mathbb{R}^n))$, $T > 0$, solve the Euler equations for a divergence-free initial velocity field $u_0 \in C^{m,\alpha}(\mathbb{R}^n;\mathbb{R}^n)$, where m is a positive integer and $\alpha \in (0,1)$. That a solution in the above space exists uniquely is for example stated and proved in [Che98, p. 77]. More information on the particle trajectories is given in the following section. In this section the associated pressure is investigated, which will lead to the unique representation of the pressure gradient. Therefore, one obtains a uniquely defined right hand side of the Euler equation.
In taking the divergence of (Ea) in the weak sense under condition (Eb), one obtains

$$\begin{aligned} & \operatorname{div}\left[\partial_t u + \sum_{i=1}^n u_i \partial_i u\right](x,t) = -\operatorname{div}(\nabla p)(x,t) \\ \Longleftrightarrow \quad & \left[\partial_t \operatorname{div} u + \sum_{j=1}^n \partial_j \sum_{i=1}^n u_i\, \partial_i\, u_j\right](x,t) = -\Delta\, p\,(x,t) \\ \Longleftrightarrow \quad & \sum_{i,j} \partial_j\,(u_i \partial_i\, u_j)(x,t) = -\Delta\, p\,(x,t) \\ \overset{(*)}{\Longleftrightarrow} \quad & \operatorname{Tr}\,(Ju)^2(x,t) = -\Delta\, p\,(x,t), \end{aligned} \tag{5.1}$$

where $J(\,.\,)$ denotes the Jacobian matrix. We will see that the pressure p is, in a later defined way, uniquely associated to u by equation (5.1). The equivalence $(*)$ is obtained by a direct calculation which follows. We have

$$\sum_{i,j} \partial_j\,(u_i \partial_i\, u_j) = \sum_{i,j} \partial_j\, u_i \partial_i\, u_j + \sum_{i,j} u_i \partial_i \partial_j\, u_j = \sum_{i,j} \partial_j\, u_i \partial_i\, u_j = \operatorname{Tr}\,(Ju)^2\,.$$

It also holds that

$$\begin{aligned}\sum_{i,j} \partial_i \partial_j \,(u_i\, u_j) &= \sum_{i,j} (\partial_i \partial_j\, u_i)\, u_j + \sum_{i,j} \partial_j u_i \partial_i\, u_j + \sum_{i,j} \partial_i\, u_i \partial_j\, u_j + \sum_{i,j} u_i (\partial_i \partial_j\, u_j) \\ &= \sum_{i,j} \partial_j (\partial_i\, u_i)\, u_j + \sum_{i,j} \partial_j u_i \partial_i\, u_j + \sum_{i,j} \partial_i\, u_i \partial_j\, u_j + \sum_{i,j} u_i \partial_i (\partial_j\, u_j),\end{aligned}$$

where only the second term does not vanish. Hence, we have the identity

$$\sum_{i,j} \partial_i \partial_j \,(u_i\, u_j) = \mathrm{Tr}\, (Ju)^2 \,. \tag{5.2}$$

Then, for a fixed $t \in [0, T)$, we write

$$u_i u_j(\,.\,, t) =: h_{ij}(\,.\,) \in C^{m,\alpha}(\mathbb{R}^n; \mathbb{R}^n) \subset W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^n) \tag{5.3}$$

and under consideration of (5.2) one obtains

$$\sum_{i,j} \partial_i \partial_j\, h_{ij} =: v \in C^{0,\alpha}(\mathbb{R}^n; \mathbb{R}^n) \subset L^\infty(\mathbb{R}^n; \mathbb{R}^n)\,. \tag{5.4}$$

We will show that (5.1) admits a unique solution and state a preliminary well known fact.

Lemma 5.1. If $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then the Newtonian potential of f, given by

$$w(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, \mathrm{d}y,$$

belongs to $C^1(\mathbb{R}^n)$ and

$$\nabla w(x) = \int_{\mathbb{R}^n} \nabla \Gamma(x-y) f(y)\, \mathrm{d}y\,.$$

Proof. [GT83, p. 55] We concentrate on the case of $n \geq 3$ spacial dimensions, whereas the case $n = 2$ is treated equally. Γ has a singularity at the origin of $\mathbb{R}^n$. In order to cut out this singularity we choose a radial function $\eta \in C^\infty(\mathbb{R}^n)$ with the following properties. For $x \in \mathbb{R}^n$: $0 \leq \eta(x) \leq 1$ and $\eta(x) = 0$, if $x \in B_1(0)$, as well as $\eta(x) = 1$, if $x \in \mathbb{R}^n \setminus B_2(0)$. For $\varepsilon > 0$ set

$$\eta_\varepsilon(x) := \eta\left(\frac{x}{\varepsilon}\right)$$

and

$$w_\varepsilon(x) := \Gamma \eta_\varepsilon * f\,(x) \;= \int_{\mathbb{R}^n} \Gamma(x-y) \eta_\varepsilon(x-y) f(y)\, \mathrm{d}y,$$

which is well defined, since $\Gamma \eta_\varepsilon \in L^\infty(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n)$. In the following fix $i \in \{1, \ldots, n\}$.

From Corollary 4.2 we get

$$w_\varepsilon \in C^1(\mathbb{R}^n) \quad \text{and} \quad \partial_i w_\varepsilon(x) = (\partial_i(\Gamma\eta_\varepsilon) * f)(x),$$

since

$$|\partial_{x_i}[\Gamma(x-y)\eta_\varepsilon(x-y)]f(y)| \leq C|f(y)| \in L^1_y(\mathbb{R}^n)$$

with a positive constant $C = C(\varepsilon)$.
In the next step we are going to show, that $\partial_i w_\varepsilon$ converges uniformly to

$$v_i := \partial_i \Gamma * f\,.$$

First, the above formula is well defined, since f is bounded and $\partial_i\Gamma$ integrable within a domain containing the origin and outside of this domain f is integrable and $\partial_i\Gamma$ is bounded. We now have for $x \in \mathbb{R}^n$ arbitrarily chosen

$$\begin{aligned} |\partial_i w_\varepsilon(x) - v_i(x)| &= |\partial_i(\Gamma\eta_\varepsilon) * f\,(x) - \partial_i\Gamma * f\,(x)| \\ &= |\partial_i[(\eta_\varepsilon - 1)\Gamma] * f\,(x)| \\ &\leq \|\partial_i[(\eta_\varepsilon - 1)\Gamma]\|_1 \, \|f\|_\infty \end{aligned}$$

and furthermore,

$$\begin{aligned} \|\partial_i[(\eta_\varepsilon - 1)\Gamma]\|_1 &= \int\limits_{\|x\|\leq 2\varepsilon} |\partial_i[(\eta_\varepsilon - 1)\Gamma](x)| \, \mathrm{d}x \\ &\leq \int\limits_{\|x\|\leq 2\varepsilon} \frac{1}{\varepsilon}\left\|\nabla\eta\left(\frac{x}{\varepsilon}\right)\right\| \frac{1}{\|x\|^{n-2}} + \underbrace{|\eta_\varepsilon(x) - 1|}_{\leq 1} \frac{1}{\|x\|^{n-1}} \, \mathrm{d}x \\ &\leq \int\limits_0^{2\varepsilon} \int\limits_{S^{n-1}} \left[\frac{c}{\varepsilon}\frac{1}{\|\lambda\omega\|^{n-2}} + \frac{1}{\|\lambda\omega\|^{n-1}}\right] \lambda^{n-1} \, \mathrm{d}\sigma(\omega) \, \mathrm{d}\lambda \\ &= \bar{c} \int\limits_0^{2\varepsilon} \left[\frac{c}{\varepsilon}\lambda + 1\right] \, \mathrm{d}\lambda \\ &= \tilde{c}\,\varepsilon, \end{aligned}$$

where $\bar{c}$ and $\tilde{c}$ are positive constants. Altogether, one obtains for any $x \in \mathbb{R}^n$

$$|\partial_i w_\varepsilon(x) - v_i(x)| \leq \tilde{c}\, \|f\|_\infty \, \varepsilon\,.$$

Hence, we showed that $\partial_i w_\varepsilon$ converges uniformly to v_i as $\varepsilon \to 0$. Now, $w_\varepsilon \in C^1(\mathbb{R}^n)$ yields $v_i \in C^0(\mathbb{R}^n)$ and by Lebesgue's theorem of dominated convergence there also holds $w_\varepsilon \to \Gamma * f$

point wise, as $\varepsilon \to 0$. Subsequently, $w \in C^1(\mathbb{R}^n)$ and

$$\nabla w(x) = \int\limits_{\mathbb{R}^n} \nabla \Gamma(x-y) f(y) \, \mathrm{d}y \,.$$

$\square$

Before we proceed, we need to define a function on a subset of $\mathbb{C}$ which will play a key role hereafter.

Definition 5.1. Let a be a holomorphic function defined on

$$S := \{z \in \mathbb{C}^n \setminus \{0\} \mid \exists x \in \mathbb{R}^n : \|z - x\| < \beta \|x\|\} \tag{5.5}$$

with the following properties:

- the restriction of a to $\mathbb{R}^n$ is real valued and radial
- $\sup_{z \in S} \{\|z\|^s |a(z)| \ + \ \|z\|^{\pm s} |D^\mu a(z)| \ + \ \|z\|^{-s} |1 - a(z)|\} < +\infty$
  for any $s \geq 0$ and $|\mu| \geq 1$.

Remark. For instance, the map

$$S \ni z \to a(z) = \bar{a}\left(\left(\sum\nolimits_{i=1}^n z_i^2\right)^{\frac{1}{2}}\right)$$

with

$$\bar{a}(y) = 1 - \exp\left(-\frac{\exp(-y)}{y}\right)$$

fulfills the above requirements.

In the next theorem all functions are real valued.

Theorem 5.1. *For $i, j = 1, \ldots, n$ let $h_{ij} \in W^{1,\infty}(\mathbb{R}^n)$ and $v := \sum_{i,j=1}^n \partial_i \partial_j h_{ij} \in L^\infty(\mathbb{R}^n)$. Then there exists $p \in C(\mathbb{R}^n)$ with*

$$\Delta p = v \qquad \text{in } D'(\mathbb{R}^n) \,. \tag{5.6}$$

It fulfills the estimates

$$|p(x_1) - p(x_2)| \leq M \|h_{ij}\|_\alpha \|x_1 - x_2\|^\alpha \,,$$

for any $x_1, x_2 \in \mathbb{R}^n$, and

$$|p(x)| \leq M' \|h_{ij}\|_\alpha \log(2 + \|x\|)$$

for any $x \in \mathbb{R}^n$. M and M' denote positive constants. Furthermore, the solution is unique in

$$N := \left\{ q \in C(\mathbb{R}^n) \,\middle|\, \lim_{\|x\| \to \infty} \frac{q(x)}{\|x\|} = 0 \,, \, q(0) = c \in \mathbb{R} \right\} \,.$$

Lemma 5.2. The gradient of the formerly found unique solution of (5.6) takes the form

$$\nabla p(x) = \nabla(a\Gamma) * v\,(x) + \sum\nolimits_{i,j} \left(\partial_i \partial_j \nabla((1-a)\Gamma) * h_{ij}\right)(x)$$

in $D'(\mathbb{R}^n)$ and belongs to $C_b(\mathbb{R}^n;\mathbb{R}^n)$.

Proof of Theorem 5.1. Existence. Fix in the following $i,j \in \{1,\dots,n\}$. The first step is to approximate the functions $h_{ij} \in W^{1,\infty}(\mathbb{R}^n)$. Note that the h_{ij} are bounded and have also bounded first derivatives. They are therefore globally Hölder-continuous to the exponent $\alpha \in (0,1)$. For $k \in \mathbb{N}$ choose a sequence $(\nu_k) \subset C_0^\infty(\mathbb{R}^n)$ such that

$$\nu_k(x) = 1, \quad \text{for all } x \in B_k(0)$$

and also, for $\mu \in \mathbb{N}_0^n$ with $|\mu| \leq 2$,

$$\|D^\mu \nu_k\|_\infty \leq C\,, \ \text{for all } k \in \mathbb{N},$$

where $C > 0$ is a fixed constant. The latter requirement is needed in order to uniformly bound the following functions. We set

$$h_{ij}^k := \nu_k \cdot h_{ij}$$

and

$$v_k(x) := \sum_{i,j=1}^n \partial_i \partial_j h_{ij}^k(x),$$

which are compactly supported. Then, $h_{ij}^k, v_k \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ (for all $i,j = 1,\dots,n\,,\ k \in \mathbb{N}$) with $\left\|h_{ij}^k\right\|_\infty \leq c\,\|h_{ij}\|_\infty$ and $\|v_k\|_\infty \leq \tilde{c}$ for some $c,\tilde{c} > 0$ and $\tilde{c} = \tilde{c}\left(\|h_{ij}\|_{W^{1,\infty}}\,, \|v\|_\infty\right)$. Additionally, we choose the functions ν_k such that

$$\left[h_{ij}^k\right]_\alpha \leq \bar{c}\,[h_{ij}]_\alpha\,, \tag{5.7}$$

where $\bar{c} > 0$ is a fixed constant independent of $k \in \mathbb{N}$ and $[\,.\,]_\alpha$ denotes the Hölder semi-norm. Thus,

$$p_k(x) = \int\limits_{\mathbb{R}^n} \Gamma(x-y) v_k(y)\,\mathrm{d}y \tag{5.8}$$

is well defined and belongs to $C^1(\mathbb{R}^n)$ in accordance with Lemma 5.1. Moreover,

$$\nabla p_k(x) = \int\limits_{\mathbb{R}^n} \nabla\Gamma(x-y) v_k(y)\,\mathrm{d}y\,. \tag{5.9}$$

Since $p_k \in L^1_{loc}(\mathbb{R}^n)$, we may write for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ in using Fubini's theorem

$$\begin{aligned}\langle \Delta p_k, \varphi\rangle &= \langle p_k, \Delta \varphi\rangle \\ &= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \Gamma(x-y)v_k(y)\,\mathrm{d}y\,\Delta\varphi(x)\,\mathrm{d}x = \int\limits_{\mathbb{R}^n} v_k(y)\int\limits_{\mathbb{R}^n}\Gamma(x-y)\Delta\varphi(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int\limits_{\mathbb{R}^n} v_k(y)\int\limits_{\mathbb{R}^n}\Gamma(x)\Delta\varphi(x+y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{\mathbb{R}^n} v_k(y)\varphi(y)\,\mathrm{d}y = \langle v_k, \varphi\rangle\,.\end{aligned}$$

Thus, it holds

$$\Delta p_k = v_k \text{ in } D'(\mathbb{R}^n) \tag{5.10}$$

and furthermore, $v_k \to v$ in $D'(\mathbb{R}^n)$ as $k \to \infty$. Also, since $\operatorname{supp} h_{ij}^k \subset O$ for $O \subset \mathbb{R}^n$ open, bounded and big enough, we may write

$$\begin{aligned}p_k(x) &= \int\limits_{\mathbb{R}^n}\Gamma(x-y)\sum_{i,j}\partial_i\partial_j h_{ij}^k(y)\,\mathrm{d}y = \sum_{i,j}\int\limits_O \Gamma(x-y)\partial_i\partial_j(h_{ij}^k(y))\,\mathrm{d}y \\ &= \sum_{i,j}\int\limits_O \Gamma(x-y)\partial_i\partial_j(h_{ij}^k(y)-h_{ij}^k(x))\,\mathrm{d}y \\ &= \sum_{i,j}\int\limits_O \partial_i\partial_j\Gamma(x-y)(h_{ij}^k(y)-h_{ij}^k(x))\,\mathrm{d}y\,.\end{aligned} \tag{5.11}$$

In the following we leave the summation sign out for convenience. The last formula (5.11) is bounded since the integrand is estimated by

$$|\partial_i\partial_j\Gamma(x-y)(h_{ij}^k(y)-h_{ij}^k(x))| \le C\left[h_{ij}^k\right]_\alpha \frac{1}{\|x-y\|^n}\|x-y\|^\alpha = C'\frac{1}{\|x-y\|^{n-\alpha}}$$

and the right hand side is a member of $L^1(O)$, where $C, C' > 0$ are constants.
We set

$$q_k(x) := p_k(x) - p_k(0) \quad , \ \forall x \in \mathbb{R}^n$$

and get as a consequence

$$\Delta q_k = \Delta p_k\,. \tag{5.12}$$

We are going to show the following estimate for $x_1, x_2 \in \mathbb{R}^n$ arbitrary:

$$|q_k(x_1) - q_k(x_2)| \le M\,\|h_{ij}\|_\alpha \inf\{\log(2+\|x_1-x_2\|), \|x_1-x_2\|^\alpha)\},$$

with a constant $M > 0$ independent of $k \in \mathbb{N}$.
To prove the Hölder-continuity of q_k we will directly estimate the necessary bounds. We state an

important fact for our subsequent approach. Let $0 < R_1 < R_2$. Then there holds

$$\begin{aligned}
\int\limits_{B_{R_2}\setminus B_{R_1}} \partial_i\partial_j \mathit{\Gamma}(y)\,\mathrm{d}y &= \int\limits_{B_{R_2}\setminus B_{R_1}} \partial_i \frac{y_j}{\|y\|}\,\mathrm{d}y \\
&= \int\limits_{\partial B_{R_2}} \frac{y_j}{\|y\|^n}\frac{y_i}{\|y\|}\,\mathrm{d}y - \int\limits_{\partial B_{R_1}} \frac{y_j}{\|y\|^n}\frac{y_i}{\|y\|}\,\mathrm{d}y \\
&= \int\limits_{S^{n-1}} \frac{R_2 y_j}{R_2^n}\frac{R_2 y_i}{R_2} R_2^{n-1}\,\mathrm{d}y - \int\limits_{S^{n-1}} \frac{R_1 y_j}{R_1^n}\frac{R_1 y_i}{R_1} R_1^{n-1}\,\mathrm{d}y \\
&= 0\,.
\end{aligned} \tag{5.13}$$

