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Hypergraph Products

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Abstract

In this work, new definitions of hypergraph products are presented. The main focus is on the generalization of the commutative standard graph products: the Cartesian, the direct and the strong graph product. We will generalize these well-known graph products to products of hypergraphs and show several properties like associativity, commutativity and distributivity w.r.t. the disjoint union of hypergraphs. Moreover, we show that all defined products of simple (hyper)graphs result in a simple (hyper)graph. We will see, for what kind of product the projections into the factors are (at least weak) homomorphisms and for which products there are similar connections between the hypergraph products as there are for graphs. Last, we give a new and more constructive proof for the uniqueness of prime factorization w.r.t. the Cartesian product than in [*Studia Sci. Math. Hungar.* **2**: 285–290 (1967)] and moreover, a product relation according to such a decomposition. That might help to find efficient algorithms for the decomposition of hypergraphs w.r.t. the Cartesian product.

Contents

Introduction

Within this diploma thesis we are concerned with hypergraph products as a generalization of graph products.

A (simple) graph G is an object consisting of vertices and edges joining two vertices with each other, edges are thus two-element subsets of the vertex set. Graph theory occurs in various fields of science. For instance, social, computational, communication or transport networks can be described as graphs. They are furthermore an important tool in life science, physics, chemistry or optimization, to name just a few examples [5].

Many graphs can be constructed from smaller, more simple graphs by operations as unions, joins or multiplications with respect to a certain product, where many properties of the constructed graph can be immediately inferred from the constituents the graph is composed of. The operations we will focus on are the graph multiplications. Of course, there are various ways to define a product of two given graphs. If we restrict this amount of graph products to those which satisfy certain algebraic properties, we have in fact only four standard products: The *Cartesian*, *direct*, *strong* and *lexicographic* product. These are the only associative *simple*[1] *products* that depend on the structure of both factors and for which at least one of the projections into a factor is a weak homomorphism, i.e. an edge is mapped either into an edge or a vertex, [26, 29].

The standard graph products have been widely investigated. The Cartesian product and besides, the strong product as well, have been introduced by G. Sabidussi (1960) in [37]. In this work the author also showed the uniqueness of prime factor decomposition of the Cartesian product. The direct product has been defined by A.N. Whitehead in 1912 as a product of binary relations [42]. In 1962, P. M. Weichsel introduced it on graphs as the *Kronecker product* [41]. The lexicographic product is due to F. Hausdorff (1914) [21], it was defined on graphs as the *composition* of graphs by Harary (1959) in [19]. It is the only non-commutative standard

[1] A graph product is a *simple* product, if the product of simple graphs is a simple graph, the vertex set of a product is the Cartesian product of the vertex sets of its factors and adjacency in the product depends on the adjacency properties of the projections of pairs of vertices into the factors.

product.

Hypergraph theory was introduced in the 1960s as a generalization of graph theory. Since then many applications for hypergraphs have been developed, for example in engineering, image processing, molecular biology or chemistry [6]. In a hypergraph more than two vertices may be linked, so the (hyper-)edges of a hypergraph are (arbitrary) subsets of the vertex set. A standard reference of this theory is due to C. Berge [3].
As for graphs, hypergraphs might be constructible from smaller hypergraphs, for example as products. One hypergraph product is given in [4], where the product of two hypergraphs is defined on the Cartesian product of their vertex sets, and the edges are the Cartesian (set) products of the edges of the factors. This product is the most common hypergraph product, see [7, 11, 35, 36]. A similar product was introduced by W. Imrich and P.F. Stadler as a product of neighborhood systems as a generalization of directed graphs in [31], where the authors also proved that these neighborhood systems have a unique prime factorization under some constraints. W. Dörfler and D. A. Waller introduced another hypergraph product 1980 [13], this is also treated by X. Zhu in 1992 [44]. However, these products are no graph products in the sense that a product of two (standard) graphs is a graph.
For consistency of hypergraph theory as a generalization of graph theory, one might consider the standard graph products and ask, how to extend them to hypergraph products, such that they fulfill certain algebraic properties that are also fulfilled in the case of graph products. Actually, there is no problem to extend the Cartesian product to hypergraphs. It has been introduced by W. Imrich in [23] as a product of set systems, i.e. hypergraphs. This hypergraph product is also considered for example in [7, 8, 11]. Besides, also the lexicographic product has been extended to hypergraphs by W. Imrich and G. Gaszt in [25] and is furthermore considered for example in [10, 38].

In this thesis, we will only focus on the commutative graph products and their generalization. Such hypergraph products are introduced in **Chapter 2**. We will be concerned with the Cartesian product of hypergraphs in *Section 2.1*. Basic properties of this product, some of them are stated in [23], are explicitly proved in that section. In the case of the direct product it becomes more difficult to define a suitable hypergraph product. The question is, how to transfer the adjacency properties of the factors to their product. In [13], the authors defined a product on r-uniform hypergraphs whose restriction to graphs coincides with the direct graph product. The problem is to extend this product to arbitrary hypergraphs. That is what we are concerned with in *Section*

2.2. We give three new definitions of hypergraph products. Two of them coincide with the hypergraph product defined in [13] in the class of r-uniform hypergraphs. The third one is modeled independently and was motivated by [31]. Furthermore, we prove basic properties of these products and we will see, for what reasons some of those products might fail for our purpose. Once found a proper definition for a direct product, there should be no problem to construct a strong product on hypergraphs by taking the union of the edge sets of the Cartesian and the direct product as the edge set of this product, as this is the case for graphs. This will be done in *Section 2.3*. Furthermore, basic properties of the strong product are shown.

In **Chapter 3**, we focus again on the Cartesian product. To be more precise, we study the prime factorization of a given hypergraph. In [23], it is shown, that the prime factor decomposition of a hypergraph is unique in the class of simple connected hypergraphs. Here we give an alternative, more constructive proof and provide a product relation according to the unique prime factor decomposition of a simple connected hypergraph.

But first we will start with some basic notions about hypergraphs and the commutative standard graph products in **Chapter 1**.

1 Basics

1.1 Hypergraphs

Within this section we provide a few basic concepts of hypergraph theory, which are needed in the main part of this thesis. Although we changed some notations, a standard reference for this theory is due to C. Berge [3].

A *(finite) hypergraph* $H = (V, \mathscr{E})$ consists of a finite set V and a family $\mathscr{E} = \{E_1, \ldots, E_m\}$, such that:

$$E_i \neq \emptyset \tag{1.1}$$

$$E_i \subseteq V \tag{1.2}$$

for all $i \in \{1, \ldots, m\}$. A hypergraph $H = (V, \mathscr{E})$, $\mathscr{E} = \{E_1, \ldots, E_m\}$ is called *simple* if

$$E_i \subseteq E_j \quad \text{implies} \quad i = j, \tag{1.3}$$

and

$$|E_i| \geq 2 \quad \text{for all } i \in \{1, \ldots, m\}. \tag{1.4}$$

In other words, there are no multiple edges, no edge of H is contained in any other edge and each edge consists of at least two vertices.

The elements $v_1, \ldots, v_n$ of V are called *vertices* and the sets $E_1, \ldots, E_m$ are the *hyperedges*, or simply *edges* of the hypergraph. For simplicity, we will refer to the family of edges $\mathscr{E}(H)$ of a hypergraph H as *edge set*, although it need not to be a usual set.

If there is a risk of confusion we will denote the vertex set and the edge family of a hypergraph H explicitly by $V(H)$ and $\mathscr{E}(H)$, respectively.

A hypergraph may be drawn as a set of points representing the vertices, an edge E_j is represented by a simple closed curve enclosing its elements if $|E_j| > 2$, for $|E_j| = 2$ by a continuous curve joining its two elements, and by a loop if $|E_j| = 1$.
Figure 1.1 shows two hypergraphs H and H' with vertices $V(H) = V(H') = \{v_1, \ldots, v_{10}\}$ and edges $\mathscr{E}(H) = \{E_1, E_2, E_3, E_4, E_5, E_6, E_7\}$ and $\mathscr{E}(H') = \{E_1, E_2, E_3, E_4, E_5\}$, respectively. The hypergraph H of the left hand side is not simple, since $E_6 \subset E_4$ and $E_7 \subset E_3$.

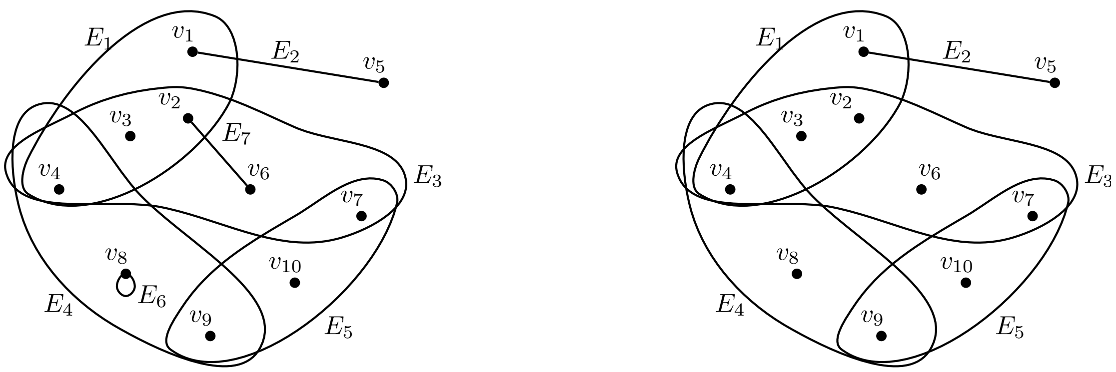


Figure 1.1: **lhs:** Hypergraph H. **rhs:** simple hypergraph H'.

We say, two vertices u and v are *adjacent* in a hypergraph $H = (V, \mathscr{E})$ if there is an edge $E \in \mathscr{E}$ such that $u, v \in E$. If for two edges $E, F \in \mathscr{E}$ holds $E \cap F \neq \emptyset$, they are said to be *incident*. A vertex v and an edge E of H are incident if $v \in E$.
The *order* of H, denoted by $n(H)$ is the number of vertices, the *number of edges* will be denoted by $m(H)$.
The *rank* of a hypergraph H is $r(H) = \max_j |E_j|$, the *anti-rank* is $s(H) = \min_j |E_j|$. A *uniform hypergraph H* is a hypergraph such that $r(H) = s(H)$. A simple uniform hypergraph of rank r will be called *r-uniform*.

Walks, Paths, Distances A *walk* in a hypergraph $H = (V, \mathscr{E})$ is a sequence

$$(v_0, E_1, v_1, E_2, \ldots, E_k, v_k), \tag{1.5}$$

where $E_1, \ldots, E_k$ are distinct edges, $v_0, \ldots, v_k$ are vertices, such that each $v_{i-1}, v_i \in E_i$. A *k-cycle* in a hypergraph is a closed walk consisting of k edges. A *path* is a walk, where the vertices $v_0, \ldots, v_k$ are all distinct. Such a path is said to *join* the vertices v_0 and v_k, we will denote it by $P_{v_0 v_k}$ and we will write $(E_1, E_2, \ldots, E_k)$ instead of (1.5) if possible. By a path between two edges

E_i, E_j we then mean a path between two vertices v_i, v_j, such that $v_i \in E_i$ and $v_j \in E_j$.
The *distance* $d_H(v,v')$ of two vertices v_0, v_k of H is the length of the shortest path joining them, i.e. the number of edges contained in the path,

$$d_H(v,v') = \min\{|E(P)| : P \text{ joins } v \text{ and } v'\}.$$

If there is no path joining them, we set $d_H(v,v') = \infty$.
A hypergraph $H = (V,\mathscr{E})$ is called *connected*, if any two vertices are joined by a path, that means for each two vertices v,v' of H we have $d_H(v,v') < \infty$.

Partial Hypergraphs For a hypergraph $H = (V,\mathscr{E})$ we call $H' = (W,\mathscr{F})$ a *partial hypergraph* of H if $W \subseteq V$ and $\mathscr{F} \subseteq \mathscr{E}$. We then write $H' \subseteq H$. A partial hypergraph $H' = (W,\mathscr{F}) \subseteq H = (V,\mathscr{E})$ is *generated by the edge set* $\mathscr{F}$ if $W = \bigcup_{E\in\mathscr{F}} E$. It is *induced by the vertex set* W if $\mathscr{F} = (E \in \mathscr{E} \mid E \subseteq W)$. Such an induced partial hypergraph will be denoted by $H' = \langle W \rangle$. Note, that a partial hypergraph of a simple hypergraph is always simple.
Figure 1.2 shows three partial hypergraphs H_1, H_2 and H_3 of the Hypergraph H in Figure 1.1, with the vertex sets $V(H_1) = V(H_2) = \{v_1,v_2,v_3,v_4,v_6,v_7,v_8\} \subset V(H)$ and $V(H_3) = \{v_1,v_2,v_3,v_4,v_6,v_7\} \subset V(H)$, respectively and with edge sets $\mathscr{E}(H_1) = \mathscr{E}(H_3) = \{E_1,E_3\} \subset \mathscr{E}(H)$ and $\mathscr{E}(H_2) = \{E_1,E_3,E_6,E_7\} \subset \mathscr{E}(H)$, respectively.

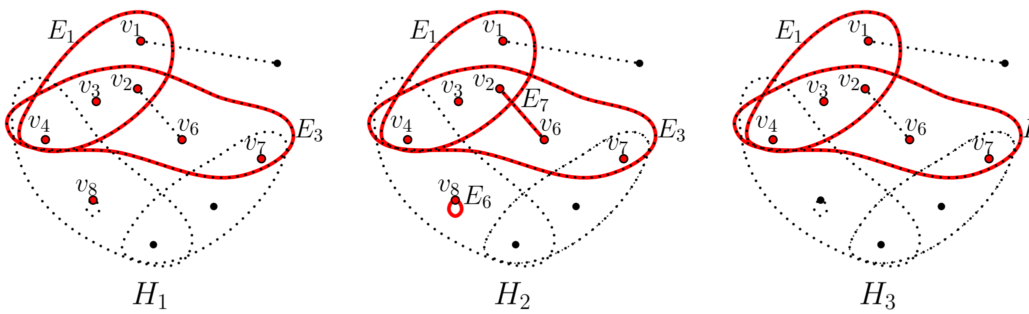


Figure 1.2: Partial hypergraphs H_1, H_2 and H_3 of the non simple hypergraph H of Figure 1.1

H_1 is neither an induced partial hypergraph, since $E_6, E_7 \subset V(H_1)$ but $E_6, E_7 \notin \mathscr{E}(H_1)$ although they are contained in $\mathscr{E}(H)$, nor is it a generated hypergraph, since $v_8 \notin E_1 \cup E_3$. While H_2 is an induced and a generated partial hypergraph, H_3 is generated but not induced, since $E_7 \notin \mathscr{E}(H_3)$, although $E_7 \subset V(H_3)$.

By a path between two partial hypergraphs H', H'' of a hypergraph H is meant a path in H between two vertices v and w such that $v \in V(H')$ and $w \in V(H'')$, respectively. The distance

$d_H(H', H'')$ between two partial hypergraphs H', H'' is the length of the shortest path between two vertices of the respective partial hypergraphs.
A partial hypergraph $H' \subseteq H$ is called *convex*, if all shortest paths in H between two vertices in H' are also contained in H'.

Homomorphisms For two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ and $H_2 = (V_2, \mathscr{E}_2)$ a *homomorphism* from H_1 into H_2 is a mapping $\varphi : V_1 \to V_2$ such that $\varphi(E) = \{\varphi(v_1), \dots, \varphi(v_r)\}$ is an edge in H_2, whenever $E = \{v_1, \dots, v_r\}$ is an edge in H_1. A mapping $\varphi : V_1 \to V_2$ is a *weak homomorphism* if edges are mapped either on edges or on vertices.
A bijective homomorphism φ is called an *isomorphism* if $\varphi(E) \in \mathscr{E}_2$ if and only if $E \in \mathscr{E}_1$. We say, H_1 and H_2 are *isomorphic*, in symbols $H_1 \cong H_2$ if there exists an isomorphism between them. In this case the two hypergraphs have the same structure.
If φ is the identity, H_1 and H_2 are said to be the same, $H_1 = H_2$, i.e., $V(H_1) = V(H_2)$ and $\mathscr{E}(H_1) = \mathscr{E}(H_2)$.

1.2 Graph Products

An *(undirected) graph* is a pair $G = (V, E)$ of *vertex set* V and a family E consisting of unordered pairs of elements of V, the *edges* of G. If an edge e of G consists of the same vertices of G, $e = \{u, u\}$, this will be called a *loop*. Such an edge will be denoted by $\{u\}$ instead of $\{u, u\}$.
A graph can be seen as a special hypergraph, whose edges are restricted to contain at most 2 elements. That is, a graph G is a hypergraph with rank $r(G) = 2$. Therefore, all definitions of the latter section may be transfered to graphs as well. A simple graph is then a 2-uniform hypergraph. The class of simple graphs will be denoted by Γ, that of simple graphs with loops by Γ_0. In the following, by a subgraph is meant a partial hypergraph of a graph. More information about graphs and graph theory can be found for example in [9] and [20].
In this section we give a short overview about the commutative standard graph products, that are the Cartesian product, the direct product and the strong product. We will restrict our considerations to their definition and a few basic algebraic properties. All assertions stated here, including their proofs and in addition, more detailed information about product graphs can be found in [29].

1.2.1 The Cartesian Product

The *Cartesian product* $G = G_1 \Box G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined on the vertex set $V(G) = V_1 \times V_2$ with edge set

$$E(G) = \big\{\{(u_1,u_2),(v_1,v_2)\} \mid \{u_1,v_1\} \in E_1, u_2 = v_2, \text{ or } u_1 = v_1, \{u_2,v_2\} \in E_2\big\}.$$

The Cartesian product satisfies several algebraic properties such like associativity, commutativity and distributivity with respect to the disjoint union, it is connected if and only if both factors are. The one vertex graph K_1 is a unit with respect to the Cartesian product, i.e. $G \Box K_1 \cong K_1 \Box G \cong G$ for any graph G. The Cartesian product $G = \Box_{i=1}^n G_i$ of arbitrary many factors G_i is well defined [28].

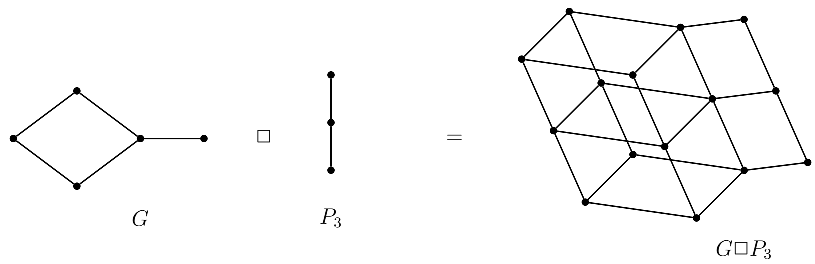


Figure 1.3: Cartesian Product of a graph G and a path P_3 on 3 vertices

The mapping $p_i : V(\Box_{i=1}^n G_i) \to V(G_i)$ defined by $p_i(v) = v_i$ for $v = (v_1, v_2, \ldots, v_n)$ is called *projection* on the i-th factor of G, and v_i is then the i-th *coordinate* of v. Each of the p_i is a weak homomorphism for all $i = i, \ldots, n$, since edges are mapped on edges or vertices. The restriction of p_i to the induced subgraph of G whose vertices differ from a vertex w only in the i-th coordinate is then an isomorphism, since

$$\big\langle \{v \in V(G) \mid p_j(v) = w_j, \text{ for all } j \neq i\} \big\rangle$$

is isomorphic to G_i. This induced subgraph is called *G_i-layer through w*, denoted by G_i^w. Notice, that G_i^w are convex subgraphs of G, while G_i is no subgraph of G. For $u \in V(G_i^w)$ we have $G_i^u = G_i^w$. If $u \notin V(G_i^w)$, then $G_i^u \neq G_i^w$, moreover $V(G_i^u) \cap V(G_i^w) = \emptyset$.
We are now in the position to give an equivalent definition of the Cartesian product in terms of projections as follows (see [37]).
For $G = \Box_{i=1}^n G_i$, $G_i = (V_i, E_i)$ and $I = \{1, \ldots, n\}$ we have

(1) $V(G) = V_1 \times V_2 \times \ldots \times V_n$

(2) $\{u, v\} \in E(G)$, $u, v \in V(G)$ iff there is an index $j \in I$, s.t. $\{p_j(u), p_j(v)\} \in E_j$ and $p_i(u) = p_i(v)$ for all $i \neq j, i \in I$.

A graph G is called *prime* with respect to a given product if it cannot be represented as a product of nontrivial factors. For the Cartesian product this means G is prime if $G = G_1 \Box G_2$ implies $G_1 = K_1$ or $G_2 = K_1$. A set $\{G_1, \ldots, G_n\}$ of graphs is called a *prime factorization* of G if $G = \Box_{i=1}^{n} G_i$, where G_i is prime and $G_i \neq K_1$ for $1 \leq i \leq n$ ([1]).

Proposition 1.1. *[29] Every graph G has a prime factor decomposition with respect to the Cartesian product. The number of prime factors is at most* $\log_2 |V(G)|$.

Cartesian Product graphs and their prime factor decomposition have been widely investigated, also from an algorithmic point of view, see [1, 14, 15, 30, 37, 40, 43].

Theorem 1.1. *[29] Prime factorization is not unique for the Cartesian product in the class of non-connected simple graphs.*

The next result was first proved by G. Sabidussi in 1960. V.G. Vizing gave an alternative proof in 1963.

Theorem 1.2. *[37, 40] Every connected graph G has a unique prime factor decomposition with respect to the Cartesian product.*

1.2.2 The Direct Product

As the Cartesian product, the direct product of two graphs $G = G_1 \times G_2$ is defined on the Cartesian product of their vertex sets $V(G) = V(G_1) \times V(G_2)$. A pair of vertices $\{(u_1, u_2), (v_1, v_2)\}$ is an edge in G if and only if $\{u_1, v_1\}$ is an edge in G_1 and $\{u_2, v_2\}$ is an edge in G_2. More formal:

$$E(G_1 \times G_2) = \big\{\{(u_1, u_2), (v_1, v_2)\} \mid \{u_1, v_1\} \in E(G_1), \{u_2, v_2\} \in E(G_2)\big\}.$$

While the factors of a connected direct product must be connected, the converse does not hold in general. As an example see Figure 1.4.

