Universität Leipzig
Fakultät für Mathematik und Informatik
Mathematisches Institut

Higher Spin Fields on Curved Spacetimes

Diplomarbeit im Studiengang Mathematik

vorgelegt von

Rainer Mühlhoff

Betreuender Hochschullehrer:

Prof. Dr. Rainer Verch
Institut für Theoretische Physik

Leipzig, im Dezember 2007

Contents

Preface

The present thesis investigates Buchdahl's equations for the description of massive particles of arbitrary spin $\frac{s}{2}$, $s \in \mathbb{N}$, on curved spacetime manifolds. This is a system of two first order linear differential equations for spinor fields, first studied by Buchdahl [Buc82a] and given by Wünsch [Wün85] in the equivalent formulation

$$\begin{cases} \nabla_{\dot{X}A} \psi^{AA_1 \dots A_{s-1}} - \mu \, \varphi_{\dot{X}}^{\;A_1 \dots A_{s-1}} = 0 \\ \nabla^{(A|\dot{X}} \varphi_{\dot{X}}^{\;|A_1 \dots A_{s-1})} - \nu \, \psi^{AA_1 \dots A_{s-1}} = 0 \, . \end{cases}$$

Assuming the spacetime manifold (M, g) to be a 4-dimensional, globally hyperbolic, oriented and time-oriended Lorentzian manifold, existence and uniqueness of advanced and retarded Green's operators for a certain quite general class of first order linear differential operators (including Buchdahl's equations and Dirac's equation) will be proved, the Cauchy problem for Buchdahl's equations will be solved globally for all $s \in \mathbb{N}$, and two possible constructions for quantizing Buchdahl fields using CAR-algebras in the fashion of [Dim82] will be given.

This is a mathematics diploma thesis on a topic in theoretical physics. It is one of the goals to present all considerations in an abstract and rigerous mathematical formulation. To this end, and to make this document accessible to both readers from a mathematical and readers from a physical background, one further main aspect of this thesis (apart from the study of Buchdahl's equations) is fully developing a comprehensive mathematical framework for dealing with spinor fields on Lorentzian manifolds.

This approach and all the in-depth preparatory work will pay off strikingly when finally starting the investigation of Buchdahl's equations in chapter 8, as they will have made the concrete situation accessible to the powerful results on normally hyperbolic linear differential operators on globally hyperbolic Lorentzian manifolds by Bär, Ginoux and Pfäffle [BGP]. Using these, the main results sketched above, which are generalisations of results by Dimock [Dim82] and Wünsch [Wün85], will be obtained in an elegant and straightforward manner.

Overview and Structure of this Thesis

As stated above, Buchdahl's equations are a system of differential equations on spinor fields on a spacetime manifold (M, g). By a spinor field we[1] mean a section of an associated vector bundle $DM := \mathcal{S}(M) \times_D \Delta$ on M, where $\mathcal{S}(M)$ denotes a spin structure (which has structure group $\mathrm{Spin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})$ in our case) and (D, Δ) is a finite-dimensional vector space representation of $\mathrm{SL}(2,\mathbb{C})$. More precisely, sections of DM are called spinor fields of type D. Spinor fields are used for the description of particles in physics, and different types of $\mathrm{SL}(2,\mathbb{C})$-representations, i. e. different types of spinor fields, generally correspond with different (classes/types of) particles.

[1] Throughout this document, "we" will be used in the sense of "the author and you—the reader".

This indicates that the representation theory of $\mathrm{SL}(2,\mathbb{C})$, notably the classification of finite-dimensional irreducible representations of $\mathrm{SL}(2,\mathbb{C})$, is of particular relevance for the physical theory. More precisely, as we will see in chapter 3, of relevance are the (equivalence classes of) finite-dimensional irreducible representations of $\mathrm{SL}(2,\mathbb{C})$ as *real* Lie group, which makes the classification procedure significantly more complicated compared to the "$\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \otimes \mathbb{C}$"-case (this is the *complex* Lie algebra of $\mathrm{SL}(2,\mathbb{C})$) presented in many introductory text books on representation theory.

This thesis will be subdivided into three parts and the first big goal in part I will be classifying finite-dimensional irreducible representations of the real Lie group $\mathrm{SL}(2,\mathbb{C})$. We will do this in chapter 3 by facilitating the well known classification of irreducible vector space representations of $\mathrm{SU}(2)$ in a way that is indicated but not carried out with satisfactory mathematical rigour in [SU01]. As a preparation for chapter 3 we collect relevant basic facts from representation theory in chapter 1. In chapter 2, we will develop a "little theory of complex algebra", i. e. of complex conjugate vector spaces, linear maps, tensor products, representations etc. This will be a core ingredient for our mathematical approach to the representation theory of $\mathrm{SL}(2,\mathbb{C})$.

In chapter 4 we will give a thorough introduction to the 2-spinor formalism in a way based on abstract indices. The 2-spinor formalism is an elementary formalism for dealing with spinors of type $D^{(\frac{1}{2},0)}$, $D^{(0,\frac{1}{2})}$, and their tensor products. It (historically) originates from physics and usually comprises a notation in components with respect to standard bases. But by adopting abstract index notation and in virtue of our considerations in chapter 2 we will be able to set up a strict formalism which deals with abstract objects.

Chapter 5 is entirely devoted to the illumination of the interrelationship of the Clifford algebra $\mathrm{Cl}_{1,3}$ of Minkowski spacetime $(\mathbb{R}^4, \eta)$, its spinor representations, Dirac matrices and the Dirac-spinor formalism. We will propose a precise notion of "a collection of Dirac matrices", learn how collections of Dirac matrices correspond with spinor representations of $\mathrm{Cl}_{1,3}$ and prove a general version of Pauli's fundamental theorem on Dirac matrices. Then we will propose a formal notion of "the standard representation", which is a realisation of the spinor representation of $\mathrm{Cl}_{1,3}$ connected with the 2-spinor formalism in a canonical way. This will then lead to the declaration of the Dirac-spinor formalism, again in the fashion of abstract index notation.

Part II of this thesis starts with chapter 6, where the construction of spinor bundles on our 4-dimensional oriented and time-oriented Lorentzian spacetime manifold (M, g) is carried out in full detail. To make this thesis sufficiently self-contained, we shall begin by reviewing relevant facts from Lorentzian geometry and about principal bundles, associated vector bundles and spin structures. Eventually we will construct the spinor bundles $D^{(j,j')}M := \mathcal{S}(M) \times_{D^{(j,j')}} \Delta_{j,j'}$ and extend the (tensor-)spinors ϵ^{AB}, ${\sigma_a}^{A\dot{X}}$ and ${\gamma_a}^{\tilde{A}}{}_{\tilde{B}}$ as given fiberwise in part I to (tensor-)spinor *fields* on M. It is a technical but very important result at the end of chapter 6 that the covariant derivatives on the spinor bundles induced by the Levi-Civita covariant derivative on TM are mutually compatible (in an appropriate sense) and that ϵ^{AB}, ${\sigma_a}^{A\dot{X}}$ and ${\gamma_a}^{\tilde{A}}{}_{\tilde{B}}$ are parallel with respect to these covariant derivatives.

In chapter 7 we will introduce linear differential operators on vector bundles. We will fix the central concept of normally hyperbolic differential operators, which are second order linear differential operators with metric principal part. These operators and the solution theory of differential equations given by them on globally hyperbolic manifolds were studied to a large extent in [BGP]. In section 7.3 we will use results from this useful reference to prove existence and uniqueness of advanced and retarded Green's operators for a first order linear differential operator P on a vector bundle $\mathcal{E}$, as soon as there is a second first order linear differential operator Q on $\mathcal{E}$, such that PQ is normally hyperbolic. This is a generalisation of [Dim82, theorem 2.1] and turns out to be applicable to the Buchdahl operator for all spin numbers $\frac{s}{2}$, $s \in \mathbb{N}$.

Part III of this thesis starts with chapter 8, where we will apply the whole theory developed so far to Buchdahl's generalised Dirac equations for spinor fields, where M is assumed globally hyperbolic. After constructing suitable bundles $D^B_s M$ of "Buchdahl spinors" for every $s \in \mathbb{N}$ we will be able to state the equations for spin $\frac{s}{2}$ in the form

$$B_s \Psi = 0, \qquad \Psi \in \Gamma(D^B_s M)\,,$$

where B_s is a first order linear differential operator ("Buchdahl's operator"). Existence and uniqueness of advanced and retarded Green's operators for B_s will be established immediately using our general theorem from chapter 7 after constructing an operator B'_s such that $B_s B'_s$ is normally hyperbolic. Subsequently, we will consider the Cauchy problem for Buchdahl's operator,

$$\begin{cases} B_s \Phi = 0, & \Phi \in \Gamma(D^B_s M) \\ \Phi|_\Sigma = \Phi_0 \end{cases}$$

where $\Sigma \subseteq M$ is a smooth spacelike Cauchy hypersurface and $\Phi_0 \in \Gamma_0(D^B_s \Sigma)$ a prescribed Cauchy datum. Existence and uniqueness of global solutions of the Cauchy problem will be proved to be equivalent to Φ_0 satisfying the constraint

$$\mathfrak{n}_{A_1}^{\dot{X}}(\tilde{\nabla}_{\dot{X}B}\psi_0{}^{BA_1\dots A_{s-1}} - \mu\varphi_{0\,\dot{X}}{}^{A_1\dots A_{s-1}}) = 0\,,$$

for $s \geq 2$, where $\mathfrak{n}$ is the future-directed unit normal vector field along Σ. Thus, we obtain a stronger version of the results by Wünsch [Wün85], who solved the Cauchy problem locally.

Finally, in chapter 9 we will investigate the possibility of performing a quantisation of Buchdahl fields by constructing the CAR-algebra generated by the space $(\mathscr{H}^s, b^s)$ of solutions of Buchdahl's equations together with a suitable Hermitian scalar product b^s. The general idea will be to generalise the quantisation procedure for the spin $\frac{1}{2}$ Dirac field as given by Dimock [Dim82], which will be briefly reviewed beforehand. The core problem will turn out to be the construction of a suitable Hermitian scalar product b^s on $\mathscr{H}^s$. Though we will find a promising candidate for b^s, we won't be able to establish that the CAR-algebra obtained by our construction is independent of a choice of Cauchy hypersurface (which has to be commited during the construction). Another possibility quantising the higher spin fields is using the framework presented by Illge [Ill93], but this turns out to be rather inappropriate as will be summarised in section 9.3.

Thanks

I'd like to thank my supervisor Prof. Dr. Rainer Verch, who gave me the opportunity of working on this exciting topic. I am grateful for Christopher Fewster's support during my stay at the University of York. Last but not least, two big bunches of flowers go to Annegret Schalke and Stephan Rave for many inspiring and supportive discussions.

Notation and Conventions

- Tensors and spinors are denoted using *abstract index notation* if not stated otherwise (for a brief summary on abstract index notation see below). Preferably, abstract tensor indices are lower case latin letters from the beginning of the alphabet, e. g. x^a, $t^a_{\ bc}$, abstract spinor indices are capital letters from the beginning or from the end of the alphabet, i. e. $A, B, C, \ldots$ or W, X, Y, Z. Abstract indices will *never* be variables for natural numbers in this document.

  For component notation of tensors and spinors we use greek indices, e. g. x^μ, $t^\alpha_{\ \mu\nu} B_\alpha \otimes B^\mu \otimes B^\nu$. Those indices are *always* variables for numbers in a suitable range. Moreover, we adopt the implicit summation convention for every diagonal pair of greek (component) indices, e. g, $x^\mu \eta_{\mu\nu}$ or $\eta^\mu_{\ \mu}$, but not $x^\mu \eta^{\mu\nu}$.

  The variables i, j, k, l, n, m are used as counting indices, this means they are variables for numbers, not abstract indices. E. g. $x_1, \ldots, x_n$ or $\sum_{i=1}^n a^i$.

  If $X_1, \ldots, X_k$ are vector spaces over the same field, $X_1 \vee \ldots \vee X_k \subseteq X_1 \otimes \ldots \otimes X_k$ denotes the symmetrised tensor product. Moreover, we set

  $$X^{\otimes k} := \underbrace{X \otimes \ldots \otimes X}_{k\text{-times}} \quad \text{and} \quad X^{\vee k} := \underbrace{X \vee \ldots \vee X}_{k\text{-times}} .$$

- Bundles and manifolds: (M, g) usuall denotes a smooth differentiable manifold with pseudo-Riemannian metric g. TM is the tangent bundle, $T^{p,q}M$ the bundle of (p, q)-tensors. TM is equipped with the Levi-Civita covariant derivative ∇.

  If $\mathcal{E}$ is a smooth vector bundle on M, $\Gamma(M) = \Gamma^\infty(M)$ denotes the space of smooth sections, $\Gamma_0(M) = \Gamma_0^\infty(M)$ the space of compactly supported smooth sections of $\mathcal{E}$. If $U \subseteq M$ is an open subset, $\mathcal{E}|_U$ denotes the restricted bundle on U.

- Integration on pseudo-Riemannian manifolds (M, g), $\int_M \ldots d\mu$ will always be performed with respect to the standard volume form μ given by the metric. We shall sometimes write μ_g to make this a bit more explicit. If $\Sigma \subseteq M$ is a submanifold we denote the induced volume form (with respect to the restricted metric) by μ_Σ.

- Linear algebra and matrices: $\mathrm{Mat}_{k\times l}(\mathbb{K})$ denotes the space of $k \times l$-matrices with entries from $\mathbb{K}$. If A is a matrix, A^{tr} denotes its transpose and $A^\dagger$ its Hermitian adjoint. We may sometimes write $(\psi, \varphi)^{tr}$ if we mean the column vector $\begin{pmatrix} \psi \\ \varphi \end{pmatrix}$ because it saves space.

  If V is a vector space and $\{x, y, z\} \subseteq V$, $\langle x, y, z\rangle_{\mathbb{K}}$ denotes the vector space span of $\{x, y, z\}$ over $\mathbb{K}$. By an orthonormal basis of a vector space with respect to a non-degenerate symmetric or Hermitian form of mixed signature we always mean a *pseudo*-orthonormal basis. Sesqui-linear or Hermitian forms on complex vector spaces are anti-linear in the first, linear in the second slot.

- If $f\colon A \to B$ is a map and $C \subseteq A$, we denote by $f|_C$ the restriction of f to C.
- $\mathbb{N}$ starts with the number 1, $\mathbb{N}_0$ starts with 0. In an expression like

  $$\sum_{\pi \in S_n} a_{\pi(1)} \otimes \cdots \otimes a_{\pi(n)} \,,$$

  S_n denotes the group of permutations of the numbers $1, \ldots, n$.

Abstract Index Notation

Abstract index notation for tensors is e. g. known from [Wal84, section 2.4], and it constitutes a reasonable compromise of classical (physical) component notation and abstract (mathematical) notation without any indices, in that it maintains the good aspects of both.

The good aspect of abstract mathematical notation is, well, that one deals with *abstract* objects, as opposed to components with respect to a basis. However, the treatment of contractions in this notation is extremely cumbersone, especially when dealing with complicated terms with many higher order tensors, as it occurs frequently in physical contexts. Contractions are much better treated in classical component notation using implicit summation: A diagonal pair of identical indices denotes a contraction and this immediately reflects the "slots-picture" of tensors: Every supscript index denotes a contravariant ("output-") slot, every subscript index a covariant ("input-") slot. A contraction connects a contravariant slot with a covariant slot.

It is not the right place here to give a full introduction to the details of abstract index notation (the reader cordially be refered to the above reference), but we shall briefly point out the basic idea: In abstract index notation for tensors, tensors are denoted using an upper index for each contravariant and a lower index for each covariant "slot" (as in usual component notation). However, these indices are *abstract* indices, which means that they are not variables for numbers $1, \ldots, n$ but rather just (abstract) *labels* for the tensor's *slots*. To make an example: If X is some finite-dimensional real vector space, $t \in X \otimes X \otimes X^*$ will get denoted by $t^{ab}{}_c$, but this is not a (basis-dependent) collection of real numbers with indices but it rather says: t is a tensor of type $(2, 1)$, i. e. an element of $X \otimes X \otimes X^*$. If $x^a \in X$ and $\varphi_a \in X^*$, an expression like $t^{ab}{}_c \varphi_b x^c$ denotes a *contraction* (still without making reference to any choice of basis).

The symbol $\otimes$ for tensor products is usually omitted in abstract index notation: $t^{ab}{}_c \varphi_d x^e$ is the tensor usually denoted by $t \otimes \varphi \otimes x$. Moreover, index order is relevant. An expression like $x^a\, y^b = x^b\, y^a$ for $x^a, y^a \in X$ means that $x \otimes y = y \otimes x$.

Symmetrisation in abstract index notation is denoted by

$$x^{(a_1 \ldots a_k)} := \frac{1}{k!} \sum_{\pi \in S_k} x^{\pi(1) \ldots \pi(k)} \,,$$

anti-symmetrisation by

$$x^{[a_1 \ldots a_k]} := \frac{1}{k!} \operatorname{sign}(\pi) \cdot \sum_{\pi \in S_k} x^{\pi(1) \ldots \pi(k)} \,.$$

The factor $\frac{1}{k!}$ gets introduced to make symmetrisation and anti-symmetrisation idempotent, i. e. $x^{((a_1 \ldots a_k))} = x^{(a_1 \ldots a_k)}$.

Part I.

Spinors and Representation Theory

1. Background in Representation Theory

We start out collecting relevant facts from representation theory in order to fix notation and to make our presentation sufficiently self-contained.

Notice that in this document, all Lie groups and Lie group maps are assumed smooth. All Lie groups are assumed real, Lie algebras may be real or complex. If G is a Lie group, $\mathrm{Lie}(G)$ denotes the Lie algebra of G. For a table of Lie groups and Lie algebras relevant to this document, cf. the appendix.

1.1. Basic Definitions and Important Theorems

1.1.1 Definition (Lie group and Lie algebra vector space representations). Let $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$ such that $\mathbb{K} \subseteq \mathbb{L}$.

(i) Let G be a real Lie group and V be a finite-dimensional $\mathbb{L}$-vector space. (ρ, V) is called a **(finite-dimensional) $\mathbb{L}$-representation of G over V,** if ρ is a Lie group homomorphism $\rho\colon G \to \mathrm{GL}_{\mathbb{L}}(V)$.

(ii) Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra and V be a finite-dimensional $\mathbb{L}$-vector space. (ρ, V) is called a **(finite-dimensional) $\mathbb{L}$-representation of $\mathfrak{g}$,** if ρ is a $\mathbb{K}$-linear homomorphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}_{\mathbb{L}}(V)$ of Lie algebras.

In both cases (representations of Lie algebras and Lie groups), V is called the **representation space,** and $\dim(\rho) := \dim_{\mathbb{L}}(V)$ the representation's **dimension**. If it is clear or unimportant, which space is the representation space, we shall sometimes denote a representation (ρ, V) just by ρ. Moreover, the image of $g \in G$ (or $g \in \mathfrak{g}$) under a representation ρ shall sometimes be denoted by ρ_g instead of $\rho(g)$.

1.1.2 Remark. We will only be considering finite-dimensional representations in this document. This is why we restricted the above definition to this (technically easier) case. Whenever we say "representation" we shall mean a representation in the sense of this definition, hence, a finite-dimensional one.

Moreover, in the case of Lie groups, we will only be considering representations of *real* Lie groups. This will not be a loss for our purposes; notice that every complex Lie group G of dimension $\dim_{\mathbb{C}}(G) = n$ may be considered a real Lie group of dimension $\dim_{\mathbb{R}}(G) = 2n$.

Just for the sake of completeness a few more basic definitions:

1.1.3 Definition (properties of representations).

(a) A representation on a finite-dimensional complex Hilbert space is called **unitary**, if every element of the represented space gets mapped to a unitary operator.

(b) An $\mathbb{L}$-representation of a Lie group or Lie algebra is called **irreducible**, if there is no non-trivial ρ-invariant $\mathbb{L}$-subspace of the representation space.

(c) Let (ρ, V) and (τ, W) be $\mathbb{L}$-representations of a Lie group or Lie algebra G. They are called **equivalent** if there is a $\mathbb{L}$-vector space isomorphism (called **intertwining isomorphism**) $\varphi\colon V \to W$, such that

$$\forall g \in G\colon \varphi\rho(g)\varphi^{-1} = \tau(g)\,.$$

In the next theorem we make an exception and consider also infinite-dimensional representations (for the reader familiar with the definitions):

1.1.4 Theorem (representations of compact Lie groups).

(a) Every complex representation of a compact Lie group on a Hilbert space is equivalent to a unitary one.

(b) Every complex, unitary, irreducible representation of a compact Lie group on a Hilbert space is finite-dimensional.

(c) Every complex, unitary representations of a compact Lie group on a Hilbert space is a direct sum of unitary, irreducible (hence, by (b), finite-dimensional) Hilbert space representations.

Proof. The finite-dimensional versions of (a) and (c) are theorem 1.7 and proposition 1.9 in [BD85, p. 68]. For the infinite-dimensional version of (a) and (c) and for (b) cf. [Rud]. □

1.1.5 Proposition (infinitesimal Lie group homomorphisms). Let G and H be Lie groups.

(a) Let $\varphi\colon G \to H$ be a Lie group homomorphism. Then

$$d\varphi_e\colon \mathrm{Lie}(G) \to \mathrm{Lie}(H)\,,$$

where $e \in G$ is the unit element, is a homomorphism of Lie algebras.

(b) Let G be simply connected and let $\varphi\colon \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ be a Lie algebra homomorphism. Then there is a unique Lie group homomorphism $\Phi\colon G \to H$ such that $d\Phi_e = \varphi$.

Proof. Cf. [Lee03, theorem 20.15] or [FH91, chapter 8]. □

The immediate application of this proposition yields:

1.1.6 Corollary (associated Lie algebra representation I). Let G be a Lie group.

(a) Let (ρ, V) be a representation of G. Then $(d\rho_e, V)$ is a representation of $\mathrm{Lie}(G)$, called the **Lie algebra representation associated with** (ρ, V)**.** It mostly will be denoted just by $d\rho$.

(b) If G is simply connected, for every Lie algebra representation (φ, V) of $\mathrm{Lie}(G)$ there is a unique Lie group representation (Φ, V) of G such that $d\Phi = \varphi$.

1.1.7 Proposition (associated Lie algebra representation II). Let G be a connected Lie group.

(a) Let ρ be a representation of G with associated Lie algebra representation $d\rho$. Then ρ is is irreducible if and only if $d\rho$ is irreducible.

(b) Let ρ and τ be representations of G with associated Lie algebra represendations $d\rho$ and $d\tau$. Then $\rho \cong \tau$ if and only if $d\rho \cong d\tau$.

Proof. For matrix Lie groups this is [Hal03, proposition 4.5]. The proof there generalises to the general case (notice that we consider only finite-dimensional representations in this document). As all Lie groups in this document shall be matrix Lie groups we refrain from presenting the generalised proof here. For (a) cf. also [FH91, exercise 8.17]. □

1.2. Forming New Representations Out of Existing Ones

1.2.1 Definition and proposition (complexification of representations).

(a) Let ρ be a $\mathbb{K}$-representation of a real Lie algebra $\mathcal{A}$ on V ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), i.e. $\rho\colon \mathcal{A} \to \mathfrak{gl}_{\mathbb{K}}(V)$ is an $\mathbb{R}$-linear Lie algebra homomorphism. Then the **complexified representation** ρ_c is the complex representation of $\mathcal{A} \otimes \mathbb{C}$ on[1] $V \otimes \mathbb{C}$ given by the $\mathbb{C}$-linearly extended $\mathbb{C}$-Lie algebra homomorphism

$$\rho_c\colon \mathcal{A} \otimes \mathbb{C} \to \mathfrak{gl}(V \otimes C)\,.$$

(b) Let ρ be an $\mathbb{L}$-representation of a real Lie algebra. Then ρ is irreducible if and only if ρ_c is irreducible.

Proof. Let $W \subseteq V$ be a ρ-invariant $\mathbb{L}$-subspace, let $\xi \otimes \lambda \in \mathcal{A} \otimes \mathbb{C}$ and $x \otimes \mu \in W \otimes \mathbb{C}$ be arbitrary. Then

$$\rho_c(\xi \otimes \lambda)(x \otimes \mu) = \lambda \cdot (\rho(\xi)(x) \otimes \mu) = \rho(\xi)(x) \otimes \lambda\mu \in W \otimes \mathbb{C}\,.$$

[1] Notice that if V is complex, by $V \otimes \mathbb{C}$ we mean $V \otimes_{\mathbb{C}} \mathbb{C} = V$. If V is real, we mean $V \otimes \mathbb{C} = V \otimes_{\mathbb{R}} \mathbb{C}$. In general, if V is a $\mathbb{K}$-vector space and W is a $\mathbb{L}$-vector space ($\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}\}$), we write $V \otimes W := V \otimes_{\min\{\mathbb{K},\mathbb{L}\}} W$ (where $\min\{\mathbb{K}, \mathbb{L}\}$ is the "minimum" with respect to $\subseteq$)

Hence, $W \otimes \mathbb{C}$ is ρ_c-invariant. By contraposition this proves that if ρ_c is irreducible, so is ρ.

To prove the converse direction, let $W \subseteq V \otimes \mathbb{C}$ be a ρ_c-invariant subspace. As it is a complex vector space, for every $x \otimes z \in W$, there is $x \otimes 1 \in W$. Hence, there is an $\mathbb{L}$-subspace $W' \subseteq V$ such that $W = W' \otimes \mathbb{C}$. Moreover,

$$\rho_c(\xi \otimes \lambda)(x \otimes \mu) = \rho(\xi)(x) \otimes \lambda\mu$$

implies that if W is ρ_c-invariant, W' must be ρ-invariant. By contraposition this proves that if ρ is irreducible, so is ρ_c. □

1.2.2 Definition and proposition (dual representation). Let (Φ, X) be a representation of a Lie group G on the vector space X. Then a representation Φ^* of G on the dual space X^* is induced by

$$\Phi^* \colon g \mapsto [\Phi(g^{-1})]^{tr}, \quad \forall g \in G\,,$$

where A^{tr} denotes the transpose of a linear map A. Φ^* is called **dual representation** or **contragredient representation of Φ.**

It is easily checked that the dual representation is constructed such that

$$\forall g \in G\, \forall \varphi \in X^*\, \forall x \in X \colon \varphi(x) = [\Phi^*(\varphi)]\Phi(x)\,.$$

The Lie algebra representation associated to Φ^* is given by

$$\varphi^* := d(\Phi^*) \colon \xi \mapsto -[\varphi(\xi)]^{tr}, \quad \forall \xi \in \mathrm{Lie}(G)\,.$$

Moreover, Φ^* is irreducible if and only Φ is irreducible.

Proof. All of this is elementary. Cf. [Hal03, 4.24]. □

1.2.3 Definition and proposition (direct sum of Lie group representations). Let (ρ, X) and (τ, Y) be real or complex representations of the Lie group G. Then a new representation of G, $\rho \oplus \tau$, called the **direct sum representation**, is obtained by

$$\begin{aligned} \rho \oplus \tau \colon G &\to \mathrm{GL}(X \oplus Y) \\ g &\mapsto [(x, y) \mapsto (\rho(x), \tau(y))]\,. \end{aligned}$$

If $\dim(\rho) > 0$ and $\dim(\tau) > 0$, a direct sum representation is never irreducible. The associated Lie algebra representation, $d(\rho \oplus \tau)$, is given by $d\rho \oplus d\tau$.

Proof. This is obvious. □

1.2.4 Definition and proposition (tensor product of Lie group representations). Let (ρ, X) and (τ, Y) be representations of a Lie group G. Then a new representation of G, called the **tensor product representation,** defined as

$$\rho \otimes \tau \colon G \to \mathrm{GL}(X \otimes Y)\,,$$

is obtained by linear extension of

$$\rho \otimes \tau \colon g \mapsto [x \otimes y \mapsto \rho_g(x) \otimes \tau_g(y)]\,.$$

The tensor product representation is, in general, not irreducible, even if ρ an τ are. The Lie algebra representation, $d(\rho \otimes \tau)$, associated with the tensor product representation is given by

$$d(\rho \otimes \tau)_\xi = d\rho_\xi \otimes \mathrm{id}_Y + \mathrm{id}_X \otimes d\tau_\xi, \qquad \forall \xi \in \mathrm{Lie}(G)\,.$$

Proof. Cf. [FH91, p. 110]. Regarding the statement about irreducibility: See the classification of SO(3)-representations for counter examples. □

Because we will make use of it later, we shall prove this (almost trivial) lemma:

1.2.5 Lemma (tensor products of equivalent representations). Let ρ, ρ', τ, τ' be representations of a Lie group G such that $\rho \cong \rho'$ and $\tau \cong \tau'$. Then $\rho \otimes \tau \cong \rho' \otimes \tau'$.

Proof. Let V' and W' be the representation spaces of ρ' and τ'. Let φ be an intertwining isomorphism such that $\forall g \in G\colon \rho'(g) = \varphi \circ \rho(g) \circ \varphi^{-1}$ and let ψ be an intertwining isomorphism such that $\forall g \in G\colon \tau'(g) = \psi \circ \tau(g) \circ \psi^{-1}$. Then we find for all $g \in G$ and $x \otimes y \in V' \otimes W'$:

$$\begin{aligned}
\big[(\varphi \otimes \psi) \circ (\rho \otimes \tau)(g) \circ (\varphi \otimes \psi)^{-1}\big](x \otimes y)& \\
&= [(\varphi \otimes \psi) \circ (\rho \otimes \tau)(g)]\left(\varphi^{-1}(x) \otimes \psi^{-1}(y)\right) \\
&= \left(\varphi \circ \rho(g) \circ \varphi^{-1}(x)\right) \otimes \left(\psi \circ \tau(g) \circ \psi^{-1}(x)\right) \\
&= [(\rho' \otimes \tau')(g)](x \otimes y)
\end{aligned}$$

This shows that $\varphi \otimes \psi$ is an intertwining isomorphism of $\rho \otimes \tau$ and $\rho' \otimes \tau'$. □

Recall that if G and H are Lie groups, then $G \times H$ is a Lie group with $\mathrm{Lie}(G \times H) = \mathrm{Lie}(G) \oplus \mathrm{Lie}(H)$.

1.2.6 Definition and proposition (external tensor product representation).

(a) Let G and H be Lie groups, (ρ, X) a representation of G and (τ, Y) a representation of H. Then there is a representation of $G \times H$ on $X \otimes Y$, called **external tensor product representation** and denoted $\rho \hat{\otimes} \tau$, defined by linear extension of

$$(g, h) \mapsto [x \otimes y \mapsto \rho_g(x) \otimes \tau_h(y)]\,.$$

(b) The associated Lie algebra representation is given by

$$d(\rho\hat{\otimes}\tau)_{(\xi,\zeta)} = d\rho_\xi \otimes \mathrm{id}_Y + \mathrm{id}_X \otimes d\tau_\zeta, \qquad \forall \xi \in \mathrm{Lie}(G), \zeta \in \mathrm{Lie}(H)\,.$$

(c) Let G and H be compact. If ρ and τ are irreducible, so is their external product representation, and if an irreducible representation of $G \times H$ is given, it is equivalent to the external product of two irreducible representations of G and H.

Proof. For (a) and (b), cf. [Ros02] and [Hal03]. For (c) cf. [BD85, proposition 4.14]. □

2. Some Complex Algebra

This section presents a little theory of "complex algebra", i. e. of complex conjugate vector spaces, their homomorphisms, tensor products, representations etc. All these considerations are rather elementary but we shall write them down in full detail as a solid basis for chapter 3, where complex conjugate Lie group representations will be arising in our program of classifying finite-dimensional irreducible complex representations of $\mathrm{SL}(2,\mathbb{C})$ as real Lie group.

The basic inspiration for the theory presented in this chapter came from the brief remarks on the definition of a complex conjugate Hilbert space in [Wal94, A.1]. What we'll do here is just carrying this idea a bit further.

In this whole section and for the rest of this document we shall call a map $\varphi\colon X \to Y$ between complex vector spaces **anti-linear,** if

$$\forall \mu \in \mathbb{C}\, \forall x, y \in X\colon \varphi(\mu x + y) = \bar{\mu}\varphi(x) + \varphi(y)\,.$$

2.1. Complex Conjugation of Vector Spaces and Lie Algebras

2.1.1 Definition (complex conjugate vector space).

(i) Let $V = (W, +, \cdot, \mathbb{C})$ be a complex vector space (W is the carrier space, i. e. the mere set of vectors without any structure, while V denotes the whole vector space structure). The **complex conjugate space** $\bar{V}$ is the vector space $\bar{V} = (W, +, \bar{\cdot}, \mathbb{C})$, consisting of the same carrier space and the same addition operation but with an altered multiplication:

$$\lambda \,\bar{\cdot}\, v := \bar{\lambda} \cdot v, \quad v \in \bar{V}, \lambda \in \mathbb{C}\,,$$

where $\cdot$ is the scalar multiplication of V. It is easily seen that $\bar{V}$ is again a complex vector space.

(ii) Let $\bar{~}\colon V \to \bar{V}$ be the map given by the identity map on the carrier space, id_W. It is called **complex conjugation** and is an *anti-linear* isomorphism. For $x \in V$, we denote by $\bar{x} \in \bar{V}$ the image of x under $\bar{~}$.

(iii) Let $(V, \langle\cdot,\cdot\rangle)$ be a unitary space. Then there is a canonical Hermitian scalar product $\langle\cdot,\cdot\rangle$ on $\bar{V}$ given by

$$\forall \bar{x}, \bar{y} \in \bar{V}\colon \langle \bar{x}, \bar{y}\rangle := \langle y, x\rangle = \overline{\langle x, y\rangle}\,.$$

If $(V, \langle\cdot,\cdot\rangle)$ is a Hilbert space, $(\bar{V}, \langle\cdot,\cdot\rangle)$ is again a Hilbert space.

The proof of these statements is really *very* elementary but a good exercise to get used to the overbar-notation. This is why we will nevertheless write it down here:

Proof. We shall briefly prove that $\bar{V}$ is again a $\mathbb{C}$-vector space: The calculation $\mu \mathbin{\bar{\cdot}} (\nu \mathbin{\bar{\cdot}} \bar{x}) = \mu \mathbin{\bar{\cdot}} (\overline{\bar{\nu} \cdot x}) = \overline{\bar{\mu} \cdot \bar{\nu} \cdot x} = \mu\nu \mathbin{\bar{\cdot}} \bar{x}$ shows associativity of scalar multiplication and multiplication in $\mathbb{C}$. The first distributivity law is derived by $(\mu + \nu) \mathbin{\bar{\cdot}} \bar{x} = \overline{\overline{(\mu + \nu)} \cdot x} = \overline{\bar{\mu} \cdot x + \bar{\nu} \cdot x} = \mu \mathbin{\bar{\cdot}} \bar{x} + \nu \mathbin{\bar{\cdot}} \bar{x}$, the second distributivity law by $\mu \mathbin{\bar{\cdot}} (\bar{x} + \bar{y}) = \overline{\bar{\mu} \cdot (x + y)} = \overline{\bar{\mu} \cdot x + \bar{\mu} \cdot y} = \mu \mathbin{\bar{\cdot}} \bar{x} + \mu \mathbin{\bar{\cdot}} \bar{y}$. All other vector space axioms are untouched, hence, $\bar{V}$ is a $\mathbb{C}$-vector space.

To show that complex conjugation is an anti-linear isomorphism, notice that bijectivity of $^-$ is given by construction and anti-linearity is immediate: $\overline{\mu \cdot x + \nu \cdot y} = \overline{\mu \cdot x} + \overline{\nu \cdot y} = \bar{\mu} \mathbin{\bar{\cdot}} \bar{x} + \bar{\nu} \mathbin{\bar{\cdot}} \bar{y}$.

To prove that $\langle \cdot, \cdot \rangle$ is a Hermitian scalar product on $\bar{V}$, we only have to check that it is anti-linear in the first, and linear in the second place: $\langle \lambda \mathbin{\bar{\cdot}} \bar{x}, \mu \mathbin{\bar{\cdot}} \bar{y} \rangle = \langle \overline{\bar{\lambda} \cdot x}, \overline{\bar{\mu} \cdot y} \rangle = \langle \bar{\mu} \cdot y, \bar{\lambda} \cdot x \rangle = \bar{\lambda}\mu \langle \bar{x}, \bar{y} \rangle$. As $\forall \bar{x} \in \bar{V} \colon \langle \bar{x}, \bar{x} \rangle = \langle x, x \rangle$ completesness of $(V, \langle \cdot, \cdot \rangle)$ implies completeness of $(\bar{V}, \langle \cdot, \cdot \rangle)$. ☐

2.1.2 Remark (properties of complex conjugation). Let V be a complex vector space.

(i) Notice the involutive character of complex conjugation:

$$\bar{\bar{V}} = V, \qquad \bar{\bar{x}} = x\,.$$

(ii) Let V have the basis $\{b_i\}_{i \in I}$. Then $\{\bar{b}_i\}_{i \in I}$ is a basis of $\bar{V}$ canonically associated with $\{b_i\}$. Moreover, if $J \subseteq I$ is a finite subset and $x = \sum_{i \in J} x^i b_i$ is a component representation of a vector $x \in V$, we find

$$\bar{x} = \overline{\sum_{i \in J} x^i \cdot b_i} = \sum_{i \in J} \bar{x}^i \mathbin{\bar{\cdot}} \bar{b}_i\,.$$

This means, the components of $\bar{x}$ with respect to the complex conjugate basis $\{\bar{b}_i\}$ are obtained by complex conjugation in $\mathbb{C}$. (Notice that when using implicit summation for a vector from $\bar{V}$ (now V is assumed finite-dimensional), e. g. $\bar{x} = x^\mu \bar{b}_\mu$, the scalar multiplication $\bar{\cdot}$ is used, i. e. $x^\mu \bar{b}_\mu = x^\mu \mathbin{\bar{\cdot}} \bar{b}_\mu$.)

2.1.3 Remark (isomorphisms of V and $\bar{V}$).

(i) There is *no* canonical[1] isomorphism of V and $\bar{V}$. In fact, the complex conjugation map, $^-\colon V \to \bar{V}$, would be a candidate (it is canonical because it is the identity on the carrier space), but it is anti-linear.

(ii) However, many (non-canonical) isomorphisms of V and $\bar{V}$ can be constructed by choosing a basis: For instance, let V be finite-dimensional[2] with basis $e_1, \ldots, e_n$.

[1] i. e. without any further arbitrary choices involved

[2] This constuction generalises to infinite-dimensional V but we restrict the exposition to finite-dimensional V for notational simplicity.

It canonically yields the basis $\bar{e}_1, \dots, \bar{e}_n$ of $\bar{V}$. Now, a linear map $\psi\colon V \to \bar{V}$ can be constructed just by linear extension of the mapping $\psi\colon e_i \mapsto \bar{e}_i$:

$$\psi(a^{\mu} \cdot e_{\mu}) := a^{\mu} \,\bar{\cdot}\, \bar{e}_{\mu} \,.$$

Check that this yields an isomorphism but which depends decisively on the choice of basis.

We recall from elementary functional analysis:

2.1.4 Remark (Riesz isomorphism).

(i) Let $(V, \langle \cdot, \cdot \rangle)$ be a Hilbert space. Then there is a canonical *anti-linear* isomorphism (Riesz's isomorphism)

$$\begin{aligned} R\colon V &\to V^* \\ x &\mapsto \langle x, \cdot \rangle \,, \end{aligned}$$

where V^* denotes the (continuous) dual space of V. Antilinearity can be seen from $R(\lambda x) = \langle \lambda x, \cdot \rangle = \bar{\lambda} \langle x, \cdot \rangle = \bar{\lambda} R(x)$.

(ii) We use the Riesz isomorphism R to induce a Hermitian scalar product on V^* by setting

$$\forall \varphi, \psi \in V^*\colon \langle \varphi, \psi \rangle := \langle R^{-1}(\psi), R^{-1}(\varphi) \rangle \,.$$

"Swapping the arguments" is necessary because R is anti-linear.

2.1.5 Remark (the complex conjugate of the dual space). Let V be a Hilbert space. Using the Riesz isomorphism R and complex conjugation $^-$, we obtain the following commutative diagram, where solid arrows are canonical *anti-linear* isomorphisms and dashed arrows are induced *linear* isomorphisms:

$$\begin{tikzcd}
 & \bar{V} \arrow[r, "R"] \arrow[dd, dashed] & \bar{V}^* \arrow[dd, dashed] \\
V \arrow[ur, "-"] \arrow[dr, "R"'] \arrow[urr, dashed] \arrow[drr, dashed] & & \\
 & V^* \arrow[r, "-"'] & \overline{V^*}
\end{tikzcd}$$

This shows, that

- $\bar{V}^* \cong \overline{V^*}$ linearly and canonically, i. e. complex conjugation and forming the (continuous) dual space commute;
- $V \cong \bar{V}^*$ linearly and canonically;
- $\bar{V} \cong V^*$ linearly and canonically.

Notice that this only works for Hilbert spaces V because otherwise we couldn't use Riesz's isomorphism as a *canonical* anti-linear isomorphism.

2.1.6 Remark (tensor products and complex conjugation).

(i) Let V and W be complex vector spaces. The mapping $x \otimes y \mapsto \bar{x} \otimes \bar{y}$ extends to a canonical anti-linear isomorphism $V \otimes W \to \bar{V} \otimes \bar{W}$ given by

$$\sum_i \lambda_i (x_i \otimes y_i) \mapsto \sum_i \overline{\lambda_i \cdot x_i} \otimes \bar{y}_i = \sum_i \bar{\lambda}_i (\bar{x}_i \otimes \bar{y}_i) .$$

Using this together with ordinary complex conjugation $V \otimes W \to \overline{V \otimes W}$, we obtain the following commutative diagram, where solid arrows are canonical *anti-linear* isomorphisms and the dashed arrow is an induced *linear* isomorphism:

$$\begin{tikzcd} & \bar{V} \otimes \bar{W} \arrow[d, dashed, "\cong"] \\ V \otimes W \arrow[ur] \arrow[dr] & \\ & \overline{V \otimes W} \end{tikzcd}$$

Thus, there is a canonical linear isomorphism $\bar{V} \otimes \bar{W} \cong \overline{V \otimes W}$ by which we shall henceforth identify both spaces. (Notice that in this remark, as both V and W are $\mathbb{C}$-vector spaces, by $V \otimes W$ we mean $V \otimes_{\mathbb{C}} W$.)

(ii) Let now V be an $\mathbb{R}$-vector space and $V \otimes \mathbb{C} = V \otimes_{\mathbb{R}} \mathbb{C}$ its complexification (we consider $V \otimes \mathbb{C}$ a complex vector space). Then the canonical mapping induced by

$$\begin{aligned} V \otimes \mathbb{C} &\to V \otimes \bar{\mathbb{C}} \\ v \otimes z &\mapsto v \otimes \bar{z} \end{aligned}$$

defines an anti-linear isomorphism: For $\lambda \in \mathbb{C}$,

$$\lambda \cdot (x \otimes z) = x \otimes (\lambda \cdot z) \mapsto x \otimes \overline{\lambda \cdot z} = x \otimes (\bar{\lambda} \cdot \bar{z}) = \bar{\lambda} \cdot (x \otimes \bar{z}) .$$

Using this and ordinary complex conjugation $V \otimes \mathbb{C} \to \overline{V \otimes \mathbb{C}}$ yields the following commutative diagram, where solid arrows are canonical *anti-linear* isomorphisms and the dashed arrow is an induced *linear* isomorphism:

$$\begin{tikzcd} & V \otimes \bar{\mathbb{C}} \arrow[d, dashed, "\cong"] \\ V \otimes \mathbb{C} \arrow[ur] \arrow[dr] & \\ & \overline{V \otimes \mathbb{C}} \end{tikzcd}$$

Thus, there is a canonical linear isomorphism $\overline{V \otimes_{\mathbb{R}} \mathbb{C}} \to V \otimes_{\mathbb{R}} \bar{\mathbb{C}}$, by which we shall henceforth identify both spaces.

2.1.7 Definition and proposition (complex conjugate Lie algebra). Let $(V, [\cdot,\cdot])$ be a complex Lie algebra.

(i) $\bar{V}$ together with the Lie bracket

$$[\bar{x}, \bar{y}] := \overline{[x, y]} , \tag{$*$}$$

is a Lie algebra, called the **complex conjugate Lie algebra**.

(ii) Let V be finite-dimensional and choose a basis $e_1, \dots, e_n$. The isomorphism $\varphi\colon V \to \bar{V}$, induced by linear extension of $\varphi\colon e_i \mapsto \bar{e}_i$, is an isomorphism of Lie algebras if and only if the structure constants[3] of $[\cdot,\cdot]$ with respect to this basis are real.

Proof.

(i) We have to prove that $(*)$ indeed defines a Lie bracket on $\bar{V}$. The only non-obvious point is bilinearity under scalar multiplications. Let $\bar{x}, \bar{y} \in \bar{V}$:

$$[\mu \mathbin{\bar{\cdot}} \bar{x}, \nu \mathbin{\bar{\cdot}} \bar{y}] = [\overline{\bar{\mu} \cdot x}, \overline{\bar{\nu} \cdot y}] = \overline{[\bar{\mu} \cdot x, \bar{\nu} \cdot y]} = \overline{\bar{\mu}\bar{\nu} \cdot [x, y]} = \mu\nu \mathbin{\bar{\cdot}} [\bar{x}, \bar{y}]\,.$$

(ii) We have to show that φ respects the Lie bracket, i. e.

$$\forall i, j\colon \varphi([e_i, e_j]) = [\bar{e}_i, \bar{e}_j]\,, \tag{$**$}$$

if and only if the structure constants C^k_{ij} of $[\cdot,\cdot]$ with respect to e_i are real.
For the left side of $(**)$ we find:

$$\varphi([e_i, e_j]) = \varphi(C^\mu_{ij} \cdot e_\mu) = C^\mu_{ij} \mathbin{\bar{\cdot}} \bar{e}_\mu\,.$$

For the right side of $(**)$ we find:

$$[\bar{e}_i, \bar{e}_j] = \overline{[e_i, e_j]} = \overline{C^\mu_{ij} \cdot e_\mu} = \overline{C^\mu_{ij}} \mathbin{\bar{\cdot}} \bar{e}_\mu$$

Hence, $(**)$ holds if and only if $\overline{C^k_{ij}} = C^k_{ij}$.

□

2.2. Complex Conjugation of Linear Maps and Representations

2.2.1 Definition (complex conjugate linear map). Let V and W be complex vector spaces and $A\colon V \to W$ a linear map. Define the **complex conjugate linear map** $\bar{A}\colon \bar{V} \to \bar{W}$ by

$$\forall x \in V\colon \bar{A}(\bar{x}) = \overline{A(x)}\,,$$

or, in other words: $\bar{A} = {}^{-} \circ A \circ ({}^{-})^{-1}$. Notice that $\bar{A}$ is again $\mathbb{C}$-linear.

2.2.2 Proposition (properties of complex conjugation of linear maps).

(i) The mapping

$$\begin{aligned} {}^{-}\colon \operatorname{Hom}(V, W) &\to \operatorname{Hom}(\bar{V}, \bar{W}) \\ A &\mapsto \bar{A} \end{aligned}$$

is an antilinear isomorphism. If V and W are unitary spaces and $A \in \operatorname{Hom}(V, W)$ is unitary, then so is $\bar{A}$.

[3]The structure constants of a Lie bracket $[\cdot,\cdot]$ with respect to a Lie algebra basis $e_1, \dots, e_n$ are the numbers C^k_{ij} given by $[e_i, e_j] = C^\mu_{ij} e_\mu$.

(ii) Using complex conjugation of linear maps and complex conjugation of vector spaces we obtain the following commutative diagram, where the solid arrows are canonical *anti-linear* isomorphisms and the dashed arrow is an induced *linear* isomorphism:

$$\begin{tikzcd}
 & \mathrm{Hom}(\bar V, \bar W) \arrow[d, dashed, "\cong"] \\
\mathrm{Hom}(V, W) \arrow[ur, "\bar{\ }"] \arrow[dr, "\bar{\ }"'] & \\
 & \overline{\mathrm{Hom}(V, W)}
\end{tikzcd}$$

Thus, there is a canonical isomorphism $\mathrm{Hom}(\bar V, \bar W) \cong \overline{\mathrm{Hom}(V, W)}$ by which we shall henceforth identify both spaces.

(iii) If V and W are finite-dimensional and if e_i resp. f_i are bases of V resp. W with corresponding bases $\bar e_i$ resp. $\bar f_i$ of $\bar V$ resp. $\bar W$, the matrix $[\bar A]$ of $\bar A$ is the complex conjugate of the matrix $[A]$ of A with respect to these bases: $[\bar A] = \overline{[A]}$.

Proof.

(i) Let $A \in \mathrm{Hom}(V, W)$, $\lambda \in \mathbb{C}$, $v \in V$. Then we find:

$$\overline{(\lambda A)}(\bar v) = \overline{\lambda A(v)} = \overline{\lambda \cdot A(v)} = \bar\lambda \,\bar\cdot\, \overline{A(v)} = (\bar\lambda \bar A)(\bar v)\,,$$

and obviously, $\bar A + \bar B = \overline{A + B}$. Thus, $\bar{\ }\colon \mathrm{Hom}(V, W) \to \mathrm{Hom}(\bar V, \bar W)$ is anti-linear: $\overline{\lambda A} = \bar\lambda \bar A$. Bijectivity follows by the obvious involutivity of $\bar{\ }$: $(\bar{\ })^2 = \mathrm{Id}$.
If A is unitary, that means, $\forall x, y \in V\colon \langle Ax, Ay\rangle = \langle x, y\rangle$, then we find:

$$\forall \bar x, \bar y \in \bar V\colon \langle \bar A \bar x, \bar A \bar y\rangle = \langle \overline{Ax}, \overline{Ay}\rangle = \langle Ay, Ax\rangle = \langle y, x\rangle = \langle \bar x, \bar y\rangle\,,$$

i. e. $\bar A$ is unitary with respect to the Hermitian inner products on $\bar V$ and $\bar W$.

(ii) There is nothing to prove here.

(iii) $\bar A(\bar e_i) = \overline{A(e_i)} = \overline{\sum_j a_{ji} \cdot f_j} = \sum_j \bar a_{ji} \,\bar\cdot\, \bar f_j$, where $[A]_{kl} =: a_{kl}$.

□

2.2.3 Definition and proposition (complex conjugate representation).

(i) Let G be a real Lie group[4] and ρ a complex representation of G on a finite-dimensional unitary space V. Then

$$\begin{aligned} \bar\rho\colon G &\to \mathrm{GL}(\overline{V}) \\ g &\mapsto \overline{\rho(g)} \end{aligned}$$

defines a smooth respresentation of G on $\overline{V}$, the **complex conjugate representation of ρ.** If ρ is unitary, so is $\bar\rho$.

[4]This definition would not work in the case of a complex Lie group with a representation which is considered a $\mathbb{C}$-Lie group homomorphism, because complex conjugation is not holomorphic. To resolve this problem one may either introduce the notion of complex conjugation of complex Lie groups or consider *real* Lie groups and stick to representations of real Lie groups only. We adopt the second approach in this thesis, which seems more appropriate in our physical context.

(ii) Let τ be a complex representation of a complex Lie algebra $\mathcal{A}$ on a finite-dimensional unitary space V. Then the complex conjugate Lie algebra homomorphism

$$\begin{aligned} \bar{\tau} \colon \bar{\mathcal{A}} &\to \overline{\mathfrak{gl}(V)} \cong \mathfrak{gl}(\overline{V}) \\ \bar{X} &\mapsto \overline{\tau(X)} \end{aligned}$$

defines a representation of the complex conjugate Lie algebra $\bar{\mathcal{A}}$ on $\overline{V}$, called **the complex conjugate representation**[5] of τ, where $\bar{\mathcal{A}}$ denotes the complex conjugate Lie algebra of $\mathcal{A}$.

Proof.

(i) That $\bar{\rho}$ is again a representation follows from $^{-}\colon \mathrm{GL}(V) \to \mathrm{GL}(\overline{V})$ being a homomorphism of real Lie groups. That $\bar{\rho}$ is unitary if ρ is unitary follows directly from proposition 2.2.2-(i).

(ii) As $\mathfrak{gl}(V) = \mathrm{End}(V)$ and, canonically, $\overline{\mathrm{End}(V)} \cong \mathrm{End}(\bar{V})$ (cf. 2.2.2-ii), we find $\overline{\mathfrak{gl}(V)} \cong \mathfrak{gl}(\bar{V})$.

□

2.2.4 Remark. Let ρ be a complex representation of a real Lie algebra $\mathcal{A}$. In general, ρ and $\bar{\rho}$ are not equivalent. A counter example will be demonstrated in remark 3.3.7.

[5]Notice that this does not give a representation of $\mathcal{A}$ itself (only of $\bar{\mathcal{A}}$), because it would be antilinear otherwise.

3. Representation Theory of $\mathrm{SL}(2, \mathbb{C})$

Premises and choices of bases

Throughout this section and for the rest of this document, the following choices of bases shall be assumed silently, if not stated otherwise.

(a) Let $(\mathbb{R}^4, \eta)$ the real 4-dimensional vector space $\mathbb{R}^4$ equipped with a Lorentz scalar product η of signature $(+---)$. Let e_0, e_1, e_2, e_3 be a choice of η-pseudo-orthonormal basis of $\mathbb{R}^4$, such that $\eta(e_0, e_0) = +1$. We shall henceforth refer to this basis as **standard basis of** $(\mathbb{R}^4, \eta)$**.** We refer to $(\mathbb{R}^4, \eta)$ as the **real Minkowski vector space.**

(b) Let the **complexified Minkowski vector space** be denoted by

$$(\mathbb{C}^4, \eta) := (\mathbb{R}^4, \eta)_{\mathbb{C}} := (\mathbb{R}^4, \eta) \otimes \mathbb{C} = (\mathbb{C}^4, \eta_{\mathbb{C}}) \,,$$

where $\eta_{\mathbb{C}}$ is the $\mathbb{C}$-bilinear extension of η. For $\mathbb{C}^4$ we take the same standard basis as for $\mathbb{R}^4 \subseteq \mathbb{C}^4$.

(c) Let $\mathbb{R}^3$ denote the Euklidean subspace $\mathbb{R}^3 = \langle e_1, e_2, e_3 \rangle_{\mathbb{R}} \subseteq \mathbb{R}^4$. We take e_1, e_2, e_3 as standard basis and equip $\mathbb{R}^3$ with the standard Euklidean scalar product $\langle \cdot, \cdot \rangle$ with respect to e_i (hence, we have $\langle x, y \rangle = -\eta(x, y)$ for alle $x, y \in \mathbb{R}^3$).

(d) Let $(\mathbb{C}^2, (\cdot, \cdot))$ be the 2-dimensional complex vector space equipped with a Hermitian scalar product $(\cdot, \cdot)$. Let E_1, E_2 be a fixed choice of orthonormal basis which we shall henceforth refer to as **standard basis of** $(\mathbb{C}^2, (\cdot, \cdot))$**.**

(e) Whenever an element of $\mathbb{R}^4$, $\mathbb{C}^4$, $\mathbb{R}^3$ or $\mathbb{C}^2$ is written as components without basis, e. g. $x = x^\mu$, components with respect to the standard bases (as declared here) are to be understood. Whenever endomorphisms of $\mathbb{R}^4$, $\mathbb{C}^4$, $\mathbb{R}^3$ or $\mathbb{C}^2$ are given as matrix without a specified basis, the standard bases are to be understood.

(f) Using the standard bases as declared here, we will freely swap between understanding the elements of $\mathcal{L}$, $\mathrm{SO}(3)$, $\mathrm{SU}(2)$ or $\mathrm{SL}(2, \mathbb{C})$ as endomorphisms on $(\mathbb{R}^4, \eta)$, $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$ or $(\mathbb{C}^2, (\cdot, \cdot))$ or as matrices (describing these endomorphisms with respect to the standard bases).

It should be emphasised that whenever we say *standard basis* in this document, we mean the bases declared here, not *the* standard bases. Of course, *the* standard bases could be one possible choice for our standard bases. But they don't have to be chosen; all we ask for is a fixed choice of (pseudo-)orthonormal bases.

3.1. Review: Representation Theory of $\mathrm{SU}(2)$

Before entering into the classification of irreducible representations of $\mathrm{SL}(2, \mathbb{C})$, we shall give a brief review of the $\mathrm{SU}(2)$- and $\mathrm{SO}(3)$-representation theory as known from quantum mechanics and fix some notation.

Lie Algebras of SO(3) and SU(2)

We shall start out by collecting some core facts about SO(3), SU(2) and their Lie algebras from the literature.

3.1.1 Reminder (Lie algebra of SO(3)). The Lie algebra of SO(3) is the real vector space spanned by

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \Lambda_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

That means, for every $0 \neq x = x^\mu \in \mathbb{R}^3$,

$$\gamma_x \colon t \mapsto \exp(t\, x^\mu \Lambda_\mu)$$

defines a one parameter subgroup of SO(3). The scaling is chosen such that if $\|x\| = 1$,

$$\gamma_x(t) = \mathbb{1} \Rightarrow t = 0, \pm 2\pi, \pm 4\pi, \ldots .$$

Notice that Λ_i is the generator of rotations around the axis spanned by e_i. All Λ_μ are anti-symmetric and $\mathfrak{so}(3)$ is the Lie algebra of real antisymmetric 3×3 matrices. The Lie bracket is given by the matrix commutator. For the generators we find:

$$[\Lambda_i, \Lambda_j] = \varepsilon_{ijk}\, \Lambda_k\, .$$

Proof. All this may be found in [CM00, section 2.2], [Str02, chapter 4.1], or [SU01, chapter 7.2]. □

3.1.2 Reminder (Lie algebra of SU(2) and Pauli matrices).

(i) $\mathfrak{su}(2)$, the Lie algebra of SU(2), is the real vector space spanned by the matrices

$$s_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad s_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad s_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

That means, for every $x = x^\mu \in \mathbb{R}^3$,

$$\gamma_x \colon t \mapsto \exp(t\, x^\mu s_\mu)$$

defines a one parameter subgroup of SU(2). The scaling is chosen such that if $\|x\| = 1$, $\gamma_x(2\pi) = -\mathbb{1}$ and $\gamma_x(4\pi) = \mathbb{1}$ (which is desirable in light of proposition 3.1.3-ii).