The following calculations are independent of $i, j \in \{1, \dots, n\}$ and $k \in \mathbb{N}$, so that we can make a few abbreviations.
We set

$$\begin{aligned}
K_1 &:= \partial_i\partial_j \mathit{\Gamma}(x_1 - y) \\
K_2 &:= \partial_i\partial_j \mathit{\Gamma}(x_2 - y) \\
h_1 &:= h_{ij}^k(x_1) \\
h_2 &:= h_{ij}^k(x_2) \\
h &:= h_{ij}^k(y)\,.
\end{aligned}$$

There holds because of (5.7)

$$|h_1 - h_2| \le C\,[h_{ij}]_\alpha\,\|x_1 - x_2\|^\alpha \le \bar{C}\,\|h_{ij}\|_\alpha\,\|x_1 - x_2\|^\alpha$$

for $\alpha \in (0,1)$, $C, \bar{C} > 0$. Fix $x_1, x_2 \in \mathbb{R}^n$ and set

$$\rho := 2\,\|x_1 - x_2\|\;.$$

Choose $R > 0$ large enough such that $\operatorname{supp} h_{ij}^k \subset B_R(x_1)$. Then with the definition of q_k and (5.11) one obtains

$$\begin{aligned}
&q_k(x_1) - q_k(x_2) \\
&\quad= \int\limits_{B_R(x_1)} K_1[h - h_1] - K_2[h - h_2]\,\mathrm{d}y \\
&\quad= \int\limits_{B_\rho(x_1)} K_1[h - h_1] - K_2[h - h_2]\,\mathrm{d}y + \int\limits_{B_R(x_1)\setminus B_\rho(x_1)} K_1[h - h_1] - K_2[h - h_2] \pm K_1\,h_1\,\mathrm{d}y \\
&\quad= \int\limits_{B_\rho(x_1)} K_1[h - h_1] - K_2[h - h_2]\,\mathrm{d}y + \int\limits_{B_R(x_1)\setminus B_\rho(x_1)} K_1[h_1 - h_2] + [K_1 - K_2][h_1 - h_2]\,\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&\leq \int\limits_{B_\rho(x_1)} K_1[h-h_1]\,\mathrm{d}y + \int\limits_{B_\rho(x_1)} K_2[h-h_2]\,\mathrm{d}y \\
&\quad + \int\limits_{B_R(x_1)\setminus B_\rho(x_1)} K_1[h_1-h_2]\,\mathrm{d}y + \int\limits_{B_R(x_1)\setminus B_\rho(x_1)} [K_1-K_2][h_1-h_2]\,\mathrm{d}y\,.
\end{aligned}
$$

The third integral vanishes because h_1 and h_2 do not depend on y and (5.13) holds.
In light of this fact, it holds that

$$
\begin{aligned}
|q_k(x_1)-q_k(x_2)| &\leq \underbrace{\int\limits_{B_\rho(x_1)} |K_1||h-h_1|\,\mathrm{d}y}_{=:I_1} + \underbrace{\int\limits_{B_\rho(x_1)} |K_2||h-h_2|\,\mathrm{d}y}_{=:I_2} \\
&\quad + \underbrace{\int\limits_{B_R(x_1)\setminus B_\rho(x_1)} |K_1-K_2||h_1-h_2|\,\mathrm{d}y}_{=:I_3} \\
&= I_1 + I_2 + I_3\,.
\end{aligned}
$$

We are going to show the Hölder property for each term separately, where $C_i, \tilde{C}_i, \hat{C}_i$ will denote positive constants for $i = 1, \dots, 6$.
For the first integral there holds

$$
\begin{aligned}
I_1 = \int\limits_{B_\rho(x_1)} |K_1||h-h_1|\,\mathrm{d}y &\leq C_1\,[h_{ij}]_\alpha \int\limits_{B_\rho(x_1)} \|y-x_1\|^{-n}\,\|y-x_1\|^{\alpha}\,\mathrm{d}y \\
&= C_1\,[h_{ij}]_\alpha \int\limits_0^\rho \int\limits_{S^{n-1}} \|(x_1+\lambda\omega)-x_1\|^{-n+\alpha}\,\lambda^{n-1}\,\mathrm{d}\sigma(\omega)\,\mathrm{d}\lambda \\
&= C_1\,[h_{ij}]_\alpha \int\limits_0^\rho \int\limits_{S^{n-1}} \|\omega\|^{-n+\alpha}\,\lambda^{-n+\alpha}\lambda^{n-1}\,\mathrm{d}\sigma(\omega)\,\mathrm{d}\lambda \\
&= C_2\,[h_{ij}]_\alpha \int\limits_0^\rho \lambda^{\alpha-1}\,\mathrm{d}\lambda \;=\; C_3\,[h_{ij}]_\alpha\,\rho^\alpha \leq C_4\,\|h_{ij}\|_\alpha\,\|x_1-x_2\|^\alpha
\end{aligned}
$$

and for the second integral

$$
\begin{aligned}
I_2 = \int\limits_{B_\rho(x_1)} |K_2||h-h_2|\,\mathrm{d}y &\leq \tilde{C}_1\,[h_{ij}]_\alpha \int\limits_{B_\rho(x_1)} \|y-x_2\|^{-n+\alpha}\,\mathrm{d}y \\
&\leq \tilde{C}_1\,[h_{ij}]_\alpha \int\limits_{B_{\frac{3\rho}{2}}(x_2)} \|y-x_2\|^{-n+\alpha}\,\mathrm{d}y
\end{aligned}
$$

$$= \tilde{C}_2 \, [h_{ij}]_\alpha \int_0^{\frac{3\rho}{2}} \lambda^{\alpha-1} \, \mathrm{d}\lambda \;\le\; \tilde{C}_3 \, [h_{ij}]_\alpha \, \rho^\alpha \le \bar{C}_3 \, \|h_{ij}\|_\alpha \, \|x_1 - x_2\|^\alpha .$$

The third integral is estimated by

$$\begin{aligned}
I_3 &= \int\limits_{B_R(x_1)\setminus B_\rho(x_1)} |K_1 - K_2||h - h_2| \, \mathrm{d}y \\
&\le [h_{ij}]_\alpha \int\limits_{B_R(x_1)\setminus B_\rho(x_1)} \|(\nabla \partial_i \partial_j \mathit{\Gamma})(x_3 - y)\| \, \|x_1 - x_2\| \, \|y - x_2\|^\alpha \, \mathrm{d}y, \\
&\qquad\qquad \text{here } x_3 = x_1 + \theta(x_2 - x_1), \text{ with } \theta \in (0,1) \\
&\le \hat{C}_1 \, [h_{ij}]_\alpha \, \|x_1 - x_2\| \int\limits_{B_R(x_1)\setminus B_\rho(x_1)} \frac{\|y - x_2\|^\alpha}{\|y - x_3\|^{n+1}} \, \mathrm{d}y, \\
&\qquad\qquad \text{now since } \|y - x_2\| \le 2 \, \|y - x_1\| \text{ and } \|y - x_3\| \ge 1/2 \, \|y - x_1\| \\
&\le \hat{C}_2 \, [h_{ij}]_\alpha \, \|x_1 - x_2\| \int\limits_{B_R(x_1)\setminus B_\rho(x_1)} \frac{\|y - x_1\|^\alpha}{\|y - x_1\|^{n+1}} \, \mathrm{d}y \\
&\le \hat{C}_2 \, [h_{ij}]_\alpha \, \|x_1 - x_2\| \int_\rho^{+\infty} \int\limits_{S^{n-1}} \|\lambda\omega\|^{-n-1+\alpha} \, \lambda^{n-1} \, \mathrm{d}\sigma(\omega) \, \mathrm{d}\lambda \\
&= \hat{C}_3 \, [h_{ij}]_\alpha \, \|x_1 - x_2\| \int_\rho^{+\infty} \lambda^{-2+\alpha} \, \mathrm{d}\lambda = \hat{C}_4 \, [h_{ij}]_\alpha \, \|x_1 - x_2\| \, \rho^{-1+\alpha} \\
&= \hat{C}_5 \, [h_{ij}]_\alpha \, \rho^\alpha \le \hat{C}_6 \, \|h_{ij}\|_\alpha \, \|x_1 - x_2\|^\alpha .
\end{aligned}$$

We thereby deduce

$$|q_k(x_1) - q_k(x_2)| \le C \, \|h_{ij}\|_\alpha \, \|x_1 - x_2\|^\alpha \tag{5.14}$$

with a positive constant C.
A logarithmic growth of q_k at infinity is yet to show. Assume for simplicity $x_1 = 0$ (the general case is treated equally) and $\|x_2\| > 1$ ($\|x_1 - x_2\| > 1$ in the general case), which implies $B_R(x_1) = B_R(0) = B_R$, $K_1 = \partial_i \partial_j \mathit{\Gamma}(y)$, $h_1 = h(0)$ and $\rho = 2 \, \|x_2\|$.

We may write for the first integral

$$
\begin{aligned}
I_1 &= \int\limits_{B_\rho} |K_1||h-h_1|\,\mathrm{d}y \\
&= \int\limits_{B_1} |K_1||h-h_1|\,\mathrm{d}y + \int\limits_{B_\rho\setminus B_1} |K_1||h-h_1|\,\mathrm{d}y \\
&\leq C_1\,[h_{ij}]_\alpha \int\limits_{B_1} \|y\|^{-n+\alpha}\ \mathrm{d}y + C_2\,\|h_{ij}\|_\infty \int\limits_{B_\rho\setminus B_1} \|y\|^{-n}\ \mathrm{d}y \\
&= C_1\,[h_{ij}]_\alpha \int\limits_0^1 \int\limits_{S^{n-1}} \frac{\lambda^{n-1}}{\|\lambda\omega\|^{n-\alpha}}\,\mathrm{d}\sigma(\omega)\,\mathrm{d}\lambda + C_2\,\|h_{ij}\|_\infty \int\limits_1^\rho \int\limits_{S^{n-1}} \frac{\lambda^{n-1}}{\|\lambda\omega\|^{n}}\,\mathrm{d}\sigma(\omega)\,\mathrm{d}\lambda \\
&= C_3\,[h_{ij}]_\alpha \int\limits_0^1 \lambda^{\alpha-1}\,\mathrm{d}\lambda + C_4\,\|h_{ij}\|_\infty \int\limits_1^\rho \lambda^{-1}\,\mathrm{d}\lambda \\
&\leq C_5\,\|h_{ij}\|_\alpha\,(1+\log(\rho)) \leq C_6\,\|h_{ij}\|_\alpha \log(2+\|x_2\|)
\end{aligned}
$$

and for the second integral

$$
\begin{aligned}
I_2 &= \int\limits_{B_\rho} |K_2||h-h_2|\,\mathrm{d}y \leq \int\limits_{B_{\frac{3\rho}{2}}(x_2)} |K_2||h-h_2|\,\mathrm{d}y \\
&= \int\limits_{B_1(x_2)} |K_2||h-h_2|\,\mathrm{d}y + \int\limits_{B_{\frac{3\rho}{2}}(x_2)\setminus B_1(x_2)} |K_2||h-h_2|\,\mathrm{d}y \\
&\leq C_1\,[h_{ij}]_\alpha \int\limits_{B_1(x_2)} \|y-x_2\|^{-n+\alpha}\ \mathrm{d}y + C_2\,\|h_{ij}\|_\infty \int\limits_{B_{\frac{3\rho}{2}}(x_2)\setminus B_1(x_2)} \|y-x_2\|^{-n}\ \mathrm{d}y \\
&= C_1\,[h_{ij}]_\alpha \int\limits_0^1 \int\limits_{S^{n-1}} \frac{\lambda^{n-1}}{\|(x_2+\lambda\omega)-x_2\|^{+n-\alpha}}\,\mathrm{d}\sigma(\omega)\,\mathrm{d}\lambda \\
&\qquad + C_2\,\|h_{ij}\|_\infty \int\limits_1^{\frac{3\rho}{2}} \int\limits_{S^{n-1}} \frac{\lambda^{n-1}}{\|(x_2+\lambda\omega)-x_2\|^{n}}\,\mathrm{d}\sigma(\omega)\,\mathrm{d}\lambda \\
&= C_3\,[h_{ij}]_\alpha \int\limits_0^1 \lambda^{\alpha-1}\,\mathrm{d}\lambda + C_4\,\|h_{ij}\|_\infty \int\limits_1^{\frac{3\rho}{2}} \lambda^{-1}\,\mathrm{d}\lambda \\
&\leq C_5\,\|h_{ij}\|_\alpha\,(1+\log(\rho)) \leq C_6\,\|h_{ij}\|_\alpha \log(2+\|x_2\|)\,.
\end{aligned}
$$

For the third integral one has

$$\begin{aligned}
I_3 &= \int\limits_{B_R\setminus B_\rho} |K_1 - K_2||h - h_2| \,\mathrm{d}y \\
&\leq 2\left\|h_{ij}\right\|_\infty \int\limits_{B_R\setminus B_\rho} \|(\nabla\partial_i\partial_j\Gamma)(x_3 - y)]\| \,\|x_2\| \,\mathrm{d}y \;, \quad (\; x_3 = \theta x_2,\, \theta \in (0,1)) \\
&\leq \hat{C}_1 \left\|h_{ij}\right\|_\infty \|x_2\| \int\limits_{B_R\setminus B_\rho} \frac{1}{\|y - x_3\|^{n+1}} \,\mathrm{d}y \,, \qquad\qquad (\|y - x_3\| \geq {}^1\!/\!_2 \,\|y\|) \\
&\leq \hat{C}_2 \left\|h_{ij}\right\|_\infty \|x_2\| \int\limits_{B_R\setminus B_\rho} \frac{1}{\|y\|^{n+1}} \,\mathrm{d}y \\
&\leq \hat{C}_3 \left\|h_{ij}\right\|_\infty \|x_2\| \int\limits_{\rho}^{+\infty} \lambda^{-2} \,\mathrm{d}\lambda \\
&= \hat{C}_5 \left\|h_{ij}\right\|_\infty \leq \hat{C}_5 \left\|h_{ij}\right\|_\alpha \log(2 + \|x_2\|) \,.
\end{aligned}$$

Altogether one obtains uniformly in $k \in \mathbb{N}$:

$$|q_k(x_1) - q_k(x_2)| \leq M \left\|h_{ij}\right\|_\alpha \|x_1 - x_2\|^\alpha \,, \tag{5.15}$$

as well as, since $q_k(0) = p_k(0) - p_k(0) = 0$,

$$|q_k(x)| \leq M' \left\|h_{ij}\right\|_\alpha \log(2 + \|x\|) \tag{5.16}$$

for any $x \in \mathbb{R}^n$ with positive constants M and M'. In the following we will extract a subsequence of (q_k) which converges uniformly on any compact subset of $\mathbb{R}^n$. Let A be a compact subset of $\mathbb{R}^n$. One has uniformly in $k \in \mathbb{N}$:

$$\sup_{x\in A}|q_k(x)| \leq \sup_{x\in A}\; M' \left\|h_{ij}\right\|_\alpha \log(2 + \|x\|) < +\infty \,.$$

Also, (q_k) is equicontinuous since for any $\varepsilon > 0$ and any $k \in \mathbb{N}$ one can choose in sight of (5.14) $\delta = (C\left\|h_{ij}\right\|_\alpha)^{-1/\alpha}\varepsilon^{1/\alpha}$, which yields

$$|q_k(x) - q_k(x')| \leq C\left\|h_{ij}\right\|_\alpha \delta^\alpha = \varepsilon \quad ,\forall \left\|x - x'\right\| \leq \delta \,.$$

Hence, (q_k) is a uniformly bounded and equicontinuous sequence in $C(A;\mathbb{R})$. The theorem of Arzelà-Ascoli ensures the existence of a subsequence (q_{k_l}) which converges uniformly in $C(A;\mathbb{R})$, i.e.

$$q_{k_l} \to q_A \in C(A;\mathbb{R}) \;\text{ as } l \to \infty \,.$$

If A' is any other compact subset of $\mathbb{R}^n$ with $A \subset A'$, then (q_{k_l}) enjoys the same properties on A' as the sequence (q_k). Thus, there exists a subsequence $(q_{k_l'}) \subseteq (q_{k_l})$ and a function $q_{A'} \in C(A';\mathbb{R})$

such that

$$q_{k'_l} \to q_{A'} \in C(A';\mathbb{R}) \quad \text{as } l \to \infty$$

and

$$q_{A'}(x) = q_A(x) \quad , \forall x \in A\,.$$

Hence, for any sequence of compact subsets (A_j) with $A_j \subset \mathbb{R}^n$ and $A_j \subseteq A_{j+1}$ one may inductively choose subsequences $(q_{k_{j,l}})_l$ which converge in $C(A_j;\mathbb{R})$ to q_{A_j} such that $q_{A_{j+1}} = q_{A_j}$ on A_j. As a consequence, we may correctly define a continuous function p on $\mathbb{R}^n$ in the following way. For $x \in \mathbb{R}^n$ choose any compact subset $A \subset \mathbb{R}^n$ with $x \in A$ and define

$$p(x) \coloneqq q_A(x),$$

where q_A is the limit of the subsequence of (q_k) which converges on A.
This function satisfies

$$|p(x_1) - p(x_2)| \leq M \left\|h_{ij}\right\|_\alpha \left\|x_1 - x_2\right\|^\alpha$$

as well as

$$|p(x)| \leq M' \left\|h_{ij}\right\|_\alpha \log(2 + \|x\|)$$

because of the uniform bounds in (5.15) and (5.16). For $\varphi \in C_0^\infty(\mathbb{R}^n)$ let $\tilde{A} = \operatorname{supp}\varphi$ and $(q_{\tilde{k}_i})$ the convergent subsequence of (q_k) with the limit function $q_{\tilde{A}}$. Then one obtains

$$\left\langle \Delta\, q_{\tilde{k}_i}, \varphi \right\rangle = \left\langle q_{\tilde{k}_i}, \Delta\,\varphi \right\rangle \to \left\langle q_{\tilde{A}}, \Delta\,\varphi \right\rangle = \left\langle p, \Delta\,\varphi \right\rangle \quad \text{as } i \to \infty\,.$$

Also, our preceding considerations in (5.10) and (5.12) yield

$$\left\langle \Delta\, q_{\tilde{k}_i}, \varphi \right\rangle = \left\langle \Delta\, p_{\tilde{k}_i}, \varphi \right\rangle = \left\langle v_{\tilde{k}_i}, \varphi \right\rangle \to \left\langle v, \varphi \right\rangle \quad , \text{as } i \to \infty\,.$$

This implies

$$\Delta\, p = v \text{ in } D'(R^n)\,.$$

Uniqueness. One verifies readily that the constructed solution p is a member of N because of the fulfilled estimates. Suppose there exists another solution $\tilde{p}$ of (5.6) in N. Then,

$$\Delta\,(p - \tilde{p}) = 0 \;\text{ in } D'(\mathbb{R}^n) \quad , \quad \lim_{\|x\| \to \infty} \frac{(p - \tilde{p})(x)}{\|x\|} = 0 \quad \text{and} \quad (p - \tilde{p})(0) = 0\,.$$

Hence, $p - \tilde{p}$ is weakly harmonic and continuous in $\mathbb{R}^n$. Weyl's lemma (see Theorem A.4) implies that there exists a harmonic function $\Phi \in C^\infty(\mathbb{R}^n)$ with $\Phi = p - \tilde{p}$ almost everywhere, therefore $p - \tilde{p}$ is harmonic in $\mathbb{R}^n$ because of its continuity. Liouville's theorem for harmonic functions (see Theorem A.5) yields that $p - \tilde{p}$ is a constant function. Then, $p - \tilde{p}$ must be identically 0 and subsequently there holds $p = \tilde{p}$. □

We continue directly with the proof of the lemma.