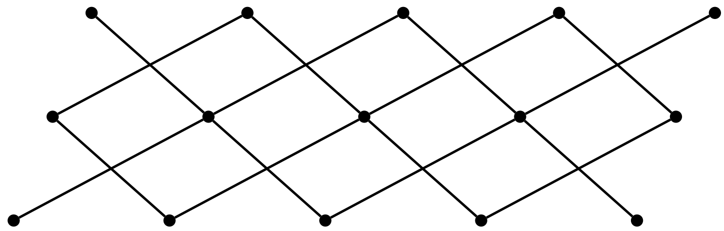


Figure 1.4: Direct product of two paths $P_3 \times P_5$. Although the factors are connected, their product consists of two connected components

As the Cartesian product the direct product satisfies several algebraic properties such like associativity, commutativity and distributivity. But it has no unit in the class of simple graphs. If we admit graphs with loops, then the one-vertex graph with a loop is a unit for the direct product [29].
The projections of a direct product into its factors are not just weak homomorphisms as for the Cartesian product, they are homomorphisms.
In terms of projections the direct product can be defined as follows:
For $G = \times_{i=1}^{n} G_i$, $G_i = (V_i, E_i)$ and $I = \{1, \ldots, n\}$ we have

(1) $V(G) = V_1 \times V_2 \times \ldots \times V_n$

(2) $\{u, v\} \in E(G)$, $u, v \in V(G)$ iff $\{p_i(u), p_i(v)\} \in E_i$ for all $i \in I$.

In distinction from the Cartesian product the prime factorization is not only non-unique in the class of non-connected graphs:

Theorem 1.3. *[29] Prime factorization with respect to the direct product is neither unique in the class of non-connected graphs with loops nor in the class of connected simple graphs.*

The next result was proved by R. McKenzie in 1971. In this work the direct product is called "cardinal product".

Theorem 1.4. *[34] Let G be a finite, non bipartite[1] connected graph in Γ_0. Then G has unique prime factor decomposition with respect to the direct product in Γ_0.*

[1] A graph G is called *bipartite* if its vertex set can be represented as the union of two disjoint sets V_1 and V_2, such that no edge of G joins vertices within V_1 and V_2, respectively.

1.2.3 The Strong Product

The *strong product* $G = G_1 \boxtimes G_2$ of two graphs G_1 and G_2 is defined on the Cartesian product of the vertex sets of the factors, $V(G) = V(G_1) \times V(G_2)$. two distinct vertices (u_1, u_2) and (v_1, v_2) are adjacent if

$$\{u_1, v_1\} \in E(G_1) \text{ and } \{u_2, v_2\} \in E(G_2), \text{ or}$$
$$\{u_1, v_1\} \in E(G_1) \text{ and } u_2 = v_2, \text{ or}$$
$$u_1 = v_1 \text{ and } \{u_2, v_2\} \in E(G_2).$$

We observe that

$$E(G_1 \boxtimes G_2) = E(G_1 \Box G_2) \cup E(G_1 \times G_2). \tag{1.6}$$

The strong product is associative, commutative, distributive w.r.t the disjoint union and has K_1 as unit. Hence the strong product $G = \boxtimes_{i=1}^{n} G_i$ of arbitrary many factors G_i is well defined. It is connected if and only if all of its factors are. The projections of a strong product graph into its factors are weak homomorphisms [29].

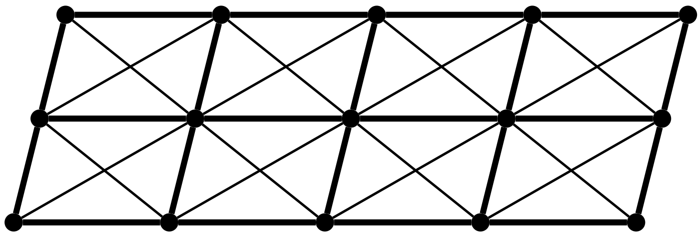


Figure 1.5: Strong product of two paths $P_3 \times P_5$. The thick edges are Cartesian edges, the thin ones are non-Cartesian edges

We can define the strong product, analogously to [37], in terms of the projection as follows: For $G = \boxtimes_{i=1}^{n} G_i$, $G_i = (V_i, E_i)$ and $I = \{1, \ldots, n\}$ we have

(1) $V(G) = V_1 \times V_2 \times \ldots \times V_n$

(2) $\{u, v\} \in E(G)$, $u, v \in V(G)$ iff there is an index set $J \subseteq I$, s.t. $\{p_j(u), p_j(v)\} \in E_j$ for all $j \in J$ and $p_i(u) = p_i(v)$ for all $i \in I \setminus J$.

As for the Cartesian product, each G_i^w-layer

$$\left\langle \{v \in V(G) \mid p_j(v) = w_j, \text{ for all } j \neq i\} \right\rangle$$

is isomorphic to G_i. But they are not convex in general, contrary to the Cartesian product.
The edges of a strong product $G = \boxtimes_{i=1}^{n} G_i$ that differ in exactly one coordinate are called *Cartesian* the others *non-Cartesian*, see Figure 1.5. In Equation (1.6) the Cartesian edges correspond to the first term $E(G_1 \Box G_2)$ on the right hand side and those which are non-Cartesian to the second term $E(G_1 \times G_2)$.

The strong product can be considered as a special case of the direct one:
For a graph $G \in \Gamma$ let $\mathcal{L}G$ denote the graph in Γ_0, which is formed from G by adding a loop to each vertex of G. On the other hand, for a graph $G' \in \Gamma_0$ let $\mathcal{N}G'$ denote the graph in Γ which emerges from G' by deleting all loops. Then we have for $G_1, G_2 \in \Gamma$:

$$G_1 \boxtimes G_2 = \mathcal{N}(\mathcal{L}G_1 \times \mathcal{L}G_2) \tag{1.7}$$

As for the Cartesian product, every graph has a prime factor decomposition with respect to the strong product. In the class of non-connected graphs, this need not be unique. The following result was proved by W. Dörfler and W. Imrich in 1970.

Theorem 1.5. *[12] Every connected graph G has unique prime factor decomposition with respect to the strong product.*

2 Hypergraph Products

A first definition of a "direct" product $H_1 \times H_2$ of two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ and $H_2 = (V_2, \mathscr{E}_2)$ on the vertex set $V(H_1 \times H_2) = V_1 \times V_2$ with edge set $\mathscr{E}(H_1 \times H_2) = \{E_1 \times E_2 \mid E_1 \in \mathscr{E}_1, E_2 \in \mathscr{E}_2\}$ can be found in [4]. Another definition of a hypergraph product is given by W. Dörfler and D. A. Waller in [13] and is also considered by X. Zhu in [44]:
For two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ and $H_2 = (V_2, \mathscr{E}_2)$ the product $H_1 \times H_2$ is the hypergraph with vertex set $V(H_1 \times H_2) = V_1 \times V_2$ and a subset $E = \{(x_1, y_1), \ldots, (x_k, y_k)\}$ of $V(H_1 \times H_2)$ is an edge of $H_1 \times H_2$ if and only if $\{x_1, \ldots, x_k\} \in \mathscr{E}_1$ and $\{y_1, \ldots, y_k\} \in \mathscr{E}_2$, where $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ need not to be distinct.
However, these products do not specialize to graphs. That is, the product of two graphs seen as 2-uniform hypergraphs is not a usual graph, but a 4-uniform hypergraph.

In this chapter we want to introduce hypergraph products, which coincide with the commutative standard graph products, defined in the previous section. In particular, we are interested in hypergraph products $\star$ that satisfy the following requirements:

1. $V(H_1 \star H_2) = V(H_1) \times V(H_2)$
2. The restriction of the product $\star$ on graphs coincides with the corresponding graph product.
3. Associativity.
4. Commutativity.
5. Distributivity with respect to the disjoint union.
6. If H_1 and H_2 are simple then $H_1 \star H_2$ is simple.
7. The projections $p_i : V(H_1 \star H_2) \to V(H_i)$ for $i \in \{1, 2\}$ are at least weak homomorphisms.

In the first section we are concerned with the Cartesian product of hypergraphs. It was introduced by W. Imrich in 1967 as the Cartesian product of set systems as a generalization of the Cartesian graph product [23].

In the second section we will introduce three direct products, and we will see, for what reasons some of them might fail for our purpose.
In the third section we will be concerned with the strong hypergraph product. As we have seen in Section 1.2.3 the edge set of the strong product of two graphs is the union of the edge sets of their Cartesian and direct product. There is no reason to change this for hypergraphs.

2.1 The Cartesian Product

Let H_1 and H_2 be two Hypergraphs. The *Cartesian product* $H = H_1 \Box H_2$ has vertex set $V(H) = V(H_1) \times V(H_2)$, that is the Cartesian product of the vertex sets of the factors and the edge set

$$\begin{aligned}\mathscr{E}(H) =& \big\{\{x\} \times F : x \in V(H_1), F \in \mathscr{E}(H_2)\big\} \\ &\cup \big\{E \times \{y\} : E \in \mathscr{E}(H_1), y \in V(H_2)\big\}.\end{aligned}$$

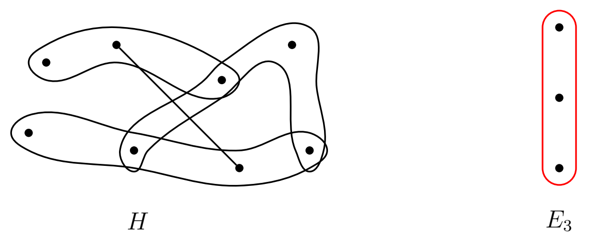
Thus, the set $\{(x_1,y_1),\dots,(x_r,y_r)\}$, $x_i \in V(H_1)$, $y_i \in V(H_2)$, $i = 1,\dots,r$, is an edge in $\mathscr{E}(H_1 \Box H_2)$ if and only if either

(i) $\{x_1,\dots,x_r\}$ is an edge in $\mathscr{E}(H_1)$ and $y_1 = \dots = y_r$, or

(ii) $\{y_1,\dots,y_r\}$ is an edge in $\mathscr{E}(H_2)$ and $x_1 = \dots = x_r$.

The Cartesian product of hypergraphs was introduced by W. Imrich [23]. We consider it for sake of completeness and prove explicitly some basic properties.

Figure 2.1 shows an example of a Cartesian product of two hypergraphs H and E_3, where E_3 is is the hypergraph which consists of a single edge with three vertices.



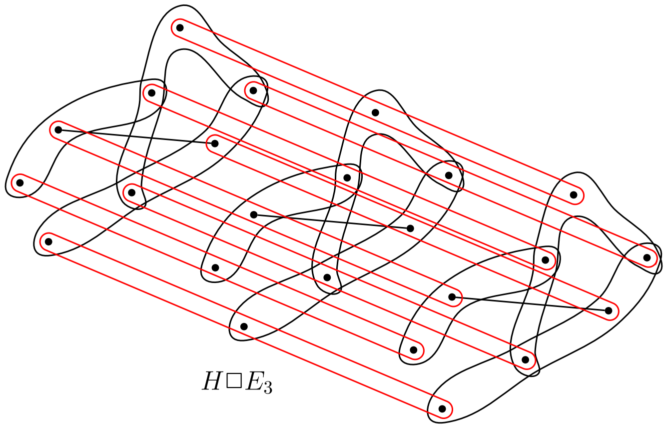


Figure 2.1: Hypergraphs H and E_3 and their Cartesian product $H\Box E_3$

Let an edge E in $\mathscr{E}(H)$ be denoted by

(i) E_{iy} if $E = E_i \times \{y\}$, where $E_i \in \mathscr{E}(H_1)$ and $y \in V(H_2)$

(ii) E_{xj} if $E = \{x\} \times E_j$, where $x \in V(H_1)$ and $E_j \in \mathscr{E}(H_2)$

We define the *projection* of a Cartesian product $H = H_1\Box H_2$ into one of the factors H_i, $i \in \{1,2\}$ by the mapping $p_i : V(H) \to V(H_i)$, with $p_i(v) = v_i$, where $v = (v_1, v_2) \in V(H)$. By $p_i(X)$ we denote the set $\{p_i(x) : x \in X\}$ for an $X \subseteq V(H)$. The projections of a Cartesian product of hypergraphs into its factors are weak homomorphisms, as edges are either mapped onto edges or onto vertices:

$$p_1(E_{xj}) = x \in V(H_1) \qquad \text{and} \qquad p_2(E_{iy}) = E_i \in \mathscr{E}(H_1)$$
$$p_1(E_{xj}) = E_j \in \mathscr{E}(H_2) \qquad \text{and} \qquad p_2(E_{iy}) = y \in V(H_2)$$

Lemma 2.1. *The Cartesian Product is associative.*

Proof. To proof associativity we need to show that the mapping $V((H_1\Box H_2)\Box H_3) \to V(H_1\Box(V_2\Box V_3))$ defined by $((x,y),z) \mapsto (x,(y,z))$ with $x \in V(H_1),\ y \in V(H_2)$ and $z \in V(H_3)$ is an isomorphism. Clearly, it is bijective.
Let $\{((x_1,y_1),z_1),\ldots,((x_r,y_r),z_r)\}$ be an edge in $(H_1\Box H_2)\Box H_3$. Then the following cases can occur:

(i) $(x_1,y_1) = \ldots = (x_r,y_r) \in V(H_1\Box H_2)$, i.e. $x_1 = \ldots = x_r \in V(H_1)$ and $y_1 = \ldots = y_r \in V(H_2)$, and therefore $\{z_1,\ldots,z_r\} \in \mathscr{E}(H_3)$, or

(ii) $z_1 = \ldots = z_r \in V(H_3)$ and therefore $\{(x_1,y_1),\ldots,(x_r,y_r)\} \in \mathscr{E}(H_1 \Box H_2)$. Then

(a) $x_1 = \ldots = x_r \in V(H_1)$ and $\{y_1,\ldots,y_r\} \in \mathscr{E}(H_2)$, or

(b) $y_1 = \ldots = y_r \in V(H_2)$ and $\{x_1,\ldots,x_r\} \in \mathscr{E}(H_1)$.

Altogether we have either

(1) $\{x_1,\ldots,x_r\}$ is an edge in H_1 and the y_i and z_i resp. are equal, or

(2) $\{y_1,\ldots,y_r\}$ is an edge in H_2 and the x_i and z_i resp. are equal, or

(3) $\{z_1,\ldots,z_r\}$ is an edge in H_3 and the x_i and y_i resp. are equal.

But this is equivalent to the following:

(i) $x_1 = \ldots = x_r \in V(H_1)$ and $\{(y_1,z_1),\ldots,(y_r,z_r)\} \in \mathscr{E}(H_2 \Box H_3)$ because of (2) and (3), respectively, or

(ii) $\{x_1,\ldots,x_r\} \in \mathscr{E}(H_1)$ and $(y_1,z_1) = \ldots = (y_r,z_r) \in V(H_2 \Box H_3)$ because of (1).

Therefore, $\{(x_1,(y_1,z_2)),\ldots,(x_r,(y_r,z_r))\}$ is an edge in $H_1 \Box (H_2 \Box H_3)$. That means the image of a subset $F \subseteq V((H_1 \Box H_2) \Box H_3)$ is an edge in $H_1 \Box (H_2 \Box H_3)$ if and only if it is an edge in$(H_1 \Box H_2) \Box H_3$.
Hence, the mapping defined above is an isomorphism, which completes the proof. $\Box$

Due to the symmetry of the definition we can state

Lemma 2.2. *The Cartesian product is commutative.*

Notice, that commutativity and associativity is meant in the sense of identifying isomorphic hypergraphs.
The one vertex hypergraph K_1 is a unit with respect to the Cartesian product, i.e., $K_1 \Box H \cong H \Box K_1 \cong H$ for any hypergraph H. The isomorphism is given by the projection into the factor H. For two vertex-disjoint hypergraphs H_1 and H_2 their *disjoint union*, denoted as $H_1 + H_2$ is defined by $V(H_1 + H_2) = V(H_1) \cup V(H_2)$ and $\mathscr{E}(H_1 + H_2) = \mathscr{E}(H_1) \cup \mathscr{E}(H_2)$.

Lemma 2.3. *The Cartesian product is left and right distributive together with the disjoint union as addition.*

Proof. Let H_1, H_2 and H_3 be hypergraphs and furthermore, let H_2 and H_3 be vertex-disjoint. Then we have for the vertex set of $H_1\Box(H_2+H_3)$:

$$\begin{aligned} V(H_1\Box(H_2+H_3)) &= V(H_1)\times(V(H_2)\cup V(H_3)) \\ &= (V(H_1)\times V(H_2))\cup(V(H_1)\times V(H_3)) \\ &= V(H_1\Box H_2+H_1\Box H_3) \end{aligned} \tag{2.1}$$

and for the edge set:

$$\begin{aligned} \mathscr{E}(H_1\Box(H_2+H_2)) =& \big\{\{x\}\times F : x\in V(H_1), F\in\mathscr{E}(H_2+H_3)\big\} \\ & \cup\big\{E\times\{y\} : E\in\mathscr{E}(H_1), y\in V(H_2+H_3)\big\} \\ =& \big\{\{x\}\times F : x\in V(H_1), F\in\mathscr{E}(H_2)\cup\mathscr{E}(H_3)\big\} \\ & \cup\big\{E\times\{y\} : E\in\mathscr{E}(H_1), y\in V(H_2)\cup V(H_3\big\} \\ =& \big\{\{x\}\times F : x\in V(H_1), F\in\mathscr{E}(H_2)\big\} \\ & \cup\big\{\{x\}\times F : x\in V(H_1), F\in\mathscr{E}(H_3)\big\} \\ & \cup\big\{E\times\{y\} : E\in\mathscr{E}(H_1), y\in V(H_2)\big\} \\ & \cup\big\{E\times\{y\} : E\in\mathscr{E}(H_1), y\in V(H_3)\big\} \\ =& \big\{\{x\}\times F : x\in V(H_1), F\in\mathscr{E}(H_2)\big\} \\ & \cup\big\{E\times\{y\} : E\in\mathscr{E}(H_1), y\in V(H_2)\big\} \\ & \cup\big\{\{x\}\times F : x\in V(H_1), F\in\mathscr{E}(H_3)\big\} \\ & \cup\big\{E\times\{y\} : E\in\mathscr{E}(H_1), y\in V(H_3)\big\} \\ =& \mathscr{E}(H_1\Box H_2)\cup\mathscr{E}(H_1\Box H_3) \\ =& \mathscr{E}(H_1\Box H_2+H_1\Box H_3) \end{aligned} \tag{2.2}$$

Hence

$$H_1\Box(H_2+H_3) = H_1\Box H_2+H_1\Box H_3,$$

From analogous considerations we can infer

$$(H_1+H_2)\Box H_3 = H_1\Box H_3+H_2\Box H_3,$$

for vertex-disjoint H_1 and H_2. $\Box$

Lemma 2.4. *The Cartesian product $H = \Box_{i=1}^{n} H_i$ of hypergraphs H_i is connected if and only if all of its factors H_i are connected.*

Proof. Because of associativity and commutativity of the Cartesian product it suffices to show the assertion for $n = 2$, therefore let $H = H_1 \Box H_2$.
First assume H_1 and H_2 to be connected. Let $v = (x,y)$ and $v' = (x',y')$ be two arbitrary vertices in $V(H)$. Consider a path $P_{xx'} = (E_1,\dots,E_r)$ from x to x' in H_1 and a path $P_{yy'} = (F_1,\dots,F_s)$ from y to y' in H_2. Then $(E_{1y},\dots,E_{ry})$ is a path from (x,y) to (x',y) in H and $(F_{x'1},\dots,F_{x's})$ is a path from (x',y) to (x',y') in H. Hence $(E_{1y},\dots,E_{ry},F_{x'1},\dots,F_{x's})$ is a path from v to v' in H.
W.l.o.g. suppose now H_1 is not connected. It is then the disjoint union of two hypergraphs, $H_1 = H_1' + H_1''$. Since the Cartesian product is distributive with respect to the disjoint union, we have $H = H_1' \Box H_2 + H_1'' \Box H_2$, that is H is the disjoint union of two hypergraphs, i.e., H is not connected. □

Lemma 2.5. *The Cartesian product $H = \Box_{i=1}^{n} H_i$ of hypergraphs H_i is simple if and only if all of its factors H_i are simple.*

Proof. Because of associativity and commutativity of the Cartesian product, it suffices to show the assertion for $n = 2$, therefore let $H = H_1 \Box H_2$.
First let H_1 and H_2 be simple and suppose H is not simple. We have to examine several cases:
Suppose H contains at least one loop $\{(x,y)\} = \{x\} \times \{y\}$. Then, it follows, $\{x\}$ is an edge in H_1, i.e., a loop, or $\{y\}$ is a loop in H_2. Both contradicts the fact, that H_1 and H_2 are simple. Thus, $|E| \geq 2$ for all $E \in \mathscr{E}(H)$.
Now, let $E_{xj} \subseteq E_{x'j'}$. It follows immediately that $x = x'$ and $E_j \subseteq E_{j'}$. Since H_2 is simple, we have $j = j'$, and therefore it follows $(xj) = (x'j')$. By commutativity, the same holds for the case $E_{iy} \subseteq E_{i'y'}$.
Now assume $E_{xj} \subseteq E_{iy}$, i.e. $(\{x\} \times E_j) \subseteq (E_i \times \{y\})$. Then we have $E_j \subseteq \{y\}$ thus $E_j = \{y\}$, contradicting that H_2 is simple. The same argumentation holds for $E_{iy} \subseteq E_{xj}$.
That means, $H_1 \Box H_2$ is simple.
Now assume (at least) one of the factors is not simple, w.l.o.g. say H_1. Then there are two edges $E_i, E_j \in \mathscr{E}(H_1)$, such that $E_i \subseteq E_j, i \neq j$ or there is an $E_i \in \mathscr{E}(H_1)$ with $|E_i| = 1$, say $E_i = \{x\}$. In the first case we have for any $y \in V(H_2)$, $E_{iy} = E_i \times \{y\} \subseteq E_j \times \{y\} = E_{jy}$ and $(iy) \neq (jy)$, hence $H_1 \Box H_2$ is not simple. In the second case, we have $|E_{iy}| = |\{(x,y)\}| = 1$ for any $y \in V(H_2)$ and $H_1 \Box H_2$ would not be simple. □

The Cartesian Product of arbitrarily many factors $H = \Box_{i=1}^{n} H_i$ is now well defined. Thus, we can extend the concept of the projections. For $H = \Box_{i=1}^{n} H_i$ and $j \in \{1,\dots,n\}$ we define $p_j : V(\Box_{i=1}^{n} H_i) \to V(H_j)$ through $p_j(v) = v_j$, for $v = (v_1,\dots,v_n)$, the *projection into the j-th factor* H_j of H. We then call v_j the j-th *coordinate* of v. Clearly, the projections p_j, $j \in \{1,\dots,n\}$, $n \geq 2$ are weak homomorphisms as well.
According to [37] the Cartesian product of hypergraphs can be described in terms of projections as follows:
For $H = \Box_{i=1}^{n} H_i$, with $H_i = (V_i, \mathscr{E}_i)$ and $I = \{1,\dots,n\}$ we have

(1) $V(H) = V_1 \times V_2 \times \dots \times V_n$,

(2) for $E \subseteq V(H)$ we have $E \in \mathscr{E}(H)$ if and only if there is an $i \in I$, s.t.