The Lie bracket of this Lie algebra is given by the matrix commutator. For the generators we find:

$$[s_i, s_j] = \varepsilon_{ijk}\, s_k\, .$$

The matrices s_μ are anti-hermitian ($s_\mu^\dagger = -s_\mu$) and traceless. The Lie algebra $\mathfrak{su}(2)$ is the Lie algebra of complex traceless anti-hermitian 2×2-matrices.

(ii) The matrices

$$\tilde{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tilde{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are called **Pauli spin matrices** (we shall use the tilde in oder to clearly distinguish the Pauli spin matrices $\tilde{\sigma}_\mu$ from the σ-tensor-spinor ${\sigma_a}^{A\dot{B}}$ introduced in definition 4.2.1). The Pauli spin matrices are both unitary ($\tilde{\sigma}_\mu^\dagger = \tilde{\sigma}_\mu^{-1}$) and hermitian ($\tilde{\sigma}_\mu^\dagger = \tilde{\sigma}_\mu$). They are related to the s_μ by

$$s_\mu = \frac{-i}{2}\tilde{\sigma}_\mu \,.$$

The matrices $\frac{1}{2}\tilde{\sigma}_\mu = is_\mu$ are sometimes called **Hermitian generators of $\mathfrak{su}(2)$ associated with** s_μ and are often used in physical contexts. For $x = x^\mu \in \mathbb{R}^3$, $\|x\| = 1$, we find

$$\gamma_x(t) = \exp(t\, x^\mu s_\mu) = \exp(\tfrac{-it}{2}\, x^\mu \tilde{\sigma}_\mu) \,.$$

The commutation relations of the Pauli matrices read:

$$\left[\tilde{\sigma}_i, \tilde{\sigma}_j\right] = 2i\varepsilon_{ijk}\,\tilde{\sigma}_k \,.$$

(iii) Consider the Pauli matrices $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$ together with the identity matrix,

$$\tilde{\sigma}_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

The real linear span of $\tilde{\sigma}_0, \dots, \tilde{\sigma}_3$ is the real 4-dimensional vector space of complex hermitian 2×2-matrices. The real linear span of $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$ is the real 3-dimensional vector space of complex *traceless* hermitian 2×2-matrices.

The complex linear span of $\tilde{\sigma}_0, \dots, \tilde{\sigma}_3$ is the complex 4-dimensional vector space of all 2×2-matrices, $\mathrm{Mat}_{2\times 2}(\mathbb{C})$. The complex linear span of $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3$ is the complex 3-dimensional vector space of *traceless* 2×2-matrices.

Proof. These results will be found in [CM00, section 2.2], [Str02, chapter 4.2], and most standard textbooks on quantum mechanics. □

3.1.3 Proposition (adjoint representations and the universal covering).

(i) Consider the isomorphism of vector spaces

$$\begin{aligned} \theta \colon \mathbb{R}^3 &\to \mathfrak{so}(3) \\ x = x^\mu &\mapsto x^\mu \Lambda_\mu \,. \end{aligned}$$

Notice that if the anti-symmetric matrices Λ_μ and their linear combinations are visualised as 2-dimensional oriented planes in $\mathbb{R}^3$, $\theta(x)$ for $x \in \mathbb{R}^3$ is the oriented plane with normal vector x.

The adjoint representation[1] of SO(3),

$$\begin{aligned}\mathrm{Ad}\colon \mathrm{SO}(3) &\to \mathrm{Aut}(\mathfrak{so}(3))\\ R &\mapsto [\xi \mapsto R\xi R^{-1}]\,,\end{aligned}$$

yields by concatenation with θ^{-1} a representation of SO(3) on $\mathbb{R}^3$. This representation happens to coincide with the *defining representation* of SO(3), i. e. for $x \in \mathbb{R}^3$ and $R \in \mathrm{SO}(3)$:

$$\theta^{-1}\,\mathrm{Ad}_R(\theta(x)) = R(x)\,.$$

(ii) Consider the isomorphism of vector spaces

$$\begin{aligned}\theta'\colon \mathbb{R}^3 &\to \mathfrak{su}(2)\\ x = x^\mu &\mapsto x^\mu s_\mu\,.\end{aligned}$$

The adjoint representation of SU(2),

$$\begin{aligned}\mathrm{Ad}\colon \mathrm{SU}(2) &\to \mathrm{Aut}(\mathfrak{su}(2))\\ U &\mapsto [\xi \mapsto U\xi U^{-1}]\,,\end{aligned}$$

yields by concatenation with θ'^{-1} a representation

$$\varphi\colon U \mapsto \theta'^{-1} \circ \mathrm{Ad}_U \circ \theta' \in \mathrm{GL}(\mathbb{R}^3)$$

of SU(2) on $\mathbb{R}^3$. It turns out that for all $U \in \mathrm{SU}(2)$, $\varphi(U) \in \mathrm{SO}(3)$, i. e. we have the representation

$$\begin{aligned}\varphi\colon \mathrm{SU}(2) &\to \mathrm{SO}(3)\\ U &\mapsto \theta'^{-1} \circ \mathrm{Ad}_U \circ \theta'\,.\end{aligned}$$

Moreover, it turns out that φ is a *universal* covering of SO(3). It is two-fold, meaning that it has kernel $\ker\varphi = \{\mathbb{1}, -\mathbb{1}\}$.

Proof. (i) is, on the matrix level, a simple and well known thing. However, on the abstract level (talking about adjoint and defining representations) the author found this result shine in a new and surprising light (isn't it remarkable that in case of SO(3), adjoint and defining representations coincide?) ...

The same goes for (ii). On the matrix level, this is well known (for a full proof cf. [Str02, chapter 4.2]), but the enlightening connection with the adjoint representation is rarely anywhere explicated.

Notice however, that instead of our isomorphism θ', physicists often use the map $x^\mu \mapsto x^\mu \tilde{\sigma}_\mu$. This has the major disadvantage that it no longer gives an isomorphism $\mathbb{R}^3 \to \mathfrak{su}(2)$ (because it doesn't even map into $\mathfrak{su}(2)$) and hence doesn't illuminate the connection of the double covering with the adjoint representation. But, on the other hand, it has the major advantage of being compatible with the $\mathfrak{sl}(2,\mathbb{C})$-case, as will be seen in proposition 3.2.3 and section 4.2. □

[1] Notice that for linear Lie groups, the adjoint representation is given by matrix conjugation.

For future references, we just write down the result on the universal covering homomorphism once again in a more concentrated form:

3.1.4 Corollary (universal covering homomorphism of SO(3)). There is a 2-fold covering map $\mathrm{SU}(2) \to \mathrm{SO}(3)$ which is a *universal* covering of SO(3). Such a map may be given as

$$\begin{aligned} \varphi\colon \mathrm{SU}(2) &\to \mathrm{SO}(3) \\ U &\mapsto \theta'^{-1} \circ \mathrm{Ad}_U \circ \theta' , \end{aligned}$$

$\theta'\colon \mathbb{R}^3 \to \mathfrak{su}(2)$, $x = x^\mu \mapsto x^\mu s_\mu$. By differentiating, this yields an isomorphism of the Lie algebras, $\hat{\varphi}\colon \mathfrak{su}(2) \to \mathfrak{so}(3)$. The generators s_μ of $\mathfrak{su}(2)$ and Λ_μ of $\mathfrak{so}(3)$ are chosen such that

$$\hat{\varphi}\colon s_\mu \mapsto \Lambda_\mu, \quad \mu = 1, 2, 3 .$$

(For notation and terminology cf. 3.1.1, 3.1.2 and 3.1.3.)

Proof. The statement about $\hat{\varphi}$ is proved in [Str02, chapter 3.2]. □

3.1.5 Remark (comment on the choice of Lie algebra bases). We shall make a brief comment on the interdependency of the choices of bases in this section: Firstly, notice that our choice of basis $\{\Lambda_1, \Lambda_2, \Lambda_3\}$ of $\mathfrak{so}(3)$ was naturally induced by our fixed standard basis $\{e_1, e_2, e_3\}$ of the Euklidean space $\mathbb{R}^3$ by demanding that Λ_i be the generator of positively oriented rotation around the i-th base vector (with respect to the orientation induced by $\{e_\mu\}$).

Secondly, once given a choice of $\{\Lambda_\mu\}$, the choice of $\{s_\mu\}$ (basis of $\mathfrak{su}(2)$) as given in reminder 3.1.2 is justified by propositions 3.1.4 and 3.1.3-ii: With our choice of $\{s_\mu\}$, the induced isomorphism $\theta'\colon \mathbb{R}^3 \to \mathfrak{su}(2)$ is particularly related to the double covering $\varphi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$, as shown in 3.1.3-ii.

In short, $\{\Lambda_\mu\}$ and $\{s_\mu\}$ are chosen such that the relations given in propositions 3.1.3 look as neat and simple as they do.

Irreducible Representations of SO(3) and SU(2)

Pulling back a representation of SO(3) through the universal covering homomorphism $\varphi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ yields a representation of SU(2). Thus, as φ is surjective, every irreducible representation of SO(3) gives an irreducible representation of SU(2). As SU(2) is simply connected, its complex irreducible representations are in one-one-correspondence with complex irreducible represenatations of $\mathfrak{su}(2)$ by corollary 1.1.6-b.

This is why the procedure of classifying irreducible complex Hilbert space representations of SO(3) and SU(2), as known e. g. from an introductory course on quantum mechanics, starts from classifying irreducible representations of the Lie algebra, $\mathfrak{su}(2) \cong \mathfrak{so}(3)$. This then gives the equivalence classes of complex irreducible representations of SU(2) itself, and the only remaining thing to do is sorting out, which of them push forward to representations of SO(3) (a priori, not all of them do!).

The next theorem sums up the result of such a classification procedure. Notice that we refrain from introducing the concept of weights here, just to spare the space and time of presenting yet another general concept without ever making use of this level of generality in our context. Instead, we shall stick to the terminology common in physics, where irreducible SU(2)-representations get labeled by their "spin numbers". Of course, the involved reader will think of twice the spin number as the representation's highest weight.

3.1.6 Theorem (irreducible SU(2)**- and** SO(3)**-representations).**

(i) The collection of equivalence classes of irreducible complex Hilbert space representations of SU(2) can be labeled by numbers

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

such that

$$\dim(D^{(j)}) = 2j + 1\,,$$

where $D^{(j)}$ denotes a representative of the equivalence class labeled by j. The number j is called **spin number of** $D^{(j)}$**.** In particular, all irreducible complex Hilbert space representations of SU(2) are finite-dimensional.

(ii) One finds that

$$D^{(j)}(2\pi\, s_i) = D^{(j)}(-\mathbb{1}) = (-1)^{2j}\mathbb{1}\,.$$

Thus, an irreducible complex Hilbert space representation of SU(2) pushes forward to an irreducible complex Hilbert space representation of SO(3) if and only if its spin number j is integral, $j = 0, 1, 2, \ldots$. Consequently, the equivalence classes of irreducible Hilbert space representations of SO(3) can be labeled by spin numbers

$$j = 0, 1, 2, \ldots$$

and again with

$$\dim(D^{(j)}) = 2j + 1\,.$$

Proof. All this is well known from introductory courses on quantum mechanics. Cf. [Str02, section 4.5] for an explicit (physical) treatment and [BD85, proposition II.5.3] for a mathematical reference. □

We quote the useful Clebsch-Gordon formula for tensor products of irreducible SU(2)-representations:

3.1.7 Theorem (Clebsch-Gordon decomposition). Let $j_1, j_2 \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ and let $D^{(j)}$ denote a representative of the equivalence class of irreducible complex SU(2)-representations of spin number j. Then we find for the tensor product representation of $D^{(j_1)}$ and $D^{(j_2)}$:

$$D^{(j_1)} \otimes D^{(j_2)} \cong D^{(j_1+j_2)} \oplus D^{(j_1+j_2-1)} \oplus \ldots \oplus D^{(|j_1-j_2|)}\,.$$

Proof. Cf. [BD85, proposition II.5.5] and [Str02, section 4.5]. $\square$

This shows that $D^{(\frac{1}{2})}$ is the *fundamental representation* of $\mathrm{SU}(2)$, meaning that all other irreducible complex representations of $\mathrm{SU}(2)$ are subrepresentations of tensor product powers of $D^{(\frac{1}{2})}$. The next proposition explicates what these subrepresentations look like and uses these circumstances to define concrete representatives for the equivalence classes of irreducible complex $\mathrm{SU}(2)$ representations for the rest of the document:

3.1.8 Definition and proposition ($D^{(j)}$ and Δ_j).

(a) We set $\Delta_{\frac{1}{2}} := \mathbb{C}^2$ and equip this space with the standard basis. Then the equivalence class of irreducible complex $\mathrm{SU}(2)$ representations of spin number $\frac{1}{2}$ shall henceforce be represented by $D^{(\frac{1}{2})}$ on $\Delta_{\frac{1}{2}}$, given by

$$D^{(\frac{1}{2})}(S)(x) := Sx = S^{\nu}{}_{\mu}x^{\mu}, \qquad S \in \mathrm{SU}(2),\ x \in \Delta_{\frac{1}{2}}\,.$$

(b) For each $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$, the equivalence class of irreducible complex $\mathrm{SU}(2)$ representations of spin number j is represendet by

$$D^{(j)} := \left(D^{(\frac{1}{2})}\right)^{\otimes 2j}\Big|_{(\Delta_{\frac{1}{2}})^{\vee 2j}} \quad \text{on} \quad \Delta_j := (\Delta_{\frac{1}{2}})^{\vee 2j}\,.$$

Here, $\vee$ denotes the symmetrised tensor product; thus, $D^{(j)}$ is the restriction of $(D^{(\frac{1}{2})})^{\otimes 2j}$ to the totally symmetric subspace $(\Delta_{\frac{1}{2}})^{\vee 2j} \subseteq (\Delta_{\frac{1}{2}})^{\otimes 2j}$. We equip Δ_j with the induced basis.

Proof. (b) may be found in [SU01, p. 244]. $\square$

3.2. Lie Algebras of $\mathcal{L}$ and $\mathrm{SL}(2,\mathbb{C})$

In order to establish notation and provide a solid basement for all subsequent considerations we shall collect some core facts about $\mathrm{SL}(2,\mathbb{C})$, $\mathcal{L}$, and their Lie algebras.

3.2.1 Reminder (Lie algebra of $\mathcal{L}$). $\mathfrak{lo}$, the Lie algebra of the Lorentz group $\mathcal{L}$, is the real vector space spanned by the matrices

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \ulcorner & & \urcorner \\ 0 & & \Lambda_1 & \\ 0 & \llcorner & & \lrcorner \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \ulcorner & & \urcorner \\ 0 & & \Lambda_2 & \\ 0 & \llcorner & & \lrcorner \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \ulcorner & & \urcorner \\ 0 & & \Lambda_3 & \\ 0 & \llcorner & & \lrcorner \end{pmatrix},$$

$$N_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

M_1, M_2, M_3 are, of course, the generators of the $\mathrm{SO}(3)$ subgroup of $\mathcal{L}^{\uparrow}_{+}$, and are called generators of (spacial) rotations. N_1, N_2, N_3 are the generators of Lorentz boosts. The

Lie bracket is given by the matrix commutator. The commutation relations for the generators read:

$$[M_i, M_j] = \varepsilon_{ijk}\, M_k, \quad [N_i, N_j] = -\varepsilon_{ijk}\, M_k, \quad [M_i, N_j] = \varepsilon_{ijk}\, N_k\,.$$

Proof. Cf. [CM00, section 3.3]. These results may also be found in [SU01, section 8.1], however, the opposite sign for the N_μ is used there. □

3.2.2 Reminder (Lie algebra of SL$(2,\mathbb{C})$). SL$(2,\mathbb{C})$ is a complex 3-dimensional Lie group. Its Lie algebra, $\mathfrak{sl}(2,\mathbb{C})$, has complex basis s_1, s_2, s_3 (notation cf. rem. 3.1.2), and hence,

$$\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2) \otimes \mathbb{C}$$

as complex Lie algebras. Considered as real Lie group, SL$(2,\mathbb{C})$ is 6-dimensional and

$$\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} = \langle s_1, s_2, s_3, (is_1), (is_2), (is_3)\rangle_{\mathbb{R}}\,,$$

where $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ denotes $\mathfrak{sl}(2,\mathbb{C})$ considered as real Lie algebra. The commutation relations of the generators read

$$[s_i, s_j] = \varepsilon_{ijk}\, s_k, \quad [(is_i), (is_j)] = -\varepsilon_{ijk}\, s_k, \quad [(is_i), s_j] = \varepsilon_{ijk}\,(is_k)\,.$$

Proof. Cf. [CM00, sections 3.4 and 4.2] for the generators (formula 4.34; notice that the opposite sign is used there). The commutation relations are immediate using 3.1.2. □

3.2.3 Proposition (universal covering homomorphism of $\mathcal{L}_+^\uparrow$). There is a 2-fold covering map $\lambda\colon \mathrm{SL}(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow$ which is a universal covering of $\mathcal{L}_+^\uparrow$. It gets realised by

$$\lambda\colon S \mapsto [x \mapsto \theta^{-1}(S\,\theta(x)\,S^\dagger)], \quad x \in \mathbb{R}^4\,,$$

where $\theta\colon \mathbb{R}^4 \to \langle\tilde{\sigma}_0, \dots, \tilde{\sigma}_3\rangle_{\mathbb{C}} \subseteq \mathrm{Mat}_{2\times 2}(\mathbb{R})$ is the map given by

$$\theta\colon x \mapsto X := x^\mu \tilde{\sigma}_\mu, \quad \mu = 0, \dots, 3$$

(we shall meet θ again in section 4.2, cf. definition 4.2.1 for a more detailed treatment). λ induces an isomorphism $\lambda_*\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{lo}$. Notice that

$$\lambda_*(s_i) = M_i, \quad \lambda_*(is_i) = N_i, \quad i = 1, 2, 3\,.$$

Proof. [CM00, section 3.4]. □

3.2.4 Remark (comment on the choice of bases). In perfect analogy with remark 3.1.5, a choice of metric and oriented orthonormal basis for $\mathbb{R}^4$ (as committed by fixing the standard basis in our case) induces a natural choice of basis $M_1, M_2, M_3, N_1, N_2, N_3$ of $\mathfrak{lo}$ by demanding that M_i be the generator of positive spacial rotation around e_i and N_i be the generator of a Lorentz boost in the e_0-e_i-coordinate plane.

In consequence, a choice of basis of $\mathfrak{sl}(2,\mathbb{C})$ as given in reminder 3.2.2 is justified by proposition 3.2.3: The s_i and is_i are just the pullbacks of M_i and N_i under the double covering, λ.

3.3. Representation Theory of $\mathrm{SL}(2,\mathbb{C})$

Towards our aim of classifying finite-dimensional irreducible complex representations of $\mathrm{SL}(2,\mathbb{C})$ (as *real* Lie group) we shall be pursuing the plan of extending the well known representation theory of $\mathrm{SU}(2)$ to $\mathrm{SL}(2,\mathbb{C})$ in a way using complex conjugate representations. Subsequently, this will naturally lead to the declaration of the 2-spinor formalism in the next chapter.

The general idea for our classification procedure is inspired by [SU01, chapter 8], but the presentation there seems to remain rather sketchy concerning mathematical aspects. We present an attempt of pursuing this plan in a more rigerous way and all our previous work, notably complex algebra and representation theory of $\mathrm{SU}(2)$, will be culminating in these considerations. However, we will have to pay for the aspired benefit in rigour by considerably more technical complexity of this section. This indicates why the presentation in [SU01] may not have been carried out in full rigour and that employing different mathematical techniques (complex structures etc.) may perhaps simplify the business. The reader who is in a hurry may just skip forward and only read theorem 3.3.6 and the subsequent remarks.

Notice that $\mathrm{SL}(2,\mathbb{C})$ may be treated as real or as complex Lie group. Of course, the universal covering map $\lambda\colon \mathrm{SL}(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow$ is a homomorphism of *real* Lie groups and this is why from a physical perspective, irreducible complex representations of $\mathrm{SL}(2,\mathbb{C})$ as real 6-dimensional Lie group are of relevance. This makes the classification prodecure more complicated than that for $\mathrm{SL}(2,\mathbb{C})$ as complex 3-dimensional Lie group, which is usually presented in mathematical textbooks on representation theory, cf. [Hal03] or [FH91].

Notice that, as $\mathrm{SL}(2,\mathbb{C})$ is simply connected, we may equivalently classify finite-dimensional irreducible complex representation of $\mathrm{SL}(2,\mathbb{C})$ or of its Lie algebra, $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}$ (cf. 1.1.6-b). To this end, one of the core artifices in this section is a clever choice of bases for $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R} \otimes \mathbb{C}$:

3.3.1 Definition and proposition (polarisation of $\mathfrak{sl}(2,\mathbb{C})_\mathbb{C}$). By complexification of the real Lie algebra $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}$,

$$\mathfrak{sl}(2,\mathbb{C})_\mathbb{C} := \mathfrak{sl}(2,\mathbb{C})_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}\,,$$

a complex 6-dimensional Lie algebra is obtained. On $\mathfrak{sl}(2,\mathbb{C})_\mathbb{C}$ we define the anti-linear automorphism (we call it “polarisation automorphism”)

$$\begin{aligned} p\colon\ &\mathfrak{sl}(2,\mathbb{C})_\mathbb{C} \to \mathfrak{sl}(2,\mathbb{C})_\mathbb{C} \\ &x^\mu s_\mu + y^\mu (i s_\mu) \mapsto \bar{x}^\mu s_\mu + \bar{y}^\mu (i s_\mu)\,. \end{aligned}$$

Consider a new basis[2] of $\mathfrak{sl}(2,\mathbb{C})_\mathbb{C}$, defined as:

$$m_\mu^+ := \frac{1}{2}\big(s_\mu - i(is_\mu)\big), \quad m_\mu^- := p(m_\mu^+) = \frac{1}{2}\big(s_\mu + i(is_\mu)\big)$$

[2]Notice that $s_1, s_2, s_3, (is_1), (is_2), (is_3)$ form a *complex* basis of $\mathfrak{sl}(s,\mathbb{C})_\mathbb{C}$ and hence are all linearly independent. Thus, $i(is_1) \neq -s_1$.

(see footnote[3]). Its commutation relations read:

$$[m_i^+, m_j^+] = \varepsilon_{ijk}\, m_k^+ , \quad [m_i^-, m_j^-] = \varepsilon_{ijk}\, m_k^- , \quad [m_i^+, m_j^-] = 0\,.$$

Hence, each of the complex 3-dimensional spaces

$$\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}^+ := \langle m_1^+, m_2^+, m_3^+ \rangle_{\mathbb{C}}, \quad \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}^- := \langle m_1^-, m_2^-, m_3^- \rangle_{\mathbb{C}}$$

is a copy of $\mathfrak{su}(2) \otimes \mathbb{C}$ and they commute with each other[4]. Moreover, the anti-linear polarisation automorphism p together with complex conjugation $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}^+ \to \overline{\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}^+}$ induce a *linear* isomorphism $\chi\colon \overline{\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}^+} \cong \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}^-$ which makes the following diagram commute:

$$\begin{tikzcd}
& \overline{\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}^+} \arrow[dd, dashed, "\cong"', "\chi"] \\
\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}^+ \arrow[ur] \arrow[dr, "p"'] & \\
& \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}^-
\end{tikzcd}$$

(Solid arrows denote anti-linear isomorphisms, the dashed arrow is the induced linear isomorphism.)

Finally, we fix the following $\mathbb{C}$-linear embeddings:

$$\begin{aligned} \iota^+ \colon \mathfrak{sl}(2,\mathbb{C}) &\to \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}^+ \subseteq \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}} \\ s_i &\mapsto m_i^+ \qquad \text{(extend } \mathbb{C}\text{-linearly from this)} \end{aligned}$$

and

$$\iota^- \colon \overline{\mathfrak{sl}(2,\mathbb{C})} \xrightarrow{\overline{\iota^+}} \overline{\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}^+} \xrightarrow{\chi} \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}^- \subseteq \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}\,,$$

where $\overline{\iota^+}\colon \overline{\mathfrak{sl}(2,\mathbb{C})} \to \overline{\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}^+}$ denotes the complex conjugate of ι^+. By using ι^+ on the first and ι^- on the second component, we obtain an isomorphism

$$\iota := (\iota^+, \iota^-) \colon \mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})} \to \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}\,.$$

3.3.2 Remark (differences between $\mathfrak{sl}(2,\mathbb{C})$, $\overline{\mathfrak{sl}(2,\mathbb{C})}$, $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ and $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$).
Just in order to sort out possible confusion, we shall make one comment on the differences between $\mathfrak{sl}(2,\mathbb{C})$, $\overline{\mathfrak{sl}(2,\mathbb{C})}$, $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ and $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$:

Notice that $\mathfrak{sl}(2,\mathbb{C})$, $\overline{\mathfrak{sl}(2,\mathbb{C})}$ and $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ all have the same carrier space and the same $\mathbb{R}$-Lie algebra structure. The only difference is their $\mathbb{C}$-Lie algebra structure, notably the way in which scalar multiplication by imaginary numbers is done: $\mathfrak{sl}(2,\mathbb{C})$ does it "the standard way", $\overline{\mathfrak{sl}(2,\mathbb{C})}$ does it "the complex conjugate of the standard way" and $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ has no $\mathbb{C}$-vector space structure at all. Thus, canonically:

$$\mathfrak{sl}(2,\mathbb{C}) \cong_{\mathbb{R}} \overline{\mathfrak{sl}(2,\mathbb{C})} \cong_{\mathbb{R}} \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \quad \text{(by the identity on the carrier space)}\,,$$

[3] It is not a mistake that $m_\mu^+ = \frac{1}{2}(s_\mu - i(is_\mu))$ (minus on the right-hand side) and $m_\mu^- = \frac{1}{2}(s_\mu + i(is_\mu))$ (plus on the right-hand side). This sign convention turns out to be of advantage later on.

[4] The author is aware that alternatively one could construct the subalgebras $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}^{\pm}$ using a complex structure J on $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$, extending it $\mathbb{C}$-linearly to $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ and taking as $\mathfrak{sl}(2,\mathbb{C})^{\pm}$ the eigenspaces for $\pm i$. However, we shall stick to the presented construction using the base vectors $m_\mu^{\pm}$ which is inspired by [SU01].

where $\cong_{\mathbb{R}}$ means isomorphic as $\mathbb{R}$-Lie algebras, but these canonical isomorphisms are not $\mathbb{C}$-linear.

$\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ is a bit detached from the others, in that it is a genuine extension of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ (it indeed has a bigger carrier space). It's just the complexification $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \otimes \mathbb{C}$ and we have the canonical $\mathbb{R}$-linear embedding *monomorphism* $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \to \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$.

As a first (and the easiest) step towards our classification aim we sort out the finite-dimensional irreducible complex $\mathbb{C}$-linear representations of $\mathfrak{sl}(2,\mathbb{C})$:

3.3.3 Remark (irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$).

(a) Let $D^{(j)}$ be an irreducible complex $\mathrm{SU}(2)$-representation with spin number j and let $\hat{D}^{(j)}$ denote the Lie algebra representation of $\mathfrak{su}(2)$ associated with $D^{(j)}$ (remember that $\mathfrak{su}(2)$ is a *real* Lie algebra). $\hat{D}^{(j)}$ is again irreducible (proposition 1.1.7-a). By complexification, $\hat{D}^{(j)}$ induces an irreducible complex represetation

$$\hat{D}_c^{(j)} := \hat{D}^{(j)} \otimes \mathbb{C}$$

of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2) \otimes \mathbb{C}$ (proposition 1.2.1-b), and, moreover, *each* finite-dimensional irreducible complex representation of $\mathfrak{sl}(2,\mathbb{C})$ is—up to equivalence—of the form $\hat{D}_c^{(j)}$ for a $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$[5].

(b) We may forget about scalar multiplication by imaginary numbers in $\mathfrak{sl}(2,\mathbb{C})$ to consider $\hat{D}_c^{(j)}$ a representation of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ (cf. remark 3.3.2). As the real Lie group $\mathrm{SL}(2,\mathbb{C})$ is simply connected and has Lie algebra $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$, each $\hat{D}_c^{(j)}$ induces a complex irreducible representation $D_c^{(j)}$ of $\mathrm{SL}(2,\mathbb{C})$, such that $dD_c^{(j)} = \hat{D}_c^{(j)}$ (corollary 1.1.6-b).

$D_c^{(j)}$ is a representation of $\mathrm{SL}(2,\mathbb{C})$ (treated as real Lie group by our convention), thus its associated Lie algebra representation $dD_c^{(j)}$ is, a priori, a representation of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$. However, as we see from the construction of $D_c^{(j)}$, the infinitesimal version, $\hat{D}_c^{(j)}$, has the *additional feature* of being compatible with the natural complex Lie algebra structure of $\mathfrak{sl}(2,\mathbb{C})$, in the sense that it may be treated as ($\mathbb{C}$-linear) representation of $\mathfrak{sl}(2,\mathbb{C})$. Thus, and this is very important to appreciate: Remark 3.3.3 *only* classifies those finite-dimensional irreducible complex representations of $\mathrm{SL}(2,\mathbb{C})$ which have this "additional feature".

3.3.4 Remark (Why is $d\bar{D}_c^{(j)} = \overline{\hat{D}_c^{(j)}}$?). Let's consider the complex conjugate representation $\bar{D}_c^{(j)}$. It is also a representation of $\mathrm{SL}(2,\mathbb{C})$ and its infinitesimal Lie algebra representation $d\bar{D}_c^{(j)}$ is, a priori, a ($\mathbb{R}$-linear) representation of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$. One might expect that it may also be considered a ($\mathbb{C}$-linear) representation of a complex Lie algebra in a canonical way analogous to the $\hat{D}_c^{(j)}$-case. However, it turns out that the complex Lie algebra structure of $\mathfrak{sl}(2,\mathbb{C})$ is not suitable in this case, as

$$\mathfrak{sl}(2,\mathbb{C}) \ni x \mapsto d\bar{D}_c^{(j)}(x) = \overline{dD_c^{(j)}(x)}$$

is anti-linear. This problem gets resolved by taking $\overline{\mathfrak{sl}(2,\mathbb{C})}$ instead, and we may canonically consider $d\bar{D}_c^{(j)}$ a ($\mathbb{C}$-linear) representation of $\overline{\mathfrak{sl}(2,\mathbb{C})}$ with $d\bar{D}_c^{(j)} = \overline{\hat{D}_c^{(j)}}$.

[5] This again is a consequence of proposition 1.2.1-b: If ρ_c is a finite-dimensional irreducible complex representation of $\mathfrak{su}(2)\otimes\mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C})$, it is easily seen that the restriction $\rho := \rho_c|_{\mathfrak{su}(2)}$ is still irreducible. Moreover, we have $\rho_c = \rho \otimes \mathbb{C}$. Proposition 1.2.1-b then does the rest.

It is possible to combine these two types of irreducible representations, $D_c^{(j)}$ and $\bar{D}_c^{(j')}$, to obtain representatives for all equivalence classes of finite-dimensional irreducible complex representations of $\mathrm{SL}(2,\mathbb{C})$ (which will be the final result, theorem 3.3.6). The next proposition presents a preliminary stage of this on the Lie algebra level:

3.3.5 Proposition (irreducible complex representations of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$). Each equivalence class of finite-dimensional irreducible complex Lie algebra representations of $\mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})}$ is given by a representative of the form

$$\hat{U}_c^{(j,j')} := \hat{D}_c^{(j)} \otimes \mathrm{Id}_{\bar{\Delta}_{j'}} + \mathrm{Id}_{\Delta_j} \otimes \overline{\hat{D}_c^{(j')}} \quad \text{on} \quad \Delta_{j,j'} := \Delta_j \otimes \bar{\Delta}_{j'}\,,$$

where $j, j' \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$. Using the isomorphism $\iota\colon \mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})} \to \mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$, this shows that the finite-dimensional irreducible complex representations of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}}$ are—up to equivalence—the representations

$$\hat{D}_c^{(j,j')} := \hat{U}_c^{(j,j')} \circ \iota^{-1}\,.$$

Notice that this proposition does not state that $\hat{D}_c^{(j,j')} \ncong \hat{D}_c^{(k,k')}$ for $(j,j') \neq (k,k')$. Of course this is true, but we don't take the effort proving it here as it will eventually be a byproduct of theorem 3.3.6.

Proof. We first show that for all $\hat{D}^{(j)}$ and $\hat{D}^{(j')}$, $\hat{U}_c^{(j,j')}$ is irreducible: Notice that canonically,

$$\mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})} \cong \mathfrak{su}(2) \otimes \mathbb{C} \oplus \mathfrak{su}(2) \otimes \overline{\mathbb{C}}$$

(cf. remark 2.1.6-ii). Hence, $\mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})}$ has the *real* subalgebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. We may restrict $\hat{U}_c^{(j,j')}$ to this subalgebra, to obtain a representation of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ on $\Delta_{j,j'}$:

$$\hat{T}^{(j,j')} := \hat{U}_c^{(j,j')}\Big|_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)} = \hat{D}^{(j)} \otimes \mathrm{Id}_{\overline{\Delta_{j'}}} + \mathrm{Id}_{\Delta_j} \otimes \overline{\hat{D}^{(j')}}\,.$$

As $\mathrm{SU}(2)$ is simply connected and compact, proposition 1.2.6 gives irreducibility of $\hat{T}^{(j,j')}$. Thus, $\hat{U}_c^{(j,j')}$ must also be irreducible.

To prove the converse direction, let T_c be an arbitrary irreducible representation of $\mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})}$ on some finite-dimensional complex representation space Δ. Consider the restriction $T := T_c|_{\mathfrak{su}(2)\oplus\mathfrak{su}(2)}$. Let $W \subseteq \Delta$ be a T-invariant subspace and pick some $\xi = (x_1 \otimes z_1, x_2 \otimes \bar{z}_2) \in \mathfrak{su}(2) \otimes \mathbb{C} \oplus \mathfrak{su}(2) \otimes \overline{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})}$ and $v \in W$. We find:

$$T_c(\xi)v = z_1\, T_c(x_1 \otimes 1, 0)v + \bar{z}_2\, T_c(0, x_2 \otimes \bar{1})v = z_1\, T(x_1, 0)v + \bar{z}_2\, T(0, x_2)v \in W\,.$$

Hence, W is T_c-invariant. Thus, we deduce irreducibility of T from irreducibility of T_c by contraposition. Moreover, using proposition 1.2.6, we find that there must be irreducible complex $\mathfrak{su}(2)$-representations ρ_1 on V_1 and ρ_2 on V_2, such that

$$T \cong \rho_1 \otimes \mathrm{Id}_{V_2} + \mathrm{Id}_{V_1} \otimes \rho_2\,.$$

By theorem 3.1.6 and corollary 1.1.6-b there are j and j' such that $\rho_1 \cong \hat{D}^{(j)}$ and $\rho_2 \cong \overline{\hat{D}^{(j')}}$ (footnote[6]) and

$$T \cong \hat{D}^{(j)} \otimes \mathrm{Id}_{\overline{\Delta_{j'}}} + \mathrm{Id}_{\overline{\Delta_j}} \otimes \overline{\hat{D}^{(j')}} \quad (\text{and } \Delta \cong \Delta_{j,j'} = \Delta_j \otimes \bar{\Delta}_{j'})\,.$$

Now we use T to express T_c and obtain our final result:

$$\begin{aligned} T_c[(\xi_1 \otimes z_1, \xi_2 \otimes \bar{z}_2)]x \otimes \bar{y} &= T_c[(\xi_1 \otimes z_1, 0)]x \otimes \bar{y} + T_c[(0, \xi_2 \otimes \bar{z}_2)]x \otimes \bar{y} \\ &= z_1\,(T[(\xi_1, 0)]x \otimes \bar{y}) + \bar{z}_2\,(T[(0, \xi_2)]x \otimes \bar{y}) \\ &= [z_1\,(\hat{D}^{(j)}(\xi_1)x)] \otimes \bar{y} + x \otimes [\bar{z}_2 \,\bar{\cdot}\, (\overline{\hat{D}^{(j')}}(\xi_2)\bar{y})] \\ &= \hat{D}_c^{(j)}(\xi_1 \otimes z_1)x \otimes \mathrm{Id}_{\overline{\Delta_{j'}}}\,\bar{y} + \mathrm{Id}_{\Delta_j}\, x \otimes \overline{\hat{D}_c^{(j')}}(\xi_2 \otimes \bar{z}_2)\bar{y} \\ &= \hat{U}_c^{(j,j')} \end{aligned}$$

This shows that an arbitrary irreducible complex representation of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})}$ must—up to equivalence—be of the form $\hat{U}_c^{(j,j')}$. $\square$

The following diagram displays the interrelationship of the various representations appearing in this chapter. It will help us prove the classification theorem for finite-dimensional complex $\mathrm{SL}(2,\mathbb{C})$-representations.

$$\begin{tikzcd}
{[\mathrm{SL}(2,\mathbb{C})]} & {[\mathrm{SL}(2,\mathbb{C})_\Delta]} & {[\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})]} & \\
D^{(j,j')} \arrow[d, mapsto, "(1)\ d"] & D_c^{(j)} \otimes \bar{D}_c^{(j')} \arrow[l, mapsto, "\circ H"', "(7)"] \arrow[d, mapsto, "(2)\ d"] & D_c^{(j)} \hat{\otimes} \bar{D}_c^{(j')} \arrow[l, mapsto, "|_{\mathrm{SL}(2,\mathbb{C})_\Delta}"', "(8)"] \arrow[d, mapsto, "(3)\ d"] & \\
\hat{D}^{(j,j')} \arrow[rrr, bend left, mapsto, "\otimes\mathbb{C}", "(9)"'] & \hat{U}^{(j,j')} \arrow[l, mapsto, "\circ \iota^{-1}|_{\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}}"', "(4)"] & \hat{U}_c^{(j,j')} \arrow[l, mapsto, "|_{\mathfrak{sl}(2,\mathbb{C})_\Delta}"', "(5)"] \arrow[r, mapsto, "\circ \iota^{-1}", "(6)"'] & \hat{D}_c^{(j,j')} \\
{[\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}]} & {[\mathfrak{sl}(2,\mathbb{C})_\Delta]} & {[\mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})}]} & {[\mathfrak{sl}(2,\mathbb{C})_\mathbb{C}]}
\end{tikzcd}$$

(Watch the diagram 3-dimensionally: imagine arrow (9) to "lay on the floor" while all other arrows lay on a vertical plane.) The Lie groups or Lie algebras written in square brackets display the object represented by the representation near by. An arrow in the diagram means that the representation at the tip gets constructed or derived from the representation at the tail in a certain way. We shall explain in detail how the various arrows work:

(a) A priori, the associated Lie algebra representation of the outer tensor product $D_c^{(j)} \hat{\otimes} \bar{D}_c^{(j')}$ is a representation of $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R} \oplus \mathfrak{sl}(2,\mathbb{C})_\mathbb{R}$. But, in consideration of remarks 3.3.2 and 3.3.4, one may equip this real Lie algebra with a complex Lie algebra

[6]Notice that by means of theorem 3.1.6, there must be an intertwining isomorphism such that $\hat{D}^{(j')} \cong \overline{\hat{D}^{(j')}}$, as both representations have the same dimension. However, this is a speciality of $\mathfrak{su}(2)$ representations; notice that a priori, a Lie algebra representation τ is not equivalent to its complex conjugate, $\bar{\tau}$. Later, remark 3.3.7 will show that in case of $\mathrm{SL}(2,\mathbb{C})$, $D_c^{(j)} \not\cong \bar{D}_c^{(j)}$.

structure which is compatible with the representation. This is how one (canonically) ends up considering $d\big[D_c^{(j)}\hat{\otimes}\bar{D}_c^{(j')}\big]$ a representation of $\mathfrak{sl}(2,\mathbb{C})\oplus\overline{\mathfrak{sl}(2,\mathbb{C})}$. And, using proposition 1.2.6, one finds that

$$d\big[D_c^{(j)}\hat{\otimes}\bar{D}_c^{(j')}\big] = \hat{U}_c^{(j,j')}\,.$$

This explains arrow (3). Arrow (6) just uses the isomorphism ι from definition 3.3.1 (cf. proposition 3.3.5).

(b) We now define

$$\mathrm{SL}(2,\mathbb{C})_\Delta := \{(S,S)\,|\,S\in\mathrm{SL}(2,\mathbb{C})\}\subseteq \mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})\,,$$

which is a real Lie group obviously isomorphic to $\mathrm{SL}(2,\mathbb{C})$ by

$$\begin{aligned} H\colon \mathrm{SL}(2,\mathbb{C}) &\to \mathrm{SL}(2,\mathbb{C})_\Delta \\ S &\mapsto (S,S)\,. \end{aligned}$$

And we define

$$\mathfrak{sl}(2,\mathbb{C})_\Delta := \{(x,\bar{x})\,|\,x\in\mathfrak{sl}(2,\mathbb{C})\}\subseteq \mathfrak{sl}(2,\mathbb{C})\oplus\overline{\mathfrak{sl}(2,\mathbb{C})}\,,$$

which inherits the structure of a *real* Lie algebra with basis

$$\big\{\big(s_1,\bar{s}_1\big),\big(s_2,\bar{s}_2\big),\big(s_3,\bar{s}_3\big),\big((is_1),\overline{(is_1)}\big),\big((is_2),\overline{(is_2)}\big),\big((is_3),\overline{(is_3)}\big)\big\}$$

and induced Lie bracket $[(\xi_1,\bar{\xi}_1),(\xi_2,\bar{\xi}_2)] = ([\xi_1,\xi_2],[\bar{\xi}_1,\bar{\xi}_2])$.
It is clear now how arrows (8), (5), and (7) work (the notation $D^{(j,j')} := D_c^{(j)}\otimes\bar{D}_c^{(j')}$ will get introduced in theorem 3.3.6).

(c) The real Lie algebra of the real Lie group $\mathrm{SL}(2,\mathbb{C})_\Delta$ is

$$\mathrm{Lie}(\mathrm{SL}(2,\mathbb{C})_\Delta) = \{(x,x)\,|\,x\in\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}\}\subseteq \mathfrak{sl}(2,\mathbb{C})_\mathbb{R}\oplus\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}\,.$$

As $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}\cong_\mathbb{R}\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}\cong_\mathbb{R}\overline{\mathfrak{sl}(2,\mathbb{C})}$ by remark 3.3.2, there is a canonical isomorphism of real Lie algebras

$$\begin{aligned} \kappa\colon \mathrm{Lie}(\mathrm{SL}(2,\mathbb{C})_\Delta) &\to \mathfrak{sl}(2,\mathbb{C})_\Delta \\ (x,x) &\mapsto (x,\bar{x})\,. \end{aligned}$$

This explains how arrow (2) is to be understood. Commutativity of the cycle (3), (5), (2), (8) is an immediate consequence of statements (a)–(c).

(d) For $x^i s_i\in\mathfrak{sl}(2,\mathbb{C})$ and $y^i\bar{s}_i\in\overline{\mathfrak{sl}(2,\mathbb{C})}$, we find:

$$\begin{aligned} \iota[(x^i s_i, y^i\bar{s}_i)] &= x^i m_i^+ + p(\bar{y}^i m_i^-) = x^i m_i^+ + y^i m_i^- \\ &= \frac{1}{2}\big[(x^i+y^i)s_i - i(x^i-y^i)(is_i)\big] \end{aligned} \tag{$*$}$$

(ι and p were introduced in definition 3.3.1). This shows that for $x^i s_i\in\mathfrak{sl}(2,\mathbb{C})$:

$$\begin{aligned} \iota[(x^i s_i,\overline{x^i s_i})] &= \frac{1}{2}\big[(x^i+\bar{x}^i)s_i - i(x^i-\bar{x}^i)(is_i)\big] \\ &= \mathrm{Re}(x^i)s_i + \mathrm{Im}(x^i)(is_i)\in\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}\,. \end{aligned} \tag{$**$}$$

and it is easily checked (using the commutation relations for s_i and (is_i) given in reminder 3.2.2) that the restriction $\iota|_{\mathfrak{sl}(2,\mathbb{C})_\Delta}\colon \mathfrak{sl}(2,\mathbb{C})_\Delta \to \mathfrak{sl}(2,\mathbb{C})_\mathbb{R}$ is an isomorphism of real Lie algebras. This explains how arrow (4) works.

For the inverse of $\iota|_{\mathfrak{sl}(2,\mathbb{C})_\Delta}$ we find:

$$\iota^{-1}|_{\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}}\colon x^i s_i + y^i(is_i) \mapsto \left((x^i+iy^i)s_i, \overline{(x^i+iy^i)s_i}\right).$$

Notice that this is just the infinitesimal version of what H does on the Lie group level, meaning that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{SL}(2,\mathbb{C}) \arrow[r, "H"] & \mathrm{SL}(2,\mathbb{C})_\Delta \\
\mathfrak{sl}(2,\mathbb{C})_\mathbb{R} \arrow[u, "\exp"] \arrow[r, "\iota^{-1}|_{\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}}"] & \mathfrak{sl}(2,\mathbb{C})_\Delta \arrow[u, "(\exp,\exp)\circ\kappa^{-1}"']
\end{tikzcd}$$

To understand arrow (1), one needs to know that $\hat{D}^{(j,j')} := \hat{U}^J \circ \iota^{-1}|_{\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}}$ (this will get introduced in theorem 3.3.6). Finally, commutativity of the cycle (2), (4), (1), (7) follows from commutativity ob the above diagram.

(e) In light of the preceding considerations, we get the following commutative diagram of $\mathbb{C}$-Lie algebra homomorphisms:

$$\begin{tikzcd}
\mathfrak{sl}(2,\mathbb{C})_\mathbb{R} \arrow[r, hook] & \mathfrak{sl}(2,\mathbb{C})_\mathbb{C} \\
\mathfrak{sl}(2,\mathbb{C})_\Delta \arrow[u, "\iota|_{\mathfrak{sl}(2,\mathbb{C})_\Delta}", "\cong_\mathbb{R}"'] \arrow[r, hook, "i"] & \mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})} \arrow[u, "\iota", "\cong_\mathbb{C}"']
\end{tikzcd}$$

The left arrow is explained in (d), the top arrow is the canonical injection and the right arrow is by by definition and proposition 3.3.1. $i\colon \mathfrak{sl}(2,\mathbb{C})_\Delta \to \mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})}$ on the bottom arrow denotes the canonical injection, it is an $\mathbb{R}$-linear Lie algebra homomorphism. Commutativity of the diagram follows from $(*)$ and $(**)$ in (d).

The direct consequence of the commutativity of the preceding diagram now is that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{sl}(2,\mathbb{C})_\mathbb{R} \otimes \mathbb{C} \arrow[r, equal] & \mathfrak{sl}(2,\mathbb{C})_\mathbb{C} \\
\mathfrak{sl}(2,\mathbb{C})_\Delta \otimes \mathbb{C} \arrow[u, "\iota|_{\mathfrak{sl}(2,\mathbb{C})_\Delta} \otimes \mathbb{C}", "\cong_\mathbb{C}"'] \arrow[r, "i\otimes\mathbb{C}", "\cong_\mathbb{C}"'] & \mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})} \arrow[u, "\iota", "\cong_\mathbb{C}"']
\end{tikzcd}$$

Finally, this together with arrows (4), (5), (6) explains arrow (9) and commutativity of the cycle (6), (5), (4), (9).

Pleasurably, we are now prepared to prove the main result:

3.3.6 Theorem (irreducible complex representations of $\mathrm{SL}(2,\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$).
The collection of equivalence classes of finite-dimensional irreducible complex representations of $\mathrm{SL}(2,\mathbb{C})$ (as *real* Lie group) is in one-one correspondence with the representations (which are to be taken as representatives)

$$D^{(j,j')} := D_c^{(j)} \otimes \bar{D}_c^{(j')} \quad \text{on} \quad \Delta_j \otimes \bar{\Delta}_{j'}$$

for $j, j' \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$. Consequently, the equivalence classes of finite-dimensional irreducible complex representations of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$ are given by $\hat{D}^{(j,j')} := dD^{(j,j')}$, for which we find

$$\hat{D}^{(j,j')} = \hat{U}^{(j,j')} \circ \iota^{-1}|_{\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}} . \tag{$*$}$$

Proof. Relation $(*)$ follows from the commutativity of the cycle (1), (4), (2), (7) in the big diagram above. So, there are three things to show: Irreducibility of $D^{(j,j')}$, that every finite-dimensional irreducible complex representation of $\mathrm{SL}(2,\mathbb{C})$ is equivalent to one of the $D^{(j,j')}$, and mutual inequivalence of $D^{(j,j')}$ and $D^{(k,k')}$ whenever $(j,j') \neq (k,k')$. By proposition 1.1.7 we may show these points for either of $D^{(j,j')}$ or $\hat{D}^{(j,j')}$ ($\mathrm{SL}(2,\mathbb{C})$ is simply connected).

To prove irreducibility of the Lie algebra representation $\hat{D}^{(j,j')}$, remember that $\hat{D}_c^{(j,j')}$ is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \otimes \mathbb{C}$. Arrow (9) from the big diagram above shows that $\hat{D}_c^{(j,j')}$ is the complexification of $\hat{D}^{(j,j')}$. Thus irreducibility of $\hat{D}^{(j,j')}$ follows from proposition 3.3.5 by proposition 1.2.1-b.

To prove that every finite-dimensional irreducible complex representation of $\mathrm{SL}(2,\mathbb{C})$ is equivalent to one if the $D^{(j,j')}$ it suffices to notice that on the Lie algebra level, this is the statement of proposition 3.3.5 in combination with proposition 1.2.1-b.

We prove the statement on inequivalence for $D^{(j,j')}$. Let $(j,j') \neq (k,k')$ and without loss of generality, $j > 0$ and $j \neq k$. To make an indirect proof, we assume $D^{(j,j')} \cong D^{(k,k')}$. Thies yields, using lemma 1.2.5:

$$D^{(j,j')} \otimes D^{(j,0)} \cong D^{(k,k')} \otimes D^{(j,0)} . \tag{$*$}$$

Using the Clebsch-Gordon formula (theorem 3.1.7), we obtain for the left-hand side:

$$D^{(j,j')} \otimes D^{(j,0)} \cong D_c^{(j)} \otimes D_c^{(j)} \otimes \bar{D}_c^{(j')} \cong \bigoplus_{\mu=0}^{2j} D_c^{(\mu)} \otimes \bar{D}_c^{(j')} = \bigoplus_{\mu=0}^{2j} D^{(\mu,j')} ,$$

and for the right-hand side:

$$D^{(k,k')} \otimes D^{(j,0)} \cong D_c^{(k)} \otimes D_c^{(j)} \otimes \bar{D}_c^{(k')} \cong \bigoplus_{\mu=|k-j|}^{j+k} D_c^{(\mu)} \otimes \bar{D}_c^{(k')} = \bigoplus_{\mu=|k-j|}^{j+k} D^{(\mu,k')} .$$

All the summands are irreducible by the first part of this proof. But as $j \neq k$, the number of summands on the left- and on the right-hand sides differ. Thus, $(*)$ can not be true because an intertwining isomorphism must be bijective and would map invariant subspaces onto invariant subspaces and thus irreducible components onto irreducible components. $(*)$ not holding contradicts the assumption and thus proves the assertion. $\square$

3.3.7 Remark ($\mathrm{SU}(2)$ vs. $\mathrm{SL}(2,\mathbb{C})$). Theorem 3.1.6 implies that all irreducible complex representations of $\mathrm{SU}(2)$ of the same dimension are equivalent. This includes $D^{(j)}$ and $\bar{D}^{(j)}$. However, in the case of $\mathrm{SL}(2,\mathbb{C})$ (which we treat as real Lie group), we just proved that $D^{(j,0)}$, which is equivalent to $D_c^{(j)}$, and $D^{(0,j)}$, which is equivalent to $\bar{D}_c^{(j)}$, are *not* equivalent. This suggests that something happens during the transition from $\mathrm{SU}(2)$ to $\mathrm{SL}(2,\mathbb{C})$. To make this a bit more plausible, we shall illuminate the example of $j=\frac{1}{2}$ in detail:

Notice that $D^{(\frac{1}{2},0)} \cong D_c^{(\frac{1}{2})}$ may be taken as

$$D_c^{(\frac{1}{2})}(S)(x) = Sx\,, \quad S \in \mathrm{SL}(2,\mathbb{C}),\ x \in \mathbb{C}^2 \cong \Delta_{\frac{1}{2}}\,.$$

To prove by contradiction that $D^{(\frac{1}{2},0)} \not\cong D^{(0,\frac{1}{2})}$, assume that $D^{(\frac{1}{2},0)} \cong D^{(0,\frac{1}{2})}$, which means that there is a $\mathbb{C}$-linear intertwiner $\varphi\colon \Delta_{\frac{1}{2}} \to \bar{\Delta}_{\frac{1}{2}}$ such that

$$\forall S \in \mathrm{SL}(2,\mathbb{C})\colon \bar{D}^{(\frac{1}{2})}(S) = \varphi \circ D_c^{(\frac{1}{2})} \circ \varphi^{-1}\,.$$

Adopting standard bases for $\Delta_{\frac{1}{2}} = \mathbb{C}^2$ and $\bar{\Delta}_{\frac{1}{2}}$, this is equivalent to the following expression on the matrix level:

$$\bar{S} = \varphi S \varphi^{-1} \quad \text{for all } S \in \mathrm{SL}(2,\mathbb{C})$$

for some invertible matrix $\varphi \in \mathrm{Mat}_{2\times 2}(\mathbb{C})$. For instance, take

$$S = \begin{pmatrix} 437\,i & 0 \\ 0 & -\frac{1}{437}i \end{pmatrix}, \quad \text{hence,} \quad \bar{S} = \begin{pmatrix} -437\,i & 0 \\ 0 & \frac{1}{437}i \end{pmatrix}.$$

S has an eigenvalue $437\,i$, thus $\varphi S \varphi^{-1}$ must have an eigenvalue $437\,i$. However, $\bar{S}$ obviously doesn't have $437\,i$ as eigenvalue. Contradiction.

Why does this argument not work in case of $\mathrm{SU}(2)$? This means: Why is there no matrix $S \in \mathrm{SU}(2)$ such that S has eigenvalue λ but $\bar{S}$ hasn't?—This follows immediately from the definition of $\mathrm{SU}(2)$:

$$\mathrm{SU}(2) = \{S \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid S^\dagger S = \mathbb{1} \text{ and } \det(S) = 1\}\,.$$

Unitaritry of S ($S^\dagger S = \mathbb{1}$) implies that there is an (orthogonal) basis of eigenvectors and that for any eigenvector λ, $|\lambda| = 1$. Unimodularity ($\det(S) = 1$) then implies that if λ_1 and λ_2 are the two eigenvectors, $1 = \det(S) = \lambda_1\lambda_2$. Together with $1 = |\lambda_1| = \lambda_1\bar{\lambda}_1$ this immediately gives $\lambda_2 = \bar{\lambda}_1$.

3.3.8 Remark (justification). We took the effort of introducing our little theory of complex algebra in chapter 2 only to have a solid framework to handle complex conjugate representations which were arising due to our polarisation construction of definition 3.3.1. Yes, indeed, notice that the core idea for our classification procedure was founded there.

So, couldn't we have avoided the need for complex conjugation by just decomposing $\mathfrak{sl}(2,\mathbb{C})_\mathbb{C}$ another way? Instead of $\iota\colon \mathfrak{sl}(2,\mathbb{C}) \oplus \overline{\mathfrak{sl}(2,\mathbb{C})} \to \mathfrak{sl}(2,\mathbb{C})_\mathbb{C}$, why not take the map

$$\begin{aligned} \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) &\to \mathfrak{sl}(2,\mathbb{C})_\mathbb{C} \\ (x^i s_i, y^i s_i) &\mapsto x^i m_i^+ + y^i m_i^-\ ? \end{aligned}$$

Using proposition 2.1.7-ii and the commutation relations for $m_i^\pm$ and s_i this would certainly be an isomorphism of complex Lie algebras and it may be used to push forward outer tensor product representations $d(D_c^{(j)}\hat{\otimes}D_c^{(j')})$ (this associated Lie algebra representation is a representation of $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$) to $\mathfrak{sl}(2,\mathbb{C})_\mathbb{C}$ (and $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}$ in the end). But, and this is the crucial point: One would just not obtain *all* finite-dimensional irreducible complex representations of $\mathfrak{sl}(2,\mathbb{C})_\mathbb{C}$ this way! Best evidence for this is that $d(D_c^{(j)}\hat{\otimes}D_c^{(0)})|_{\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}}$ and $d(D_c^{(0)}\hat{\otimes}D_c^{(j)})|_{\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}}$ would be equivalent, as is easily checked. Resolving this problem one will always be—more or less explicitly—ending up with considering complex conjugate representations.

3.3.9 Remark. We sort out relationships between the representations $D_c^{(j)}$ from remark 3.3.3 and let $D^{(j,j')}$ from theorem 3.3.6:

(i) For all spin numbers j, there is a canonical equivalence $D^{(j,0)}\cong D_c^{(j)}$: Notice that $D^{(j,0)}=D_c^{(j)}\otimes D_c^{(0)}$, which is a represenation on $\Delta_j\otimes\bar{\mathbb{C}}$. The canonical intertwining isomorphism $\Delta_j\otimes\bar{\mathbb{C}}\cong\Delta_j$ is given by ($\mathbb{C}$-linear extension of)

$$x\otimes\bar{z}=x\otimes\overline{(z\cdot 1)}=x\otimes(\bar{z}\,\bar{\cdot}\,\bar{1})=\bar{z}\,(z\otimes\bar{1})\mapsto\bar{z}\cdot x$$

for $x\in\Delta_j$ and $\bar{z}\in\bar{\mathbb{C}}$.