Proof of Lemma 5.2. Choose p_k as in the previous proof, i.e. for $k \in \mathbb{N}$

$$p_k = \Gamma * v_k \quad \text{and} \quad v_k = \sum_{i,j=1}^{n} \partial_i \partial_j h_{ij}^k(x),$$

whereas $h_{ij}^k = \nu_k \cdot h_{ij}$. The $\nu_k \in C_0^\infty(\mathbb{R}^n)$ ensure that $h_{ij}^k, v_k \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ (for all $i,j = 1, \dots, n$, $k \in \mathbb{N}$) with

$$\left\| h_{ij}^k \right\|_\infty \leq c \left\| h_{ij} \right\|_\infty \quad \text{and} \quad \left\| v_k \right\|_\infty \leq \tilde{c}$$

for some $c, \tilde{c} > 0$. And further that

$$h_{ij}^k \to h_{ij} \text{ and } v_k \to v \quad \text{in } L^\infty(\mathbb{R}^n) \text{ as } k \to \infty\,.$$

Fix $k \in \mathbb{N}$. As in (5.9) the gradient of p_k takes the form

$$\nabla p_k(x) = \int_{\mathbb{R}^n} \nabla \Gamma(x-y) v_k(y) \,\mathrm{d}y$$

in accordance with Lemma 5.1. In fact, we had $q_k(x) \coloneqq p_k(x) + p_k(0)$ which yields $\nabla q_k = \nabla p_k$ and we may directly argue for the sequence (∇p_k). We claim that (∇p_k) converges locally uniformly. Let a be the function from Definition 5.1 with $a(0) = 1$ and $D^\mu a\,(0) = 0$ for any $\mu \in \mathbb{N}_0^n$, $|\mu| \geq 1$. One may write

$$\nabla p_k = \nabla \Gamma * v_k = \nabla(a\Gamma + (1-a)\Gamma) * v_k = \nabla(a\Gamma) * v_k + \nabla((1-a)\Gamma) * v_k$$

Consider the second term of this expression in $D'(\mathbb{R}^n)$. We get for any $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\begin{aligned}
\langle \nabla((1-a)\Gamma) * v_k, \varphi \rangle &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \nabla((1-a)\Gamma)(y)\, v_k(x-y) \,\mathrm{d}y \,\varphi(x) \,\mathrm{d}x \\
&= \sum_{i,j} \int_{\mathbb{R}^n} \nabla((1-a)\Gamma)(y) \int_{\mathbb{R}^n} \partial_i \partial_j h_{ij}^k(x-y) \varphi(x) \,\mathrm{d}x \,\mathrm{d}y \\
&= \sum_{i,j} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \nabla((1-a)\Gamma)(x-y)\, h_{ij}^k(y) \,\mathrm{d}y \,\partial_i \partial_j \varphi(x) \,\mathrm{d}x \\
&= \sum_{i,j} \int_{\mathbb{R}^n} h_{ij}^k(y) \int_{\mathbb{R}^n} \partial_i \partial_j \nabla((1-a)\Gamma)(x-y)\, \varphi(x) \,\mathrm{d}x \,\mathrm{d}y \\
&= \left\langle \textstyle\sum_{i,j} \left(\partial_i \partial_j \nabla((1-a)\Gamma) * h_{ij}^k \right), \varphi \right\rangle .
\end{aligned}$$

Note that the preceding calculations hold since, for any $\mu \in \mathbb{N}_0^n$, $|\mu| \geq 1$, the derivative

$D^\mu((1-a)\Gamma)$ stays bounded because of the properties of the function a. Thus, there holds

$$\nabla p_k = \nabla(a\Gamma) * v_k + \sum_{i,j} \left(\partial_i\partial_j \nabla((1-a)\Gamma) * h_{ij}^k\right)$$

in $D'(\mathbb{R}^n)$. In order to proceed we need to show that $\nabla(a\Gamma)$ and $D^\alpha\nabla((1-a)\Gamma)$, $|\alpha| = 2$, belong to $L^1(\mathbb{R}^n)^n$. Fix $i, j, l \in \{1, \dots, n\}$ and $r > 0$ arbitrarily, then by the properties of the function a there holds

$$\begin{aligned}\|\partial_i(a\Gamma)(x)\| &\leq \|(\partial_i a)\Gamma(x)\| + \|a(\partial_i\Gamma)(x)\| \\ &\leq \mathbb{1}_{\mathbb{R}^n\setminus B_r} C_1 \frac{1}{\|x\|^{n+1}} + \mathbb{1}_{B_r} C_2 \frac{1}{\|x\|^{n-1}}, \quad C_1, C_2 > 0,\end{aligned}$$

which is a member of $L^1(\mathbb{R}^n)$. The function $\mathbb{1}_{B_r}$ is identically zero outside of B_r and $\mathbb{1}_{B_r}(x) = 1$ for $x \in B_r$, it is also called the characteristic function on B_r. Furthermore, it holds that

$$\|D_{ijl}((1-a)\Gamma)(x)\| \leq \sum_{|\mu+\tilde{\mu}|=3} \left\|D^\mu(1-a)D^{\tilde{\mu}}\Gamma(x)\right\| .$$

Two cases arise for $\mu \in \mathbb{N}_0^n$, $|\mu| \leq 3$. In the first case $|\mu| \geq 1$ and in the second case $|\mu| = 0$. For $|\mu| \geq 1$ there holds

$$\begin{aligned}\left\|D^\mu(1-a)D^{\tilde{\mu}}\Gamma(x)\right\| &= \left\|D^\mu a\, D^{\tilde{\mu}}\Gamma(x)\right\| \\ &\leq \mathbb{1}_{\mathbb{R}^n\setminus B_r} \bar{C}_1 \frac{1}{\|x\|^{n+1}} + \mathbb{1}_{B_r} \bar{C}_2 \frac{1}{\|x\|^{n-1}},\end{aligned}$$

with $\bar{C}_1,\ \bar{C}_2 > 0$, which belongs to $L^1(\mathbb{R}^n)$. For $|\mu| = 0$ it holds that

$$\left\|(1-a)D^{\tilde{\mu}}\Gamma(x)\right\| \leq \mathbb{1}_{\mathbb{R}^n\setminus B_r} \tilde{C}_1 \frac{1}{\|x\|^{n+1}} + \mathbb{1}_{B_r} \tilde{C}_2, \quad \tilde{C}_1, \tilde{C}_2 > 0,$$

which also belongs to $L^1(\mathbb{R}^n)$. Subsequently, we showed that

$$\nabla(a\Gamma), D^\alpha\nabla((1-a)\Gamma) \in L^1(\mathbb{R}^n)^n \quad \text{with } |\alpha| = 2\,.$$

In light of this fact we get in using Young's inequality (see Theorem A.9)

$$\begin{aligned}\|\nabla p_k\|_\infty &\leq \|\nabla(a\Gamma) * v_k\|_\infty + \sum_{i,j} \left\|\left(\partial_i\partial_j\nabla((1-a)\Gamma) * h_{ij}^k\right)\right\|_\infty \\ &\leq \|\nabla(a\Gamma)\|_1 \|v_k\|_\infty + \sum_{i,j} \|(\partial_i\partial_j\nabla((1-a)\Gamma))\|_1 \left\|h_{ij}^k\right\|_\infty \\ &\leq \tilde{c} + c\sum_{i,j} \|h_{ij}\|_\infty =: C < +\infty\end{aligned}$$

with $c, \tilde{c} > 0$ and $\tilde{c} = \tilde{c}\left(\|h_{ij}\|_{W^{1,\infty}}, \|v\|_{\infty}\right)$. Hence, there exists $C > 0$ such that

$$\|\nabla p_k\|_{\infty} \leq C \quad \text{for all } k \in \mathbb{N}\,.$$

We are going to examine the behavior for $k \to \infty$. One observes, that the integrands of ∇p_k which are

$$\nabla(a\Gamma)(x-y)v_k(y) \quad \text{and} \quad \partial_i\partial_j\nabla((1-a)\Gamma)(x-y)h_{ij}^k(y)$$

tend to

$$\nabla(a\Gamma)(x-y)v(y) \quad \text{and} \quad \partial_i\partial_j\nabla((1-a)\Gamma)(x-y)h_{ij}(y)$$

respectively as $k \to \infty$.
The preceding inspections admit the application of Lebesgue's theorem of dominated convergence and we get

$$\begin{aligned}\lim_{k\to\infty} \nabla p_k(x) &= \int\limits_{\mathbb{R}^n} \nabla(a\Gamma)(x-y)v(y)\,\mathrm{d}y + \sum_{i,j}\int\limits_{\mathbb{R}^n} \partial_i\partial_j\nabla((1-a)\Gamma)(x-y)h_{ij}(y)\,\mathrm{d}y\\ &=: g(x)\end{aligned}$$

which belongs to $C_b(\mathbb{R}^n)^n$ as a consequence of Theorem 4.3 (the bound is the same as for $\|\nabla p_k\|_{\infty}$). The gradients of any subsequence $(p_{k_l}) \subseteq (p_k)$ converge to g in $D'(\mathbb{R}^n)$ which, by the definition of q_k, also holds for any subsequence $(q_{k_l}) \subseteq (q_k)$. Hence, the construction of our solution implies

$$\nabla p = g\,.$$

□

Remark. In fact, for our proceeding inspections we will only need the representation of the pressure gradient as stated in the last lemma. A necessary condition for a solution (u, p) to the n-dimensional Euler equation is that $u = (u_1, \ldots, u_n)$ and p satisfy $-\Delta p = v$ in $D'(\mathbb{R}^n)$ with $v = \mathrm{Tr}\,(Ju)^2 = \sum_{i,j} \partial_i\partial_j(u_iu_j)$. The in Theorem 5.1 obtained uniqueness of p assures that the right hand side of the Euler equation takes the in Lemma 5.2 stated form for $u(\,.\,,t) \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)$ with $\operatorname{div} u = 0$. More precisely, for $h_{ij} = u_iu_j$, $i,j \in \{1,\ldots,n\}$, as in (5.3) and in sight of Lemma 5.2 under consideration of Theorem 5.1, a solution u of the incompressible Euler equations in $L^{\infty}_{loc}([0,T), W^{1,\infty}(\mathbb{R}^n,\mathbb{R}^n)$, $T > 0$, satisfies

$$\begin{aligned}\partial_t u + (u\cdot\nabla)u &= \nabla(a\Gamma) * v \;+ \textstyle\sum_{i,j}\left(\partial_i\partial_j\nabla((1-a)\Gamma) * h_{ij}\right)\\ \operatorname{div} u &= 0\,.\end{aligned}$$

5.2. The PDE for the characteristic curves

The solution u verifies for any $x \in \mathbb{R}^n$: $u(x, .) \in L^\infty_{loc}([0,T))$, which yields for any $T^* < T$ that $u(x, .) \in W^{1,\infty}([0,T^*])$ since it solves the Euler equations. Thus, u is continuous in time and we may therefore assume $u \in C_b([0,T); C^{m,\alpha}(\mathbb{R}^n;\mathbb{R}^n))$ (with T possibly smaller). The solution to (U), denoted by X, is such that $X - Id \in C^1([0,T), C^{m,\alpha}(\mathbb{R}^n;\mathbb{R}^n))$, $Id(t) = Id$ and is unique in $C^1([0,T); F_1)$, where

$$F_1 := \left\{ f \in \mathrm{L}^\infty\left((1+\|x\|)^{-1}, \mathbb{R}^n; \mathbb{R}^n\right) \mid Jf \in C^{m-1,\alpha}(\mathbb{R}^n; \mathbb{R}^{n\times n}) \right\} .$$

We will see that X is even C^2 in time. A differentiation of the characteristic equation yields

$$JX_t(x,t) = Ju(X(x,t),t)JX(x,t) \quad \text{with} \quad JX(\,.\,,0) = I_n\,. \tag{5.17}$$

It has the solution

$$JX(x,t) = \exp \int_0^t Ju(X(x,s),s)\,\mathrm{d}s$$

which is invertible and bounded for any $t \in [0,T)$. Thus we may write

$$Ju(X(x,t),t) = \left(JX_t \cdot (JX)^{-1}\right)(x,t)\,. \tag{5.18}$$

Also, the following result applies.

Lemma 5.3. For the solution X of the characteristic equation (U) with $\operatorname{div} u = 0$, there holds

$$|JX|(x,t) = \det JX(x,t) = 1$$

for all $(x,t) \in \mathbb{R}^n \times [0,T)$.

Proof. Set

$$F := JX,$$

then

$$F_t = JX_t$$

and in applying Jacobi's formula for the derivative of the determinant function there holds with (5.18)

$$\begin{aligned}
\partial_t \det F(x,t) &= \mathrm{Tr}\,(\mathrm{cof}\,(F)^\top F_t)(x,t) \\
&= [\det(F)\,\mathrm{Tr}\,(F^{-1}F_t)](x,t) \\
&= \det(F(x,t))\,\mathrm{Tr}\,((Ju)(X(x,t),t)) \\
&= \det(F(x,t))\,(\operatorname{div} u)(X(x,t),t) \\
&= 0\,.
\end{aligned}$$

Thus, we deduce, since F is continuous in t and $\det F(x,0) = \det I_n = 1$ for any $x \in \mathbb{R}^n$, that

$$\det JX(x,t) = 1$$

for all $t \in [0,T)$. $\square$

As a consequence, one obtains with (5.18)

$$\begin{aligned} Ju(X(x,t),t) &= \left(JX_t \cdot (JX)^{-1}\right)(x,t) \\ &= \left(JX_t \cdot \det(JX)^{-1} \cdot \operatorname{cof}(JX)^\top\right)(x,t) \\ &= \left(JX_t \cdot \operatorname{cof}(JX)^\top\right)(x,t) \end{aligned} \tag{5.19}$$

and we may write

$$\begin{aligned} X_{tt}(x,t) &= u_t(X(x,t),t) + (Ju)(X(x,t),t) \cdot X_t(x,t) \\ &= u_t(X(x,t),t) + (Ju)(X(x,t),t) \cdot u(X(x,t),t) \\ &= u_t(X(x,t),t) + (u \cdot \nabla)u\,(X(x,t),t) \\ &= -\nabla p(X(x,t),t)\,. \end{aligned}$$

Hence, there holds $X \in C^2([0,T);F_1)$ since $\nabla p(x,t)$ is continuous in $t \in [0,T)$. One acquires by Theorem 5.1 with $h_{ij} := u_i u_j$ together with (5.1) and (5.2):

$$\begin{aligned} X_{tt}(x,t) = &\int_{\mathbb{R}^n} \nabla(a\Gamma)(X(x,t)-y)\operatorname{Tr}\,(Ju)^2(y,t)\,\mathrm{d}y \\ &+ \sum_{i,j}\int_{\mathbb{R}^n} \partial_i\partial_j\nabla((1-a)\Gamma)(X(x,t)-y)(u_iu_j)(y,t)\,\mathrm{d}y\,. \end{aligned}$$

By a change of variables $y \mapsto X(y,t)$ and (5.19) inserted, we obtain that X solves the PDE

$$\begin{aligned} \tilde{X}_{tt}(x,t) &= \tilde{G}(\tilde{X},\tilde{X}_t)(x,t), \\ (\tilde{X},\tilde{X}_t)(0) &= (Id, u_0) \end{aligned} \tag{5.20}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T)$, whereas

$$\begin{aligned} \tilde{G}(X,Y)(x,t) = &\int_{\mathbb{R}^n} \nabla(a\Gamma)(X(x,t)-X(y,t))\operatorname{Tr}\left(JY \cdot \operatorname{cof}(JX)^\top\right)^2(y,t)|JX|(y,t)\,\mathrm{d}y \\ &+ \sum_{i,j}\int_{\mathbb{R}^n} \partial_i\partial_j\nabla((1-a)\Gamma)(X(x,t)-X(y,t))(Y_iY_j)(y,t)|JX|(y,t)\,\mathrm{d}y \end{aligned}$$

with $|J(\,.\,)| := \det J(\,.\,)$ and $X,Y : \mathbb{R}^n \times [0,T) \to \mathbb{R}^n$.

6. Holomorphic trajectories

6.1. Derivation of a complex ODE

We begin with a technical lemma and outline therein some properties of the elements of the set

$$S \coloneqq \{z \in \mathbb{C}^n \setminus \{0\} \mid \exists x \in \mathbb{R}^n : \|z - x\| < \beta \|x\|\},$$

which will allow a complex extension of the Newtonian potential from $\mathbb{R}^n$ to S.