  (i) $p_i(E) \in \mathscr{E}_i$ and

  (ii) $|p_j(E)| = 1$ for all $j \in I \setminus \{i\}$.

Notice, that $|p_i(E)| = |E|$ holds.
Let $w \in V(H)$ be a vertex of H. The partial hypergraph of H induced by all vertices of H which differ from w exactly in the j-th coordinate is isomorphic to H_j, more formal

$$\langle \{v \in V(H) \mid p_k(v) = w_k \text{ for } k \neq j\} \rangle \cong H_j.$$

We will call this partial hypergraph the *H_j-layer through w*, denoted as H_j^w. The isomorphism $H_j^w \to H_j$ is then the projection p_j. For $u \in V(H_j^w)$ we have $H_j^u = H_j^w$ and moreover $V(H_j^u) \cap V(H_j^w) = \emptyset$ if and only if $u \notin V(H_j^w)$.

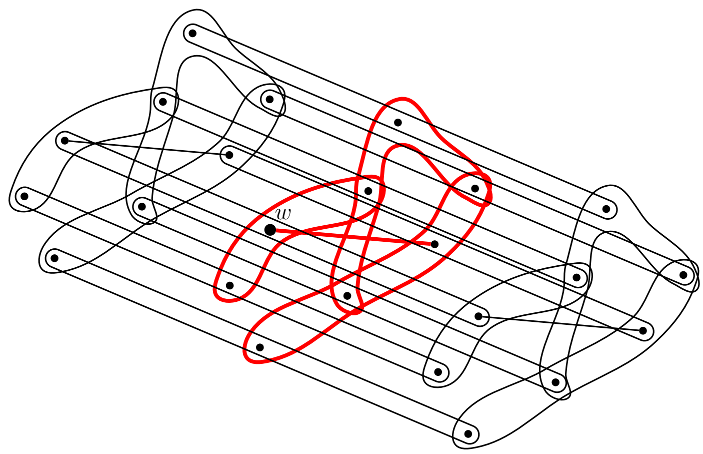


Figure 2.2: Cartesian product $H \Box E_3$ (cf. Figure 2.1), red H^w-layer

Figure 2.2 shows the Cartesian product of a hypergraph H and the hypergraph E_3 consisting of a single edge with three vertices, see Figure 2.1 and a layer of the hypergraph H through the vertex w, H^w, highlighted by red edges.

Summary

To conclude this section, we summarize the preceding results. For the Cartesian product holds:

1. The restriction of this product on graphs is the Cartesian graph product.
2. Associativity.
3. Commutativity.
4. Distributivity with respect to the disjoint union.
5. The product of two simple hypergraphs is simple.
6. The projections of a product hypergraph onto its factors are weak homomorphisms.

As an open problem, it remains to examine if this product has a unique prime factorization. This will be done in Chapter 3.

2.2 The Direct Product

To find suitable direct hypergraph products, we have to claim that they are closed under the restriction on 2-uniform hypergraphs. The most simple way to ensure this is to define a product which preserves the rank of one of its factors, and therefore is closed on r-uniform hypergraphs in general. Besides, it is also possible to define a direct hypergraph product, which does not need to be r-uniform although its factors are.
In this section we will introduce three different direct products. The direct product $\check{\times}$, which preserves the minimal rank of two factors, the direct product $\hat{\times}$, which preserves the maximal rank of two factors and the direct product $\tilde{\times}$, which does not preserve any rank of its factors.

2.2.1 r-Uniformity Preserving Direct Product

Recall, that for two given graphs, G_1 and G_2, $e = \{(x_1,x_2),(y_1,y_2)\}$ is an edge in $G_1 \times G_2$, for $x_i, y_i \in V(G_i)$, $i = 1,2$, if and only if $\{x_1,y_1\}$ is an edge in G_1 and $\{x_2,y_2\}$ is an edge in G_2.

There is no problem to extend this definition to r-uniform hypergraphs:
Let H_1 and H_2 be two r-uniform hypergraphs. We define their direct product $H_1 \times H_2$ by

$$\mathscr{E}(H_1 \times H_2) := \big\{ \{(x_1,y_1),\ldots,(x_r,y_r)\} \mid \{x_1,\ldots,x_r\} \in \mathscr{E}(H_1), \{y_1,\ldots,y_r\} \in \mathscr{E}(H_2) \big\}. \quad (2.3)$$

Thus $E = \{(x_1,y_1),\ldots,(x_r,y_r)\}$, $x_i \in V(H_1)$, $y_i \in V(H_2)$, $i = 1,\ldots,r$, is an edge in $H_1 \times H_2$ if and only if

(i) $\{x_1,\ldots,x_r\}$ is an edge in H_1 and

(ii) $\{y_1,\ldots,y_r\}$ is an edge in H_2.

Figure 2.3 shows a direct product of two hypergraphs that consists only of a single edge with three vertices.

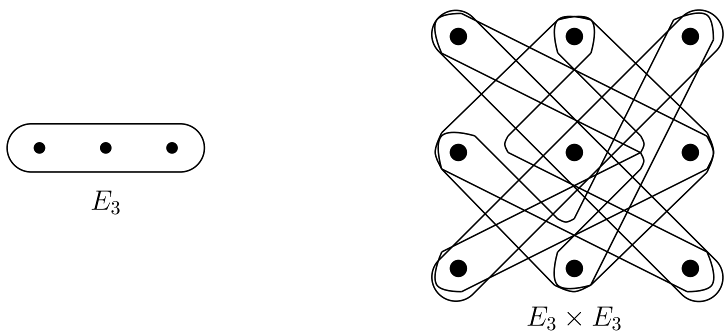


Figure 2.3: Direct product $E_3 \times E_3$

Clearly, for $r = 2$ this is the direct graph product.

This product was introduced by W. Dörfler and D. A. Waller in [13]. However, it is only defined on r-uniform hypergraphs. A natural question is how to extend this to a product between two arbitrary, non-uniform hypergraphs.

Minimal Rank Preserving Direct Product

Now we introduce a direct hypergraph product which preserves the the minimal rank of one of its factors.
Let $H_1 = (V_1, \mathscr{E}_1)$ and $H_2 = (V_2, \mathscr{E}_2)$ be two hypergraphs. Then their *direct product* $\check{\times}$ is defined on the Cartesian product of the vertex set, and a subset $E = \{(x_1,y_1),\ldots,(x_r,y_r)\}$ of $V_1 \times V_2$ is an edge in $H_1 \check{\times} H_2$ if and only if

(i) $\{x_1,\dots,x_r\}$ is an edge in H_1 and $\{y_1,\dots,y_r\}$ is the subset of an edge in H_2, or

(ii) $\{x_1,\dots,x_r\}$ is the subset of an edge in H_1 and $\{y_1,\dots,y_r\}$ is an edge in H_2

More formal, for two hypergraphs $H_1 = (V_1,\mathscr{E}_1), H_2 = (V_2,\mathscr{E}_2)$, we define their direct product $\check{\times}$ by the edge set

$$\mathscr{E}(H_1 \check{\times} H_2) := \Big\{ \bigcup_{x\in E} \{(x,\pi(x))\} \mid \pi : E \to F \text{ injective, } E \in \mathscr{E}_1, F \in \mathscr{E}_2 \Big\} \\ \cup \Big\{ \bigcup_{y\in F} \{(\pi'(y),y)\} \mid \pi' : F \to E \text{ injective, } E \in \mathscr{E}_1, F \in \mathscr{E}_2 \Big\}.$$

Let an edge $\{(x_1,y_1),\dots,(x_r,y_r)\}$ in $H = H_1 \check{\times} H_2$ be denoted by

(i) $E_{i\pi}$ if $\{x_1,\dots,x_r\} = E_i \in \mathscr{E}_1$ and $\{y_1,\dots,y_r\} \subseteq F_j \in \mathscr{E}_2$, where $\pi : E_i \to F_j$ is the injective mapping defined by $\pi(x_k) = y_k$, for all $k \in \{1,\dots,r\}$, and

(ii) $E_{\pi' j}$ if $\{y_1,\dots,y_r\} = F_j \in \mathscr{E}_2$ and $\{x_1,\dots,x_r\} \subseteq E_i \in \mathscr{E}_1$, where $\pi' : F_j \to E_i$ is the injective mapping defined by $\pi'(y_k) = x_k$ for all $k \in \{1,\dots,r\}$.

Notice, if there exists no edge of the form $E_{i\pi}$, all edges of H are of the form $E_{\pi' j}$ and vice versa.

Figure 2.4 shows a direct product $\check{\times}$ of a hypergraph H and the hypergraph E_3, which consists only of a single edge with three vertices.

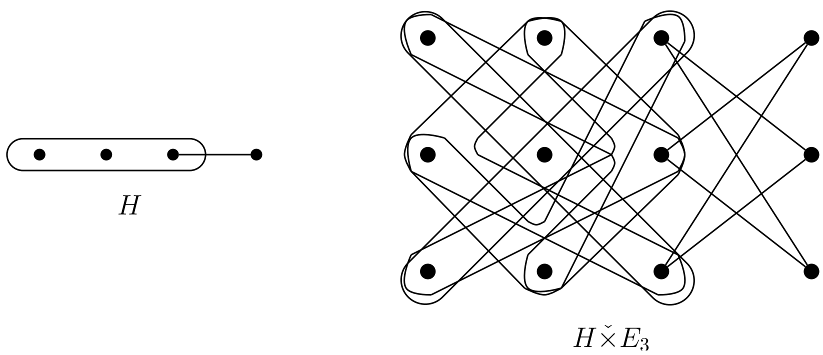


Figure 2.4: Direct product $\check{\times}$, $H \check{\times} E_3$

The restriction of the direct product $\check{\times}$ to r-uniform hypergraphs is the product defined in Equation (2.3), hence the restriction of this product to simple graphs coincides with the direct

graph product.
We have:

$$r(H_1 \check{\times} H_2) = \min\{r(H_1), r(H_2)\}$$

and

$$s(H_1 \check{\times} H_2) = \min\{s(H_1), s(H_2)\}$$

where $r(H)$ and $s(H)$ denote the rank and the anti-rank, respectively of a hypergraph H.

Lemma 2.6. *The direct product $\check{\times}$ is associative.*

Proof. Let $H_1 = (V_1 \mathscr{E}_1), H_2 = (V_2, \mathscr{E}_2), H_3 = (V_3, \mathscr{E}_3)$ be given hypergraphs. As for the Cartesian product we proof associativity by showing that the mapping $\psi : V(H_1 \check{\times}(H_2 \check{\times} H_3)) \rightarrow V((H_1 \check{\times} H_2)\check{\times} H_3)$ defined by $(x,(y,z)) \mapsto ((x,y),z)$, with $x \in V_1$, $y \in V_2$ and $z \in V_3$, is an isomorphism. Obviously ψ is bijective, hence, it remains to show the homomorphism property, i.e., we have to show that E is an edge in $H_1 \check{\times}(H_2 \check{\times} H_3)$ if and only if ψE is an edge in $(H_1 \check{\times} H_2)\check{\times} H_3$.
Let $E = \{((x_1,y_1),z_1), \ldots, ((x_r,y_r),z_r)\}$ be an edge in $(H_1 \check{\times} H_2)\check{\times} H_3$.
There are two cases which can occur. First, $\{z_1,\ldots,z_r\}$ is an edge in H_3 and $\{(x_1,y_1),\ldots,(x_r,y_r)\}$ is thus a subset of an edge in $H_1 \check{\times} H_2$, hence $\{x_1,\ldots,x_r\}$ and $\{y_1 \ldots, y_r\}$ must be subsets of edges in H_1 and H_2 respectively. But then $\{(y_1,z_1),\ldots,(y_r,z_r)\}$ is an edge in $H_2 \check{\times} H_3$, thus $\psi E = \{(x_1,(y_1,z_1)),\ldots,(x_r,(y_r,z_r))\}$ is an edge in $H_1 \check{\times}(H_2 \check{\times} H_3)$.
Second, $\{(x_1,y_1),\ldots,(x_r,y_r)\}$ is an edge in $H_1 \check{\times} H_2$ and $\{z_1,\ldots,z_r\}$ is a subset of an edge in H_3. Then, $\{x_1,\ldots,x_r\}$ is an edge in H_1 and $\{y_1 \ldots, y_r\}$ is a subset of an edge in H_2, or vice versa. In the first case $\{(y_1,z_1),\ldots,(y_r,z_r)\}$ is a subset of an edge in $H_2 \check{\times} H_3$, hence, ψE is an edge in $H_1 \check{\times}(H_2 \check{\times} H_3)$, and in the second case $\{(y_1,z_1),\ldots,(y_r,z_r)\}$ is an edge in $H_2 \check{\times} H_3$ and thus ψE is an edge in $H_1 \check{\times}(H_2 \check{\times} H_3)$.
This implies that, whenever E is an edge in $(H_1 \check{\times} H_2)\check{\times} H_3$, then ψE is an edge in $H_1 \check{\times}(H_2 \check{\times} H_3)$. The converse, i.e., if ψE is an edge in $H_1 \check{\times}(H_2 \check{\times} H_3)$ then E is an edge in $(H_1 \check{\times} H_2)\check{\times} H_3$, is shown analogously. Hence, it holds $(H_1 \check{\times} H_2)\check{\times} H_3 \cong H_1 \check{\times}(H_2 \check{\times} H_3)$. □

As for the Cartesian product, we have due to the symmetry of its definition:

Lemma 2.7. *The direct product $\check{\times}$ is commutative*

Distributivity with respect to the disjoint union as addition follows by set-theoretic considerations similar to those as in the case of the Cartesian product.

Lemma 2.8. *The direct product $\check{\times}$ is left and right distributive with respect to the disjoint union as addition.*

Proof. Let $H_1 = (V_1, \mathscr{E}_1)$, $H_2 = (V_2, \mathscr{E}_2)$ and $H_3 = (V_3, \mathscr{E}_3)$ be hypergraphs and let H_2 and H_3 be vertex-disjoint. Then we have

$$\begin{aligned}
\mathscr{E}(H_1 \check{\times} (H_2 + H_3)) =& \Big\{ \bigcup_{x\in E} \{(x,\pi(x))\} \mid \pi : E \to F \text{ injective}, E \in \mathscr{E}_1, F \in \mathscr{E}_2 \cup \mathscr{E}_3 \Big\} \\
& \cup \Big\{ \bigcup_{y\in F} \{(\pi'(y),y)\} \mid \pi' : F \to E \text{ injective}, E \in \mathscr{E}_1, F \in \mathscr{E}_2 \cup \mathscr{E}_3 \Big\} \\
=& \Big\{ \bigcup_{x\in E} \{(x,\pi(x))\} \mid \pi : E \to F \text{ injective}, E \in \mathscr{E}_1, F \in \mathscr{E}_2 \Big\} \\
& \cup \Big\{ \bigcup_{x\in E} \{(x,\pi(x))\} \mid \pi : E \to F \text{ injective}, E \in \mathscr{E}_1, F \in \mathscr{E}_3 \Big\} \\
& \cup \Big\{ \bigcup_{y\in F} \{(\pi'(y),y)\} \mid \pi' : F \to E \text{ injective}, E \in \mathscr{E}_1, F \in \mathscr{E}_2 \Big\} \\
& \cup \Big\{ \bigcup_{y\in F} \{(\pi'(y),y)\} \mid \pi' : F \to E \text{ injective}, E \in \mathscr{E}_1, F \in \mathscr{E}_3 \Big\} \\
=& \Big\{ \bigcup_{x\in E} \{(x,\pi(x))\} \mid \pi : E \to F \text{ injective}, E \in \mathscr{E}_1, F \in \mathscr{E}_2 \Big\} \\
& \cup \Big\{ \bigcup_{y\in F} \{(\pi'(y),y)\} \mid \pi' : F \to E \text{ injective}, E \in \mathscr{E}_1, F \in \mathscr{E}_2 \Big\} \\
& \cup \Big\{ \bigcup_{x\in E} \{(x,\pi(x))\} \mid \pi : E \to F \text{ injective}, E \in \mathscr{E}_1, F \in \mathscr{E}_3 \Big\} \\
& \cup \Big\{ \bigcup_{y\in F} \{(\pi'(y),y)\} \mid \pi' : F \to E \text{ injective}, E \in \mathscr{E}_1, F \in \mathscr{E}_3 \Big\} \\
=& \mathscr{E}(H_1 \check{\times} H_2) \cup \mathscr{E}(H_1 \check{\times} H_3) \\
=& \mathscr{E}((H_1 \check{\times} H_2) + (H_1 \check{\times} H_3))
\end{aligned}$$

With the same arguments as in Equation (2.1) we can conclude $V(H_1 \check{\times} (H_2 + H_3)) = V((H_1 \check{\times} H_2) + (H_1 \check{\times} H_3))$ and hence, it follows $H_1 \check{\times} (H_2 + H_3) = (H_1 \check{\times} H_2) + (H_1 \check{\times} H_3)$. Analogously we can conclude $(H_1 + H_2) \check{\times} H_3 = (H_1 + H_3) \check{\times} (H_2 + H_3)$ for vertex disjoint hypergraphs H_1 and H_2. □

Note, that the direct product $\check{\times}$ of two connected hypergraphs need not to be connected.

Lemma 2.9. *The direct product $\check{\times}$, $H = \check{\times}_{i=1}^{n} H_i$, of simple hypergraphs H_i is simple.*

Proof. Because of associativity and commutativity of the direct product $\check{\times}$, it suffices to prove the assertion for $n = 2$. Therefore, let $H_1 = (V_1, \mathscr{E}_1)$ and $H_2 = (V_2, \mathscr{E}_2)$ be two simple hypergraphs

and suppose $H = H_1 \check{\times} H_2$ is not simple. Then several cases can occur.
Suppose first H contains at least one loop $\{(x,y)\}$. Then, it follows, $\{x\}$ is an edge in H_1, i.e., a loop, or $\{y\}$ is a loop in H_2, contradicting the fact, that H_1 and H_2 are simple. Thus, $|E| \geq 2$ for all $E \in \mathscr{E}(H)$.
Now,assume there is an edge $E_{i_1\pi_1}$ contained in an edge $E_{i_2\pi_2}$. If there are no edges of the form $E_{i\pi}$ in H, we consider the hypergraph $H_2 \check{\times} H_1$.
We have $E_{i_s\pi_s} = \bigcup_{x \in E_{i_s}} \{(x, \pi_s(x))\}$, where $\pi_s : E_{i_s} \to F_{j_s}$ is an injective mapping, with $E_{i_s} \in \mathscr{E}_1$ and $F_{j_s} \in \mathscr{E}_2$, for $s = 1,2$. It follows

$$E_{i_1} = \bigcup_{x \in E_{i_1}} \{x\} \subseteq \bigcup_{x \in E_{i_2}} \{x\} = E_{i_2}$$

and since H_1 is simple we conclude $i_1 = i_2$. Furthermore we have

$$\pi_1(E_{i_1}) = \bigcup_{x \in E_{i_1}} \{\pi_1(x)\} \subseteq \bigcup_{x \in E_{i_2}} \{\pi_2(x)\} \subseteq F_{j_2},$$

hence π_1 is a mapping from E_{i_1} into F_{j_2} and since $\pi_1(x) = \pi_2(x)$ must hold for all $x \in E_{i_1} = E_{i_2}$ we have $\pi_1 = \pi_2$. Thus, $E_{i_1\pi_1} \subseteq E_{i_2\pi_2}$ implies $(i_1\pi_1) = (i_2\pi_2)$. Analogously we can conclude that $E_{\pi'_1 j_1} \subseteq E_{\pi'_2 j_2}$ implies $(\pi'_1 j_1) = (\pi'_2 j_2)$.
Now assume we have $E_{i\pi} \subseteq E_{\pi' j}$ with $E_{i\pi} = \bigcup_{x \in E_i} \{(x, \pi(x))\}$, where $\pi : E_i \to F_{j'}$, and $E_{\pi' j} = \bigcup_{y \in F_j} \{(\pi'(y), y)\}$, $\pi' : F_j \to E_{i'}$, respectively. Remark that $\pi(E_i) \subseteq F_{j'}$ as well as $\pi'(F_j) \subseteq E_{j'}$. It follows

$$E_i = \bigcup_{x \in E_i} \{x\} \subseteq \bigcup_{y \in F_j} \{\pi'(y)\} = \pi'(F_j) \subseteq E_{i'}$$

and since H_1 is simple, we conclude $i = i'$. But then it follows $E_i = \pi'(F_j)$, i.e. $\pi' : F_j \to E_i$ is surjective and therefore bijective. Thus we can write $E_{\pi' j} = \bigcup_{y \in F_j} \{(\pi'(y), y)\} = \bigcup_{x \in E_i} \{(x, \pi'^{-1}(x))\} = E_{i\pi'^{-1}}$. Since $\pi(x) = y$ if and only if $\pi'(y) = x$ for all $x \in E_i$ we obtain $\pi'^{-1} = \pi$. Hence $E_{i\pi} \subseteq E_{\pi' j}$ with $E_{i\pi} = \bigcup_{x \in E_i} \{(x, \pi(x))\}$ implies $(i\pi) = (\pi' j)$.
The fact that $E_{\pi' j} \subseteq E_{i\pi}$ implies $(i\pi) = (\pi' j)$ as well is shown analogously. Thus, H is simple. □

The direct product $\check{\times}$ does not have a unit in the class of simple hypergraphs, since the direct product has no unit for simple graphs. Also in the class of non simple hypergraphs, there exists no unit. To be more precise, neither the one vertex hypergraph K_1 without edges, nor the one with a loop, $\mathcal{L}K_1$, is a unit for the direct product $\check{\times}$:

Example 2.1. *Consider the (hyper)graphs* $K_2 = \big(\{a,b\},\{\{a,b\}\}\big)$*, consisting of two vertices and one single edge containing these vertices and* $\mathcal{L}K_1 = \big(\{x\},\{\{x\}\}\big)$*, respectively. Then:*

$$V(K_2 \check{\times} \mathcal{L}K_1) = \{(a,x),(b,x)\} = V(K_2 \times \mathcal{L}K_1)$$

but

$$\mathscr{E}(K_2 \check{\times} \mathcal{L}K_1) = \big\{\{(a,x)\},\{(b,x)\}\big\} \neq \big\{\{(a,x),(b,x)\}\big\} = \mathscr{E}(K_2 \times \mathcal{L}K_1),$$

where $\check{\times}$ *denotes the direct product* $\check{\times}$ *of hypergraphs, and* $\times$ *denotes the (usual) direct graph product. Therefore* $K_2 \check{\times} \mathcal{L}K_1 \neq K_2 \times \mathcal{L}K_1$.