(ii) For all spin numbers j, there is a canonical equivalence $D^{(0,j')}\cong\bar{D}_c^{(j')}$: Notice that $D^{(0,j')}=D_c^{(0)}\otimes\bar{D}_c^{(j')}$, which is a representation on $\mathbb{C}\otimes\bar{\Delta}_{j'}$. The canonical intertwining isomorphism $\mathbb{C}\otimes\bar{\Delta}_{j'}\cong\bar{\Delta}_{j'}$ is given by ($\mathbb{C}$-linear extension of)

$$z\otimes\bar{x}=z\,(1\otimes\bar{x})\mapsto z\,\bar{\cdot}\,\bar{x}$$

for $z\in\mathbb{C}$ and $\bar{x}\in\bar{\Delta}_{j'}$.

(iii) For all spin numbers j and j' there is a canonical equivalence $\overline{D^{(j,j')}}\cong D^{(j',j)}$. The intertwining isomorphism $\bar{\Delta}_{j,j'}\cong\Delta_{j',j}$ is induced by

$$\overline{\Delta_{j,j'}}\cong\bar{\Delta}_j\otimes\Delta_{j'}\cong\Delta_{j'}\otimes\bar{\Delta}_j=\Delta_{j',j}\,,$$

where the first isomorphism is given in remark 2.1.6 and the second isomorphism is “by swapping arguments”.

3.3.10 Corollary (Clebsch-Gordon decomposition). It is immediately derived from theorem 3.1.7 and the previous remark that for all spin numbers $j,j',k,k'\in\{0,\frac{1}{2},1,\frac{3}{2},\ldots\}$:

$$D^{(j,j')}\otimes D^{(k,k')}\cong\bigoplus_{\mu=|j-k|}^{j+k}\;\bigoplus_{\mu'=|j'-k'|}^{j'+k'}D^{(\mu,\mu')}\,.$$

3.3.11 Remark (irreducible complex representations of $\mathcal{L}_+^\uparrow$). It can be shown that an SL$(2,\mathbb{C})$-representation $D^{(j,j')}$ pushes forward to an $\mathcal{L}_+^\uparrow$-representation if and only if $j+j'$ is integral.[7]

Before we conclude this section, we shall write down some of the most important representations of SL$(2,\mathbb{C})$, which we shall meet again later. We equip Δ_j and $\bar{\Delta}_j$ with the standard bases.

[7]We shall not give a proof here as this result is not central for our purposes. The reader be refered to [SU01], even though a rigerous proof won't be found there, either.

<table>
<tr><td>$D^{(\frac{1}{2},0)}$, $D^{(0,\frac{1}{2})}$</td><td>positive/negative Weyl spinor representations of $\mathrm{SL}(2,\mathbb{C})$. The elements of $\Delta_{\frac{1}{2},0}$ resp. $\Delta_{0,\frac{1}{2}}$ are called positive resp. negative Weyl spinors.
$\dim D^{(\frac{1}{2},0)} = \dim D^{(0,\frac{1}{2})} = 2$.
These are the two *fundamental representations* of $\mathrm{SL}(2,\mathbb{C})$; all other $D^{(j,j')}$ are irreducible components of tensor products of these.
Concrete realisations:
$$\begin{aligned} D^{(\frac{1}{2},0)}(S)(x) &= Sx, \qquad S \in \mathrm{SL}(2,\mathbb{C}),\ x \in \Delta_{\frac{1}{2},0}, \\ D^{(0,\frac{1}{2})}(S)(x) &= \overline{S}x, \qquad S \in \mathrm{SL}(2,\mathbb{C}),\ x \in \Delta_{0,\frac{1}{2}}, \end{aligned}$$
where Sx and $\overline{S}x$ denote ordinary matrix multiplication.</td></tr>
<tr><td>$D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$</td><td>Dirac spinor representation of $\mathrm{SL}(2,\mathbb{C})$ on $\Delta_{\frac{1}{2},0} \oplus \Delta_{0,\frac{1}{2}}$. The elements of $\Delta_{\frac{1}{2},0} \oplus \Delta_{0,\frac{1}{2}}$ are called Dirac spinors.
$\dim(D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}) = 4$
Notice that this representation is obviously not irreducible.
Concrete realisation:
$$\begin{aligned} (D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})})(S)(x,x') &= \left(D^{(\frac{1}{2},0)}(S)(x), D^{(0,\frac{1}{2})}(S)(x')\right) \\ &= (Sx, \overline{S}x), \qquad S \in \mathrm{SL}(2,\mathbb{C}),\ (x,x') \in \Delta_{\frac{1}{2},0} \oplus \Delta_{0,\frac{1}{2}}\,. \end{aligned}$$</td></tr>
<tr><td>$D^{(\frac{1}{2},\frac{1}{2})}$</td><td>Vector representation of $\mathrm{SL}(2,\mathbb{C})$ and $\mathcal{L}_+^\uparrow$. For $\mathcal{L}_+^\uparrow$ this is the defining and the fundamental representation.
$\dim D^{(\frac{1}{2},\frac{1}{2})} = 4$
Conrete realisation:
$$D^{(\frac{1}{2},\frac{1}{2})}(S)(X) = \underbrace{S^C{}_A\, \bar{S}^{\dot{D}}{}_{\dot{B}}\, X^{A\dot{B}}}_{\text{notation of chapter 4}} = \underbrace{SXS^\dagger}_{\text{matrix notation from prop. 3.2.3}}$$
where $S \in \mathrm{SL}(2,\mathbb{C})$, $X = X^{A\dot{B}} \in \Delta_{\frac{1}{2},\frac{1}{2}}$.
$$D^{(\frac{1}{2},\frac{1}{2})}(L)(x) = Lx, \qquad L \in \mathcal{L}_+^\uparrow,\ x \in \mathbb{C}^4 \cong \Delta_{\frac{1}{2},\frac{1}{2}}\,.$$
In light of proposition 3.2.3, a neat choice for the isomorphism $\mathbb{C}^4 \cong \Delta_{\frac{1}{2},\frac{1}{2}}$ would be $X^{A\dot{B}} = x^\mu \sigma_\mu{}^{A\dot{B}}$. Cf. chapter 4 for more details.</td></tr>
</table>

4. The 2-Spinor Formalism

In this chapter we will introduce an elementary formalism for dealing with 2-spinors and their tensor products, which is historically grown from the physical theory of spin $\frac{1}{2}$ particles. 2-spinors are the elements of the representation spaces of the 2-dimensional irreducible (fundamental) spinor-representations of $\mathrm{SL}(2,\mathbb{C})$, i. e. spinors of types $D^{(\frac{1}{2},0)}$ and $D^{(0,\frac{1}{2})}$.

As every finite-dimensional irreducible representation of $\mathrm{SL}(2,\mathbb{C})$ is a subrepresentation of a tensor product representation of the fundamental representations,

$$\Delta_{j,j'} = (\Delta_{\frac{1}{2},0})^{\vee 2j} \otimes (\Delta_{0,\frac{1}{2}})^{\vee 2j'} \subseteq (\Delta_{\frac{1}{2},0})^{\otimes 2j} \otimes (\Delta_{0,\frac{1}{2}})^{\otimes 2j'}$$

(cf. proposition 3.1.8), all $\mathrm{SL}(2,\mathbb{C})$-spinors of type $D^{(j,j')}$ and their tensor products or direct sums can be handled using the 2-spinor formalism, which therefore is the most universal but also most elementary spinor formalism.

Mostly, 2-spinor notation occurs in physical context as a component-based formalism, i. e. spinors get represented as components with respect to a basis, indexed by 2-spinor indices. However, in order to avoid this base dependency and to make 2-spinor notation available for dealing with spinors as abstract mathematical objects, we will develop the 2-spinor formalism using (base independent) abstract index notation. The reader is assumed to be familiar with abstract index notation for tensors, compare the section on notations and conventions at the beginning of this document.

4.1. Foundations of 2-Spinor Notation

4.1.1 Definition ($\mathrm{SL}(2,\mathbb{C})$**-spinors).**

(a) Let D be a representation of $\mathrm{SL}(2,\mathbb{C})$ with representation space Δ. Then the elements of Δ are called $\mathrm{SL}(2,\mathbb{C})$**-spinors of type** D. If D^* denotes the dual representation on Δ^*, then the elements of Δ^* are called $\mathrm{SL}(2,\mathbb{C})$**-co-spinors of type** D.[1]

(b) (Co-)Spinors of type $D^{(\frac{1}{2},0)}$ are called **positive Weyl (co-)spinors,** (co-)spinors of type $D^{(0,\frac{1}{2})}$ are called **negative Weyl (co-)spinors.** All positive and negative Weyl (co-)spinors are also called **2-(co-)spinors** or just 2-spinors.

(c) (co-)Spinors of type $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})*}$, i. e. the elements of $\Delta_{\frac{1}{2},0} \oplus \Delta^*_{0,\frac{1}{2}}$ (or the dual space), are called **Dirac (co-)spinors.**

Now we start declaring 2-spinor notation:

[1] Notice that in this terminology, co-spinors of type D are spinors of type D^*.

1. The elements of $\Delta_{\frac{1}{2},0}$ and $\Delta_{0,\frac{1}{2}}$ are denoted using **abstract indices,** if not explicitly stated otherwise. We use capital letters written as supscripts for spinors of type $D^{(\frac{1}{2},0)}$ and dotted capital letters written as supscripts for spinors of type $D^{(0,\frac{1}{2})}$. E. g.

$$\psi = \psi^A \in \Delta_{\frac{1}{2},0}, \quad \bar{\varphi} = \bar{\varphi}^{\dot{A}} \in \Delta_{0,\frac{1}{2}} .$$

2. Co-spinors of types $\Delta_{\frac{1}{2},0}$ and $\Delta_{0,\frac{1}{2}}$ are the elements of the dual spaces, $\Delta^*_{\frac{1}{2},0}$ and $\Delta^*_{0,\frac{1}{2}}$, where (canonically)

$$\Delta^*_{0,\frac{1}{2}} = (\overline{\Delta_{\frac{1}{2},0}})^* \cong \overline{\Delta^*_{\frac{1}{2},0}}$$

(cf. remark 2.1.5). **Co-spinors** are then denoted with abstract indices written as subscript (undottet/dottet) capital letters, e. g.

$$\psi_A \in \Delta^*_{\frac{1}{2},0}, \quad \bar{\varphi}_{\dot{A}} \in \Delta^*_{0,\frac{1}{2}} .$$

3. One may form arbitrary **tensor products** of $\Delta_{\frac{1}{2},0}$, $\Delta_{0,\frac{1}{2}}$, $\Delta^*_{\frac{1}{2},0}$ and $\Delta^*_{0,\frac{1}{2}}$. Particularly, the elements of

$$(\Delta_{\frac{1}{2},0})^{\otimes k} \otimes (\Delta_{0,\frac{1}{2}})^{\otimes l} = \underbrace{\Delta_{\frac{1}{2},0} \otimes \cdots \otimes \Delta_{\frac{1}{2},0}}_{k\text{-times}} \otimes \underbrace{\Delta_{0,\frac{1}{2}} \otimes \cdots \otimes \Delta_{0,\frac{1}{2}}}_{l\text{-times}}$$

are called **2-spinors of type** (k,l)**,** or "(k,l)-spinors" in short, (for possible notational confusion see footnote[2]). The elements of

$$(\Delta^*_{\frac{1}{2},0})^{\otimes k} \otimes (\Delta^*_{0,\frac{1}{2}})^{\otimes l} = \underbrace{\Delta^*_{\frac{1}{2},0} \otimes \cdots \otimes \Delta^*_{\frac{1}{2},0}}_{k\text{-times}} \otimes \underbrace{\Delta^*_{0,\frac{1}{2}} \otimes \cdots \otimes \Delta^*_{0,\frac{1}{2}}}_{l\text{-times}}$$

are called **2-co-spinors of type** (k,l)**,** or "(k,l)-co-spinors" in short.
An element of $\Delta_{\frac{1}{2},0} \otimes \Delta_{\frac{1}{2},0} \otimes \Delta^*_{0,\frac{1}{2}} \otimes \Delta_{0,\frac{1}{2}}$, for instance, gets denoted by $\psi^{AB}{}_{\dot{C}}{}^{\dot{D}}$. Extend this scheme to arbitrary tensor products of the four spaces, $\Delta_{\frac{1}{2},0}$, $\Delta_{0,\frac{1}{2}}$, $\Delta^*_{\frac{1}{2},0}$ and $\Delta^*_{0,\frac{1}{2}}$. Notice that the ordering of factors and indices is relevant, as tensor products generally don't commute.

4. **Contractions** are, as usual in (abstract) index notation, written like

$$\rho_{\dot{C}} = \psi^{AB}{}_{\dot{C}}{}^{\dot{D}} \varphi_A x_{B\dot{D}} .$$

Notice that this is not an implicit summation, as we are using abstract indices.

[2]Notice that, as (k,l)-spinors are precisely those spinors "with $k+l$ indices, the first k undotted and the remaining l dotted," not every (k,l) spinor is of type $D^{(j,j')}$ for any (j,j'), because in general,

$$\Delta_{j,j'} = (\Delta_{\frac{1}{2},0})^{\vee 2j} \otimes (\Delta_{0,\frac{1}{2}})^{\vee 2j'} \subsetneq (\Delta_{\frac{1}{2},0})^{\otimes 2j} \otimes (\Delta_{0,\frac{1}{2}})^{\otimes 2j'}$$

(cf. proposition 3.1.8). This means: not every (k,l)-spinor is an element of an *irreducible* representation of $\mathrm{SL}(2,\mathbb{C})$, especially, (k,l)-spinors are not to be confused with spinors of type $D^{(k,l)}$!

5. The canonical anti-linear **complex conjugation** maps $^{-}\colon \Delta_{\frac{1}{2},0} \to \Delta_{0,\frac{1}{2}}$ and $^{-}\colon \Delta^*_{\frac{1}{2},0} \to \Delta^*_{0,\frac{1}{2}}$ can be used to transform a $(1,0)$-(co-)spinor into a $(0,1)$-(co-)spinor (and vice versa). The complex conjugate of ψ^A would get denoted by $\bar{\psi}^{\dot{A}}$.
   Complex conjugation may in the obvious way be extended multi-anti-linearly to arbitrary tensor products of 2-(co-)spinors. For instance, complex conjugation maps a spinor
   $$\Psi^{A}{}_{\dot{B}}{}^{\dot{C}} := (\alpha_1)^A \otimes (\bar{\beta}_1)_{\dot{B}} \otimes (\bar{\gamma}_1)^{\dot{C}} + (\alpha_2)^A \otimes (\bar{\beta}_2)_{\dot{B}} \otimes (\bar{\gamma}_2)^{\dot{C}}$$
   to the spinor
   $$\bar{\Psi}^{\dot{A}}{}_{B}{}^{C} := (\bar{\alpha}_1)^{\dot{A}} \otimes (\beta_1)_{B} \otimes (\gamma_1)^{C} + (\bar{\alpha}_2)^{\dot{A}} \otimes (\beta_2)_{B} \otimes (\gamma_2)^{C}$$
6. In cases were abstract index notation is not appropriate we may fall back to usual **component notation:** Let $\{B_\mu\}$, $\mu = 1, 2$, be a basis of $\Delta_{\frac{1}{2},0}$. Then $\{\bar{B}_{\dot{\mu}}\}$ is an induced basis of $\Delta_{0,\frac{1}{2}}$. We denote the induced dual bases of $\Delta^*_{\frac{1}{2},0}$ and $\Delta^*_{0,\frac{1}{2}}$ by $\{B^\mu\}$ and $\{\bar{B}^{\dot{\mu}}\}$[3].
   Elements of $\Delta_{\frac{1}{2},0}$, $\Delta_{0,\frac{1}{2}}$ and their dual spaces may be denoted in components with respect to the bases $\{B_\mu\}$, $\{\bar{B}_{\dot{\mu}}\}$, $\{B^\mu\}$ and $\{\bar{B}^{\dot{\mu}}\}$ in the usual way, using greek indices and implicit summation. E. g.
   $$\psi^{A}{}_{\dot{B}}{}^{\dot{C}} = \psi^{\mu}{}_{\dot{\nu}}{}^{\dot{\kappa}} B_\mu \otimes \bar{B}^{\dot{\nu}} \otimes \bar{B}_{\dot{\kappa}} \,.$$
   Notice that complex conjugation of $\psi^{A}{}_{\dot{B}}{}^{\dot{C}}$ in component notation is just given by complex conjugation of each single component (cf. remark 2.1.2):
   $$\bar{\psi}^{\dot{A}}{}_{B}{}^{C} = \bar{\psi}^{\dot{\mu}}{}_{\nu}{}^{\kappa} \bar{B}_{\dot{\mu}} \otimes B^{\nu} \otimes B_{\kappa} \,.$$
7. Let E_μ, $\mu = 1, 2$, denote the standard basis for $\Delta_{\frac{1}{2},0}$ (recall that $\Delta_{\frac{1}{2},0} = \mathbb{C}^2$ (definition 3.1.8, remark 3.3.9) and cf. point (d) at the beginning of chapter 3). $\{E_\mu\}$ induces bases $\{\bar{E}_{\dot{\mu}}\}$, $\{E^\mu\}$ and $\{\bar{E}^{\dot{\mu}}\}$ of $\Delta_{0,\frac{1}{2}}$, $\Delta^*_{\frac{1}{2},0}$ and $\Delta^*_{0,\frac{1}{2}}$ in the way previously described. These bases are called the **standard bases of $\Delta_{\frac{1}{2},0}$, $\Delta_{0,\frac{1}{2}}$, $\Delta^*_{\frac{1}{2},0}$ and $\Delta^*_{0,\frac{1}{2}}$, respectively.**
8. Let ϵ^{AB} denote the **totally antisymmetric** $(2,0)$**-spinor** given by the component matrix
   $$[\epsilon^{\mu\nu}] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{$*$}$$
   with respect to the standard bases (μ: row index, ν: column index). Let ϵ_{AB} denote its inverse, such that $\epsilon^{AB}\epsilon_{CB} = \mathrm{Id}^A_C$, where $\mathrm{Id}^A_C \in \Delta_{\frac{1}{2},0} \otimes \Delta^*_{\frac{1}{2},0}$ is the tensor representation of the identity map on $\Delta_{\frac{1}{2},0}$. It is easily seen on the component level that ϵ_{AB} also has component matrix $(*)$ with respect to the standard bases.

[3] This means that $\{B^\mu\}$ and $\{\bar{B}^{\dot{\mu}}\}$ denote the bases defined by $B^\mu(B_\nu) = \delta^\mu_\nu$ and $\bar{B}^{\dot{\mu}}(\bar{B}_{\dot{\nu}}) = \delta^{\dot{\mu}}_{\dot{\nu}}$, as usual.

Moreover, by $\epsilon^{\dot{A}\dot{B}} := \bar{\epsilon}^{\dot{A}\dot{B}}$ and $\epsilon_{\dot{A}\dot{B}} := \bar{\epsilon}_{\dot{A}\dot{B}}$ we denote the complex conjugates of ϵ^{AB} and ϵ_{AB}. Their component matrices with respect to the standard bases are again given by $(*)$. This is why one usually omits the bars on $\bar{\epsilon}^{\dot{A}\dot{B}}$ and $\bar{\epsilon}_{\dot{A}\dot{B}}$ ("on the component level, it's all the same").

The ϵ-spinors play a fundamental role in the 2-spinor formalism because they are used for **"indix shifting",** which is to be performed according to the following rules:

$$\begin{aligned} \psi_B &= \epsilon_{AB}\psi^A, & \psi_{\dot{B}} &= \epsilon_{\dot{A}\dot{B}}\psi^{\dot{A}} \\ \psi^A &= \epsilon^{AB}\psi_B, & \psi^{\dot{A}} &= \epsilon^{\dot{A}\dot{B}}\psi_{\dot{B}} \end{aligned}$$

"Lowering with the first, raising with the second index of ϵ."

Notice that due to the anti-symmetry of ϵ,

$$\psi_B\varphi^B = \epsilon_{AB}\psi^A\varphi^B = -\epsilon_{BA}\psi^A\varphi^B = -\psi^A\varphi_A\,,$$

and, analogously, $\bar{\psi}_{\dot{B}}\bar{\varphi}^{\dot{B}} = -\bar{\psi}^{\dot{B}}\bar{\varphi}_{\dot{B}}$.

$\mathrm{SL}(2,\mathbb{C})$-Representations in the 2-Spinor Formalism

4.1.2 Remark ($D^{(\frac{1}{2},0)}$ in 2-spinor notation).

(i) Let $\psi^A \in \Delta_{\frac{1}{2},0}$ and $\bar{\varphi}^{\dot{A}} \in \Delta_{0,\frac{1}{2}}$. Then for $S = S^A{}_B \in \mathrm{SL}(2,\mathbb{C})$,

$$D^{(\frac{1}{2},0)}(S)\psi^A = S^A{}_B\,\psi^B, \qquad D^{(0,\frac{1}{2})}(S)\bar{\varphi}^{\dot{A}} = \bar{S}^{\dot{A}}{}_{\dot{B}}\,\bar{\varphi}^{\dot{B}}$$

(cf. point (f) at the beginning of chapter 3, definition 3.1.8 and remarks 3.3.3 and 3.3.9). Notice that $D^{(\frac{1}{2},0)}\psi^A$ actually means $(D^{(\frac{1}{2},0)}\psi)^A$, but we may omit the brackets; analogously for $D^{(0,\frac{1}{2})}\bar{\varphi}^{\dot{A}}$.

(ii) Let

$$\psi^{A_1\ldots A_k} = \psi^{A_1} \otimes \cdots \otimes \psi^{A_k} \in (\Delta_{\frac{1}{2},0})^{\otimes k}$$

be a *simple* $(k,0)$-spinor. Then $S \in \mathrm{SL}(2,\mathbb{C})$ acts on it according to the tensor product representation $(D^{(\frac{1}{2},0)})^{\otimes k}$:

$$\begin{aligned} (D^{(\frac{1}{2},0)})^{\otimes k}(S)\psi^{A_1\ldots A_k} &= (D^{(\frac{1}{2},0)}(S)\psi)^{A_1} \otimes \cdots \otimes (D^{(\frac{1}{2},0)}(S)\psi)^{A_k} \\ &= S^{A_1}{}_{B_1}\cdots S^{A_k}{}_{B_k}\,\psi^{B_1\ldots B_k}\,. \end{aligned}$$

The analogous result holds for dotted spinors:

$$\begin{aligned} (D^{(0,\frac{1}{2})})^{\otimes k}(S)\bar{\psi}^{\dot{A}_1\ldots \dot{A}_k} &= (D^{(0,\frac{1}{2})}(S)\bar{\psi})^{\dot{A}_1} \otimes \cdots \otimes (D^{(0,\frac{1}{2})}(S)\bar{\psi})^{\dot{A}_k} \\ &= \bar{S}^{\dot{A}_1}{}_{\dot{B}_1}\cdots \bar{S}^{\dot{A}_k}{}_{\dot{B}_k}\,\bar{\psi}^{\dot{B}_1\ldots \dot{B}_k}\,. \end{aligned}$$

4.1.3 Remark ($\mathrm{SL}(2,\mathbb{C})$-action on co-spinors). The $\mathrm{SL}(2,\mathbb{C})$-action on co-spinors is given by the dual representations (cf. definition 1.2.2), e. g. $D^{(\frac{1}{2},0)*}$, in the following way:

$$D^{(\frac{1}{2},0)}\psi_A = (S^{-1})^C{}_A\psi_C\,.$$

It has the nice feature that scalars (for example, $\psi_A \varphi^A$) remain constant:

$$D^{(\frac{1}{2},0)}(S)\psi_A \, D^{(\frac{1}{2},0)}(S)\varphi^A = (S^{-1})^C{}_A \psi_C \, S^A{}_B \varphi^B = \psi_A \varphi^A \,.$$

On the other hand, it would fit into our formalism ever so neatly, if the action of $\mathrm{SL}(2,\mathbb{C})$ on ψ_A was just given by the action on ψ^A and a subsequent index shift. This would mean:

$$D^{(\frac{1}{2},0)}(S)\psi_A = \epsilon_{BA} \, S^B{}_C \, \psi^C = -S_A{}^C \, \psi_C \,.$$

It turns out, that both approaches coincide (see proof). Hence, summing up, the action of $D^{(\frac{1}{2},0)}$ on co-spinors is given by:

$$D^{(\frac{1}{2},0)}(S)\psi_A = (S^{-1})^C{}_A \psi_C = -S_A{}^C \, \psi_C \,.$$

Analogously for dotted spinors:

$$D^{(\frac{1}{2},0)}(S)\bar{\psi}_{\dot{A}} = (\bar{S}^{-1})^{\dot{C}}{}_{\dot{A}} \bar{\psi}_{\dot{C}} = -\bar{S}_{\dot{A}}{}^{\dot{C}} \, \bar{\psi}_{\dot{C}} \,.$$

Proof. We use component notation with respect to the standard basis of $\mathbb{C}^2$ and $\Delta_{\frac{1}{2},0}$. To see the asserted coincidence, notice that for $S \in \mathrm{SL}(2,\mathbb{C})$,

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \quad \Rightarrow \quad S^{-1} = \begin{pmatrix} s_{22} & -s_{12} \\ -s_{21} & s_{11} \end{pmatrix} ,$$

and hence, on the matrix level,

$$(S^{-1})^\nu{}_\mu \psi_\nu = (\psi_1, \psi_2) \cdot [S^{-1}] = (\psi_1 s_{22} - \psi_2 s_{21}, -\psi_1 s_{12} + \psi_2 s_{11})$$

and

$$-S_\mu{}^\nu \, \psi_\nu = -\epsilon_{\kappa\mu} \, S^\kappa{}_\lambda \, \epsilon^{\nu\lambda} \, \psi_\nu = (-s_{21}\psi_2 + s_{22}\psi_1, s_{11}\psi_2 - s_{12}\psi_1) \,.$$

Both expressions equal, which completes the proof. □

4.1.4 Proposition (coordinate independence of ϵ^{AB}). $\epsilon^{AB} \in (\Delta_{\frac{1}{2},0})^{\otimes 2}$ is invariant under the $(D^{(\frac{1}{2},0)})^{\otimes 2}$-action. Thus, the inverse ϵ_{AB} must also be invariant under the $(D^{(\frac{1}{2},0)*})^{\otimes 2}$-action.

Considering the action of $D^{(\frac{1}{2},0)}(S)$ for $S \in \mathrm{SL}(2,\mathbb{C})$ as passive (coordinate) transformation, this thows that the component representation of ϵ^{AB} with respect to a basis $\{B_\mu\}$ of $\Delta_{\frac{1}{2},0}$ is invariant under base transformastions $B_\mu \to B'_\mu$ given by $B'_\mu := D^{(\frac{1}{2},0)}(S)B_\mu$. Analogous statements hold for ϵ_{AB}, $\epsilon^{\dot{A}\dot{B}}$ and $\epsilon_{\dot{A}\dot{B}}$.

Proof. For arbitrary $S \in \mathrm{SL}(2,\mathbb{C})$, the assertion reads:

$$(D^{(\frac{1}{2},0)})^{\otimes 2}(S)\epsilon^{AB} = S^A{}_C S^B{}_D \, \epsilon^{CD} = \epsilon^{AB} \,.$$

In component notation with restect to the standard bases of $\mathbb{C}^2$ and $\Delta_{\frac{1}{2},0}$, this statement is equivalent to $\det(S) = 1$, as can be checked by simple calculation. □

Symmetrisation and Anti-symmetrisation in the 2-Spinor Formalism

4.1.5 Definition (symmetrisation and anti-symmetrisation). Let $\psi^{A_1\ldots A_k} \in (\Delta_{\frac{1}{2},0})^{\otimes k}$. Then

$$\psi^{(A_1\ldots A_k)} := \frac{1}{k!}\sum_{\pi\in S_k} \psi^{A_{\pi(1)}\ldots A_{\pi(k)}}$$

is called **symmetrisation of** ψ and

$$\psi^{[A_1\ldots A_k]} := \frac{1}{k!}\sum_{\pi\in S_k} \mathrm{sign}(\pi)\, \psi^{A_{\pi(1)}\ldots A_{\pi(k)}}$$

is called **anti-symmetrisation of** ψ**.**

Analogous definitions hold for dotted spinors $\psi^{\dot{A}_1\ldots\dot{A}_k}$, co-spinors $\psi_{A_1\ldots A_k}$, $\psi_{\dot{A}_1\ldots\dot{A}_k}$ or mixed expressions like $\psi^{A(B_1\ldots B_k)C}{}_{\dot{D}\dot{E}(\dot{F}_1\ldots\dot{F}_l)}$.

4.1.6 Remark (properties of symmetrisation and anti-symmetrisation). All of these statements hold in an analogous fashion for dotted spinors.

(i) Notice that symmetrisation and anti-symmetrisation are idempotent. Hence, for symmetric spinors $\psi^{A_1\ldots A_k}$ or anti-symmetric spinors $\varphi^{A_1\ldots A_k}$ we find that $\psi^{A_1\ldots A_k} = \psi^{(A_1\ldots A_k)}$ and $\varphi^{A_1\ldots A_k} = \varphi^{[A_1\ldots A_k]}$.

(ii) Let $\vee$ denote the **symmetrised** and $\wedge$ the **anti-symmetrised tensor product** of vector spaces. Then we may write

$$\psi^{A_1\ldots A_k} \in (\Delta_{\frac{1}{2},0})^{\vee k} \;\Leftrightarrow\; \psi^{A_1\ldots A_k} = \psi^{(A_1\ldots A_k)}$$

and

$$\psi^{A_1\ldots A_k} \in (\Delta_{\frac{1}{2},0})^{\wedge k} \;\Leftrightarrow\; \psi^{A_1\ldots A_k} = \psi^{[A_1\ldots A_k]}\,.$$

(iii) Notice that for simple combinatorical reasons,

$$k > 2 \Rightarrow \psi^{[A_1\ldots A_k]} = 0\,.$$

Thus, the dimension of $(\Delta_{\frac{1}{2},0})^{\wedge k}$ is 0 if $k > 2$ and 1 for $k = 2$. In this case, ϵ_{AB} is a base vector for this space. From this, one can easily deduce:

$$\psi^{[AB]} = \frac{1}{2}\psi_C{}^C\,\epsilon^{AB} \quad \text{as well as} \quad \psi_{[AB]} = \frac{1}{2}\psi_C{}^C\,\epsilon_{AB}\,.$$

(iv) We immediately find

$$\psi^{AB} = \psi^{(AB)} + \psi^{[AB]}\,.$$

Proof of (iii). Use component notation with respect to the standard basis $\{E_\mu\}$. Notice that $\psi_\nu{}^\nu = \psi^{\lambda\nu}\epsilon_{\lambda\nu} = \psi^{1\,2} - \psi^{2\,1}$. Moreover,

$$\psi^{[1\,1]} = 0 \qquad \psi^{[1\,2]} = \frac{1}{2}(\psi^{1\,2} - \psi^{2\,1})$$

$$\psi^{[2\,1]} = \frac{1}{2}(\psi^{2\,1} - \psi^{1\,2}) \qquad \psi^{[2\,2]} = 0$$

Comparing coefficients eventually yields $\psi^{[\mu\kappa]} = \frac{1}{2}\psi_\nu{}^\nu\,\epsilon^{\mu\kappa}$. ☐

4.2. Vector-Spinor-Correspondence

Let $(\mathbb{R}^4, \eta)$ and $(\mathbb{R}^4, \eta)_{\mathbb{C}} = (\mathbb{C}^4, \eta)$ be the real resp. complex four-dimensional (Minkowski) vector spaces equipped with the standard basis and standard Lorentz scalar product of signature $(+ - --)$ as declared at the beginning of chapter 3.

4.2.1 Definition and proposition (tensor-spinor ${\sigma_a}^{A\dot{B}}$). Let ${\sigma_a}^{A\dot{B}} \in (\mathbb{C}^4)^* \otimes \Delta_{\frac{1}{2},\frac{1}{2}}$ be the tensor given in component notation with respect to the standard bases of $(\mathbb{C}^4, \eta)$, $\Delta_{\frac{1}{2},0}$ and $\Delta_{0,\frac{1}{2}}$ by

$$\begin{aligned} {\sigma_0}^{\mu\dot{\nu}} &= \frac{1}{\sqrt{2}}\,{\tilde{\sigma}_0}^{\mu\dot{\nu}}, & {\sigma_1}^{\mu\dot{\nu}} &= \frac{1}{\sqrt{2}}\,{\tilde{\sigma}_1}^{\mu\dot{\nu}} \\ {\sigma_2}^{\mu\dot{\nu}} &= \frac{1}{\sqrt{2}}\,{\tilde{\sigma}_2}^{\mu\dot{\nu}}, & {\sigma_3}^{\mu\dot{\nu}} &= \frac{1}{\sqrt{2}}\,{\tilde{\sigma}_3}^{\mu\dot{\nu}}, \end{aligned}$$

where $\tilde{\sigma}_\kappa$ denote the Pauli spin matrices as in reminder 3.1.2-ii (thus, ${\tilde{\sigma}_\kappa}^{\mu\dot{\nu}}$ means the entry of $\tilde{\sigma}_\kappa$ at row index μ and column index $\dot{\nu}$)[4]. We shall refer to ${\sigma_a}^{A\dot{B}}$ as **the "tensor-spinor"** (everything needs a name, doesn't it?).

${\sigma_a}^{A\dot{B}}$ can be considered a $\mathbb{C}$-linear map $\sigma\colon \mathbb{C}^4 \to \Delta_{\frac{1}{2},\frac{1}{2}}$ which is actually an isomorphism with inverse

$$\sigma^{-1}\colon X^{A\dot{B}} \mapsto x^a = X^{A\dot{B}} \sigma^a_{\ A\dot{B}}\,, \tag{$*$}$$

where all indices of $\sigma^a_{\ A\dot{B}}$ were shifted according to the rules, i. e. tensor indices using η_{ab} and spinor indices using ϵ_{AB} and $\epsilon_{\dot{A}\dot{B}}$.

Proof. We have to prove formula $(*)$ for σ^{-1}. Firstly, σ is injective because $\tilde{\sigma}_0, \ldots, \tilde{\sigma}_3$ are $\mathbb{C}$-linearly independent, as is quickly checked (cf. also reminder 3.1.2-iii). Thus, as $\dim(\mathbb{C}^4) = \dim(\Delta_{\frac{1}{2},\frac{1}{2}}) = 4$, bijectivity of σ follows. Secondly, $\sigma^a_{\ A\dot{X}}{\sigma_b}^{A\dot{X}} = \mathrm{Id}^a_{\ b}$ is to be checked by calculation on the matrix level using component notation with respect to the standard bases (notice that $(\tilde{\sigma}_i)^2 = \mathbb{1}$ for the Pauli matrices $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_2$). ☐

4.2.2 Remark (on notational confusion concerning σ). Notice that σ occurs in three different ways:

1. ${\sigma_a}^{A\dot{B}}$ alsways denotes the tensor-spinor as in the preceding definition 4.2.1,
2. σ denotes the linear map induced by this tensor-spinor, and
3. $\tilde{\sigma}_\mu$ $(\mu = 0, \ldots, 3)$ denote the Pauli spin matrices.

If the components of $\tilde{\sigma}_\mu$ are meant, we shall write additional indices for the row and for the column number: $\tilde{\sigma}^{\alpha\beta}_\mu$ or $\tilde{\sigma}^{\alpha\dot{\beta}}_\mu$. Notice that the components $\tilde{\sigma}^{\alpha\dot{\beta}}_\mu$ differ from the components ${\sigma_\mu}^{\alpha\dot{\beta}}$ (which are the components of ${\sigma_a}^{A\dot{B}}$ with respect to the standard bases) by factor $\frac{1}{\sqrt{2}}$. This is is why we introduced the tilde for the Pauli matrices.

[4]This definition is in accordance with [SU01, p. 247] but [Wal84, p. 353] uses another sign convention: ${\sigma_0}^{\mu\dot{\nu}} \mapsto {\sigma_0}^{\mu\dot{\nu}}$, ${\sigma_i}^{\mu\dot{\nu}} \mapsto -{\sigma_i}^{\mu\dot{\nu}}$, $i = 1, 2, 3$.

4.2.3 Remark (vector-spinor correspondence). The tensor-spinor ${\sigma_a}^{A\dot{B}}$ plays a central role in the 2-spinor formalism as we shall demonstrate now:

Let $\lambda\colon \mathrm{SL}(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow$ be the universal covering map as given in proposition 3.2.3. From the same proposition we know that the Pauli matrices $\tilde{\sigma}_\mu$ have a special relationship with λ, and in our new terminology, we may rewrite proposition 3.2.3 for $S \in \mathrm{SL}(2,\mathbb{C})$ and $x = x^a \in \mathbb{C}^4$ as:[5]

$$\lambda(S)(x) = \sigma^{-1}\left[D^{(\frac{1}{2},\frac{1}{2})}(S)\,\sigma(x)\right] = {\sigma^a}_{A\dot{X}}\, S^A{}_B\, \bar{S}^{\dot{X}}{}_{\dot{Y}}\, {\sigma_b}^{B\dot{Y}} x^b\,.$$

This shows: If x gets mapped into $\Delta_{\frac{1}{2},\frac{1}{2}}$ by σ, then the action of $D^{(\frac{1}{2},\frac{1}{2})}(S)$ on $\sigma(x)$ is given by applying the *Lorentz transformation* $\lambda(S)$ to x and subsequently applying σ to the transformation result; this means, the following diagram commutes:

$$\begin{tikzcd}
(\mathbb{R}^4,\eta)_{\mathbb{C}} \arrow[r, "\lambda(S)"] \arrow[d, "\sigma"] & (\mathbb{R}^4,\eta)_{\mathbb{C}} \arrow[d, "\sigma"] \\
\Delta_{\frac{1}{2},\frac{1}{2}} \arrow[r, "D^{(\frac{1}{2},\frac{1}{2})}(S)"] & \Delta_{\frac{1}{2},\frac{1}{2}}
\end{tikzcd}$$

It is in this sense that ${\sigma_a}^{A\dot{B}}$ consitutes a correspondence of vectors with $(1,1)$-spinors. Therefore, if x^a is a vector, we shall sometimes use the abbreviating notation

$$x^{A\dot{X}} := x^a {\sigma_a}^{A\dot{X}}\,.$$

In remark 4.2.5 we will see that there is a canonical Lorentz metric on $\Delta_{\frac{1}{2},\frac{1}{2}}$ with respect to which $\sigma\colon (\mathbb{R}^4,\eta)_{\mathbb{C}} \to \Delta_{\frac{1}{2},\frac{1}{2}}$ is an isometry.

4.2.4 Remark (regarding the index order of ${\sigma_a}^{A\dot{B}}$). As the Pauli matrices, $\tilde{\sigma}_\mu$, are selfadjoint, it is easly checked on the matrix level that for $x \in (\mathbb{R}^4,\eta)$ and $\varphi_A \in \Delta^*_{\frac{1}{2},0}$, $\psi_{\dot{A}} \in \Delta^*_{0,\frac{1}{2}}$,

$$x^{A\dot{B}}\varphi_A\,\psi_{\dot{B}} = x^a{\sigma_a}^{A\dot{B}}\varphi_A\,\psi_{\dot{B}} = x^a{\bar{\sigma}_a}{}^{\dot{B}A}\varphi_A\,\psi_{\dot{B}} = \bar{x}^{\dot{B}A}\varphi_A\,\psi_{\dot{B}} =: x^{\dot{B}A}\varphi_A\,\psi_{\dot{B}}\,.$$

We shall sometimes omit the bar on $\bar{x}^{\dot{B}A}$ in favour of $x^{\dot{B}A}$ if $x^a \in \mathbb{R}^4$ (as opposed to $\mathbb{C}^4$), and $x^{A\dot{B}}$ and $\bar{x}^{\dot{B}A} = x^{\dot{B}A}$ may be freely substituted for eachother.

Notice that all this is in general not possible for a general $(1,1)$-spinor $x^{A\dot{B}}$ or for $x^a \in \mathbb{C}^4 \setminus \mathbb{R}^4$. "$x^a$ begin real" (i. e. $x^a \in \mathbb{R}^4$) is a core ingredient here.

4.2.5 Remark (spinor metric on $\Delta_{\frac{1}{2},\frac{1}{2}}$). Let η_{ab} denote the metric tensor of our Minkowski vector space $(\mathbb{R}^4,\eta)$. Then, as is easily seen,

$$\eta_{A\dot{X}B\dot{Y}} = \eta_{ab}\,{\sigma^a}_{A\dot{X}}{\sigma^b}_{B\dot{Y}} \tag{A}$$

[5]Notice that the θ in prop. 3.2.3 and the σ in this remark differ by factor $\sqrt{2}$. However, this has no consequences upon our statement here, as this additional factor cancels out.

is a Lorentz scalar product on $\Delta_{\frac{1}{2},\frac{1}{2}}$ of signature $(+---)$. Actually, this is no surprise because $\eta_{A\dot{X}B\dot{Y}}$ is just the pushforward of η_{ab} by σ.
But it *is* surprising that one finds:

$$\eta_{A\dot{X}B\dot{Y}} = \epsilon_{AB}\epsilon_{\dot{X}\dot{Y}}\,. \tag{B}$$

By rising the indices of $\eta_{A\dot{X}B\dot{Y}}$ by means of ϵ^{AB} and $\epsilon^{\dot{X}\dot{Y}}$, one obtains the inverse metric on $\Delta^*_{\frac{1}{2},\frac{1}{2}}$,

$$\eta^{A\dot{X}B\dot{Y}} = \epsilon^{AB}\epsilon^{\dot{X}\dot{Y}}\,, \tag{C}$$

which coincides with $\eta^{ab}\,\sigma_a{}^{A\dot{X}}\sigma_b{}^{B\dot{Y}}$.
Summing up: $\sigma\colon (\mathbb{R}^4,\eta)_{\mathbb{C}} \to \Delta_{\frac{1}{2},\frac{1}{2}}$ *is an isometry with respect to* η_{ab} *and* $\eta_{A\dot{X}B\dot{Y}} := \epsilon_{AB}\epsilon_{\dot{X}\dot{Y}}$. It commutes with index shifting by η_{ab} and ϵ_{AB}, i.e. for $x \in (\mathbb{R}^4,\eta)_{\mathbb{C}}$, the following diagram commutes:

$$\begin{tikzcd}
x_a \arrow[r, mapsto, "\sigma^a{}_{A\dot{X}}"] & x_{A\dot{X}} \\
x^a \arrow[u, mapsto, "\eta_{ab}"'] \arrow[r, mapsto, "\sigma_a{}^{A\dot{X}}"] & x^{A\dot{X}} \arrow[u, mapsto, "\epsilon_{AB}\epsilon_{\dot{X}\dot{Y}} = \eta_{A\dot{X}B\dot{Y}}"']
\end{tikzcd}$$

Proof. We make (B) a definition, i.e. we set $\eta_{A\dot{X}B\dot{Y}} := \epsilon_{AB}\epsilon_{\dot{X}\dot{Y}}$, and first prove

$$\eta_{A\dot{X}B\dot{Y}}\,\sigma_a{}^{A\dot{X}}\sigma_b{}^{B\dot{Y}} = \eta_{ab}\,. \tag{$*$}$$

by elementary calculation on the matrix level using component notation with respect to the standard bases:[6]

$$\begin{array}{ll}
\epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_0{}^{\mu\dot\kappa}\sigma_0{}^{\nu\dot\lambda} = 1 & \epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_1{}^{\mu\dot\kappa}\sigma_0{}^{\nu\dot\lambda} = 0 \\
\epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_0{}^{\mu\dot\kappa}\sigma_1{}^{\nu\dot\lambda} = 0 & \epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_1{}^{\mu\dot\kappa}\sigma_1{}^{\nu\dot\lambda} = -1 \\
\epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_0{}^{\mu\dot\kappa}\sigma_2{}^{\nu\dot\lambda} = 0 & \epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_1{}^{\mu\dot\kappa}\sigma_2{}^{\nu\dot\lambda} = 0 \\
\epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_0{}^{\mu\dot\kappa}\sigma_3{}^{\nu\dot\lambda} = 0 & \epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_1{}^{\mu\dot\kappa}\sigma_3{}^{\nu\dot\lambda} = 0 \\[1ex]
\epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_2{}^{\mu\dot\kappa}\sigma_0{}^{\nu\dot\lambda} = 0 & \epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_3{}^{\mu\dot\kappa}\sigma_0{}^{\nu\dot\lambda} = 0 \\
\epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_2{}^{\mu\dot\kappa}\sigma_1{}^{\nu\dot\lambda} = 0 & \epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_3{}^{\mu\dot\kappa}\sigma_1{}^{\nu\dot\lambda} = 0 \\
\epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_2{}^{\mu\dot\kappa}\sigma_2{}^{\nu\dot\lambda} = -1 & \epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_3{}^{\mu\dot\kappa}\sigma_2{}^{\nu\dot\lambda} = 0 \\
\epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_2{}^{\mu\dot\kappa}\sigma_3{}^{\nu\dot\lambda} = 0 & \epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_3{}^{\mu\dot\kappa}\sigma_3{}^{\nu\dot\lambda} = -1
\end{array}$$

This shows $(*)$. Using the inverse formula from proposition 4.2.1, $(*)$ implies

$$\eta_{ab}\,\sigma^a{}_{A\dot{X}}\sigma^b{}_{B\dot{Y}} = \epsilon_{AB}\epsilon_{\dot{X}\dot{Y}}$$

and thus, (A). This also shows that σ is an isometry with respect to η_{ab} and $\eta_{A\dot{X}B\dot{Y}}$. Finally, commutativity of the diagram is an immediate consequence of all this. $\square$

[6] Notice that $\epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_\alpha{}^{\mu\dot\mu}\sigma_\beta{}^{\nu\dot\lambda} = (-\epsilon_{\nu\mu})(-\epsilon_{\dot\lambda\dot\kappa})\,\sigma_\alpha{}^{\mu\dot\kappa}\sigma_\beta{}^{\nu\dot{Y}} = \epsilon_{\mu\nu}\epsilon_{\dot\kappa\dot\lambda}\,\sigma_\beta{}^{\mu\dot\mu}\sigma_\alpha{}^{\nu\dot\lambda}$, hence not all of the following expressions have to get calculated separately on the matrix level.

4.2.6 Remark (vectors, spinors and coordinate transformations). "2-spinors are objects that change sign under a rotation by 2π"—this is often said in physical context. Viewed from a slightly different perspective, this just sais that a Lorentz transformation does not contain all the information about a coordinate transformation of a physical system if the mathematical model of the system uses 2-spinors.

The fundamental transformation group for "systems with spin", i. e. for systems whose mathematical model uses (2-)spinors, is $\mathrm{SL}(2,\mathbb{C})$. We shall briefly describe here how such a coordinate transformation, given by $S \in \mathrm{SL}(2,\mathbb{C})$, works:

Let a physical system be described by elements of the Minkowski vector space $(\mathbb{R}^4, \eta)$ and spinors of type D, where D is any representation of $\mathrm{SL}(2,\mathbb{C})$ on a space Δ. Let reference frames for a coordinate description of the system be given by a basis $\{b_\mu\}$ of $(\mathbb{R}^4,\eta)$ and a basis $\{B_\nu\}$, $\nu = 1,\dots,\dim(\Delta)$ of Δ. Moreover, let $\lambda\colon \mathrm{SL}(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow$ denote the universal covering map as given in proposition 3.2.3.

Then an element $S \in \mathrm{SL}(2,\mathbb{C})$ induces transformations $b_\mu \to b'_\mu$ and $B_\nu \to B'_\nu$ of our reference frames, given by

$$b'_\mu := \lambda(S)(b_\mu) \quad \text{and} \quad B'_\nu := D(S)B_\nu\,.$$

Surprisingly, using the commutative diagram from remark 4.2.3, it turns out that for $D = D^{(\frac{1}{2},\frac{1}{2})}$ the component representation of ${\sigma_a}^{A\dot{B}}$ with respect to the reference frames $\{b_\mu\}$, $\{B_\mu\}$ and $\{\bar{B}_{\dot{\mu}}\}$ is invariant under the above transformations, though generally, the coordinate representation of such an object may of course change.

In remark 6.4.5 we will see that the reference frame transformation mechanism discussed here occurs *fiberwise*, when the reference frames of a system described by vector *fields* and spinor *fields* on a Lorentzian spacetime manifold get transformed by an element of $\mathrm{SL}(2,\mathbb{C})$ (in a suitable way).

4.2.7 Lemma (some useful spinor identities).

(i) $\sigma_{aA\dot{X}}{\sigma_b}^{B\dot{X}} + \sigma_{bA\dot{X}}{\sigma_a}^{B\dot{X}} = \eta_{ab}\delta_A^B$.

(ii) For $x \in (\mathbb{R}^4,\eta)$,

$$x_{A\dot{X}}\bar{x}^{\dot{X}B} = -\frac{1}{2}\, x^a x^b\, \eta_{ab}\, \delta_A^B\,.$$

$\delta_A^B := \mathrm{Id}_A^B$ denotes the identity on $\Delta_{\frac{1}{2},0}$.

Proof.

(i) At first, we notice:

$$\begin{aligned} &\sigma_{aA\dot{X}}{\sigma_b}^{B\dot{X}} + \sigma_{bA\dot{X}}{\sigma_a}^{B\dot{X}} = \eta_{ab}\delta_A^B \\ \Leftrightarrow\ &\epsilon^{BC}\epsilon^{\dot{X}\dot{Y}}\sigma_{aA\dot{X}}\sigma_{bC\dot{Y}} + \epsilon^{BC}\epsilon^{\dot{X}\dot{Y}}\sigma_{bA\dot{X}}\sigma_{aC\dot{Y}} = \eta_{ab}\delta_A^B \\ \Leftrightarrow\ &\epsilon^{\dot{X}\dot{Y}}\sigma_{aA\dot{X}}\sigma_{bB\dot{Y}} + \epsilon^{\dot{X}\dot{Y}}\sigma_{bA\dot{X}}\sigma_{aB\dot{Y}} = \eta_{ab}\epsilon_{AB} \end{aligned}$$

Then we make the following deductions:

$$
\begin{aligned}
&\eta_{ab} + \eta_{ba} = 2\eta_{ab} \\
\Rightarrow\ &\epsilon^{AB}\epsilon^{\dot{X}\dot{Y}}\sigma_{aA\dot{X}}\sigma_{bB\dot{Y}} + \epsilon^{BA}\epsilon^{\dot{X}\dot{Y}}\sigma_{aB\dot{X}}\sigma_{bA\dot{Y}} = \eta_{ab}\,\epsilon_{AB}\epsilon^{AB} \\
\Rightarrow\ &\epsilon^{AB}\epsilon^{\dot{X}\dot{Y}}\sigma_{aA\dot{X}}\sigma_{bB\dot{Y}} - \epsilon^{AB}\epsilon^{\dot{X}\dot{Y}}\sigma_{aB\dot{X}}\sigma_{bA\dot{Y}} = \eta_{ab}\,\epsilon_{AB}\epsilon^{AB}
\end{aligned}
$$

Because $\epsilon^{\dot{X}\dot{Y}}\sigma_{aA\dot{X}}\sigma_{bB\dot{Y}} - \epsilon^{\dot{X}\dot{Y}}\sigma_{aB\dot{X}}\sigma_{bA\dot{Y}} = 2\epsilon^{\dot{X}\dot{Y}}\sigma_{a[A|\dot{X}}\sigma_{b|B]\dot{Y}}$ (anti-symmetry in A and B), we can continue:

$$
\begin{aligned}
\Rightarrow\ &\epsilon^{\dot{X}\dot{Y}}\sigma_{aA\dot{X}}\sigma_{bB\dot{Y}} - \epsilon^{\dot{X}\dot{Y}}\sigma_{aB\dot{X}}\sigma_{bA\dot{Y}} = \eta_{ab}\epsilon_{AB} \\
\Rightarrow\ &\epsilon^{\dot{X}\dot{Y}}\sigma_{aA\dot{X}}\sigma_{bB\dot{Y}} + \epsilon^{\dot{Y}\dot{X}}\sigma_{aB\dot{X}}\sigma_{bA\dot{Y}} = \eta_{ab}\epsilon_{AB} \\
\Rightarrow\ &\epsilon^{\dot{X}\dot{Y}}\sigma_{aA\dot{X}}\sigma_{bB\dot{Y}} + \epsilon^{\dot{X}\dot{Y}}\sigma_{bA\dot{X}}\sigma_{aB\dot{Y}} = \eta_{ab}\epsilon_{AB}
\end{aligned}
$$

(ii) We calculate:

$$
\begin{aligned}
x_{A\dot{X}}\bar{x}^{\dot{X}B} &= \frac{1}{2}(x^a x^b \sigma_{aA\dot{X}}\sigma_b{}^{B\dot{X}} + x^b x^a \sigma_{bA\dot{X}}\sigma_a{}^{B\dot{X}}) \\
&= \frac{1}{2}(x^a x^b \sigma_{aA\dot{X}}\sigma_b{}^{B\dot{X}} + x^a x^b \sigma_{bA\dot{X}}\sigma_a{}^{B\dot{X}})\,, \\
&= \frac{1}{2}x^a x^b \eta_{ab}\delta_A^B
\end{aligned}
$$

where we used (i) for the last step.

□

5. Clifford Algebras and Dirac Spinors

This chapter intends to illuminate the interrelationship of the Clifford algebra $\mathrm{Cl}_{1,3} := \mathrm{Cl}(\mathbb{R}^4, \eta)$ of the Minkowski vector space, its spinor representation, Dirac matrices and the Dirac spinor formalism.

In section 5.1 we will start out briefly collecting relevant background knowledge on Clifford algebras to fix notation and to make our presentation self-contained. In section 5.2 we will propose a precise formal notion of "collection of Dirac marices", which fits into out mathematical framework described so far and which discloses an illuminating relationship of the (historical) concept of Dirac matrices with the spinor representations of $\mathrm{Cl}_{1,3}$ and $\mathrm{Cl}^c_{1,3} := \mathrm{Cl}_{1,3} \otimes \mathbb{C}$. The core aspect of this relationship will be reflected in a general version of Pauli's fundamental theorem on Dirac matrices.

Finally, in section 5.3 we will single out one particular spinor representation of $\mathrm{Cl}_{1,3}$ which we will call "the standard representation" and which brings about a canonical connection of the Dirac spinor formalism and the 2-spinor formalism.

5.1. Clifford Algebras and Related Definitions

5.1.1 Definition and proposition (Clifford algebra). Let (V, q) be a $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) with a quadratic form q.

(a) An associative unital $\mathbb{K}$-algebra $\mathrm{Cl}(V, q)$ together with a linear map $\iota\colon V \to \mathrm{Cl}(V, q)$ is called **Clifford algebra of** (V, q), if

  (i) $\forall x \in V: (\iota(x))^2 = q(x) \cdot \mathbb{1}$,

  (ii) (universal property:) if there is a second associative unital $\mathbb{K}$-algebra $\mathcal{A}$ and a linear map $\varphi\colon V \to \mathcal{A}$ such that $(\varphi(x))^2 = q(x) \cdot \mathbb{1}$ for all $x \in V$, then there exists a unique $\mathbb{K}$-algebra homomorphism $\Phi\colon \mathrm{Cl}(V, q) \to \mathcal{A}$, such that $\varphi = \Phi \circ \iota$.

(b) For every (V, q), there exists a Clifford algebra. Any two Clifford algebras $\mathrm{Cl}(V, q)$ and $\mathrm{Cl}'(V, q)$ of (V, q) are isomorphic.

(c) A concrete realisation of $\mathrm{Cl}(V, q)$ is given by the tensor algebra of V modulo the relation $\forall x \in V: x \otimes x = q(x)$, i. e.,

$$\mathrm{Cl}(V, q) := \mathcal{T}(V)/I\,,$$

where

$$\mathcal{T}(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k}\,,$$

and I is the ideal

$$I := \{x \otimes x - q(x) \,|\, x \in V\}\,.$$

(d) Every Clifford algebra $\mathrm{Cl}(V,q)$ has a $\mathbb{Z}_2$-grading with *grading automorphism* α given by the extension of $\alpha(v) = -v$ for $v \in V$ to all of $\mathrm{Cl}(V,q)$. It is easily seen that α is involutive ($\alpha^2 = \mathrm{Id}$) and thus there is the eigenspace decomposition

$$\mathrm{Cl}(V,q) = \mathrm{Cl}^+(V,q) \oplus \mathrm{Cl}^-(V,q)\,,$$

where $\mathrm{Cl}^+(V,q)$ is the subalgebra given by the eigenspace for eigenvalue 1 and $\mathrm{Cl}^-(V,q)$ is the subspace given by the eigenspace for eigenvalue -1. $\mathrm{Cl}^+(V,q)$ is called **even part,** $\mathrm{Cl}^-(V,q)$ is called **odd part.**

Proof. All this is well known; cf. [Fri97] and [LM89]. □

5.1.2 Lemma (complexification of Clifford algebra). Let V be an $\mathbb{R}$-vector space with quadratic form q. Let $q_\mathbb{C}$ be the $\mathbb{C}$-quadratic extension of q to the complexification $V \otimes \mathbb{C}$. Then canonically:

$$\mathrm{Cl}(V \otimes \mathbb{C}, q_\mathbb{C}) \cong \mathrm{Cl}(V,q) \otimes \mathbb{C}\,.$$

Proof. Cf. [Fri97, p. 12]. □

5.1.3 Notation (standard Clifford algebras).

- $\mathrm{Cl}_{r,s} := \mathrm{Cl}(\mathbb{R}^n, x_1^2 + \ldots + x_r^2 - x_{r+1}^2 - \ldots - x_{r+s}^2)$, $n = r+s$
- $\mathrm{Cl}_n := \mathrm{Cl}_{n,0}$
- $\mathrm{Cl}^c_{r,s} := \mathrm{Cl}_{r,s} \otimes \mathbb{C}$
- $\mathrm{Cl}^c_n := \mathrm{Cl}_n \otimes \mathbb{C}$

Notice that for all r, s such that $r + s = n$, $\mathrm{Cl}^c_n \cong \mathrm{Cl}^c_{r,s}$ (non-canonically), because if $e_1, \ldots, e_r, e_{r+1}, \ldots, e_{r+s}$ is a basis of $\mathbb{C}^n$ with $q(e_1) = \ldots = q(e_r) = 1$ and $q(e_{r+1}) = \ldots = q(e_{r+s}) = -1$, $ie_1, \ldots, ie_r, e_{r+1}, \ldots, e_{r+s}$ is a basis such that $q = 1$ for all base vectors (cf. [Fri97, p. 11]).