Lemma 6.1. Let $0 < \beta < 1/2$, then there holds, for any $z = (z_i)_{i=1}^n$ in S and $x \in \mathbb{R}^n$ such that $\|z - x\| < \beta \|x\|$,

$$(1 - \beta) \|x\| < \|z\| < (1 + \beta) \|x\| , \tag{6.1}$$

$$\frac{1}{c} \|x\| < \left| \sum\nolimits_{i=1}^n z_i^2 \right|^{\frac{1}{2}} < c \|x\| \tag{6.2}$$

for a fixed $c > 0$ and finally

$$\Big(\sum\nolimits_{i=1}^n z_i^2 \Big)^{\frac{1}{2}} \in S_{\pi/4} \coloneqq \{z = \varrho \, \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \mid \varrho > 0 \ , \ |\theta| < \pi/4\}. \tag{6.3}$$

Proof. For $z \in S$ there exists $x \in \mathbb{R}^n$ such that $\|z - x\| < \beta \|x\|$. We define $\operatorname{Re} z \coloneqq (\operatorname{Re} z_1, \dots, \operatorname{Re} z_n)^T$ and $\operatorname{Im} z \coloneqq (\operatorname{Im} z_1, \dots, \operatorname{Im} z_n)^T$. Then inequality (6.1) follows immediately from the triangle inequality. We also have

$$\|z - x\|^2 = \|\operatorname{Re} z - x\|^2 + \|\operatorname{Im} z\|^2 < \beta^2 \|x\|^2 \ .$$

This yields

$$\|\operatorname{Re} z - x\| < \beta \|x\| \qquad \text{and} \qquad \|\operatorname{Im} z\| < \beta \|x\|$$

and so

$$(1 - \beta) \|x\| < \|\operatorname{Re} z\| < (1 + \beta) \|x\| \ .$$

One calculates directly that

$$\begin{aligned} \operatorname{Re} \sum\nolimits_{i=1}^n z_i^2 &= \sum\nolimits_{i=1}^n \operatorname{Re} z_i{}^2 - \sum\nolimits_{i=1}^n \operatorname{Im} z_i{}^2 = \|\operatorname{Re} z\|^2 - \|\operatorname{Im} z\|^2 \\ &> (1 - \beta)^2 \|x\|^2 - \beta^2 \|x\|^2 = (1 - 2\beta) \|x\|^2 > 0 \, . \end{aligned}$$

Subsequently, there exists a fixed constant $\tilde{c} > 0$ depending only on β such that for any $z \in S$ and $x \in \mathbb{R}^n$ with $\|z - x\| < \beta \|x\|$ there holds

$$\begin{aligned} 0 &< \frac{1}{\tilde{c}} \|x\|^2 < \operatorname{Re} \sum\nolimits_{i=1}^n z_i^2 \\ &< \left| \sum\nolimits_{i=1}^n z_i^2 \right| < \left[\Big(\|\operatorname{Re} z\|^2 - \|\operatorname{Im} z\|^2 \Big)^2 + \Big(2 \|\operatorname{Re} z\|^2 \cdot \|\operatorname{Im} z\|^2 \Big)^2 \right]^{\frac{1}{2}} < \tilde{c} \|x\|^2 , \end{aligned}$$

which proves (6.2). Furthermore, it has been shown that the real part in the previous equation is positive, which implies

$$\left|\arg\left(\sum_{i=1}^n z_i^2\right)\right| < \frac{\pi}{2}\,.$$

Then $\left(\sum_{i=1}^n z_i^2\right)^{\frac{1}{2}}$ is well defined and we get

$$\left|\arg\left(\left(\sum_{i=1}^n z_i^2\right)^{\frac{1}{2}}\right)\right| < \frac{\pi}{4},$$

yielding (6.3).

□

In order to obtain a complex ODE for the particle trajectories, we need to define an appropriate Banach space in which we can embed the formerly found system of PDEs (5.20). Let $m \geq 1$ as well as $M, R > 0$ and set

$$E := E_1 \times E_2$$

with

$$\begin{aligned} E_1 &:= \{f \in \mathrm{L}^\infty((1+\|x\|)^{-1}, \mathbb{R}^n; \mathbb{C}^n) \mid Jf \in C^{m-1,\alpha}(\mathbb{R}^n, \mathbb{C}^{n\times n})\}, \\ E_2 &:= C^{m,\alpha}(\mathbb{R}^n, \mathbb{C}^n) \end{aligned}$$

and

$$E_R := H(D_R, E)$$

which are Banach spaces equipped with the norms

$$\begin{aligned} \|(X,Y)\|_E &:= \left\|X(\,.\,)(1+\|\,.\,\|)^{-1}\right\|_{L^\infty} + \|JX(\,.\,)\|_{C^{m-1,\alpha}} + \|Y\|_{C^{m,\alpha}}\,, \\ \|\,.\,\|_{E_R} &:= \|\,.\,\|_{L^\infty(D_R,E)} \end{aligned}$$

respectively. Later we want to apply the Cauchy-Lipschitz existence Theorem (4.6) and so we define for a fixed $0 < \beta < 1/2$ the open and convex subsets

$$\begin{aligned} \Omega &:= \{(X,Y) \in E \mid \|JX - \mathrm{I}_n\|_\infty < \beta\,;\, \|(X,Y)\|_E < M\}, \\ \Omega_R &:= H(D_R, \Omega) \end{aligned}$$

in E, E_R respectively. The norm $\|\,.\,\|_\infty$ denotes the operator norm in $\mathrm{L}^\infty(\mathbb{R}^n; \mathbb{C}^{n\times n}) = L^\infty(\mathbb{R}^n)^{n\times n}$ and is given by

$$\|A\|_\infty := \|A\|_{L^\infty(\mathbb{R}^n)^{n\times n}} := \sup_{x\in\mathbb{R}^n} \sup_{\substack{v\in\mathbb{R}^n \\ \|v\|=1}} \|A(x)\cdot v\|_{\mathbb{C}^n}$$

for $A \in L^\infty(\mathbb{R}^n)^{n\times n}$. In sight of our proceedings we state some more important facts.

Lemma 6.2. For $(X, .) \in \Omega_R$, $R > 0$, and arbitrary $x, y \in \mathbb{R}^n, x \neq y$, set

$$Z \coloneqq X(x,r) - X(y,r) \quad \text{and} \quad \tilde{x} \coloneqq x - y\,.$$

Then one has

$$\|Z - \tilde{x}\| < \beta \, \|\tilde{x}\| \,. \tag{6.4}$$

Thus, Z belongs to S and Lemma 6.1 may be applied.

Proof. For any $(X, .) \in \Omega_R$ and $x, y \in \mathbb{R}^n$, $x \neq y$, $r \in D_R$ arbitrary, there holds

$$\begin{aligned} X_i(x,r) - X_i(y,r) &= \int_0^1 \nabla X_i(\lambda x + (1-\lambda) y, r) \cdot (x-y) \,\mathrm{d}\lambda \\ &= (x_i - y_i) + \int_0^1 (\nabla X_i(\lambda x + (1-\lambda)\, y, r) - e_i) \cdot (x-y) \,\mathrm{d}\lambda \end{aligned}$$

for all $i = 1, \ldots, n$ ($(e_i)_{i=1}^n$ denotes the canonical base in $\mathbb{R}^n$). Equivalently,

$$X(x,r) - X(y,r) - (x-y) = \int_0^1 (JX(\lambda x + (1-\lambda)\, y, r) - \mathrm{I}_n) \cdot (x-y) \,\mathrm{d}\lambda$$

and therefore,

$$\begin{aligned} \|(X(x,r) - X(y,r)) - (x-y)\| &\leq \int_0^1 \|(JX(\lambda x + (1-\lambda)\, y, r) - \mathrm{I}_n) \cdot (x-y)\| \,\mathrm{d}\lambda \\ &\leq \int_0^1 \|JX - \mathrm{I}_n\|_\infty \, \|(x-y)\| \,\mathrm{d}\lambda < \beta \, \|x-y\| \,. \end{aligned}$$

We deduce that if $x, y \in \mathbb{R}^n, x \neq y$, $(X, .) \in \Omega_R$ and $Z \coloneqq X(x,r) - X(y,r)$, $\tilde{x} \coloneqq x - y$, then

$$\|Z - \tilde{x}\| < \beta \, \|\tilde{x}\| \,, \tag{6.5}$$

which implies $Z \in S$. ☐

Also, the following technical lemma will be invoked before we proceed. In the following we abbreviate

$$\|A\|_{\infty,R} \coloneqq \sup_{r \in D_R} \|A(r)\|_\infty$$

for $A : D_R \to L^\infty(\mathbb{R}^n)^{n \times n}$.

Lemma 6.3. Let $X : D_R \to C^1(\mathbb{R}^n; \mathbb{C}^n)$ such that $\|JX - \mathrm{I}_n\|_{\infty,R} < \beta < 1$ and set $X_\kappa \coloneqq \operatorname{Re} X + \kappa \operatorname{Im} X$ for $\kappa \in \mathbb{C}$. Then the following propositions hold.

a.) There exists $s > 0$ such that

$$\|JX_\kappa - \mathrm{I}_n\|_{\infty,R} < \beta$$

for any

$$\kappa \in W \coloneqq \{z \in \mathbb{C} \mid |\operatorname{Re} z| < s \text{ and } |\operatorname{Im} z| < 1 + s\}\,.$$

b.) For W as before, $\kappa \in W$ implies $\det(JX_\kappa(x,r)) \neq 0$ for any $(x,r) \in \mathbb{R}^n \times D_R$.

c.) Whenever $X_\kappa(\,.\,,r)$ is real valued for fixed $r \in D_R$ and $\kappa \in W$, then $(JX_\kappa)^{-1}$ is bounded and $X_\kappa(\,.\,,r)$ is a global homeomorphism on $\mathbb{R}^n$.

Remark. We have $X_\mathrm{i} = X$ ($\kappa = \mathrm{i} \in W$) and if $\kappa \in W \cap \mathbb{R}$, then $X_\kappa(\,.\,,r)$ is real valued and hence a global homeomorphism on $\mathbb{R}^n$ for any $r \in D_R$. In addition, the lemma holds equally for $X : [0,R) \to C^1(\mathbb{R}^n;\mathbb{R}^n)$ such that $\|JX - \mathrm{I}_n\|_{\infty,R} < \beta < 1$.

Proof. (a.) For $r \in D_R$ and $x \in \mathbb{R}^n$ set $A := \operatorname{Re} JX(x,r)$ and $B := \operatorname{Im} JX(x,r)$. The following implications hold because of the norm definitions. We have for arbitrary $(x,r) \in \mathbb{R}^n \times D_R$:

$$\begin{aligned}
\|JX - \mathrm{I}_n\|_{\infty,R} < \beta &\Rightarrow \sup_{\substack{v\in\mathbb{R}^n\\ \|v\|=1}} \|(JX(x,r) - \mathrm{I}_n)v\|^2 < \beta^2 \\
&\Rightarrow \sup_{\substack{v\in\mathbb{R}^n\\ \|v\|=1}} \|(A - \mathrm{I}_n + \mathrm{i}B)v\|^2 < \beta^2 \\
&\Rightarrow \sup_{\substack{v\in\mathbb{R}^n\\ \|v\|=1}} \|(A - \mathrm{I}_n)v\|^2 + \|Bv\|^2 < \beta^2 \,.
\end{aligned}$$

Then there exists $s > 0$ such that $b \in [-(1+s), 1+s]$ implies

$$\sup_{\substack{v\in\mathbb{R}^n\\ \|v\|=1}} \|(A - \mathrm{I}_n)v\|^2 + \|bBv\|^2 < \beta^2 \,.$$

and therefore with $\kappa = b\mathrm{i}$,

$$\|JX_\kappa(x,r) - \mathrm{I}_n\|^2 = \sup_{\substack{v\in\mathbb{R}^n\\ \|v\|=1}} \|(A + \kappa B - \mathrm{I}_n)v\|^2 = \sup_{\substack{v\in\mathbb{R}^n\\ \|v\|=1}} \|(A - \mathrm{I}_n)v\|^2 + \|bBv\|^2 < \beta^2 \,.$$

There exists a neighborhood of $\{b\mathrm{i} \mid b \in [-(1+s), 1+s]\}$ in which the proposition also holds and because $[-(1+s), 1+s]$ is closed, one can choose $\tilde{s} > 0$ sufficiently small such that $\{z \in \mathbb{C} \mid |\operatorname{Re} z| < \tilde{s} \text{ and } |\operatorname{Im} z| < 1 + \tilde{s}\}$ is contained in this neighborhood.

(b.) Let $\kappa \in W$, then $\|JX_\kappa - \mathrm{I}_n\|_{\infty,R} < \beta$ is verified in the operator norm. Hence, the Neumann series (see Theorem A.6)

$$\sum_{k\geq 0} (JX_\kappa(x,r) - \mathrm{I}_n)^k$$

converges with respect to the operator norm for any $(x,r) \in \mathbb{R}^n \times D_R$ and represents therefore $(JX_\kappa(x,r))^{-1}$. That means, $JX_\kappa(x,r)$ is invertible for any $(x,r) \in \mathbb{R}^n \times D_R$ and so its determinant does not vanish.

(c.) Let $X_\kappa(\,.\,,r)$ be real valued for $r \in D_R$ and $\kappa \in W$. It follows from (a.) that $\|JX_\kappa - \mathrm{I}_n\| < \beta$ and from (b.) that $(JX_\kappa)^{-1}(\,.\,,r)$ exists globally. Because of the norm-bounded Neumann series representation, $(JX_\kappa)^{-1}(\,.\,,r)$ must be norm-bounded too and the claimed result follows at once from Hadamard's theorem (see Theorem A.8). $\square$

Now, consider the operator

$$
\begin{aligned}
G(X,Y)(x,r) = & \int\limits_{\mathbb{R}^n} \nabla(a\Gamma)(X(x,r)-X(y,r))\mathrm{Tr}\left(JY\cdot \mathrm{cof}\,(JX)^\top\right)^2(y,r)|JX|(y,r)\,\mathrm{d}y \\
& + \sum_{i,j} \int\limits_{\mathbb{R}^n} \partial_i\partial_j\nabla((1-a)\Gamma)(X(x,r)-X(y,r))(Y_iY_j)(y,r)|JX|(y,r)\,\mathrm{d}y
\end{aligned}
$$

for $(x,r)\in\mathbb{R}^n\times D_R$, $R>0$ and $X,Y:\mathbb{R}^n\times D_R\to\mathbb{C}^n$. Then we will see in the next section (more precisely in Lemma 6.5) that $G(X,Y)$ is well defined for $(X,Y)\in\Omega_R$. In the following, we want to investigate the ODE

$$
\boxed{\begin{aligned} \hat{X}''(\tau) &= G(\hat{X},\hat{X}')(\tau), \tau\in D_R \\ (\hat{X},\hat{X}')(0) &= (Id,u_0), \end{aligned}} \tag{6.6}
$$

where we abbreviate $\hat{X}'(\tau) := \frac{\mathrm{d}}{\mathrm{d}\tau}\hat{X}(\tau)$. This equation corresponds to the PDE (5.20) in the way that $\tilde{G}(X,Y)$ coincides with $G(X,Y)$ for functions $X,Y:\mathbb{R}^n\times[0,R)\to\mathbb{R}^n$, $R>0$. We will eventually (in the proof of Theorem 6.1) show that the ODE (6.6) admits a unique solution. In addition, this solution is real valued for real values of τ and must coincide with the formerly found time evolving vector field which corresponds to the particle trajectories. This will be particularly outlined in the proof of our main theorem (Theorem 3.1).
In the above definition of $G(X,Y)$,

$$
\Gamma(z) := C_n\left(\sum\nolimits_{j=1}^n {z_j}^2\right)^{-\frac{1}{2}(n-2)}, \text{ for } z\in S \text{ and } C_n>0,
$$

is a holomorphic function on S which coincides with the Newtonian potential whenever $z\in\mathbb{R}^n\setminus\{0\}$. $\Gamma(z)$ is well defined for $z\in S$ because of Lemma 6.1, and Lemma 6.2 implies that $\Gamma(X(\,.\,)-X(\,.\,))$ is well defined whenever X is such that $(X,\,.\,)\in\Omega_R$. In other words, Γ is by an abuse of notation the complex continuation of the Newtonian potential to S. Especially, if Z and $\tilde{x}$ are as in Lemma 6.2, then there holds with $|\mu|\geq 0$ arbitrary:

$$
|D^\mu\Gamma(Z)| \leq C_1\left|\sum\nolimits_{j=1}^n {Z_j}^2\right|^{-\frac{1}{2}(n-2+|\mu|)} \leq C_2\left\|\tilde{x}\right\|^{-n+2-|\mu|}, \tag{6.7}
$$

where $C_1,C_2>0$ and D^μ is taken with respect to Z.

6.2. Boundedness of the operator G

A crucial requirement for an application of the Cauchy-Lipschitz existence theorem is the boundedness of the operator G in a neighborhood of the initial data. In order to achieve this, we will prove the following Lemma.

Lemma 6.4. Consider for $(X,Y) \in \Omega_R$, $g \in H(D_R, L^\infty(\mathbb{R}^n))$ and $f \in H(D_R, C^{m-1,\alpha}(\mathbb{R}^n))$ (g and f will depend on (X,Y)) the two operators

$$G_1(X,f)(x,r) := \int_{\mathbb{R}^n} \nabla(a\Gamma)(X(x,r)-X(y,r))\, f(y,r)\, |JX|(y,r)\, \mathrm{d}y\,,$$
$$G_2(X,g)(x,r) := \int_{\mathbb{R}^n} D_{ij}\nabla((1-a)\Gamma)(X(x,r)-X(y,r))\, g(y,r)\, \mathrm{d}y\,.$$

Then $G_1(X,f)$ and $G_2(X,g)$ are well defined members of $H(D_R, C^{m,\alpha}(\mathbb{R}^n)^n)$ and there exist constants $C = C(M)$ and $\tilde{C} = \tilde{C}(M, \|g\|_{\infty,R})$, where $\|g\|_{\infty,R} = \|g\|_{H(D_R,L^\infty(\mathbb{R}^n))}$, such that

$$\|G_1(X,f)\|_{H(D_R,C^{m,\alpha}(\mathbb{R}^n)^n)} \leq C(M)\, \|f\|_{H(D_R,C^{m-1,\alpha}(\mathbb{R}^n))}\,,$$
$$\|G_2(X,g)\|_{H(D_R,C^{m,\alpha}(\mathbb{R}^n)^n)} \leq \tilde{C}(M, \|g\|_{\infty,R})\,.$$

Remark. Throughout the following proofs we will make a convention concerning constant factors. $C_{1,2,3,4}, C, \tilde{C}, \bar{C}, \hat{C}$ or any other varieties of C will always denote positive constants and an interchange of these placeholders will only indicate an effect on the constant factor in a certain term.