The latter example implies, that the direct product $\check{\times}$ does not coincide with the direct graph product in the class of graphs with loops. Furthermore, it turns out, that the direct product $\check{\times}$ has no unique prime factorization in general, as shown in the following example:

Example 2.2. *Let* E_3 *be the hypergraph consisting of three vertices and one edge containing these vertices. Then we have*

$$K_2 \check{\times} E_3 \cong K_2 \check{\times} E_3.$$

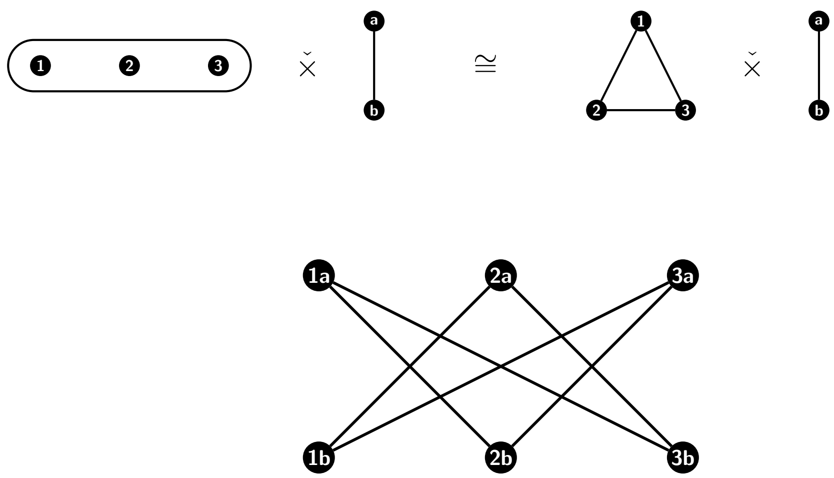


Figure 2.5: Two different pairs of hypergraphs whose direct product $\check{\times}$ is isomorphic

Furthermore, consider the projections of such a product hypergraph into its factors:

Example 2.3. *The direct product* $\check{\times}$ *as in Figure 2.5,* $H = E_3 \check{\times} K_2$*, has edge set*

$$\mathscr{E}(H) = \big\{\{(1,a),(2,b)\},\{(1,a),(3,b)\},\{(2,a),(1,b)\},\{(2,a),(3,b)\},\{(3,a),(1,b)\},\{(3,a),(2,b)\}\big\}.$$

The Factor E_3 *has edge set* $\mathscr{E}(E_3) = \big\{\{1,2,3\}\big\}$*. For the projection* p_{E_3} *into this factor holds:*

$$p_{E_3}(\{(1,a),(2,b)\}) = p_{E_3}(\{(2,a),(1,b)\}) = \{1,2\} \notin \mathscr{E}(E_3)$$
$$p_{E_3}(\{(1,a),(3,b)\}) = p_{E_3}(\{(3,a),(1,b)\}) = \{1,3\} \notin \mathscr{E}(E_3)$$
$$p_{E_3}(\{(2,a),(3,b)\}) = p_{E_3}(\{(3,a),(2,b)\}) = \{2,3\} \notin \mathscr{E}(E_3).$$

In general, we have

$$p_1(E_{i\pi}) = E_i \in \mathscr{E}_1 \qquad \text{and} \qquad p_2(E_{\pi' j}) = F_j \in \mathscr{E}_2$$

but

$$p_1(E_{\pi' j}) = \pi'(F_j)$$

and this need neither be a vertex nor an edge in H_1, as well as

$$p_2(E_{i\pi}) = \pi(E_i)$$

and this need neither be a vertex nor an edge in H_2, too.
So the projections indeed preserve adjacency, i.e., two vertices in a direct product $\check{\times}$ hypergraph are adjacent, whenever they are adjacent in both of the factors. However, the projections need not to be (weak) homomorphisms in general.
Thus, we not consider this product further.

Maximal Rank Preserving Direct Product

In the last paragraph we defined a product where the size of two multiplied edges is reduced to the size of the smaller one. This might be the reason for the non-uniqueness of the prime factor decomposition as seen in Example 2.2, and that the projections of a product into its factors need not to be weak homomorphisms.
Now we will define a direct product, such that an edge, we get by multiplying two edges, preserves the size of the bigger edge of the factors.
For this product most of the basic properties are shown analogously as for the direct product $\check{\times}$.

Let $H_1 = (V_1, \mathscr{E}_1)$ and $H_2 = (V_2, \mathscr{E}_2)$ be two hypergraphs. Then their *direct product* $\hat{\times}$ is defined on the Cartesian product of the vertex set, and a subset $E = \{(x_1,y_1),\dots,(x_r,y_r)\}$ of $V_1 \times V_2$ is an edge in $H_1 \hat{\times} H_2$ if and only if

(i) $\{x_1,\ldots,x_r\}$ is an edge in H_1 and there is an edge $F \in \mathscr{E}_2$ of H_2 such that $\{y_1,\ldots y_r\}$ is a family of elements of F, and $F \subseteq \{y_1,\ldots,y_r\}$, or

(ii) $\{y_1,\ldots,y_r\}$ is an edge in H_2 and there is an edge $E \in \mathscr{E}_1$ of H_1 such that $\{x_1,\ldots x_r\}$ is a family of elements of E, and $E \subseteq \{x_1,\ldots,x_r\}$.

More formal, we define the direct product $\hat{\times}$ of two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ and $H_2 = (V_2, \mathscr{E}_2)$ by the edge set

$$\begin{aligned}\mathscr{E}(H_1 \hat{\times} H_2) := \Big\{ \bigcup_{x\in E} \{(x,\varphi(x))\} \mid \varphi : E \to F \text{ surjective, } E \in \mathscr{E}_1, F \in \mathscr{E}_2 \Big\} \\ \cup \Big\{ \bigcup_{y\in F} \{(\varphi'(y),y)\} : \varphi' : F \to E \text{ surjective, } E \in \mathscr{E}_1, F \in \mathscr{E}_2 \Big\}.\end{aligned}$$

Figure 2.6 shows the direct product $\hat{\times}$ of the hypergraph E_3, which consists of one single edge of size 3 and the K_2, consisting of one single edge with two vertices. Another example can be seen in Figure 2.3 at the beginning of this section.

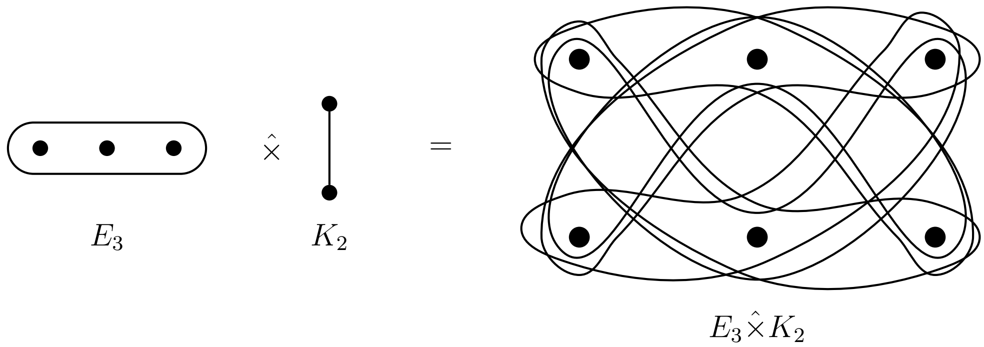


Figure 2.6: Direct product $\hat{\times}$, $E_3 \hat{\times} K_2$

Let an edge $\{(x_1,y_1),\ldots,(x_r,y_r)\}$ in $H = H_1 \hat{\times} H_2$ be denoted by

(i) $E_{i\varphi}$ if $\{x_1,\ldots,x_r\} = E_i \in \mathscr{E}_1$ and $\mathscr{E}_2 \ni F_j \subseteq \{y_1,\ldots,y_r\} \subseteq F_j^r$, where $\varphi : E_i \to F_j$ is the surjective mapping defined by $\varphi(x_k) = y_k$ for all $k \in \{1,\ldots,r\}$, and

(ii) $E_{\varphi' j}$ if $\{y_1,\ldots,y_r\} = F_j \in \mathscr{E}_2$ and $\mathscr{E}_1 \ni E_i \subseteq \{x_1,\ldots,x_r\} \subseteq E_i^r$, where $\varphi' : F_j \to E_i$ is the surjective mapping defined by $\varphi'(y_k) = x_k$ for all $k \in \{1,\ldots,r\}$,

where X^r denotes a family of elements of a set X, that contains each element of X with multiplicity r. Notice, if there exists no edge of the form $E_{i\varphi}$, all edges of H are of the form $E_{\varphi' j}$ and vice versa. With this notations, for the projections p_1 and p_2 of a product hypergraph $H = H_1 \hat{\times} H_2$ into its factors H_1 and H_2, respectively, holds:

$$p_1(E_{i\varphi}) = E_i \in \mathscr{E}(H_1) \quad \text{and} \quad p_2(E_{i\varphi}) = \varphi(E_i) = F_j \in \mathscr{E}(H_2),$$

since $\varphi : E_i \to F_j$ is surjective, and

$$p_2(E_{\varphi' j}) = F_j \in \mathscr{E}(H_2) \quad \text{and} \quad p_1(E_{\varphi' j}) = \varphi'(F_j) = E_i \in \mathscr{E}(H_1),$$

since $\varphi' : F_j \to E_i$ is surjective. Hence, p_1 and p_2 are homomorphisms.
We can state:
For two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ and $H_2 = (V_2, \mathscr{E}_2)$ a subset $E \subseteq V_1 \times V_2$ of the Cartesian product of their vertex sets is an edge in their direct product $\hat{\times}$, $H = H_1 \hat{\times} H_2$ if and only if

(i) $p_1(E)$ is an edge in H_1 and

(ii) $p_2(E)$ is an edge in H_2 and

(iii) $|E| = \max\{|p_1(E)|, |p_2(E)|\}$.

If we restrict this product to r-uniform hypergraphs, we get the product defined by Equation (2.3), i.e,

$$\mathscr{E}(H_1 \hat{\times} H_2) := \big\{\{(x_1,y_1),\ldots,(x_r,y_r)\} \mid \{x_1,\ldots,x_r\} \in \mathscr{E}(H_1),\ \{y_1,\ldots,y_r\} \in \mathscr{E}(H_2)\big\}$$

for r-uniform hypergraphs H_1 and H_2. Thus, this product coincides with the direct graph product in the class of simple graphs.
We have:

$$r(H_1 \hat{\times} H_2) = \max\{r(H_1), r(H_2)\}$$

and

$$s(H_1 \hat{\times} H_2) = \max\{s(H_1), s(H_2)\}$$

where $r(H)$ and $s(H)$ denote the rank and the anti-rank, respectively of a hypergraph H.
From analogous considerations as for the direct product $\check{\times}$, we can state:

Lemma 2.10. *The direct product $\hat{\times}$ is associative, commutative and distributive with respect to the disjoint union.*

The one vertex hypergraph with a loop, $\mathcal{L}K_1$ is a unit for the direct product $\hat{\times}$ in the class of hypergraphs with loops. In the class of simple hypergraphs, this product has no unit. The direct product $\hat{\times}$ of two connected hypergraphs need not to be connected, since it need not to be connected in the class of graphs. It remains to show the following property:

Lemma 2.11. *The direct product $\hat{\times}$, $H = \hat{\times}_{i=1}^{n} H_i$ of simple hypergraphs H_i is simple*

Proof. Because of associativity and commutativity of the direct product $\hat{\times}$, it suffices to prove the assertion for $n = 2$. Therefore, let $H_1 = (V_1, \mathscr{E}_1)$ and $H_2 = (V_2, \mathscr{E}_2)$ be two simple hypergraphs and suppose $H = H_1 \hat{\times} H_2$ is not simple. Then several cases can occur.
Suppose H contains at least one loop $\{(x,y)\}$. Then, it follows, $\{x\}$ is an edge in H_1, i.e., a loop and $\{y\}$ is a loop in H_2, contradicting the fact, that H_1 and H_2 are simple. Thus, $|E| \geq 2$ for all $E \in \mathscr{E}(H)$.
Now assume there is an edge $E_{i_1\varphi_1}$ that is contained in an edge $E_{i_2\varphi_2}$, where $E_{i_1\varphi_1} = \{(x_1,y_1),\ldots,(x_r,y_r)\}$, such that $\{x_1,\ldots,x_r\} = E_{i_1} \in \mathscr{E}_1$ and $\varphi_1 : E_{i_1} \to F_{j_1}$ through $x_k \mapsto y_k$, for all $k \in \{1,\ldots,r\}$ and $E_{i_2\varphi_2} = \{(x'_1,y'_1),\ldots,(x'_s,y'_s)\}$, $s \geq r$ such that $\{x'_1,\ldots,x'_s\} = E_{i_2} \in \mathscr{E}_1$ and $\varphi_2 : E_{i_2} \to F_{j_2}$ through $x'_k \mapsto y'_k$, for all $k \in \{1,\ldots,s\}$. If there is no edge of the form $E_{i\varphi}$ contained in H, we consider the hypergraph $H_2 \hat{\times} H_1$. Then it immediately follows

$$E_{i_1} = \{x_1,\ldots,x_r\} \subseteq \{x'_1,\ldots,x'_s\} = E_{i_2} \tag{2.4}$$

$$\{y_1,\ldots,y_r\} \subseteq \{y'_1,\ldots,y'_s\} \tag{2.5}$$

Equation (2.4) implies $i_1 = i_2$, since H_1 is simple. On the other hand $\varphi_1(x) = \varphi_2(x)$ must hold for all $x \in E_{i_1} = E_{i_2}$, hence $\varphi_1 = \varphi_2$. Thus, we can conclude that $E_{i_1\varphi_1} \subseteq E_{i_2\varphi_2}$ implies $(i_1\varphi_1) = (i_2\varphi_2)$.
The fact that $E_{\varphi'_1 j_1} \subseteq E_{\varphi'_2 j_2}$ implies $(\varphi'_1 j_1) = (\varphi'_2 j_2)$ is shown analogously.
Now suppose we have $E_{i\varphi} \subseteq E_{\varphi' j}$, where $E_{i\varphi} = \{(x_1,y_1),\ldots,(x_r,y_r)\}$, such that $\{x_1,\ldots,x_r\} = E_i \in \mathscr{E}_1$ and $\varphi : E_i \to F_m$, $F_m \in \mathscr{E}_2$, through $x_k \mapsto y_k$, for all $k \in \{1,\ldots,r\}$, respectively $E_{\varphi' j} = \{(x'_1,y'_1),\ldots,(x'_s,y'_s)\}$, $s \geq r$ such that $\{y'_s,\ldots,y'_s\} = F_j \in \mathscr{E}_2$ and $\varphi' : F_j \to E_l$, $E_l \in \mathscr{E}_1$, through $y'_k \mapsto x'_k$, for all $k \in \{1,\ldots,s\}$. Notice that $F_m \subseteq \{y_1,\ldots,y_r\} \subseteq F_m^r$ and $E_l \subseteq \{x'_1,\ldots,x'_s\} \subseteq E_l^s$, respectively. It follows

$$E_i = \{x_1,\ldots,x_r\} \subseteq \{x'_1,\ldots,x'_s\} \subseteq E_l^s \tag{2.6}$$

$$F_m \subseteq \{y_1,\ldots,y_r\} \subseteq \{y'_1,\ldots,y'_s\} = F_j \tag{2.7}$$

Equation (2.7) implies $m = j$, since H_2 is simple. In particular holds $F_m = \{y_1,\ldots,y_r\}$, hence $y_k \neq y_{k'}$, and therefore $\varphi(x_k) \neq \varphi(x'_k)$ for all $k \neq k'$ and as $x_k \neq x'_k$ for all $k \neq k'$ we can conclude

that φ is injective, thus bijective. We can now define $\varphi^{-1} : F_m = F_j \to E_i$ through $\varphi^{-1}(y_k) = x_k$ for all $k \in \{1,\dots,r\}$ and denote $E_{i\varphi}$ by $E_{\varphi^{-1}j}$. On the other hand $\varphi'(y) = \varphi^{-1}(y)$ must hold for all $y \in F_m = F_j$, hence $\varphi' = \varphi^{-1}$, and therefore $E_{i\varphi} \subseteq E_{\varphi' j}$ implies $(i\varphi) = (\varphi^{-1} j) = (\varphi' j)$.
The fact that $E_{\varphi' j} \subseteq E_{i\varphi}$ implies $(\varphi' j) = (i\varphi)$ is shown analogously as well.
Thus H is simple. □

As mentioned, the restriction of the direct product $\hat{\times}$ to simple graphs is the direct graph product. Moreover, the restriction of this product to graphs coincides with the direct graph product in general, also in the class of not necessarily simple graphs with loops.

Since this product is associative and commutative, the direct product $\hat{\times}$ of arbitrary many factors $H = \hat{\times}_{i=1}^{n} H_i$ is well defined, and we can state:
$E \subseteq V(H)$ is an edge in H if and only if

(1) $p_i(E) \in \mathscr{E}(H_i)$ for all $i \in \{1,\dots,n\}$ and

(2) $|E| = |p_j(E)|$ for a $j \in \{1,\dots,n\}$.

Notice, that item (2) implies $|E| = \max_j |p_j(E)|$.

2.2.2 A Direct Product that does not preserve Rank

For the sake of completeness, we want to introduce a hypergraph product here, whose restriction on 2-uniform hypergraphs coincides with the direct graph product, but which does not preserve r-uniformity in general.

For two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ and $H_2 = (V_2, \mathscr{E}_2)$, we define their *direct product* $\tilde{\times}$ by the edge set

$$\mathscr{E}(H_1 \tilde{\times} H_2) := \Big\{\{(x,y)\} \cup \big((E_i \setminus \{x\}) \times (F_j \setminus \{y\})\big) \mid (x,y) \in V(H) \text{ and } x \in E_i \in \mathscr{E}_1;\ y \in F_j \in \mathscr{E}_2\Big\}$$

Let an edge E in $H = H_1 \tilde{\times} H_2$ be denoted as

$$E_{(xy),(ij)} \text{ if } E = \{(x,y)\} \cup ((E_i \setminus \{x\}) \times (F_j \setminus \{y\})),$$

where $E_i \in \mathscr{E}(H_1)$ and $F_j \in \mathscr{E}(H_2)$.

The direct product $\tilde{\times}$ is motivated by [31], where the authors introduced the concept of $\mathscr{N}$-*systems*. An $\mathscr{N}$-system $(X, \mathscr{N})$ consists of a nonempty finite set X and a system $\mathscr{N}$ that associates to each $x \in X$ a collection $\mathscr{N}(x) = \{N^1(x), \dots, N^{d(x)}(x)\}$ of subsets $N^i(x)$ of X that contain x. The collection $\mathscr{N}(x)$ is then called the *neighborhood of x*. The *direct product* $(X,\mathscr{N}) = (X_1,\mathscr{N}_1)\times(X_2,\mathscr{N}_2)$ of two $\mathscr{N}$-systems $(X_1,\mathscr{N}_1)$ and $(X_2,\mathscr{N}_2)$ is defined by

(1) $X = X_1 \times X_2$

(2) The neighborhoods $\mathscr{N}((x_1,x_2))$ are the sets $\{N' \times N'' \mid N' \in \mathscr{N}_1(x_1),\ N'' \in \mathscr{N}_2(x_2)\}$.

A (simple) hypergraph $H = (V,\mathscr{E})$ can be described as an $\mathscr{N}$-system $(V,\mathscr{N})$ if we set

$$\mathscr{N}(v) = \{E \in \mathscr{E} \mid v \in E\}$$

for all $v \in V$.
Conversely, we can construct a system $(V,\mathscr{N}^o)$ of *"open"* neighborhoods by

$$\mathscr{N}^o(v) = \{E \setminus \{v\} \mid v \in E \in \mathscr{E}\}.$$

We define the product of such systems analogously. The represented hypergraph of a product of such $\mathscr{N}^o$-systems is then the direct product $\tilde{\times}$ of the represented hypergraphs of the factors.