5.1.4 Remark (different sign conventions in the literature). [LM89, p. 8, p. 19] uses $x^2 = -q(x) \cdot \mathbb{1}$ in the definition of a Clifford algebra such that what they call $\mathrm{Cl}_{r,s}$ is $\mathrm{Cl}_{s,r}$ in our setting. [Bau81] uses our sign convention for the definition of a Clifford algebra but writes $C_{n,r}$ for what we denote $\mathrm{Cl}_{r,n-r}$. [Fri97] uses our sign convention for the definition of a Clifford algebra but uses $\mathcal{C}_n := \mathrm{Cl}_n = \mathrm{Cl}_{0,n}$ and $\mathcal{C}'_n := \mathrm{Cl}_{n,0}$.

In our (physical) context we shall mostly deal with $\mathrm{Cl}(\mathbb{R}^4, \eta)$ with η of signature $(+---)$. This Clifford algebra is denoted by $\mathrm{Cl}_{1,3}$ in this document, $\mathrm{Cl}_{3,1}$ in [LM89], $C_{4,1}$ in [Bau81] and doesn't occur in [Fri97].

5.1.5 Lemma. Let (V, η) be a $\mathbb{K}$-vector space with a symmetric bilinear form η. Let $\mathrm{Cl}(V, \eta)$ be its Clifford algebra. Then for all $x, y \in V \subseteq \mathrm{Cl}(V, \eta)$:

$$x \cdot y + y \cdot x = 2\,\eta(x, y)\,.$$

Proof. This is a standard formula. Cf. [LM89, p. 8]. □

The aim for the remaining part of this section is introducing the covering homomorphisms $\mathrm{Pin}(r, s) \to \mathrm{O}(r, s)$ and $\mathrm{SL}(2, \mathbb{C}) \cong \mathrm{Spin}^+(1, 3) \to \mathrm{SO}^+(1, 3) = \mathcal{L}_+^\uparrow$. In order to do this on a rigerous mathematical level we shall briefly introduce basic concepts like the group of units, the Clifford group and the twisted adjoint representation. Consider this a crash course on relevant facts (mostly) from [LM89, section I.2].

5.1.6 Definition (group of units). Let (V, q) be a $\mathbb{K}$-vector space with quadratic form q. The multiplicative subgroup

$$\mathrm{Cl}^\times(V, q) := \{x \in \mathrm{Cl}(V, q) \,|\, \exists x^{-1} \in \mathrm{Cl}(V, q)\colon x^{-1}x = xx^{-1} = \mathbb{1}\} \subseteq \mathrm{Cl}(V, q)$$

is called **group of units of** $\mathrm{Cl}(V, q)$**.** If V is finite-dimensional with $\dim_{\mathbb{K}}(V) = n$, $\mathrm{Cl}^\times(V, q)$ is a $\mathbb{K}$-Lie group of dimension 2^n. We write $\mathrm{Cl}_{r,s}^\times$, $\mathrm{Cl}_n^\times$, $\mathrm{Cl}_{r,s}^{c\times}$, $\mathrm{Cl}_n^{c\times}$, for the groups of units of $\mathrm{Cl}_{r,s}$, Cl_n, $\mathrm{Cl}_{r,s}^c$, Cl_n^c, respectively.

The representation of $\mathrm{Cl}^\times(V, q)$ on $\mathrm{Cl}(V, q)$, $\widetilde{\mathrm{Ad}}$, given by

$$\widetilde{\mathrm{Ad}}_\varphi(x) = \alpha(\varphi) \cdot x \cdot \varphi^{-1}, \quad \varphi \in \mathrm{Cl}^\times(V, q),\ x \in \mathrm{Cl}(V, q)$$

will be refered to as **twisted adjoint representation of** $\mathrm{Cl}^\times(V, q)$**.** (Cf. [LM89, p. 14].)

5.1.7 Definition and proposition (Clifford group). Let (V, q) be a vector space with quadratic form q. The subgroup

$$\Gamma(V, q) := \{\varphi \in \mathrm{Cl}^\times(V, q) \,|\, \widetilde{\mathrm{Ad}}_\varphi(V) = V\}$$

of $\mathrm{Cl}^\times(V, q)$ is called the **Clifford group of** (V, q)[1]. We write $\Gamma_{r,s}$, Γ_n, $\Gamma_{r,s}^c$, Γ_n^c, for the Clifford groups of $\mathrm{Cl}_{r,s}$, Cl_n, $\mathrm{Cl}_{r,s}^c$, Cl_n^c, respectively.

Let V be finite-dimensional and q non-degenerate. Then one finds that for all $\varphi \in \Gamma(V, q)$, $\widetilde{\mathrm{Ad}}_\varphi$ preserves the quadratic form q, i. e. $\forall x \in V \subseteq \mathrm{Cl}(V, q)\colon q(\widetilde{\mathrm{Ad}}_\varphi\, x) = q(x)$. Hence, the twisted adjoint representation $\widetilde{\mathrm{Ad}}$ restricts to

$$\widetilde{\mathrm{Ad}}\colon \Gamma(V, q) \to \mathrm{O}(V, q)\,,$$

where $\mathrm{O}(V, q)$ denotes the orthogonal group of (V, q). (Cf. [LM89, p. 16].)

[1] Notation in [LM89, p. 14]: $\tilde{P}(V, q)$

5.1.8 Definition (Pin **and** Spin **groups).** For a Clifford algebra $\mathrm{Cl}(V,q)$, $\mathrm{Pin}(V,q)$ denotes the multiplicative group generated by $\{x \in V \mid q(x) = \pm 1\}$. It is a subgroup of $\Gamma(V,q)$. Moreover, we define

$$\mathrm{Spin}(V,q) := \mathrm{Pin}(V,q) \cap \mathrm{Cl}^+(V,q)\,.$$

We write $\mathrm{Spin}(r,s)$, $\mathrm{Spin}(n)$, $\mathrm{Spin}^c(r,s)$, $\mathrm{Spin}^c(n)$, for the spin groups of $\mathrm{Cl}_{r,s}$, Cl_n, $\mathrm{Cl}^c_{r,s}$, Cl^c_n, respectively. Putting everything together we have this inclusion chain of subgroups:

$$\mathrm{Spin}(V,q) \subseteq \mathrm{Pin}(V,q) \subseteq \Gamma(V,q) \subseteq \mathrm{Cl}^\times(V,q) \subseteq \mathrm{Cl}(V,q)\,.$$

5.1.9 Proposition (covering homomorphisms). Let (V,q) be a $\mathbb{K}$-vector space with $\dim(V) = n < \infty$ and with non-degenerate quadratic form q. There is a short exact sequence

$$0 \to \mathbb{F} \to \mathrm{Pin}(V,q) \overset{\widetilde{\mathrm{Ad}}}{\to} \mathrm{O}(V,q) \to 1$$

restricting to

$$0 \to \mathbb{F} \to \mathrm{Spin}(V,q) \overset{\widetilde{\mathrm{Ad}}}{\to} \mathrm{SO}(V,q) \to 1\,,$$

where $\mathbb{F} = \{1,-1\} \cong \mathbb{Z}_2$ if $\mathbb{K} = \mathbb{R}$ and $\mathbb{F} = \{1,-1,i,-i\} \cong \mathbb{Z}_4$ if $\mathbb{K} = \mathbb{C}$. These sequences give (Lie) group covering maps $\mathrm{Pin}(V,q) \overset{\widetilde{\mathrm{Ad}}}{\to} \mathrm{O}(V,q)$. The covering

$$0 \to \mathbb{Z}_2 \to \mathrm{Pin}(r,s) \overset{\widetilde{\mathrm{Ad}}}{\to} \mathrm{O}(r,s) \to 1$$

is universal if and only if $r \in \{0,1\}$ and $s \geq 3$.
In the physically interesting case $r = 1$, $s = 3$, the restriction to $\mathrm{Spin}^+(1,3)$ (the connected component of the unit element in $\mathrm{Spin}(1,3)$) gives a universal covering homomorphism of $\mathcal{L}^\uparrow_+ = \mathrm{SO}^+(1,3)$,

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}^+(1,3) \overset{\widetilde{\mathrm{Ad}}}{\to} \mathrm{SO}^+(1,3) \to 1\,.$$

In light of proposition 3.2.3, this shows that $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Spin}^+(1,3)$.

Proof. The first statement is [LM89, theorem 2.9]. The criteria for the covering maps' being universal are [Bau81, Folgerung 1.2]. For the last statement also cf. [Bau81, pp. 60]. ☐

5.2. The Spinor Representation of $\mathrm{Cl}_{1,3}$ and Dirac-Matrices

The representation theory of $\mathrm{Cl}_{r,s}$ depends crucially on r and s. This is why, from now on, our exposition shall be restricted to the special case of $\mathrm{Cl}_{1,3}$ and $\mathrm{Cl}^c_{1,3}$, i. e. to the real Clifford algebra of the Minkowski vector space and its complexification. For more general results the reader be referred to [Bau81] and [LM89, pp. 30].

As already stated at the beginning of section 3,

$$(\mathbb{R}^4, \eta)_{\mathbb{C}} := (\mathbb{R}^4, \eta) \otimes \mathbb{C} = (\mathbb{C}^4, \eta_{\mathbb{C}}) =: (\mathbb{C}^4, \eta)$$

denotes the **complexified Minkowski vector space**, where $\eta_{\mathbb{C}}$ is the $\mathbb{C}$-bilinear extension of η. We then have $\mathrm{Cl}^c_{1,3} = \mathrm{Cl}\big((\mathbb{R}^4, \eta)_{\mathbb{C}}\big)$.

Notice that $\mathrm{Cl}^c_{1,3} \cong \mathrm{Cl}^c_4$, as stated in 5.1.3. However, we will always work with $\mathrm{Cl}^c_{1,3}$ here as it comes with the canonical embedding

$$\mathrm{Cl}_{1,3} \to \mathrm{Cl}_{1,3} \otimes \mathbb{C} = \mathrm{Cl}^c_{1,3} \,.$$

5.2.1 Definition (representation of a Clifford algebra). Let $\mathrm{Cl}(V, q)$ be a Clifford algebra, where V is a $\mathbb{K}$-vector space ($\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$) with quadratic form q. Let W be a finite-dimensional $\mathbb{L}$-vector space, where $\mathbb{L} = \mathbb{C}$ or $\mathbb{L} = \mathbb{R}$ and $\mathbb{L} \supseteq \mathbb{K}$. A **Clifford algebra representation of** $\mathrm{Cl}(V, q)$ **on** W is a $\mathbb{K}$-algebra homomorphism

$$\rho \colon \mathrm{Cl}(V, q) \to \mathrm{End}_{\mathbb{K}}(W) \,.$$

W, the representation space of ρ, is then called a $\mathrm{Cl}(V, q)$**-module** (over $\mathbb{K}$) and the action of $\mathrm{Cl}(V, q)$ on the elements of W is refered to as **Clifford multiplication.** We may write

$$\rho(\varphi)(x) =: \varphi \cdot x, \qquad \varphi \in \mathrm{Cl}(V, q),\ x \in W \,.$$

5.2.2 Definition and proposition (spinor representation).

(i) $\mathrm{Cl}^c_{1,3}$ is $\mathbb{C}$-algebra-isomorphic to $\mathrm{End}(\mathbb{C}^4)$:

$$\mathrm{Cl}^c_{1,3} \cong \mathrm{End}(\mathbb{C}^4) \,.$$

Such an isomorphism considered as representation of Cl^c_4 on $\mathbb{C}^4$ is called a **spinor representation of** $\mathrm{Cl}^c_{1,3}$**.** It is irreducible and up to equivalence the only irreducible complex representation of $\mathrm{Cl}^c_{1,3}$.

(ii) A representation $\kappa_{1,3}$ of $\mathrm{Cl}_{1,3}$ is called a **spinor representation of** $\mathrm{Cl}_{1,3}$**,** if $\kappa^c_{1,3} := \kappa_{1,3} \otimes \mathbb{C}$ is a spinor representation of $\mathrm{Cl}^c_{1,3}$. Every spinor representation of $\mathrm{Cl}_{1,3}$ is irreducible and up to equivalence the only irreducible representation of $\mathrm{Cl}_{1,3}$.

(iii) As $\mathrm{Cl}^c_{1,3} \cong \mathrm{Cl}_{1,3} \otimes \mathbb{C}$, all this shows that spinor representations $\kappa_{1,3}$ of $\mathrm{Cl}_{1,3}$ and spinor representations $\kappa^c_{1,3}$ of $\mathrm{Cl}^c_{1,3}$ are in an one-one correspondence given by $\kappa^c_{1,3} = \kappa_{1,3} \otimes \mathbb{C}$.

Proof. Existence of an isomorphism $\mathrm{End}(\mathbb{C}^4) \cong \mathrm{Cl}^c_4 (\cong \mathrm{Cl}^c_{1,3})$ is proven in [Fri97, p. 14]. Irreducibility of spinor representations of $\mathrm{Cl}^c_{1,3}$ is trivial and uniqueness up to isomorphism follows from [LM89, theorem 5.7]. Irreducibility of spinor representations $\kappa_{1,3}$ of $\mathrm{Cl}_{1,3}$ is a consequence of $\kappa_{1,3} \otimes \mathbb{C}$ being irreducible (by a result similar to 1.2.1-b). Uniqueness again follows from [LM89, theorem 5.7]. $\square$

With respect to a fixed basis of $\mathbb{C}^4$, spinor representations $\kappa^c_{1,3}$ and $\kappa_{1,3}$ yield representations of $\mathrm{Cl}^c_{1,3}$ and $\mathrm{Cl}_{1,3}$ as *matrices*. In order to deal with such matrix representations more conveniently we now introduce the historical concept of **Dirac matrices**.

5.2.3 Definition (Dirac matrices). Let $b_0, \dots, b_3$ be a basis of $(\mathbb{R}^4, \eta)_{\mathbb{C}}$ and let $\eta_{\mu\nu}$ be the component representation of η with respect to this basis. A collection of complex 4×4-matrices $\gamma_0, \dots, \gamma_3$ is called a **collection of Dirac matrices with respect to the basis** $\{b_\mu\}$, if

$$\forall \mu, \nu \colon \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} \cdot \mathbb{1} \,. \tag{$*$}$$

Notice that the basis $\{b_\mu\}$ is not unique, meaning that there might be another basis of $(\mathbb{R}^4, \eta)_{\mathbb{C}}$ with respect to which $\{\gamma_\mu\}$ comprises a collection of Dirac matrices.

5.2.4 Remark. Notice that the statement "$\{\gamma_\mu\}$ is a collection of Dirac matrices" is meaningless without the information *with respect to which basis* of $(\mathbb{R}^4, \eta)_{\mathbb{C}}$, because the term $\eta_{\mu\nu}$ in equation $(*)$ depends on a choice of basis.

5.2.5 Theorem (Dirac matrices and spinor representations).

(a) Every collection of Dirac matrices $\{\gamma_\mu\}$ with respect to a basis $b_0, \dots, b_3$ of $(\mathbb{R}^4, \eta)_{\mathbb{C}}$ induces an unique $\mathbb{C}$-algebra-isomorphism

$$\gamma \colon \mathrm{Cl}^c_{1,3} \to \mathrm{Mat}_{4\times 4}(\mathbb{C})$$

such that for $x = x^\mu b_\mu \in \mathbb{C}^4 \subseteq \mathrm{Cl}^c_{1,3}$,

$$\gamma(x) = x^\mu \gamma_\mu \,.$$

Conversely, every $\mathbb{C}$-algebra-isomorphism $\gamma \colon \mathrm{Cl}^c_{1,3} \to \mathrm{Mat}_{4\times 4}(\mathbb{C})$ may be described by a collection of Dirac matrices: After choosing a basis $b_0, \dots, b_3$ of $\mathbb{C}^4 \subseteq \mathrm{Cl}^c_{1,3}$, just set $\gamma_\mu := \gamma(b_\mu)$.

(b) Let a choice of Dirac matrices $\{\gamma_\mu\}$ with respect to a basis $\{b_\mu\}$ be given and let $\gamma \colon \mathrm{Cl}^c_{1,3} \to \mathrm{Mat}_{4\times 4}(\mathbb{C})$ be the induced isomorphism according to (a). After choosing a basis $E_1, \dots, E_4$ of $\mathbb{C}^4$ this defines a *spinor representation* $\mathrm{Cl}^c_{1,3} \to \mathrm{End}(\mathbb{C}^4)$, by setting:

$$\gamma(x)\xi := [\gamma(x)]^\nu{}_\mu \xi^\mu E_\nu, \qquad x \in \mathrm{Cl}^c_{1,3},\ \xi = \xi^\mu E_\mu \in \mathbb{C}^4 \,.$$

(Notice that $E_1, \dots, E_4$ is meant to be a basis of the representation space, $\mathbb{C}^4$, while $b_0, \dots, b_3$ is a basis of $(\mathbb{R}^4, \eta)_{\mathbb{C}}$.)

Conversely, every spinor representation $\kappa^c_{1,3} \colon \mathrm{Cl}^c_{1,3} \to \mathrm{End}(\mathbb{C}^4)$ defines a collection of Dirac matrices after choosing a basis $b_0, \dots, b_3$ of $\mathbb{C}^4 \subseteq \mathrm{Cl}^c_{1,3}$ and a basis $E_1, \dots, E_4$ of the representation space $\mathbb{C}^4$: Just let γ_μ be the component matrix of $\kappa^c_{1,3}(b_\mu)$ with respect to $\{E_\nu\}$.

(c) Using proposition 5.2.2-iii all this shows that after choosing a basis $\{E_\nu\}$ of $\mathbb{C}^4$, each collection of Dirac matrices $\{\gamma_\mu\}$ with respect to a basis $\{b_\mu\}$ of $(\mathbb{R}^4, \eta)$ gives a spinor representation of $\mathrm{Cl}_{1,3}$; conversely, every spinor representation of $\mathrm{Cl}_{1,3}$ may be described by a collection of Dirac matrices with respect to any basis $\{b_\mu\}$ of $(\mathbb{R}^4, \eta)$.

Before we enter into the proof, we make a brief comment on what we just learned:

5.2.6 Remark.

(a) Notice that (a) just shows that a choice of a collection of Dirac matrices $\{\gamma_\mu\}$ with respect to a basis $\{b_\mu\}$ of $(\mathbb{R}^4, \eta)_\mathbb{C}$ is equivalent to a choice of $\mathbb{C}$-algebra-isomorphism $\mathrm{Cl}^c_{1,3} \to \mathrm{Mat}_{4\times 4}(\mathbb{C})$, in the sense that both choices "contain the same amount of information".

(b) From (b) we learn that after two choices of bases (one for $(\mathbb{R}^4, \eta)_\mathbb{C}$ and one for the representation space, $\mathbb{C}^4$), *a choice of Dirac matrices fixes one concrete spinor representation* $\kappa^c_{1,3} \colon \mathrm{Cl}^c_{1,3} \to \mathrm{End}(\mathbb{C}^4)$.

Observe the similarity of the relationship between Dirac matrices and the spinor representation on the one hand and the Pauli spin matrices $\tilde{\sigma}_\mu$ and the tensor-spinor ${\sigma_a}^{A\dot{X}}$ from chapter 4 on the other hand. We shall see in the next section (5.3) that this similarity is not merely coincidence.

The following proposition is a preparation for the proof of theorem 5.2.5. It addresses the question "what happenes to the Dirac matrices under a change of bases?" Of course, as there are two kinds of bases involved, there are two kinds of base transformations:

5.2.7 Proposition (base transformations and Dirac matrices).

(a) Base transformation in $(\mathbb{R}^4, \eta)_\mathbb{C}$: Let $\{\gamma_\mu\}$ be a collection of Dirac matrices with respect to a basis $\{b_\mu\}$ of $(\mathbb{R}^4, \eta)_\mathbb{C}$ and let ${A^\nu}_\mu$ be an invertible complex 4×4-matrix (base transformation matrix). Then

$$\gamma'_\mu := {A^\nu}_\mu \gamma_\nu$$

is a set of Dirac matrices with respect to the transformed basis

$$b'_\mu = {A^\nu}_\mu b_\nu, \quad \mu = 0, \dots, 3 .$$

(b) **Pauli's fundamental theorem for Dirac matrices;** base transformation in the representation space: Let $\{\gamma_\mu\}$ be a set of Dirac matrices with respect to some basis $\{b_\mu\}$ of $(\mathbb{R}^4, \eta)$. A second collection $\gamma'_0, \dots, \gamma'_3 \in \mathrm{Mat}_{4\times 4}(\mathbb{C})$ is also a collection of Dirac matrices with respect to $\{b_\mu\}$ if and only if there is an invertible matrix $S \in \mathrm{Mat}_{4\times 4}(\mathbb{C})$ (base transformation matrix) such that

$$\forall \mu \colon \gamma'_\mu = S \gamma_\mu S^{-1} .$$

Proof.

(a) Let $b_0, \dots, b_3$ and $b'_0, \dots, b'_3$ be two bases of $(\mathbb{R}^4, \eta)_\mathbb{C}$ and let $A \in \mathrm{Mat}_{4\times 4}(\mathbb{C})$ be the base transformation matrix, i. e. the matrix such that

$$\forall \mu \colon b'_\mu = {A^\nu}_\mu b_\nu .$$

Let $[\eta]_{\mu\nu}$ be the component representation of η with respect to $\{b_\mu\}$ and let $[\eta]'_{\mu\nu}$ be the component representation of η with respect to $\{b'_\mu\}$.

We have to show that for $\gamma'_\mu := A^\nu{}_\mu \gamma_\nu$,

$$\forall \mu, \nu \colon \gamma'_\mu \gamma'_\nu + \gamma'_\nu \gamma'_\mu = 2[\eta]'_{\mu\nu} \cdot \mathbb{1}\,.$$

Starting with the left hand side, we calculate:

$$\begin{aligned}
\gamma'_\mu \gamma'_\nu + \gamma'_\nu \gamma'_\mu &= \sum_{i,j} A^i{}_\mu A^j{}_\nu \gamma_i \gamma_j + \sum_{i,j} A^j{}_\nu A^i{}_\mu \gamma_j \gamma_i \\
&= \sum_{i,j} A^i{}_\mu A^j{}_\nu (\gamma_i \gamma_j + \gamma_j \gamma_i) \\
&= \sum_{i,j} A^i{}_\mu A^j{}_\nu \cdot 2[\eta]_{ij} \cdot \mathbb{1} \\
&= 2[\eta]'_{\mu\nu} \cdot \mathbb{1}
\end{aligned}$$

(b) The backward implication ($S \in \mathrm{Mat}_{4\times 4}(\mathbb{C})$ is invertible $\Rightarrow \gamma'_\mu := S\gamma_\mu S^{-1}$ is again a collection of Dirac matrices) is trivial. The forward direction is a tricky historical result by Pauli, *provided that* $\{\gamma_\mu\}$ *are Dirac matrices with respect to a pseudo-orthonormal basis of* $(\mathbb{R}^4, \eta)$ (cf. [Pau36], [?] or [Mes58, ch. XX, §10]). The generalisation of Pauli's version to collections of Dirac matrices with respect to arbitrary bases of $(\mathbb{R}^4, \eta)_\mathbb{C}$ may be performed using part (a):

Let $\{\gamma_\mu\}$ and $\{\gamma'_\mu\}$ be two collections of Dirac matrices with respect to a basis $\{b_\mu\}$ of $(\mathbb{R}^4, \eta)_\mathbb{C}$. Let $\{\tilde{b}_\mu\}$ be a pseudo-orthonormal basis of $(\mathbb{R}^4, \eta)$ and let $A^\nu{}_\mu$ be the base transformation matrix such that $\forall \mu \colon \tilde{b}_\mu = A^\nu{}_\mu b_\nu$. Define $\tilde{\gamma}_\mu := A^\nu{}_\mu \gamma_\nu$ and $\tilde{\gamma}'_\mu := A^\nu{}_\mu \gamma'_\nu$. According to (a), $\tilde{\gamma}_\mu$ and $\tilde{\gamma}'_\mu$ are collections of Dirac matrices with respect to $\tilde{b}_\mu$. Using the version of Pauli's fundamental theorem on Dirac matrices as given in [Mes58], there is an invertible $S \in \mathrm{Mat}_{4\times 4}(\mathbb{C})$, such that $\forall \mu \colon \tilde{\gamma}'_\mu = S\tilde{\gamma}_\mu S^{-1}$. Looking at this equation a bit more closely, we find:

$$A^\nu{}_\mu \gamma'_\nu = S\, A^\nu{}_\mu \gamma_\nu\, S^{-1} = A^\nu{}_\mu\, S\gamma_\nu S^{-1}\,.$$

By multiplication of A^{-1} from the left we find that $\gamma'_\mu = S\gamma_\mu S^{-1}$. Thus, the transformation matrix S also works for the $\{\gamma_\mu\}$ and $\{\gamma'_\mu\}$, which are our original collections of Dirac matrices with respect to $\{b_\mu\}$.

□

5.2.8 Remark. Part (a) of this proposition is why indices of Dirac matrices are to be written as subscripts: At a change of basis in $(\mathbb{R}^4, \eta)_\mathbb{C}$, Dirac matrices transform covariantly. This, of course, fits well into the picture of theorm 5.2.5.

Proof of theorem 5.2.5.

(i) To prove the first part of (a), let $\{\gamma_\mu\}$ be a collection of Dirac matrices with respect to a basis $\{b_\mu\}$ and extend the mapping $b_\mu \mapsto \gamma_\mu$ linearly to a vector space homomorphism $\varphi \colon \mathbb{C}^4 \to \mathrm{Mat}_{4\times 4}(\mathbb{C})$. According the the universal property in definition 5.1.1, there is a $\mathbb{C}$-algebra homomorphism $\gamma \colon \mathrm{Cl}^c_{1,3} \to \mathrm{Mat}_{4\times 4}(\mathbb{C})$ with $\varphi = \gamma \circ \iota$ ($\iota \colon \mathbb{C}^4 \to \mathrm{Cl}^c_{1,3}$ is the embedding map), as soon as

$$\forall x \in \mathbb{C}^4 \colon (\varphi(x))^2 = \eta(x,x) \cdot \mathbb{1}\,. \tag{$*$}$$

Notice that γ is just the map we are searching for.
So, let some $x = x^\mu b_\mu \in \mathbb{C}^4$ be given. We find in the algebra $\mathrm{Mat}_{4\times4}(\mathbb{C})$:

$$\begin{aligned}
\varphi(x)\cdot\varphi(x) &= \sum_{\mu,\nu} x^\mu x^\nu b_\mu \cdot b_\nu \\
&= \sum_{\mu<\nu} x^\mu x^\nu (b_\mu b_\nu + b_\nu b_\mu) + \sum_\mu x^\mu x^\mu b_\mu b_\mu \\
&= \sum_{\mu<\nu} x^\mu x^\nu\, 2\cdot \eta_{\mu\nu}\cdot \mathbb{1} + \sum_\mu x^\mu x^\mu \eta_{\mu\mu}\cdot \mathbb{1} \\
&= \eta(x,x)\cdot \mathbb{1}\,,
\end{aligned}$$

where we made use of the defining property of Dirac matrices in the third step. This shows relation $(*)$ and thus the first part of (a), up to the fact that γ is an isomorphism, which will be step (iv) of this proof.

(ii) To prove the second part of (b), let $\kappa^c_{1,3}\colon \mathrm{Cl}^c_{1,3} \to \mathrm{End}(\mathbb{C}^4)$ be a spinor representation. Choose bases $\{b_\mu\}$ of $(\mathbb{R}^4,\eta)_\mathbb{C}$ and $\{E_\nu\}$ of $\mathbb{C}^4$. Then we shall show that $\gamma_\mu := [\kappa^c_{1,3}(b_\mu)]$, where $[\kappa^c_{1,3}(b_\mu)]$ means the representation of the endomorphism $\kappa^c_{1,3}(b_\mu)$ as component matrix with respect to $\{E_\nu\}$, constitute a collection of Dirac matrices, i. e. $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$. Of course, this is completely independent of the choice of basis $\{E_\nu\}$. We may equivalently show on the level of $\mathrm{End}(\mathbb{C}^4)$ that

$$\forall \mu,\nu\colon \kappa^c_{1,3}(b_\mu)\kappa^c_{1,3}(b_\nu) + \kappa^c_{1,3}(b_\nu)\kappa^c_{1,3}(b_\mu) = 2\eta_{\mu\nu}\,\mathrm{Id}\ .$$

But as $\kappa^c_{1,3}$ is an algebra-homomorphism, this statement follows immediately from lemma 5.1.5.

(iii) The first part of (b) is a consequence of the second part of (b) (step (ii) of this proof) and Pauli's theorem (5.2.7-b): Let $\{\gamma_\mu\}$ be a choice of Dirac matrices with respect to $\{b_\mu\}$. Let $\{E_\nu\}$ be a basis of the representation space, $\mathbb{C}^4$. Let $\kappa^c_{1,3}\colon \mathrm{Cl}^c_{1,3} \to \mathrm{End}(\mathbb{C}^4)$ be a spinor representation. Let γ'_μ be the collection of Dirac matrices induced by $\kappa^c_{1,3}$ with respect to $\{b_\mu\}$ and $\{E_\nu\}$. Using Pauli's theorem, there is a regular matrix S such that $\gamma_\mu = S\gamma'_\mu S^{-1}$. Considering all matrices as endomorphisms of $\mathbb{C}^4$ (represented with respect to $\{E_\nu\}$), this shows that $\gamma = S\kappa^c_{1,3}S^{-1}$, where $\gamma\colon \mathrm{Cl}^c_{1,3} \to \mathrm{End}(\mathbb{C}^4)$ is the homomorphism induced by $\{\gamma_\mu\}$. This shows that γ is an irreducible representation of $\mathrm{Cl}^c_{1,3}$ and that γ and $\kappa^c_{1,3}$ are equivalent.

(iv) That the homomorphism $\gamma\colon \mathrm{Cl}^c_{1,3} \to \mathrm{End}(\mathbb{C}^4)$ as constucted in (a) is indeed an isomorphism follows from the previous consideration, where we showed that γ is a spinor representation, together with proposition 5.2.2, where we learned that $\mathrm{Cl}^c_{1,3} \cong \mathrm{End}(\mathbb{C}^4)$ by means of any spinor representation.
The second part of (a) (the converse direction) now is immediate as well: An isomorphism $\gamma\colon \mathrm{Cl}^c_{1,3} \to \mathrm{End}(\mathbb{C})$ is a spinor representation according to proposition 5.2.2 and thus may be given by Dirac matrices with respect to a basis of $(\mathbb{R}^4,\eta)_\mathbb{C}$, as we learned in (iii).

(v) Finally, (c) is an immediate consequence of proposition 5.2.2-iii and the previous results.

□

5.3. The Dirac-Spinor Formalism

The spinor representation of $\mathrm{Cl}_{1,3}$ is of particular intererst for our purposes because of its relation to irreducible $\mathrm{SL}(2,\mathbb{C})$-representations which shall be illuminated now. Recall from proposition 5.1.9 that $\mathrm{Spin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})$. Moreover, the Minkowski vector space $(\mathbb{R}^4,\eta)$ be equipped with the standard bases as declared at the beginning of chapter 3.

5.3.1 Definition and proposition (Dirac spinor representation). The restriction of a spinor representation $\kappa_{1,3}$ of $\mathrm{Cl}_{1,3}$ to $\mathrm{Spin}^+(1,3) \subseteq \mathrm{Cl}_{1,3}$ is equivalent to the representation

$$D^D := D^{(\frac{1}{2},0)} \oplus \tilde{D}^{(0,\frac{1}{2})*} \quad \text{on} \quad \Delta_D := \Delta_{\frac{1}{2},0} \oplus \Delta^*_{0,\frac{1}{2}} (\cong \mathbb{C}^4)\,.$$

D^D is called **bi-spinor representation or Dirac spinor representation of** $\mathrm{SL}(2,\mathbb{C})$**,** its representation space Δ_D is called the **space of bi-spinors or Dirac spinors.** Setting $\Delta_D^+ := \Delta_{\frac{1}{2},0}$ and $\Delta_D^- := \Delta^*_{0,\frac{1}{2}}$, one also writes $\Delta_D = \Delta_D^+ \oplus \Delta_D^-$. The spaces $\Delta_D^\pm$ are then refered to as the **positive/negative chiral parts** of Δ_D.

We shall postpone the proof of this central result until after the introduction of **Dirac spinor notation,** also refered to as **4-spinor or bi-spinor notation:**

1. The elements of Δ_D are called Dirac spinors (or bi-spinors or 4-spinors) and are denoted using **abstract indices**, if not explicitly stated otherwise. For Dirac spinors we use capital letters with a tilde ($\tilde{\ }$), written as supscripts. The elements of the dual space $\Delta_D^* = \Delta^*_{\frac{1}{2},0} \oplus \Delta_{0,\frac{1}{2}}$, called Dirac co-spinors, are denoted using capital letter with a tilde written as subscript (as usual). E. g.

$$\Psi = \Psi^{\tilde{A}} \in \Delta_D, \quad \Phi = \Phi_{\tilde{A}} = \Delta_D^*\,.$$

2. Of course, one may form arbitrary **tensor products** of Dirac (co-)spinors. They are denoted using multiple indices similar to ordinary tensor calculus or 2-spinor formalism.
3. **Contractions** are, as usual in (abstract) index notation, written like

$$\Psi_{\tilde{C}} = \Phi^{\tilde{A}}{}_{\tilde{B}\tilde{C}} x_{\tilde{A}} y^{\tilde{B}}\,.$$

   Notice that this is not an implicit summation, as we are using abstract indices.
4. There is a natural anti-linear isomorphism

$$\begin{aligned} {}^+ \colon \Delta_D &\to \Delta_D^* \\ \Psi^{\tilde{A}} &\mapsto \Psi^+_{\tilde{A}} \end{aligned}$$

   given by complex conjugation on $\Delta_{\frac{1}{2},0}$ and $\Delta^*_{0,\frac{1}{2}}$:

$$\begin{pmatrix} \psi^A \\ \varphi_{\dot{A}} \end{pmatrix}^+ := \begin{pmatrix} \bar{\varphi}_A \\ \bar{\psi}^{\dot{A}} \end{pmatrix}.$$

   For $\Psi \in \Delta_D$, Ψ^+ is called the **Dirac adjoint.** The inverse map is denoted the same way: For $\zeta \in \Delta_D^*$, $\zeta^+ \in \Delta_D$ is the preimage under this isomorphism. Of course, $\forall \Psi \in \Delta_D \colon \Psi^{++} = \Psi$.

5. In appropriate cases we may fall back to usual **component notation:** Let $\{B_\mu\}$ denote a basis of $\Delta_{\frac{1}{2},0}$. By complex conjugation and subsequently forming the dual basis, we obtain a basis $\{\bar{B}^{\dot{\mu}}\}$ of $\Delta^*_{0,\frac{1}{2}}$. $\{B_\mu\}$ together with $\{\bar{B}^{\dot{\mu}}\}$ induce a basis $\{\tilde{B}_{\tilde{\mu}}\}$ of Δ_D defined as

$$\tilde{B}_{\tilde{1}} := (B_1, 0), \quad \tilde{B}_{\tilde{2}} := (B_2, 0), \quad \tilde{B}_{\tilde{3}} := (0, \bar{B}^{\dot{1}}), \quad \tilde{B}_{\tilde{4}} := (0, \bar{B}^{\dot{2}})\,.$$

We shall refer to $\{\tilde{B}_{\tilde{\mu}}\}$ as the basis of Δ_D induced by $\{B_\mu\}$. The dual basis $\{\tilde{B}^{\tilde{\mu}}\}$ may be refered to as the basis of Δ^*_D induced by $\{B_\mu\}$.
6. Let $\{E_\mu\}$, $A = 1, 2$, denote the standard basis of $\Delta_{\frac{1}{2},0}$. Then the induced basis $\{\tilde{E}_{\tilde{\mu}}\}$ is called **standard bases of Δ_D,** and the induced basis $\{\tilde{E}^{\tilde{\mu}}\}$ is called **standard basis of Δ^*_D.**
7. In the literature, Dirac spinors are also called 4-spinors because Δ_D is 4-dimensional. They are also called bi-spinors because every Dirac spinor "consists of two ordinary (2-)spinors".

Proof of proposition 5.3.1. We did not fully develop all the background theory for an elegant version of this proof. We just present a sketch for a proof, in order to give an idea of why this result holds, but possibly leaving out detailed calculations and arguments.

Let $\{\gamma_\mu\}$ denote the collection of Dirac matrices given in example 5.3.3. All spinor representations of $\mathrm{Cl}_{1,3}$ are equivalent by proposition 5.2.2-ii, therefore it suffices to prove the assertion for the concrete spinor representation $\kappa_{1,3}$ induced by the $\{\gamma_\mu\}$ with respect to the standard bases $\{e_\mu\}$ of $(\mathbb{R}^4, \eta)$ and $\{\tilde{E}_\mu\}$ of Δ_D.

The Lie algebra of $\mathrm{Spin}(1,3)^+$ is given by the vector space of bivectors (cf. [Cno02, pp. 32]),

$$\mathfrak{spin}(1,3) = \langle e_{10}, e_{20}, e_{30}, e_{12}, e_{23}, e_{31}\rangle_{\mathbb{R}}\,,$$

where $e_{\mu\nu} := e_\mu \cdot e_\nu$, and it can be shown that the Lie group isomorphism $\varphi\colon \mathrm{Spin}(1,3)^+ \cong \mathrm{SL}(2,\mathbb{C})$ mentioned in proposition 5.1.9 is on the Lie algebra level given by

$$\begin{aligned} d\varphi\colon \mathfrak{spin}(1,3) &\longrightarrow \mathfrak{sl}(2,\mathbb{C}) \\ e_{i0} &\mapsto 2is_i \\ e_{ij} &\mapsto \sum_k \varepsilon_{ijk}\, 2s_k, \qquad i,j = 1,2,3,\ i<j\,. \end{aligned}$$

Using the matrix representation of $\mathrm{Cl}_{1,3}$ given by $\{\gamma_\mu\}$, the bivectors $e_{\mu\nu}$ get represented by $\gamma_\mu \cdot \gamma_\nu$. It is easily calculated that for $i,j = 1,2,3,\ i<j$,

$$\begin{aligned} \gamma_i \cdot \gamma_0 &= \begin{pmatrix} \tilde{\sigma}_i\,\tilde{\sigma}_0 & 0 \\ 0 & -\tilde{\sigma}_i\,\tilde{\sigma}_0 \end{pmatrix} = \begin{pmatrix} \tilde{\sigma}_i & 0 \\ 0 & -\tilde{\sigma}_i \end{pmatrix} = \begin{pmatrix} 2i\,s_i & 0 \\ 0 & -2i\,s_i \end{pmatrix} \\ \gamma_i \cdot \gamma_j &= \begin{pmatrix} -\tilde{\sigma}_i\,\tilde{\sigma}_j & 0 \\ 0 & -\tilde{\sigma}_i\,\tilde{\sigma}_j \end{pmatrix} = \begin{pmatrix} -i\,\sigma_k & 0 \\ 0 & -i\,\sigma_k \end{pmatrix} = \begin{pmatrix} 2\,s_k & 0 \\ 0 & 2\,s_k \end{pmatrix} \end{aligned}$$

Let $[\xi]$ denote the matrix, representing an element $\xi \in \mathfrak{spin}(1,3)$ in the matrix representation of $\mathfrak{spin}(1,3)$ induced by $\{\gamma_\mu\}$. Then notice that the action of ξ upon an element $x \in \Delta_D$ by means of the Lie algebra representation associated with $\kappa_{1,3}|_{\mathrm{Spin}^+(1,3)}$

is given by matrix multiplication $[\xi] \cdot [x]$, where $[x]$ is the column vector representing x with respect to $\{\tilde{E}_\mu\}$.

Our arbitrary element ξ is a sum of the $\gamma_i \cdot \gamma_0$ and $\gamma_i \cdot \gamma_j$ and thus is block diagonal. This shows that the representation space may be splitted into two invariant 2-dimensional subspaces. It is easily checked on the matrix level that these 2-dimensional subrepresentations are irreducible.

Finally, it can be seen on the matrix level that the first irreducible subrepresentation (given by the top-left part of the matrices $\gamma_\mu \cdot \gamma_\nu$) is canonically equivalent to the Lie algebra representation $\hat{D}^{(\frac{1}{2},0)}$ (use the standard basis $\{E_\mu\}$ of $\Delta_{\frac{1}{2},0}$, notice that $\tilde{E}_1$ and $\tilde{E}_2$ are given by E_1 and E_2). Analogously, it turns out that the second irreducible subrepresentation (given by the bottom-right part of the matrices $\gamma_\mu \cdot \gamma_\nu$) equals the dual representation of $\hat{D}^{(0,\frac{1}{2})}$ (use the standard basis $\{\bar{E}^{\dot{\mu}}\}$ of $\Delta^*_{0,\frac{1}{2}}$ but notice that we assume the elements of $\Delta^-_D = \Delta^*_{0,\frac{1}{2}}$ to be given as *column* vectors).

In this way one proves the assertion for the Lie algebra representation associated with D^D. Then using proposition 1.1.7 completes the proof. □

The Standard Representation

We may use proposition 5.3.1 to single out a special kind of Dirac matrices which is connected to the Dirac spinor formalism in a particular way.

5.3.2 Definition (standard representation).

(a) Using the tensor-spinor ${\sigma_a}^{A\dot{B}}$ from definition 4.2.1, we define the **"Dirac tensor-spinor"**

$$\gamma_a{}^{\tilde{A}}{}_{\tilde{B}} \in (\mathbb{C}^4)^* \otimes \Delta_D \otimes \Delta_D^* \cong (\mathbb{C}^4)^* \otimes \mathrm{End}(\Delta_{\frac{1}{2},0} \oplus \Delta^*_{0,\frac{1}{2}})$$

by

$$\gamma_a{}^{\tilde{A}}{}_{\tilde{B}} = \begin{pmatrix} 0 & {\sigma_a}^{A\dot{B}} \\ \bar{\sigma}_{a\dot{A}B} & 0 \end{pmatrix}.$$

(Cf. remark 5.3.4 for an explanation of this notation.)

(b) Let $\{\gamma_\mu\}$ be a collection of Dirac matrices with respect to a basis $\{b_\mu\}$ of $(\mathbb{R}^4, \eta)_{\mathbb{C}}$, and let $\{B_\mu\}$ be a basis of $\Delta_{\frac{1}{2},0}$. Then $\{\gamma_\mu\}$ is said to **belong to the standard representation with respect to** $\{B_\mu\}$[2], if

$$\gamma_\mu = \sqrt{2} \cdot [\gamma_\mu{}^{\tilde{A}}{}_{\tilde{B}}], \quad \forall \mu = 0, \ldots, 3, \tag{$*$}$$

where $[\gamma_\mu{}^{\tilde{A}}{}_{\tilde{B}}]$ denotes the component matrix of $\gamma_a{}^{\tilde{A}}{}_{\tilde{B}}$ with respect to $\{b_\mu\}$ and $\{\tilde{B}_{\tilde{\mu}}\}$.[3]

[2]This definition differs from the "usual" definition. Frequently, as e. g. in [FV02], a collection of Dirac matrices is said to belong to the standard representation if $\gamma_0^\dagger = \gamma_0$ and $\gamma_i^\dagger = -\gamma_i$ for $i = 1, 2, 3$, where $\gamma_\mu^\dagger$ denotes the Hermitian adjoint. This is equivalent to our definition if one only considers collections of Dirac matrices *with respect to the standard basis of* $(\mathbb{R}^4, \eta)$.

[3]The factor $\sqrt{2}$ arises due to the factor $\sqrt{2}^{-1}$ in definition 4.2.1 and is, in the end, to guarantee compatibility with historical customs.

(c) Let $\varepsilon_1, \ldots, \varepsilon_4$ denote the standard basis of the coordinate space $\mathbb{C}^4$.[4] Let $\kappa^c_{1,3}$ be a spinor representation of $\mathrm{Cl}^c_{1,3}$. Let $\{b_\mu\}$ be a basis of $(\mathbb{R}^4, \eta)_\mathbb{C}$ and let $\{\gamma_\mu\}$ be the collection of Dirac matrices induced by $\kappa^c_{1,3}$ with respect to the bases $\{b_\mu\}$ and $\{\varepsilon_i\}$. Then $\kappa^c_{1,3}$ is called **standard spinor representation,** if the Dirac matrices $\{\gamma_\mu\}$ belong to the standard representation with respect to $\{E_\mu\}$.

Equivalently, this means: $\kappa^c_{1,3}$ is called standard representation if the intertwining isomorphism $\mathbb{C}^4 \to \Delta_D$ for the representations $\kappa^c_{1,3}$ and D^D is given by linear extension of $\varepsilon_\mu \mapsto \tilde{E}_\mu$ and therefore is canonical[5]. In this sense, the standard representation is the spinor representation which is *given by the Dirac tensor spinor* $\gamma_a{}^{\tilde{A}}{}_{\tilde{B}}$ (these circumstances are the very essence of the standard representation).

5.3.3 Example. The following is a collection of Dirac matrices with respect to the standard basis of $(\mathbb{R}^4, \eta)$, which belongs to the standard representation with respect to the standard basis of $\Delta_{\frac{1}{2},0}$:

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

5.3.4 Remark (standard repr. and relationship of 2- and 4-spinors). The preferable feature of the standard representation is its canonical connection to the γ-tensor-spinor (as was pointed out at the end of definition 5.3.2), who in turn is directly related to the 2-spinor formalism. Thus, there is a connection between the Dirac pinor formalism (using the standard representation) and the 2-spinor formalism, which is encoded in the special nature of then tensor-spinor $\gamma_a{}^{\tilde{A}}{}_{\tilde{B}}$. We will illuminate this with an example:

Let $x = x^a \in (\mathbb{R}^4, \eta)$ be a vector. By the standard representation $\kappa_{1,3}$, x gets represented as an endomorphism on $\Delta_D \cong \mathbb{C}^4$ given by

$$\kappa_{1,3}(x) = x^a \gamma_a{}^{\tilde{A}}{}_{\tilde{B}} \,.$$

Choose some

$$\Psi^{\tilde{A}} = \begin{pmatrix} \varphi^A \\ \xi_{\dot{A}} \end{pmatrix} \in \Delta_D = \Delta_{\frac{1}{2},0} \oplus \Delta^*_{0,\frac{1}{2}} \,.$$

Then we find:

$$\kappa_{1,3}(x)\Psi = x^a \gamma_a{}^{\tilde{A}}{}_{\tilde{B}} \Psi^{\tilde{B}} = x^a \begin{pmatrix} 0 & \sigma_a{}^{A\dot{B}} \\ \sigma_{a\dot{A}B} & 0 \end{pmatrix} \begin{pmatrix} \varphi^B \\ \chi_{\dot{B}} \end{pmatrix} = \begin{pmatrix} x^a \sigma_a{}^{A\dot{B}} \chi_{\dot{B}} \\ x^a \sigma_{a\dot{A}B} \varphi^B \end{pmatrix}$$

[4]We mean $\varepsilon_1 = (1,0,0,0)^{tr}$, $\varepsilon_2 = (0,1,0,0)^{tr}$, ...

[5]after fixation of standard bases; "canonical" in the sense of "no further arbitrary choices involved"

This shows how a Dirac spinor equation can be translated into a system of two 2-spinor equations, and moreover, how "in the background", the Dirac spinor formalism still relies on the 2-spinor formalism, using the standard representation. Frequently, when proving a statement involving Dirac spinors, one has to "fall back" to the more elementary 2-spinor formalism using this translation procedure.

5.3.5 Example (Dirac's equation). Let's have a look at the classical Dirac equation on Minkowski spacetime using the techniques introduced in this section. Let

$$\Psi^{\tilde{A}} = \begin{pmatrix} \varphi^A \\ \psi_{\dot{X}} \end{pmatrix} \in C^\infty(\mathbb{R}^4, \Delta_D)$$

be a smooth Dirac spinor-valued field. Let $\{\gamma_\mu\}$ be a collection of Dirac matrices with respect to the standard basis of $(\mathbb{R}^4, \eta)$ which belongs to the standard representation with respect to $\{E_\mu\}$. Then Dirac's equation reads

$$i\gamma^\mu \partial_\mu \Psi + k\, \Psi = 0\,.$$

Transforming this into invariant notation usign abstract indices and the tensor-spinor $\gamma_a{}^{\tilde{A}}{}_{\tilde{B}}$ (and setting $\kappa := \sqrt{2} \cdot k$), this may be written as:

$$i\gamma^{a\tilde{A}}{}_{\tilde{B}} (\partial_a \Psi)^{\tilde{B}} + \kappa\, \Psi^{\tilde{A}} = 0\,.$$

We may transform this into a system of two 2-spinor equations:

$$0 = i\gamma^{a\tilde{A}}{}_{\tilde{B}} (\partial_a \Psi)^{\tilde{B}} + \kappa\, \Psi^{\tilde{A}} = i \begin{pmatrix} 0 & \sigma^{aB\dot{X}} \\ \bar{\sigma}^a{}_{\dot{Y}A} & 0 \end{pmatrix} \begin{pmatrix} \partial_a \varphi^A \\ \partial_a \psi_{\dot{X}} \end{pmatrix} + \kappa \begin{pmatrix} \varphi^A \\ \psi_{\dot{X}} \end{pmatrix}$$

which is equivalent to

$$\begin{cases} 0 = i\,\sigma^{aB\dot{X}} (\partial_a \psi)_{\dot{X}} + \kappa\, \varphi^B =: i\, \partial^{B\dot{X}} \psi_{\dot{X}} + \kappa\, \varphi^B \\ 0 = i\, \bar{\sigma}^a{}_{\dot{Y}A} (\partial_a \varphi)^A + \kappa\, \psi_{\dot{Y}} =: i\, \partial_{\dot{Y}A} \varphi^A + \kappa\, \psi_{\dot{Y}}\,. \end{cases}$$

Part II.

Differential Operators on Spinor Fields

6. Spinor Fields on Curved Spacetime

This thesis' main objects are spinor fields on spacetime manifolds and differential operators acting on them. In part one, we introduced the concept of $\mathrm{SL}(2,\mathbb{C})$-spinors, which form the representation spaces of finite-dimensional complex $\mathrm{SL}(2,\mathbb{C})$-representations (cf. definition 4.1.1). Now, the next step is introducing spinor-valued *fields* on spacetime manifolds. This will be done by constructing a *spinor bundle* for each type of $\mathrm{SL}(2,\mathbb{C})$-representation, which is a special vector bundle with fiber equal to the representation's spinor space. Then, as usual, a spinor field is a section of such a spinor bundle. Finally, in order to define differential operators acting on sections of spinor bundles, we need to equip these bundles with covariant derivatives, which is possible in a canonical way (after the manifold was equipped with some extra structure).

To make this document accessible to both readers with physical and readers with mathematical background, and in order to fix notation and terminology, we will give quick crash crouses on relevant concepts from Lorentzian geometry in section 6.1, principal bundles and associated vector bundles in section 6.2, and connections on principal bundles in section 6.3. Finally, in section 6.4 we come to the declaration of the concrete mathematical framework we will be working with for the rest of this thesis.

Sections 6.1–6.3 will be a digestion of [Bau81], [BGV92], [Fri97], [LM89], [O'N83] and [Roe98], where "digestion" means that the author tried to present only a careful selection of necessary definitions and propositions, but in such a way that consistency and completeness of the presented theory is maintained. The Construction presented in section 6.4 is inspired by [FV02].

6.1. Pseudo-Riemannian and Lorentzian Manifolds

If not stated otherwise, alle manifolds, bundles, maps and vector fields are assumed to be smooth.

6.1.1 Definition (pseudo-Riemannian Manifold). Let M be a smooth n-dimensional manifold and let $g \in \Gamma(T^{0,2}M)$ be a (smooth) metric of signature (p,q) on M.[1]

- If $p = n$ and $q = 0$, (M,g) is called a **Riemannian manifold,**

[1] This means that on each fiber, G is a symmetric, non-degenerate bilinear form of signature (p,q). We say a symmetric, non-degenerate bilinear form b on a real vector space V is of signature (p,q), if there is a b-orthogonal decomposition $V = X \oplus Y$, where $\dim(X) = p$ and $\dim(Y) = q$ and b restricted to X and $-b$ restricted to Y are positive definite.

- if $p \geq 1$ and $1 \leq q \leq n-1$, (M, g) is called a **pseudo-Riemannian manifold,** and
- if $p = 1$ and $q = n-1$, (M, q) is called a **Lorentzian manifold.**

It can be shown that every smooth manifold may be equipped with a Riemannian metric, and a smooth manifold has a pseudo-Riemannian metric of signature (p, q) if and only if there is a rank q subbundle of TM. Moreover, we find in [Bau81, pp. 43]:

6.1.2 Definition and proposition (timelike and spacelike subbundles). Let (M, g) be a pseudo-Riemannian manifold of signature (p, q). Then there is a g-orthogonal decomposition $TM = \tau \oplus \xi$, where τ is a rank p subbundle and ξ is a rank q subbundle of TM, such that the restriction of g to τ and the restriction of $-g$ to ξ are both positive-definite.

We call τ a **timelike** and ξ a **spacelike subbundle** of TM.

6.1.3 Remark (signature conventions). Notice that our signature convention for Lorentzian manifolds and particularly, spacetime manifolds, is $(+-\ldots-)$. This choice is convenient when dealing with spinors (which we shall be doing extensively), because the spinor metric given by

$$\eta_{A\dot{A}B\dot{B}} = \epsilon_{AB}\epsilon_{\dot{A}\dot{B}}$$

has signature $(+---)$, cf. remark 4.2.5.

Orientations of Pseudo-Riemannian Manifolds

6.1.4 Definition (orientability of a pseudo-Riemannian manifold). Let (M, g) be a pseudo-Riemannian manifold and let $TM = \tau \oplus \xi$ be an orthogonal decomposition of TM into a timelike and a spacelike subbundle. (M, g) is called

(i) **orientable,** if TM is orientable,

(ii) **time-orientable,** if τ is orientable,

(iii) **space-orientable,** if ξ is orientable.

The following proposition illuminates the interdependence of the different orientability concepts:

6.1.5 Proposition. Let (M, g) be a pseudo-Riemannian manifold and let $TM = \tau \oplus \xi$ be an orthogonal decomposition of TM into a timelike and a spacelike subbundle.

(i) If (M, g) is orientable and time-orientable then it is space-orientable.

(ii) If (M, g) is orientable and space-orientable then it is time-orientable.

(iii) If (M, g) is both space- and time-orientable then it is orientable.

The converse directions of these implications are in general not true.

Proof. Recall that a vector bundle is orientable if and only if its first Stiefel-Whitney class vanishes (cf. [LM89, theorem 1.2] or [Mil74] for a general introduction to characteristic classes). Notice, moreover, that using the Whitney product rule, $w_1(\tau \oplus \xi) = w_1(\tau) + w_1(\xi)$. Using this, all statements follow by simple calculations. $\square$

6.1.6 Definition. Let (M, g) be a Lorentzian manifold.

(i) Fix a point $m \in M$. A tangent vector $\xi \in T_m M$ is called
- **timelike** if $g_m(\xi, \xi) > 0$,
- **lightlike** if $g_m(\xi, \xi) = 0$ and $\xi \neq 0$,
- **spacelike** if $g_m(\xi, \xi) < 0$ or $\xi = 0$ (see footnote[2]),
- **causal** if ξ is timelike, lightlike or $\xi = 0$.

(ii) A parametrised C^1-curve on M is called timelike, spacelike, causal, respectively, if its velocity vector is everywhere timelike, spacelike, causal, respectively.

6.1.7 Remark (time orientations on Lorentzian manifolds). Notice that a time-orientation on a time-orientable Lorentzian manifold (M, g) may be given by a global, nowhere vanishing timelike vector field χ on M, because timelike subbundles of TM are 1-dimensional.

6.1.8 Definition. Let (M, g) be a Lorentzian manifold with time-orientation given by a global, timelike, nowhere vanishing unit vector field χ on M.

(a) For $m \in M$, a causal vector $x \in T_m M$ is called **future-directed,** if $g_m(x, \chi_m) \geq 0$. It is called **past-directed,** if $g_m(x, \chi_m) \leq 0$.

Notice that each causal tangent vector is either future- or past-directed, except for the 0-vector, which is both.

(b) A parametrised causal C^1-curve on M is called future resp. past directed if its velocity vector is everywhere future resp. past directed.