Proof. We start with G_2 and check the boundedness of the integrand. We recall the properties of the holomorphic function a :
For $s \geq 0$, $|\mu| \geq 1$ and $Z := X(x,r) - X(y,r)$ there holds under consideration of (6.4) and (6.1)

$$\sup_{x\neq y} \{\|x-y\|^s\, |a(Z)|\} < +\infty \tag{6.8}$$
$$\sup_{x\neq y} \{\|x-y\|^{\pm s}\, |D^\mu a(Z)|\} < +\infty \tag{6.9}$$
$$\sup_{x\neq y} \left\{\frac{|1-a(Z)|}{\|x-y\|^s}\right\} < +\infty\,. \tag{6.10}$$

The coordinate functions of the integrand of G_2 are a sum of two forms:

1.) $(D^\beta(1-a)\, D^{\beta'}\Gamma)(Z)\, g(y,r)$,with $|\beta| \geq 1$, $|\beta'| \geq 0$

2.) $(D^\beta(1-a)\, D^{\beta'}\Gamma)(Z)\, g(y,r)$,with $|\beta| = 0$, $|\beta'| \geq 3$.

In the first case we have, with (6.7), (6.9), $g \in H(D_R, L^\infty(\mathbb{R}^n))$ and $\tilde{R} > 0$ fixed

$$\begin{aligned}\left\|(D^{\beta}(1-a)\,D^{\beta'}\Gamma)(Z)g(y,r)\right\| &\leq C_1\frac{\left\|(D^{\beta}a)(Z)\right\|}{\|Z\|^{n-2+|\beta'|}}\leq C_2\frac{\left\|(D^{\beta}a)(Z)\right\|}{\|x-y\|^{n-2+|\beta'|}}\\ &\leq C_3\mathbb{1}_{B_{\tilde{R}}(x)}(y)+C_4\mathbb{1}_{\mathbb{R}^n\setminus B_{\tilde{R}}(x)}(y)\frac{1}{\|x-y\|^{n-2+|\beta'|+s}}\end{aligned}\tag{6.11}$$

which is a member of $L^1_y(\mathbb{R}^n)$, where $\mathbb{1}_{B_{\tilde{R}}(x)}$ denotes the characteristic function on $B_{\tilde{R}}(x)\subseteq\mathbb{R}^n$ and $s\in\mathbb{R}$ is such that $s>|\beta'|-2$. In the second case we have with (6.7), (6.10), $g\in H(D_R,L^\infty(\mathbb{R}^n))$ and $\tilde{R}>0$ fixed

$$\begin{aligned}\left\|((1-a)\,D^{\beta'}\Gamma)(Z)g(y,r)\right\| &\leq C_1\frac{\|(1-a)(Z)\|}{\|Z\|^{n-2+|\beta'|}}\leq C_2\frac{\|(1-a)(Z)\|}{\|x-y\|^{n-2+|\beta'|}}\\ &\leq C_3\mathbb{1}_{B_{\tilde{R}}(x)}(y)+C_4\mathbb{1}_{\mathbb{R}^n\setminus B_{\tilde{R}}(x)}(y)\frac{1}{\|x-y\|^{n-2+|\beta'|}}\end{aligned}\tag{6.12}$$

which is also a member of $L^1_y(\mathbb{R}^n)$ since $|\beta'|\geq 3$. In summary, we established an $L^1_y(\mathbb{R}^n)$-bound, uniform in r but non-uniform in x, for $(D^{\beta}(1-a)\,D^{\beta'}\Gamma)(Z)g(y,r)$, for any $\beta,\beta'\in\mathbb{N}_0^n$ with $|\beta+\beta'|\geq 3$. Hence, $G_2(X,g)$ is well defined and a holomorphic function on D_R for fixed x because its integrand is holomorphic and bounded by an $L^1(\mathbb{R}^n)$ function, see Theorem 4.4 for details.
We turn to the differentiability in $x\in\mathbb{R}^n$. For $\varepsilon>0$, consider the following integral kernels

$$J_\varepsilon := [(1-a(\tfrac{\cdot}{\varepsilon}))D^{\beta}(1-a)\,D^{\beta'}\Gamma](Z)\qquad ,0\leq|\mu|\leq m,$$

such that

$$J_0 := \lim_{\varepsilon\to 0}J_\varepsilon = [D^{\beta}(1-a)\,D^{\beta'}\Gamma](Z)$$

where $D^{\beta},D^{\beta'}$ are taken with respect to $(X_1,\dots,X_n)$ and $|\beta+\beta'|\geq 3$. We set $W_0 := \int_{\mathbb{R}^n}J_0\,\mathrm{d}y$ and claim that

$$\partial_i W_0(x)=\partial_i\int_{\mathbb{R}^n}J_0\,\mathrm{d}y=\int_{\mathbb{R}^n}\partial_i J_0\,\mathrm{d}y =: V(x)\quad ,i\in\{1,\dots,n\}\,.\tag{6.13}$$

In fact, one may not directly apply Corollary 4.2, since the bounds in (6.11) and (6.12) are not uniform in $x\in\mathbb{R}^n$. Thus we need to show the differentiability as in the proof for the derivative of the Newtonian potential in Lemma 5.1. The integral $W_\varepsilon(x) := \int_{\mathbb{R}^n}J_\varepsilon\,\mathrm{d}y$ is differentiable with the partial derivatives

$$\partial_i W_\varepsilon(x)=\partial_i\int_{\mathbb{R}^n}J_\varepsilon\,\mathrm{d}y=\int_{\mathbb{R}^n}\partial_i J_\varepsilon\,\mathrm{d}y,\tag{6.14}$$

because of the bounds (6.11), (6.12), $0<1-a(Z/\varepsilon)<1$ and $(\partial_i a)(Z/\varepsilon)(\partial_i X(x,r)\varepsilon^{-1}\leq C<+\infty$ for all $x\neq y$, $\varepsilon>0$ and Corollary 4.2.

From this, we obtain

$$
\begin{aligned}
V(x) - \partial_i W_\varepsilon(x) &= \int\limits_{\mathbb{R}^n} \partial_i [a(\tfrac{\cdot}{\varepsilon}) D^\beta (1-a)\, D^{\beta'} \Gamma](Z)\, \mathrm{d}y\,. \\
&= \partial_i X(x,r) \int\limits_{\mathbb{R}^n} [(\partial_i a(\tfrac{\cdot}{\varepsilon})\varepsilon^{-1}) D^\beta (1-a)\, D^{\beta'}\Gamma](Z) + [a(\tfrac{\cdot}{\varepsilon})\partial_i D^\beta (1-a) D^{\beta'}\Gamma](Z) \\
&\qquad\qquad + [a(\tfrac{\cdot}{\varepsilon}) D^\beta (1-a) \partial_i D^{\beta'} \Gamma](Z)\, \mathrm{d}y\,.
\end{aligned}
$$

The terms $(\partial_i a)(\frac{Z}{\varepsilon})\varepsilon^{-1}$ and $a(\frac{Z}{\varepsilon})$ are bounded in $x \neq y$ and $\varepsilon > 0$ and tend to 0 for $\varepsilon \to 0$. The bounds (6.11) and (6.12) allow then the application of Lebesgue's theorem of dominated convergence (Theorem A.2) to conclude the convergence of the difference $V(x) - \partial_i W_\varepsilon(x)$ to 0 locally uniformly. As $W_\varepsilon \to W_0$ pointwise for $\varepsilon \to 0$, the proof of our claim is established. Taking into account that $\|D_x^\gamma X\|_\infty \le M$, for $1 \le |\gamma| \le m$, together with the proved claim above (which is reapplied m times) and $g \in H(D_R, L^\infty(\mathbb{R}^n))$ one verifies readily the membership

$$
\int\limits_{\mathbb{R}^n} (D^\beta (1-a)\, D^{\beta'}\Gamma)(X(\,.\,,r) - X(y,r))\, g(y,r)\, \mathrm{d}y \in C_b^m(\mathbb{R}^n) \subset C^\alpha(\mathbb{R}^n) \tag{6.15}
$$

for any $r \in D_R$. This holds especially for $\beta, \beta' \in \mathbb{N}_0^n$ with $|\beta + \beta'| = 3$ which yields

$$
G_2(X,g)(\,.\,,r) \in C_b^m(\mathbb{R}^n)^n\,.
$$

To check the Hölder-continuity of the m-th derivatives of G_2 we may, in using (6.15), directly calculate, with $|\mu| = m$ and $r \in D_R$,

$$
\begin{aligned}
&[D^\mu G_2(X,g)(\,.\,,r)]_{C^\alpha(\mathbb{R}^n)^n} \\
&\quad = \left[D^\mu \int_{\mathbb{R}^n} D_{ij} \nabla ((1-a)\Gamma)(X(\,.\,,r) - X(y,r))\, g(y,r)\, \mathrm{d}y \right]_{C^\alpha(\mathbb{R}^n)^n} \\
&\quad = \max_{k=1,\dots,n} \left[\int_{\mathbb{R}^n} D_x^\mu\, D_{ijk}((1-a)\Gamma)(X(\,.\,,r) - X(y,r))\, g(y,r)\, \mathrm{d}y \right]_\alpha \\
&\quad \le C \sum_{\beta,\beta'} \Bigg\{ \left\| \int_{\mathbb{R}^n} (D^\beta (1-a)\, D^{\beta'}\Gamma)(X(\,.\,,r) - X(y,r))\, g(y,r)\, \mathrm{d}y \right\|_\infty [P_{\beta,\beta'}]_\alpha \\
&\qquad\qquad + \left[\int_{\mathbb{R}^n} (D^\beta (1-a)\, D^{\beta'}\Gamma)(X(\,.\,,r) - X(y,r))\, g(y,r)\, \mathrm{d}y \right]_\alpha \|P_{\beta,\beta'}\|_\infty \Bigg\} \\
&\quad \le 2C \sum_{\beta,\beta'} \left\{ \left\| \int_{\mathbb{R}^n} (D^\beta (1-a)\, D^{\beta'}\Gamma)(X(\,.\,,r) - X(y,r))\, g(y,r)\, \mathrm{d}y \right\|_{C^\alpha(\mathbb{R}^n)} \|P_{\beta,\beta'}\|_{C^\alpha(\mathbb{R}^n)} \right\} \\
&\quad \le \tilde{C} \tilde{P}(M)
\end{aligned}
$$

where the sum is taken over $4 \le |\beta + \beta'| \le m+3$ and $P_{\beta,\beta'} = P_{\beta,\beta'}\left((D^\gamma X(x,r))_{1 \le |\gamma| \le m} \right)$ denotes a polynomial of degree $\leq m$ in dependence of β, β'. $\tilde{P}$ is a polynomial in M. Hence, there truly holds

$$G_2(X,g) \in H(D_R, C^{m,\alpha}(\mathbb{R}^n)^n)$$

with

$$\|G_2(X,g)\|_{H(D_R, C^{m,\alpha}(\mathbb{R}^n)^n)} \leq C(M, \|g\|_{\infty,R}),$$

where the constant C is a polynomial in M and $\|g\|_{\infty,R}$.
The exact reasoning for the holomorphy property was added later, more precisely after a conversation of the thesis author with Philippe Serfati in October 2017 over the Internet platform Researchgate.net. It goes as follows.
Any derivative up to the order m of $G_2(X,g)$ exists and has, for fixed $x \in \mathbb{R}^n$, a holomorphic integral kernel which is uniformly (in r) bounded by an integrable function and for this reason, they are $\mathbb{C}$-valued holomorphic functions in the sense of Theorem 4.4. For a fixed but arbitrary $r \in D_R$ the Cauchy integral formula applies and thus

$$D^\mu G_2(X,g)(r,x) - \int\limits_{|r-r'|=d} \frac{D^\mu G_2(X,g)(r',x)}{r-r'} \mathrm{d}r' = 0 \quad \text{in } \mathbb{C}^n,$$

for any $x \in \mathbb{R}^n$, any multi-index $0 \leq |\mu| \leq m$ and $d < \operatorname{dist}(r, \partial D_R)$. This identity is preserved in the L^∞-norm and therefore, there holds

$$D^\mu G_2(X,g) \in H(D_R; L^\infty(\mathbb{R}^n; \mathbb{C}^n))\,. \tag{6.16}$$

With the same reasoning we obtain

$$\frac{D^\mu_x G_2(X,g)(x,\,.\,) - D^\mu_x G_2(X,g)(x',\,.\,)}{\|x-x'\|^\alpha} \in H(D_R; L^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_{x'}; \mathbb{C}^n)) \quad ,\text{for } |\mu| = m$$

and subsequently $G_2(X,g) \in H(D_R, C^{m,\alpha}(\mathbb{R}^n)^n)$.

Another method to confirm holomorphy is given in [Her17], where the author proves time-analyticity of the Lagrangian trajectories in 2D in reasoning similarly to P. Serfati in [Ser95].
By definition, a function K belongs to $H(D_R; E)$, E being a complex Banach space, if the limit of the difference quotient $(K(z) - K(z_0))(z-z_0)^{-1}$ exists in E for any $z_0 \in D_R$. For fixed $x \in \mathbb{R}^n$ $G_2(X,g)(x,\,.\,)$ belongs to $H(D_R, \mathbb{C}^n)$ as a consequence of Theorem 4.4 and with Theorem 4.1 we have

$$\partial_z^l G_2(X,g)(x,z) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|\xi|=R} \frac{G_2(X,g)(x,\xi)}{(\xi - z)^{l+1}} \,\mathrm{d}\xi \quad ,\text{for } z \in D_R,\ l \in \mathbb{N}_0\,.$$

For $z_0, z_1 \in D_R$ one calculates

$$
\begin{aligned}
&\frac{G_2(X,g)(x,z_1)-G_2(X,g)(x,z_0)}{(z_1-z_0)} - \partial_z G_2(X,g)(x,z_0) \\
&\quad= \frac{1}{(z_1-z_0)}\sum_{j=0,1}(-1)^j \frac{1}{2\pi\mathrm{i}}\int\limits_{|\xi|=R} \frac{G_2(X,g)(x,\xi)}{(\xi-z_j)}\,\mathrm{d}\xi - \frac{1}{2\pi\mathrm{i}}\int\limits_{|\xi|=R}\frac{G_2(X,g)(x,\xi)}{(\xi-z_0)^2}\,\mathrm{d}\xi \\
&\quad= \frac{1}{2\pi\mathrm{i}}\int\limits_{|\xi|=R}\frac{G_2(X,g)(x,\xi)(z_1-z_0)}{(\xi-z_0)^2(\xi-z_1)}\,\mathrm{d}\xi
\end{aligned}
$$

and one obtains in applying the $C^{m,\alpha}$-norm

$$
\begin{aligned}
\left\|\frac{G_2(X,g)(x,z_1)-G_2(X,g)(x,z_0)}{(z_1-z_0)} - \partial_z G_2(X,g)(x,z_0)\right\|_{C^{m,\alpha}} \\
\le \frac{1}{2\pi}\int\limits_{|\xi|=R}\frac{\|G_2(X,g)(\,.\,,\xi)\|_{C^{m,\alpha}}\,|z_1-z_0|}{|\xi-z_0|^2|\xi-z_1|}\,\mathrm{d}\xi \\
= C|z_1-z_0|\frac{R}{\min\{\mathrm{dist}(z_0,\partial D_R)^2;\mathrm{dist}(z_1,\partial D_R)\}}
\end{aligned}
$$

where $C=C(M,\|g\|_{\infty,R})$ is the constant from above.
We conclude that the limit $z_1\to z_0$ of the difference quotient exists in $C^{m,\alpha}(\mathbb{R}^n;\mathbb{C}^n)$ which implies the membership of $G_2(X,g)$ to $H(D_R,C^{m,\alpha}(\mathbb{R}^n;\mathbb{C}^n))$.