Remark 2.1. *In the definition of the direct product $\tilde{\times}$ the following case might occur: Consider two hypergraphs H_1 and H_2 such that there are edges $E_i \in \mathscr{E}(H_1)$ and $F_j \in \mathscr{E}(H_2)$, such that $E_i = \{x,a\}$ and $F_j = \{y,b\}$. For the product $H = H_1 \tilde{\times} H_2$ holds:*

$$\begin{aligned}\mathscr{E}(H) \ni E_{(xy),(ij)} =& \{(x,y)\} \cup \big((E_i \setminus \{x\}) \times (F_j \setminus \{y\})\big)\\ =& \{(a,b)\} \cup \big((E_i \setminus \{a\}) \times (F_j \setminus \{b\})\big) = E_{(ab),(ij)} \in \mathscr{E}(H)\end{aligned}$$

That is, we might get some multiple edges in the product hypergraph. In this case, we will consider those edges as one *single edge.*
Conversely, this does not imply that we consider two edges $\{(x,y)\} \cup \big((E_i \setminus \{x\}) \times (F_j \setminus \{y\})\big)$ and $\{(a,b)\} \cup \big((E_k \setminus \{a\}) \times (F_l \setminus \{b\})\big) \in \mathscr{E}(H)$, with $E_i = \{x,a\},\ E_k = \{x,a\} \in \mathscr{E}(H_1)$ and $F_j = \{y,b\},\ F_l = \{y,b\} \in \mathscr{E}(H_2)$, as one single edge if $i \neq k$ or $j \neq l$, i.e., in the case that H_1 or H_2 are non simple hypergraphs.

Figure 2.7 shows the direct product $\tilde{\times}$ of two hypergraphs, both consisting of a single edge with three vertices. As this product is horrible to visualize, two hypergraphs H_1 and H_2 are depicted, whose union of the edge sets are the edges of the product hypergraph H.

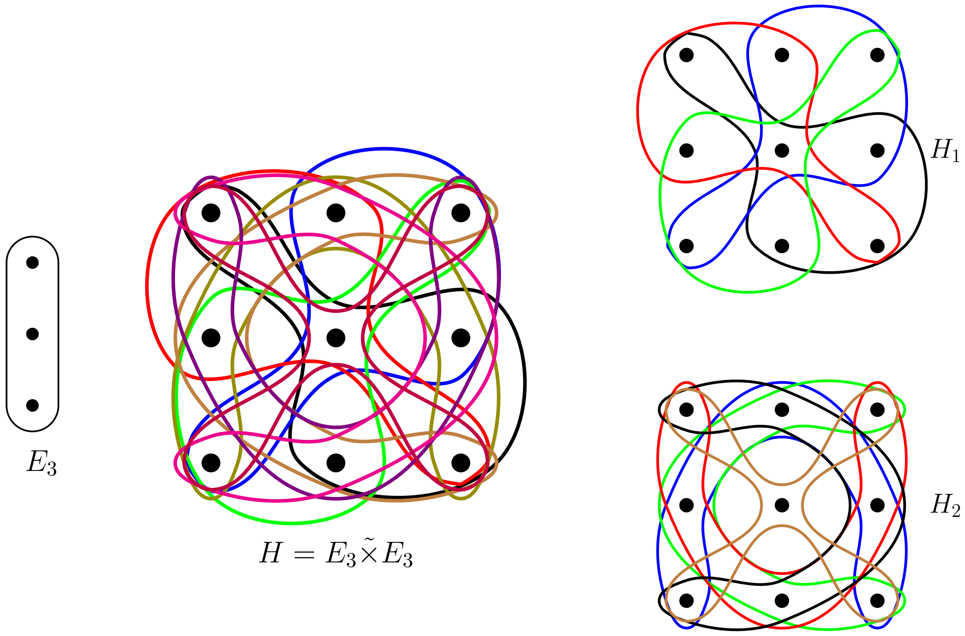


Figure 2.7: Direct product $\tilde{\times}$, $H = E_3 \tilde{\times} E_3$ with edge set $\mathscr{E}(H) = \mathscr{E}(H_1) \cup \mathscr{E}(H_2)$

For the projections p_1 and p_2 of a product hypergraph $H = H_1 \tilde{\times} H_2$ into its factors H_1 and H_2, respectively, holds:

$$p_1(E_{(xy),(ij)}) = \{p_1\big((x,y)\big)\} \cup p_1\big((E_i \setminus \{x\}) \times (F_j \setminus \{y\})\big) = \{x\} \cup (E_i \setminus \{x\}) = E_i \in \mathscr{E}(H_1),$$
$$p_2(E_{(xy),(ij)}) = \{p_2\big((x,y)\big)\} \cup p_2\big((E_i \setminus \{x\}) \times (F_j \setminus \{y\}\big) = \{y\} \cup (F_j \setminus \{y\}) = F_j \in \mathscr{E}(H_2).$$

Thus, they are homomorphisms.

If we restrict the definition of this product to 2-uniform hypergraphs, i.e. simple graphs, we have: $E \subseteq V(G_1 \tilde{\times} G_2)$ is an edge in $G_1 \tilde{\times} G_2$ iff $E = \{(x,y)(x',y')\}$ and $\{x,x'\}$ is an edge in G_1 and $\{y,y'\}$ is an edge in G_2. That is exactly the direct graph product.

For the direct product $\tilde{\times}$ we have:

$$r(H_1 \tilde{\times} H_2) = (r(H_1) - 1)(r(H_2) - 1) + 1$$

and

$$s(H_1 \tilde{\times} H_2) = (s(H_1) - 1)(s(H_2) - 1) + 1$$

where $r(H)$ and $s(H)$ denote the rank and the anti-rank, respectively of a hypergraph H. Thus the (anti-)rank of a product might not be the (anti-)rank of one of its factors in general.

Lemma 2.12. *The direct product $\tilde{\times}$ is associative.*

Proof. Let $H_1 = (V_1, \mathscr{E}_1), H_2 = (V_2, \mathscr{E}_2)$ and $H_3 = (V_3, \mathscr{E}_3)$ be hypergraphs. As before we need to show that the mapping $\psi : V((H_1 \tilde{\times} H_2) \tilde{\times} H_3) \to V(H_1 \tilde{\times} (H_2 \tilde{\times} H_3))$ defined by $((x,y)z) \mapsto (x,(y,z))$ is an isomorphism. There is nothing to show about bijectivity. It remains to show that for a subset $E \subseteq V((H_1 \tilde{\times} H_2) \tilde{\times} H_3)$, E is an edge in $(H_1 \tilde{\times} H_2) \tilde{\times} H_3$, if and only if ψE is an edge in $H_1 \tilde{\times} (H_2 \tilde{\times} H_3)$. First, we will examine, how an edge in $(H_1 \tilde{\times} H_2) \tilde{\times} H_3$ and $H_1 \tilde{\times} (H_2 \tilde{\times} H_3)$, respectively, look like. Recall that an edge E' of $H_1 \tilde{\times} H_2$ must be of the form $E' = \{(x,y)\} \cup ((D_i \setminus \{x\}) \times (E_j \setminus \{y\}))$ where $x \in D_i \in \mathscr{E}_1$ and $y \in E_j \in \mathscr{E}_2$. Thus, we have $E \subseteq V((H_1 \tilde{\times} H_2) \tilde{\times} H_3)$ is an edge in $(H_1 \tilde{\times} H_2) \tilde{\times} H_3$, iff

$$\begin{aligned} E &= \{(w,z)\} \cup \big[(E' \setminus \{w\}) \times (F_k \setminus \{z\})\big] \\ &= \{\big((x,y),z\big)\} \cup \Big[\big[\big(\{(x,y)\} \cup ((D_i \setminus \{x\}) \times (F_j \setminus \{y\}))\big) \setminus \{(x,y)\}\big] \times \big[F_k \setminus \{z\}\big]\Big] \\ &= \{\big((x,y),z\big)\} \cup \Big[\big[(D_i \setminus \{x\}) \times (E_j \setminus \{y\})\big] \times \big[F_k \setminus \{z\}\big]\Big], \end{aligned} \tag{2.8}$$

where $w = (x,y) \in V_1 \times V_2$ and $z \in V_3$, and D_i, E_j and F_k are edges in $\mathscr{E}_1, \mathscr{E}_2$ and $\mathscr{E}_3$, respectively, such that $x \in D_i$, $y \in E_j$ and $z \in F_k$.
On the other hand we have $F \subseteq V(H_1 \tilde{\times} (H_2 \tilde{\times} H_3))$ is an edge in $H_1 \tilde{\times} (H_2 \tilde{\times} H_3)$, iff

$$\begin{aligned} F &= \{(x,v)\} \cup \big[(D_i \setminus \{x\}) \times (F' \setminus \{v\})\big] \\ &= \{\big(x,(y,z)\big)\} \cup \Big[\big[D_i \setminus \{x\}\big] \times \big[\big(\{(y,z)\} \cup ((E_j \setminus \{y\}) \times (F_k \setminus \{z\}))\big) \setminus \{(y,z)\}\big]\Big] \\ &= \{(x,(y,z))\} \cup \Big[\big[D_i \setminus \{x\}\big] \times \big[(E_j \setminus \{y\}) \times (F_k \setminus \{z\})\big]\Big], \end{aligned} \tag{2.9}$$

where $F' = \{(y,z)\} \cup ((E_j \setminus \{y\}) \times (F_k \setminus \{z\}))$ is an edge in $H_2 \tilde{\times} H_3$, $v = (y,z) \in V_2 \times V_3$ and D_i, E_j and F_k are as in Equation (2.8).
We observe, that whenever $E \in \mathscr{E}((H_1 \tilde{\times} H_2) \tilde{\times} H_3)$, then it has the form as in Equation (2.8) and for its image $\psi(E)$ we have $\psi(E) = F \in \mathscr{E}(H_1 \tilde{\times} (H_2 \tilde{\times} H_3))$.
Conversely, an edge F of $H_1 \tilde{\times} (H_2 \tilde{\times} H_3)$ must have the form as in Equation (2.9), and if we set $F := \psi(E)$, its preimage E is an edge in $(H_1 \tilde{\times} H_2) \tilde{\times} H_3$.
Therefore the mapping ψ is an isomorphism and the assertion is true. □

Lemma 2.13. *The direct product $\tilde{\times}$ is commutative.*

Proof. Let $H_1 = (V_1, \mathscr{E}_1)$ and $H_2 = (V_2, \mathscr{E}_2)$ be two hypergraphs. We need to show, the mapping $\psi : V(H_1 \tilde{\times} H_2) \to V(H_2 \tilde{\times} H_1)$ defined by $(x,y) \mapsto (y,x)$ for $x \in V_1$, $y \in V_2$ is an isomorphism. For a subset E of $V(H_1 \tilde{\times} H_2) = V_1 \times V_2$, E is an edge in $H_1 \times H_2$ if and only if $E = \{(x,y)\} \cup ((E_i \setminus \{x\}) \times (F_j \setminus \{y\}))$, for $x \in E_i \in \mathscr{E}_1$ and $y \in F_j \in \mathscr{E}_2$.
This is equivalent to $\psi(E) = \{(y,x)\} \cup ((F_j \setminus \{y\}) \times (E_i \setminus \{x\}))$. Hence, $E \in \mathscr{E}(H_1 \tilde{\times} H_2)$ if and only if $\psi(E)$ is an edge in $H_2 \tilde{\times} H_1$. □

Lemma 2.14. *The direct product $\tilde{\times}$ is left and right distributive together with the disjoint union as addition.*

Proof. Let H_1, H_2 and H_3 be hypergraphs and furthermore, let H_2 and H_3 be vertex-disjoint. Then we have for the edge set of $H_1 \tilde{\times} (H_2 + H_3)$:

$$\begin{aligned}
\mathscr{E}(H_1 \tilde{\times} (H_2 + H_3)) =& \big\{\{(x,y)\} \cup \big[(E \setminus \{x\}) \times (F \setminus \{y\})\big] \mid x \in E,\ E \in \mathscr{E}_1;\ y \in F,\ F \in \mathscr{E}_2 \cup \mathscr{E}_3\big\} \\
=& \big\{\{(x,y)\} \cup \big[(E \setminus \{x\}) \times (F \setminus \{y\})\big] \mid x \in E,\ E \in \mathscr{E}_1;\ y \in F,\ F \in \mathscr{E}_2\big\} \\
& \cup \big\{\{(x,y)\} \cup \big[(E \setminus \{x\}) \times (F \setminus \{y\})\big] \mid x \in E,\ E \in \mathscr{E}_1;\ y \in F,\ F \in \mathscr{E}_3\big\} \\
=& \mathscr{E}(H_1 \tilde{\times} H_2) \cup \mathscr{E}(H_1 \tilde{\times} H_3) \\
=& \mathscr{E}((H_1 \tilde{\times} H_2) + (H_1 \tilde{\times} H_3)). \qquad (2.10)
\end{aligned}$$

With the same arguments as in Equation (2.1) we can conclude $V(H_1 \tilde{\times} (H_2 + H_3)) = V((H_1 \tilde{\times} H_2) + (H_1 \tilde{\times} H_3))$ and hence, it follows

$$H_1 \tilde{\times} (H_2 + H_3) = (H_1 \tilde{\times} H_2) + (H_1 \tilde{\times} H_3).$$

Analogously, it is shown

$$(H_1 + H_2) \tilde{\times} H_3 = (H_1 \tilde{\times} H_3) + (H_2 \tilde{\times} H_3)$$

for vertex-disjoint H_1 and H_2. □

Next, we will examine on which conditions the direct product $\tilde{\times}$ is a simple hypergraph.

Lemma 2.15. *The direct product $\tilde{\times}$, $H = \tilde{\times}_{i=1}^{n} H_i$, of simple hypergraphs H_i is simple.*

Proof. Because of associativity and commutativity of the direct product $\tilde{\times}$, it suffices to prove the assertion for $n = 2$. Therefore, let $H_1 = (V_1, \mathscr{E}_1)$ and $H_2 = (V_2, \mathscr{E}_2)$ be two simple hypergraphs and suppose $H = H_1 \tilde{\times} H_2$ is not simple.
Assume first H contains at least one loop $\{(x,y)\}$. Then, $\{(x,y)\} = \{(x',y')\} \cup \left[(E \setminus \{x'\}) \times (F \setminus \{y'\})\right]$ holds for an $x' \in E \in \mathscr{E}_1$ and an $y' \in F \in \mathscr{E}_2$. It follows $x = x'$ and $y = y'$ and $(E \setminus \{x'\}) \times (F \setminus \{y'\}) = \emptyset$. Hence, $E \setminus \{x'\} = \emptyset$ or $F \setminus \{y'\} = \emptyset$ must hold. We conclude that $E = \{x\}$ is an edge in H_1, i.e., a loop, or $F = \{y\}$ is a loop in H_2. Both contradicts the fact, that H_1 and H_2 are simple. Thus, $|E| \geq 2$ for all $E \in \mathscr{E}(H)$.
Now suppose $E_{(xy),(ij)} \subseteq E_{(x'y'),(i'j')}$, i.e.

$$\{(x,y)\} \cup \left[(E_i \setminus \{x\}) \times (F_j \setminus \{y\})\right] \subseteq \{(x',y')\} \cup \left[(E_{i'} \setminus \{x'\}) \times (F_{j'} \setminus \{y'\})\right] \tag{2.11}$$

Then it immediately follows

$$(x,y) \in \{(x',y')\} \cup \left[(E_{i'} \setminus \{x'\}) \times (F_{j'} \setminus \{y'\})\right] \subseteq E_{i'} \times F_{j'} \tag{2.12}$$

and hence

$$x \in E_{i'} \qquad \text{and} \qquad y \in F_{j'}. \tag{2.13}$$

and therefore we have

$$E_i \times F_j \subseteq E_{i'} \times F_{j'}, \tag{2.14}$$

thus

$$E_i \subseteq E_{i'} \qquad \text{and} \qquad F_j \subseteq F_{j'} \tag{2.15}$$

and since H_1 and H_2 are both simple, it follows

$$i = i' \qquad \text{as well as} \qquad j = j' \tag{2.16}$$

It remains to show that $(x,y) = (x',y')$. We will prove this indirect. Assume $(x,y) \neq (x',y')$. Now from (2.11) and (2.16) we have

$$\underbrace{(E_i \setminus \{x\}) \times (F_j \setminus \{y\})}_{=:X} \subseteq \underbrace{\{(x',y')\} \cup \left[(E_i \setminus \{x'\}) \times (F_j \setminus \{y'\})\right]}_{=:Y} \tag{2.17}$$

If we compute the intersection of these two sets we get

$$
\begin{aligned}
A :=& \big[(E_i \setminus \{x\}) \times (F_j \setminus \{y\})\big] \cap \Big[\{(x',y')\} \cup \big[(E_i \setminus \{x'\}) \times (F_j \setminus \{y'\})\big]\Big] \\
=& \underbrace{\Big[\big[(E_i \setminus \{x\}) \times (F_j \setminus \{y\})\big] \cap \{(x',y')\}\Big]}_{=:B} \\
& \cup \underbrace{\Big[\big[(E_i \setminus \{x\}) \times (F_j \setminus \{y\})\big] \cap \big[(E_i \setminus \{x'\}) \times (F_j \setminus \{y'\})\big]\Big]}_{=:C}
\end{aligned} \tag{2.18}
$$

For short, it holds

$$
X \cap Y = A = B \cup C \overset{(2.17)}{=} X \tag{2.19}
$$

Now we have to distinguish several cases: First suppose $x \neq x'$ and $y = y'$ Then it follows for the sets B and C in (2.18)

$$
\begin{aligned}
B &= \emptyset \\
C &= (E_i \setminus \{x,x'\}) \times (F_j \setminus \{y\})
\end{aligned} \tag{2.20}
$$

But Equation (2.19) implies

$$
(E_i \setminus \{x\}) \times (F_j \setminus \{y\}) = (E_i \setminus \{x,x'\}) \times (F_j \setminus \{y\}) \tag{2.21}
$$

and hence in particular

$$
E_i \setminus \{x\} = E_i \setminus \{x,x'\} \tag{2.22}
$$

which is a contradiction, so the case $x' \neq x$ and $y = y'$ cannot occur. The case $x = x'$ and $y \neq y'$ leads analogously to a contradiction.

Now suppose $x \neq x'$ and $y \neq y'$. We then have for B and C in (2.18):

$$
\begin{aligned}
B &= \{(x',y')\} \\
C &= (E_i \setminus \{x,x'\}) \times (F_j \setminus \{y,y'\})
\end{aligned} \tag{2.23}
$$

And again from (2.19)

$$
(E_i \setminus \{x\}) \times (F_j \setminus \{y\}) = \{(x',y')\} \cup \big[(E_i \setminus \{x,x'\}) \times (F_j \setminus \{y,y'\})\big] \tag{2.24}
$$

must hold. But this can only be fulfilled if $E_i = \{x,x'\}$ and $F_j = \{y,y'\}$.

Together with Remark 2.1 we can conclude that H is simple. □

The direct product $\tilde{\times}$ of arbitrary many hypergraphs $H = \tilde{\times}_{i=1}^{n} H_i$ is well defined, since it is associative and commutative. Also the projections of a hypergraph product of arbitrary many hypergraphs into its factors are homomorphisms. However, the direct product $\tilde{\times}$ does not have a unit, neither in the class of simple hypergraphs, since the direct graph product has no unit in the class of simple graphs, nor in the class of non simple hypergraphs.

Example 2.4. *Consider the (hyper)graphs $K_2 = \big(\{a,b\}, \big\{\{a,b\}\big\}\big)$, consisting of two vertices and one single edge containing these vertices and $\mathcal{L}K_1 = \big(\{x\}, \big\{\{x\}\big\}\big)$, respectively. Then:*

$$V(K_2 \tilde{\times} \mathcal{L}K_1) = \{(a,x),(b,x)\} = V(K_2 \times \mathcal{L}K_1)$$

but

$$\mathscr{E}(K_2 \tilde{\times} \mathcal{L}K_1) = \big\{\{(a,x)\},\{(b,x)\}\big\} \neq \big\{\{(a,x),(b,x)\}\big\} = \mathscr{E}(K_2 \times \mathcal{L}K_1),$$

where $\tilde{\times}$ denotes the direct product $\tilde{\times}$ of hypergraphs, and $\times$ denotes the (usual) direct graph product. Thus, $K_2 \tilde{\times} \mathcal{L}K_1 \neq K_2 \times \mathcal{L}K_1$.

Hence, $\mathcal{L}K_1$ is not a unit with respect to the direct product $\tilde{\times}$, and the direct product $\tilde{\times}$ does not coincide with the direct graph product in the class of graphs with loops.
For this reasons we will not consider this product further.

Summary

Three hypergraph products were defined in this section. The direct product $\check{\times}$, the direct product $\hat{\times}$ and the direct product $\tilde{\times}$. For all these products, the following results hold:

1. The restriction of these products on simple graphs is the direct graph product.
2. Associativity.
3. Commutativity.
4. Distributivity with respect to the disjoint union.
5. The product of two simple hypergraphs is simple.

Furthermore, for the direct product $\hat{\times}$ and the direct product $\tilde{\times}$ holds:

6. The projections of a product hypergraph onto its factors are homomorphisms.

And in addition, the direct product $\hat{\times}$ coincides with the direct graph product in the class of non simple graphs and graphs with loops as well.

Remark 2.2. *In the following, we will refer to the direct product* $\hat{\times}$ *as* direct product*, denoted by* $\times$.

As an open problem, it remains to examine if the direct product has a unique prime factorization. But that would go beyond the scope of this thesis.