(c) For $m \in M$, define

$$I(m) := \{\xi \in T_m M \mid \xi \text{ timelike}\} \subseteq T_m M$$

and

$$J(m) := \overline{I(m)} \qquad (\text{closure of } I(m))\,.$$

$I(m)$ is open in $T_m M$ and called the **open light cone at** m**.**

(d) The open light cone $I(m)$ at $m \in M$ has two connected components, $I_+(m)$ and $I_-(m)$, given by

$$I_+(m) = \{x \in I(m) \mid x \text{ future-directed}\}$$

and

$$I_-(m) = \{x \in I(m) \mid x \text{ past-directed}\}\,.$$

We then define $J_+(m) := \overline{I_+(m)}$ and $J_-(m) := \overline{I_-(m)}$.

[2] By considering the zero vector spacelike we stick to [BGP, p. 8]. However, not all of the references agree with this.

(e) Let $A \subsetneq M$ be a subset. The **causal future** resp. **causal past** of A in M is the set $J_\pm^M(A) \subseteq M$, $+$ for future, $-$ for past, of all points that can be reached by future resp. past directed causal parametrised C^1-curves starting at a point in A. For $m \in M$ we write $J_\pm^M(m) := J_\pm^M(\{m\})$. Notice that $J_\pm^M(m) \subseteq M$ while $J_\pm(m) \subseteq T_mM$.

Globally Hyperbolic Manifolds

In this thesis we will be considering differential equations on Lorentzian manifolds. For both physical reasons (e. g. in order to avoid spacetimes that have paradoxical causality features) and mathematical reasons (concerning solvability of the differential equations involved), one typically restricts the attention to a special class of Lorentzian manifolds, *globally hyperbolic* manifolds. We are now in the position to present the core definitions.

6.1.9 Definition (Cauchy hypersurface). Let M be a connected time-oriented Lorentzian manifold. A subset $S \subseteq M$ is called a **Cauchy hypersurface in** M if every inextendible timelike curve[3] in M meets S in one and only one point.

6.1.10 Remark.

(a) A Cauchy hypersurface is always a connected, closed C^0-hypersurface in M.

(b) If $S \subseteq M$ is a Cauchy hypersurface, every inextendible *causal* curve intersects with S, but possibly more then once (at points where it is lightlike).

For proofs cf. [O'N83, chapter 14, lemma 29, proposition 31].

6.1.11 Definition (strong causality). A Lorentzian manifold M is said to satisfy the **strong causality condition** if for all $p \in M$ and for each open neighbourhood $U \subseteq M$ of p there exists an open neighbourhood $V \subseteq U$ of p such that each causal curve starting and ending in V is entirely contained in U. ("There are no almost closed causal loops.")

6.1.12 Definition (globally hyperbolic manifold). A Lorentzian manifold M is called **globally hyperbolic** if it is connected and time-oriented, if it satisfies the strong causality condition and if for all $p, q \in M$ the intersection $J_+(p) \cap J_-(q)$ is compact.

6.1.13 Theorem. Let M be a connected, time-oriented Lorentzian manifold. The following statements are equivalent:

(a) M is globally hyperbolic.

(b) There exists a Cauchy hypersurface in M.

(c) There exists a smooth, spacelike Cauchy hypersurface in M.

[3] A timelike curve $\gamma\colon (t_1, t_2) \to M$ is called **inextendible,** if there is no timelike curve $\tilde\gamma\colon (s_1, s_2) \to M$ with $(t_1, t_2) \subsetneq (s_1, s_2)$, such that $\tilde\gamma|_{(t_1,t_2)} = \gamma$.

(d) M is isometric to the smooth product manifold $\mathbb{R}\times S$ with metric $\beta dt^2 - g_t$, where β is a smooth positive function on M, S is a smooth manifold, g_t is a Riemannian metric on S depending smoothly on $t \in \mathbb{R}$ and each $\{t\} \times S$ is a smooth spacelike Cauchy hypersurface in M.

Proof. (a)⇒(d) is essentially [BS05, theorem 1.1], however, we chose a formulation similar to that of [BGP, theorem 1.3.10]. (d)⇒(c) is trivial, as well as (c)⇒(b). (b)⇒(a) is given by [O'N83, chapter 14, corollary 39]. (The converse direction, (a)⇒(c), by the way, was given in [BS03, theorem 1.1]; however, our proof is already completed without using this result.) □

Finally, for later use, we quote the technical lemmas [BGP, A.5.4, A.5.7]:

6.1.14 Lemma. Let M be a globally hyperbolic Lorentzian manifold.

(a) Let $K, K' \subseteq M$ be two compact subsets of M. Then $J_+^M(K)\cap J_-^M(K')$ is compact.

(b) Let $K \subseteq M$ be a compact subset and let $\Sigma \subseteq M$ be a Cauchy hypersurface. Then $J_\pm^M(K) \cap \Sigma$ and $J_\pm^M(K) \cap J_\mp^M(\Sigma)$ are compact.

6.2. Principal Bundles and Associated Bundles

This section is a crash course on principal bundles and associated vector bundles. The main references for this section are: [Bau81, pp. 34], [BGV92, p. 14], [Roe98, p. 23]; further references are: [Fri97, pp. 170], [HV00, pp. 164], [LM89, p. 370]. Many of the definitions and propositions stated here may be found in those references in a more or less similar form. It is the purpose and the (non-trivial) goal of this section to present a certain amount of mathematical beckground knowledge in a concentrated and consistent form. We shall not give proofs if they may as well be found in the references.

If not stated otherwise, all manifolds will be smooth, fiber bundles and Lie groups included. All maps between manifolds and bundles will be considered smooth, Lie group actions included. M denotes a manifold, G a Lie group. For the definitions of a fiber bundle and a vector bundle we refer to [BGV92, p. 13].

Recall that a group action $G \times M \to M$ is called **free** if $\forall g \in G\colon (\exists m \in M\colon g \cdot m = m) \Rightarrow g = e$, where e is the unit element of G. It is called **transitive** if $\forall m_1, m_2 \in M\, \exists g \in G\colon m_2 = g \cdot m_1$.

6.2.1 Definition (principal G-bundle). Let M be a smooth manifold and G a Lie Group. A smooth fiber bundle $(\mathcal{P}, \pi, M)$ with typical fiber isomorphic to G and a smooth right action $\mathcal{P} \times G \to \mathcal{P}$ is called **(smooth) principal G-bundle**, if

(i) $\forall p \in \mathcal{P}\, \forall g \in G\colon \pi(p \cdot g) = \pi(p)$ (G acts on the fibers)

(ii) the G-action is free,

(iii) $\forall p, q \in \mathcal{P}\colon \pi(p) = \pi(q) \Rightarrow \exists g \in G\colon p \cdot g = q$ (transitivity on the fibers).

G is called **structure group** of the principal bundle $\mathcal{P}$.

Notice that for a principal G-bundle $(\mathcal{P}, \pi, M)$, the base M is diffeomorphic to the quotient $\mathcal{P}/G$ as the fibers are diffeomorphic to G.

6.2.2 Definition (reduction of a principal bundle). Let $(\mathcal{P}, p, M)$ be a principal G-bundle and let $\lambda\colon H \to G$ be a smooth homomorphism of Lie groups. A **λ-reduction of $\mathcal{P}$** is a principal H-bundle $(\mathcal{Q}, q, M)$ together with a bundle map $f\colon \mathcal{Q} \to \mathcal{P}$ such that the following diagram commutes (cf. [Bau81, p. 35]):

$$\begin{tikzcd}
\mathcal{Q} \times H \arrow[r, "\cdot"] \arrow[dd, "{(f, \lambda)}"] & \mathcal{Q} \arrow[dd, "f"] \arrow[dr, "q"] & \\
& & M \\
\mathcal{P} \times G \arrow[r, "\cdot"] & \mathcal{P} \arrow[ur, "p"'] &
\end{tikzcd}$$

where $\cdot$ on the horizontal arrows denotes the right action of the group on the bundle. If $H \subseteq G$ is a Lie subgroup and $\lambda\colon H \to G$ the canonical embedding map, we may call a λ-reduction of $\mathcal{P}$ just a H-reduction of $\mathcal{P}$.

We will need the concept of principal bundle reductions when dealing with frame bundles at the end of this section.

6.2.3 Definition (fibered product). Let G be a group, let M be a right G-space (i. e. M is a set and there is a right G-action on M) and let N be a left G-space.

(a) Define a right G-action on $M \times N$:

$$(m, n) \cdot g := (m \cdot g, g^{-1} \cdot n)\,.$$

Then the **fibered product** $M \times_G N$ is the quotient

$$M \times_G N := (M \times N)/G\,.$$

We shall denote the elements of $M \times_G N$ (which are equivalence classes) by

$$[m, n]_G \in M \times_G N\,.$$

(b) Let the right G-action on M be transitive and free, and let ρ be a representation of G on a $\mathbb{K}$-vector space V (considered a left G-action). We then denote the fibered product $M \times_G V$ by $M \times_\rho V$ and the equivalence classes by $[g, v]_\rho$.
In such a situation, the fibered product $M \times_\rho V$ inherits a vector space structure from V, given by

$$\lambda \cdot [g, v]_\rho := [g, \lambda v]_\rho \quad \text{and} \quad [g, v]_\rho + [g, w]_\rho := [g, v + w]_\rho\,.$$

Check that this is well defined because the left action of G on V is linear and because due to the transitivity and freeness of the right action of G on M, for every $g \in G$ and for every equivalence class $[h, v]_\rho$, there exists one and only one $w \in V$ such that $[g, w]_\rho = [h, v]_\rho$.

This motivates the following definition:

6.2.4 Definition and proposition (associated vector bundle).

(a) Let $(\mathcal{P}, \pi, M)$ be a principal G-bundle and let $\rho\colon G \to GL(V)$ be a representation of G on a finite dimensional vector space V, taken as left G-action. Then the fibered product $\mathcal{P} \times_\rho V$ forms a vector bundle on M, called **the vector bundle associated to $\mathcal{P}$ by ρ.**

(b) Let $\mathcal{V} := \mathcal{P} \times_\rho V$ be the associated bundle from (a). Then the dual bundle $\mathcal{V}^*$ is given by

$$\mathcal{V}^* = \mathcal{P} \times_{\rho^*} V^* .$$

Moreover, if V is a complex vector space with complex conjugate $\bar{V}$, we denote by $\bar{\mathcal{V}}$ the associated bundle

$$\bar{\mathcal{V}} := \mathcal{P} \times_{\bar{\rho}} \bar{V} .$$

The anti-linear complex conjugation isomorphism (cf. definition 2.1.1) $\bar{\ }\colon V \to \bar{V}$ canonically induces a bundle map $\bar{\ }\colon \mathcal{V} \to \bar{\mathcal{V}}$ (of real vector bundles, given by the identity map on the total space) which is fiberwise anti-linear.

(c) Let again $\mathcal{V} := \mathcal{P} \times_\rho V$ be the associated bundle from (a). If τ is a representation of G on a second vector space W, and if $\mathcal{W} := \mathcal{P} \times_\tau W$ is the bundle associated to $\mathcal{P}$ by τ, then the tensor product bundle $\mathcal{V} \otimes \mathcal{W}$ is given by

$$\mathcal{V} \otimes \mathcal{W} = \mathcal{P} \times_{\rho \otimes \tau} (V \otimes W) ,$$

and the direct sum $\mathcal{V} \oplus \mathcal{W}$ is given by:

$$\mathcal{V} \oplus \mathcal{W} = \mathcal{P} \times_{\rho \oplus \tau} (V \oplus W) .$$

An intuitive understanding of what exactly the fibered product does and what an associated bundle is, may become clearer by considering frame bundles and their associated vector bundles:

Frame Bundles

6.2.5 Definition (frame bundle). Let $\mathcal{E}$ be a $\mathbb{K}$-vector bundle on M of rank k ($\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$). The bundle

$$\mathcal{F}(\mathcal{E}) := \{(m, (v_1, \ldots, v_k)) \,|\, m \in M,\ v_i \in \mathcal{E}_m,\ (v_1, \ldots, v_k) \text{ linear independent}\}$$

with projection map $(m, (v_1, \ldots, v_k)) \mapsto m$ over M is called the **frame bundle** or **repère bundle of $\mathcal{E}$.** By $\mathcal{F}(M)$ we denote the frame bundle of TM.

Let $\mathcal{F}(\mathcal{E})$ be the frame bundle of a $\mathbb{K}$-vector bundle $\mathcal{E}$. Every point in the fiber over $m \in M$, $p = (m, (v_1, \ldots, v_k)) \in \mathcal{F}(\mathcal{E})_m$, gives rise to an isomorphism

$$\varphi_p\colon \mathbb{R}^k \to \mathcal{E}_m, \quad (x^1, \ldots, x^k) \mapsto \sum_i x^i v_i ,$$

where $\mathcal{E}_m$ denotes the fiber of $\mathcal{E}$ over m. After once choosing a fixed basis for $\mathcal{E}_m$ this shows that the fiber $\mathcal{F}(\mathcal{E})_m$ is isomorphic to $\mathrm{GL}(k, \mathbb{K})$. Moreover, there is a right action of $\mathrm{GL}(k, \mathbb{K})$ on $\mathcal{F}(\mathcal{E})$, given on each fiber by

$$\begin{aligned}\big(m,(v_1,\dots,v_k)\big)\cdot A &:= \Big(m, \underbrace{(v_1,\dots,v_k)\cdot A}_{\substack{\text{“row vector times matrix”;}\\ \text{base transformation}}}\Big) = \underbrace{\Bigg(m, \Big(\sum_{i=1}^{k} v_i A_{i1}, \dots, \sum_{i=1}^{k} v_i A_{ik}\Big)\Bigg)}_{\substack{A\text{ taken as base transf. matrix,}\\ \text{equivalently:}\\ A\text{ considered as endomorphism}\\ \text{of } \mathcal{F}(\mathcal{E})_m \text{ represented with}\\ \text{respect to the bases } (v_1,\dots,v_k);\\ \text{then, } \sum_i^k v_i A_{ij} = A(v_j)}} \\ &= \Big(m, \big(A(v_1),\dots,A(v_k)\big)\Big)\,.\end{aligned}$$

It can be shown that with this right $\mathrm{GL}(k, \mathbb{K})$-action, $\mathcal{F}(\mathcal{E})$ forms a principal $\mathrm{GL}(k, \mathbb{K})$-bundle on M.

We now closely follow [Bau81, pp. 45]. If M is a pseudo-Riemannian manifold of signature (p, q) and with frame bundle $\mathcal{F}(M)$ one can consider the subbundle[4] of orthonormal frames, $\mathcal{O}(M) \leq \mathcal{F}(M)$, called **orthonormal frame bundle of M.** It is easily seen that $\mathcal{O}(M)$ is an $\mathrm{O}(p, q)$-reduction of the frame bundle. Moreover, we have the following results:

6.2.6 Proposition. Let (M, g) be a connected pseudo-Riemannian manifold of signature (p, q) with orthonormal frame bundle $\mathcal{O}(M)$.

(i) If (M, g) is orientable but neither space- nor time-orientable then $\mathcal{O}(M)$ has two connected components and is reducible to a principal $\mathrm{SO}(p, q)$-bundle with connected total space.

One may think of such a reduction as the subbundle $\mathcal{SO}(M) \leq \mathcal{O}(M)$ of oriented orthonormal frames.

(ii) If (M, g) is space-orientable and not time-orientable resp. time-orientable and not space-orientable then $\mathcal{O}(M)$ has two connected components and is reducible to a principal $\mathrm{O}_s(p, q)$- resp. $\mathrm{O}_t(p, q)$-bundle with connected total space[5].

One may think of such a reduction as the subbundle $\mathcal{O}_s(M) \leq \mathcal{O}(M)$ resp. $\mathcal{O}_t(M) \leq \mathcal{O}(M)$ of orthonormal frames whose spacelike resp. timelike part is oriented.

(iii) If (M, g) is both space- and time-orientable then $\mathcal{O}(M)$ has four connected components and is reducible to a principal $\mathrm{SO}^+(p, q)$-bundle with connected total space.

One may think of such a reduction as the subbundle $\mathcal{SO}^+(M) \leq \mathcal{O}(M)$ of space-oriented and time-oriented frames.

(iv) If (M, g) is not orientable in any way then $\mathcal{O}(M)$ is connected.

Proof. Cf. [Bau81, Satz 0.51]. □

[4] $\mathcal{F} \leq \mathcal{E}$ for vector bundles $\mathcal{E}$ and $\mathcal{F}$ means that $\mathcal{F}$ is a subbundle of $\mathcal{E}$.

[5] For the definitions of $\mathrm{O}_t(p, q)$, $\mathrm{O}_s(p, q)$ and $\mathrm{SO}^+(p, q)$ cf. the appendix or [Bau81, pp. 45].

Notice that the alternatives given in this proposition are mutually exclusive. Notice further that using proposition 6.1.5, it is immediate that the alternatives are exhaustive, meaning that to every pseudo-Riemannian manifold one of the alternatives applies. This enables us to state the following definition which will be of importance later:

6.2.7 Definition (connected frame bundle). Let (M, g) be a connected pseudo-Riemannian manifold of signature (p, q). Reduce $\mathcal{O}(M)$ as far as possible to the principal $\mathrm{O}(p,q)$-, $\mathrm{O}_s(p,q)$-, $\mathrm{O}_t(p,q)$-, $\mathrm{SO}(p,q)$- or $\mathrm{SO}^+(p,q)$-bundle $\mathcal{O}(M)$, $\mathcal{O}_s(M)$, $\mathcal{O}_t(M)$, $\mathcal{SO}(M)$ or $\mathcal{SO}^+(M)$, using the previous proposition.[6] The resulting bundle, which has connected total space, is called **connected frame bundle of** M and shall henceforth be denoted by $\mathcal{F}_c(M)$.

After these considerations we conclude this section by presenting one remak on the promised intuitive understanding of associated vector bundles:

6.2.8 Remark (on the concept of associated vector bundles). Check that for an orientable and time-orientable pseudo-Riemannian manifold of signature (p, q), the following relations hold:

$$\begin{aligned} TM &\cong \mathcal{F}(M) \times_{\mathrm{GL}(n)} \mathbb{R}^n\,, &\quad TM &\cong \mathcal{O}(M) \times_{\mathrm{O}(p,q)} \mathbb{R}^n\,, \\ TM &\cong \mathcal{SO}(M) \times_{\mathrm{SO}(p,q)} \mathbb{R}^n\,, &\quad TM &\cong \mathcal{SO}^+(M) \times_{\mathrm{SO}^+(p,q)} \mathbb{R}^n\,. \end{aligned}$$

Taking the first one of these isomorphisms as an example we may be able to get a better intuition for associated vector bundles:

According to the definition, the fibers of the associated vector bundle $\mathcal{F}(M) \times_{\mathrm{GL}(n)} \mathbb{R}^n$ consist of equivalence classes of tuples (f, x), where f is a frame and $x = x^\mu \in \mathbb{R}^n$ is a "column vector". Thus, $TM \cong \mathcal{F}(M) \times_{\mathrm{GL}(n)} \mathbb{R}^n$ entails that each tangent vector $\xi \in T_m M$, $m \in M$, corresponds with an equivalence class $[f, x]_{\mathrm{GL}(n)}$. Now, this relationship between tangent vectors and equivalent classes may visualised like this: Consider the frame, f, as a *basis* of the tangent space $T_m M$, and x as the column vector representing ξ with respect to this basis.

Of course, there exist several bases for $T_m M$ and with respect to each basis, ξ gets represented as another column vector. According to definition 6.2.3, two such tuples (f, x) and (f', x') of bases and column vectors are equivalent if and only if there is a group element $A \in \mathrm{GL}(n, \mathbb{R})$, such that

$$(f', x') = (f, x) \cdot A = (f \cdot A, A^{-1} \cdot x)\,.$$

But the transition from (f, x) to $(f \cdot A, A^{-1} \cdot x)$ is just a basis and coordinate transformation! As usual, the reference frame f gets transformed by multiplying the base transformation matrix A from the right, while the column vector x transforms by multiplying the inverse of the base transformation matrix from the left.

Thus: *all the tuples (f, x) in the equivalence class $[f, x]_{\mathrm{GL}(n)}$ may be thought of as base and column vector representations of one and the same "abstract vector".*

[6] This means: we reduce $\mathcal{O}(M)$ to $\mathrm{SO}^+(M)$ if M is space-orientable and time-orientable, to $\mathcal{SO}(M)$ if M is only orientable, to $\mathrm{O}_s(M)$ resp. $\mathrm{O}_t(M)$ if M is space- but not time- resp. time- but not space-orientable and we don't reduce it at all if M is not orientable.

All this may be done in an analogous fashion for the three other isomorphisms given above and in general for all associated vector bundles. Last but not least, these considerations may serve as belated motivation for the above definition of the fibered product (which might have seemed a bit odd at first sight).

6.3. Connections on Principal Bundles

It is the purpose of this section to briefly introduce connections on principal bundles, the lifting of a covariant derivative on a vector bundle to its frame bundle and induced covariant derivatives on associated vector bundles. We need all this in order to equip spinor bundles on spacetime manifolds with a covariant derivative in the next section.

To save space and time we shall be rather minimalistic as regarding the amount of mathematical background theory presented. For instance, we shall refrain from introducing vector bundle valued differential forms (which would make parts of the exposition much more elegant, e. g. induced covariant derivatives on associated bundles). In case of induced principal bundle connections on a frame bundles and in case of induced covariant derivatives of associated bundles, our general guidline will be to give an idea of how the constructions are performed but to skip the proofs that they indeed yield the desired objects.

This section's main references are [BGV92, p. 14–36], [Roe98, p. 23–25]. We only give proofs if they may not be found in the references.

Again, if not stated otherwise, all manifolds will be smooth, fiber bundles and Lie groups included. All maps between manifolds and bundles will be considered smooth, Lie group actions included. M denotes a manifold, G a Lie group.

For the definition of a covariant derivatives on vector bundles we refer to [BGV92, p. 21].

Principal Bundle Connections

6.3.1 Definition (vertical bundle). Let $(\mathcal{P}, \pi, M)$ be a fiber bundle on M. The kernel of $d\pi\colon T\mathcal{P} \to TM$ is a subbundle of $T\mathcal{P}$ and is called **vertical bundle of $\mathcal{P}$,** denoted by $V\mathcal{P}$. A vector $\xi \in T_p\mathcal{P}$, $p \in \mathcal{P}$, is called **vertical**, if $d\pi_p(\xi) = 0$.

Consider the case of a principal G-bundle $(\mathcal{P}, \pi, M)$. Differentiating the G-action $\eta_p\colon G \to \mathcal{P}$, $p \in \mathcal{P}$, yields a map $d\eta_p\colon \mathfrak{g} \to T_p\mathcal{P}$ where $\mathfrak{g}$ denotes the Lie algebra of G. It turns out that $d\eta_p\colon \mathfrak{g} \to V_p\mathcal{P}$ is a (canonical) isomorphism (because the G-action on the principal bundles is transitive and free on the fibers). Moreover, for $X \in \mathfrak{g}$, the vector field $X_\mathcal{P}$ on $\mathcal{P}$, in each fiber given by $(X_\mathcal{P})_p = d\eta_p(X)$, is G-invariant and called **Killing vector field corresponding to X on $\mathcal{P}$.**

Recall that a **splitting** of a short exact sequence ([Lan02, p. 132])

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, shift left, "f"] & B \arrow[l, shift left, dashed, "\varphi"] \arrow[r, shift left, "g"] & C \arrow[l, shift left, dashed, "\psi"] \arrow[r] & 0 \end{tikzcd},$$

is a coice of map $\psi\colon C \to B$ such that $g \circ \psi = \mathrm{id}$, or equivalently a choice of map $\varphi\colon B \to A$ such that $\varphi \circ f = \mathrm{id}$. This then yields isomorphisms

$$B \cong \operatorname{Im} f \oplus \ker \varphi, \qquad B \cong \ker g \oplus \operatorname{Im} \psi, \qquad B \cong A \oplus C\,.$$

Let $(\mathcal{P}, \pi, M)$ be a principal G-bundle and let ρ denote the right action of G on $\mathcal{P}$. A splitting of the short exact sequence of subbundles

$$0 \to V\mathcal{P} \to T\mathcal{P} \to \pi^* TM \to 0 \tag{$*$}$$

gives a subbundle $H\mathcal{P}$ of $T\mathcal{P}$, called **horizontal bundle**, such that $T\mathcal{P} = V\mathcal{P} \oplus H\mathcal{P}$. A such a splitting is called G-equivariant, if

$$\forall p \in \mathcal{P}\, \forall g \in G \colon H_{p \cdot g}\mathcal{P} = d\rho_g(H_p\mathcal{P}),$$

where $d\rho_g$ just denotes the differential map of the G-action $\rho_g \colon \mathcal{P} \to \mathcal{P}$.

6.3.2 Definition (fiber bundle and principal bundle connections).

(a) Let $(\mathcal{P}, \pi, M)$ be a fiber bundle. A smooth splitting of the short exact sequence of vector bundles

$$0 \to V\mathcal{P} \to T\mathcal{P} \to \pi^* TM \to 0$$

is called **connection** on $(\mathcal{P}, \pi, M)$.

(b) A G-equivariant fiber bundle connection on a principal G-bundle $(\mathcal{P}, \pi, M)$ is called **principal bundle connection on $\mathcal{P}$.**

Connection 1-Forms

An alternative approach to constructing a principal bundle connection is via Cartan's connection 1-forms: In case of a principal G-bundle $(\mathcal{P}, \pi, M)$, the fibers $V_p\mathcal{P}$ of the vertical bundle $V\mathcal{P}$ are canonically isomorphic to $\mathfrak{g} = \mathrm{Lie}(G)$, as introduced before. Let a connection on $\mathcal{P}$ be given as splitting of the short exact sequence $(*)$, and we assume this splitting to be given by the map $\varphi \colon T\mathcal{P} \to V\mathcal{P}$. Using in each fiber the isomorphism $V_p\mathcal{P} \cong \mathfrak{g}$, φ may be considered a $\mathfrak{g}$-valued 1-form on $\mathcal{P}$. This 1-form contains all the information of the connection. This motivates the following definition:

6.3.3 Definition (principal bundle connection 1-form). Let $(\mathcal{P}, \pi, M)$ be a principal G-bundle. A **principal bundle connection 1-form** on $\mathcal{P}$ is a smooth $\mathfrak{g}$-valued 1-form ω on $\mathcal{P}$ with the properties

1. $\forall g \in G \colon (\rho_g)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$ (Ad-equivariance of ω),
2. $\forall X \in \mathfrak{g} \colon \omega(X_{\mathcal{P}}) = X$ (vertical projection property),

where ρ denotes the right action of G on $\mathcal{P}$, Ad denotes the adjoint representation of G on $\mathfrak{g}$ and $X_{\mathcal{P}}$ denotes the Killing vector field on $\mathcal{P}$ associated with X.

The Ad-equivariance condition on ω corresponds to the G-equivariance condition in the definition of a principal bundle connection (6.3.2-b) and is the reason why one can prove (cf. the references given above):

6.3.4 Proposition. Let $(\mathcal{P}, \pi, M)$ be a principal G-bundle. There is a one-to-one correspondence between principal bundle connections on $\mathcal{P}$ and principal bundle connection 1-forms on $\mathcal{P}$.

The construction of the principal bundle connection 1-form for a given principal bundle connection was indicated above. Conversely, if a principal bundle connection 1-form is given, the horizontal bundle $H\mathcal{P}$ of the induced principal bundle connection is just the kernel of ω. (Of course one would have to give more detailed proofs for all this. Cf. also [Roe98, pp. 24].)

Lifted Connections on Frame Bundles

For slightly different but equivalent approaches to constructing the lifted connection on a frame bundle (using bundle valued differential forms) cf. [Roe98, p. 24–25], [BGV92, p. 23].

Let $(\mathcal{E}, \pi, M)$ be a rank k vector bundle on M with frame bundle $\mathcal{F}(\mathcal{E})$. Let $\mathcal{E}$ be equipped with a covariant derivative. We endeavour to construct a principal bundle connection on $\mathcal{F}(\mathcal{E})$, induced by the covariant derivative on $\mathcal{E}$:

Fix an arbitrary point $e \in \mathcal{F}(\mathcal{E})$ with foot point $m := \pi(e) \in M$. Let $x \in T_e\mathcal{F}(\mathcal{E})$ be a tangent vector. Represent x by a smooth curve $\tilde{\gamma}\colon (-\varepsilon, \varepsilon) \to \mathcal{F}(\mathcal{E})$ with $\tilde{\gamma}(0) = e$ and $\dot{\tilde{\gamma}}(0) = x$. We can write the curve $\tilde{\gamma}$ as

$$\tilde{\gamma}(t) = (\gamma(t), (\tilde{\gamma}_1(t), \ldots, \tilde{\gamma}_k(t))\,,$$

where $\gamma = \pi\tilde{\gamma}$ is the "curve of foot points" and $\tilde{\gamma}_i$ are suitable curves in $\mathcal{E}$. The curves $\tilde{\gamma}_i$ may also be considered as sections of $\mathcal{E}$ along the curve γ, which allows to define that x be called horizontal if for all i, the section $\tilde{\gamma}_i$ along γ is parallel according to the covariant derivative on $\mathcal{E}$. By this procedure we may construct the horizontal subspace $H_e\mathcal{F}(\mathcal{E}) \subseteq T_e\mathcal{F}(\mathcal{E})$ over each point $e \in \mathcal{F}(\mathcal{E})$.

One may now prove in detail that the collection of horizontal spaces $H_e\mathcal{F}(\mathcal{E})$ constitutes a principal bundle connection on $\mathcal{F}(\mathcal{E})$. Moreover, it is easily visualised using our construction that a covariant derivative on the tangent bundle TM only induces a connection on the reduced bundle $\mathcal{O}(M)$, if it is metric compatible (i. e. $\nabla_X\, g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for all $X, Y, Z \in \Gamma(TM)$).

All in all, one finds:

6.3.5 Proposition (lifted connections on frame bundles).

(a) Let $\mathcal{E}$ be a vector bundle on M with frame bundle $\mathcal{F}(\mathcal{E})$. Then every covariant derivative on $\mathcal{E}$ canonically induces a principal bundle connection on $\mathcal{F}(\mathcal{E})$, and vice versa.

(b) Let (M, g) be a connected pseudo-Riemannian manifold of signature (p, q) and with connected frame bundle $\mathcal{F}_c(M)$ (cf. definition 6.2.7). Let G denote the structure group of $\mathcal{F}_c(M)$. Then every metric compatible covariant derivative on TM canonically induces a principal G-bundle connection on $\mathcal{F}_c(M)$, and vice versa.

We call the principal bundle connection on $\mathcal{F}(\mathcal{E})$ resp. $\mathcal{F}_c(M)$, induced by a covariant derivative on $\mathcal{E}$ resp. TM **the lifted connection on $\mathcal{F}(\mathcal{E})$ resp. $\mathcal{F}_c(M)$.**

Proof. (a) can be found in [BGV92, p. 23, p. 26]. Statement (b) can be deduced from what is mentioned in [Roe98, p. 24–25] using a decomposition $TM = \tau \oplus \xi$ in the sense of proposition 6.1.5. □

Induced Covariant Derivatives on Associated Vector Bundles

We will not be following the references in this section.

Let $\mathcal{P}$ be a principal G-bundle on M, equipped with a principal bundle connection. let (V, ρ) be a vector space representation of G and let $\mathcal{E} = \mathcal{P} \times_\rho V$ be the vector bundle associated to $\mathcal{P}$ by ρ.

Our aim is constructing an induced covariant derivative on the associated bundle $\mathcal{E}$. As the first ingredient to this we quote the following standard lemma:

6.3.6 Lemma (horizontal lifting of curves). Let $(\mathcal{P}, \pi, M)$ be a principal bundle on M with connection given by a horizontal bundle $H\mathcal{P}$. Let $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be a smooth curve on M. Then for every starting point $p \in \mathcal{P}_{\gamma(0)}$ there is a unique curve $\tilde{\gamma}\colon (-\varepsilon, \varepsilon) \to \mathcal{P}$ such that

1. $\tilde{\gamma}(0) = p$,
2. $\pi \circ \tilde{\gamma} = \gamma$,
3. $\forall t \in (-\varepsilon, \varepsilon)\colon \dot{\tilde{\gamma}}(t) \in H_{\tilde{\gamma}(t)}\mathcal{P}$.

The last condition states that the curve $\tilde{\gamma}$ has everywhere horizontal velocity vector. This is why we call $\tilde{\gamma}$ the **horizontal lift** of γ with respect to the connection $H\mathcal{P}$.

Proof. Cf. [Roe98, p. 25]. □

Let X be a vector field on M and let s be a smooth section of $\mathcal{E}$. We will now construct the covariant derivative $\nabla_X s$ on $\mathcal{E}$ induced by the principal bundle connection on $\mathcal{P}$: Pick a point $m \in M$ and represent X_m as a curve $\gamma\colon (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = m$, $\dot{\gamma}(0) = X_m$. Pick a point $p \in \mathcal{P}_m$ and let $\tilde{\gamma}$ be the horizontal lift of γ on $\mathcal{P}$ with $\tilde{\gamma}(0) = p$. Then there is a smooth curve $v\colon (-\varepsilon, \varepsilon) \to V$, such that $\forall t\colon s_{\gamma(t)} = [\tilde{\gamma}(t), v(t)]_\rho$. As V is a vector space, there is a canonical way of forming the derivative $\dot{v} = \frac{d}{dt}v$ of the curve v. Thus, we may define:

$$(\nabla_X s)_m := [\tilde{\gamma}(0), \dot{v}(0)]_\rho\,. \tag{$*$}$$

As we will prove below, this defines a covariant derivative on $\mathcal{E}$.

6.3.7 Remark. To develop a pictorial understanding of the construction we just performed, put it in the light of remark 6.2.8:

We represented $s_{\gamma(t)}$ as tuples $(\tilde{\gamma}(t), v(t))$ of reference frames and “component vectors”. $\tilde{\gamma}(t)$ is a collection of reference frames along $\gamma(t)$, which is by construction *parallel,* meaning that the frames do not “drift” as t increases. This notion of “parallel” or “not drifting” is derived from the connection on $\mathcal{P}$.

Then, if we think of $v(t)$ as the components of $s_{\gamma(t)}$ with respect to these (non-drifting) reference frames as suggested by remark 6.2.8 (for simplicity, imagine $V = \mathbb{R}^n$), it is clear that the “infinitesimal change of $s_{\gamma(t)}$ for increasing t” is reduced to the change $\dot{v}(t)$ of the coordinates.

6.3.8 Proposition (induced covariant derivative). Let $\mathcal{P}$ be a principal G-bundle on M and let (V,ρ) be a vector space representation of G. Then every connection on $\mathcal{P}$ induces a covariant derivative on the associated vector bundle $\mathcal{P}\times_\rho V$ by the construction indicated above.

Proof. As the above construction is not to be found in any of the references, we give a full proof here. It is immediately seen that for every $s\in\Gamma(\mathcal{E})$ there is a smooth function $\psi^s\colon\mathcal{P}\to V$ such that

$$\forall p\in\mathcal{P}\colon [p,\psi^s(p)]_\rho = s_{\pi(p)}\,.$$

Notice that ψ^s has the following equivariance property:

$$\forall g\in G\,\forall p\in\mathcal{P}\colon \eta_g^*(\psi^s)_p = \psi^s_{p\cdot g} = \rho_{g^{-1}}(\psi^s_p)\,,$$

where $\eta_g\colon\mathcal{P}\to\mathcal{P}$ denotes the right action of $g\in G$ on $\mathcal{P}$.
Moreover, for every $X\in\Gamma(TM)$ there is a unique vector field $X^{\mathcal{P}}\in\Gamma(H\mathcal{P})$ such that $\forall p\in\mathcal{P}\colon d\pi_p(X^{\mathcal{P}}_p) = X_{\pi(p)}$ (just represent X pointwise by curves and apply lemma 6.3.6). $X^{\mathcal{P}}$ is called the horizontal lift of X to $\mathcal{P}$. It is a standard result that $X^{\mathcal{P}}$ is right-invariant:

$$\forall g\in G\,\forall p\in\mathcal{P}\colon \eta_g^*(X^{\mathcal{P}}) = X^{\mathcal{P}}\,.$$

We can express our above construction of $(\nabla_X s)_m$ in this new terminology. Notice that for $p=\tilde\gamma(0)\in\mathcal{P}_m$, $X^{\mathcal{P}}_p=\dot{\tilde\gamma}(0)$ and $\dot v(0) = d\psi^s_p(X^{\mathcal{P}}_p)$. Thus, we find

$$(\nabla_X s)_m = [p, d\psi^s_p(X^{\mathcal{P}}_p)]_\rho\,. \tag{$*$}$$

We now can show independence of the starting point $p\in\mathcal{P}_m$ of γ (and hence, completely forget about γ by making $(*)$ the definition of $(\nabla_X s)_m$): Let $p'\in\mathcal{P}_m$ be a second point and let $g\in G$ be such that $p'=p\cdot g$.

$$\begin{aligned}[p', d\psi^s_{p'}(X^{\mathcal{P}}_{p'})]_\rho &= [p\cdot g, d\psi^s_{p\cdot g}(X^{\mathcal{P}}_{p\cdot g})]_\rho = [p\cdot g, d(\eta_g^*\psi^s)_p(\eta_g^* X^{\mathcal{P}})_p]_\rho\\ &= [p\cdot g, d(\rho_{g^{-1}}\psi^s)_p(X^{\mathcal{P}}_p)]_\rho = [p\cdot g, \rho_{g^{-1}} d\psi^s_p(X^{\mathcal{P}}_p)]_\rho = [p, d\psi^s_p(X^{\mathcal{P}}_p)]_\rho\,.\end{aligned}$$

Finally, $C^\infty(M)$-linearity of $\nabla_X s$ in X, $\mathbb{K}$-linearity in s and Leibnitz's rule are immediate now, as all this holds for $d\psi^s\colon T\mathcal{P}\to V$: Pick $f\in C^\infty(M)$, $X,Y\in\Gamma(TM)$ and $s,u\in\Gamma(\mathcal{E})$. As $\psi^{s+f\,u}=\psi^s + f\,\psi^u$, we find:

$$\begin{aligned}\big(\nabla_{X+fY}(s)\big)_m &= [p, d\psi^s_p(X^{\mathcal{P}}_p + f_m\,Y^{\mathcal{P}}_p)]_\rho\\ &= [p, d\psi^s_p(X^{\mathcal{P}}_p)]_\rho + [p, f_m\, d\psi^s_p(Y^{\mathcal{P}}_p)]_\rho\\ &= (\nabla_X s)_m + (f_m\,\nabla_Y s)_m\end{aligned}$$

and

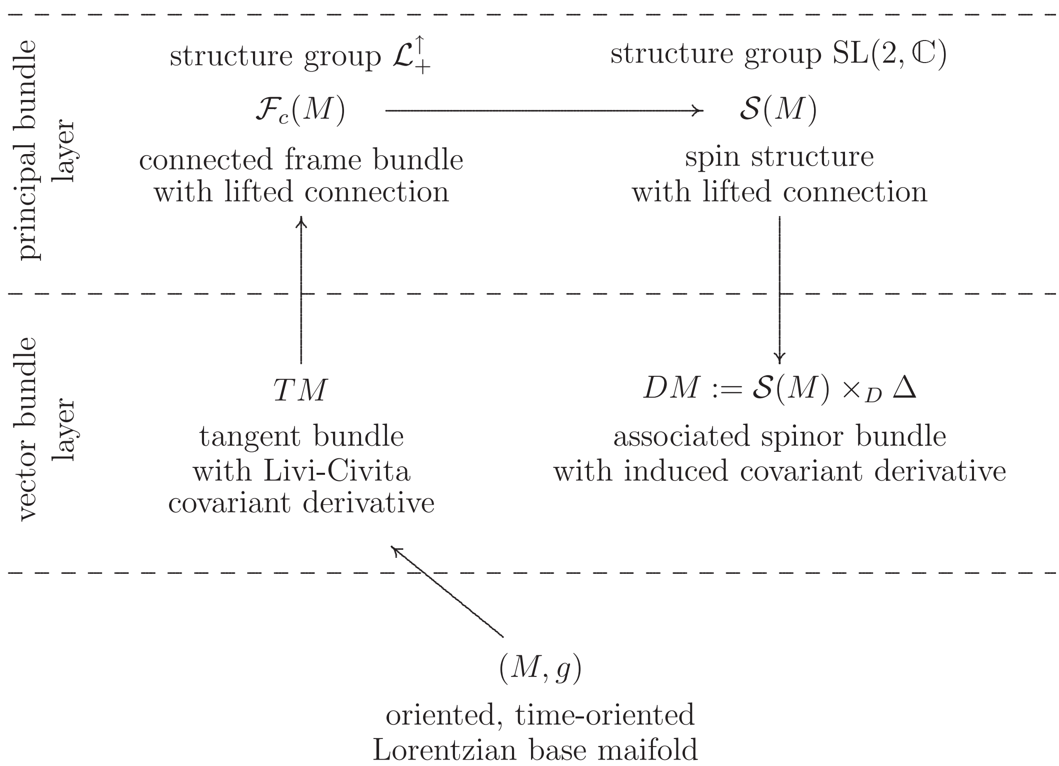
$$\begin{aligned}\big(\nabla_X(s+f\,u)\big)_m &= [p, d\psi^{s+fu}_p(X^{\mathcal{P}}_p)]_\rho\\ &= [p, d\psi^s_p(X^{\mathcal{P}}_p) + d(\pi^* f\cdot\psi^u)_p(X^{\mathcal{P}}_p)]_\rho\\ &= [p, d\psi^s_p(X^{\mathcal{P}}_p)]_\rho + [p, f_m\, d(\psi^u)_p(X^{\mathcal{P}}_p)]_\rho + [p, d(\pi^* f)_p(X^{\mathcal{P}}_p)\,\psi^u_p]_\rho\\ &= [p, d\psi^s_p(X^{\mathcal{P}}_p)]_\rho + [p, f_m\, d(\psi^u)_p(X^{\mathcal{P}}_p)]_\rho + [p, df_m(X_m)\,\psi^u_p]_\rho\\ &= (\nabla_X s)_m + (f\,\nabla_X u)_m + (X.f\,u)_m\end{aligned}$$

Smoothness of $\nabla_X s$ is a consequence of all involved maps and bundles being smooth. $\square$

6.4. Spin Structures and Spinor Bundles

After all the preceding preparatory considerations we will now construct a spinor bundle DM for every finite-dimensional vector space representation of $\mathrm{SL}(2,\mathbb{C})$ and equip these bundles with covariant derivatives which are induced by the Levi-Civita covariant derivative on TM.

Our overall construction is shown in the diagram below. In order to construct a spinor bundle DM we have to take the big detour from (M,g) to TM to $\mathcal{F}_c(M)$ to the spin structure $\mathcal{S}(M)$) and then to the associated bundle DM.



We shall start out defining the yet missing link—the spin structure of the connected frame bundle. For this, cf. [Bau81, p. 70].

6.4.1 Definition (spin structure). Let (M,g) be a connected pseudo-Riemannian manifold of signature (p,q) with connected frame bundle $\mathcal{F}_c(M)$ (cf. definition 6.2.7). Let G be the structure group of $\mathcal{F}_c(M)$[7]. Let $\lambda\colon \mathrm{Pin}_{p,q} \to \mathrm{O}(p,q)$ denote the 2-fold covering map as given in proposition 5.1.9, and let $\tilde{G} := \lambda^{-1}(G)$ be the λ-preimage of G.

A smooth principal $\tilde{G}$-bundle $\mathcal{S}(M)$ on M together with a bundle map $\Lambda\colon \mathcal{S}(M) \to \mathcal{F}_c(M)$ is called a **spin structure on** M**,** if the pair $(\mathcal{S}(M),\Lambda)$ is a λ-reduction of $\mathcal{F}_c(M)$.

[7] we have $G \in \{\mathrm{O}(p,q), \mathrm{O}_t(p,q), \mathrm{O}_s(p,q), \mathrm{SO}(p,q), \mathrm{SO}^+(p,q)\}$

6.4.2 Remark.

(a) Spin structures do not necessarily exist and are not necessarily unique. In [Bau81] one finds that a space- or time-orientable pseudo-Riemannian manifold has a spin structure if and only if its second Stiefel-Whitney class vanishes (Folgerung 2.1), and that every simply connected pseudo-Riemannian manifold has (up to isomorphism) at most one spin structure (Folgerung 2.2). Geroch showed in [Ger70b] that every 4-dimensional globally hyperbolic Lorentzian manifold has a spin structure (this is the relevant case for our considerations).

(b) If $\mathcal{F}_c(M)$ is equipped with a principal bundle connection given by a horizontal bundle $H\mathcal{F}_c(M)$, a spin structure $\Lambda\colon \mathcal{S}(M) \to \mathcal{F}_c(M)$ canonically induces a connection on $\mathcal{S}(M)$ by setting in each fiber over $m \in M$ for every point $p \in \mathcal{S}(M)_m$ and every $\xi \in T_p\mathcal{S}(M)$:

$$\xi \in H_p\mathcal{S}(M) \;:\Leftrightarrow\; d\Lambda_p(\xi) \in H_{\Lambda(p)}\mathcal{F}_c(M)\,.$$

One can show that this yields a principal bundle connection on $\mathcal{S}(M)$, called the **induced spin connection** (cf. [Bau81, p. 119]).

Finally, we can now build spinor bundles as vector bundles associated to a chosen spin structure by means of spinor representations:

6.4.3 Definition (spinor bundles). Let (M, g) be a connected pseudo-Riemannian manifold of signature (p, q) with connected frame bundle $\mathcal{F}_c(M)$ and spin structure $\mathcal{S}(M)$. Let the structure group of $\mathcal{S}(M)$ be denoted by $\tilde{G}$. We have $\tilde{G} \subseteq \mathrm{Pin}(p, q)$.

(a) Let D be a representation of $\tilde{G}$ on a vector space Δ. Then the associated bundle

$$DM := \mathcal{S}(M) \times_D \Delta$$

is called the **spinor bundle**[8] **of type** D or the **bundle of** D**-spinors** on M with respect to the spin structure $\mathcal{S}(M)$. The (smooth) sections of DM are called **spinor fields of type** D**.**

(b) Of course, this thesis is all about the special case of (M, g) being a time-oriented and space-oriented, globally hyperbolic (thus, connected), 4-dimensional, Lorentzian manifold of signature $(+ - - -)$. In this case, the connected frame bundle is $\mathcal{SO}^+(M)$, it has structure group $\mathrm{SO}^+(1,3) = \mathcal{L}_+^\uparrow$ and a spin structure $\mathcal{S}(M)$ on (M, g) is a principal $\mathrm{SL}(2,\mathbb{C})$-bundle.

Then for every finite-dimensional representation D of $\mathrm{SL}(2,\mathbb{C})$ on a vector space Δ we may form the **bundles of** D**-spinors on** M**:**

$$DM := \mathcal{S}(M) \times_D \Delta\,.$$

Particularly, for the irreducible spinor representatios $D^{(j,j')}$ of $\mathrm{SL}(2,\mathbb{C})$ on $\Delta_{j,j'}$ we may form the **bundles of** (j, j')**-spinors on** M**:**

$$D^{(j,j')}M := \mathcal{S}(M) \times_{D^{(j,j')}} \Delta_{j,j'}$$

[8] Notice that in the context of Clifford algebra representations one might find a different usage of the term "spinor bundle" in the literature, meaning a vector bundle associated by a spinor representation of a Clifford algebra, cf. [BGV92, p. 111], [LM89, p. 96].

and the bundles of (j, j')**-co-spinors on** M**:**

$$D^{(j,j')*}M := (D^{(j,j')}M)^* = \mathcal{S}(M) \times_{D^{(j,j')*}} \Delta^*_{j,j'}$$

Analogously, for D^D the Dirac spinor representation,

$$D^D M := \mathcal{S}(M) \times_{D^D} \Delta_D, \quad \text{and} \quad D^{D*}M := \mathcal{S}(M) \times_{D^{D*}} \Delta^*_D$$

are the **bundles of Dirac spinors and Dirac co-spinors on** M**.**

For the rest of the section we shall restrict our attention to the (physical) setting as mentioned in part (b) of the preceding definition. **Thus, for the rest of this section, we make the following assumptions:**

Let (M, g) be a time-oriented and space-oriented, 4-dimensional, globally hyperbolic, Lorentzian manifold of signature $(+ - --)$. (This implies that (M, g) is oriented and connected, it satisfies the strong causality condition and it has a spin structure). Let both space and time orientations be chosen.

Let, moreover, $\mathcal{F}_c(M)$ denote its connected frame bundle, which equals $\mathcal{SO}^+(M)$ and has structure group $\mathrm{SO}^+(1, 3) = \mathcal{L}^\uparrow_+$. Let $\Lambda\colon \mathcal{S}(M) \to \mathcal{F}_c(M)$ denote a spin structure on (M, g) (then $\mathcal{S}(M)$ is a principal $\mathrm{SL}(2, \mathbb{C})$-bundle).

6.4.4 Remark (Clebsch-Gordon decomposition on the bundle level). It is easily seen using the Clebsch-Gordon formula (corollary 3.3.10) together with definition and proposition 6.2.4-c that for all spin numbers $j, j', k, k' \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$:

$$D^{(j,j')}M \otimes D^{(k,k')}M \cong \bigoplus_{\mu=|j-k|}^{j+k} \; \bigoplus_{\mu'=|j'-k'|}^{j'+k'} D^{(\mu,\mu')}M$$

as vector bundles.

6.4.5 Remark (reference frame transformations on spacetime manifolds). In remark 4.2.6 we were considering transformations of reference frames of a physical system modelled by vectors from $(\mathbb{R}^4, \eta)$ and spinors from an $\mathrm{SL}(2, \mathbb{C})$ spinor representation (D, Δ). Let a new physical system be given which is modelled by sections of TM (i. e. vector fields on M) and sections of DM (i. e. spinor fields on M). We shall now see that for every $S \in \mathrm{SL}(2, \mathbb{C})$, the spin structure $\Lambda\colon \mathcal{S}(M) \to \mathcal{SO}^+(M)$ communicates a transformation of reference frames of this new system, which fiberwise equals the transformation mechanism from remark 4.2.6.

Let $U \subseteq M$ be an open neighbourhood on which local reference frames of the system are given by a local tetraed field $(\hat{b}_0, \hat{b}_1, \hat{b}_2, \hat{b}_3)$ (this is a local section of $\mathcal{SO}^+(M)$) and by linearly independent local sections $\hat{B}_1, \ldots, \hat{B}_{\dim(D)} \in \Gamma(DM|_U)$. Then there is a local section F of $\mathcal{S}(M)$ such that $(\hat{b}_0, \ldots, \hat{b}_3) = \Lambda \circ F$, and one may choose local functions $B_\nu\colon U \to \Delta$, such that

$$\hat{B}_\nu = [F, B_\nu]_D, \quad \nu = 1, \ldots, \dim(D)\,.$$

Now, an element $S \in \mathrm{SL}(2,\mathbb{C})$ induces transformations $\hat{b}_\mu \to \hat{b}'_\mu$ and $\hat{B}_\nu \to \hat{B}'_\nu$ of these reference frames, given by

$$(\hat{b}'_0, \dots, \hat{b}'_3) := \underbrace{\Lambda \circ (F \cdot S) = (\hat{b}_0, \dots, \hat{b}_3) \cdot \lambda(S) = (\lambda(S)\hat{b}_0, \dots, \lambda(S)\hat{b}_3)}_{\text{cf. comm. diagram in def. 6.2.2 and text on frame bdls. below def. 6.2.5}} ,$$

where λ is the double covering $\lambda\colon \mathrm{SL}(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow$, and

$$\hat{B}'_\nu := [F \cdot S, B_\nu] = [F, D(S)B_\nu] .$$

In words: In each fiber, S transforms the local reference frame $\{\hat{b}_\mu\}$ by the Lorentz transformation $\Lambda(S) \in \mathcal{L}_+^\uparrow$ and the local reference frame $\{\hat{B}_\mu\}$ by $D(S)$. This is just what we had in remark 4.2.6. (It is easily seen that this construction is independent of the choice of F.)

Our next aim is constructing spinor fields $\hat{\sigma}_a{}^{A\dot{X}}$, $\hat{\epsilon}^{AB}$ and $\hat{\gamma}_a{}^{\tilde{A}}{}_{\tilde{B}}$ on M which are fiberwise given by $\sigma_a{}^{A\dot{X}}$, ϵ^{AB} and $\gamma_a{}^{\tilde{A}}{}_{\tilde{B}}$ as known from the previous sections. It is amazing to see that, after the choice of a spin structure, this can be done in a canonical way[9].

6.4.6 Definition and proposition ($\sigma_a{}^{A\dot{X}}$, ϵ^{AB} and $\gamma_a{}^{\tilde{A}}{}_{\tilde{B}}$ on a manifold).

(a) Let F be a local section of $\mathcal{S}(M)$ on an open neighbourhood $U \subseteq M$. By Λ, F gets mapped to a local section in $\mathcal{SO}^+(M)$, which constitutes a local tetraed field $(\hat{b}_0, \hat{b}_1, \hat{b}_2, \hat{b}_3) := \Lambda \circ F$. Moreover, let $\{E_\nu\}$, $\{\bar{E}_{\dot{\nu}}\}$, $\{E^\nu\}$, $\{\bar{E}^{\dot{\nu}}\}$, $\{\tilde{E}_{\tilde{\nu}}\}$ and $\{\tilde{E}^{\tilde{\nu}}\}$ denote the standard bases of $\Delta_{\frac{1}{2},0}$, $\Delta_{0,\frac{1}{2}}$, $\Delta^*_{\frac{1}{2},0}$, $\Delta^*_{0,\frac{1}{2}}$, Δ_D and Δ^*_D (cf. point 7 in section 4.1 and point 6 in section 5.3). Then F induces local sections $\{\hat{E}_\nu\}$ of $D^{(\frac{1}{2},0)}M$, $\{\hat{\bar{E}}_{\dot{\nu}}\}$ of $D^{(0,\frac{1}{2})}M$, $\{\hat{E}^\nu\}$ of $D^{(\frac{1}{2},0)*}M$, $\{\hat{\bar{E}}^{\dot{\nu}}\}$ of $D^{(0,\frac{1}{2})*}M$, $\{\hat{\tilde{E}}_{\tilde{\nu}}\}$ of $D^D M$, and $\{\hat{\tilde{E}}^{\tilde{\nu}}\}$ of $D^{D*}M$ by setting

$$\begin{aligned} \hat{E}_\nu &:= [F, E_\nu]_{D^{(\frac{1}{2},0)}}, & \hat{\bar{E}}_{\dot{\nu}} &:= [F, \bar{E}_{\dot{\nu}}]_{D^{(0,\frac{1}{2})}}, & \hat{\tilde{E}}_{\tilde{\nu}} &:= [F, \tilde{E}_{\tilde{\nu}}]_{D^D}, \\ \hat{E}^\nu &:= [F, E^\nu]_{D^{(\frac{1}{2},0)*}}, & \hat{\bar{E}}^{\dot{\nu}} &:= [F, \bar{E}^{\dot{\nu}}]_{D^{(0,\frac{1}{2})*}}, & \hat{\tilde{E}}^{\tilde{\nu}} &:= [F, \tilde{E}^{\tilde{\nu}}]_{D^{D*}} \end{aligned}$$

for all $\nu, \dot{\nu} = 1, 2,\ \tilde{\nu} = 1, \dots, 4$.

(b) Let $\sigma_\kappa{}^{\mu\dot{\nu}}$ be the coordinate representation of the spinor-tensor $\sigma_a{}^{A\dot{X}}$ from definition 4.2.1 with respect to the standard bases $\{e^\mu\}$, $\{E_\nu\}$ and $\{\bar{E}_{\dot{\nu}}\}$. Then we define the tensor-spinor field

$$\hat{\sigma}_a{}^{A\dot{X}} \in \Gamma\big(T^*_{\mathbb{C}}M \otimes D^{(\frac{1}{2},\frac{1}{2})}M\big), \qquad T^*_{\mathbb{C}}M := T^*M \otimes \mathbb{C}$$

locally by

$$\hat{\sigma}_a{}^{A\dot{X}} = \sigma_\kappa{}^{\mu\dot{\nu}}\, \hat{b}^\kappa \otimes \hat{E}_\mu \otimes \hat{\bar{E}}_{\dot{\nu}} . \tag{A}$$

[9] Strictly speaking, the construction further depends upon the choice of standard basis of $\Delta_{\frac{1}{2},0}$, as done at the beginning of chapter 3, but still we call this canonical, in the sense of "no *further* arbitrary choices involved".

(c) Analogously, let $\epsilon^{\mu\nu}$ be the totally anti-symmetric spinor ϵ^{AB} from section 4.1 given in components with respect to the standard basis $\{E_\nu\}$. Then we define the spinor field

$$\hat{\epsilon}^{AB} \in \Gamma\big(D^{(\frac{1}{2},0)}M \otimes D^{(\frac{1}{2},0)}M\big)$$

locally by

$$\hat{\epsilon}^{AB} = \epsilon^{\mu\nu}\, \hat{E}_\mu \otimes \hat{E}_\nu\,. \tag{B}$$

(d) Finally, let $\gamma_a{}^{\tilde{A}}{}_{\tilde{B}}$ be the Dirac tensor-spinor as defined in definition 5.3.2-a. Let $\gamma_\kappa{}^{\tilde{\mu}}{}_{\tilde{\nu}}$ be its component representation with respect to $\{e_\mu\}$ and $\{\tilde{E}_{\tilde{\nu}}\}$. Then we define the tensor-spinor field

$$\hat{\gamma}_a{}^{\tilde{A}}{}_{\tilde{B}} \in \Gamma\big(T^*_{\mathbb{C}}M \otimes D^D M \otimes D^{D*}M\big)$$

locally by

$$\hat{\gamma}_a{}^{\tilde{A}}{}_{\tilde{B}} = \gamma_\kappa{}^{\tilde{\mu}}{}_{\tilde{\nu}}\, \hat{b}^\kappa \otimes \hat{\tilde{E}}_{\tilde{\mu}} \otimes \hat{\tilde{E}}^{\tilde{\nu}}, \qquad T^*_{\mathbb{C}}M := T^*M \otimes \mathbb{C}\,. \tag{C}$$

(A), (B) and (C) do not depend on the choice of local section F and thus all these objects are well defined. We shall only write the hat on $\hat{\sigma}_a{}^{A\dot{X}}$, $\hat{\epsilon}^{AB}$ and $\hat{\gamma}_a{}^{\tilde{A}}{}_{\tilde{B}}$ if it is necessary to distinguish these tensor-/spinor-fields from the corresponding objects $\sigma_a{}^{A\dot{X}}$, ϵ^{AB} and $\gamma_a{}^{\tilde{A}}{}_{\tilde{B}}$ on a single fiber.