Consider now the operator G_1 and set temporarily $m=1$. If we want to differentiate $G_1(X,f)$ once, we will obtain integrals with the following kernels

$$
(D^{\beta}aD^{\beta'}\mathit{\Gamma})\,(X(x,r)-X(y,r))
$$

with $\beta,\beta'\in\mathbb{N}_0^n$ and $|\beta+\beta'|=2$. In the case where $|\beta|\neq 0$ the Hölder-estimates are concluded as in the treatment of G_2 (first case). The difficulty lies in the case where $|\beta|=0$ and $|\beta'|=2$, since the integral kernel above is not integrable anymore. To navigate around that obstacle we will show in analogy with [Ser95], that the first derivative of $G_1(X,f)$ takes the form

$$
D_x\,G_1(X,f)(x,r)=\int\limits_{\mathbb{R}^n} D_x(\nabla(a\mathit{\Gamma})\,(X(x,r)-X(y,r)))\,(f(y,r)-f(x,r))\,|JX|(y,r)\,\mathrm{d}y\,. \tag{6.17}
$$

In order to achieve this, we are going to show with $Z\coloneqq X(x,r)-X(y,r)$ and for any $\varepsilon>0$

$$
\int\limits_{\mathbb{R}^n} D_x\left[\left(1-a\left(\tfrac{Z}{\varepsilon}\right)\right)\nabla(a\mathit{\Gamma})(Z)\right]\,|JX|(y,r)\,\mathrm{d}y=0\,, \tag{6.18}
$$

as we will then have

$$
\begin{aligned}
&D_x\, G_1(X,f)(x,r)\\
&\quad = \lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^n} D_x \left[\left(1-a\left(\tfrac{Z}{\varepsilon}\right)\right)\nabla(a\Gamma)(Z)\right]\left(f(y,r)-f(x,r)+f(x,r)\right)|JX|(y,r)\,\mathrm{d}y\\
&\quad = \lim_{\varepsilon\to 0} \int\limits_{\mathbb{R}^n} D_x \left[\left(1-a\left(\tfrac{Z}{\varepsilon}\right)\right)\nabla(a\Gamma)(Z)\right]\left(f(y,r)-f(x,r)\right)|JX|(y,r)\,\mathrm{d}y\,.
\end{aligned}
\tag{6.19}
$$

In fact, $G_1(X,f)$ is the locally-uniform limit of

$$
G_{1,\varepsilon} \coloneqq \int\limits_{\mathbb{R}^n} \left(1-a\left(\tfrac{Z}{\varepsilon}\right)\right)\nabla(a\Gamma)(Z)\, f(y,r)\,|JX|(y,r)\,\mathrm{d}y,
$$

whose derivative is

$$
D_x\, G_{1,\varepsilon} = \int\limits_{\mathbb{R}^n} D_x \left[\left(1-a\left(\tfrac{Z}{\varepsilon}\right)\right)\nabla(a\Gamma)(Z)\right] f(y,r)\,|JX|(y,r)\,\mathrm{d}y,
$$

because of the former computations. We show (6.18) : From Lemma 6.3 follows the existence of $s>0$ such that $X_\kappa \coloneqq \operatorname{Re} X + \kappa \operatorname{Im} X$ fulfills $\|JX_\kappa - \mathrm{I}_n\| < \beta$ for any $\kappa \in W \coloneqq \{z\in\mathbb{C} \mid |\operatorname{Re} z| < s\ ;\ |\operatorname{Im} z| < 1+s\}$. Also, X_κ is real valued and invertible in $x\in\mathbb{R}^n$ for $\kappa \in W\cap\mathbb{R}$. We further set $Z_\kappa \coloneqq X_\kappa(x,r) - X_\kappa(y,r)$ and obtain by a change of variables

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^n} D_x \left[\left(1-a\left(\tfrac{Z_\kappa}{\varepsilon}\right)\right)\nabla(a\Gamma)(Z_\kappa)\right]|JX_\kappa|(y,r)\,\mathrm{d}y\\
&\quad = \int\limits_{\mathbb{R}^n} D_x \left[\left(1-a\left(\frac{x-X_\kappa(y,r)}{\varepsilon}\right)\right)\nabla(a\Gamma)(x-X_\kappa(y,r))\right]\underbrace{JX_\kappa(X_\kappa^{-1}(x,r),r)}_{=:h(x,r)}|JX_\kappa|(y,r)\,\mathrm{d}y\\
&\quad = h(x,r)\int\limits_{\mathbb{R}^n} D_x \left[\left(1-a\left(\tfrac{x-y}{\varepsilon}\right)\right)\nabla(a\Gamma)(x-y)\right]\underbrace{|JX_\kappa(X_\kappa^{-1}(y,r),r)||JX_\kappa^{-1}(y,r)|}_{=1}\,\mathrm{d}y\\
&\quad = h(x,r)\int\limits_{\mathbb{R}^n} D_y \left[\left(1-a\left(\tfrac{y}{\varepsilon}\right)\right)\nabla(a\Gamma)(y)\right]\,\mathrm{d}y,
\end{aligned}
$$

which is well defined (see properties of a). For $R>0$ one calculates

$$
\begin{aligned}
h(R) &:= \int\limits_{R>|y|>\frac{1}{R}} D_y \left[\left(1 - a\left(\tfrac{y}{\varepsilon}\right)\right) \nabla(a\Gamma)(y) \right] \mathrm{d}y \\
&= \int\limits_{|y|=R} \left[\left(1 - a\left(\tfrac{y}{\varepsilon}\right)\right) \nabla(a\Gamma)(y) \right] \cdot \frac{y^T}{R} \, \mathrm{d}\sigma(y) \\
&\quad - \int\limits_{|y|=\frac{1}{R}} \left[\left(1 - a\left(\tfrac{y}{\varepsilon}\right)\right) \nabla(a\Gamma)(y) \right] \cdot R y^T \, \mathrm{d}\sigma(y) \\
&= R^{-1} \int\limits_{|y|=1} \left(1 - a\left(\tfrac{Ry}{\varepsilon}\right)\right) \left[\left((\nabla a)\Gamma + a\nabla\Gamma\right)(Ry) \right] \cdot y^T \, R^n \, \mathrm{d}\sigma(y) \\
&\quad - R^{-1} \int\limits_{|y|=1} \left(1 - a\left(\tfrac{y}{R\varepsilon}\right)\right) \left[\left((\nabla a)\Gamma + a\nabla\Gamma\right)\left(\tfrac{y}{R}\right) \right] \cdot y^T \, R^{2-n} \, \mathrm{d}\sigma(y) \\
&= CR^{-1} \Bigg(\int_{|y|=1} \left(1 - a\left(\tfrac{\cdot}{\varepsilon}\right)\right) \left(\nabla a(\,.\,) \cdot R^2 y^T + a(\,.\,) Ry \cdot y^T \right) (Ry) \, \mathrm{d}\sigma(y) \\
&\quad - \int_{|y|=1} \Big[\left(1 - a\left(\tfrac{\cdot}{\varepsilon}\right)\right) \nabla a(\,.\,) \cdot y^T + \left(1 - a\left(\tfrac{\cdot}{\varepsilon}\right)\right) a(\,.\,) Ry \cdot y^T \Big] \left(\tfrac{y}{R}\right) \, \mathrm{d}\sigma(y) \Bigg).
\end{aligned}
$$

Because of the properties (6.8), (6.9) and (6.10) of a, the integrals in the last equation are bounded for any $R > 0$ and stay bounded as we pass to the limit $R \to +\infty$, hence

$$
\lim_{R \to +\infty} h(R) = 0
$$

and this yields on the one hand, since $Z_\kappa = \operatorname{Re} Z + \kappa \operatorname{Im} Z$, that

$$
F(\kappa) := \int\limits_{\mathbb{R}^n} D_x \left[\left(1 - a\left(\tfrac{\operatorname{Re} Z + \kappa \operatorname{Im} Z}{\varepsilon}\right)\right) \nabla(a\Gamma)(\operatorname{Re} Z + \kappa \operatorname{Im} Z) \right] |JX_\kappa|(y,r) \, \mathrm{d}y = 0
$$

and on the other hand, that we can extend F to $W := \{ z \in \mathbb{C} \mid |\operatorname{Re} z| < s \,;\, |\operatorname{Im} z| < 1 + s \}$ (provided s sufficiently small) to be a holomorphic function on W. The above identity then holds by analytic continuation for any $\kappa \in W$, hence, (6.18) follows for $\kappa = \mathrm{i}$.
Furthermore, the integrand of

$$
\int\limits_{\mathbb{R}^n} D_x \left[\left(1 - a\left(\tfrac{Z}{\varepsilon}\right)\right) \nabla(a\Gamma)(Z) \right] (f(y,r) - f(x,r)) |JX|(y,r) \, \mathrm{d}y,
$$

$0 < 1 - a(Z/\varepsilon) < 1$ and $(\partial_i a)(Z/\varepsilon)(\partial_i X(x,r)\varepsilon^{-1} \leq C < +\infty$ for all $x \neq y$, $\varepsilon > 0$, is bounded by

$$
C \left\| \nabla(a\Gamma)(Z) \right\| \cdot 2 \left\| f \right\|_\infty + \left\| D_x \left[\nabla(a\Gamma)(Z) \right] (f(y,r) - f(x,r)) \right\|,
$$

which belongs to $L^1_y(\mathbb{R}^n)$ since the properties of the function a control the singularities of Γ and $\nabla\Gamma$. For instance, $|\beta'| = 2$ and $\tilde{R} > 0$ imply

$$
\begin{aligned}
|(aD^{\beta'}\Gamma)(Z)(f(y,r) - f(x,r))| &\leq C\,\|f\|_{\alpha,R}\,|a(Z)|\frac{1}{\|x-y\|^{n-\alpha}} \\
&\leq C'\left(\mathbb{1}_{B_{\tilde{R}}(x)}(y)\frac{1}{\|x-y\|^{n-\alpha}} + \mathbb{1}_{\mathbb{R}^n\setminus B_{\tilde{R}}(x)}(y)\frac{1}{\|x-y\|^{n+1}}\right),
\end{aligned}
$$

which is integrable in $\mathbb{R}^n$ with respect to y. Here we also used the fact that $f \in C^{m-1,\alpha}(\mathbb{R}^n)_R = C^{0,\alpha}(\mathbb{R}^n)_R$ and so

$$\|f(y,r) - f(x,r)\| \leq \|f\|_{\alpha,R}\,\|y-x\|^{\alpha}$$

with $\|f\|_{\alpha,R} = \|f\|_{H(D_R,C^{\alpha}(\mathbb{R}^n))}$. The integrand is bounded uniformly in $\varepsilon > 0$, allowing the application of the theorem of dominated convergence. Thus, we may calculate the limit $\varepsilon \to 0$ in (6.19) and deduce (6.17).

Next we will show the Hölder-continuity of the derivative of $G_1(X,f)$ which is given by (6.17). As mentioned before, the only case which was not yet shown is the case for the integrand

$$(aD^{\beta'}\Gamma)(Z)(f(y,r) - f(x,r))|JX|(y,r) \qquad , |\beta'| = 2\,.$$

We may set

$$T(x,r) := \int\limits_{\mathbb{R}^n} (a\,\Gamma_{ij})(X(x,r) - X(y,r))(f(x,r) - f(y,r))|JX|(y,r)\,\mathrm{d}y$$

where we used the abbreviation Γ_{ij} for the term $\partial_i\partial_j\Gamma$, $i,j = 1,\ldots,n$. Since the following calculations are independent of $i,j \in \{1,\ldots,n\}$ we will allow ourself to make a few more abbreviations. We set

$$
\begin{aligned}
K_1 &:= (a\Gamma_{ij})(X(x_1,r) - X(y,r))|JX|(y,r), \\
K_2 &:= (a\Gamma_{ij})(X(x_2,r) - X(y,r))|JX|(y,r), \\
f_1 &:= f(x_1,r), \\
f_2 &:= f(x_2,r), \\
f &:= f(y,r)\,.
\end{aligned}
$$

Then we have for any $\rho > 0$

$$\left|\int_{\mathbb{R}^n\setminus B_\rho(x_l)} K_l\,\mathrm{d}y\right| \leq C, \tag{6.20}$$

where $l = 1,2$ and $C > 0$ is independent of ρ. This is shown below in following again [Ser95]. We express the kernel, in abbreviating $Z := X(x_l,r) - X(y,r)$, as

$$(a\partial_i\partial_j\Gamma)(Z) = \partial_i(a\partial_j\Gamma)(Z) - (\partial_i a\partial_j\Gamma)(Z)\,.$$

The second term in the sum is integrable because of the properties of a. The first term we write as

$$\partial_i((1-a(\cdot/\rho))a\partial_j\Gamma)(Z) + \partial_i(a(\cdot/\rho)\,a\partial_j\Gamma)(Z)\,.$$

As in (6.18) the integral over $\mathbb{R}^n$ of the first term in the last sum vanishes and, in writing $\int_{\mathbb{R}^n\setminus B_\rho(x_l)} = \int_{\mathbb{R}^n} - \int_{B_\rho(x_l)}$, we get

$$\int\limits_{\mathbb{R}^n\setminus B_\rho(x_l)} \partial_i(a\partial_j\Gamma)(Z)\,\mathrm{d}y = -\int\limits_{B_\rho(x_l)} \partial_i((1-a(\cdot/\rho))a\partial_j\Gamma)(Z) + \int\limits_{\mathbb{R}^n\setminus B_\rho(x_l)} \partial_i(a(\cdot/\rho)\,a\partial_j\Gamma)(Z)\,\mathrm{d}y\,.$$

The bounds for a ensure that the above integrals exist uniformly in ρ and (6.20) follows from the representation

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus B_\rho(x_l)} K_l\,\mathrm{d}y &= \int_{\mathbb{R}^n\setminus B_\rho(x_l)} \partial_i(a\partial_j\Gamma)(Z) - (\partial_i a\partial_j\Gamma)(Z)\,\mathrm{d}y \\
&= -\int\limits_{B_\rho(x_l)} \partial_i((1-a(\cdot/\rho))a\partial_j\Gamma)(Z) + \int\limits_{\mathbb{R}^n\setminus B_\rho(x_l)} \partial_i(a(\cdot/\rho)\,a\partial_j\Gamma)(Z)\,\mathrm{d}y \\
&\quad - \int_{\mathbb{R}^n\setminus B_\rho(x_l)} (\partial_i a\partial_j\Gamma)(Z)\,\mathrm{d}y\,.
\end{aligned}$$

We now fix

$$\rho := 2\,\|x_1 - x_2\|$$

and may write

$$\begin{aligned}
&T(x_1,r) - T(x_2,r) \\
&\quad= \int\limits_{\mathbb{R}^n} K_1[f-f_1] - K_2[f-f_2]\,\mathrm{d}y \\
&\quad= \int\limits_{B_\rho(x_1)} K_1[f-f_1] - K_2[f-f_2]\,\mathrm{d}y + \int\limits_{\mathbb{R}^n\setminus B_\rho(x_1)} K_1[f-f_1] - K_2[f-f_2] \pm K_1\,f_2\,\mathrm{d}y \\
&\quad= \int\limits_{B_\rho(x_1)} K_1[f-f_1] - K_2[f-f_2]\,\mathrm{d}y + \int\limits_{\mathbb{R}^n\setminus B_\rho(x_1)} K_1[f_2-f_1] + [K_1-K_2][f-f_2]\,\mathrm{d}y \\
&\quad= \underbrace{\int\limits_{B_\rho(x_1)} K_1[f-f_1]\,\mathrm{d}y}_{=:I_1} + \underbrace{\int\limits_{B_\rho(x_1)} K_2[f-f_2]\,\mathrm{d}y}_{=:I_2} \\
&\qquad + \underbrace{\int\limits_{\mathbb{R}^n\setminus B_\rho(x_1)} K_1[f_2-f_1]\,\mathrm{d}y}_{=:I_3} + \underbrace{\int\limits_{\mathbb{R}^n\setminus B_\rho(x_1)} [K_1-K_2][f-f_2]\,\mathrm{d}y}_{=:I_4} \\
&\quad= I_1 + I_2 + I_3 + I_4\,.
\end{aligned}$$

The following estimations are, except for the third term, essentially the same as in the proof of Theorem 5.1, we may therefore abridge those calculations. The needed estimates will be shown for each term separately. For the first term we have

$$\begin{aligned}|I_1| \le \int_{B_\rho(x_1)} |K_1||f-f_1|\,\mathrm{d}y &\le C_1\,\|f\|_{\alpha,R}\int_{B_\rho(x_1)} \|y-x_1\|^{-n}\,\|y-x_1\|^{\alpha}\;\mathrm{d}y\\ &= C_2\,\|f\|_{\alpha,R}\int_0^\rho \lambda^{\alpha-1}\,\mathrm{d}\lambda \;=\; C_3\,\|f\|_{\alpha,R}\,\rho^\alpha\end{aligned}$$

and for the second term

$$\begin{aligned}|I_2| \le \int_{B_\rho(x_1)} |K_2||f-f_2|\,\mathrm{d}y &\le \tilde{C}_1\,\|f\|_{\alpha,R}\int_{B_\rho(x_1)} \|y-x_2\|^{-n+\alpha}\;\mathrm{d}y\\ &\le \tilde{C}_2\,\|f\|_{\alpha,R}\int_0^{\frac{3\rho}{2}} \lambda^{\alpha-1}\,\mathrm{d}\lambda \;\le\; \tilde{C}_3\,\|f\|_{\alpha,R}\,\rho^\alpha\,.\end{aligned}$$

In case of the third term there holds

$$|I_3| \le |f_2-f_1|\left|\int_{\mathbb{R}^n\setminus B_\rho(x_1)} K_1\,\mathrm{d}y\right| \le \bar{C}_1\,\|f\|_{\alpha,R}\,\|x_1-x_2\|^\alpha \underbrace{\left|\int_{\mathbb{R}^n\setminus B_\rho(x_1)} K_1\,\mathrm{d}y\right|}_{\le C,\ (6.20)} \le \bar{C}_2\,\|f\|_{\alpha,R}\,\rho^\alpha$$

and finally

$$\begin{aligned}|I_4| &\le \int_{\mathbb{R}^n\setminus B_\rho(x_1)} |K_1-K_2||f-f_2|\,\mathrm{d}y\\ &\le \|f\|_{\alpha,R}\int_{\mathbb{R}^n\setminus B_\rho(x_1)} \|D[(a\Gamma_{ij})(X(x_3,r)-X(y,r))]\|\,\|x_1-x_2\|\,\|y-x_2\|^\alpha\;\mathrm{d}y,\\ &\qquad\qquad\qquad\text{with } x_3 = x_1+\theta(x_2-x_1),\ \theta\in(0,1)\\ &\le \hat{C}_1\,\|f\|_{\alpha,R}\,\|x_1-x_2\|\int_{\mathbb{R}^n\setminus B_\rho(x_1)} \frac{\|y-x_2\|^\alpha}{\|y-x_3\|^{n+1}}\,\mathrm{d}y,\\ &\qquad\qquad\qquad\text{since } \|y-x_2\|\le 2\,\|y-x_1\| \text{ and } \|y-x_3\|\ge 1/2\,\|y-x_1\|\\ &\le \hat{C}_2\,\|f\|_{\alpha,R}\,\|x_1-x_2\|\int_{\mathbb{R}^n\setminus B_\rho(x_1)} \frac{\|y-x_1\|^\alpha}{\|y-x_1\|^{n+1}}\,\mathrm{d}y\\ &= \hat{C}_3\,\|f\|_{\alpha,R}\,\|x_1-x_2\|\int_\rho^{+\infty} \lambda^{-2+\alpha}\,\mathrm{d}\lambda \;=\; \hat{C}_4\,\|f\|_{\alpha,R}\,\rho^\alpha\,.\end{aligned}$$