2.3 The Strong Product

In graph theory, the edge set of a strong product of two graphs is the union of the edge sets of their Cartesian and direct product. For hypergraphs, we will proceed in the same way.
With the direct products of the last section, there are three possible definitions of a strong hypergraph product, whose restrictions to simple graphs result in the strong graph product:
Let the edge set of a strong product $H = H_1 \overset{*}{\boxtimes} H_2$, $* = \vee, \wedge, \sim$ of two hypergraphs H_1, H_2 be

$$\mathscr{E}(H_1 \overset{*}{\boxtimes} H_2) = \mathscr{E}(H_1 \Box H_2) \cup \mathscr{E}(H_1 \overset{*}{\times} H_2),$$

where $\mathscr{E}(H_1 \overset{*}{\times} H_2)$ corresponds to the respective direct product from Section 2.2.
For the same reasons as in Section 2.2, the respective strong products resulting from the direct product $\tilde{\times}$ and the direct product $\check{\times}$ do not coincide with the strong graph product if we admit loops. Therefore, we will only consider the strong product which belongs to the direct product, i.e., the direct product $\hat{\times}$.
We define the *strong product* $H = H_1 \boxtimes H_2$ of two hypergraphs $H_1 = (V_1, \mathscr{E}_1)$ and $H_2 = (V_2, \mathscr{E}_2)$ by the edge set

$$\mathscr{E}(H_1 \boxtimes H_2) = \mathscr{E}(H_1 \Box H_2) \cup \mathscr{E}(H_1 \times H_2), \tag{2.25}$$

In other words, a subset $E = \{(x_1, y_1), \ldots, (x_r, y_r)\}$ of the vertex set $V(H) = V_1 \times V_2$ is an edge in $H = H_1 \boxtimes H_2$ if and only if

(i) $\{x_1, \ldots, x_r\}$ is an edge in $\mathscr{E}(H_1)$ and $y_1 = \ldots = y_r$, or

(ii) $\{y_1, \ldots, y_r\}$ is an edge in $\mathscr{E}(H_2)$ and $x_1 = \ldots = x_r$, or

(iii) $\{x_1, \ldots, x_r\}$ is an edge in H_1 and there is an edge $F \in \mathscr{E}_2$ of H_2 such that $\{y_1, \ldots y_r\}$ is a family of elements of F, and $F \subseteq \{y_1, \ldots, y_r\}$, or

(iv) $\{y_1,\dots,y_r\}$ is an edge in H_2 and there is an edge $F \in \mathscr{E}_1$ of H_1 such that $\{x_1,\dots x_r\}$ is a family of elements of E, and $E \subseteq \{x_1,\dots,x_r\}$.

The projections p_1 and p_2 of a strong product $H = H_1 \boxtimes H_2$ of two hypergraphs into its factors are weak homomorphisms, since edges included in the second term of the right hand side of Equation (2.25) are mapped into edges, and those included in the first term are mapped into edges or vertices.
We have for an edge $E \in \mathscr{E}(H)$:

- E is *Cartesian* if and only if

  $|p_1(E)| = 1$ and $p_2(E) \in \mathscr{E}(H_2)$, or

  $p_1(E) \in \mathscr{E}(H_1)$ and $|p_2(E)| = 1$ and

- E is *non-Cartesian* if and only if

  $p_1(E) \in \mathscr{E}(H_1)$, $p_2(E) \in \mathscr{E}(H_2)$ and $|p_i(E)| = |E|$ for an $i \in \{1,2\}$.

Let a Cartesian edge $E = \{(x_1,y_1),\dots,(x_r,y_r)\}$ in $H = H_1 \boxtimes H_2$ be denoted by

(i) E_{iy} if $E = E_i \times \{y\}$, where $\{x_1,\dots,x_r\} = E_i \in \mathscr{E}(H_1)$ and $y_1 = \dots = y_r = y \in V(H_2)$, and

(ii) E_{xj} if $E = \{x\} \times E_j$, where $x_1 = \dots = x_r = x \in V(H_1)$ and $\{y_1,\dots,y_r\} = E_j \in \mathscr{E}(H_2)$

and a non-Cartesian edge by

(iii) $E_{i\varphi}$ if $\{x_1,\dots,x_r\} = E_i \in \mathscr{E}_1$ and $\mathscr{E}_2 \ni F_j \subseteq \{y_1,\dots,y_r\} \subseteq F_j^r$, where $\varphi : E_i \to F_j$ is the surjective mapping defined by $\varphi(x_k) = y_k$ for all $k \in \{1,\dots,r\}$, and

(iv) $E_{\varphi' j}$ if $\{y_1,\dots,y_r\} = F_j \in \mathscr{E}_2$ and $\mathscr{E}_1 \ni E_i \subseteq \{x_1,\dots,x_r\} \subseteq E_i^r$, where $\varphi' : F_j \to E_i$ is the surjective mapping defined by $\varphi'(y_k) = x_k$ for all $k \in \{1,\dots,r\}$.

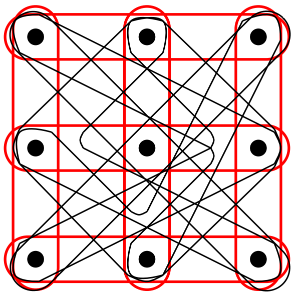
If H_i, $i \in \{1,2\}$ is not prime, i.e., $H_i = H_i' \boxtimes H_i''$, then an edge $E \in \mathscr{E}(H)$ is Cartesian in $(H_1' \boxtimes H_1'') \boxtimes H_2$ iff (i) holds or (ii) is fulfilled and $p_1(E)$ is Cartesian in $H_1' \boxtimes H_1''$. Similar E is Cartesian in $H_1 \boxtimes (H_2' \boxtimes H_2'')$ iff (i) is fulfilled and $p_2(E)$ is Cartesian in $H_2' \boxtimes H_2''$ or (ii) holds. Otherwise, if the respective condition would be fulfilled, but $p_i(E)$ is non-Cartesian in $H_i' \boxtimes H_i''$, then E is non-Cartesian in $(H_1' \boxtimes H_1'') \boxtimes H_2$ or $H_1 \boxtimes (H_2' \boxtimes H_2'')$, respectively.

In graph theory, the strong product can be seen as a special case of the direct one. There is no reason not to claim this for strong hypergraph products.

Let $\mathcal{N}H$ denote the partial hypergraph of H with edges $\mathscr{E}(\mathcal{N}H) = \{E \in \mathscr{E}(H) \mid |E| \geq 2\}$. On the other hand let $\mathcal{L}H'$ denote the hypergraph, which arises from H' by assigning a loop to each vertex of H'. For a hypergraph $H = (V, \mathscr{E})$ without loops we have $\mathscr{E}(\mathcal{L}H) = \mathscr{E} \cup \big\{\{v\} \mid v \in V\big\}$. Then for the strong and the direct product holds

$$H_1 \boxtimes H_2 = \mathcal{N}(\mathcal{L}H_1 \times \mathcal{L}H_2) \tag{2.26}$$

for hypergraphs H_1, H_2 without loops.



$E_3 \boxtimes E_3$

Figure 2.8: Strong product $E_3 \boxtimes E_3$, red Cartesian edges

Figure 2.8 shows the strong product of the hypergraph E_3, which consists of a single edge with three vertices, multiplied with itself. The Cartesian edges of this product are highlighted red.

Lemma 2.16. *The strong product is associative*

Proof. Let $H_1 = (V_1, \mathscr{E}_1), H_2 = (V_2, \mathscr{E}_2)$ and $H_3 = (V_3, \mathscr{E}_3)$ be hypergraphs. Again we have to show, that the bijective mapping $\psi : V((H_1 \boxtimes H_2) \boxtimes H_3) \to V(H_1 \boxtimes (H_2 \boxtimes H_3))$, defined by $((x,y),z) \mapsto (x,(y,z))$ is an isomorphism.

In the following, let $p_{H_i \boxtimes H_{i+1}}$ denote the projection from $(H_1 \boxtimes H_2) \boxtimes H_3$ into $H_1 \boxtimes H_2$ for $i = 1$ and from $H_1 \boxtimes (H_2 \boxtimes H_3)$ into $H_2 \boxtimes H_3$ for $i = 2$, respectively, whereas p_j denotes the projection into H_j for all $j \in \{1,2,3\}$.

Consider first a Cartesian edge E of $(H_1 \boxtimes H_2) \boxtimes H_3$. One observe that for E one of the following cases must be fulfilled

(i) $p_1(E) \in \mathscr{E}_1$, $|p_1(E)| = |E|$ and $|p_2(E)| = |p_3(E)| = 1$, or

(ii) $p_2(E) \in \mathscr{E}_2$, $|p_2(E)| = |E|$ and $|p_1(E)| = |p_3(E)| = 1$, or

(iii) $p_3(E) \in \mathscr{E}_3$, $|p_3(E)| = |E|$ and $|p_1(E)| = |p_2(E)| = 1$

But since $p_i(\psi(E)) = p_i(E)$ for all $E \subseteq V((H_1 \boxtimes H_2) \boxtimes H_3)$ and all $i \in \{1,2,3\}$, as one can easily see, this is equivalent to $\psi(E)$ being a Cartesian edge in $H_1 \boxtimes (H_2 \boxtimes H_3)$.
Now let E be non-Cartesian. Then one of the following must hold:

(i) $p_{H_1 \boxtimes H_2}(E) \in \mathscr{E}(H_1 \boxtimes H_2)$ and non-Cartesian, $|p_{H_1 \boxtimes H_2}(E)| = |E|$ and $|p_3(E)| = 1$, or

(ii) $p_{H_1 \boxtimes H_2}(E) \in \mathscr{E}(H_1 \boxtimes H_2)$, $p_3(E) \in \mathscr{E}_3$ and $|p_{H_1 \boxtimes H_2}(E)| = |E|$ or $|p_3(E)| = |E|$.

Then (i) is equivalent to $p_1(E) \in \mathscr{E}_1$, $p_2(E) \in \mathscr{E}_2$, $|p_i(E)| = |E|$ for an $i \in \{1,2\}$ and $|p_3(E) = 1|$. Condition (ii) is equivalent to the following:

(iia) $p_i(E) \in \mathscr{E}_i$ for $i = 1,2,3$ and $|p_j(E)| = |E|$ for a $j \in \{1,2,3\}$, or

(iib) $p_1(E) \in \mathscr{E}_1$, $|p_2(E)| = 1$, $p_3(E) \in \mathscr{E}_3$ $|p_1(E)| = |E|$ or $|p_3(E)| = |E|$, or

(iic) $|p_1(E)| = 1$, $p_2(E) \in \mathscr{E}_2$, $p_3(E) \in \mathscr{E}_3$ $|p_2(E)| = |E|$ or $|p_3(E)| = |E|$.

And again, since $p_i(\psi(E)) = p_i(E)$ for all $E \subseteq V((H_1 \boxtimes H_2) \boxtimes H_3)$ and $i \in \{1,2,3\}$, it follows (iic) is equivalent to

(i') $|p_1(\psi(E)) = 1|$, $p_{H_2 \boxtimes H_3}(\psi(E)) \in \mathscr{E}(H_2 \boxtimes H_3)$ and non-Cartesian and $|p_{H_2 \boxtimes H_3}(\psi(E))| = |\psi(E)|$,

whereas conditions (i),(iia) and (iib) are equivalent to

(ii') $p_1(\psi(E)) \in \mathscr{E}_1$, $p_{H_2 \boxtimes H_3}(\psi(E)) \in \mathscr{E}(H_2 \boxtimes H_3)$ and $|p_1(\psi(E))| = |\psi(E)|$ or $|p_{H_2 \boxtimes H_3}(\psi(E))| = |\psi(E)|$.

This is equivalent to $\psi(E)$ being a non-Cartesian edge in $H_1 \boxtimes (H_2 \boxtimes H_3)$.
Altogether we can state, E is an edge in $(H_1 \boxtimes H_2) \boxtimes H_3$ if and only if $\psi(E)$ is an edge in $H_1 \boxtimes (H_2 \boxtimes H_3)$. Hence $(H_1 \boxtimes H_2) \boxtimes H_3 \cong H_1 \boxtimes (H_2 \boxtimes H_3)$. □

Because of the symmetry of the definition of the strong product, we can state:

Lemma 2.17. *The strong product is commutative.*

The strong product has K_1 as unit. From distributivity of the Cartesian and the strong product, respectively, together with the disjoint union we can infer:

Lemma 2.18. *The strong product is left and right distributive together with the disjoint union as addition.*

Proof. Let $H_1 = (V_1, \mathscr{E}_1)$, $H_2 = (V_2, \mathscr{E}_2)$ and $H_3 = (V_3, \mathscr{E}_3)$ be hypergraphs and let H_2 and H_3 be vertex-disjoint. Then the following holds

$$\begin{aligned}\mathscr{E}(H_1 \boxtimes (H_2 + H_3)) &= \mathscr{E}(H_1 \Box (H_2 + H_3)) \cup \mathscr{E}(H_1 \times (H_2 + H_3)) \\ &= \mathscr{E}(H_1 \Box H_2) \cup \mathscr{E}(H_1 \Box H_3) \cup \mathscr{E}(H_1 \times H_2) \cup \mathscr{E}(H_1 \times H_3) \\ &= \mathscr{E}(H_1 \boxtimes H_2) \cup \mathscr{E}(H_1 \boxtimes H_3) = \mathscr{E}(H_1 \boxtimes H_2 + H_1 \boxtimes H_3).\end{aligned}$$

With the same arguments as in Equation (2.1) we can conclude $V(H_1 \boxtimes (H_2 + H_3)) = V((H_1 \boxtimes H_2) + (H_1 \boxtimes H_3))$ and hence, it follows $H_1 \boxtimes (H_2 + H_3) = (H_1 \boxtimes H_2) + (H_1 \boxtimes H_3)$.
Analogously we can conclude $(H_1 + H_2) \boxtimes H_3 = (H_1 + H_3) \boxtimes (H_2 + H_3)$ for vertex disjoint hypergraphs H_1 and H_2.

□

Lemma 2.19. *The strong product, $H = \boxtimes_{i=1}^{n} H_i$, of simple hypergraphs H_i is simple.*

Proof. Since the strong product is associative and commutative, it suffices to prove the assertion for $n = 2$. Therefore, let $H_1 = (V_1, \mathscr{E}_1)$ and $H_2 = (V_2, \mathscr{E}_2)$ be two simple hypergraphs and consider their strong product $H = H_1 \boxtimes H_2$. Due to the fact that the Cartesian product and the direct product of simple hypergraphs is simple, as shown in Lemma 2.5 and Lemma 2.11, respectively, it remains to show, that no Cartesian edge is contained in any non-Cartesian edge or vice versa.

We show first, that no Cartesian edge is contained in any non-Cartesian edge if the factors are simple.
Assume $E_{iy} = \bigcup_{x \in E_i} \{(x, y)\} \subseteq E_{j\varphi} = \bigcup_{x \in E_j} \{(x, \varphi(x))\}$, where φ is a surjective mapping $\varphi : E_j \to F_k$. It immediately follows $E_i \subseteq E_j$, which implies $i = j$, hence $|E_{iy}| = |E_{j\varphi}|$, and therefore we have $E_{iy} = E_{j\varphi}$ and thus $F_j = \{y\}$, which implies that H_2 is not simple.
Analogously, it is shown that the case $E_{xj} \subseteq E_{\varphi' i}$ cannot occur if H_1 and H_2 are simple.
Now suppose $E_{iy} = \{(x_1, y), \ldots, (x_r, y)\} \subseteq E_{\varphi' j} = \{(\varphi'(y_1), y_1), \ldots, (\varphi'(y_s), y_s)\}$ for $\mathscr{E}_1 \ni E_i = \{x_1, \ldots, x_r\}$ and $\mathscr{E}_2 \ni F_j = \{y_1, \ldots, y_s\}$, where φ' is a surjective mapping $\varphi' : F_j \to E_k$. This can only occur if $F_j = \{y\}$, hence H_2 would not be simple.
Analogously $E_{xj} \subseteq E_{i\varphi}$ can only be fulfilled if H_1 would not be simple.
Now suppose it holds $E \subseteq F$ where E is non-Cartesian and F is a Cartesian edge. Thus $|p_i(F)| = 1$ for an $i \in \{1, 2\}$ and therefore $|p_i(E)| = 1$, but $p_i(E)$ must be an edge in H_i. Hence, one of the factors would not be simple. □

Because of associativity and commutativity, the strong product $H = \boxtimes_{i=1}^{n} H_i$ of arbitrary many factors H_i is well defined. We can define this product in terms of projections as follows:
For $H = \boxtimes_{i=1}^{n} H_i$, $H_i = (V_i, \mathscr{E}_i)$ and $I = \{1,\dots,n\}$ we have

(1) $V(H) = V_1 \times V_2 \times \ldots \times V_n$

(2) $E \in \mathscr{E}(H), E \subseteq V(H)$ if and only if there is a nonempty index set $J \subseteq I$, s.t.

  (i) $p_j(E) \in E_j$ for all $j \in J$ and

  (ii) $|E| = |p_k(E)|$ for a $k \in J$ and

  (iii) $|p_i(E)| = 1$ for all $i \in I \setminus J$.

The vertices of Cartesian edges differ in exactly one coordinate, i.e. $|J| = 1$, the other edges, i.e. for which $|J| > 1$ are non-Cartesian.
The set of Cartesian edges of a strong product $H = \boxtimes_{i=1}^{n} H_i$ generates a partial hypergraph H' of H with $V(H') = V(H)$. This partial hypergraph is indeed the Cartesian product of the factors of H, $H' = \Box_{i=1}^{n} H_i$. We will call such a hypergraph H' the *Cartesian skeleton* of H. Therefore, it is clear, that the strong product is connected if and only if all of its factors are.
As for the Cartesian product each H_j-layer through an arbitrary vertex w of a strong product $H = \boxtimes_{i=1}^{n} H_i$ is isomorphic to the factor H_j,

$$\langle \{v \in V(H) \mid p_k(v) = w_k \text{ for } k \neq j\} \rangle \cong H_j.$$

Summary

To conclude this section, we summarize the preceding results. For the strong product holds:

1. The restriction of this product on graphs is the strong graph product.
2. Associativity.
3. Commutativity.
4. Distributivity with respect to the disjoint union.
5. The product of two simple hypergraphs is simple.
6. The projections of a product hypergraph onto its factors are weak homomorphisms.

As an open problem, it remains to examine if this product has a unique prime factorization. But that would go beyond the scope of this thesis.

3 Unique Prime Factorization with Respect to the Cartesian Product

In this chapter we are concerned with the decomposition, more exactly the prime factor decomposition of a given hypergraph H with respect to the Cartesian product.
Uniqueness of prime factorization of simple and connected hypergraphs was proved by W. Imrich in [23]. A factorization algorithm for a small class of hypergraphs, the *conformal*[1] *hypergraphs* is given by A. Bretto et al. in [8]. They showed that the prime factorization of a Cartesian product of conformal hypergraphs can be reduced to prime factorization of a Cartesian product graph, namely its 2-*section* and then used the factorization algorithm for Cartesian graph products given in [30]. Here we give an alternative, more constructive proof for uniqueness of prime factor decomposition of simple connected hypergraphs and provide a product relation according to the unique prime factorization of a given simple connected hypergraph. The proof is modeled after the proof of unique prime factor decomposition of a Cartesian product graph by W. Imrich and J. Žerovnik in [32].

First, we have to introduce some further notations.
A Hypergraph H is called *prime* with respect to a given product if it cannot be represented as the product of two nontrivial hypergraphs, i.e. for the Cartesian product, $H = H_1 \Box H_2$ implies $H_1 = K_1$ or $H_2 = K_1$. By a *prime factorization* of a hypergraph H is meant a representation of H as a Cartesian product hypergraph $H = \Box_{i=1}^{n} H_i$ such that the H_i are prime and $H_i \neq K_1$ for all $i \in \{1, \ldots, n\}$.
Let $H = A \Box B$ be a Cartesian product such that A and B are both nontrivial Hypergraphs and let $A = A_1 \Box A_2$ be a nontrivial representation of A. Then we call the product representation $A_1 \Box A_2 \Box B$ a *refinement* of $A \Box B$. Every sequence of refinements has to terminate as a product

[1] A hypergraph H is *conformal* if, for every $E \subseteq V(H)$, E is a maximal clique of $[H]_2$ iff E is a hyperedge of H. The 2-*section* $[H]_2$ of a hypergraph H is the graph whose vertices are the vertices of H, and where two vertices are adjacent iff they belong to a same hyperedge.

$H_1 \Box H_2 \Box \ldots \Box H_n$ of prime hypergraphs, since every factor of a nontrivial hypergraph product has fewer vertices than the product itself. Thus we can state the following proposition.

Proposition 3.1. *Every hypergraph H has a prime factorization with respect to the Cartesian product. The number of prime factors is at most* $\log_2 |V(H)|$.

The latter statement comes from the fact that every nontrivial hypergraph has at least two vertices.

By Proposition 3.1, every hypergraph H can be represented as a product $H = \Box_{i=1}^{n} H_i$ of prime factors H_i. If H itself is prime, we have $n = 1$ and $H = H_1$. Since graphs are a special class of hypergraphs, prime factorization of non-connected hypergraphs with respect to the Cartesian product is not unique (see Theorem 1.1).

Remark 3.1. *In the following we are concerned with simple connected hypergraphs.*

Let $H = \Box_{i=1}^{n} H_i$ be a Cartesian product of hypergraphs H_i. Recall that $E = \{v_1, \ldots, v_r\}$ is an edge in H if and only if there is an $j \in \{1 \ldots, n\}$, such that $p_j(E) = \{p_j(v_1), \ldots, p_j(v_r)\} \in \mathscr{E}(H_j)$ and $p_i(v_1) = \ldots = p_i(v_r)$ for all $i \neq j$.
A *product coloring* on the edge set of $H = \Box_{i=1}^{n} H_i$ is given by the mapping $c : \mathscr{E}(H) \to \{1, \ldots, n\}$ defined by $c(E) = j$ if the vertices of E differ in the j-th coordinate.
An equivalence relation γ on the edge set $\mathscr{E}(H)$ of a Cartesian product $H = \Box_{i=1}^{n} H_i$ of (not neccessairily prime) hypergraphs H_i is a *product relation* if E and F are in relation γ if and only if there exists an $j \in \{1, \ldots, n\}$, such that

$$|p_j(E)| > 1 \quad \text{and} \quad |p_j(F)| > 1,$$

for $E, F \in \mathscr{E}(H)$. It is clear, that $|p_i(E)| = |p_i(F)| = 1$ holds for all $i \neq j$. If the H_i are all prime, we denote this relation by σ. And in this case we have E and F are in relation σ if and only if $c(E) = c(F)$. Thus, each equivalence class of σ belongs to a prime factor of H. Moreover, let Σ_i, $i = 1, \ldots, n$ be the equivalence classes of σ. Every connected component of a partial hypergraph generated by the edges of an equivalence class Σ_i is isomorphic to H_i. Consider now the connected components of a partial hypergraph generated by the union of arbitrary equivalence classes of σ, $\bigcup_{j \in J} \Sigma_j$, $J \subseteq \{1, \ldots, n\}$. Each connected component of this partial hypergraph is then isomorphic to $H_J := \Box_{j \in J} H_j$.