Proof. Let F and F' be two smooth local sections of $\mathcal{S}(M)$ on an open neighbourhood $U \subseteq M$. Then there is a smooth function $T\colon U \to \mathrm{SL}(2,\mathbb{C})$ such that $F' = F \cdot T$. Using remark 6.4.5, the local reference frames $(\hat{b}'_0, \dots, \hat{b}'_3)$ and $(\hat{E}'_1, \dots, \hat{E}'_4)$ induced by F' according to the procedure described in (a) are then related to the $(\hat{b}_0, \dots, \hat{b}_3)$ and $(\hat{E}_1, \dots, \hat{E}_4)$ by

$$(\hat{b}'_0, \dots, \hat{b}'_3) = \Lambda \circ (F \cdot T) = (\lambda(T)\hat{b}_0, \dots, \lambda(T)\hat{b}_3)$$

(where in the right-hand expression, λ is the double covering $\lambda\colon \mathrm{SL}(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow$), and

$$\hat{E}'_\nu = T \cdot \hat{E}_\nu = [F \cdot T, E_\nu] = [F, D^{(\frac{1}{2},0)}(T)E_\nu]\,.$$

Analogous results hold for the transformations of the other local reference frames constructed in (a) and we shall refrain from writing them all down here.

Now, notice that in each fiber over $m \in U$, this constitutes precisely such a collection of basis transformations given by $T(m)$, as described in remark 4.2.6. According to the same remark, coordinate representations of $\sigma_a{}^{A\dot{X}}$ are invariant under basis transformations of this kind. This shows that in each fiber over U,

$$\hat{\sigma}_a{}^{A\dot{X}} = \sigma_\kappa{}^{\mu\dot{\nu}}\, \hat{b}_\kappa \otimes \hat{E}_\mu \otimes \hat{\bar{E}}_{\dot{\nu}} = \sigma_\kappa{}^{\mu\dot{\nu}}\, \hat{b}'_\kappa \otimes \hat{E}'_\mu \otimes \hat{\bar{E}}'_{\dot{\nu}}\,.$$

In a completely analogous way, application of proposition 4.1.4, shows that in each fiber over U,

$$\hat{\epsilon}^{AB} = \epsilon^{\mu\nu}\, \hat{E}_\mu \otimes \hat{E}_\nu = \epsilon^{\mu\nu}\, \hat{E}'_\mu \otimes \hat{E}'_\nu\,.$$

The case of $\hat{\gamma}_a{}^{\tilde{A}}{}_{\tilde{B}}$ can be reduced to the cases of $\hat{\sigma}_a{}^{A\dot{X}}$ and $\hat{\epsilon}^{AB}$ using the definition of $\gamma_a{}^{\tilde{A}}{}_{\tilde{B}}$ (5.3.2-a). ∎

6.4.7 Remark (induced covariant derivative on spinor bundles).

(a) Notice that as soon as there is a metric compatible covariant derivative ∇ on TM, this induces a connection on the connected frame bundle $\mathcal{SO}^+(M)$, which induces a connection on the spin structure $\mathcal{S}(M)$, which induces a covariant derivative on associated bundles, especially on all spinor bundles (cf. propositions 6.3.5, 6.3.8, remark 6.4.2-c). We shall denote these covariant derivatives again by ∇.

(b) It is easily checked that the covariant derivatives on different spinor bundles are compatible in the following sense:

Let (only for the moment) $\nabla^{(j,j')}$ denote the covariant derivative on $D^{(j,j')}M$. Then for all $S \in \Gamma(D^{(j,0)}M)$ and all $T \in \Gamma(D^{(0,j')})$ one finds:

$$\nabla_a^{(j,j')}(T \otimes S) = \nabla_a^{(j,0)} S \otimes T + S \otimes \nabla_a^{(0,j')} T \,.$$

(The proof is just a matter of chasing definitions.)

(c) It is important to notice that with respect to the induced covariant derivatives on $D^{(\frac{1}{2},0)}M \otimes D^{(\frac{1}{2},0)}M$, $T^*_{\mathbb{C}}M \otimes D^{(\frac{1}{2},\frac{1}{2})}M$ and $T^*_{\mathbb{C}}M \otimes D^D M \otimes D^{D*}M$, the fields ϵ^{AB}, ${\sigma_a}^{A\dot{X}}$ and ${\gamma_a}^{\tilde{A}}{}_{\tilde{A}}$ are *parallel*, i. e.

$$\nabla_a \epsilon^{AB} = 0, \quad \nabla_a {\sigma_b}^{A\dot{X}} = 0, \quad \nabla_a {\gamma_b}^{\tilde{A}}{}_{\tilde{B}} = 0 \,.$$

Proof. We sketch the proof for ${\sigma_a}^{A\dot{X}}$; the other proofs are completely analogous. To prove $x^a(\nabla_a {\sigma_b}^{A\dot{X}})_m = 0$ for an arbitrary $m \in M$ and $x \in T_mM$, let $\gamma\colon (-\varepsilon,\varepsilon) \to M$, $\gamma(0) = m$, $\dot{\gamma}(0) = x$ be a geodesic. Fix any $p \in (\mathcal{SO}^+(M))_m$ and let $\tilde{\gamma}\colon (-\varepsilon,\varepsilon) \to \mathcal{SO}^+(M)$ be the horizontal lift of γ with $\tilde{\gamma}(0) = p$. Moreover, choose a curve $\hat{\gamma}\colon (-\varepsilon,\varepsilon) \to \mathcal{S}(M)$ such that $\Lambda(\hat{\gamma}(t)) = \tilde{\gamma}(t)$. In analogy with definition 6.4.6-a define the reference frames

$$\hat{E}_\nu(t) := [\hat{\gamma}(t), E_\nu]_{D^{(\frac{1}{2},0)}} \in D^{(\frac{1}{2},0)}M$$
$$\hat{\bar{E}}_{\dot{\nu}}(t) := [\hat{\gamma}(t), \bar{E}_{\dot{\nu}}]_{D^{(0,\frac{1}{2})}} \in D^{(0,\frac{1}{2})}M$$

(these are "reference frames along a curve"). According to proposition 6.4.6, ${\sigma_a}^{A\dot{X}}$ along γ is given by

$$({\sigma_a}^{A\dot{X}})_{\gamma(t)} = {\sigma_\kappa}^{\mu\dot{\nu}}\, (e^\kappa)_{\gamma(t)} \otimes (\hat{E}_\mu)_{\gamma(t)} \otimes (\hat{\bar{E}}_{\dot{\nu}})_{\gamma(t)} \,,$$

where the coordinates ${\sigma_\kappa}^{\mu\dot{\nu}}$ do not depend on t and $\{e^\kappa\}$ is the reference frame of T^*M along γ given by $\tilde{\gamma}$. Thus, we have in the fiber over m:

$$\begin{aligned}\nabla_x {\sigma_a}^{A\dot{X}} &= \nabla_x({\sigma_\kappa}^{\mu\dot{\nu}}) e_\kappa \otimes \hat{E}_\mu \otimes \hat{\bar{E}}_{\dot{\nu}} + {\sigma_\kappa}^{\mu\dot{\nu}}\, (\nabla_x e_\kappa) \otimes \hat{E}_\mu \otimes \hat{\bar{E}}_{\dot{\nu}} \\ &\quad + {\sigma_\kappa}^{\mu\dot{\nu}}\, e_\kappa \otimes (\nabla_x \hat{E}_\mu) \otimes \hat{\bar{E}}_{\dot{\nu}} + {\sigma_\kappa}^{\mu\dot{\nu}}\, e_\kappa \otimes \hat{E}_\mu \otimes \nabla_x(\hat{\bar{E}}_{\dot{\nu}}) \\ &= 0\end{aligned}$$

because by construction, $\hat{E}_\mu$, $\hat{\bar{E}}_{\dot{\mu}}$ and e^κ are parallel. $\square$

Notice that parts (b) and (c) of this remark are of core importance for everyday life when using 2- and 4-spinors on manifolds. Without them one could not write stuff like

$$\epsilon_{CD}\epsilon_{AB}\nabla_a\psi^{ABC} = \nabla_a\psi_B{}^B{}_D\,.$$

All calculations like those in sections 8.2–8.4 and many physics papers and text books silently assume these results!

7. Normally Hyperbolic Differential Operators

This chapter is the last step on our road towards a mathematically full-featured framework for dealing with differential equations for spinor fields on curved spacetime manifolds and for tackeling Buchdahl's generalised Dirac equations in an effective and elegant way.

For the reader who is not familiar with the formal definitions, we will start out in section 7.1 introducing the mathematical concept of differential operators on general smooth vector bundles and the principal symbol. In section 7.2 we will define the physically relevant class of normally hyperbolic differential operators.

Finally, in section 7.3 we will prove existence and uniqueness of advanced and retarded Green's operators for a certain class of first order differential operators (including the Dirac operator on Dirac spin $\frac{1}{2}$ fields and the Buchdahl operator on higher spin fields) using results from [BGP]. Thereby we obtain a generalisation of a central theorem from [Dim82].

7.1. Differential Operators on Manifolds

Let in this section M denote an n-dimensional smooth manifold with a smooth volume density $d\mu$. If not stated otherwise, all vector bundles, maps and sections are assumed smooth. If $\mathcal{E}$ is a vector bundle on M, $\Gamma(\mathcal{E}) := \Gamma^\infty(\mathcal{E})$ denotes the space of smooth sections.

7.1.1 Definition (multi index notation). An **k-multi index** is an k-tuple of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_k)$. We write $|\alpha| := \sum_{i=1}^{k} \alpha_i$ (legth of α) and

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} := \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^k}\right)^{\alpha_k} .$$

7.1.2 Definition. Let $\pi\colon \mathcal{E} \to M$ be a smooth vector bundle on M with typical fiber E. Let $U \subseteq M$ be an open subset on which there is a local coordinate chart $\psi\colon U \to V$ for an open subset $V \subseteq \mathbb{R}^n$, and such that there is a local trivialisation $\chi\colon \pi^{-1}(U) \to U \times E$ of $\mathcal{E}$. Let the local coordinates induced by ψ be denoted by $x^1, \ldots, x^n$ and the induced local coordinate vector fields by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^\nu}$.

Then by $\frac{\partial_\chi}{\partial x^1}, \ldots, \frac{\partial_\chi}{\partial x^n}$ we denote the differential operators on $\mathcal{E}|_U$[1], given through χ by

[1] $\mathcal{E}|_U$ denotes the vector bundle $\mathcal{E}$, restricted to $U \subseteq M$.

differentiation in E, i.e. for $x \in U$ and $s \in \Gamma(\mathcal{E}|_U)$,

$$\frac{\partial_\chi}{\partial x^\mu} s\Big|_x := \chi^{-1}\Big(x, \frac{\partial}{\partial x^\mu}\Big|_{\psi(x)} \underbrace{(\mathrm{pr}_2 \circ \chi \circ s \circ \psi^{-1})}_{\mathbb{R}^n \supseteq V \to E}\Big),$$

where $\mathrm{pr}_2 \colon U \times E \to E$ denotes the projection onto the second component. We refer to these operators as (local) **componentwise derivative operators on $\mathcal{E}$ with respect to χ and $\{x^\mu\}$.**

7.1.3 Remark (on componentwise derivative operators). Why are the $\frac{\partial_\chi}{\partial x^\mu}$ called *componentwise* derivative operators?—Let $\mathcal{E}$ be of rank p and choose an isomorphism $E \cong \mathbb{R}^p$, such that $\mathcal{E}$ has typical fiber $\mathbb{R}^p$. With respect to the local trivialisation χ on U, every section $s \in \Gamma(\mathcal{E}|_U)$ may then be written as

$$s(x) = \begin{pmatrix} s^1(x) \\ \vdots \\ s^p(x) \end{pmatrix}, \quad s^1, \dots, s^p \in C(U), \quad x \in U.$$

In this notation, $\frac{\partial_\chi}{\partial x^\mu}$ is given by

$$\frac{\partial_\chi}{\partial x^\mu} s(x) = \begin{pmatrix} \frac{\partial}{\partial x^\mu} s^1(x) \\ \vdots \\ \frac{\partial}{\partial x^\mu} s^p(x) \end{pmatrix}.$$

This is just componentwise differentiation in the local trivialisation χ. One may also form higher order componentwise derivatives: Let α be a p-multi index. Then one has:

$$\frac{\partial_\chi^{|\alpha|}}{\partial x^\alpha} s(x) = \begin{pmatrix} \frac{\partial^{|\alpha|}}{\partial x^\alpha} s^1(x) \\ \vdots \\ \frac{\partial^{|\alpha|}}{\partial x^\alpha} s^p(x) \end{pmatrix}.$$

As this is, in each component, differentiation of an $\mathbb{R}$-valued function, it is easily seen that the local componentwise derivative operators $\frac{\partial_\chi}{\partial x^\mu}$ commute.

7.1.4 Definition (differential operator on vector bundles). Let $\mathcal{E}$ and $\mathcal{F}$ be be smooth $\mathbb{K}$-vector bundles on M with typical fibers E and F. A **linear differential operator of order at most k from $\mathcal{E}$ to $\mathcal{F}$** is a $\mathbb{K}$-linear map

$$P \colon \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F}),$$

which has the following property: Around every point on M there is an open neighbourhood $U \subseteq M$, with local coordinates $x^1, \dots, x^n$ on U, and there is a local trivialisation $\chi \colon \pi^{-1}(U) \to U \times E$ of $\mathcal{E}$, such that for all $s \in \Gamma(\mathcal{E})$:

$$(Ps)(m) = \sum_{|\alpha| \le k} \Big(A^\alpha \frac{\partial_\chi^{|\alpha|}}{\partial x^\alpha} s\Big)(m) \qquad \forall m \in U,$$

where for every multi index α with $|\alpha| \le k$, $A^\alpha \in \Gamma(\mathrm{Hom}(\mathcal{E}|_U, \mathcal{F}|_U))$.

P is said to be of order k, if for no $l < k$ it is a linear differential operator of oder at most l.

7.1.5 Remark (change of coordinates and trivialisations). Let $\mathcal{E}$ and $\mathcal{F}$ be defined as in definition 7.1.4, fix local coordinates $x^1, \ldots, x^n$ on $U \subseteq^{\text{open}} M$ and fix a local trivialisation $\chi\colon \pi^{-1}(U) \to U \times E$ of $\mathcal{E}$.

(a) Let the linear differential operator P of order k be given with respect to our local coordinates and our local trivialisation by

$$P = \sum_{|\alpha| \le k} A^\alpha(x) \frac{\partial_\chi^{|\alpha|}}{\partial x^\alpha} ,$$

where A^α are smooth sections of $\mathrm{Hom}(\mathcal{E}|_U, \mathcal{F}|_U)$.
For every $0 \le \kappa \le k$ we define a local section Γ_κ of $\vee^\kappa TM|_U \otimes \mathrm{Hom}(\mathcal{E}|_U, \mathcal{F}|_U)$ (cf. footnote[2]) by setting in our local coordinates:

$$\Gamma_\kappa := \sum_{|\alpha|=\kappa} (\tfrac{\partial}{\partial x^1})^{\vee \alpha_1} \vee \ldots \vee (\tfrac{\partial}{\partial x^n})^{\vee \alpha_n} \otimes A^\alpha .$$

As argued in remark 7.1.3, the $\frac{\partial_\chi}{\partial x^\mu}$ commute. Therefore we can locally write P as:

$$P = \sum_{\kappa=0}^{k} (\Gamma_\kappa)^{\mu_1 \cdots \mu_\kappa} \frac{\partial_\chi}{\partial x^{\mu_1}} \cdots \frac{\partial_\chi}{\partial x^{\mu_\kappa}} , \tag{$*$}$$

here we use implicit summation over $\mu_1, \ldots, \mu_\kappa$ and for all $\mu_1, \ldots, \mu_\kappa = 1, \ldots, n$, $(\Gamma_\kappa)^{\mu_1 \cdots \mu_\kappa}$ is an element of $\mathrm{Hom}(\mathcal{E}|_U, \mathcal{F}|_U)$.

(b) Now, let $y^1, \ldots, y^n$ be a second collection of local coordinates on U. Again, $\frac{\partial}{\partial y^\mu}$ denote the local coordinate vector fields and $\frac{\partial_\chi}{\partial y^\mu}$ the local differential operators on $\mathcal{E}$ according to definition 7.1.2. It is easily seen that we have:

$$\frac{\partial_\chi}{\partial x^\mu} = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial_\chi}{\partial y^\nu} .$$

This gives:

$$\begin{aligned} P &= \sum_{\kappa=0}^{k} (\Gamma_\kappa)^{\mu_1 \cdots \mu_\kappa} \frac{\partial_\chi}{\partial x^{\mu_1}} \cdots \frac{\partial_\chi}{\partial x^{\mu_\kappa}} \\ &= \sum_{\kappa=0}^{k} (\Gamma_\kappa)^{\mu_1 \cdots \mu_\kappa} \left(\frac{\partial y^{\nu_1}}{\partial x^{\mu_1}} \frac{\partial_\chi}{\partial y^{\nu_1}} \right) \cdots \left(\frac{\partial y^{\nu_\kappa}}{\partial x^{\mu_\kappa}} \frac{\partial_\chi}{\partial y^{\nu_\kappa}} \right) \\ &= \sum_{\kappa=0}^{k} \Bigg[\underbrace{(\Gamma_\kappa)^{\mu_1 \cdots \mu_\kappa} \frac{\partial y^{\nu_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\nu_\kappa}}{\partial x^{\mu_\kappa}}}_{\substack{\text{The tensor } \Gamma_\kappa \text{ transformed into} \\ \text{the new coordinate basis}}} \frac{\partial_\chi}{\partial y^{\nu_1}} \cdots \frac{\partial_\chi}{\partial y^{\nu_\kappa}} + \text{lower order derivatives} \Bigg] \end{aligned}$$

---

[2] By $\vee$ we denote the symmetrised tensor product,

$$(x_1 \vee \ldots \vee x_k)^{a_1 \ldots a_k} := \frac{1}{k!} \sum_{\pi \in S_k} x_1^{a_{\pi(1)}} \otimes \ldots \otimes x_k^{a_{\pi(k)}} .$$

By "lower order derivatives" we mean the differential operators of oder $< \kappa$ on $\mathcal{E}$ arising from using the Leibniz product rule on an expression like

$$\left(\frac{\partial y^{\nu_1}}{\partial x^{\mu_1}}\frac{\partial_\chi}{\partial y^{\nu_1}}\right)\left(\frac{\partial y^{\nu_2}}{\partial x^{\mu_2}}\frac{\partial_\chi}{\partial y^{\nu_2}}\right)\cdots\left(\frac{\partial y^{\nu_\kappa}}{\partial x^{\mu_\kappa}}\frac{\partial_\chi}{\partial y^{\nu_\kappa}}\right).$$

The appearance of these lower order derivative terms is crucial. It shows that transforming expression $(*)$ into the local coordinates $y^1, \ldots, y^n$ can *not* be done by just transforming each of the tensors Γ_κ separately! This means, in general,

$$P \overset{!}{\neq} \sum_{\kappa=0}^{k} (\tilde{\Gamma}_\kappa)^{\nu_1\ldots\nu_\kappa} \frac{\partial_\chi}{\partial y^{\nu_1}} \cdots \frac{\partial_\chi}{\partial y^{\nu_\kappa}}, \quad \text{if} \quad (\tilde{\Gamma}_\kappa)^{\nu_1\ldots\nu_\kappa} := (\Gamma_\kappa)^{\mu_1\ldots\mu_\kappa} \frac{\partial y^{\nu_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\nu_\kappa}}{\partial x^{\mu_\kappa}}.$$

(c) Let's return to our original local coordinates $\{x^\mu\}$ but let $\varphi\colon \pi^{-1}(U) \to U \times E$ be a second local trivialisation of $\mathcal{E}$ on U. This new local trivialisation induces new local differential operators $\frac{\partial_\varphi}{\partial x^\mu}$, given according to definition 7.1.2.
Notice that both the $\frac{\partial_\chi}{\partial x^\mu}$ and the $\frac{\partial_\varphi}{\partial x^\mu}$ constitute covariant derivatives on $\mathcal{E}|_U$. It is known that two covariant derivatives differ by a linear map on the fibers, i. e. there is a section $C \in T^*M|_U \otimes \mathrm{End}(\mathcal{E}|_U)$, such that

$$\forall s \in \Gamma(\mathcal{E}|_U)\colon \frac{\partial_\chi}{\partial x^\mu}(s) = \frac{\partial_\varphi}{\partial x^\mu}(s) + C_\mu(s).$$

For our differential operator P this gives:

$$\begin{aligned} P &= \sum_{\kappa=0}^{k} (\Gamma_\kappa)^{\mu_1\ldots\mu_\kappa} \frac{\partial_\chi}{\partial x^{\mu_1}} \cdots \frac{\partial_\chi}{\partial x^{\mu_\kappa}} \\ &= \sum_{\kappa=0}^{k} (\Gamma_\kappa)^{\mu_1\ldots\mu_\kappa} \left(\frac{\partial_\varphi}{\partial x^{\mu_1}} + C_{\mu_1}\right) \cdots \left(\frac{\partial_\varphi}{\partial x^{\mu_\kappa}} + C_{\mu_\kappa}\right) \\ &= \sum_{\kappa=0}^{k} \left[(\Gamma_\kappa)^{\mu_1\ldots\mu_\kappa} \frac{\partial_\varphi}{\partial x^{\mu_1}} \cdots \frac{\partial_\varphi}{\partial x^{\mu_\kappa}} + \text{lower order derivatives}\right] \end{aligned}$$

By "lower order derivatives" we mean the differential operators of oder $< \kappa$ on $\mathcal{E}$ arising from expanding

$$\left(\frac{\partial_\varphi}{\partial x^{\mu_1}} + C_{\mu_1}\right) \cdots \left(\frac{\partial_\varphi}{\partial x^{\mu_\kappa}} + C_{\mu_\kappa}\right) = \frac{\partial_\varphi}{\partial x^{\mu_\kappa}} \cdots \frac{\partial_\varphi}{\partial x^{\mu_\kappa}} + \underbrace{C_{\mu_1}\frac{\partial_\varphi}{\partial x^{\mu_2}} \cdots \frac{\partial_\varphi}{\partial x^{\mu_\kappa}} + \ldots + C_{\mu_1}\cdots C_{\mu_\kappa}}_{\text{“lower order derivatives”}}$$

Again, we have the appearance of crucial lower order derivative terms. This shows that transforming expression $(*)$ into an analog expression with respect to a new local trivialisation φ can, in general, not be attained by mere substituting $\frac{\partial_\varphi}{\partial x^\mu}$ for $\frac{\partial_\chi}{\partial x^\mu}$: In general,

$$P \overset{!}{\neq} \sum_{\kappa=0}^{k} (\Gamma_\kappa)^{\mu_1\ldots\mu_\lambda} \frac{\partial_\varphi}{\partial x^{\mu_1}} \cdots \frac{\partial_\varphi}{\partial x^{\mu_\lambda}}.$$

(d) Notice, however, that there is one exception: If P is of order k, then there will never arise an additional derivative term of order k (all the additional terms are of order $< k$, for both coordinate transformations and transformations of the local trivialisation). Therefore, it is clear that if

$$P = \sum_{\kappa=0}^{k} (\Gamma'_\kappa)^{\nu_1 \ldots \nu_\kappa} \frac{\partial_\varphi}{\partial y^{\nu_1}} \cdots \frac{\partial_\varphi}{\partial y^{\nu_\kappa}} ,$$

is a representation of P with respect to the local coordinates $y^1, \ldots, y^n$ and the local trivialisation φ, we find

$$(\Gamma'_k)^{\nu_1 \ldots \nu_k} = (\tilde{\Gamma}_k)^{\nu_1 \ldots \nu_k} := (\Gamma_k)^{\mu_1 \ldots \mu_k} \frac{\partial y^{\nu_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\nu_k}}{\partial x^{\mu_k}}$$

(for k only, not generally for κ). So, Γ_k, the local section of $\vee^k TM \otimes \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ "which handles the highest order derivative term in a local coordinate representation of P" has the special property of transforming contravariantly under change of local coordinates and being invariant under change of local trivialisations. This is why Γ_k also has a special name: the **principal symbol of P,** denoted by σ_P. Because of it's transformation behaviour, the principal symbol is a *global* section of $\vee^k TM \otimes \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ and, with respect to *every* local coordinate chart $x^1, \ldots, x^n$ and local trivialisation χ, P may be written locally as

$$P = \underbrace{(\sigma_P)^{\mu_1 \ldots \mu_n} \frac{\partial_\chi}{\partial x^{\mu_1}} \cdots \frac{\partial_\chi}{\partial x^{\mu_n}}}_{\{x^\mu\} \text{ and } \chi \text{ independent part}} + \underbrace{\text{lower order derivatives}}_{\{x^\mu\} \text{ and } \chi \text{ dependent part}} .$$

(The highest order term $(\sigma_P)^{\mu_1 \ldots \mu_n} \frac{\partial_\chi}{\partial y^{\mu_1}} \cdots \frac{\partial_\chi}{\partial y^{\mu_n}}$ is often called the **principal part of P in the local coordinates x^μ and the local trivialisation χ.**)

7.1.6 Definition (principal symbol). Let P be a linear differential operator of order k from $\mathcal{E}$ to $\mathcal{F}$ given locally by

$$P = \sum_{|\alpha| \leq k} A^\alpha \frac{\partial_\chi^{|\alpha|}}{\partial x^\alpha} ,$$

as in definition 7.1.4. The **principal symbol σ_P of P** is the section of $\vee^k TM \otimes \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ locally defined by:

$$\sigma_P := \sum_{|\alpha|=k} (\tfrac{\partial}{\partial x^1})^{\vee \alpha_1} \vee \ldots \vee (\tfrac{\partial}{\partial x^n})^{\vee \alpha_n} \otimes A^\alpha .$$

We learned in the preceding remark that this definition is independent of the choice of local coordinates x^μ and local trivialisation χ.

7.1.7 Lemma. Let $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ be smooth vector bundles on M. Let $P\colon \Gamma(\mathcal{E}_1) \to \Gamma(\mathcal{E}_2)$ and $Q\colon \Gamma(\mathcal{E}_2) \to \Gamma(\mathcal{E}_3)$ be linear differential operators. Let P be of order p and let Q

be of order q. Then $QP\colon \Gamma(\mathcal{E}_1) \to \Gamma(\mathcal{E}_3)$ is a linear differential operator of order at most $p+q$ and the principal symbol

$$\sigma_{QP} \in \vee^{p+q} TM \otimes \operatorname{Hom}(\mathcal{E}_1, \mathcal{E}_3)$$

of QP is given by

$$\begin{aligned}(\sigma_{QP})^{a_1 \ldots a_{p+q}} &= (\sigma_Q)^{(a_1 \ldots a_p} \circ (\sigma_P)^{a_{p+1} \ldots a_{p+q})} \\ &= \sum_{\pi \in S_{p+q}} (\sigma_Q)^{a_{\pi(1)} \ldots a_{\pi(p)}} \circ (\sigma_P)^{a_{\pi(p+1)} \ldots a_{\pi(p+q)}} .\end{aligned}$$

Proof. That QP is a linear differential operator of order at most $p+q$ is immediate. To prove the formula for σ_{QP} we argue with respect to local coordinates $x^1, \ldots, x^n$ on $U \subseteq M$ and with respect to local trivialisations χ and φ of $\mathcal{E}_1$ and $\mathcal{E}_2$ on U. Adopting terminology and notation from remark 7.1.5, the principal parts of P and Q are given by

$$(\sigma_P)^{\mu_1 \ldots \mu_n} \frac{\partial_\chi}{\partial x^{\mu_1}} \cdots \frac{\partial_\chi}{\partial x^{\mu_n}} \quad \text{and} \quad (\sigma_Q)^{\mu_1 \ldots \mu_n} \frac{\partial_\varphi}{\partial x^{\mu_1}} \cdots \frac{\partial_\varphi}{\partial x^{\mu_n}} .$$

Then we find ("l. o. t." meaning "lower order terms"):

$$\begin{aligned}QP &= \left((\sigma_Q)^{\nu_1 \ldots \nu_n} \frac{\partial_\varphi}{\partial x^{\nu_1}} \cdots \frac{\partial_\varphi}{\partial x^{\nu_n}} + \text{l. o. t} \right) \left((\sigma_P)^{\mu_1 \ldots \mu_n} \frac{\partial_\chi}{\partial x^{\mu_1}} \cdots \frac{\partial_\chi}{\partial x^{\mu_n}} + \text{l. o. t.} \right) \\ &= (\sigma_Q)^{\nu_1 \ldots \nu_n} \frac{\partial_\varphi}{\partial x^{\nu_1}} \cdots \frac{\partial_\varphi}{\partial x^{\nu_n}} \; (\sigma_P)^{\mu_1 \ldots \mu_n} \frac{\partial_\chi}{\partial x^{\mu_1}} \cdots \frac{\partial_\chi}{\partial x^{\mu_n}} + \text{l. o. t}\end{aligned}$$

Now, $\frac{\partial_\chi}{\partial x^\mu}$ and $\frac{\partial_\varphi}{\partial x^\mu}$ together induce local differential operators $\frac{\partial_{\chi,\varphi}}{\partial x^\mu}$ on $\operatorname{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ such that for every local section $s \in \Gamma(\mathcal{E}_1|_U)$ and for all $\mu_1, \ldots, \mu_n$ we find:

$$\frac{\partial_\varphi}{\partial x^\mu} \big((\sigma_P)^{\mu_1 \ldots \mu_n} s \big) = \frac{\partial_{\chi,\varphi}}{\partial x^\mu} \big((\sigma_P)^{\mu_1 \ldots \mu_n} \big) \, s + (\sigma_P)^{\mu_1 \ldots \mu_n} \, \frac{\partial_\chi}{\partial x^\mu} (s) .$$

Using this we continue the above calculation:

$$= (\sigma_Q)^{\nu_1 \ldots \nu_n} (\sigma_P)^{\mu_1 \ldots \mu_n} \frac{\partial_\chi}{\partial x^{\nu_1}} \cdots \frac{\partial_\chi}{\partial x^{\nu_n}} \frac{\partial_\chi}{\partial x^{\mu_1}} \cdots \frac{\partial_\chi}{\partial x^{\mu_n}} + \text{new l. o. t} + \text{l. o. t}$$

Using the argument from remark 7.1.5-a, this equals:

$$= (\sigma_Q)^{(\nu_1 \ldots \nu_n} (\sigma_P)^{\mu_1 \ldots \mu_n)} \frac{\partial_\chi}{\partial x^{\nu_1}} \cdots \frac{\partial_\chi}{\partial x^{\nu_n}} \frac{\partial_\chi}{\partial x^{\mu_1}} \cdots \frac{\partial_\chi}{\partial x^{\mu_n}} + \text{new l. o. t} + \text{l. o. t} .$$

□

7.1.8 Definition and proposition (formal adjoint of a differential operator). Let $\mathcal{E}$ and $\mathcal{F}$ be smooth $\mathbb{K}$-vector bundles on M wich dual bundles $\mathcal{E}^*$ and $\mathcal{F}^*$. Let $P\colon \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$ be a linear differential operator. Then there is a unique linear differential operator $P^*\colon \Gamma(\mathcal{F}^*) \to \Gamma(\mathcal{E}^*)$, called **formal adjoint of** P, such that

$$\forall \varphi \in \Gamma_0(\mathcal{E}), \psi \in \Gamma_0(\mathcal{F}^*)\colon \int_M \psi(P\varphi) \, d\mu = \int_M (P^*\psi)(\varphi) \, d\mu .$$

We find that $(P^*)^* = P$, and P^* is of order k if P is of order k.

Proof. Cf. [BGP, p. 4]. □

7.2. Normally Hyperbolic Differential Operators

Let (M, g) denote an n-dimensional smooth Lorentzian manifold. If not stated otherwise, all vector bundles, maps and sections are assumed smooth.

7.2.1 Definition (local componentwise Laplace operator). Let $\mathcal{E}$ be a smooth vector bundle with typical fiber E on (M, g). Choose an open subset $U \subseteq M$ such that there is a local coordinate chart $\xi\colon U \to V \subseteq \mathbb{R}^n$ and a local trivialisation $\chi\colon \pi^{-1}(U) \to U \times E$ of $\mathcal{E}$. Let $x^1, \dots, x^n$ be the local coordinate functions induced by ξ.

Then the **local componentwise Laplace operator with respect to ξ and χ on $\mathcal{E}$** is the linear differential operator $\Delta_{\xi,\chi}\colon \Gamma(\mathcal{E}|_U) \to \Gamma(\mathcal{E}|_U)$ given by:

$$\Delta_{\xi,\chi}(s) := g^{\mu\nu} \frac{\partial_\chi^2}{\partial x^\mu \partial x^\nu} s, \quad s \in \Gamma(\mathcal{E}|_U)\,.$$

Notice that by remark 7.1.5, $\Delta_{\xi,\chi}$ indeed depends on the choices of ξ and χ. For every ξ and χ there is such a local Laplace operator and they may differ be terms of order at most 1.

Local componentwise Laplace operators are the very prototypes of a bigger and physically important class of second order linear differential operators—*normally hyperbolic differential operators.* From the mathematical perspective, they have a particularly accessible solution theory. All second order operators appearing in this thesis will be of normally hyperbolic type from this point on.

7.2.2 Definition (normally hyperbolic differential operator). A second order linear differential operator P acting on sections of a smooth real or complex vector bundle $\mathcal{E}$ over a Lorentzian manifold (M, g) is called **normally hyperbolic,** if its principal symbol is given by the metric:

$$\forall x \in M\, \forall \varphi, \psi \in \Gamma(T_x^*M)\colon \sigma_P(\varphi, \psi) = g(\varphi, \psi)\, \mathrm{Id}_{\mathcal{E}}\ .$$

Equivalently, P is normally hyperbolic if, for every $m \in M$, after choosing arbitrary local coordinates $x^1, \dots, x^n$ and a local trivialisation χ of $\mathcal{E}$ around m, we have for every $s \in \Gamma(\mathcal{E})$:

$$(Ps)(m) = \sum_{\mu,\nu=1}^{n} g^{\mu\nu}(m) \frac{\partial_\chi^2}{\partial x^\mu \partial x^\nu} s(m) + \sum_{\mu=1}^{n} B_\mu(m) \frac{\partial_\chi}{\partial x^\mu} s(m) + C(m)s(m)\,,$$

where $g^{\mu\nu}$ are the inverse metric components and B_μ, C are smooth local sections of $\mathrm{End}(\mathcal{E})$ around m.

7.2.3 Remark (sign convention). Notice that in [BGP, section 1.5] they use the opposite sign convention $\sigma_P(\cdot,\cdot) = -g(\cdot,\cdot)\,\mathrm{Id}_{\mathcal{E}}$. But they also use the opposite metric signatur $(-+++)$, such that the definitions are equivalent.

7.2.4 Proposition. The formal adjoint of a normally hyperbolic differential operator is normally hyperbolic.

7.2.5 Lemma. Let $\mathcal{E}$ be a smooth vector bundle on (M,g), let $U \subseteq M$ be an open neighbourhood with a local coordinate chart $\xi\colon U \to V \subseteq \mathbb{R}^n$ and a local trivialisation $\chi\colon \pi^{-1}(U) \to U \times E$ of $\mathcal{E}$ on U. Let $x^1, \ldots, x^n$ be the local coordinate functions induced by ξ and let χ^* be the local trivialisation of $\mathcal{E}^*$ dual to χ.[3]

Then the formal adjoint of $\Delta_{\xi,\chi}\colon \Gamma(\mathcal{E}|_U) \to \Gamma(\mathcal{E}|_U)$ is given by Δ_{ξ,χ^*}.

Proof. Pick arbitrary $\psi \in \Gamma_0(\mathcal{E}^*|_U)$ and $\varphi \in \Gamma_0(\mathcal{E}|_U)$. Choose a basis $e_1, \ldots, e_k$ of E and let $e^1, \ldots, e^k$ be the dual basis of E^*. By means of χ and χ^* these bases induce tetraed fields on $\mathcal{E}|_U$ and $\mathcal{E}^*|_U$ with respect to which φ and ψ can be written in components $\varphi = \varphi^\mu$ and $\psi = \psi_\mu$, such that $\psi(\varphi) = \psi_\mu \varphi^\mu$. Now we calculate:

$$
\begin{aligned}
\int_U \psi(\Delta_{\xi,\chi}\varphi)\, d\mu &= \int_U \psi\Big(g^{\mu\nu} \frac{\partial_\chi^2}{\partial x^\mu \partial x^\nu} \varphi\Big)\, d\mu \\
&= \int_U \sum_{i=1}^k \psi_i\, g^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} \varphi^i \, d\mu = \sum_{i=1}^k \int_U \psi_i\, \Delta_\xi(\varphi^i)\, d\mu
\end{aligned}
$$

As all of the scalar functions φ^i and ψ_i have compact support with $\operatorname{supp}(\varphi^i) \subseteq U$ and $\operatorname{supp}(\psi_i) \subseteq U$, we may use Green's second formula to obtain:

$$
= \sum_{i=1}^k \int_U \Delta_\xi(\psi_i)\, \varphi^i \, d\mu = \int_U \Big(g^{\mu\nu} \frac{\partial_{\chi^*}^2}{\partial x^\mu \partial x^\nu} \psi\Big)(\varphi)\, d\mu = \int_U (\Delta_{\xi,\chi^*}\psi)(\varphi)\, d\mu
$$

As on general grounds, for $x \in M$ and $\psi \in \Gamma(\mathcal{E}^*|_U)$, $\Delta^*_{\xi,\chi}(\psi)(x)$ depends only locally on ψ, this shows that $\Delta^*_{\xi,\chi} = \Delta_{\xi,\chi^*}$. □

Proof of proposition 7.2.4. Let $\mathcal{E}$ be a smooth vector bundle with typical fiber E on a Lorentzian manifold (M,g). Let $P\colon \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ be a normally hyperbolic linear differential operator and let P^* be its formal adjoint.

We have to show that for every point $m \in M$ there is an open neighbourhood $m \in U \subseteq M$ with local coordinates x^μ on U and a local trivialisation χ^* of $\mathcal{E}^*$, such that for every $\psi \in \Gamma(\mathcal{E}^*)$ we have

$$
(P^*\psi)(m) = \Big(g^{\mu\nu} \frac{\partial_{\chi^*}^2}{\partial x^\mu \partial x^\nu} + \text{l. o. d.}\Big)(\psi)(m)\,, \tag{$*$}
$$

where "l. o. d." means lower order derivative terms.

Pick $m \in M$. Then there is an open neighbourhood $m \in U \subseteq M$ with a local coordinate chart $\xi\colon U \to \mathbb{R}^n$ and a local trivialisation $\chi\colon \pi^{-1}(U) \to U \times E$ of $\mathcal{E}$ on U. Let $x^1, \ldots, x^n$ be the local coordinate functions induced by ξ and let χ^* be the local trivialisation of $\mathcal{E}^*$ dual to χ.

Pick an arbitrary $\psi \in \Gamma_0(\mathcal{E}^*|_U)$ and also an arbitrary $\varphi \in \Gamma_0(\mathcal{E}|_U)$. There are sections $B^\mu, C \in \Gamma(\operatorname{End}(\mathcal{E}|_U))$, such that

$$
(P\varphi)(x) = \underbrace{g^{\mu\nu} \frac{\partial_\chi^2}{\partial x^\mu \partial x^\nu} \varphi(x)}_{=\Delta_{\xi,\chi}(\varphi)} + \underbrace{B^\mu \frac{\partial_\chi}{\partial x^\mu} \varphi(x) + C(\varphi)(x)}_{:=R_\chi(\varphi)}\,, \quad \forall x \in U\,,
$$

[3] The local trivialisation of $\mathcal{E}^*$ dual to χ is the unique local trivialisation $\chi^*\colon \pi^{-1}(U) \to U \times E^*$ such that $\forall \varphi \in \Gamma(\mathcal{E}|_U)\, \forall \psi \in \Gamma(\mathcal{E}^*|_U)\colon \psi(\varphi) = \big(\mathrm{pr}_2 \circ \chi^*(\psi)\big)(\mathrm{pr}_2 \circ \chi(\varphi))$.

where $\Delta_{\xi,\chi}$ is the componentwise Laplace operator on $\Gamma(\mathcal{E}|_U)$ with respect to χ and ξ. This implies:

$$\int_U \psi(P\varphi)\, d\mu = \int_U \psi\big(\Delta_{\xi,\chi}(\varphi) + R_\chi(\varphi)\big)\, d\mu = \int_U (\Delta^*_{\xi,\chi}\psi)(\varphi)\, d\mu + \int_U (R^*_\chi\psi)(\varphi)\, d\mu\,.$$

Thus, we must have $P^*(\psi) = (\Delta^*_{\xi,\chi} + R^*_\chi)(\psi)$. R^*_χ is the formal adjoint of a linear differential operator of order at most 1 and thus is itself of order at most 1 by definition and proposition 7.1.8. Using lemma 7.2.5 we therefore find:

$$P^*(\psi)(m) = \Delta_{\xi,\chi^*}(\psi)(m) + R^*_\chi(\psi)(m) = g^{\mu\nu}\frac{\partial^2_{\chi^*}}{\partial x^\mu \partial x^\nu}(\psi)(m) + \mathrm{l.\,o.\,d.}\ (\psi)(m)\,.$$

Again, on general grounds, $(P^*\psi)(m)$ depends on ψ only locally for $\psi \in \Gamma(\mathcal{E}^*)$. Thus, we proved equation $(*)$. $\square$

7.2.6 Proposition. Let P and Q be first order linear differential operators on sections of a smooth vector bundle $\mathcal{E}$ on a Lorentzian manifold M such that PQ is normally hyperbolic. Then QP is normally hyperbolic, too.

Proof. We have to show that

$$(\sigma_{QP})^{ab} = g^{ab} \otimes \mathrm{Id}_{\mathcal{E}}\ , \tag{+}$$

where σ_{QP} denotes the principal symbol of QP.

Let $n = \dim(M)$ and let k be the rank of the vector bundle $\mathcal{E}$. Fix some arbitrary point $m \in M$ and an open neighbourhood $m \in U \subseteq M$ on which there are *normal* coordinates $x^1, \dots, x^n$ at m and a local tetraed field $(e_1, \dots, e_k)$ of $\mathcal{E}$. Notice that $g^{\mu\nu}(m)$ is diagonal as the coordinates are normal at m. We will now that (+) is valid at our point m. As m was arbitrary, this procedure can be repeated for every $m \in M$.

Represent the principal symbols of P, Q, PQ and QP locally as components with respect to the local coordinates $\{x^\mu\}$ and the local tetraed field (e_μ) on U. Using lemma 7.1.7, we have to show that

$$\forall \mu,\nu\colon (\sigma_{QP})^{\mu\nu}(m) = \frac{1}{2}\left[(\sigma_Q)^\mu \cdot (\sigma_P)^\nu + (\sigma_Q)^\nu \cdot (\sigma_P)^\mu\right](m) = g^{\mu\nu}(m)\,\mathbb{1}\,, \tag{++}$$

while we assume that PQ is normally hyperbolic, i. e.

$$\forall \mu,\nu\colon (\sigma_{PQ})^{\mu\nu}(m) = \frac{1}{2}\left[(\sigma_P)^\mu \cdot (\sigma_Q)^\nu + (\sigma_P)^\nu \cdot (\sigma_Q)^\mu\right](m) = g^{\mu\nu}(m)\,\mathbb{1}\,. \tag{$*$}$$

Notice that these are equations of $k \times k$-matrices and $\cdot$ denotes matrix multiplication. By construction, $g^{\mu\nu}(m)$ is diagonal; therefore equation $(*)$ for $\mu = \nu$ specialises to

$$\forall \mu\colon (\sigma_P)^\mu(m) \cdot (\sigma_Q)^\mu(m) = g^{\mu\mu}(m)\,\mathbb{1}\,,$$

which shows that $(\sigma_P)^\mu(m)$ and $(\sigma_Q)^\mu(m)$ are (up to the factor $g^{\mu\mu}(m)$) inverses of each other. Especially, they are elements of $\mathrm{GL}(k)$, and as left-inverses in groups are also right-inverses and right-inverses are also left-inverses and all inverses are unique, we can deduce:

$$\forall \mu \colon (\sigma_Q)^\mu(m) \cdot (\sigma_P)^\mu(m) = g^{\mu\mu}(m)\, \mathbb{1}\,. \tag{$**$}$$

For $\mu \neq \nu$, $(*)$ specialises to:

$$(\sigma_P)^\mu(m) \cdot (\sigma_Q)^\nu(m) + (\sigma_P)^\nu(m) \cdot (\sigma_Q)^\mu(m) = 0\,.$$

Multiplying by

$$[(\sigma_P)^\mu(m)]^{-1} = [g^{\mu\mu}(m)]^{-1}(\sigma_Q)^\mu(m)$$

from the left and by

$$[(\sigma_Q)^\mu(m)]^{-1} = [g^{\mu\mu}(m)]^{-1}(\sigma_P)^\mu(m)$$

from the right, one obtains for $\mu \neq \nu$:

$$(\sigma_Q)^\mu(m) \cdot (\sigma_P)^\nu(m) + (\sigma_Q)^\nu(m) \cdot (\sigma_P)^\mu(m) = 0\,. \tag{$***$}$$

Finally, combining $(**)$ and $(***)$ yields $(++)$. As the choice of m was arbitrary this implies $(+)$. □

7.3. Existence of Green's Operators

In this section we will prove existence and uniqueness of advanced and retarded Green's operators for a certain (quite big) class of first order linear differential operators including the Dirac operator on Dirac spinor fields and Buchdahl's operator on higher spin fields. In section 8.4 this result will play an important role for solving Cauchy's problem for Buchdahl's equations.

Following the terminology from [BGP, p. 72], we first define:

7.3.1 Definition (advanced and retarded Green's operators). Let (M, g) be a time-ordiented connected Lorentzian manifold. Let P be a differential operator acting on sections of a smooth vector bundle $\mathcal{E}$ over M. Linear maps $G_\pm \colon \Gamma_0(\mathcal{E}) \to \Gamma(\mathcal{E})$ are called **advanced $(+)$ resp. retarded $(-)$ Green's operators for P**, if

(i) $P \circ G_\pm = \mathrm{id}_{\Gamma_0(\mathcal{E})}$,

(ii) $G_\pm \circ P = \mathrm{id}_{\Gamma_0(\mathcal{E})}$,

(iii) $\forall \varphi \in \Gamma_0(\mathcal{E})\colon\ \mathrm{supp}(G_\pm\varphi) \subseteq J^M_\pm(\mathrm{supp}(\varphi))$.

In Dimock's paper on Dirac quantum fields on a manifold [Dim82], it is one of the central steps showing that unique advanced and retarded Green's operators (in Dimock's terminology: fundamental solutions) always exist for the Dirac operator acting on Dirac spinor fields on a globally hyperbolic space-time manifold (cf. [Dim82, theorem 2.1]).

The goal of this section is generalising Dimock's result to a wider class of first order differential operators utilising results on normally hyperbolic second order differential operators from [BGP]. These generalised results can then be applied to Buchdahl's equations to immediately yield advanced and retarded Green's operators for our concrete system of field equations, which contain the Dirac equation as a special case.

The central result of this section is:

7.3.2 Theorem. Let P and Q be first order linear differential operators on sections of a smooth vector bundle $\mathcal{E}$ on a time-oriented, globally hyperbolic Lorentzian manifold (M, g), such that PQ is normally hyperbolic. Then there exist unique advanced and retarded Green's operators $S_\pm \colon \Gamma_0(\mathcal{E}) \to \Gamma(\mathcal{E})$ for P. Moreover, $S_\pm$ are independent of the choice of Q.

Proof. We set $L := PQ$ and $L' := P^*Q^*$. L is normally hyperbolic by assumption, normal hyperbolicity of L' follows by $L' = P^*Q^* = (QP)^*$ from propositions 7.2.6 and 7.2.4. This enables us to apply [BGP, corollary 3.4.3], which says that there exist unique advanced and retarded Green's operators $G_\pm \colon \Gamma_0(\mathcal{E}) \to \Gamma(\mathcal{E})$ for L and $G'_\pm \colon \Gamma_0(\mathcal{E}^*) \to \Gamma(\mathcal{E}^*)$ for L'.

Define $S_\pm := QG_\pm$ and $S'_\pm := Q^*G'_\pm$. We will show that $S_\pm$ are advanced/retarded Green's operators for P be checking conditions (i)–(iii) from definition 7.3.1. As a byproduct we will find that $S'_\pm$ are advanced/retarded Green's operators for P^*:

(i) It is immediate that $PS_\pm = LG_\pm = \mathrm{id}_{\Gamma_0(\mathcal{E})}$ and $P^*S'_\pm = L'G'_\pm = \mathrm{id}_{\Gamma_0(\mathcal{E}^*)}$.

(iii) As $G_\pm$ are Green's operators, $\mathrm{supp}(G_\pm\varphi) \subseteq J^M_\pm(\mathrm{supp}\,\varphi)$ for all $\varphi \in \Gamma_0(\mathcal{E})$. Moreover, Q being a differential operator implies $\mathrm{supp}(QG_\pm\varphi) \subseteq \mathrm{supp}(G_\pm\varphi)$ on general grounds. Thus, $S_\pm = QG_\pm$ fulfill the support criterion from definition 7.3.1. The argument for $S'_\pm$ is the same.

(ii) To show $S_\pm P|_{\Gamma_0(\mathcal{E})} = \mathrm{id}_{\Gamma_0(\mathcal{E})}$ and $S'_\pm P^*|_{\Gamma_0(\mathcal{E}^*)} = \mathrm{id}_{\Gamma_0(\mathcal{E}^*)}$, consider for $\psi \in \Gamma_0(\mathcal{E}^*)$ and $f \in \Gamma_0(\mathcal{E})$:[4]

$$\langle S'_\mp\psi, f\rangle = \langle S'_\mp\psi, PS_\pm f\rangle \overset{(2)}{=} \langle P^*S'_\mp\psi, S_\pm f\rangle = \langle \psi, S_\pm f\rangle\,. \tag{$*$}$$

The equality (2) may be performed because $\mathrm{supp}(S'_\mp\psi) \cap \mathrm{supp}(S_\pm f)$ is a subset of a compact set, which is a consequence of global hyperbolicity: $\mathrm{supp}(S_\pm f) \subseteq J_\pm(\mathrm{supp}(f))$ and $\mathrm{supp}(S'_\pm f) \subseteq J_\pm(\mathrm{supp}(\psi))$, as we learned from (iii) above. As f and ψ are compactly supported, we can apply lemma 6.1.14-a to deduce that $J_\pm(\mathrm{supp}(f)) \cap J_\mp(\mathrm{supp}(\psi))$ is compact.

We found that $S_\pm = (S'_\mp)^*$. From $P^*S'_\mp = \mathrm{id}_{\Gamma_0(\mathcal{E}^*)}$ we get, by forming the adjoint, $\mathrm{id}_{\Gamma_0(\mathcal{E})} = (S'_\mp)^*P = S_\pm P$. In a totally analogous way one can deduce $\mathrm{id}_{\Gamma_0(\mathcal{E}^*)} = S'_\pm P^*$ ($PS_\mp = \mathrm{id}_{\Gamma_0(\mathcal{E})} \Rightarrow \mathrm{id}_{\Gamma_0(\mathcal{E}^*)} = S^*_\mp P^* = (S'^*_\pm)^*P^* = S'_\pm P^*$).

Summing up, we have so far proved that $S_\pm$ are advanced/retarded Green's operators for P. It remains to show uniqueness:

Let $T_\pm \neq S_\pm$ be a second pair of advanced and retarded Green's operators for P. This implies that $PT_\pm = \mathrm{id}_{\Gamma_0(\mathcal{E})}$ and therefore we find for every $\psi \in \Gamma_0(\mathcal{E}^*)$ and $f \in \Gamma_0(\mathcal{E})$:

$$\langle \psi, S_\pm f\rangle \overset{(*)}{=} \langle S'_\mp\psi, f\rangle = \langle S'_\mp\psi, PT_\pm f\rangle \overset{(3)}{=} \langle P^*S'_\mp\psi, T_\pm f\rangle = \langle \psi, T_\pm f\rangle\,,$$

where step (3) was performed by an analogous argument to step (2) in (ii) above. As ψ and f were arbitrary, this shows $T_\pm = S_\pm$. As the choice of $T_\pm$ did not depend on Q, this argument also shows that the construction of $S_\pm$ does not depend on Q. ☐

[4] We now set $\langle \psi, f\rangle := \int_M \psi(f)\,d\mu$.

7.3.3 Remark. Notice that indeed, [Dim82, theorem 2.1] is the special case of our theorem 7.3.2 for $P = -i\nabla\!\!\!/ + m$, $Q = i\nabla\!\!\!/ + m$: Let $\varphi^{\tilde{A}}$ be a Dirac spinor field.

$$\begin{aligned}
(-i\nabla\!\!\!/ + m)(i\nabla\!\!\!/ + m)\varphi &= (-i\gamma^{a\tilde{A}}{}_{\tilde{B}}\nabla_a + m)(i\gamma^{b\tilde{B}}{}_{\tilde{C}}\nabla_b + m)\,\varphi^{\tilde{C}} \\
&= \gamma^{a\tilde{A}}{}_{\tilde{B}}\gamma^{b\tilde{B}}{}_{\tilde{C}}\nabla_a\nabla_b\,\varphi^{\tilde{C}} + m^2\,\varphi \\
&= \frac{1}{2}\gamma^{a\tilde{A}}{}_{\tilde{B}}\gamma^{b\tilde{B}}{}_{\tilde{C}}\nabla_a\nabla_b\,\varphi^{\tilde{C}} \\
&\qquad\qquad + (\eta^{ab}\,\mathrm{Id}^{\tilde{A}}{}_{\tilde{C}} - \frac{1}{2}\gamma^{b\tilde{A}}{}_{\tilde{B}}\gamma^{a\tilde{B}}{}_{\tilde{C}})\nabla_a\nabla_b\,\varphi^{\tilde{C}} + m^2\,\varphi \\
&\stackrel{*}{=} \frac{1}{2}\gamma^{a\tilde{A}}{}_{\tilde{B}}\gamma^{b\tilde{B}}{}_{\tilde{C}}\nabla_a\nabla_b\,\varphi^{\tilde{C}} + \eta^{ab}\,\mathrm{Id}^{\tilde{A}}{}_{\tilde{C}}\,\nabla_a\nabla_b\,\varphi^{\tilde{C}} \\
&\qquad\qquad - \frac{1}{2}\gamma^{b\tilde{A}}{}_{\tilde{B}}\gamma^{a\tilde{B}}{}_{\tilde{C}}\nabla_b\nabla_a\,\varphi^{\tilde{C}} + \mathrm{l.\,o.\,t.} + m^2\,\varphi \\
&= \eta^{ab}\,\mathrm{Id}^{\tilde{A}}{}_{\tilde{C}}\,\nabla_a\nabla_b\,\varphi^{\tilde{C}} + \mathrm{l.\,o.\,t.}
\end{aligned}$$

where the "lower order terms" in the equality $*$ arise due to the non-commutativity of the covariant derivative on curved space-times. Hence, we see that $PQ = (-i\nabla\!\!\!/ + m)(i\nabla\!\!\!/ + m)$ is normally hyperbolic. Thus, the situation of [Dim82, theorem 2.1] fullfills the premises of our theorem 7.3.2.

Part III.

Buchdahl's Generalised Dirac Equations

8. Solving Buchdahl's Generalised Dirac Equations

At last, all preparations are done to tackle Buchdahl's [Buc82a] generalised Dirac equations for particles of higher spin. After giving a quick historical review of the development of higher spin wave equations and stating Buchdahl's equations as differential operators on suitable spinor bundles in section 8.1, and after proving a couple of technical lemmas in section 8.2, we will show existence of unique advanced and retarded Green's operators for Buchdahl's equations in section 8.3, basically just applying our central theorem 7.3.2 to the concrete situation. Finally, in section 8.4 we will solve the global Cauchy problem for Buchdahl's equations and thereby obtain an improved version of the results on existence of local solutions by [Wün85].

8.1. Statement of Buchdahl's Generalised Dirac Equations

In this section we will fix notation for dealing with Buchdahl's generalised Dirac equations. We start out fixing the physical setting for the rest of this thesis:

8.1.1 Premises (physical setting). Let (M, g) be a time-oriented and space-oriented, 4-dimensional, globally hyperbolic, Lorentzian manifold of signature $(+---)$. (This implies that (M, g) is oriented and connected, satisfies the strong causality condition and has a spin structure). Let both space and time orientations be chosen.

Let, moreover, $\mathcal{F}_c(M)$ denote the connected frame bundle on (M, g), which equals $\mathcal{SO}^+(M)$ and has structure group $\mathrm{SO}^+(1,3) = \mathcal{L}_+^\uparrow$. Let $\Lambda\colon \mathcal{S}(M) \to \mathcal{F}_c(M)$ denote a spin structure on (M, g) (then $\mathcal{S}(M)$ is an $\mathrm{SL}(2,\mathbb{C})$-principal bundle).