We thereby deduce

$$|T(x_1,r)-T(x_2,r)| \leq C' \|f\|_{\alpha,R}\, \rho^\alpha \leq C \|f\|_{\alpha,R} \|x_1-x_2\|^\alpha \, .$$

In summary, we have shown that $G_1(X,f) \in H(D_R, C^1(\mathbb{R}^n))$ and for any fixed $r \in D_R$

$$\begin{aligned}
&[D_x\, G_1(X,f)(\,.\,,r)]_\alpha \\
&= \left[\int_{\mathbb{R}^n} D_x(\nabla(a\mathit{\Gamma})\,(X(\,.\,,r)-X(y,r)))\,(f(y,r)-f(\,.\,,r))\,|JX|(y,r)\,\mathrm{d}y \right]_\alpha \\
&\leq \left\| \int_{\mathbb{R}^n} (\nabla\cdot\nabla^T(a\mathit{\Gamma})\,(X(\,.\,,r)-X(y,r)))\,(f(y,r)-f(\,.\,,r))\,|JX|(y,r)\,\mathrm{d}y \right\|_\alpha \cdot \|JX(\,.\,,r)\|_\alpha \\
&\leq C(M)\, \|f\|_{\alpha,R} \, .
\end{aligned}$$

Hence,

$$\|G_1(X,f)\|_{H(D_R,C^{1,\alpha}(\mathbb{R}^n)^n)} \leq C(M)\, \|f\|_{H(D_R,C^{0,\alpha}(\mathbb{R}^n))} \, . \tag{6.21}$$

The integrands of

$$G_1(X,f)(x,r) \ , \quad D_xG_1(X,f)(x,r) \ \text{ and } \quad \frac{D_xG_1(X,f)(x,r)-D_xG_1(X,f)(x',r)}{\|x-x'\|^\alpha}$$

are holomorphic for fixed $x, x' \in \mathbb{R}^n$, $x \neq x'$. Hence the assertion $G_1(X,f) \in H(D_R, C^{m,\alpha}(\mathbb{R}^n)^n)$, here $m=1$, is fulfilled in reasoning in the same way as for $G_2(X,g)$.
We finally switch to the general case $m \in \mathbb{N}$ and argue first with $X_\kappa = \mathrm{Re}\, X + \kappa \mathrm{Im}\, X$. As before, X_κ is invertible for $\kappa \in W \cap \mathbb{R}$, $W = \{z \in \mathbb{C} \mid |\mathrm{Re}\, z| < s; |\mathrm{Im}\, z| < 1+s\}$, $s>0$ sufficiently small, and we may write in abbreviating $X_\kappa(x,r) = X_\kappa(x)$

$$\begin{aligned}
G_1(X_\kappa,f)(X_\kappa^{-1}(x),r) &= \int_{\mathbb{R}^n} \nabla(a\mathit{\Gamma})(x-X_\kappa(y))\, f(y,r)\, |JX_\kappa|(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \nabla(a\mathit{\Gamma})(x-y)\, f(X_\kappa^{-1}(y),r)\, \underbrace{|JX_\kappa(y)||JX_\kappa(X_\kappa^{-1}(y))|}_{=1}\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} \nabla(a\mathit{\Gamma})(y)\, f(X_\kappa^{-1}(x-y),r)\,\mathrm{d}y \, .
\end{aligned}$$

One obtains

$$\begin{aligned}
G_1(X_\kappa,f)(x,r) &= G_1(X_\kappa,f)(X_\kappa^{-1}(X_\kappa(x)),r) \\
&= \int_{\mathbb{R}^n} \nabla(a\mathit{\Gamma})(y)\, f(X_\kappa^{-1}(X_\kappa(x)-y),r)\,\mathrm{d}y
\end{aligned}$$

and so

$$D_x[G_1(X_\kappa,f)(x,r)] = \int_{\mathbb{R}^n} \nabla(a\mathit{\Gamma})(y)\, D_x[f(X_\kappa^{-1}(X_\kappa(x)-y),r)]\,\mathrm{d}y$$

$$= \int_{\mathbb{R}^n} \nabla(a\Gamma)(y) \cdot (\nabla f)^\top (X_\kappa^{-1}(X_\kappa(x) - y), r) \cdot \big(JX_\kappa^{-1}(X_\kappa(x) - y) \cdot JX_\kappa(x)\big)\,\mathrm{d}y\,.$$

For $\mu \in \mathbb{N}_0^n$ with $|\mu| \leq m-2$ one verifies that after an execution of D^μ and a subsequent change of variables $y \mapsto X_\kappa(x) - \tilde{y}$ and $\tilde{y} \mapsto X_\kappa^{-1}(\bar{y})$ it holds that

$$D^\mu D_x G_1(X_\kappa, f)(x,r) = G_1(X_\kappa, P)(x,r),$$

where P is a polynomial depending on the following entries

$$P = P\left(\big(D^{\tilde{\mu}} JX_\kappa(x)\big)_{|\tilde{\mu}|\leq|\mu|}, \Big(D_x^l f(x)\Big)_{l\leq m-1}, \big(D^{\bar{\mu}} JX_\kappa^{-1}(X_\kappa(x))\big)_{|\bar{\mu}|\leq|\mu|} \right).$$

The last entry may be rewritten as

$$D^{\bar{\mu}} JX_\kappa^{-1}(X_\kappa(x)) = D^{\bar{\mu}} (JX_\kappa)^{-1}(x) = D^{\bar{\mu}}[\mathrm{cof}\,(JX_\kappa)^\top |JX_\kappa|^{-1}](x)\,.$$

which implies

$$P = P\left(\big(D^{\tilde{\mu}} JX_\kappa(x)\big)_{|\tilde{\mu}|\leq|\mu|}, \Big(D_x^l f(x)\Big)_{l\leq m-1}, \Big(D^{\bar{\mu}}[\mathrm{cof}\,(JX_\kappa)^\top |JX_\kappa|^{-1}](x)\Big)_{|\bar{\mu}|\leq|\mu|} \right).$$

We can now proceed with an analytic continuation of $X_\kappa = \mathrm{Re}\,X + \kappa \mathrm{Im}\,X$ and set $\kappa = \mathrm{i}$, yielding $X_\kappa = X_\mathrm{i} = X$. Note that this can be done since $|JX_\kappa|(x,r) \neq 0$ for all $\kappa \in W$ and any $(x,r) \in \mathbb{R}^n \times D_R$, which has been proved in Lemma 6.3. The assumptions made on f and X imply $P \in H(D_R, C^{0,\alpha}(\mathbb{R}^n)^{n\times n})$. Thus, under consideration of (6.21) we have

$$\begin{aligned} \|G_1(X,P)\|_{H(D_R, C^{1,\alpha}(\mathbb{R}^n)^{n\times n})} &\leq C\,\|P\|_{H(D_R, C^{0,\alpha}(\mathbb{R}^n)^{n\times n})} \\ &\leq \tilde{C}\,\|f\|_{H(D_R, C^{m-1,\alpha}(\mathbb{R}^n))}\,, \end{aligned}$$

where C and $\tilde{C}$ depend only on M. But that means, since $G_1(X,P) = D^\mu D_x G_1(X,f)$ with $\mu \in \mathbb{N}_0^n$ and $|\mu| \leq m-2$, that $G_1(X,f)$ is a member of $C^{m,\alpha}(\mathbb{R}^n)^n$ and fulfills the claimed inequality. □

Lemma 6.5. The operator G is a bounded map from $H(D_R,\Omega)$ into $H(D_R,C^{m,\alpha}(\mathbb{R}^n;\mathbb{C}^n))$. There exists a constant $C'=C'(M)$ such that

$$\|G(X,Y)\|_{H(D_R,C^{m,\alpha}(\mathbb{R}^n)^n)} \leq C' . \tag{6.22}$$

Proof. In fact, if we choose, for $(X,Y)\in\Omega_R$, $(y,r)\in\mathbb{R}^n\times D_R$ and $i,j\in\{1,\dots,n\}$,

$$\begin{aligned} f(y,r) &= \operatorname{Tr}\left(JY\cdot\operatorname{cof}(JX)^\top\right)^2(y,r)\\ g_{ij}(y,r) &= (Y_iY_j)(y,r)|JX|(y,r) \end{aligned}$$

then the requirements for Lemma 6.4 are fulfilled by the definition of Ω_R and since G may be written as the sum

$$G(X,Y)(x,r) = G_1(X,f)(x,r) + \sum_{i,j} G_2(X,g_{ij})\,(x,r),$$

it is well defined on Ω_R and a bounded member of $H(D_R,C^{m,\alpha}(\mathbb{R}^n;\mathbb{C}^n))$ because of Lemma 6.4. The norm of $(X,Y)\in\Omega_R$ is bounded by M and $f,g_{i,j}$ depend only on (X,Y), hence f and $g_{i,j}$ are bounded by a constant depending only on M. Therefore the norm of G in $H(D_R,C^{m,\alpha}(\mathbb{R}^n;\mathbb{C}^n))$ is bounded by $C'=C'(M)$. $\square$

6.3. Unique solution of the complex ODE

We have finally collected all the necessary results which concern the operator G and we may now formulate and prove the key theorem. We recall the definition of the needed Banach spaces. As before we define for $R>0$, $M>0$, and $0<\beta<1/2$

$$E := E_1\times E_2 \tag{6.23}$$

with

$$\begin{aligned} E_1 &:= \{f\in \mathrm{L}^\infty((1+\|x\|)^{-1},\mathbb{R}^n;\mathbb{C}^n)\mid Jf\in C^{m-1,\alpha}(\mathbb{R}^n,\mathbb{C}^{n\times n})\},\\ E_2 &:= C^{m,\alpha}(\mathbb{R}^n,\mathbb{C}^n) \end{aligned}$$

as well as

$$\Omega := \{(X,Y)\in E \mid \|JX-\mathrm{I}_n\|_\infty < \beta\,;\ \|(X,Y)\|_E < M\}$$

and

$$E_R := H(D_R,E) \quad , \quad \Omega_R := H(D_R,\Omega)\,.$$

Theorem 6.1. *Let $u_0 \in C^{m,\alpha}(\mathbb{R}^n)$. Then there exists $\hat{T} > 0$ such that the ordinary differential equation, obtained in section 6.1,*

$$\begin{aligned} \hat{X}''(\tau) &= G(\hat{X}, \hat{X}')(\tau) \\ (\hat{X}, \hat{X}')(0) &= (Id, u_0), \end{aligned} \tag{6.24}$$

admits a unique solution in $H(D_{\hat{T}}, E_1)$. In (6.24), the right hand side is defined for $(X, Y) \in \Omega_R$, $R > 0$, and $(x, r) \in \mathbb{R}^n \times D_R$ as

$$\begin{aligned} G(X,Y)(x,r) = &\int_{\mathbb{R}^n} \nabla(a\Gamma)(X(x,r) - X(y,r)) \operatorname{Tr}\left(JY \cdot \operatorname{cof}(JX)^\top\right)^2(y,r)|JX|(y,r)\,\mathrm{d}y \\ &+ \sum_{i,j} \int_{\mathbb{R}^n} \partial_i \partial_j \nabla((1-a)\Gamma)(X(x,r) - X(y,r))(Y_i Y_j)(y,r)|JX|(y,r)\,\mathrm{d}y\,. \end{aligned}$$

Proof. The ODE is autonomous and of second order, it may therefore be equivalently transformed into a differential-algebraic equation, namely

$$\begin{aligned} &Z'(\tau) = K(Z(\tau)) \quad \text{with } Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \text{ and } K(Z) = \begin{pmatrix} Z_2 \\ G(Z_1, Z_2) \end{pmatrix} \\ &Z(0) = (Id, u_0), \end{aligned} \tag{6.25}$$

which we are going to solve in E_R. We verify at first that $Z^0 = Z(0)$ is an element of Ω. For $M > 0$ sufficiently large in the definition of Ω, it holds that

$$\left\|Z^0\right\|_E = \left\|\frac{Id(\,.\,)}{1 + \|\,.\,\|}\right\|_{L^\infty} + \|\mathrm{I}_n\|_{C^{m-1,\alpha}} + \|u_0\|_{C^{m,\alpha}} < M,$$

where $\|u_0\|_{C^{m,\alpha}}$ is bounded since u_0 is a member of $C^{m,\alpha}(\mathbb{R}^n; \mathbb{R}^n)$ and $JId = \mathrm{I}_n$ is a constant map in $H(D_R, C^{m-1,\alpha}(\mathbb{R}^n; \mathbb{C}^{n\times n}))$. Additionally,

$$\left\|\frac{Id(\,.\,)}{1 + \|\,.\,\|}\right\|_{L^\infty} = 1 \quad \text{and} \quad \|\mathrm{I}_n - \mathrm{I}_n\|_\infty = 0 < \beta\,.$$

Hence, for M chosen big enough, the norm requirements of Ω are fulfilled by Z^0. Also, $K(Z)$ is a member of E for any $Z = (Z_1, Z_2) \in \Omega$, because $G(Z)$ belongs to $C^{m,\alpha}(\mathbb{R}^n; \mathbb{C}^n)$ by Lemma 6.5 and Z_2 is likewise a member of $C^{m,\alpha}(\mathbb{R}^n; \mathbb{C}^n)$. Z_2 is therefore bounded which yields

$$\left\|\frac{Z_2(\,.\,)}{1 + \|\,.\,\|}\right\|_\infty = \|Z_2\|_\infty < +\infty \quad \text{as well as} \quad \|JZ_2\|_{C^{m-1,\alpha}} < +\infty\,.$$

So, $(Z_2, G(Z_1, Z_2))^\top$ is norm-bounded in E, hence $K(Z)$ is a member of E for any $Z \in \Omega$. Choose $M > 0$ such that the above properties are satisfied and that for any $0 < r < \operatorname{dist}(Z^0, \partial\Omega)$ there holds

$$\overline{B_r(Z^0)} \subset \Omega\,.$$

In order to apply the Cauchy-Lipschitz existence Theorem (4.6), we need to show that K is bounded, locally Lipschitz-continuous in Ω and $K(\Omega_R) \subset E_R$ for any $R > 0$. The boundedness follows directly from the definition of Ω and Lemma 6.5. Holomorphy is also a consequence of Lemma 6.5 and the definition of Ω_R. The operator K now fulfills the assumptions of Theorem 4.5 and is therefore locally Lipschitz-continuous. Thus, the requirements for the Cauchy-Lipschitz existence Theorem are matched and in setting

$$\hat{T} := \frac{r}{\|K\|_\infty},$$

where $\|K\|_\infty$ denotes the bound of K, we obtain a unique solution $Z \in H(D_{\hat{T}}, E)$ of (6.25) which yields a unique solution of (6.24) in the space $H(D_{\hat{T}}, E_1)$. $\square$

7. Proof of the main theorem

7.1. Time-analytic trajectories

In this section we want to give the proof of our main theorem (3.1). Let $u \in C_b([0,T); C^{m,\alpha}(\mathbb{R}^n;\mathbb{R}^n))$ solve the Euler equations for some $T > 0$ with $u(\,.\,,0) = u_0 \in C^{m,\alpha}(\mathbb{R}^n;\mathbb{R}^n)$, $m \geq 1$. The particle trajectories X, i.e. the solution of

$$X_t(x,t) = u(X(x,t),t) \quad \text{with} \quad X(x,0) = x \tag{U}$$

is unique in $C^2([0,T); F_1)$, where

$$F_1 \coloneqq \left\{ f \in \mathrm{L}^\infty\left((1+\|x\|)^{-1}, \mathbb{R}^n;\mathbb{R}^n\right) \mid Jf \in C^{m-1,\alpha}(\mathbb{R}^n;\mathbb{R}^{n\times n}) \right\}.$$

Then local time-analyticity of X is the claim of our main theorem (3.1).