Definition 3.1. *Let $E_1, \ldots, E_s$ and $F_1, \ldots, F_r$ be edges of a hypergraph H. We say they form an* $r \times s$-grid *if* *(i)* $|E_i \cap F_j| = 1$, *and*

(ii) $E_i \cap E_{i'} = F_j \cap F_{j'} = \emptyset$,

for all $i, i' \in \{1, \dots, s\}$, $j, j' \in \{1, \dots, r\}$, *with* $i \neq i'$, $j \neq j'$. *If there is an edge* $D \in \mathscr{E}(H)$, *such that*

(iii) $E_k \cap F_l \cap D \neq \emptyset$ *and* $E_{k'} \cap F_{l'} \cap D \neq \emptyset$

holds for $k, k' \in \{1, \dots, s\}$ *and* $l, l' \in \{1, \dots, r\}$ *with* $k \neq k'$ *and* $l \neq l'$, *we call* D *a* diagonal *of this* $r \times s$*-grid.*

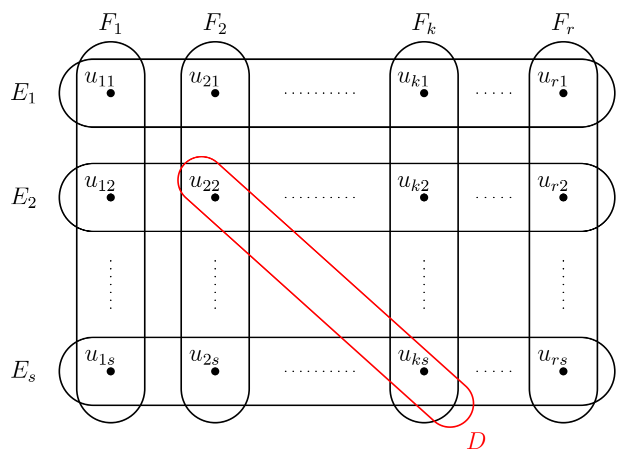


Figure 3.1: $r \times s$-Grid with diagonal D

Note, that for the edges $E_1, \dots, E_r$ and $F_1, \dots, F_s$ we have $|E_i| = s$ and $|F_j| = r$ for all $i \in \{1, \dots, r\}$, $j \in \{1, \dots, s\}$.
Such a grid emerges, whenever two edges of two hypergraphs are multiplied with respect to the Cartesian product. This leads us to a relation δ, defined as follows.

Definition 3.2. *Let* H *be a connected hypergraph. For* $E, F \in \mathscr{E}(H)$ *we say* E *and* F *are in* relation δ *if one of the following conditions holds:*

(i) $E \cap F = \emptyset$ *and* E *and* F *are opposite edges of a four-cycle*

(ii) $E \cap F \neq \emptyset$ *and there is no* $(|E| \times |F|)$*-grid without diagonals containing them.*

Remark, that for $|E \cap F| > 1$ there is no such $(|E| \times |F|)$-grid, hence $E \delta F$.
Obviously, the relation δ is reflexive and symmetric. The *transitive closure* δ^*, i.e., the smallest transitive relation containing δ, is then an equivalence relation. Condition (ii) implies, that any two incident edges E,F with $(E,F) \notin \delta$ span an $(|E| \times |F|)$-grid without diagonals. Let this grid consist of the edges $E, E_1, \ldots, E_s$ and $F, F_1, \ldots, F_r$ with $E \delta E_i$ and $F \delta F_j$ for all $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, r\}$, respectively. Suppose now there is another $(|E| \times |F|)$-grid consisting of edges $E, E'_1, \ldots, E'_s$ and $F, F'_1, \ldots, F'_r$.

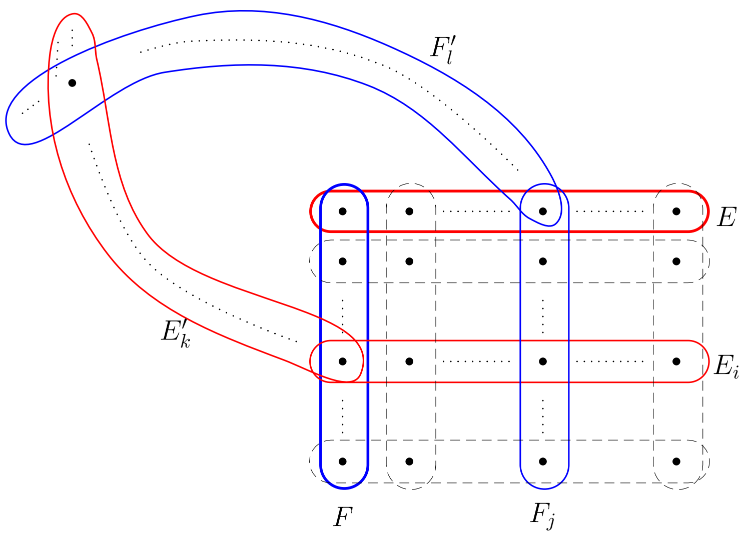


Figure 3.2: Two edges E and F, with $E \delta F$, which span more than one $|E| \times |F|$-grid

There must be a $k \in \{1, \ldots, s\}$ and an $l \in \{1, \ldots, r\}$ such that $E'_k \notin \{E_1, \ldots, E_s\}$ and $F'_l \notin \{F_1, \ldots, F_r\}$, respectively, see Figure 3.2. Then there exists an $E_i \in \{E_1, \ldots, E_s\}$ as well as an $F_j \in \{F_1, \ldots, F_r\}$ with $|E'_k \cap E_i| \neq 0$ and $|F'_l \cap F_j| \neq 0$. Thus, there is a four cycle $E'_k E_i F_j F'_l$, where E'_k and F_j as well as E_i and F'_l are opposite edges. Hence $E'_k \delta F_j$ and $E_i \delta F'_l$, and therefore $(E,F) \in \delta^*$. Thus, if E and F belong to distinct δ^*-equivalence classes, they span exactly one $(|E| \times |F|)$-grid. This leads us to the following definition:

Definition 3.3. *Let γ be an equivalence relation on the edge set $\mathscr{E}(H)$ of a hypergraph H. We say γ has the* grid property *if any two adjacent edges E and F of H of distinct γ-equivalence classes span exactly one diagonal free $|E| \times |F|$-grid.*

As seen before, δ^* has the grid property. Let γ be an arbitrary equivalence relation on the edge set of a hypergraph H that contains δ^*. For any two edges E and F with $(E,F) \notin \gamma$ holds

$(E,F) \notin \delta^*$ and therefore, they span exactly one $|E| \times |F|$-grid. Thus, any equivalence relation γ with $\delta^* \subseteq \gamma$ satisfies the grid property.

Another relevant property satisfied by δ^* and any equivalence relation γ with $\delta \subseteq \gamma$ is introduced in the next lemma.

Lemma 3.1. *Let γ be an equivalence relation on the edge set $\mathscr{E}(H)$ of a connected hypergraph H with equivalence classes $\Gamma_1, \dots, \Gamma_k$, which satisfies the grid property. Then every vertex of $V(H)$ is incident to edges of Γ_i for all $i \in \{1, \dots, k\}$.*

Proof. Suppose there is an equivalence class Γ_i of γ such that there are vertices in $V(H)$ which are not contained in a Γ_i-edge. As H is connected, there exists a pair of vertices $u, v \in V(H)$ and an edge $E \in \mathscr{E}(H)$ with $\{u, v\} \subseteq E$, such that u belongs to a Γ_i-edge, say F and there is no Γ_i-edge containing v. Then clearly, E is not in Γ_i, it is in Γ_k with $\Gamma_k \neq \Gamma_i$. But then E and F are two adjacent edges belonging to different equivalence classes of γ and thus by the grid property there must be a Γ_i-edge containing v, which contradicts the assumption. □

Lemma 3.2. *Let $H = \Box_{i=1}^n H_i$ be a Cartesian product of prime hypergraphs H_i and let $E, F \in \mathscr{E}(H)$. If E and F are in relation δ, they are in relation σ.*

Proof. Let the first condition in Definition 3.2 be satisfied, i.e., $E \cap F = \emptyset$ and there are edges $E', F' \in \mathscr{E}(H)$ such that $EE'FF'$ build a four-cycle. Let $E = \{x_1, \dots, x_r\}$ and $F = \{y_1, \dots, y_s\}$. W.l.o.g. assume $x_1 \in E \cap E'$, $x_r \in E \cap F'$, $y_1 \in F \cap E'$, $y_s \in F \cap F'$. Let $c(E) = i$, $c(F) = j$, $c(E') = i'$, $c(F') = j'$.

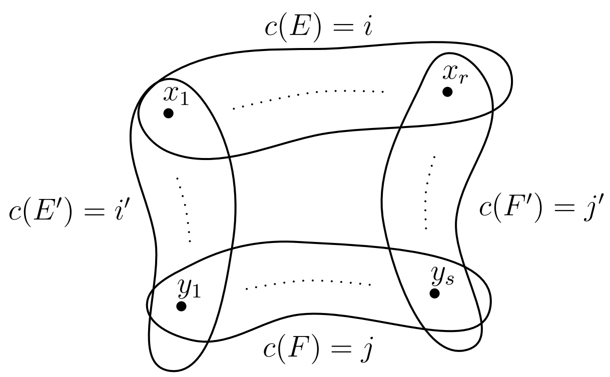


Figure 3.3: 4-cycle $EE'FF'$

Then we have:

$$p_k(x_1) = p_k(x_r) \qquad \text{for all } k \neq i \tag{3.1}$$

$$p_k(y_1) = p_k(y_s) \qquad \text{for all } k \neq j \tag{3.2}$$

$$p_k(x_1) = p_k(y_1) \qquad \text{for all } k \neq i' \tag{3.3}$$

$$p_k(x_r) = p_k(y_s) \qquad \text{for all } k \neq j'. \tag{3.4}$$

It follows from (3.1) and (3.3) that

$$p_k(x_1) = p_k(y_s) \qquad \text{for all } k \neq i, j' \tag{3.5}$$

and from (3.2) and (3.4)

$$p_k(x_1) = p_k(y_s) \qquad \text{for all } k \neq j, i'. \tag{3.6}$$

Therefore we have either $i = j$ and $i' = j'$ or $i = i'$ and $j' = j$. Assume $i \neq j$. Then the latter case must hold and we have $p_k(x_r) = p_k(x_1) = p_k(y_i)$ for all $k \neq i$, and since $x_r \neq y_1$ holds, $p_i(x_r) \neq p_i(y_1) = p_i(y_s)$ and $p_j(x_r) \neq p_j(y_s)$. So x_r and y_s differ in more than one coordinate, thus they cannot lie in the same edge F', which contradicts the assumption, so $i = j$ must hold, i.e. $c(E) = c(F)$ hence $E\sigma F$.

Now let E and F be adjacent edges of a hypergraph H, i.e., $|E \cap F| > 0$ and suppose, there is no $|E| \times |F|$-grid without diagonals containing them.
First, consider the case $|E \cap F| > 1$. There is an $i \in \{1, \ldots, n\}$ such that $|p_i(E)| > 1$ and in particular, $|p_i(E')| > 1$ holds for all $E' \subseteq E$ with $|E'| > 1$. Since $(E \cap F) \subseteq E$ and $|E \cap F| > 1$, it follows $|p_i(E \cap F)| > 1$. But as $(E \cap F) \subseteq F$ we have $|p_i(F)| > 1$ as well and therefore $E\sigma F$.
Now let $|E \cap F| = 1$ and suppose E and F are not in relation σ. Let $E \cap F = \{v\}$, say $E = \{v, x_1, \ldots, x_r\}$ and $c(E) = i$ as well as $F = \{v, y_1, \ldots, y_s\}$ and $c(F) = j \neq i$. For all $x_a \in E$, $a \in \{1, \ldots, r\}$, there exists vertices $z_{ab} \in V(H)$, such that for all $b \in \{1, \ldots, s\}$ hold

$$p_i(z_{ab}) = p_i(x_a) \tag{3.7}$$

$$p_k(z_{ab}) = p_k(y_b) \qquad \text{for all } k \neq i. \tag{3.8}$$

Then we have for the set $F_a = \{x_a, z_{a1}, \ldots, z_{as}\}$:

$$p_j(F_a) = \{p_j(x_a), p_j(z_{a1}), \ldots, p_j(z_{as})\} = \{p_j(v), p_j(y_1), \ldots, p_j(y_s)\} = p_j(F) \tag{3.9}$$

as well as

$$p_k(z_{a1}) = \ldots = p_k(z_{as}) = p_k(y_b) = p_k(v) = p_k(x_a) \text{ for all } k \neq i, j \tag{3.10}$$

from (3.8) and the fact, that $\{v, y_b\} \subseteq F$ for all $b \in \{1, \ldots, s\}$ and $\{v, x_a\} \subseteq E$.
Now from (3.7) and (3.10) we gain

$$p_k(z_{a1}) = \ldots = p_k(z_{as}) = p_k(x_a) \qquad \text{for all } k \neq i \tag{3.11}$$

and therefore, from (3.9) and (3.11) it follows that F_a is an edge in H with $|F_a| = |F|$, for arbitrary $a \in \{1, \ldots, r\}$.
On the other hand, for the set $E_b = \{y_b, z_{1b}, \ldots, z_{rb}\}$ holds

$$p_i(E_b) = \{p_i(y_b), p_i(z_{1b}), \ldots, p_i(z_{rb})\} = \{p_i(v), p_i(x_1), \ldots, p_i(x_r)\} = p_i(E), \tag{3.12}$$

again from (3.7) and the fact that $\{v, y_b\} \subseteq F$ for all $b \in \{1, \ldots, s\}$. Together with (3.8) it follows that the E_b for $b \in \{1, \ldots, s\}$ are edges in H.

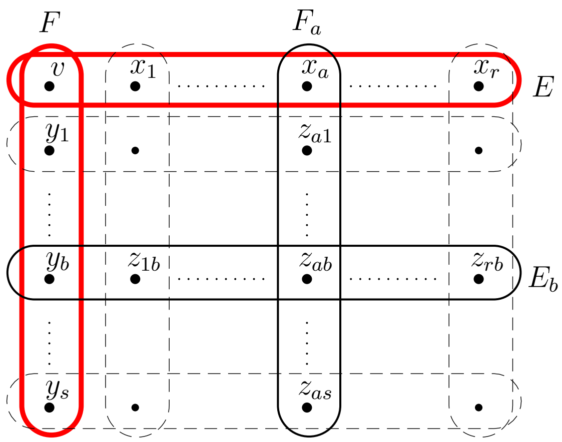


Figure 3.4: Edges E and F (red edges) with $(E, F) \notin \sigma$ and the $|E| \times |F|$-grid they span

Furthermore, if we relabel v as z_{00}, E as E_0, F as F_0 and the x_l as z_{l0} for $l \in \{1, \ldots, r\}$, the y_l as z_{0l} for $l \in \{1, \ldots, s\}$ respectively, we have $E_b \cap F_a = \{z_{ab}\}$ for all $a \in \{0, \ldots, r\}$ and $b \in \{0, \ldots, s\}$ respectively. Obviously, the intersection of more then two edges is empty. That means, whenever two adjacent edges E and F are not in relation σ, they span such an $|E| \times |F|$-grid with $|E| \times |F|$ vertices.
It remains to show that these grids have no diagonals. Therefore, we need to show that there is

no $D \in \mathscr{E}(H)$ such that $\{z_{ab}, z_{a'b'}\} \subseteq D$ for all $a, a' \in \{0, \ldots, r\}$, $b, b' \in \{0, \ldots, s\}$ with $a \neq a'$ and $b \neq b'$. For $z_{ab}, z_{a'b'}$ we have:

$$p_i(z_{ab}) \overset{(3.7)}{=} p_i(z_{a0}) \neq p_i(z_{a'0}) \overset{(3.7)}{=} p_i(z_{a'b'}) \tag{3.13}$$

$$p_j(z_{ab}) \overset{(3.8)}{=} p_j(z_{0b}) \neq p_j(z_{0b'}) \overset{(3.8)}{=} p_j(z_{a'b'}) \tag{3.14}$$

The inequalities follow from the fact, that $\{z_{a0}, z_{a'0}\} \subseteq E_0$ and $\{z_{0b}, z_{0b'}\} \subseteq F_0$.
That means that z_{ab} and $z_{a'b'}$ differ in more than one coordinate, hence, they cannot be contained in the same edge, which completes the proof. □

Lemma 3.2 implies that $\delta \subseteq \sigma$ and since σ is an equivalence relation, even $\delta^* \subseteq \sigma$ holds, thus, σ has the grid property.

In a hypergraph product $H = \Box_{i=1}^n H_i$, the H_k-layers are convex partial hypergraphs, as we will see next. Even more, we can state:

Lemma 3.3. *Let $H = \Box_{i=1}^n H_i$ be a Cartesian product of connected hypergraphs H_i. Then each $H_J = \Box_{j \in J} H_j$-layer is convex for any index set $J \subseteq \{1, \ldots, n\}$.*

Proof. It suffices to show that whenever there is a path P between two arbitrary vertices u and v of the same H_J-layer H_J^u, containing no edges of this layer, then there exists a path Q which entirely lies in H_J^u such that $|Q| < |P|$.
Suppose $P = (u = u_0, E_1, u_1, E_2, \ldots, u_{k-1}, E_k, u_k = v)$. Since u and v belong to the same H_j-layer, $p_l(u) = p_l(v)$ holds for all $l \in I_n \setminus J$. There must be an edge E_i of P such that E_i is contained in some H_J layer, by assumption different from H_J^u. Otherwise we would have $p_l(u) = p_l(v)$ for all $l \in J$, hence $p_l(u) = p_l(v)$ for all $l \in I_n$, i.e. $u = v$.
Let $\{E_{j_1}, \ldots, E_{j_r}\}$ be a subset of edges of P, with $j_1, j_2, \ldots, j_r \in \{1, \ldots, k\}$, $j_1 < j_2 < \ldots < j_r$, that are in some H_J-layer different from H_J^u, and no edge is the copy of another. To be more precise, for each j_i there is a $k_i \in J$ with

$$p_{k_i} E_{j_i} \in \mathscr{E}(H_{k_i}) \qquad \text{and} \qquad p_{k_a} E_{j_a} \neq p_{k_b} E_{j_b} \quad \text{for } a \neq b \tag{3.15}$$

and j_r is maximal. Notice, that $a \neq b$ does not imply $k_a \neq k_b$.
By assumption E_1 is not contained in any H_J-layer, thus $r < k$. Without loss of generality, we can assume that the E_{j_i} are not incident. In the following we will denote the vertices u_{j_i} by j_i. If

we set $j_0 := u_0$, we have

$$p_l(j_{i-1}) = p_l(j_i - 1) \quad \text{for all } l \in J \text{ and all } i \in \{1,\dots,r\} \tag{3.16}$$

and since $j_i - 1, j_i \in E_{j_i}$,

$$p_l(j_i) = p_l(j_i - 1) \tag{3.17}$$

holds for all $l \neq k_i$, $k_i \in J$.

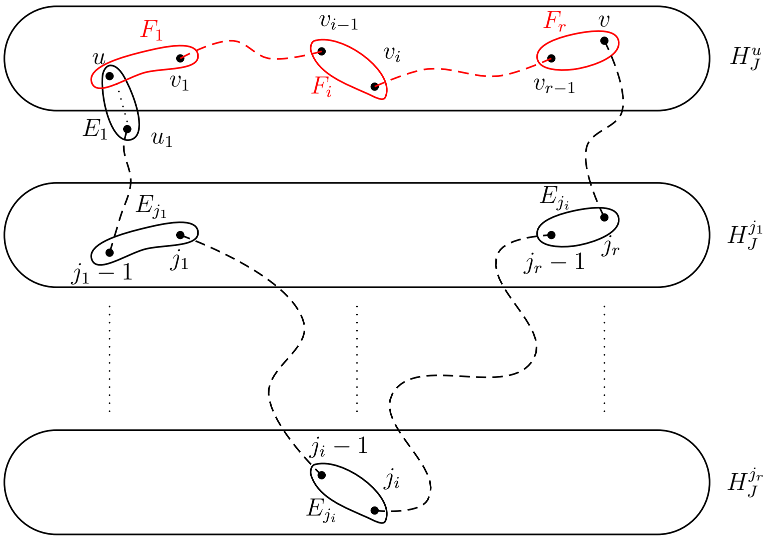


Figure 3.5: **Idea of the proof:** Path P (black) and Path Q (red) which we got by shifting the edges E_{j_i} in the H_J^u-layer

Furthermore, for each j_i, $i \in \{1,\dots,r\}$, there exists a $v_i \in V(H)$, such that

$$p_l(v_i) = p_l(u) \quad \text{for all } l \in I_n \setminus J \tag{3.18}$$

$$p_l(v_i) = p_l(j_i) \quad \text{for all } l \in J \tag{3.19}$$

In particular, Equation (3.18) implies $v_i \in V(H_J^u)$ for all $i \in \{1,\dots,r\}$. It follows

$$p_l(v_{i-1}) \overset{(3.19)}{=} p_l(j_{i-1}) \overset{(3.16),(3.17)}{=} p_l(j_i) \overset{(3.19)}{=} p_l(v_i) \quad \text{for all } l \in J \setminus \{k_i\} \tag{3.20}$$

and by Equation (3.18) we have

$$p_l(v_{i-1}) = p_l(v_i) \quad \text{for all } l \in I_n \setminus J \tag{3.21}$$

hence by Equations (3.20) and (3.21)

$$p_l(v_{i-1}) = p_l(v_i) \quad \text{for all } l \neq k_i \tag{3.22}$$

In other words, each two vertices v_{i-1}, v_i lie in the same H_{k_i} layer for some $k_i \in J$.
Next we show that there are edges $F_i \in \mathscr{E}(H_J^u)$ containing both v_{i-1} and v_i. From Equations (3.16) and (3.19) it follows

$$p_{k_i}(v_{i-1}) \overset{(3.19)}{=} p_{k_i}(j_{i-1}) \overset{(3.16)}{=} p_{k_i}(j_i - 1) \tag{3.23}$$

$$p_{k_i}(v_i) \overset{(3.19)}{=} p_{k_i}(j_i) \tag{3.24}$$

Thus we have by Equations (3.23), (3.24) and (3.15)

$$p_{k_i}(v_{i-1}),\ p_{k_i}(v_i) \in p_{k_i}(E_{j_i}) \tag{3.25}$$

Hence by Equations (3.18), (3.22) and (3.25), for each $i \in \{1,\ldots,r\}$ there exists an edge F_i in H_J^u, such that $v_{i-1},\ v_i \in F_i$.
Consider now v_r. Since there is no more edge E_j, $j > j_r$, of P that is contained in any H_J-layer, j_r and v belong to the same $\widehat{H_J} := \Box_{i\in I_n\setminus J} H_i$-layer, and therefore we have

$$p_l(v_r) \overset{(3.19)}{=} p_l(j_r) = p_l(v) \quad \text{for all } l \in J \tag{3.26}$$

and from the definition of v_r and the fact that u and v are in the same H_J-layer, it follows

$$p_l(v_r) \overset{(3.18)}{=} p_l(v) \quad \text{for all } l \in I_n \setminus J. \tag{3.27}$$

Therefore, we can conclude $v_r = v$, and we found a path $Q = (u = v_0, F_1, v_1, \ldots, v_{r-1}, F_r, v_r = v)$ from u to v, whose edges entirely lie in H_J^u and for which holds

$$|Q| = r < k = |P|,$$

which completes the proof. □

This gives rise to the next definition.