The spinor bundles $D^{(j,j')}M$ and $D^D M$ are declared as in definition 6.4.3; ${\sigma_a}^{A\dot{X}}$, ${\gamma_a}^{\tilde{A}}{}_{\tilde{B}}$ and ϵ^{AB} as in definition 6.4.6. ∇ denotes the Levi-Civita covariant derivative on TM and the induced covariant derivatives on all spinor bundles. For all this keep in mind remark 6.4.7-b and -c.

Finally, notice that we shall often be using the abbreviating notations $x^{A\dot{X}} := x^a {\sigma_a}^{A\dot{X}}$ for $x \in \Gamma(TM)$ and $\nabla_{A\dot{X}} := \sigma^a{}_{A\dot{X}} \nabla_a$. Keep in mind remark 4.2.4.

8.1.2 Definition (bundles of Buchdahl spinors). We assume premises 8.1.1

(i) For $s \in \mathbb{N}$, define the (auxiliary) $\mathrm{SL}(2,\mathbb{C})$-representations

$$\begin{aligned}
D^{+,s} &:= D^{(\frac{1}{2},0)} \otimes D^{(\frac{s-1}{2},0)} && \text{on} \quad \Delta_{+,s} := \Delta_{(\frac{1}{2},0)} \otimes \Delta_{(\frac{s-1}{2},0)} \\
D^{s} &:= D^{(\frac{s}{2},0)} && \text{on} \quad \Delta_{s} := \Delta_{(\frac{s}{2},0)} \\
D^{-,s} &:= D^{(0,\frac{1}{2})*} \otimes D^{(\frac{s-1}{2},0)} && \text{on} \quad \Delta_{-,s} := \Delta^*_{(0,\frac{1}{2})} \otimes \Delta_{(\frac{s-1}{2},0)}\,,
\end{aligned}$$

and the associated spinor bundles

$$D^{+,s}M := \mathcal{S}(M) \times_{D^{+,s}} \Delta_{+,s}$$
$$D^{s}M := \mathcal{S}(M) \times_{D^{s}} \Delta_{s}$$
$$D^{-,s}M := \mathcal{S}(M) \times_{D^{-,s}} \Delta_{-,s}\,.$$

Sections of these bundles may be written using 2-spinor index notation. E. g.

$$\psi^{AA_1\dots A_{s-1}} = \psi^{A(A_1\dots A_{s-1})} \in \Gamma(D^{+,s}M)$$
$$\psi_{\dot{X}}{}^{A_1\dots A_{s-1}} = \psi_{\dot{X}}{}^{(A_1\dots A_{s-1})} \in \Gamma(D^{-,s}M)\,.$$

(ii) For $s \in \mathbb{N}$ we call the $\mathrm{SL}(2,\mathbb{C})$-representation

$$D^B_s := D^s \oplus D^{-,s} \quad \text{on} \quad \Delta^B_s := \Delta_s \oplus \Delta_{-,s}$$

the **Buchdahl spinor representation of spin $\frac{s}{2}$.** The associated bundle

$$D^B_s M := D^s M \oplus D^{-,s}M$$

is called the **Buchdahl spinor bundle for spin $\frac{s}{2}$.**
Notice that

$$\Delta_D \otimes \Delta_{s-1} = (\Delta_{\frac{1}{2},0} \oplus \Delta^*_{0,\frac{1}{2}}) \otimes \Delta_{s-1} \cong \Delta_{+,s} \oplus \Delta_{-,s} \supseteq \Delta_s \oplus \Delta_{-,s}$$

and thus, $\Delta^B_s \subseteq \Delta_D \otimes \Delta_{s-1}$. This is why we will write sections of $D^B_s M$ using a hybride Dirac-/2-spinor notation, e. g.:

$$\Psi^{\tilde{A}A_1\dots A_{s-1}}\,.$$

This may be broken down to the 2-spinor level like:

$$\Psi = \Psi^{\tilde{A}A_1\dots A_{s-1}} = \begin{pmatrix} (\psi_1)^{AA_1\dots A_{s-1}} \\ (\psi_2)_{\dot{X}}{}^{A_1\dots A_{s-1}} \end{pmatrix} = \begin{pmatrix} (\psi_1)^{A(A_1\dots A_{s-1})} \\ (\psi_2)_{\dot{X}}{}^{(A_1\dots A_{s-1})} \end{pmatrix}\,.$$

As a generalisation of his spin $\frac{1}{2}$ equations, Dirac [Dir36] proposed the following system of first order linear differential equations for particles of higher spin:

$$\begin{cases} \partial^A{}_{\dot{X}_0}\psi_{AA_1\dots A_k\dot{X}_1\dots\dot{X}_l} + \mu\,\varphi_{A_1\dots A_k\dot{X}_0\dots\dot{X}_l} = 0 \\ \partial^{\dot{X}}{}_{A_0}\varphi_{A_1\dots A_k\dot{X}\dot{X}_1\dots\dot{X}_l} - \nu\,\psi_{A_0\dots A_k\dot{X}_1\dots\dot{X}_l} = 0\,, \end{cases} \tag{D}$$

for $\psi \in \Gamma(D^{(\frac{k+1}{2},\frac{l}{2})}M)$ and $\varphi \in \Gamma(D^{(\frac{k}{2},\frac{l+1}{2})}M)$, where $(M,g) = (\mathbb{R}^4,\eta)$ is flat Minkowski spacetime and $\mu,\nu \in \mathbb{C}$ are constants. Fierz and Pauli [FP39] investigated the minimal coupling of these equations to an electromagnetic field but found the minimally coupled system to be inconsistent for spin $\frac{k+l}{2} > 1$, where "inconsistent" means that only under (unacceptably) strong assumptions the system of differential equations is integrable.

The minimal coupling to a gravitational field modelled by a curved general-relativistic spactime manifold (M,g) is achieved by substituting the Levi-Civita covariant derivative for partial derivatives:

$$\begin{cases} \nabla^A{}_{\dot{X}_0}\psi_{AA_1\dots A_k\dot{X}_1\dots\dot{X}_l} + \mu\,\varphi_{A_1\dots A_k\dot{X}_0\dots\dot{X}_l} = 0 \\ \nabla^{\dot{X}}{}_{A_0}\varphi_{A_1\dots A_k\dot{X}\dot{X}_1\dots\dot{X}_l} - \nu\,\psi_{A_0\dots A_k\dot{X}_1\dots\dot{X}_l} = 0\,. \end{cases} \tag{PF}$$

This system was studied by Buchdahl [Buc62] and again found to be inconsistent for spin $k+l>1$; for $k \geq 2$ and $l=0$ it is consistent only if (M,g) is conformally flat. As the next important step of the development, Buchdahl [Buc82a] suggested the following modified equations to describe a particle of spin $\frac{s}{2}$:

$$\begin{cases} \nabla_{\dot{X}A}\psi^{AA_1\ldots A_{s-1}} - \mu\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}} = 0 \\ \nabla^{A\dot{X}}\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}} - \frac{(s-1)(s-2)}{\mu s}\epsilon^{A(A_1}\Psi^{|PQD|A_2}\varphi_{PQD}{}^{A_3\ldots A_{s-1})} - \nu\psi^{AA_1\ldots A_{s-1}} = 0\,, \end{cases} \tag{B'}$$

for $(\psi,\varphi)^{tr} \in \Gamma(D^B_s M)$ (cf. footnote[1]). Here, Ψ_{ABCD} is the Weyl spinor, [Wal84, pp. 370], and $\mu,\nu \in \mathbb{C}$ are constants[2]. Notice that (B') and (PF) are equivalent for $l=0$ and $k=0,1$ or $\Psi_{ABCD}=0$.

The striking feature of (B') is that it is consistent for all $s \in \mathbb{N}$, if $\mu \neq 0$ (ν may still be $\in \mathbb{C}$). Massiveness of the field ($\mu \neq 0$) is an essential premise to this, which is why in this thesis, we will be considering the massive case only. (For an overview of the development of the massless case, cf. [IS99, section 7].)

At first sight, the extra term in (B') seems to be added "by hand". Buchdahl [Buc82a] himself considered this a violation of the principle of minimal coupling and therefore proposed its abandonment. However—and this is the next step of the historical development—Wünsch [Wün85] found that (B') can equivalently and much simpler be written as

$$\begin{cases} \nabla_{\dot{X}A}\psi^{AA_1\ldots A_{s-1}} - \mu\,\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}} = 0 \\ \nabla^{(A|\dot{X}}\varphi_{\dot{X}}{}^{|A_1\ldots A_{s-1})} - \nu\,\psi^{AA_1\ldots A_{s-1}} = 0\,. \end{cases} \tag{BW'}$$

The symmetrisation just takes care that "we stay in the same irreducible $\mathrm{SL}(2,\mathbb{C})$-representation": If $(\psi,\varphi)^{tr} \in \Gamma(D^B_s M)$, which implies that $\varphi \in \Gamma(D^{-,s}M)$ (thus, φ is a section of a spinor bundle for an *irreducible* $\mathrm{SL}(2,\mathbb{C})$-representation, hence it is symmetric in its undotted indices), then why in general would $\nabla^{A\dot{X}}\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}}$ still be symmetric in its undotted indices? So, it seems rather obvious that $\nabla^{A\dot{X}}\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}} - \nu\,\psi^{AA_1\ldots A_{s-1}} = 0$ will rarely be fulfilled for $(\psi,\varphi)^{tr} \in \Gamma(D^B_s M)$. But after adding the symmetrisation, which means: after projecting $\nabla^{A\dot{X}}\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}}$ back onto $\Gamma(D^{-,s}M)$, this seems much more likely.

After this brief historical review we shall define some auxiliary differential operators which will allow to state equations (B') and (BW') more elegantly.

8.1.3 Definition (auxiliary differential operators). Assume premises 8.1.1. For $s \in \mathbb{N}$, we define the following differential operatos:

(i) $M_s\colon \Gamma(D^{+,s}M) \to \Gamma(D^{-,s}M)$,

$$(M_s\psi)_{\dot{X}}{}^{A_1\ldots A_{s-1}} := \nabla_{\dot{X}A}\psi^{AA_1\ldots A_{s-1}}\,,$$

[1] Buchdahl and Wünsch usually denoted all indices of ψ and φ as *subscript* indices, thus they actually considered co-spinor fields (ψ,φ), $\psi \in \Gamma(D^{(\frac{s}{2},0)*}M)$, $\varphi \in \Gamma(D^{(0,\frac{1}{2})*}M \otimes D^{(\frac{s-1}{2},0)*}M)$. However, in this thesis we shall adopt the hybride Dirac- and 2-spinor framework as outlined in definition 8.1.2 because it seems to offer more conceptual insight. Therefore, we translated most of the equations quoted from papers by Buchdahl and Wünsch into our notation. E. g. the $-\mu$ (instead of $+\mu$) arising in the first equation of (B') is one consequence of this.

[2] Wünsch [Wün85] uses $\mu = \nu = i\frac{c\,m_0}{\sqrt{2}\,\hbar}$, with c the speed of light, m_0 the rest mass.

(ii) $N_s\colon \Gamma(D^{-,s}M) \to \Gamma(D^{+,s}M)$,

$$(N_s\psi)^{AA_1\ldots A_{s-1}} := \nabla^{A\dot{X}}\psi_{\dot{X}}{}^{A_1\ldots A_{s-1}}\,,$$

(iii) $\overset{\vee}{N}_s\colon \Gamma(D^{-,s}M) \to \Gamma(D^{s}M)$,

$$(\overset{\vee}{N}_s\psi)^{AA_1\ldots A_{s-1}} := (N_s\psi)^{(AA_1\ldots A_{s-1})}\,,$$

(iv) $P_s\colon \Gamma(D^{+,s}M) \to \Gamma(D^{+,s}M)$,

$$(P_s\varphi)^{AA_1\ldots A_{s-1}} := \frac{(s-1)(s-2)}{s}\epsilon^{A(A_1}\Psi^{|PQD|A_2}\varphi_{PQD}{}^{A_3\ldots A_{s-1})}$$

Here, Ψ is the Weyl spinor, cf. [Wal84, pp. 370]. Notice that $P_1 = 0$ and $P_2 = 0$.

8.1.4 Definition (Buchdahl's equations). We assume premises 8.1.1.

(a) For $s \in \mathbb{N}$, **Buchdahl's generalised Dirac equation for spin $\frac{s}{2}$** is the system of differential equations

$$\begin{cases} M_s\,\psi - \mu\varphi = 0 \\ N_s\,\varphi - \frac{1}{\mu}P_s\,\psi - \nu\psi = 0\,, \end{cases} \tag{B}$$

where $(\psi,\varphi)^{tr} \in \Gamma(D^B_sM)$, $\mu,\nu \in \mathbb{C}$, $\mu \neq 0$.
Defining **Buchdahl's operator** $B_s\colon \Gamma(D^B_sM) \to \Gamma(D^B_sM)$,

$$B_s := \begin{pmatrix} -\nu - \frac{1}{\mu}P_s & N_s \\ M_s & -\mu \end{pmatrix},$$

we can write (B) for $\Psi \in \Gamma(D^B_sM)$ simply as

$$B_s(\Psi) = 0\,.$$

(b) For $s \in \mathbb{N}$, **Buchdahl's generalised Dirac equations for spin $\frac{s}{2}$ in the form as given by Wünsch** [Wün85] is the system of differential equations

$$\begin{cases} M_s\,\psi - \mu\varphi = 0 \\ \overset{\vee}{N}_s\,\varphi - \nu\psi = 0\,, \end{cases} \tag{BW}$$

where $(\psi,\varphi)^{tr} \in \Gamma(D^B_sM)$, $\mu,\nu \in \mathbb{C}$, $\mu \neq 0$.
Define the operators $B^W_s\colon \Gamma(D^B_sM) \to \Gamma(D^B_sM)$,

$$B^W_s := \begin{pmatrix} -\nu & \overset{\vee}{N}_s \\ M_s & -\mu \end{pmatrix},$$

which we call **Buchdahl's operators in the form as given by Wünsch.** Using them we can write (BW) for $\Psi \in \Gamma(D^B_sM)$ simply as

$$B^W_s(\Psi) = 0\,.$$

8.2. Lemmas for Buchdahl's Equations in 2-Spinor Notation

This section is, in essence, devoted to showing the equivalence of equations (B), i. e. Buchdahl's equations, and (BW), i. e. Buchdahl's equations as given by Wünsch (cf. corollary 8.2.5). This is a technical result needed in the sequel, and we shall give a far more detailed version of the proof presented in [Wün85], using the theoretic framework developed so far.

For the whole section we assume premises 8.1.1. The central result of this section is:

8.2.1 Proposition. Assume Buchdahl's equations as given by Wünsch, (BW), hold for $(\psi, \varphi)^{tr} \in \Gamma(D^B_s M)$, $s \in \mathbb{N}$. Then we find:

$$\overset{\vee}{N}_s\, \varphi = N_s\, \varphi - \frac{1}{\mu} P_s\, \psi\,.$$

8.2.2 Remark. Notice that this result is trivial for $s = 1$ but indeed surprising for $s = 2$: Remember that $P_2 = 0$. Thus, the assertion is: Whenever $(\psi, \varphi)^{tr}$ solves (B) or (BW) for $s = 2$, $N_2\, \varphi = \nabla^{A\dot{X}} \varphi_{\dot{X}}{}^{A_1}$ is already symmetric. Notice that this is not a priori the case for a general $\varphi \in \Gamma(D^{\frac{1}{2},\frac{1}{2}} M)$.

We first prove an auxiliary lemma:

8.2.3 Lemma.

$$(N_s \varphi)^{AA_1 \dots A_{s-1}} = (\overset{\vee}{N}_s \varphi)^{AA_1 \dots A_{s-1}} - \frac{s-1}{s}\, \epsilon^{A(A_1} \nabla^{|U\dot{V}} \varphi_{\dot{V}U}{}^{|A_2 \dots A_{s-1})}$$

for all $\varphi \in \Gamma(D^{-,s} M)$ and $s \geq 1$.

Proof. The assertion is equivalent to [Wün85, eq. 21], but which is left unproven in the reference. So we shall prove the equation from scratch here.

Notice that for $s = 1$ the assertion is trivial. We assume $s \geq 2$ henceforth. Starting with the right hand side of the assertion, we find:

$$\begin{aligned}
&\nabla^{(A\dot{X}} \varphi_{\dot{X}}{}^{A_1 \dots A_{s-1})} - \frac{s-1}{s} \epsilon^{A(A_1} \nabla^{|U\dot{V}} \varphi_{\dot{V}U}{}^{|A_2 \dots A_{s-1})} \\
&\quad = \frac{1}{(s-1)!} \sum_{\pi \in S_{s-1}} \Big[\frac{1}{s}\big(\nabla^{A\dot{X}} \varphi_{\dot{X}}{}^{A_{\pi(1)} \dots A_{\pi(s-1)}} + \nabla^{A_{\pi(1)}\dot{X}} \varphi_{\dot{X}}{}^{AA_{\pi(2)} \dots A_{\pi(s-1)}} + \dots \\
&\qquad + \nabla^{A_{\pi(1)}\dot{X}} \varphi_{\dot{X}}{}^{A_{\pi(2)} \dots A_{\pi(s-1)}A}\big) - \frac{s-1}{s} \epsilon^{AA_{\pi(1)}} \nabla^{U\dot{V}} \varphi_{\dot{V}U}{}^{A_{\pi(2)} \dots A_{\pi(s-1)}}\Big]
\end{aligned}$$

We notice that $\varphi_{\dot{X}}{}^{A_1 \dots A_{s-1}}$ is symmetric in the indices $A_1 \dots A_{s-1}$ and that in consequence, $\nabla^{A\dot{X}}\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}}$ is also symmetric in $A_1\ldots A_{s-1}$. Hence, we continue:

$$
\begin{aligned}
&= \frac{(s-1)!}{s!}\nabla^{A\dot{X}}\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}} + \frac{1}{s!}\sum_{\pi\in S_{s-1}}\big[(s-1)\nabla^{A_{\pi(1)}\dot{X}}\varphi_{\dot{X}}{}^{A_{\pi(2)}\ldots A_{\pi(s-1)}A} \\
&\quad - (s-1)\epsilon^{AA_{\pi(1)}}\nabla^{U\dot{V}}\varphi_{\dot{V}U}{}^{A_{\pi(2)}\ldots A_{\pi(s-1)}}\big] \\
&= \frac{1}{s}\nabla^{A\dot{X}}\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}} + \frac{s-1}{s!}\sum_{\pi\in S_{s-1}}\big[\nabla^{A_{\pi(1)}\dot{X}}\varphi_{\dot{X}}{}^{AA_{\pi(2)}\ldots A_{\pi(s-1)}} \\
&\quad + \epsilon^{AA_{\pi(1)}}\nabla_U{}^{\dot{X}}\varphi_{\dot{X}}{}^{UA_{\pi(2)}\ldots A_{\pi(s-1)}}\big]
\end{aligned}
$$

As a trivial consequence of remark 4.1.6-(iii) we obtain:

$$
\begin{aligned}
&= \frac{1}{s}\nabla^{A\dot{X}}\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}} + \frac{s-1}{s!}\sum_{\pi\in S_{s-1}}\nabla^{A\dot{X}}\varphi_{\dot{X}}{}^{A_{\pi(1)}\ldots A_{\pi(s-1)}} \\
&= \frac{1}{s}\nabla^{A\dot{X}}\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}} + \frac{s-1}{s}\nabla^{A\dot{X}}\varphi_{\dot{X}}{}^{(A_1\ldots A_{s-1})} \\
&= \nabla^{A\dot{X}}\varphi_{\dot{X}}{}^{A_1\ldots A_n}
\end{aligned}
$$

□

We just quote here:

8.2.4 Lemma (spinorial Ricci identities). For $\varphi\in\Gamma\big((D^{(\frac{1}{2},0)}M)^{\otimes s}\big)$ and $s\in\mathbb{N}$:

$$
\nabla_{\dot{X}(A}\nabla_{B)}{}^{\dot{X}}\varphi_{C_1\ldots C_s} = \sum_{i=1}^{s}\big[\Psi_{ABC_i}{}^{D}\varphi_{C_1\ldots D\ldots C_s} - 2\Lambda\epsilon_{C_i(A|}\varphi_{C_1\ldots|B)\ldots C_s}\big]
$$

$$
\nabla_{A(\dot{X}}\nabla_{\dot{Y})}{}^{A}\varphi_{C_1\ldots C_s} = \sum_{i=1}^{s}\Phi_{\dot{X}\dot{Y}C_i}{}^{D}\varphi_{C_1\ldots D\ldots C_s}\,,
$$

where here, $\Psi_{ABCD}=\Psi_{(ABCD)}$ (Weyl spinor), $\Phi_{\dot{X}\dot{Y}CD}=\Phi_{(\dot{X}\dot{Y})(CD)}$ and Λ are certain spinors derived from the spinor equivalent of the Riemann tensor as demonstrated in [Wal84, pp. 370].

Proof. For a comprehensive treatment cf. [Wal84, pp. 370]. The first formula is derived there explicitly; for the second formula apply the described procedure for the derivation of (13.2.28) to (13.2.13) instead of (13.2.11). □

Proof of proposition 8.2.1. Notice that for $s=1$ the assertion is trivial. Thus, we set $s\geq 2$. Let $(\psi,\varphi)^{tr}\in\Gamma(D_s^B M)$. We have to show:

$$
(\overset{\vee}{N}_s\,\varphi)^{AA_1\ldots A_{s-1}} = (N_s\,\varphi)^{AA_1\ldots A_{s-1}} - \frac{(s-1)(s-2)}{\mu s}\,\epsilon^{A(A_1}\Psi^{|EFG|A_2}\psi_{EFG}{}^{A_3\ldots A_{s-1})}\,.
$$

By assumption, equations (BW) hold. This yields:

$$
\varphi = \frac{1}{\mu}M_s\,\psi\,.
$$

Substituting this into 8.2.3 yields:

$$\begin{aligned}(\overset{\vee}{N}_s\,\varphi)^{AA_1\dots A_{s-1}} &= (N_s\,\varphi)^{AA_1\dots A_{s-1}} - \frac{s-1}{\mu s}\,\epsilon^{A(A_1|}\nabla^{U\dot V}(M_s\,\psi)_{\dot V U}{}^{|A_2\dots A_{s-1})}\\ &= (N_s\,\varphi)^{AA_1\dots A_{s-1}} + \frac{s-1}{\mu s}\,\epsilon^{A(A_1|}\nabla_U{}^{\dot V}\nabla_{\dot V B}\psi^{BU|A_2\dots A_{s-1})}\\ &= (N_s\,\varphi)^{AA_1\dots A_{s-1}} - \frac{s-1}{\mu s}\,\epsilon^{A(A_1|}\nabla_{U\dot V}\nabla^{\dot V}{}_B\psi^{BU|A_2\dots A_{s-1})}\,.\end{aligned}$$

Now we notice that ψ is symmetric in all indices[3]. Hence, we can just add the following symmetrisation:

$$= (N_s\,\varphi)^{AA_1\dots A_{s-1}} - \frac{s-1}{\mu s}\epsilon^{A(A_1|}\nabla_{(U|\dot V}\nabla^{\dot V}{}_{|B)}\psi^{BU|A_2\dots A_{s-1})}\,. \qquad (*)$$

We now apply the spinorial Ricci identities (lemma 8.2.4) on the last part:

$$\begin{aligned}&\epsilon^{A(A_1|}\nabla_{(U|\dot V}\nabla^{\dot V}{}_{|B)}\psi^{BU|A_2\dots A_{s-1})}\\ &\quad= \epsilon^{A(A_1|}\epsilon^{BC}\epsilon^{UD}\epsilon^{|A_2|B_2}\cdots\epsilon^{|A_{s-1})B_{s-1}}\nabla_{(U|\dot V}\nabla^{\dot V}{}_{|B)}\psi_{CDB_2\dots B_{s-1}}\\ &\quad= \epsilon^{A(A_1|}\epsilon^{BC}\epsilon^{UD}\epsilon^{|A_2|B_2}\cdots\epsilon^{|A_{s-1})B_{s-1}}\\ &\qquad\cdot\big[\Psi_{UBC}{}^P\psi_{PDB_2\dots B_{s-1}} - 2\Lambda\epsilon_{C(U}\psi_{B)DB_2\dots B_{s-1}}\\ &\qquad\;\; + \Psi_{UBD}{}^P\psi_{CPB_2\dots B_{s-1}} - 2\Lambda\epsilon_{D(U|}\psi_{C|B)B_2\dots B_{s-1}} \qquad\qquad (**)\\ &\qquad\;\; + \underbrace{\sum_{i=2}^{s-1}\big(\Psi_{UBB_i}{}^P\psi_{CDB_2\dots P\dots B_{s-1}} - 2\Lambda\epsilon_{B_i(U|}\psi_{CDB_2\dots|B)\dots B_{s-1}}\big)}_{\text{this term is non-existent if } s=2}\big]\,.\end{aligned}$$

Let's look at the summands separately. Using the symmetries of Ψ and ψ, we get:

$$\epsilon^{BC}\Psi_{UBC}{}^P\psi_{PDB_2\dots B_{s-1}} = 0$$

$$\begin{aligned}\epsilon^{BC}\epsilon^{UD}2\Lambda\epsilon_{C(U}\psi_{B)DB_2\dots B_{s-1}} &= \epsilon^{BC}\epsilon^{UD}(\Lambda\epsilon_{CU}\psi_{BDB_2\dots B_{s-1}} + \Lambda\epsilon_{CB}\psi_{UDB_2\dots B_{s-1}})\\ &= -\Lambda\epsilon^{BD}\psi_{BDB_2\dots B_{s-1}})\\ &= 0\end{aligned}$$

$$\epsilon^{UD}\Psi_{UBD}{}^P\psi_{CPB_2\dots B_{s-1}} = 0$$

$$\begin{aligned}\epsilon^{BC}\epsilon^{UD}2\Lambda\epsilon_{D(U|}\psi_{C|B)B_2\dots B_{s-1}} &= \epsilon^{BC}\epsilon^{UD}(\Lambda\epsilon_{DU}\psi_{CBB_2\dots B_{s-1}} + \Lambda\epsilon_{DB}\psi_{CUB_2\dots B_{s-1}})\\ &= -\epsilon^{UC}\Lambda\epsilon_{DB}\psi_{CUB_2\dots B_{s-1}}\\ &= 0\end{aligned}$$

$$\begin{aligned}&\epsilon^{A(A_1|}\epsilon^{BC}\epsilon^{UD}\epsilon^{|A_2|B_2}\cdots\epsilon^{|A_{s-1})B_{s-1}}\sum_{i=2}^{s-1}\Psi_{UBB_i}{}^P\psi_{CDB_2\dots P\dots B_{s-1}}\\ &\qquad= -\epsilon^{A(A_1}\sum_{i=2}^{s-1}\Psi_{UB}{}^{A_i|}{}_P\psi^{BU|A_2\dots|P|\dots A_{s-1})}\\ &\qquad= -\epsilon^{A(A_1}\sum_{i=2}^{s-1}\Psi_{UB}{}^{A_i}{}_P\psi^{|BUP|A_2\dots\hat A_i\dots A_{s-1})}\end{aligned}$$

[3]This step wouldn't work if ψ was in $\Gamma(D^{+,s}M)$ but not in $\Gamma(D^sM)$.

$$
\begin{aligned}
&= -\epsilon^{A(A_1}(s-2)\Psi_{UB}{}^{A_2}{}_{P}\psi^{|BUP|A_3\dots A_{s-1})}\\
&= (s-2)\epsilon^{A(A_1}\Psi^{|UBP|A_2}\psi_{BUP}{}^{A_3\dots A_{s-1})}
\end{aligned}
$$

$$
\begin{aligned}
&\epsilon^{BC}\epsilon^{UD}2\Lambda\epsilon_{B_i(U|}\psi_{CDB_2\dots|B)\dots B_{s-1}}\\
&\quad= \epsilon^{BC}\epsilon^{UD}(\Lambda\epsilon_{B_iU}\psi_{CDB_2\dots B\dots B_{s-1}} + \Lambda\epsilon_{B_iB}\psi_{CDB_2\dots U\dots B_{s-1}})\\
&\quad= 0
\end{aligned}
$$

Hence, we can rewrite $(**)$ as

$$
\epsilon^{A(A_1|}\nabla_{(U|\dot V}\nabla^{\dot V}{}_{|B)}\psi^{BU|A_2\dots A_{s-1})} = (s-2)\epsilon^{A(A_1}\Psi^{|UBP|A_2}\psi_{BUP}{}^{A_3\dots A_{s-1})},
$$

and finally get, by resubstitution into $(*)$:

$$
(\overset{\vee}{N}_s\,\varphi)^{AA_1\dots A_{s-1}} = (N_s\,\varphi)^{AA_1\dots A_{s-1}} - \frac{(s-2)(s-1)}{\mu s}\epsilon^{A(A_1}\Psi^{|UBP|A_2}\psi_{BUP}{}^{A_3\dots A_{s-1})}.
$$

□

8.2.5 Corollary (equivalence of (B) and (BW)). Using proposition 8.2.1, Buchdahl's equations (B) and Buchdahl's equations as given by Wünsch (BW) are equivalent.

8.3. Normal Hyperbolicity and Existence of Green's Operators

Now we will show existence and uniqueness of advanced and retarded Green's operators for Buchdahl's operator B_s using our preparations from section 7.3. Notably, we want to apply theorem 7.3.2, and in order to fullfill its premises, for every $s \in \mathbb{N}$ we need to find a second first order linear differential operator B_s' on $\Gamma(D_s^B M)$, such that $B_s B_s'$ is normally hyperbolic.

8.3.1 Lemma. Setting

$$
B_s' := 2\begin{pmatrix} -\nu + \frac{1}{\mu}P_s & N_s \\ M_s & -\mu \end{pmatrix}
$$

on $\Gamma(D_s^B M)$, $s \in \mathbb{N}$, $B_s B_s'$ is normally hyperbolic.

Proof. We calculate:

$$
\begin{aligned}
B_s B_s' &= \begin{pmatrix} -\nu - \frac{1}{\mu}P_s & N_s \\ M_s & -\mu \end{pmatrix} \cdot 2\begin{pmatrix} -\nu + \frac{1}{\mu}P_s & N_s \\ M_s & -\mu \end{pmatrix}\\
&= 2\begin{pmatrix} \nu^2 - \frac{1}{\mu^2}P_s^2 + N_sM_s & -(\nu+\mu)N_s - \frac{1}{\mu}P_sN_s \\ -(\nu+\mu)M_s + \frac{1}{\mu}M_sP_s & M_sN_s + \mu^2 \end{pmatrix}.
\end{aligned}
$$

This is a linear differential operator of order 2. In order to prove normal hyperbolicity, we have to show that its second order term (i. e. its principal part) is of metric type (cf. definition 7.2.2). We may decompose $B_s B'_s$ into

$$B_s B'_s = \begin{pmatrix} 2N_s M_s & 0 \\ 0 & 2M_s N_s \end{pmatrix} + \text{terms of order} \leq 1\,.$$

Therefore it suffices proving normal hyperbolicity of $2N_s M_s \colon \Gamma(D^{+,s}M) \to \Gamma(D^{+,s}M)$ and $2M_s N_s \colon \Gamma(D^{-,s}M) \to \Gamma(D^{-,s}M)$. For this purpose, pick $\psi \in \Gamma(D^{(\frac{s}{2},0)}M)$ and calculate:

$$\begin{aligned}
(2N_s M_s \psi)^{AA_1\ldots A_{s-1}} &= 2\,\nabla^{A\dot{X}}\nabla_{\dot{X}B}\psi^{BA_1\ldots A_{s-1}} \\
&= 2\,\epsilon^{AD}\epsilon^{BC}\nabla_D{}^{\dot{X}}\nabla_{\dot{X}B}\psi_C{}^{A_1\ldots A_{s-1}} \\
&= 2\,\epsilon^{AD}\epsilon^{BC}\nabla_{[D|}{}^{\dot{X}}\nabla_{\dot{X}|B]}\psi_C{}^{A_1\ldots A_{s-1}} + 2\,\epsilon^{AD}\epsilon^{BC}\nabla_{(D|}{}^{\dot{X}}\nabla_{\dot{X}|B)}\psi_C{}^{A_1\ldots A_{s-1}} \\
&\overset{*}{=} -\epsilon^{AD}\epsilon^{BC}\epsilon_{DB}\nabla^{E\dot{X}}\nabla_{\dot{X}E}\psi_C{}^{A_1\ldots A_{s-1}} + 2\,\epsilon^{AD}\epsilon^{BC}\nabla_{(D|}{}^{\dot{X}}\nabla_{\dot{X}|B)}\psi_C{}^{A_1\ldots A_{s-1}} \\
&\overset{**}{=} \nabla^a\nabla_a\psi^{AA_1\ldots A_{s-1}} + \text{terms of order zero}
\end{aligned}$$

where for $*$ we used 4.1.6-iii and for $**$ we used the spinorial Ricci identities (lemma 8.2.4). Hence, the principal part of $2N_s M_s$ is $g^{ab}\nabla_a\nabla_b$, which shows that $2N_s M_s$ is normally hyperbolic. Next, pick $\varphi \in \Gamma(D^{-,s}M)$ and calculate completely analogously:

$$\begin{aligned}
(2M_s N_s \varphi)_{\dot{X}}{}^{A_1\ldots A_{s-1}} &= 2\,\nabla_{\dot{X}B}\nabla^{B\dot{Y}}\varphi_{\dot{Y}}{}^{A_1\ldots A_{s-1}} \\
&= 2\,\epsilon^{\dot{Y}\dot{Z}}\nabla_{\dot{X}B}\nabla^B{}_{\dot{Z}}\varphi_{\dot{Y}}{}^{A_1\ldots A_{s-1}} \\
&= 2\,\epsilon^{\dot{Y}\dot{Z}}\nabla_{[\dot{X}|B}\nabla^B{}_{|\dot{Z}]}\varphi_{\dot{Y}}{}^{A_1\ldots A_{s-1}} + 2\,\epsilon^{\dot{Y}\dot{Z}}\nabla_{(\dot{X}|B}\nabla^B{}_{|\dot{Z})}\varphi_{\dot{Y}}{}^{A_1\ldots A_{s-1}} \\
&= \epsilon^{\dot{Y}\dot{Z}}\epsilon_{\dot{X}\dot{Z}}\nabla_{\dot{W}B}\nabla^{B\dot{W}}\varphi_{\dot{Y}}{}^{A_1\ldots A_{s-1}} + 2\,\epsilon^{\dot{Y}\dot{Z}}\nabla_{(\dot{X}|B}\nabla^B{}_{|\dot{Z})}\varphi_{\dot{Y}}{}^{A_1\ldots A_{s-1}} \\
&= \nabla_a\nabla^a\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}} + \text{terms of order zero}
\end{aligned}$$

Hence, the principal part of $2M_s N_s$ is $g^{ab}\nabla_a\nabla_b$, which shows that $2M_s N_s$ is normally hyperbolic. □

8.3.2 Corollary (advanced and retarded Green's operators for B_s). Applying theorem 7.3.2 for $P = B_s$ and $Q = B'_s$ shows that there are unique advanced and retarded Green's operators $G_\pm \colon \Gamma_0(D^B_s M) \to \Gamma(D^B_s M)$ for B_s, $s \in \mathbb{N}$, and this construction does not depend on the choice of B'_s.

8.4. The Cauchy Problem

8.4.1 Premises. We assume premises 8.1.1. and in addition, let $\Sigma \subseteq M$ be a smooth spacelike Cauchy hypersurface with future-directed unit normal vector field $\mathfrak{n}$.

In this section we will consider the Cauchy problem for Buchdahl's operator,

$$\begin{cases} B_s\Phi = 0, \quad \Phi \in \Gamma(D^B_s M) \\ \Phi|_\Sigma = \Phi_0 \end{cases}$$

for given $\Phi_0 \in \Gamma_0(D_s^B\Sigma)$, $s \in \mathbb{N}$. In the case $s \geq 2$ it turns out (theorem 8.4.6) that this is well posed only if the Cauchy datum Φ_0 satisfies certain constraints. We will achieve an improved version of results by Wünsch [Wün85], who proved existence and uniqueness of *local* solutions for this Cauchy problem (local means: only on a neighbourhood of Σ).

This section's general idea is to facilitate the theory developed in [BGP], notably theorem 3.2.11, which we shall cite here in a slightly modified version:

8.4.2 Theorem (Bär, Ginoux, Pfäffle 2007). Let (M, g) be a time-oriented, globally hyperbolic Lorentzian manifold and let Σ be a smooth spacelike Cauchy hypersurface with future-directed unit normal vector field $\mathfrak{n}$. Let K be a normally hyperbolic differential operator acting on sections of a smooth vector bundle $\mathcal{E}$ on M. Then the Cauchy problem

$$\begin{cases} K\Phi = 0 \\ \Phi|_\Sigma = \Phi_0 \\ (\nabla_\mathfrak{n}\Phi)|_\Sigma = \Psi_0\,, \end{cases}$$

has a unique solution for all $\Phi_0, \Psi_0 \in \Gamma_0(\mathcal{E}|_\Sigma)$ and we have

$$\operatorname{supp}(\Phi) \subseteq J^M\big(\operatorname{supp}(\Phi_0) \cup \operatorname{supp}(\Psi_0)\big)\,.$$

The first part of our argument still works for two general first order linear differential operators P and Q on sections of a vector bundle $\mathcal{E}$ such that PQ is normally hyperbolic. Of course, later we will set $P = B_s$ and $Q = B_s'$ (cf. lemma 8.3.1).

8.4.3 Definition (various Cauchy problems). Let (M, g) be a time-oriented, globally hyperbolic Lorentzian manifold and let Σ be a smooth spacelike Cauchy hypersurface with future-directed unit normal vector field $\mathfrak{n}$. Let P and Q be first order differential operators on sections of a smooth vector bundle $\mathscr{E}$ on M, such that PQ is normally hyperbolic. Set $L := PQ$ and $L' := QP$.

We define the following Cauchy problems:

$$(P) \quad \begin{cases} P\Phi = 0, \quad \Phi \in \Gamma(\mathcal{E}) \\ \Phi|_\Sigma = \Phi_0 \end{cases}$$

for given $\Phi_0 \in \Gamma_0(\mathcal{E}|_\Sigma)$.

$$(L) \quad \begin{cases} L\Phi = 0, \quad \Phi \in \Gamma(\mathcal{E}) \\ \Phi|_\Sigma = \Phi_0 \\ (\nabla_\mathfrak{n}\Phi)|_\Sigma = \Psi_0 \end{cases}$$

for given $\Phi_0, \Psi_0 \in \Gamma_0(\mathcal{E}|_\Sigma)$.

$$(L') \quad \begin{cases} L'\Phi = 0, \quad \Phi \in \Gamma(\mathcal{E}) \\ \Phi|_\Sigma = \Phi_0 \\ (\nabla_\mathfrak{n}\Phi)|_\Sigma = \Psi_0 \end{cases}$$

for given $\Phi_0, \Psi_0 \in \Gamma_0(\mathcal{E}|_\Sigma)$.

$$(\tilde{L}) \quad \begin{cases} L\Phi = 0, \quad \Phi \in \Gamma(\mathcal{E}) \\ \Phi|_\Sigma = \Phi_0 \\ (Q\Phi)|_\Sigma = 0 \end{cases}$$

for given $\Phi_0 \in \Gamma_0(\mathcal{E}|_\Sigma)$.

$$(\tilde{L}') \quad \begin{cases} L'\Phi = 0, \quad \Phi \in \Gamma(\mathcal{E}) \\ \Phi|_\Sigma = \Phi_0 \\ (P\Phi)|_\Sigma = 0 \end{cases}$$

for given $\Phi_0 \in \Gamma_0(\mathcal{E}|_\Sigma)$.

8.4.4 Lemma. In the context of the preceding definition, the following diagram holds:

$$\begin{tikzcd} (P) \arrow[r, shift left] & (\tilde{L}') \arrow[l, shift left] \arrow[r, shift right] & (L') \arrow[l, dashed, shift right] \end{tikzcd}$$

Here, a solid arrow means: If the Cauchy problem at the tail has solution Φ with $\Phi|_\Sigma = \Phi_0$ for a given function $\Phi_0 \in \Gamma_0(\mathcal{E}|_\Sigma)$ as (part of the) Cauchy data, then Φ is also a solution of the Cauchy problem at the tip. In the case where the Cauchy problem at the tip is (L'), the missing Cauchy datum Ψ_0 can be uniquely constructed from Φ and Φ_0.

The dashed arrow is only valid if the Cauchy data Φ_0 and Ψ_0 fulfill certain *constraints* (if $s \geq 2$), which depend on the differential operator P. This will be treated in proposition 8.4.5.

Proof. First notice that normal hyperbolicity of PQ implies normal hyperbolicity of QP by proposition 7.2.6.

$(\tilde{L}') \to (L')$: the task is to construct a $\Psi_0 \in \Gamma_0(\mathcal{E}|_\Sigma)$ such that Φ is a solution of (L') for the Cauchy data Φ_0 and Ψ_0. It is immediate that $\Psi_0 := (\nabla_{\mathfrak{n}}\Phi)|_\Sigma$. That this Ψ_0 has compact support follows from the last statement of theorem 8.4.2. Analogously, $(\tilde{L}) \to (L)$ holds; this is not part of the diagram but will be needed in one of next parts of the proof.

$(P) \to (\tilde{L}')$: this is immediate, as $P\Phi = 0$ implies $(P\Phi)|_\Sigma = 0$.

$(\tilde{L}') \to (P)$: we are given $\Phi_0 \in \Gamma_0(\mathcal{E}|_\Sigma)$ and $\Phi \in \Gamma(\mathcal{E})$ such that

$$L'\Phi = QP\Phi = 0, \quad (P\Phi)|_\Sigma = 0, \quad \Phi|_\Sigma = \Phi_0\,.$$

This implies:

$$PL'\Phi = L(P\Phi) = 0, \quad Q(P\Phi)|_\Sigma = 0, \quad (P\Phi)|_\Sigma = 0\,,$$

which means that $P\Phi$ is a solution of the Cauchy problem $\tilde{L}$ for initial datum $\equiv 0$.

Using theorem 8.4.2, solutions of (L) are unique. Combining this with $(\tilde{L}) \to (L)$, uniqueness of solutions of $(\tilde{L})$ is immediate. Using a standard argument, this implies that solutions of $(\tilde{L})$ with initial datum $\equiv 0$ must be 0 everywhere. Hence, $P\Phi = 0$, which completes the proof. □

At this point our general considerations end and we set $P = B_s$ and $Q = B'_s$. The aim is this: We want to show the equivalence of Cauchy problems (P) and (L') in the sense of Lemma 8.4.4, because (P) (for $P = B_s$) is the Cauchy problem for Buchdahl's operator (which we want to solve) and (L') is a Cauchy problem where existence and uniqueness of solutions is already estabilshed by theorem 8.4.2. The yet missing bit is the dashed arrow in the diagram of lemma 8.4.4. In our concrete setting, this is only valid if the Cauchy datum $\Phi_0 \in \Gamma(D^B_s\Sigma)$ fulfills certain extra conditions:

8.4.5 Proposition (constraints for $(L') \to (\tilde{L}')$). We assume premises 8.4.1 and use the notation introduced in definition 8.4.3 with $P = B_s$ and $Q = B'_s$ for $s \in \mathbb{N}$.

For $\Phi_0 \in \Gamma_0(D^B_s\Sigma)$ and $\Phi \in \Gamma(D^B_s M)$, the following statements are equivalent:

(i) There is a $\Psi_0 = (\psi_0, \varphi_0)^{tr} \in \Gamma_0(D^B_s\Sigma)$ which satisfies[4]

$$\mathfrak{n}_{A_1}^{\ \dot{X}}(\tilde{\nabla}_{\dot{X}B}\psi_0{}^{BA_1\dots A_{s-1}} - \mu\varphi_{0\,\dot{X}}{}^{A_1\dots A_{s-1}}) = 0 \quad \text{(only if } s \geq 2) \tag{$*$}$$

and such that Φ is a solution of the Cauchy problem (L') with Cauchy data Φ_0 and Ψ_0.

(ii) Φ is a solution of the Cauchy problem $(\tilde{L}')$ with the Cauchy datum Φ_0.

Moreover, in a situation where this equivalence holds, Ψ_0 is uniquely fixed by Φ_0.

Proof. For $\Phi = (\psi, \varphi)^{tr} \in \Gamma(D^B_s M)$ and $\Phi_0 = (\psi_0, \varphi_0)^{tr} \in \Gamma_0(D^B_s\Sigma)$ we want to show:

$$\begin{cases} L'\Phi = 0 \\ \Phi|_\Sigma = \Phi_0 \\ (\nabla_{\mathfrak{n}}\Phi)|_\Sigma = \Psi_0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} L'\Phi = 0 \\ \Phi|_\Sigma = \Phi_0 \\ (B_s\Phi)|_\Sigma = 0 \end{cases} \tag{$**$}$$

for a $\Psi_0 \in \Gamma_0(\mathcal{E}|_\Sigma)$ satisfying $(*)$

First of all, we investigate the condition $(B_s\Phi)|_\Sigma = 0$ a bit more closely (the following calculation is a more detailed version of a calculation in [Wün85]):

$$\begin{aligned}
&(B_s\Phi)|_\Sigma = 0 \\
\Leftrightarrow &\begin{cases} (\nabla_{\dot{X}B}\psi^{BA_1\dots A_{s-1}} - \mu\varphi_{\dot{X}}{}^{A_1\dots A_{s-1}})|_\Sigma = 0 \\ (\nabla^{B\dot{X}}\varphi_{\dot{X}}{}^{A_1\dots A_{s-1}} - \frac{1}{\mu}P_s\psi^{BA_1\dots A_{s-1}} - \nu\psi^{BA_1\dots A_{s-1}})|_\Sigma = 0 \end{cases} \\
\Leftrightarrow &\begin{cases} (\sigma^a{}_{\dot{X}B}(\tilde{\nabla}_a + \mathfrak{n}_a\nabla_{\mathfrak{n}})\psi^{BA_1\dots A_{s-1}} - \mu\varphi_{\dot{X}}{}^{A_1\dots A_{s-1}})|_\Sigma = 0 \\ (\sigma^{aB\dot{X}}(\tilde{\nabla}_a + \mathfrak{n}_a\nabla_{\mathfrak{n}})\varphi_{\dot{X}}{}^{A_1\dots A_{s-1}} - \frac{1}{\mu}P_s\psi^{BA_1\dots A_{s-1}} - \nu\psi^{BA_1\dots A_{s-1}})|_\Sigma = 0 \end{cases} \\
\Leftrightarrow &\begin{cases} \mathfrak{n}_{\dot{X}B}(\nabla_{\mathfrak{n}}\psi^{BA_1\dots A_{s-1}})|_\Sigma = -\tilde{\nabla}_{\dot{X}B}\psi_0{}^{BA_1\dots A_{s-1}} + \mu\varphi_{0\,\dot{X}}{}^{A_1\dots A_{s-1}} \\ \mathfrak{n}^{B\dot{X}}(\nabla_{\mathfrak{n}}\varphi_{\dot{X}}{}^{A_1\dots A_{s-1}})|_\Sigma = -\tilde{\nabla}^{B\dot{X}}\varphi_{0\,\dot{X}}{}^{A_1\dots A_{s-1}} + \frac{1}{\mu}P_s\psi_0{}^{BA_1\dots A_{s-1}} + \nu\psi_0{}^{BA_1\dots A_{s-1}} \end{cases} \\
\overset{*}{\Leftrightarrow} &\begin{cases} (\nabla_{\mathfrak{n}}\psi^{AA_1\dots A_{s-1}})|_\Sigma = -2\mathfrak{n}^{A\dot{X}}(\tilde{\nabla}_{\dot{X}B}\psi_0{}^{BA_1\dots A_{s-1}} - \mu\varphi_{0\,\dot{X}}{}^{A_1\dots A_{s-1}}) \\ (\nabla_{\mathfrak{n}}\varphi_{\dot{X}}{}^{A_1\dots A_{s-1}})|_\Sigma = -2\mathfrak{n}_{\dot{X}B}(\tilde{\nabla}^{B\dot{X}}\varphi_{0\,\dot{X}}{}^{A_1\dots A_{s-1}} - \frac{1}{\mu}P_s\psi_0{}^{BA_1\dots A_{s-1}} - \nu\psi_0{}^{BA_1\dots A_{s-1}}), \end{cases}
\end{aligned}$$

where step $*$ follows in both directions by contraction with $\mathfrak{n}^{A\dot{X}}$ respectively $\mathfrak{n}_{\dot{X}B}$ and subsequent application of lemma 4.2.7-ii.

This calculation shows: if there is a Φ_0 which enables the equivalence $(**)$, then it must be of the form

$$\Psi_0 = \begin{pmatrix} -2\mathfrak{n}^{A\dot{X}}(\tilde{\nabla}_{\dot{X}B}\psi_0{}^{BA_1\dots A_{s-1}} - \mu\varphi_{0\,\dot{X}}{}^{A_1\dots A_{s-1}}) \\ -2\mathfrak{n}_{\dot{X}B}(\tilde{\nabla}^{B\dot{X}}\varphi_{0\,\dot{X}}{}^{A_1\dots A_{s-1}} - \frac{1}{\mu}P_s\psi_0{}^{BA_1\dots A_{s-1}} - \nu\psi_0{}^{BA_1\dots A_{s-1}}) \end{pmatrix} \tag{$+$}$$

because according to $(**)$, we must have

$$\Psi_0 = (\nabla_{\mathfrak{n}}\Phi)|_\Sigma = \begin{pmatrix} (\nabla_{\mathfrak{n}}\psi^{AA_1\dots A_{s-1}})|_\Sigma \\ (\nabla_{\mathfrak{n}}\varphi_{\dot{X}}{}^{A_1\dots A_{s-1}})|_\Sigma \end{pmatrix}.$$

[4] In this section, $\tilde{\nabla}_a := \nabla_a - \mathfrak{n}^b\nabla_b$ is the "tangential covariant derivative along Σ" (for fields on Σ).

(As a byproduct this already shows the asserted uniqueness of Ψ_0, because $(+)$ depends only on Φ_0.) However, it is not clear that $(+)$ always yields an "admitted" Cauchy datum Ψ_0, i. e. an element of $\Gamma_0(D_s^B\Sigma)$. Compactness of the support is already guaranteed because Φ_0 has compact support, but for $s \geq 2$ we need to take care of symmetry in the undotted indices. As Ψ_0 as given by $(+)$ is already symmetric in the indices $A_1 \ldots A_{s-1}$ because $\Phi_0 = (\psi_0, \varphi_0) \in \Gamma_0(D_s^B\Sigma)$, the only thing to take care of is symmetry between A and A_1 of the upper component of $(+)$, because $x^{AA_1\ldots A_{s-1}} = x^{(AA_1)\ldots A_{s-1}}$ and $x^{AA_1\ldots A_{s-1}} = x^{A(A_1\ldots A_{s-1})}$ implies $x^{AA_1\ldots A_{s-1}} = x^{(AA_1\ldots A_{s-1})}$ for every spinor x. Thus, Ψ_0 as given by $(+)$ being an element of $\Gamma_0(D_s^B\Sigma)$ is equivalent to the condition

$$\begin{aligned}0 &= \epsilon_{AA_1}\mathfrak{n}^{A\dot{X}}(\tilde{\nabla}_{\dot{X}B}\psi_0{}^{BA_1\ldots A_{s-1}} - \mu\varphi_{0\,\dot{X}}{}^{A_1\ldots A_{s-1}}) \\ &= \mathfrak{n}_{A_1}^{\;\dot{X}}(\tilde{\nabla}_{\dot{X}B}\psi_0{}^{BA_1\ldots A_{s-1}} - \mu\varphi_{0\,\dot{X}}{}^{A_1\ldots A_{s-1}})\,.\end{aligned}$$

Summing up, we showed that assuming $L'\Phi = 0$ and $\Phi|_\Sigma = \Phi_0$, $(B_s\Phi)|_\Sigma = 0$ if and only if $\Psi_0 := (\nabla_{\mathfrak{n}}\Phi)|_\Sigma$ satisfies condition $(*)$. This proves $(**)$. □

8.4.6 Theorem (Cauchy problem for Buchdahl's equations). We assume premises 8.4.1. For $s \in \mathbb{N}$, the Cauchy problem

$$(B_s) \qquad \begin{cases} B_s\Phi = 0, & \Phi \in \Gamma(D_s^B M) \\ \Phi|_\Sigma = \Phi_0 \end{cases}$$

with Cauchy datum $\Phi_0 \in \Gamma_0(D_s^B\Sigma)$ has a solution if and only if $\Phi_0 = (\psi_0, \varphi_0)^{tr}$ satisfies the constraint

$$\mathfrak{n}_{A_1}^{\;\dot{X}}(\tilde{\nabla}_{\dot{X}B}\psi_0{}^{BA_1\ldots A_{s-1}} - \mu\varphi_{0\,\dot{X}}{}^{A_1\ldots A_{s-1}}) = 0 \quad (\text{only if } s \geq 2)\,.$$

Moroever, solutions of (B_s) are unique.

Proof. "⇒": Let (B_s) have a solution Φ for some $\Phi_0 \in \Gamma_0(D_s^B\Sigma)$. Then, using lemma 8.4.4 for $P = B_s$ and $Q = B_s'$, Φ is also a solution of $(\tilde{L}')$ for Φ_0. Then, using proposition 8.4.5, it follows that Φ_0 satisfies the constraints.

"⇐": Let Φ_0 be given such that it satisfies the constraints. Then, for $P = B_s$ and $Q = B_s'$, theorem 8.4.2 together with proposition 8.4.5 in combination with $(\tilde{L}') \to (P)$ from lemma 8.4.4 guarantees the existence of a solution Φ of (B_s).

To prove uniqueness, take a second solution Φ' of (B_s) for Φ_0 and again use lemma 8.4.4 to show that Φ' must also be a solution of (L'). But solutions of (L') are unique according to theorem 8.4.2, hence $\Phi' = \Phi$. □

8.4.7 Corollary (compatibility of the constraints). Assuming premises 8.4.1, let $\Sigma' \subseteq M$ be a second smooth spacelike Cauchy hypersurface with future-directed timelike unit normal field $\mathfrak{m}$. For $s \in \mathbb{N}$, let Φ be a solution of the Cauchy problem (B_s) for some Cauchy datum $\Phi_0 \in \Gamma_0(D_s^B\Sigma)$. Then Φ is also a solution of the Cauchy problem

$$\begin{cases} B_s\Phi = 0, & \Phi \in \Gamma(D_s^B M) \\ \Phi|_{\Sigma'} = \Phi_0' \end{cases}$$

for $\Phi'_0 := \Phi|_{\Sigma'}$, which means that $\Phi'_0 = (\psi'_0, \varphi'_0)^{tr}$ is a proper Cauchy datum, i. e., $\Phi'_0 := \Phi|_{\Sigma'} \in \Gamma_0(D^B_s\Sigma')$ and Φ'_0 satisfies the constraint condition

$$\mathfrak{m}_{A_1}{}^{\dot{X}}(\tilde{\nabla}_{\dot{X}B}\psi_0{}^{BA_1\ldots A_{s-1}} - \mu\varphi_{0\,\dot{X}}{}^{A_1\ldots A_{s-1}}) = 0 \quad \text{(only if } s \geq 2) \,.$$

Proof. That Φ'_0 has compact support follows from the additional statement about the support in theorem 8.4.2 together with lemma 6.1.14-b. That Φ'_0 satisfies the constraints follows immediately from proposition 8.4.5 in combination with lemma 8.4.4. □

9. CAR-Algebraic Quantisation of Buchdahl Fields

Our final goal now is performing a CAR-algebraic quantisation of fields given by solutions of Buchdahl's generalised Dirac equations. This was done before e. g. by Dimock [Dim82] for the special case of the spin $\frac{1}{2}$ Dirac field and in sections 9.2 and 9.3 we will investigate two possible ways of generalising this construction to the case of arbitrary spin.

9.1. The General Idea

In this section we shall briefly review one well established quantisation procedure for the classical Dirac field using CAR-algebras, which was given by Dimock in [Dim82]. This procedure will be our guideline for a subsequent attempt of quantisation of general Buchdahl fields.

We assume premises 8.1.1 and first make the following definition:

9.1.1 Definition. Let $\Sigma \subseteq M$ be a smooth spacelike Cauchy hypersurface in M with future directed unit normal vector field $\mathfrak{n}$. For $s \in \mathbb{N}$ we define the following spaces:

$$\begin{aligned}\mathscr{H}^s_\Sigma &:= \{\Phi_0 \in \Gamma_0(D^B_s\Sigma) \,|\, \Phi_0 \text{ is an admitted Cauchy datum for } B_s\Phi = 0 \text{ on } \Phi\} \\ \mathscr{H}^s &:= \{\Phi \in \Gamma(D^B_s M) \,|\, B_s\Phi = 0 \text{ and } \Phi|_\Sigma \in \mathscr{H}^s_\Sigma\}\,,\end{aligned}$$

where B_s is Buchdahl's operator for spin $\frac{s}{2}$ (cf. definition 8.1.4) and a Cauchy datum is called "admitted" if it satisfies the constraints, cf. theorem 8.4.6. We call $\mathscr{H}^s$ **the space of solutions with localised Cauchy data** and $\mathscr{H}^s_\Sigma$ the **space of localised Cauchy data on Σ.**

Notice that the definition of $\mathscr{H}^s$ is independent of the choice of Σ by corollary 8.4.7. Both spaces are complex vector spaces, as Buchdahl's operator B_s is a linear differential operator and the constraints are linear. For every smooth spacelike Cauchy hypersurface Σ, the canonical map

$$\begin{aligned}\mathscr{H}^s &\to \mathscr{H}^s_\Sigma \\ \Phi &\mapsto \Phi|_\Sigma\,,\end{aligned}$$

is a $\mathbb{C}$-vector space isomorphism using theorem 8.4.6; the inverse is given by assigning to Φ_0 the solution of the Cauchy problem from theorem 8.4.6 with Cauchy datum Φ_0. Thus, if Σ' is a second smooth spacelike Cauchy hypersurface, there is a canonical $\mathbb{C}$-vector space isomorphism

$$\mathscr{H}^s_\Sigma \to \mathscr{H}^s_{\Sigma'}\,.$$

Now, the general idea for our quantisation procedure is the following: Imagine $\mathscr{H}^s$ was equipped with a Hermitian scalar product to form a pre-Hilbert space, $(\mathscr{H}^s, (\cdot,\cdot))$. Then the system could be quantised (cf. [Dim82] for the spin $\frac{1}{2}$ case) by assigning to this pre-Hilbert space the algebra of canonical anti-commutation relations, $\mathrm{CAR}(\mathscr{H}^s)$, which is defined as follows:

9.1.2 Definition and proposition (CAR-algebra). Let $(\mathscr{H}, (\cdot,\cdot))$ be a complex pre-Hilbert space. A unital C^*-algebra $\mathrm{CAR}(\mathscr{H})$ together with a $\mathbb{C}$-anti-linear embedding $\chi\colon \mathscr{H} \to \mathrm{CAR}(\mathscr{H})$ is called **CAR-algebra for** $\mathscr{H}$ (CAR = canonical anti-commutation relations),

(1) if $\mathrm{CAR}(\mathscr{H})$ is generated[1] by $\chi(\mathscr{H}) \cup \{\mathbb{1}\}$, where $\mathbb{1}$ denotes the unit element,

(2) $\forall x, y \in \mathscr{H}\colon \chi(x)\chi(y)^* + \chi(y)^*\chi(x) = (x, y) \cdot \mathbb{1}$,

(3) $\forall x, y \in \mathscr{H}\colon \chi(x)\chi(y) + \chi(y)\chi(x) = 0$.