Proof of the main theorem. The vector field X exists because of the properties of u and solves (U) uniquely. As shown in section 5.2, a differentiation of the above equation yields that X also solves

$$X_{tt}(x,t) = -(\nabla p)(X(x,t),t) \quad , \quad (X, X_t)(x,0) = (x, u_0(x)) \tag{7.1}$$

for $(x,t) \in \mathbb{R}^n \times [0,T)$. If E_1 is the space defined in (6.23), then Theorem 6.1 assures that $\hat{T} > 0$ exists such that the complex ODE

$$\hat{X}''(\tau) = G(\hat{X},\hat{X}')(\tau) \qquad , \quad (\hat{X},\hat{X}')(0) = (Id, u_0) \tag{7.2}$$

admits a unique solution in $H(D_{\hat{T}}, E_1)$. G is the well defined and bounded map $H(D_R, \Omega) \to H(D_R, C^{m,\alpha}(\mathbb{R}^n;\mathbb{C}^n))$, $R > 0$, given by (6.25). The solution to (7.2) is obtained in solving

$$Z'(\tau) = (Z_2(\tau), G(Z)(\tau)) \quad , \quad Z(0) = Z^0 = (Id, u_0),$$

$Z = (Z_1, Z_2) \in \Omega$, which is the limit of the Picard iteration which corresponds to the integral equation (IE) in the proof of the Cauchy Lipschitz existence theorem 4.6. This iteration is given by

$$Z^{k+1}(\tau) = Z^0 + \int_0^\tau (Z_2^k(\omega), G(Z^k)(\omega))\,\mathrm{d}\omega \quad , k \in \mathbb{N}_0\,.$$

For $t \in (-\hat{T},\hat{T})$ the integral can be taken over the real line segment $[0,t]$ (resp. $[t,0]$) and therefore,

$$Z^1(t) = (Id, u_0) + \int_0^t (Id(s), G(Id,u_0)(s))\,\mathrm{d}s$$

is real valued, since, for $s \in [0,t]$, $(Id, u_0)(s) = (Id, u_0)$ and $G(Id, u_0)$ is real valued. This implies inductively that, for any $k \in \mathbb{N}$, Z^k is real valued. Hence, the solution to (7.2) takes real values for $t \in (-\hat{T}, \hat{T})$. Consequently, $\hat{X}$ must be an analytic function from $(-\hat{T}, \hat{T})$ to F_1. Also, $\hat{X}_t(\,.\,, t)$ is a member of $C^{m,\alpha}(\mathbb{R}^n; \mathbb{R}^n)$ and an integration over $[0,t]$ (resp. $[t,0]$) yields

$$(\hat{X} - Id)(\,.\,, t) \in C^{m,\alpha}(\mathbb{R}^n; \mathbb{R}^n)$$

for $t \in (-\hat{T}, \hat{T})$. Furthermore, we claim that there exists $\tilde{T} > 0$ such that the PDE (5.20), namely

$$\begin{aligned} \tilde{X}_{tt}(x,t) &= \tilde{G}(\tilde{X}, \tilde{X}_t)(x,t), \\ (\tilde{X}, \tilde{X}_t)(0) &= (Id, u_0) \end{aligned}$$

admits a unique solution in

$$C^2\left((-\tilde{T}, \tilde{T}); F_1\right) .$$

We therefore consider the ODE

$$\begin{aligned} \tilde{X}''(t) &= \tilde{G}(\tilde{X}, \tilde{X}')(t), \\ (\tilde{X}, \tilde{X}')(0) &= (Id, u_0) \,. \end{aligned} \tag{7.3}$$

We set

$$F = F_1 \times C^{m,\alpha}(\mathbb{R}^n; \mathbb{R}^n) \,.$$

F is the subspace of real valued functions in the Banach space E given in (6.23). Furthermore, we set

$$O \coloneqq \{(X,Y) \in F \mid \|JX - \mathrm{I}_n\|_\infty < \beta \,;\, \|(X,Y)\|_E < M\} \subset \Omega,$$

such that O is an open, bounded and convex subset of F. Now $\tilde{G}$ equals G on F and the same calculations as in the proof of Lemma 6.4 yield the following fact. $\tilde{G}$ maps $C((-R,R); O)$ to $C((-R,R); C^{m,\alpha}(\mathbb{R}^n; \mathbb{R}^n))$, $R > 0$. The proof is equal to that of Lemma 6.4, only that the holomorphy requirement is replaced by continuity. Furthermore, $\tilde{G}$ is locally Lipschitz-continuous on O because it is the restriction of G to $O \subset \Omega$ and G is locally Lipschitz-continuous on Ω. Following the same argumentation as in the proof of Theorem 6.1, in applying the Cauchy-Lipschitz existence theorem in its real version, one obtains the existence of $\tilde{T} > 0$ such that

$$\tilde{X} \in C^2((-\tilde{T}, \tilde{T}), F_1)$$

uniquely solves equation (7.3). In conclusion, the formerly found analytic function $\hat{X}$ must equal $\tilde{X}$ on the interval $(-T^*, T^*)$, where $T^* = \min\{\hat{T}, \tilde{T}\}$, because they both satisfy (7.3). If X is the unique solution to the characteristic equation (U), then it satisfies as a function in $C^2([0,T^*); F_1)$ also the ODE (7.3) since $\tilde{G}(X, X_t)(t)$ equals $-\nabla p(X(t), t)$ which has been discussed in section 5.2. Hence, X is the restriction of $\hat{X}$ to $[0, T^*)$ and so, X may be analytically continued to $(-T^*, T^*)$. Hence, analyticity in a neighborhood of $t_0 = 0$ holds for the particle trajectories. Subsequently, we may choose any $t_0 \in [0, T)$ and set

$$u^*(x,t) \coloneqq u(x, t - t_0)$$

for $t \in (t_0 - s, t_0 + s) \subset [0,T)$ with $s > 0$ small enough. Then we have $u^*(\,.\,,t_0) = u_0^* \in C^{m,\alpha}(\mathbb{R}^n;\mathbb{R}^n)$, $\operatorname{div} u^* = 0$ and $X(x, t - t_0)$ solves the characteristic equation of u^*. The same line of reasoning yields that there exists a neighborhood of t_0 in which the characteristic curves of u^* are analytic. Thus, X is locally analytic in time. □

7.2. Final remarks

The approach of this thesis is slightly different to that of P.Serfati in [Ser95]. We depart directly from a solution of (E) which is in $C_b([0,T);C^{m,\alpha}(\mathbb{R}^n;\mathbb{R}^n))$ and show that the corresponding particle trajectories are locally analytic in time. The proof in the cited paper assumes that $u \in C([0,T');W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n))$, $T' > 0$, solves the Euler equations. Incompressibility is invoked in order to obtain the operator $\tilde{G}$ in setting $|JX| = 1$ and one derives the complex ODE (7.1). Then it is additionally shown that $\operatorname{div} u_0 = 0$ implies that the solution $\hat{X}$ to (7.1) admits $|J\hat{X}| = 1$. Subsequently, one obtains an existence and uniqueness result for the Euler equations in setting

$$u(x,t) = \hat{X}_t(\hat{X}^{-1}(x,t),t)$$

with locally time-analytic particle trajectories $\hat{X}$. It holds $u(\,.\,,t) \in C^{m,\alpha}(\mathbb{R}^n;\mathbb{R}^n)$ for $t \in (-T^*,T^*)$, $T^* > 0$. This solution (u,p), where p is associated by (5.6) with $h_{i,j} := u_i u_j$, is unique in

$$L^\infty_{loc}\left((-T^*,T^*);C^{1,\alpha}(\mathbb{R}^n;\mathbb{R}^n) \times N\right),$$

where

$$N := \left\{ q \in C(\mathbb{R}^n) \,\middle|\, \lim_{\|x\|\to\infty} \frac{q(x)}{\|x\|} = 0\,,\; q(0) = c \right\}$$

with $c \in \mathbb{R}$. A slightly differing version of this result is obtained alternatively by J.-Y. Chemin [Che98], which served as a starting point for our approach.

List of symbols

Notation	**Description**
$\mathbb{N};\mathbb{N}_0$	set of natural numbers beginning from 1; resp. 0
$\mathbb{R};\mathbb{R}_+;\mathbb{R}_+^0$	set of real numbers; resp. real positive numbers; resp. real non-negative numbers
$E,F;E',F'$	complex Banach spaces; resp. their dual spaces, i.e. $E'=\{\varphi \mid \varphi$ is a bounded, linear map $E\to\mathbb{C}\}$
$\Omega;\overline{\Omega};\partial\Omega$	open subset of a Banach space; resp. closure of Ω ; resp. boundary of Ω
$B_R=B_R(0);B_R^*$	open ball in $\mathbb{R}^n$ around 0 with radius $R>0$; resp. without 0: $B_R^*=B_R\setminus\{0\}$
$D_R,D_{\boldsymbol{r}}$	complex disc around $0\in\mathbb{C}$ with radius $R>0$; resp. polydisc around 0 in $\mathbb{C}^n$ with multi-radius $\boldsymbol{r}\in\mathbb{R}_+^n$
$C(E;F)$	continuous maps $E\to F$
$C^m(\mathbb{R}^n);C^{m,\alpha}(\mathbb{R}^n)$	with $m\in\mathbb{N}_0$ and $\alpha\in(0,1)$; m-times continuously differentiable , complex valued scalar fields $\mathbb{R}^n\to\mathbb{C}$; resp. whose m-th order derivative is bounded and α-Hölder continuous, see chapter 2 for a complete definition
$C_0^\infty(\mathbb{R}^n)$	infinitely often differentiable scalar fields with compact support
$D'(\mathbb{R}^n)$	dual space of $C_0^\infty(\mathbb{R}^n)$ - space of test functions
$C_b(\mathbb{R}^n),C_b^m(\mathbb{R}^n)$	space of bounded, continuous scalar fields, resp. whose derivatives up to the m-th order exist and are continuous and bounded
$D_x;\nabla;J$	total derivative with respect to the variable $x\in\mathbb{R}^n$; resp. nabla operator; resp. symbol for the Jacobi matrix
$\operatorname{cof}(JX)^\top;\lvert JX\rvert$	transposed cofactor matrix of JX; resp. Jacobi-determinant of a vector field X
$\partial_{x_i}=\partial_i;D^\mu$	partial derivative with respect to the variable x_i ; resp. partial derivatives in multi-index notation, i.e. for $\mu\in\mathbb{N}_0^n$ is $D^\mu=\partial_{x_1}^{\mu_1}\partial_{x_2}^{\mu_2}\dots\partial_{x_n}^{\mu_n}$
$\mathbb{1}_\Omega$	real valued characteristic function on Ω i.e. $\mathbb{1}_\Omega(x)=1$ if $x\in\Omega$, $\mathbb{1}_\Omega(x)=0$ if $x\notin\Omega$
$L^\infty(\mathbb{R}^n);L^1(\mathbb{R}^n)$	Lebesgue space of essentially bounded, complex valued functions on $\mathbb{R}^n$; resp. Lebesgue space of complex valued, integrable functions on $\mathbb{R}^n$, more precisely explained in chapter 2
$L_y^1(\mathbb{R}^n);L_{loc}^1(\mathbb{R}^n)$	$L^1(\mathbb{R}^n)$ with respect to the variable $y\in\mathbb{R}^n$; resp. locally integrable functions on $\mathbb{R}^n$
$L^\infty(q(x),\mathbb{R}^n)$	functions which are essentially bounded if multiplied by $q(x)$
$W^{m,p}(\mathbb{R}^n)$	Sobolev space on $\mathbb{R}^n$, i.e. complex valued functions whose weak derivatives up to the order of $m\in\mathbb{N}_0$ are members of $L^p(\mathbb{R}^n),p\in[0,\infty]$

Notation	Description
$H(D_{\boldsymbol{r}}, \Omega) = \Omega_{\boldsymbol{r}}$	maps $\overline{D_{\boldsymbol{r}}} \to \Omega$ which are holomorphic in $D_{\boldsymbol{r}}$ and continuous on $\overline{D_{\boldsymbol{r}}}$, $\boldsymbol{r} \in \mathbb{R}_{+}^{n}$, see section 4.1.
$\mathcal{L}^n(E; F)$	n-linear maps $E \to F$
$\operatorname{dist}(a, \partial\Omega)$	distance between a and $\partial\Omega$, $\Omega \subset E$ bounded, i.e. $\inf\{\|a - x\| \mid x \in \partial\Omega\}$
$\operatorname{Tr}(A); \det A = \|A\|$	trace , resp. determinant of a matrix A

Bibliography

[Ber77] Melvyn Stuart Berger. *Nonlinearity and functional analysis: Lectures on nonlinear problems in mathematical analysis*. Vol. v. 74. Pure and applied mathematics, a series of monographs and textbooks. New York: Academic Press, 1977. ISBN: 0-12-090350-4. URL: http://site.ebrary.com/lib/alltitles/docDetail.action?docID=10258849.

[Che92] Jean-Yves Chemin. "Régularité de la trajectoire des particules d'un fluide parfait incompressible remplissant l'espace". In: *Journal Math. Pures et Appl.* (9), 71(5) (1992), pp. 407–417.

[Che98] Jean-Yves Chemin. *Perfect incompressible fluids*. Vol. 14. Oxford lecture series in mathematics and its applications. Oxford: Clarendon, 1998. ISBN: 0198503970. URL: http://www.loc.gov/catdir/enhancements/fy0606/98035385-d.html.

[Eva02] Lawrence C. Evans. *Partial differential equations*. Reprinted with corr. Vol. 19. Graduate studies in mathematics. Providence, RI: American Math. Soc, 2002. ISBN: 9780821807729.

[FZ14] Uriel Frisch and Vladislav Zheligovsky. "A Very Smooth Ride in a Rough Sea". In: *Communications in Mathematical Physics* 326.2 (2014), pp. 499–505. ISSN: 0010-3616. DOI: 10.1007/s00220-013-1848-1.

[GT83] David Gilbarg and Neil S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1983. DOI: 10.1007/978-3-642-61798-0.

[Her17] M. Hernandez. "Mechanisms of Lagrangian analyticity in fluids". In: *ArXiv e-prints* (May 2017). arXiv: 1705.06238 [math.AP].

[Her89] Michel A. Hervé. *Analyticity in infinite dimensional spaces*. Vol. 10. De Gruyter studies in mathematics. Berlin, New York: W. de Gruyter, 1989. ISBN: 0-89925-205-2. DOI: 10.1515/9783110856941. URL: http://site.ebrary.com/lib/alltitles/docDetail.action?docID=10599132.

[Kön04] Konrad Königsberger. *Analysis 2*. 5., korrigierte Aufl. Springer-Lehrbuch. Berlin: Springer, 2004. ISBN: 978-3540203896.

[PT87] Jürgen Pöschel and Eugene Trubowitz. *Inverse spectral theory*. Vol. 130. Pure and applied mathematics. Boston: Academic Press, 1987. ISBN: 0-12-563040-9. URL: http://site.ebrary.com/lib/alltitles/docDetail.action?docID=10244124.

[Ser92] Philippe Serfati. *Thèse: Étude mathématique de fammes infiniment minces en combustion. Résultats de structure et de régularité pour l'équation d'Euler incompressible.* Universitè Paris VI, 1992.

[Ser95] Philippe Serfati. "Structure holomorphes à faible régularité spatiale en mécanique des fluides". In: *Journal Math. Pures et Appl.* 74 (1995), pp. 95–104.

[Shn12] A. Shnirelman. “On the Analyticity of Particle Trajectories in the Ideal Incompressible Fluid”. In: *ArXiv e-prints* (May 2012). arXiv: 1205.5837 [math.AP].

[Wer11] Dirk Werner. *Funktionalanalysis*. 7., korrigierte und erw. Aufl. Springer-Lehrbuch. Berlin, Heidelberg: Springer-Verlag Berlin Heidelberg, 2011. ISBN: 978-3-642-21016-7. DOI: 10.1007/978-3-642-21017-4. URL: http://dx.doi.org/10.1007/978-3-642-21017-4.

A. Used theorems

This section is to state a list of important theorems and facts which have been used throughout this work. All of these assertions are well known and therefore no proof will be given here. For each theorem the referenced book will provide further information.

Lemma A.1 (Consequence of Hahn-Banach Theorem)**.** [Wer11, p. 98] Let V be a normed space, then for any $v \in V \setminus \{0\}$ there exists a functional $v' \in V'$ such that

$$\|v'\| = 1 \quad \text{and} \quad v'(v) = \|v\| \, .$$

Hence, for $v_1, v_2 \in V$ with $v_1 \neq v_2$ there exists $v' \in V'$ such that $v'(v_1) \neq v'(v_2)$.

Theorem A.1 (Fubini's Theorem)**.** *[Kön04, p. 289] Let $f \in L^1(\mathbb{R}^n \times \mathbb{R}^m)$, where $n, m \in \mathbb{N}$, then $y \mapsto f(x, y)$ is a member of $L^1(\mathbb{R}^m)$ and*

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f(x, y) \, \mathrm{d}(x, y) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x, y) \, \mathrm{d}x \right) \mathrm{d}y \, .$$

Theorem A.2 (Lebesgue's Theorem of dominated convergence)**.** *[Kön04, p. 278] Assume a sequence $(f_k) \subset L^1(\mathbb{R}^n)$. If there exists a measurable function f and an integrable function $g \in L^1(\mathbb{R}^n)$ such that*

$$\begin{aligned} f_k &\to f \quad && \text{,a.e. as } k \to \infty, \\ |f_k| &\leq g \quad && \text{,a.e. } \forall k \in \mathbb{N}, \end{aligned}$$

then $f \in L^1(\mathbb{R}^n)$ and

$$\int\limits_{\mathbb{R}^n} f_k(x) \, \mathrm{d}x \to \int\limits_{\mathbb{R}^n} f(x) \, \mathrm{d}x \qquad \text{,as } k \to \infty \, .$$

Theorem A.3 (Banach's fixed point Theorem)**.** *[Wer11, p. 166] Let X be a Banach space and $F : X \to X$ a contraction, i.e. for any $x, y \in X$ there exists $0 < q < 1$ such that $\|F(x) - F(y)\|_X \leq q \, \|x - y\|_X$. Then there exists a unique $z \in X$ such that $F(z) = z$.*

Theorem A.4 (Weyl's Lemma)**.** *If $u \in L^1_{loc}(\mathbb{R}^n; \mathbb{R})$ is weakly harmonic, i.e. for any $\varphi \in C_0^\infty(\mathbb{R}^n)$ holds*

$$\int\limits_{\mathbb{R}^n} u(x) \Delta \varphi(x) \, \mathrm{d}x = 0,$$

then there exists a harmonic function $\tilde{u} \in C^\infty(\mathbb{R}^n; \mathbb{R})$ such that $u = \tilde{u}$ almost everywhere.

Theorem A.5 (A general version of Liouville's Theorem)**.** *A harmonic function $u \in C^2(\mathbb{R}^n;\mathbb{R})$ which verifies*

$$\lim_{\|x\|\to\infty} \frac{\|u(x)\|}{\|x\|} = 0,$$

is a constant function.

Theorem A.6 (Neumann Series)**.** *[Wer11, p. 56] Let X be a normed space and $T : X \to X$ a linear map. If $\sum_{n\geq 0} T^n$ is convergent with respect to the operator norm, then $Id - T$ is invertible and*

$$(Id - T)^{-1} = \sum\nolimits_{n\geq 0} T^n .$$

Theorem A.7 (Inverse Function Theorem)**.** *[Eva02, p. 716] Let $F \in C^m(\Omega;\mathbb{R}^n)$ for an open set $\Omega \subseteq \mathbb{R}^n$ and let $x_0 \in \Omega$. If $\det(JF(x_0)) \neq 0$, then there exist open sets $V, W \in \mathbb{R}^n$ with $x_0 \in V$ and $F(x_0) \in W$ such that $F|_V : V \to W$ has an inverse which belongs to $C^m(W;V)$.*

Theorem A.8 (Hadamard's Theorem)**.** *[Ber77, p. 222] If $\Phi \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ is such that $(J\Phi)^{-1}$ exists globally and is norm-bounded, then Φ is a homeomorphism.*

Theorem A.9 (Young's Inequality)**.** *[Wer11, p. 78] Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. Then for every $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$ it follows that $f * g$ is well defined in $\mathbb{R}^n$ and*

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)} .$$

Acknowledgements

I would like to thank Professor Dr. Székelyhidi for giving me the opportunity to write this thesis and for supporting me patiently with his knowledgeable advise.

I would also like to thank Clemens Förster with whom i could discuss any occurring problem.