Definition 3.4. *An equivalence relation γ on the edge set $\mathscr{E}(H)$ of a hypergraph H with equivalence classes Γ_i, $i \in I$, is called* convex *if for any $J \subseteq I$ every connected component of the partial hypergraph generated by $\bigcup_{i\in J} \Gamma_i$ is convex.*

By Lemma 3.3, the product relation σ is a convex relation. Moreover, any product relation must be convex and has to satisfy the grid property. On the other hand, we have the following statement

Theorem 3.1. *Let γ be a convex equivalence relation on the edge set $\mathscr{E}(H)$ of a connected Hypergraph H which satisfies the grid property. Then γ induces a factorization of H with respect to the Cartesian product.*

To prove this theorem, we first have to show the validity of the next two lemmas.

Lemma 3.4. *Let γ be an equivalence relation on the edge set $\mathscr{E}(H)$ of a connected hypergraph H which satisfies the grid property. Let Γ be an equivalence class of γ. If all connected components of the partial hypergraph of H generated by Γ are convex, they are isomorphic.*

Proof. Let H_Γ be the partial hypergraph generated by Γ with connected components $C_1,\ldots,C_r$ and let $\widehat{\Gamma}$ denote the union of all equivalence classes of γ, distinct from Γ, i.e., $\widehat{\Gamma} = \bigcup_{\Gamma' \neq \Gamma} \Gamma'$. It suffices to show, that any two components C_1, C_2 which are connected by a $\widehat{\Gamma}$-edge are isomorphic. We define a mapping $\varphi : V(C_1) \to V(C_2)$, through $x \mapsto \varphi x$, whenever x and φx are connected by a $\widehat{\Gamma}$-edge. From the grid property and Lemma 3.1, it follows that for all $x \in V(C_1)$ there exists a $\varphi x \in V(C_2)$. The grid property ensures that adjacent vertices in C_1 have different images in C_2 and edges in C_1 map onto edges in C_2. By convexity we have, that non adjacent vertices in C_1 have different images in C_2 as well, i.e. the mapping φ is injective. On the other hand we can extend φ^{-1} to a mapping $\psi : V(C_2) \to V(C_1)$. Analogously, it follows that for all $y \in V(C_2)$ there is a ψy in $V(C_1)$, hence $\varphi^{-1} = \psi$, i.e. φ is bijective, and every edge in C_2 maps onto an edge in C_1, thus φ is an isomorphism between C_1 and C_2. □

Sometimes the transitive closure of δ is already convex. If this is the case, then each path between two vertices of the same connected component of an equivalence class of δ^* must contain at least one edge of this equivalence class (see Lemma 3.3).

Figure 3.6 shows a hypergraph where δ^* is not convex and thus, the mapping φ defined in the proof of Lemma 3.4 is no isomorphism. The connected component C_1 of the black equivalence class is mapped via the red edges onto the connected component C_2. Although φ preserves adjacency and non-adjacency, the mapping is not isomorphic, since it is not injective.

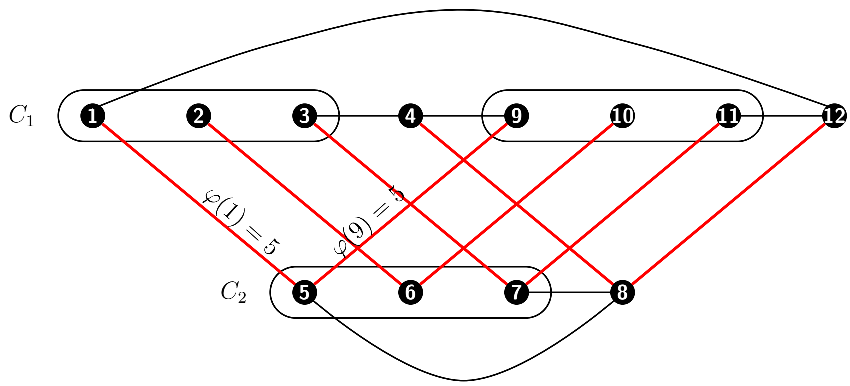


Figure 3.6: Hypergraph and its equivalence classes (red and black, respectively) of δ^*, and the connected components C_1 and C_2 of the hypergraph generated by the black equivalence class. We have $d_H(1,9) = 2$, while $d_{C_1}(1,9) = 3$, hence C_1 is not convex.

Lemma 3.5. *Let γ be an equivalence relation on the edge set $\mathscr{E}(H)$ of a connected hypergraph H satisfying the grid property with only two equivalence classes Γ and $\widehat{\Gamma}$. Let H_Γ and $H_{\widehat{\Gamma}}$ be the subgraphs generated by Γ and $\widehat{\Gamma}$, with connected components $C_1,\ldots,C_r$ and $\widehat{C}_1,\ldots,\widehat{C}_s$, respectively. Then*

$$V(C_i)\cap V(\widehat{C}_j) \neq \emptyset \quad \textit{for all } i \in \{1,\ldots,r\},\ j \in \{1,\ldots,s\}.$$

In particular,

$$|V(C_i)\cap V(\widehat{C}_j)| = 1$$

holds if C_i and $\widehat{C}_j$ are convex.

Proof. Suppose there are components C_i, $\widehat{C}_j$ with $V(C_i)\cap V(\widehat{C}_j) = \emptyset$, such that they have minimal distance. Let $P = (v_0, E_1, v_1, E_2, \ldots, E_k, v_k)$ be a shortest path from C_i to $\widehat{C}_j$, such that $v_0 \in V(C_i)$ and $v_k \in V(\widehat{C}_j)$. Obviously, the first edge E_1 must lie in $\widehat{\Gamma}$ and the vertex v_1 is not in C_i, otherwise E_1 would be in Γ which contradicts the minimality of P. Lemma 3.1 implies, that v_1 must be contained in a Γ-component, say C_k. Since the distance from C_k to $\widehat{C}_j$ is smaller than $|P|$, we have $V(C_k)\cap V(\widehat{C}_j) \neq \emptyset$. Let w be a vertex in $V(C_k)\cap V(\widehat{C}_j)$ and let P' be a path from v_1 to w in C_k. By repeated application of the grid property we gain a vertex u in $V(C_i)$ connected to w by a $\widehat{\Gamma}$-edge. But then u must be in $V(\widehat{C}_j)$ and thus $|V(C_i)\cap V(\widehat{C}_j)| \geq 1$.
Now assume $|V(C_i)\cap V(\widehat{C}_j)| \geq 2$. Let $u,w \in V(C_i)\cap V(\widehat{C}_j)$. By connectivity we have a path Q from u to w in C_i and a path Q' from u to w in $\widehat{C}_j$ as well. Therefore either $|Q| > |Q'|$ or

$|Q'| > |Q|$ or $|Q| = |Q'|$ holds. Hence either C_i or $\widehat{C}_j$ or both are not convex. And therefore the second proposition holds. □

We are now able to prove Theorem 3.1:

Proof of Theorem 3.1. First assume γ has only two equivalence classes Γ and $\widehat{\Gamma}$ with connected components $C_1,\ldots,C_r$ and $\widehat{C}_1,\ldots,\widehat{C}_s$ respectively, of the generated partial hypergraphs. By Lemma 3.5 we can assign uniquely determined coordinates (i,j) to each vertex of H, whenever $\{v\} = V(C_i) \cap V(\widehat{C}_j)$, $i \in \{1,\ldots,r\}$, $j \in \{1,\ldots,s\}$. On the other hand for all such coordinates there exists a uniquely determined vertex in $V(H)$, since $|V(C_i) \cap V(\widehat{C}_j)| = 1$.
In the following we will identify each vertex of H with its coordinates. Obviously we have $V(C_i) = \{(i,1),\ldots,(i,s)\}$ for all $i \in \{1,\ldots,r\}$ and $V(\widehat{C}_j) = \{(1,j),\ldots,(r,j)\}$ for all $j \in \{1,\ldots,s\}$. Recall that Lemma 3.4 implies that the C_i are isomorphic for all $i \in \{1,\ldots,r\}$. In particular $C_1 \cong C_i$ holds for all $i \in \{1,\ldots,r\}$. The isomorphism is given by the mapping

$$(1,j) \mapsto (i,j) \quad \text{for all} \quad j \in \{1,\ldots,s\}.$$

If C_1 and C_i are connected by an edge, it is an isomorphism as in the proof of Lemma 3.4. If they are connected by a path, it is an isomorphism by induction on the length of the path. Analogously we have $\widehat{C}_1 \cong \widehat{C}_j$ for all $j \in \{1,\ldots,s\}$ given by the isomorphism

$$(i,1) \mapsto (i,j) \quad \text{for all} \quad i \in \{1,\ldots,r\}.$$

A set of vertices $\{(i_1,j_1),\ldots,(i_q,j_q)\}$, $1 \le i_1,\ldots,i_q \le r$, $1 \le j_1,\ldots,j_q \le s$, is an edge in H if and only if either

(i) it is in the same C_i, hence $i_1 = \ldots = i_q = i$ and $\{(1,j_1),\ldots,(1,j_q)\}$ is an edge in C_1, or

(ii) it is in the same $\widehat{C}_j$, hence $j_1 = \ldots = j_q = j$ and $\{(i_1,1),\ldots,(i_q,1)\}$ is an edge in $\widehat{C}_1$.

That is, H is isomorphic to $C_1 \Box \widehat{C}_1$.
Now define hypergraphs H_1 and H_2 by setting $V(H_1) = \{i : (i,1) \in V(C_1)\}$ and $V(H_2) = \{j : (1,j) \in V(\widehat{C}_1)\}$. H_1 and H_2 are isomorphic to C_1 and $\widehat{C}_1$ by the isomorphic mappings $i \mapsto (i,1)$ and $j \mapsto (1,j)$ respectively, thus $H = H_1 \Box H_2$.
Assume now γ has arbitrarily many equivalence classes Γ_i, $i = 1,\ldots,n$. Let γ_i be the equivalence relation with the two equivalence classes Γ_i and $\widehat{\Gamma}_i = \bigcup_{k=1,k\neq i}^{n} \Gamma_k$ for arbitrary $i \in \{1,\ldots,n\}$. As already shown, we get a factorization of H into two factors $H_i \Box \widehat{H}_i$ where H_i and $\widehat{H}_i$ belong to Γ_i

and $\widehat{\Gamma}_i$, respectively. We will call the projection, to be more precise, the image of the projection of a vertex v in $H_i \Box \widehat{H}_i$ into the factor H_i the i-th coordinate of v, denoted by v^i.
Now it is clear, that we can assign coordinates to each vertex. If two vertices u, v have the same i-th coordinate, then, by convexity, there is no Γ_i-edge on any shortest path between them. Thus, if u and v have the same coordinates, there is no nontrivial shortest path between them, hence $u = v$. Therefore the assignment of coordinates to vertices of a connected hypergraph H is bijective.
A subset $\{v_1, \ldots, v_r\}$ of $V(H)$ is an edge of H if and only if the v_k differ in the same coordinate, say the i-th, for all $k \in \{1, \ldots, r\}$ and $\{v_1^i, \ldots, v_r^i\}$ is an edge in H_i.
Thus we have $H = \Box_{i=1}^{n} H_i$. □

The equivalence relation whose only equivalence class is the whole edge set of a hypergraph H is trivially convex and satisfies the grid property and is therefore a product relation. And this relation always exists. By Theorem 3.1 we can conclude that any convex relation on the edge set of a connected hypergraph that satisfies the grid property is a product relation and induces a factorization of this hypergraph. The smallest convex relation satisfying the grid-property, if such a relation exists at all, must therefore induce a prime factorization with respect to the Cartesian product.

The following lemma is true for graphs, see [32], but it can immediately be transfered to hypergraphs.

Lemma 3.6. *Let γ_j, $j \in J$ be an arbitrary set of convex relations on the Edge set $\mathscr{E}(H)$ of a hypergraph H containing δ. Then $\gamma = \bigcap_{j \in J} \gamma_j$ is convex.*

It is clear that for arbitrary equivalence relations on the edge set of a hypergraph, which satisfy the grid-property, their intersection also has the grid-property. Therefore Lemma 3.6 implies that there is exactly one finest convex equivalence relation on the edge set $\mathscr{E}(H)$ of a hypergraph H satisfying the grid property, namely the intersection of all convex relations on $\mathscr{E}(H)$ containing δ, that is its *convex hull*, $\mathfrak{C}(\delta)$. Conversely, any product relation must be convex and contains δ. Thus we have proved the following theorems.

Theorem 3.2. *Every connected hypergraph H has a unique prime factor decomposition with respect to the Cartesian product.*

Theorem 3.3. *The relation corresponding to the unique prime factorization of a connected hypergraph H is the convex hull of δ, i.e. $\sigma = \mathfrak{C}(\delta)$.*

4 Conclusion

4.1 Additional Notes

Only finite hypergraphs and products of finitely many factors are treated in this thesis. It is possible to extend the definitions of the Cartesian, the direct and the strong product to infinitely many graphs.

In particular, we have for the Cartesian product:
Let $\{H_i \mid i \in I\}$ be a set of (finite or infinite) hypergraphs. Then their Cartesian product, $\Box_{i\in I} H_i$, is the following hypergraph:

(1) $V(\Box_{i\in I} H_i) = \times_{i\in I} V(H_i)$,

(2) for $E \subseteq \times_{i\in I} V(H_i)$ we have $E \in \mathscr{E}(\Box_{i\in I} H_i)$ if and only if there is an $i \in I$, s.t.

  (i) $p_i(E) \in \mathscr{E}(H_i)$ and

  (ii) $|p_j(E)| = 1$ for all $j \in I \setminus \{i\}$.

While a Cartesian product hypergraph of finitely many connected hypergraphs is connected, whether they are finite or not, this does not hold for the product of infinitely many hypergraphs. Since in a product of infinitely many factors there are vertices that differ in infinitely many coordinates and thus, cannot be connected by a path of finite length, as vertices of the same edge differ in only one coordinate. An infinite connected hypergraph can have infinitely many prime factors with respect to the Cartesian product. In this case it cannot be the Cartesian product of these factors, since the product is not connected, but a connected component of this product. Therefore it might be useful to define a so called *weak Cartesian product*.
Let $\{H_i \mid i \in I\}$ be a family of hypergraphs and let $a_i \in V(H_i)$ for $i \in I$. The weak Cartesian product $H = \Box_{i\in I}(H_i, a_i)$ of the rooted hypergraphs (H_i, a_i) is defined by

$$V(H) = \{v \in \bigtimes_{i\in I} V(H_i) \mid p_i(v) \neq a_i \text{ for at most finitely many } i \in I\}$$

$$\mathscr{E}(H) = \{E \subseteq V(H) \mid p_j(E) \in \mathscr{E}(H_j) \text{ for exactly one } j \in I, \text{ and } |p_i(E)| = 1 \text{ for } i \neq j\}.$$

We will also write $\square_{i\in I}^{a} H_i$ instead of $\square_{i\in I}(H_i, a_i)$, with $a \in V(H)$, such that $p_i(a) = a_i$ for all $i \in I$. The weak Cartesian product of connected hypergraphs is connected. In this case, it is the connected component of the Cartesian product of the hypergraphs H_i which contains a. One observes that the weak Cartesian product does not depend on the a_i and is the Cartesian product if I is finite. It is shown in [24] by W. Imrich, that every connected graph has a unique prime factor decomposition with respect to the weak Cartesian product. In this contribution, the author also extends the results to set systems, i.e. hypergraphs.
The proof of uniqueness of the prime factorization with respect to the Cartesian product of hypergraphs in Chapter 3 is modeled after the proof of uniqueness of prime factorization with respect to the weak Cartesian product of graphs in [32]. Therefore, with small modifications of the notations in Chapter 3, we can extend Theorem 3.2 to the following statement:

Theorem 4.1. *Every connected hypergraph has a unique representation as a weak Cartesian product.*

4.2 Summary

In this diploma thesis we studied hypergraph products as a generalization of the commutative standard graph products. The Cartesian product, which we were concerned with in Section 2.1 was already defined by W. Imrich in 1967 [23].

In Section 2.2 and Section 2.3 we defined some new hypergraph products. That are the *direct product* $\check{\times}$, the *direct product* $\times$, the *direct product* $\tilde{\times}$, and the *strong product* $\boxtimes$. Table 4.1 shows what kind of hypergraph products, treated in this thesis, fulfills which of the following properties:

1. $V(H_1 \star H_2) = V(H_1) \times V(H_2)$.
2. If H_1 and H_2 are simple then $H_1 \star H_2$ is simple.
3. The adjacency properties of a product depends on those of its factors.
4. Associativity.
5. Commutativity.
6. Distributivity with respect to the disjoint union.
7. The projections $p_i : V(H_1 \star H_2) \to V(H_i)$ for $i \in \{1,2\}$ are (at least weak) homomorphisms.

8. The restriction of the product $\star$ on graphs is the corresponding graph product.

9. The product $H_1 \star H_2$ is connected whenever the factors H_1 and H_2 are connected.

10. Unique prime factorization in the class of simple connected hypergraphs (or more special hypergraph classes).

Properties	Cartesian $\Box$	Direct $\check{\times}$	Direct $\times$	Direct $\tilde{\times}$	Strong $\boxtimes$
1.	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
2.	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
3.	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
4.	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
5.	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
6.	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
7.	weak	–	$\surd$	$\surd$	weak
8.	$\surd$	only for simple graphs	$\surd$	only for simple graphs	$\surd$
9.	$\surd$	–	–	–	$\surd$
10.	$\surd$	(?)*	?	?	?

Table 4.1: Properties of the hypergraph products
*: Prime factorization w.r.t. the direct product $\check{\times}$ is not unique in the class of conformal hypergraphs (cf. Example 2.2).

According to graph products, we will call hypergraph products which satisfy the first three conditions *simple hypergraph products*.

Uniqueness of the prime factor decomposition of simple connected hypergraphs was first proved by W. Imrich in [23]. We gave an alternative proof and showed, that the product relation corresponding to the unique prime factorization is the convex hull of a starting relation δ on the edge set of a given hypergraph.

4.3 Outlook

To conclude this thesis some open problems will be listed in the following. This listing makes no claim to be complete.

Other Products As seen in Section 2.2 of this thesis, there are not fewer than three hypergraph products that coincide with the direct graph product at least in the class of graphs without loops. Also three different definitions of hypergraph products that coincide with the strong graph product in this graph class may arise, here we considered only one product. Besides, it is shown for graphs, that there exists 256 simple graph products. Four of them are associative and the projections of a product into at least one of its factors are weak homomorphisms [26]. Obviously one question is if there are more hypergraph products than mentioned here, generalizing the common graph products. Moreover, how many simple hypergraph products do exist at all?

Prime Factorization A product relation σ, that belongs to the unique prime factorization of a simple connected hypergraph with respect to the Cartesian product, is provided in this thesis. This could be the basis for developing decomposition algorithms. One might ask in this context, which complexity of time and space is needed to compute the prime factors of a given hypergraph.

Also prime factor decomposition with respect to the strong and direct product of hypergraphs should be considered. At first, it remains to examine if the prime factorization with respect to the direct product and the strong product, respectively, is unique in special classes of hypergraphs. In graph theory, direct and strong graph products can be factorized by identifying their Cartesian skeleton and decompose it into its prime factors, see [16, 22]. The question is if this works for hypergraphs as well, and if it is easy to identify the Cartesian skeleton in this case.

Partial Hypergraphs of Product Hypergraphs In graph theory also subgraphs of product graphs are studied. The focus lies on isometric embeddings and retracts of the Cartesian and strong products, see for example [18, 27, 39]. It is shown, that every graph can be embedded isometrically into a strong product of paths [29].

Also graphs which can be represented as nontrivial subgraphs (i.e., each projection of this subgraphs into the factors contains at least two vertices) of Cartesian product graphs are of interest. Those graphs, for which such a representation is not possible are called *S-prime* and an infinite family of such graphs is classified for example in [33].
It is to examine if similar results also hold for hypergraphs.

Directed Hypergraphs In this thesis, only undirected hypergraphs are treated. But also directed hypergraphs are applied in various fields, see [2] for a survey. There are several ways to generalize the concept of directed graphs to directed hypergraphs. We give here a definition that

can be found in [17], since most of the others can be seen as a special case of this definition.
A directed hyperedge or *hyperarc* is an ordered pair, $E = (X,Y)$, of (possibly empty) disjoint subsets of vertices; X is the tail of E while Y is its head. The tail and the head of hyperarc E will be denoted by T(E) and H(E), respectively. A *directed hypergraph* is a hypergraph with directed hyperedges.
One might ask if the results of this thesis can be easily extended to directed hypergraphs.

List of Figures

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Erklärung

Ich versichere, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe.

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