If $\mathrm{CAR}(\mathscr{H})$ and $\mathrm{CAR}'(\mathscr{H})$ are two CAR-algebras for $(\mathscr{H}, (\cdot,\cdot))$ with embedding maps χ and χ', then there exists a unique $*$-isomorphism $\alpha\colon \mathrm{CAR}(\mathscr{H}) \to \mathrm{CAR}'(\mathscr{H})$, such that $\forall x \in \mathscr{H}\colon \alpha(\chi(x)) = \chi'(x)$. In this sense, $\mathrm{CAR}(\mathscr{H})$ is unique and we call it **the CAR-algebra of** $(\mathscr{H}, (\cdot,\cdot))$.

Moreover, $\mathrm{CAR}(\mathscr{H}) = \mathrm{CAR}(\bar{\mathscr{H}})$, i. e. it doesn't matter whether we take the pre-Hilbert space or its Hilbert space completion to generate a CAR-algebra from its elements.

Proof. This is well known from [BR96, section 5.2.2]. □

As an example, we demonstrate the construction of a CAR-algebra for the classical Dirac field ($s = 1$ in our terminology, i. e. spin $= \frac{1}{2}$). Our construction is closely related and equivalent to [Dim82, section III]:

First, notice that for $s = 1$, the constraints on the admissible Cauchy data from theorem 8.4.6 vanish and hence,

$$\mathscr{H}^1_\Sigma = \Gamma_0(D^B_1\Sigma) = \Gamma_0(D^D\Sigma)\,.$$

In order to equip $\mathscr{H}^1$ with a Hermitian scalar product, one may utilise the classical **Dirac current,**

$$j^a := \Phi^+_{\tilde{A}}\, \gamma^{a\tilde{A}}{}_{\tilde{B}}\, \Phi^{\tilde{B}} \in \Gamma(TM)\,,$$

(footnote[2]) for a general Dirac spinor field $\Phi^{\tilde{A}} \in \Gamma(D^D_M)$ with Dirac adjoint $\Phi^+_{\tilde{A}}$ (cf. point 4 in section 5.3; for a general treatment of the Dirac current cf. [Rol03, pp. 349], [Mes58, pp. 894]). The Dirac current is $\mathbb{R}$-quadratic in Φ and therefore motivates constructing the sesqui-linear map $\Gamma(D^DM) \times \Gamma(D^DM) \to \Gamma(TM \otimes \mathbb{C})$, given by

$$j^a(\Psi, \Phi) := \Psi^+_{\tilde{A}}\, \gamma^{a\tilde{A}}{}_{\tilde{B}}\, \Phi^{\tilde{B}}\,.$$

[1] We say a C^*-algebra $\mathfrak{A}$ is generated by a subset $X \subseteq \mathfrak{A}$, if there is no genuine C^*-subalgebra $\mathfrak{B} \subsetneq \mathfrak{A}$ such that $X \subseteq \mathfrak{B}$.

[2] Notice that a priori, j^a is a section of $TM \otimes \mathbb{C}$, cf. definition 6.4.6. However, it turns out that this expression for j^a always yields a section of TM.

Using this, one may build a Hermitian form $b^1_\Sigma \colon \mathscr{H}^1_\Sigma \times \mathscr{H}^1_\Sigma \to \mathbb{C}$ by setting

$$b^1_\Sigma(\Psi_0, \Phi_0) := \int\limits_\Sigma \iota\mathfrak{n}_a\, j^a(\Psi_0, \Phi_0)\, d\mu_\Sigma\,, \tag{$*$}$$

where $\iota\colon TM \to TM \otimes \mathbb{C}$ is the canonical embedding and for the contraction $\iota\mathfrak{n}_a j^a$ we use the complexified metric $g \otimes \mathbb{C}$ on $TM \otimes \mathbb{C}$. It can be shown (cf. [Dim82]) that this form is a Hermitian *scalar product* on $\mathscr{H}^1_\Sigma$; in particular, for every smooth spacelike Cauchy hypersurface Σ, b^1_Σ is positive definite.

Actually, we wanted a scalar product on $\mathscr{H}^1$, not on $\mathscr{H}^1_\Sigma$. But as $\mathscr{H}^1 \cong \mathscr{H}^1_\Sigma$ as $\mathbb{C}$-vector spaces in the canonical way described above, we may define a scalar product b^1 on $\mathscr{H}^1$ by pulling back b^1_Σ through this isomorphism:

$$b^1(\Psi, \Phi) := b^1_\Sigma(\Psi|_\Sigma, \Phi|_\Sigma)\,.$$

It is very surprising that one can prove that b^1 does not depend on the choice of Σ [Dim82, section III]. Thus, we have the following isometric isomorphisms of $\mathbb{C}$-inner product spaces:

$$(\mathscr{H}^1, b^1) \cong (\mathscr{H}^1_\Sigma, b^1_\Sigma) \cong (\mathscr{H}^1_{\Sigma'}, b^1_{\Sigma'}) \cong \ldots$$

and one may use this structure to quantise the system by building $\mathrm{CAR}(\mathscr{H}^1, b^1)$.

In the next section we will try to generalise this procedure to the general case of $s \in \mathbb{N}$. As it turns out, the hard bit is to find a scalar product on $\mathscr{H}^s$ which is positive definite and "as litte ambiguous as possible".

9.2. Generalisation of Dimock's Quantisation

In this section we present an immediate generalisation attempt of the spin $= \frac{1}{2}$ construction recalled in the previous section to general $s \in \mathbb{N}$. We will construct a scalar product b^s_Σ on $\mathscr{H}^s_\Sigma$ for every smooth spacelike Cauchy hypersurface Σ and prove positivity. However, for $s > 1$ it could not be established that there are (canonical) isometries $(\mathscr{H}^s_\Sigma, b^s_\Sigma) \cong (\mathscr{H}^s_{\Sigma'}, b^s_{\Sigma'})$, such that there is no unique "pulled back" scalar product on $\mathscr{H}^s$. We therefore obtain only a quantisation which depends on a choice of Cauchy surface.

9.2.1 Definition. We adopt premises 8.1.1 and the notation introduced in section 8.1.

(i) Let

$$\Phi = \begin{pmatrix} \psi^{AA_1\ldots A_{s-1}} \\ \varphi_{\dot{X}}^{\;A_1\ldots A_{s-1}} \end{pmatrix} \in \Gamma(D^B_s M)$$

be a Buchdahl spinor. Then we define a **generalised Dirac adjoint** Φ^+:

$$\Phi^+ := \begin{pmatrix} \bar{\varphi}_A^{\;\dot{X}_1\ldots\dot{X}_{s-1}} \\ \bar{\psi}^{\dot{X}\dot{X}_1\ldots\dot{X}_{s-1}} \end{pmatrix}.$$

Notice that this is a section of $\bar{D}^B_s M \subseteq D^{D*}M \otimes D^{(0,\frac{s-1}{2})}M$, which is why we write Φ^+ using abstract indices as

$$\Phi^+ = (\Phi^+)_{\tilde{A}}^{\;\dot{X}_1\ldots\dot{X}_{s-1}}\,.$$

(ii) Let $\Sigma \subseteq M$ be a smooth spacelike Cauchy hypersurface with future-directed unit normal vector field $\mathfrak{n}^a$. Set

$$\mathfrak{n}^{\tilde{A}}{}_{\tilde{B}} := \mathfrak{n}^a \gamma_a{}^{\tilde{A}}{}_{\tilde{B}} \quad \text{and} \quad \mathfrak{n}_{\dot{X}A} := \mathfrak{n}^a \sigma_{a\dot{X}A}\,.$$

Then on the space $\Gamma_0(D^B_s\Sigma) \supseteq \mathscr{H}^s_\Sigma$ we define the sesquilinear form

$$\begin{aligned} b^s_\Sigma(\Phi,\Psi) &:= \int_\Sigma (\Phi^+)_{\tilde{A}}{}^{\dot{X}_1\ldots\dot{X}_{s-1}}\, \mathfrak{n}^{\tilde{A}}{}_{\tilde{B}}\, \mathfrak{n}_{\dot{X}_1A_1}\cdots \mathfrak{n}_{\dot{X}_{s-1}A_{s-1}}\, \Psi^{\tilde{A}A_1\ldots A_{s-1}}\, d\mu_\Sigma \\ &= \int_\Sigma \Big[(\bar{\varphi}_1)^{\dot{X}\dot{X}_1\ldots\dot{X}_{s-1}} \mathfrak{n}_{\dot{X}A}\mathfrak{n}_{\dot{X}_1A_1}\cdots \mathfrak{n}_{\dot{X}_{s-1}A_{s-1}} (\psi_1)^{AA_1\ldots A_{s-1}} \\ &\qquad + (\bar{\varphi}_2)_A{}^{\dot{X}_1\ldots\dot{X}_{s-1}} \mathfrak{n}^{A\dot{X}} \mathfrak{n}_{\dot{X}_1A_1}\cdots \mathfrak{n}_{\dot{X}_{s-1}A_{s-1}} (\psi_2)_{\dot{X}}{}^{A_1\ldots A_{s-1}} \Big] d\mu_\Sigma\,, \end{aligned}$$

whereby we assumed $\Phi, \Psi \in \Gamma_0(D^B_s\Sigma)$ with

$$\Phi = \begin{pmatrix} (\varphi_1)^{AA_1\ldots A_{s-1}} \\ (\varphi_2)_{\dot{X}}{}^{A_1\ldots A_{s-1}} \end{pmatrix}, \quad \Psi = \begin{pmatrix} (\psi_1)^{AA_1\ldots A_{s-1}} \\ (\psi_2)_{\dot{X}}{}^{A_1\ldots A_{s-1}} \end{pmatrix}.$$

Notice that this generalises the sequilinear form given by formula $(*)$ in section 9.1 (or [Dim82, section III.A]). Of course, this is not the only generalisation one could think of, but after all it seemed to be the most promising one (most attempts failed being positive definite). It is non-trivial that b^s_Σ is still a scalar product for $s > 1$, but this is indeed the case:

9.2.2 Theorem (inner product on $\mathscr{H}^s_\Sigma$). For every smooth spacelike Cauchy hypersurface $\Sigma \subseteq M$ and for every $s \geq 1$, b^s_Σ as given in definition 9.2.1 is a Hermitian scalar product on $\Gamma_0(D^B_s\Sigma) \supseteq \mathscr{H}^s_\Sigma$.

To prove this we will show that b^s_Σ is postitive definite. As it turns out, this is the case even fiberwise, i. e. we show that $b^s_\Sigma(\Phi,\Phi)$ is an integral over positive numbers if $\Phi \neq 0$. We start out with some lemmas:

9.2.3 Lemma. For $\psi \in \Delta_{\frac{1}{2},0}$, $\psi \neq 0$,

$$x^a := \sigma^a{}_{A\dot{X}} \psi^A \bar{\psi}^{\dot{X}}$$

is always an element of $(\mathbb{R}^4, \eta)$ (see footnote[3]), which is lightlike and future-directed. Notice that $\sigma_a{}^{A\dot{X}}$ is now the tensor-spinor "on the fiber" (definition 4.2.1).

Proof. Represent all vectors with respect to the standard (orthonormal) basis $\{e_\mu\}$ of $(\mathbb{R}^4, \eta)$ (and $(\mathbb{R}^4, \eta)_\mathbb{C}$) and represent all spinors with respect to the standard basis $\{E_\mu\}$ of $\Delta_{\frac{1}{2},0}$ (cf. the beginning of chapter 3 and point 7 in section 4.1). Using the

[3]Notice that a priori, x is an element of $(\mathbb{R}^4, \eta)_\mathbb{C}$; cf. definition 4.2.1.

component matrix representing ${\sigma^a}_{A\dot{X}}$ with respect to the standard bases (cf. definition 4.2.1), we find for $x^a = {\sigma^a}_{A\dot{X}}\psi^A\bar{\psi}^{\dot{X}}$:

$$\begin{aligned}
x^0 &= \frac{1}{2}(\psi^1\bar{\psi}^1 + \psi^2\bar{\psi}^2) = \frac{1}{2}(|\psi^1|^2 + |\psi^2|^2)\,,\\
x^1 &= \frac{1}{2}(\psi^1\bar{\psi}^2 + \psi^2\bar{\psi}^1) = \mathrm{Re}(\psi^1\bar{\psi}^2)\,,\\
x^2 &= \frac{1}{2}\frac{1}{i}(\psi^1\bar{\psi}^2 - \psi^2\bar{\psi}^1) = \mathrm{Im}(\psi^1\bar{\psi}^2)\,,\\
x^3 &= \frac{1}{2}(\psi^1\bar{\psi}^1 - \psi^2\bar{\psi}^2) = \frac{1}{2}(|\psi^1|^2 - |\psi^2|^2)\,.
\end{aligned}$$

Thus, all of the x^μ are real numbers, hence, $x^a \in \mathbb{R}^4$. Moreover, $x^0 > 0$, thus x is future-directed.

x^a being lightlike is equivalent to

$$(x^0)^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$$

Starting with the right hand side, we calculate:

$$\begin{aligned}
4\left[(x^1)^2 + (x^2)^2 + (x^3)^2\right] &= (\psi^1\bar{\psi}^2 + \psi^2\bar{\psi}^1)^2 + (i\psi^2\bar{\psi}^1 - i\psi^1\bar{\psi}^2)^2 + (\psi^1\bar{\psi}^1 - \psi^2\bar{\psi}^2)^2\\
&= (\psi^1\bar{\psi}^2)^2 + 2\psi^1\bar{\psi}^2\psi^2\bar{\psi}^1 + (\psi^2\bar{\psi}^1)^2\\
&\quad - (\psi^2\bar{\psi}^1)^2 + 2\psi^2\bar{\psi}^1\psi^1\bar{\psi}^2 - (\psi^1\bar{\psi}^2)^2\\
&\quad + (\psi^1\bar{\psi}^1)^2 - 2\psi^1\bar{\psi}^1\psi^2\bar{\psi}^2 + (\psi^2\bar{\psi}^2)^2\\
&= (\psi^1\bar{\psi}^1)^2 + 2\psi^1\bar{\psi}^1\psi^2\bar{\psi}^2 + (\psi^2\bar{\psi}^2)^2\\
&= (\psi^1\bar{\psi}^1 + \psi^2\bar{\psi}^2)^2\\
&= 4\,(x^0)^2
\end{aligned}$$

□

9.2.4 Lemma. Every lightlike and future-directed vector $x \in (\mathbb{R}^4, \eta)$ may be written as

$$x^a = {\sigma^a}_{A\dot{X}}\psi^A\bar{\psi}^{\dot{X}}$$

for some $\psi \in \Delta_{\frac{1}{2},0}$. Again, ${\sigma_a}^{A\dot{X}}$ is the tensor-spinor "on the fiber" (definition 4.2.1).

Proof. Let all vectors be represented with respect to the standard orthonormal basis $\{e_\mu\}$ of $(\mathbb{R}^4, \eta)$ and let all spinors be represented with respect to the standard basis $\{E_\mu\}$ of $\Delta_{\frac{1}{2},0}$ (cf. the beginning of chapter 3 and point 7 in section 4.1).

Pick an arbitrary $0 \neq \varphi \in \Delta_{\frac{1}{2},0}$. Then $y^a := {\sigma^a}_{A\dot{X}}\varphi^A\bar{\varphi}^{\dot{X}}$ is lightlike and future-directed. Moreover, without loss of generality, we may assume that $y^0 = x^0$: If this is not the case, pick $\mu \in \mathbb{R}_{>0}$ such that $x^0 = \mu y^0$ (this is always possible as $x^0 \in \mathbb{R}_{>0}$ and $y^0 \in \mathbb{R}_{>0}$ because x and y are future-directed) and scale φ by $\sqrt{\mu}$.

Now, as $x^0 = y^0$ and both x and y are lightlike and future-directed, there is a Lorentz transformation $L \in \mathcal{L}_+^\uparrow$ such that $x = L(y)$ (notice that L is just a spacial rotation).

Let $S \in \mathrm{SL}(2,\mathbb{C})$ be such that $\lambda(S) = L$, where λ denotes the universal covering $\lambda\colon \mathrm{SL}(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow$, cf. 3.2.3.
$L(y)$ on the level of spinors is given by the vector representation $D^{(\frac{1}{2},\frac{1}{2})}$ of $\mathcal{L}_+^\uparrow$:

$$x = L(y) = \sigma^{-1}\left(D^{(\frac{1}{2},\frac{1}{2})}(S)\sigma(y)\right)$$

which means,

$$x^a = \sigma^a{}_{B\dot{Y}} S^B{}_A \bar{S}^{\dot{Y}}{}_{\dot{X}} \varphi^A \bar{\varphi}^{\dot{X}} = \sigma^a{}_{B\dot{Y}} (S\varphi)^B (\overline{S\varphi})^{\dot{Y}} .$$

Thus, setting $\psi := S\varphi$, we find $x^a = \sigma^a{}_{A\dot{X}} \psi^A \bar{\psi}^{\dot{X}}$. □

9.2.5 Proposition. For $0 \neq \varphi^{A_1\ldots A_r\dot{Y}_1\ldots\dot{Y}_s} \in (\Delta_{\frac{1}{2},0})^{\otimes r} \otimes (\Delta_{0,\frac{1}{2}})^{\otimes s}$ define:

$$\Phi^{a_1\ldots a_r b_1\ldots b_s} := \sigma^{a_1}{}_{A_1\dot{X}_1} \cdots \sigma^{a_r}{}_{A_r\dot{X}_r} \sigma^{b_1}{}_{B_1\dot{Y}_1} \cdots \sigma^{b_s}{}_{B_s\dot{Y}_s} \varphi^{A_1\ldots A_r\dot{Y}_1\ldots\dot{Y}_s} \bar{\varphi}^{\dot{X}_1\ldots\dot{X}_r B_1\ldots B_s} .$$

Let $\mathfrak{n}^a \in (\mathbb{R}^4, \eta)$ be future-directed and timelike. Then

$$0 < \mathfrak{n}_{a_1} \ldots \mathfrak{n}_{a_r} \mathfrak{n}_{b_1} \ldots \mathfrak{n}_{b_s} \Phi^{a_1\ldots a_r b_1\ldots b_s} \in \mathbb{R} .$$

Again, $\sigma_a{}^{A\dot{X}}$ is the tensor-spinor "on the fiber" (definition 4.2.1).

Proof.

(a) Consider the map

$$\begin{aligned} b_{\mathfrak{n}}\colon \Delta_{\frac{1}{2},0} \times \Delta_{\frac{1}{2},0} &\to \mathbb{C} \\ (\varphi, \psi) &\mapsto \mathfrak{n}_{A\dot{X}} \psi^A \bar{\varphi}^{\dot{X}} . \end{aligned} \tag{$*$}$$

It is obviously sesquilinear ($\mathfrak{n} \neq 0$ as timelike vector). As $\mathfrak{n}^a \in \mathbb{R}^4$, we find $\bar{\mathfrak{n}}^{\dot{X}A} = \mathfrak{n}^{A\dot{X}}$ (in the sense of remark 4.2.4) and therefore:

$$\overline{b_{\mathfrak{n}}(\psi,\varphi)} = \overline{\mathfrak{n}_{A\dot{X}} \varphi^A \bar{\psi}^{\dot{X}}} = \bar{\mathfrak{n}}_{\dot{X}A} \bar{\varphi}^{\dot{X}} \psi^A = b_{\mathfrak{n}}(\varphi,\psi) .$$

Hence, $b_{\mathfrak{n}}$ is a Hermitian sesquilinear form on the 2-dimensional complex vector space $\Delta_{\frac{1}{2},0}$. Morover, we find for all $\psi \in \Delta_{\frac{1}{2},0}$:

$$b_{\mathfrak{n}}(\psi,\psi) = \mathfrak{n}_{A\dot{X}} \psi^A \bar{\psi}^{\dot{X}} = \mathfrak{n}_a \sigma^a_{A\dot{X}} \psi^A \bar{\psi}^{\dot{X}} > 0 ,$$

where "> 0" follows using lemma 9.2.3 and the fact that $\eta(x,y) > 0$ for x future directed and timelike and y future-directed and lightlike.
Summing up: $\mathfrak{n}$ naturally gives rise to a Hermitian scalar product $b_{\mathfrak{n}}$ on $\Delta_{\frac{1}{2},0}$. Thus, there is a basis $\{B_1, B_2\}$ of $\Delta_{\frac{1}{2},0}$, such that $b_{\mathfrak{n}}(B_\mu, B_\nu) = \delta_{\mu\nu}$.
Notice that $b_{\mathfrak{n}}$ induces a Hermitian scalar product $\bar{b}_{\mathfrak{n}}$ on $\Delta_{0,\frac{1}{2}} = \bar{\Delta}_{\frac{1}{2},0}$ by

$$\bar{b}_{\mathfrak{n}}(\bar{\varphi},\bar{\psi}) := b_{\mathfrak{n}}(\psi,\varphi) = \mathfrak{n}_{A\dot{X}} \varphi^A \bar{\psi}^{\dot{X}}, \quad \bar{\varphi},\bar{\psi} \in \Delta_{0,\frac{1}{2}} , \tag{$**$}$$

cf. 2.1.1-iii.

(b) Let $\{B_1, B_2\}$ be a fixed choice of such a $b_{\mathfrak{n}}$-orthonormal basis of $\Delta_{\frac{1}{2},0}$ and let $\{\bar{B}_1, \bar{B}_2\}$ be the complex conjugate basis of $\Delta_{0,\frac{1}{2}}$. Represent $\varphi^{A_1\ldots A_r\dot{Y}_1\ldots\dot{Y}_s}$ as component matrix with respect to these bases:

$$\varphi = \varphi^{\alpha_1\ldots\alpha_r\dot{\nu}_1\ldots\dot{\nu}_s} B_{\alpha_1}\otimes\ldots\otimes B_{\alpha_r}\otimes\bar{B}_{\dot{\nu}_1}\otimes\ldots\otimes\bar{B}_{\dot{\nu}_s}\,.$$

Notice that $\bar{\varphi}^{\dot{X}_1\ldots\dot{X}_r B_1\ldots B_s}$ has complex conjugate components with respect to our bases (point 6 in section 4.1):

$$\bar{\varphi} = \bar{\varphi}^{\dot{\mu}_1\ldots\dot{\mu}_r\beta_1\ldots\beta_s} \bar{B}_{\dot{\mu}_1}\otimes\ldots\otimes\bar{B}_{\dot{\mu}_r}\otimes B_{\beta_1}\otimes\ldots\otimes B_{\beta_s}\,.$$

Then we finally calculate using $(*)$ from above:

$$\begin{aligned}
&\mathfrak{n}_{A_1\dot{X}_1}\ldots\mathfrak{n}_{A_r\dot{X}_r}\mathfrak{n}_{B_1\dot{Y}_1}\ldots\mathfrak{n}_{B_s\dot{Y}_s}\varphi^{A_1\ldots A_r\dot{Y}_1\ldots\dot{Y}_s}\bar{\varphi}^{\dot{X}_1\ldots\dot{X}_s B_1\ldots B_s}\\
&\quad=\mathfrak{n}_{A_1\dot{X}_1}\ldots\mathfrak{n}_{A_r\dot{X}_r}\mathfrak{n}_{B_1\dot{Y}_1}\ldots\mathfrak{n}_{B_s\dot{Y}_s}\varphi^{\alpha_1\ldots\alpha_r\dot{\nu}_1\ldots\dot{\nu}_s}\bar{\varphi}^{\dot{\mu}_1\ldots\dot{\mu}_r\beta_1\ldots\beta_s}\\
&\qquad\qquad\cdot(B_{\alpha_1})^{A_1}\otimes\ldots\otimes(B_{\alpha_r})^{A_r}\otimes(\bar{B}_{\dot{\nu}_1})^{\dot{Y}_1}\otimes\ldots\otimes(\bar{B}_{\dot{\nu}_s})^{\dot{Y}_s}\\
&\qquad\qquad\otimes(\bar{B}_{\dot{\mu}_1})^{\dot{X}_1}\otimes\ldots\otimes(\bar{B}_{\dot{\mu}_r})^{\dot{X}_r}\otimes(B_{\beta_1})^{B_1}\otimes\ldots(\otimes B_{\beta_s})^{B_s}\\
&\quad=\varphi^{\alpha_1\ldots\alpha_r\dot{\nu}_1\ldots\dot{\nu}_s}\bar{\varphi}^{\dot{\mu}_1\ldots\dot{\mu}_r\beta_1\ldots\beta_s}\cdot\\
&\qquad\qquad\underbrace{b_{\mathfrak{n}}(B_{\dot{\mu}_1},B_{\alpha_1})\cdot\ldots\cdot b_{\mathfrak{n}}(B_{\dot{\mu}_r},B_{\alpha_r})\cdot\bar{b}_{\mathfrak{n}}(\bar{B}_{\beta_1},\bar{B}_{\dot{\nu}_1})\cdot\ldots\cdot\bar{b}_{\mathfrak{n}}(\bar{B}_{\beta_s},\bar{B}_{\dot{\nu}_s})}_{=0\text{ unless }\forall i:\,\dot{\mu}_i=\alpha_i\text{ and }\beta_i=\dot{\nu}_i}\\
&\quad=\sum_{\substack{\alpha_1,\ldots,\alpha_r=1\\ \dot{\nu}_1,\ldots,\dot{\nu}_s=1}}^{2}\underbrace{|\varphi^{\alpha_1\ldots\alpha_r\dot{\nu}_1\ldots\dot{\nu}_s}|^2}_{\geq 0}\\
&\qquad\qquad\underbrace{b_{\mathfrak{n}}(B_{\alpha_1},B_{\alpha_1})\cdot\ldots\cdot b_{\mathfrak{n}}(B_{\alpha_r},B_{\alpha_r})\cdot\bar{b}_{\mathfrak{n}}(\bar{B}_{\dot{\nu}_1},\bar{B}_{\dot{\nu}_1})\cdot\ldots\cdot\bar{b}_{\mathfrak{n}}(\bar{B}_{\dot{\nu}_s},\bar{B}_{\dot{\nu}_s})}_{>0}\\
&\quad>0
\end{aligned}$$

where in the very last step (">0") we do not have to include equalness because φ was assumed to be $\neq 0$ which means that at least one of the summands is non-vanishing.

□

Proof of theorem 9.2.2. We have to show for all $0\neq\Phi=(\psi,\varphi)^{tr}\in\Gamma_0(D^B_s\Sigma)$ that $b^s_\Sigma(\Phi,\Phi)>0$, i. e.

$$\begin{aligned}
0<b^s_\Sigma(\Phi,\Phi)=\int_\Sigma\Big[&\bar{\psi}^{\dot{X}\dot{X}_1\ldots\dot{X}_{s-1}}\mathfrak{n}_{\dot{X}A}\mathfrak{n}_{\dot{X}_1A_1}\cdots\mathfrak{n}_{\dot{X}_{s-1}A_{s-1}}\psi^{AA_1\ldots A_{s-1}}\\
&+\bar{\varphi}_A{}^{\dot{X}_1\ldots\dot{X}_{s-1}}\mathfrak{n}^{A\dot{X}}\mathfrak{n}_{\dot{X}_1A_1}\cdots\mathfrak{n}_{\dot{X}_{s-1}A_{s-1}}\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}}\Big]d\mu_\Sigma\,.
\end{aligned}$$

This may be accomplished by showing positivity in each fiber, i. e. for every timelike future-directed vector $\mathfrak{n}^a\in(\mathbb{R}^4,\eta)$, for all $\psi^{AA_1\ldots A_{s-1}}\in\Delta_{\frac{s}{2},0}$ and for all $\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}}\in\Delta^*_{0,\frac{1}{2}}\otimes\Delta_{\frac{s-1}{2},0}$, we have to show:

$$\begin{aligned}
0<\;&\bar{\psi}^{\dot{X}\dot{X}_1\ldots\dot{X}_{s-1}}\mathfrak{n}_{\dot{X}A}\mathfrak{n}_{\dot{X}_1A_1}\cdots\mathfrak{n}_{\dot{X}_{s-1}A_{s-1}}\psi^{AA_1\ldots A_{s-1}}\\
&+\bar{\varphi}_A{}^{\dot{X}_1\ldots\dot{X}_{s-1}}\mathfrak{n}^{A\dot{X}}\mathfrak{n}_{\dot{X}_1A_1}\cdots\mathfrak{n}_{\dot{X}_{s-1}A_{s-1}}\varphi_{\dot{X}}{}^{A_1\ldots A_{s-1}}\,.
\end{aligned}$$

This is obtained immediately applying proposition 9.2.5 separately to each of the two summands. ◻

What we achieved so far is constructing a scalar product b^s_Σ on $\mathscr{H}^s_\Sigma$ for all $s \in \mathbb{N}$ and all smooth spacelike Cauchy hypersurfaces $\Sigma \subseteq M$. The next step would be proving that the induced scalar product b^s on $\mathscr{H}^s$, given by

$$b^s(\Psi, \Phi) := b^s_\Sigma(\Psi|_\Sigma, \Phi|_\Sigma)$$

is independent of the choice of Σ, such that we obtain isometric isomorphisms

$$(\mathscr{H}^s, b^s) \cong (\mathscr{H}^s_\Sigma, b^s_\Sigma) \cong (\mathscr{H}^s_{\Sigma'}, b^s_{\Sigma'}) \cong \dots\,,$$

such that we could use $(\mathscr{H}^s, b^s)$ to construct a unique CAR-algebra for the system. However, it turned out that proving independence of Σ for $s \geq 2$ is much more complicated than expected. Generalising the $s = 1$ proof in [Dim82] seems, in the end, to require a kind of divergence theorem for higher order contravariant tensor fields. Maybe this could get carried further using cohomological methods; in any case this hints at a presumably deep and interesting mathematical background yet to be illuminated by future research.

9.3. Quantisation in Illge's Framework

Illge in [Ill93] presented a description of massive higher spin fields on general-relativistic spacetime manifolds, which uses four "degrees of freedom" instead of two (see below). He gives a Langrangian for the fields and the associated Euler-Lagrange equations turn out to be given by two independent systems of Buchdahl equations. Moreover, minimal coupling to an electromagnetic field is carried out and yields a U(1)-gauge invariance of the Lagrangian and a generalised Dirac current vector field. We tried using this current vector to perform a CAR-algebraic quantisiation (for the Fermionic case) as outlined in section 9.1. However, it turned out that the sesquilinear form constructed by the current is not positive definite and thus doesn't form a Hermitian scalar product.

Review of [Ill93]

We shall start out presenting the core results from [Ill93] in order to make expicit what we mean by "Illge's framework". We assume premises 8.1.1.

In [Ill93, theorem 1], Illge presents a certain Lagrangian density $L_{\frac{s}{2}}$ ($s \in \mathbb{N}$) which depends on four spinor fields,

$$\varphi^{AA_1\dots A_{s-1}} \in \Gamma(D^{(\frac{s}{2},0)}M), \qquad \xi^{\dot X\dot X_1\dots\dot X_{s-1}} \in \Gamma(D^{(0,\frac{s}{2})}M),$$
$$\chi_{\dot X}{}^{A_1\dots A_{s-1}} \in \Gamma(D^{(0,\frac{1}{2})*}M \otimes D^{(\frac{s-1}{2},0)}M), \qquad \vartheta_A{}^{\dot X_1\dots\dot X_{s-1}} \in \Gamma(D^{(\frac{1}{2},0)*}M \otimes D^{(0,\frac{s-1}{2})}M)\,.$$

The Euler-Lagrange equations for the variational principal associated with $L_{\frac{s}{2}}$ are the equations

$$0 = B_s\begin{pmatrix}\varphi^{AA_1\dots A_{s-1}}\\ \chi_{\dot X}{}^{A_1\dots A_{s-1}}\end{pmatrix}, \qquad 0 = B_s\begin{pmatrix}\bar\xi^{AA_1\dots A_{s-1}}\\ \bar\vartheta_{\dot X}{}^{A_1\dots A_{s-1}}\end{pmatrix}$$

(two copies of Buchdahl's equations).

The question, of course, is: What is the physical relevance of the *four* fields $\varphi, \chi, \xi, \vartheta$ and of the Lagrangian density $L_{\frac{s}{2}}$? [Ill93] presents the following answers:

(a) From [Ill93, proposition 4.3] we learn: The case spin $\frac{s}{2} = \frac{1}{2}$ describes a classical Dirac field if $\varphi^A = \vartheta^A$ and $\chi_{\dot{X}} = \xi_{\dot{X}}$. Setting

$$\Psi^{\tilde{A}} := \begin{pmatrix} \varphi^A \\ \chi_{\dot{X}} \end{pmatrix} \in \Gamma(D^D M)\,,$$

the Lagrangian density $L_{\frac{1}{2}}$ can be shown to be equivalent to the well known Lagrangian density of the Dirac field, [Ill93, eq. 4.30]

$$\tilde{L}_{\frac{1}{2}} = \frac{1}{2}(\Psi^+ \overrightarrow{\nabla\!\!\!/}\, \Psi - \Psi^+ \overleftarrow{\nabla\!\!\!/}\, \Psi) + m\, \Psi^+ \Psi\,.$$

(b) From [Ill93, proposition 4.4] we learn: The case of an arbitrary odd s (i. e. half integral spin) describes a general Fermionic field of higher spin. Such a field is given by the four spinor fields $\varphi, \chi, \xi, \vartheta$ but can also be written as a single hybride tensor-spinor field by setting

$$\Psi^{\tilde{A}a_1\ldots a_{s-1}} := \begin{pmatrix} \varphi^{AA_1\ldots A_{s-1}}\epsilon^{\dot{X}_1\ldots\dot{X}_{s-1}} + \vartheta^{A\dot{X}_1\ldots\dot{X}_{s-1}}\epsilon^{A_1\ldots A_{s-1}} \\ \chi_{\dot{X}}{}^{A_1\ldots A_{s-1}}\epsilon^{\dot{X}_1\ldots\dot{X}_{s-1}} + \xi_{\dot{X}}{}^{\dot{X}_1\ldots\dot{X}_{s-1}}\epsilon^{A_1\ldots A_{s-1}} \end{pmatrix}, \tag{9.1}$$

(where we set $\epsilon_{A_1\ldots A_k} := \epsilon_{A_1A_2}\cdot\ldots\cdot\epsilon_{A_{k-1}A_k}$ for even k, and we silently make identifications $\psi^{A\dot{X}} = \sigma_a{}^{A\dot{X}}\psi^a$, cf. remark 4.2.3), and

$$\Psi^{+}{}_{\tilde{A}}{}^{a_1\ldots a_{s-1}} = \begin{pmatrix} \bar{\chi}_A{}^{\dot{X}_1\ldots\dot{X}_{s-1}}\epsilon^{A_1\ldots A_{s-1}} + \bar{\xi}_A{}^{A_1\ldots A_{s-1}}\epsilon^{\dot{X}_1\ldots\dot{X}_{s-1}} \\ \bar{\varphi}^{\dot{X}\dot{X}_1\ldots\dot{X}_{s-1}}\varepsilon^{A_1\ldots A_{s-1}} + \bar{\vartheta}^{\dot{X}A_1\ldots A_{s-1}}\epsilon^{\dot{X}_1\ldots\dot{X}_{s-1}} \end{pmatrix}.$$

Notice that this is a true generalisation of (a) in the sense that (a) is the special case for $s = 1$. For the Lagrangian density $L_{\frac{s}{2}}$ written out using Ψ, cf. [Ill93, eq. 4.49], for the field equations cf. [Ill93, eq. 4.50].

(c) From [Ill93, proposition 4.1] we learn: The case spin $= \frac{s}{2} = 1$ describes a classical Proca field if $\chi_{\dot{X}A} = \vartheta_{A\dot{X}}$. Constructing a tensor field U_a as

$$U_a := \chi_{\dot{X}A}$$

and H_{ab} as [Ill93, eq. 4.3]

$$H_{ab} := \varphi_{AB}\epsilon_{\dot{X}\dot{Y}} + \xi_{\dot{X}\dot{Y}}\epsilon_{AB}\,,$$

the Lagrangian density L_1 is equivalent to [Ill93, eq. 4.1]

$$\tilde{L}_1 = -\frac{1}{2}\bar{H}_{ab}H^{ab} + m^2\bar{U}_aU^a\,.$$

(d) From [Ill93, proposition 4.2] we learn: (c) can be generalised to arbitrary even s (i. e. integral spin) to describe a Bosonic field of higher spin. Such a field again is given by four spinor fields $\varphi, \chi, \xi, \vartheta$ and may equivalently be written as field tensors $H_{a_1 \dots a_n}$ and $U_{ba_3 \dots a_n}$ which are given by [Ill93, eq. 4.15]:

$$H_{a_1 \dots a_n} := \varphi_{A_1 \dots A_n} \epsilon_{\dot{X}_1 \dots \dot{X}_n} + \xi_{\dot{X}_1 \dots \dot{X}_n} \epsilon_{A_1 \dots A_n} \tag{9.2}$$

and

$$U_{ba_3 \dots a_n} := \chi_{\dot{Y} B A_3 \dots A_n} \epsilon_{\dot{X}_3 \dots \dot{X}_n} + \vartheta_{B \dot{Y} \dot{X}_3 \dots \dot{X}_n} \epsilon_{A_3 \dots A_n} \,. \tag{9.3}$$

Illge's Generalised Dirac Current

In [Ill93, theorem 2], Illge considers the minimal coupling of the higher spin wave equations to an electromagnetic field which yields an U(1)-gauge invariance of the Lagrangian. Out of these considerations he introduces a generalised Dirac current associated with a field given by $\varphi, \chi, \xi, \vartheta$:

$$\begin{aligned} j_{A\dot{X}} = ie\big[& k(\varphi_{AA_1 \dots A_{s-1}} \bar{\vartheta}_{\dot{X}}{}^{A_1 \dots A_{s-1}} + \chi_{\dot{X} A_1 \dots A_{s-1}} \bar{\xi}_A{}^{A_1 \dots A_{s-1}}) \\ & - \bar{k}(\bar{\varphi}_{\dot{X}\dot{X}_1 \dots \dot{X}_{s-1}} \vartheta_A{}^{\dot{X}_1 \dots \dot{X}_{s-1}} + \bar{\chi}_{A\dot{X}_1 \dots \dot{X}_{s-1}} \xi_{\dot{X}}{}^{\dot{X}_1 \dots \dot{X}_{s-1}})\big] \,, \end{aligned}$$

where $e \in \mathbb{R}$ (charge) and $k \in \mathbb{C}$ (cf. the reference for details on the constant k). Illge states that for all s, the generalised Dirac current is conserved, i. e. $\nabla_a j^a = 0$.

In [Ill93, eq. 5.11] we find that for the general Fermionic case (s odd) this current may be written by means of Ψ as

$$j^a = e \, 2^{-\frac{s-1}{2}} \cdot \Psi^+_{\tilde{A} a_1 \dots a_{s-1}} \gamma^{a\tilde{A}}{}_{\tilde{B}} \, \Psi^{\tilde{B} a_1 \dots a_{s-1}} \,. \tag{9.4}$$

Notice that for spin $\frac{s}{2} = \frac{1}{2}$ this reduces to the well known special case

$$j^a = e \cdot \Psi^+_{\tilde{A}} \gamma^{a\tilde{A}}_{\tilde{B}} \, \Psi^{\tilde{B}} \,.$$

Quantisation within Illge's Framework

In order to perform a quantisation procedure analogous to the one presented for the Dirac field in section 9.1, we restrict our attention to the Fermionic case (s odd). Then the first step is to transform the expression for the generalised Dirac current vector field, which depends $\mathbb{R}$-quadratically upon *one* field, into an expression which depends sesquilineарly upon two fields.

For the current as given in eq. (9.4), the formal analogy with the spin $\frac{1}{2}$ case of section 9.1 is immediate and motivates the Ansatz

$$j^a(\Psi, \Phi) := e \, 2^{-\frac{s-1}{2}} \cdot \Psi^+_{\tilde{A} a_1 \dots a_{s-1}} \gamma^{a\tilde{A}}{}_{\tilde{B}} \, \Phi^{\tilde{B} a_1 \dots a_{s-1}} \,, \tag{9.5}$$

where Ψ and Φ are two solutions of the variational principal associated with the Lagrangian density $L_{\frac{s}{2}}$, each of which is given by *four* spinor fields $\varphi, \chi, \xi, \vartheta$.

Now let $\Sigma \subseteq M$ be a smooth spacelike Cauchy hypersurface. Then we may define the sesquilinear form

$$b^s_\Sigma(\Psi,\Phi) := \int\limits_\Sigma \iota\mathfrak{n}_{\mathfrak{a}}\, j^a(\Psi,\Phi)\, d\mu_\Sigma\,,$$

where $\iota\colon TM \to TM \otimes \mathbb{C}$ is the canonical inclusion (moreover, notice that $\Phi|_\Sigma$ and $\Psi|_\Sigma$ have compact support).

It turns out that b^s_Σ is Hermitian.

Proof. The proof is an easy but lengthy calculation. Using equations (9.5) and (9.1) we obtain with $\kappa = e\,2^{-\frac{s-1}{2}}$:

$$\begin{aligned}
\kappa^{-1}\cdot j^a(\Psi,\Phi) &= \Psi^+_{\tilde{A}a_1\dots a_{s-1}}\,\gamma^{a\tilde{A}}{}_{\tilde{B}}\,\Psi^{\tilde{B}a_1\dots a_{s-1}}\\
&= \begin{pmatrix} \bar{\chi}_{1\,A\dot{X}_1\dots\dot{X}_{s-1}}\epsilon_{A_1\dots A_{s-1}} + \bar{\xi}_{1\,AA_1\dots A_{s-1}}\epsilon_{\dot{X}_1\dots\dot{X}_{s-1}}\\ \bar{\varphi}_1{}^{\dot{X}}{}_{\dot{X}_1\dots\dot{X}_{s-1}}\varepsilon_{A_1\dots A_{s-1}} + \bar{\vartheta}_1{}^{\dot{X}}{}_{A_1\dots A_{s-1}}\epsilon_{\dot{X}_1\dots\dot{X}_{s-1}}\end{pmatrix}\\
&\quad\cdot\begin{pmatrix} 0 & \sigma^{aA\dot{X}}\\ \sigma^a{}_{\dot{X}A} & 0\end{pmatrix}\begin{pmatrix}\varphi_2{}^{AA_1\dots A_{s-1}}\epsilon^{\dot{X}_1\dots\dot{X}_{s-1}} + \vartheta_2{}^{A\dot{X}_1\dots\dot{X}_{s-1}}\epsilon^{A_1\dots A_{s-1}}\\ \chi_{2\,\dot{X}}{}^{A_1\dots A_{s-1}}\epsilon^{\dot{X}_1\dots\dot{X}_{s-1}} + \xi_{2\,\dot{X}}{}^{\dot{X}_1\dots\dot{X}_{s-1}}\epsilon^{A_1\dots A_{s-1}}\end{pmatrix}\\
&= (\bar{\chi}_{1\,A\dot{X}_1\dots\dot{X}_{s-1}}\epsilon_{A_1\dots A_{s-1}} + \bar{\xi}_{1\,AA_1\dots A_{s-1}}\epsilon_{\dot{X}_1\dots\dot{X}_{s-1}})\\
&\qquad\sigma^{aA\dot{X}}\,(\chi_{2\,\dot{X}}{}^{A_1\dots A_{s-1}}\epsilon^{\dot{X}_1\dots\dot{X}_{s-1}} + \xi_{2\,\dot{X}}{}^{\dot{X}_1\dots\dot{X}_{s-1}}\epsilon^{A_1\dots A_{s-1}})\\
&\quad+ (\bar{\varphi}_1{}^{\dot{X}}{}_{\dot{X}_1\dots\dot{X}_{s-1}}\varepsilon_{A_1\dots A_{s-1}} + \bar{\vartheta}_1{}^{\dot{X}}{}_{A_1\dots A_{s-1}}\epsilon_{\dot{X}_1\dots\dot{X}_{s-1}})\\
&\qquad\sigma^a{}_{\dot{X}A}\,(\varphi_2{}^{AA_1\dots A_{s-1}}\epsilon^{\dot{X}_1\dots\dot{X}_{s-1}} + \vartheta_2{}^{A\dot{X}_1\dots\dot{X}_{s-1}}\epsilon^{A_1\dots A_{s-1}})\\
&= \bar{\chi}_{1\,A\dot{X}_1\dots\dot{X}_{s-1}}\,\sigma^{aA\dot{X}}\,\xi_{2\,\dot{X}}{}^{\dot{X}_1\dots\dot{X}_{s-1}} + \bar{\xi}_{1\,AA_1\dots A_{s-1}}\,\sigma^{aA\dot{X}}\,\chi_{2\,\dot{X}}{}^{A_1\dots A_{s-1}}\\
&\quad+ \bar{\varphi}_1{}^{\dot{X}}{}_{\dot{X}_1\dots\dot{X}_{s-1}}\,\sigma^a{}_{\dot{X}A}\,\vartheta_2{}^{A\dot{X}_1\dots\dot{X}_{s-1}} + \bar{\vartheta}_1{}^{\dot{X}}{}_{A_1\dots A_{s-1}}\,\sigma^a{}_{\dot{X}A}\,\varphi_2{}^{AA_1\dots A_{s-1}}
\end{aligned}$$

where in the last step we took into consideration the spinor fields' symmetry properties. Next, we find:

$$\begin{aligned}
\overline{\kappa^{-1}\cdot j^a(\Psi,\Phi)} &= \chi_{1\,\dot{X}A_1\dots A_{s-1}}\,\sigma^{a\dot{X}A}\,\bar{\xi}_{2\,A}{}^{A_1\dots A_{s-1}} + \xi_{1\,\dot{X}\dot{X}_1\dots\dot{X}_{s-1}}\,\sigma^{a\dot{X}A}\,\bar{\chi}_{2\,A}{}^{\dot{X}_1\dots\dot{X}_{s-1}}\\
&\quad+ \varphi_1{}^A{}_{A_1\dots A_{s-1}}\,\sigma^a{}_{A\dot{X}}\,\bar{\vartheta}_2{}^{\dot{X}A_1\dots X_{s-1}} + \vartheta_1{}^A{}_{\dot{X}_1\dots\dot{X}_{s-1}}\,\sigma^a{}_{A\dot{X}}\,\bar{\varphi}_2{}^{\dot{X}\dot{X}_1\dots\dot{X}_{s-1}}\\
&= \chi_{1\,\dot{X}}{}^{A_1\dots A_{s-1}}\,\sigma^{a\dot{X}A}\,\bar{\xi}_{2\,AA_1\dots A_{s-1}} + \xi_{1\,\dot{X}}{}^{\dot{X}_1\dots\dot{X}_{s-1}}\,\sigma^{a\dot{X}A}\,\bar{\chi}_{2\,A\dot{X}_1\dots\dot{X}_{s-1}}\\
&\quad+ \varphi_1{}^{AA_1\dots A_{s-1}}\,\sigma^a{}_{A\dot{X}}\,\bar{\vartheta}_2{}^{\dot{X}}{}_{A_1\dots\dot{X}_{s-1}} + \vartheta_1{}^{A\dot{X}_1\dots\dot{X}_{s-1}}\,\sigma^a{}_{A\dot{X}}\,\bar{\varphi}_2{}^{\dot{X}}{}_{\dot{X}_1\dots\dot{X}_{s-1}}\\
&= \bar{\chi}_{2\,A\dot{X}_1\dots\dot{X}_{s-1}}\,\sigma^{aA\dot{X}}\,\xi_{1\,\dot{X}}{}^{\dot{X}_1\dots\dot{X}_{s-1}} + \bar{\xi}_{2\,AA_1\dots A_{s-1}}\,\sigma^{aA\dot{X}}\,\chi_{1\,\dot{X}}{}^{A_1\dots A_{s-1}}\\
&\quad+ \bar{\varphi}_2{}^{\dot{X}}{}_{\dot{X}_1\dots\dot{X}_{s-1}}\,\sigma^a{}_{\dot{X}A}\,\vartheta_1{}^{A\dot{X}_1\dots\dot{X}_{s-1}} + \bar{\vartheta}_2{}^{\dot{X}}{}_{A_1\dots A_{s-1}}\,\sigma^a{}_{\dot{X}A}\,\varphi_1{}^{AA_1\dots A_{s-1}}\\
&= \kappa^{-1}\cdot j^a(\Phi,\Psi)\,,
\end{aligned}$$

where for the second equality we used that $s-1$ is even, and for the third equality we kept in mind remark 4.2.4. ∎

So far everything looks promising that we could perform a CAR-algebraic quantisation (at least for s odd) using b^s_Σ as a scalar product. But the crucial point now is positivity, and we find:

9.3.1 Proposition. b^s_Σ is *not* positive definite for $s \geq 3$.

Proof. Let Ψ be given by the four spinor fields $\varphi, \chi, \xi, \vartheta$. From the first big calculation in the preceding proof we learn:

$$\begin{aligned}\kappa^{-1} \cdot j^a(\Psi, \Psi) = \bar{\chi}_{A\dot{X}_1\ldots\dot{X}_{s-1}}\, \sigma^{aA\dot{X}}\, \xi_{\dot{X}}{}^{\dot{X}_1\ldots\dot{X}_{s-1}} + \bar{\xi}_{AA_1\ldots A_{s-1}}\, \sigma^{aA\dot{X}}\, \chi_{\dot{X}}{}^{A_1\ldots A_{s-1}} \\ + \bar{\varphi}^{\dot{X}}{}_{\dot{X}_1\ldots\dot{X}_{s-1}}\, \sigma^a{}_{\dot{X}A}\, \vartheta^{A\dot{X}_1\ldots\dot{X}_{s-1}} + \bar{\vartheta}^{\dot{X}}{}_{A_1\ldots A_{s-1}}\, \sigma^a{}_{\dot{X}A}\, \varphi^{AA_1\ldots A_{s-1}} \,.\end{aligned}$$

Remember that $(\varphi, \chi)^{tr}$ and $(\bar{\xi}, \bar{\vartheta})^{tr}$ are solutions of Buchdahl's generalised Dirac equation. Moreover, notice that if $(\varphi, \chi)^{tr}$ is a solution of Buchdahl's generalised Dirac equation, $(-\varphi, -\chi)^{tr}$ is as well a solution of Buchdahl's generalised Dirac equation. Let Ψ' be the spinor-tensor field given by $-\varphi, -\chi, \xi, \vartheta$ according to equation (9.1). Then we immediately find, using the expression for $j^a(\Psi, \Psi)$ derived above:

$$j^a(\Psi', \Psi') = -j^a(\Psi, \Psi) \,.$$

This (fiberwise) shows that if there exists a solution Ψ such that $b^s_\Sigma(\Psi, \Psi) > 0$, then there is also a solution Ψ' such that $b^s_\Sigma(\Psi', \Psi') < 0$. Thus, b^s_Σ cannot be positive.

Notice how the spin $\frac{1}{2}$ case is not affected by these considerations because for Ψ being called a solution for $s = 1$, we imposed the additional conditions $\varphi = \bar{\vartheta}$ and $\chi = \bar{\xi}$. Therefore, we cannot construct the modified solution Ψ' if $s = 1$ and our argument doesn't work then. □

This proof illustrates that the non-positivity of b^s_Σ in the Ille framework is due to the fact that there is "(too?) much freedom" in that a physical field is described by *four* spinor fields $\varphi, \chi, \xi, \vartheta$ instead of only two: The proof made use of the very fact that such a field consists of two independent parts.

So, one idea of rescuing the situation would be not to consider all solutions $(\varphi, \chi, \xi, \vartheta)$ of Illge's Euler-Lagrange equations but only a subspace. For instance, one could embed $\mathscr{H}^s$ into this space, by setting either $\mathscr{H}^s \ni (\psi, \varphi)^{tr} \mapsto (\psi, \varphi, \bar{\psi}, \bar{\varphi})$ or $\mathscr{H}^s \ni (\psi, \varphi)^{tr} \mapsto (\psi, \varphi, 0, 0)$. However, it can be seen that our sesquilinear form for the first suggestion still is not positive definite and for the second suggestion it is always 0.

All this indicates that despite its advantages (Lagrangian formulation, U(1)-gauge invariance), Illge's framework seems not to be accessible for the CAR-algebraic quantisation strategy outlined in section 9.1.

Table of Relevant Lie Groups

$\mathrm{GL}(n,\mathbb{K})$	general linear group of $\mathbb{K}^n$
definition:	$\mathrm{GL}(n,\mathbb{K}) := \{Q \in \mathrm{Mat}_{n\times n}(\mathbb{K}) \,\vert\, Q \text{ is invertible}\}$
eq. definition:	$\mathrm{GL}(n,\mathbb{K}) = \{Q \in \mathrm{Mat}_{n\times n}(\mathbb{K}) \mid \det(Q) \neq 0\}$
structure:	non-compact $\mathbb{K}$-Lie group. $\mathrm{GL}(n,\mathbb{R})$ is not connected, $\mathrm{GL}(n,\mathbb{C})$ is connected but not simply connected.
dimension:	n^2, as $\mathbb{K}$-Lie group.
Lie algebra:	$\mathfrak{gl}(n,\mathbb{K}) := \mathrm{Mat}_{n\times n}(\mathbb{K})$ with matrix commutator as Lie bracket.
$\mathrm{GL}^+(n)$	general linear group with positiv determinant
definition:	$\mathrm{GL}^+(n) := \{Q \in \mathrm{GL}(n,\mathbb{R}) \mid \det Q > 0\}$
structure:	real Lie group, connected, not simply connected for $n > 1$, simply connected for $n = 1$
dimension:	n^2
Lie algebra:	$\mathfrak{gl}(n,\mathbb{K}) = \mathrm{Mat}_{n\times n}(\mathbb{R})$
$\mathrm{SL}(n,\mathbb{K})$	special linear group, group of unimodular ($\det = 1$) matrices
definition:	$\mathrm{SL}(n,\mathbb{K}) := \{Q \in \mathrm{GL}(n,\mathbb{K}) \mid \det Q = 1\}$
structure:	connected $\mathbb{K}$-Lie group; for $n \geq 2$: simply connected if and only if $\mathbb{K} = \mathbb{C}$, non-compact; $n = 1$: compact and simply connected.
dimension:	$n^2 - 1$ as $\mathbb{K}$-Lie group
Lie algebra:	$\mathfrak{sl}(n,\mathbb{K}) = \{Q \in \mathrm{Mat}_{n\times n}(\mathbb{K}) \,\vert\, \mathrm{tr}(Q) = 0\}$
remark:	normal subgroup of $\mathrm{GL}(n,\mathbb{K})$
$\mathrm{O}(n)$, $\mathrm{O}(n,\mathbb{K})$	orthogonal group
definition:	$\mathrm{O}(n) := \mathrm{O}(n,\mathbb{R})$ $\mathrm{O}(n,\mathbb{K}) := \{Q \in \mathrm{GL}(n,\mathbb{K}) \,\vert\, Q^{tr}Q = QQ^{tr} = \mathbb{1}\}$
eq. definition:	all $Q \in \mathrm{Mat}_{n\times n}(\mathbb{K})$ such that all column vectors are orthogonal with respect to the standard Euclidean metric $x_1y_1 + \ldots + x_ny_n$
eq. definition:	all $Q \in \mathrm{GL}(n,\mathbb{K})$ such that $\forall x,y \in \mathbb{K}^n\colon \langle Qx, Qy\rangle = \langle x, y\rangle$ where $\langle\cdot,\cdot\rangle$ is strictly the *Euclidean* inner product $\sum_i x_iy_i$, even if $\mathbb{K} = \mathbb{C}$
relation:	$Q \in \mathrm{O}(n,\mathbb{K}) \Rightarrow \det Q = \pm 1$
structure:	non-connected $\mathbb{K}$-Lie group, non-compact if $n \geq 2$ and $\mathbb{K} = \mathbb{C}$, compact otherwise
dimension:	$\frac{1}{2}n(n-1)$ ($\mathbb{K}$-dimension)
Lie algebra:	$\mathfrak{so}(n,\mathbb{K})$
remark:	$\mathrm{O}(n)$ is the maximal compact subgroup of $\mathrm{GL}(n,\mathbb{R})$
$\mathrm{SO}(n)$, $\mathrm{SO}(n,\mathbb{K})$	special orthogonal group
definition:	$\mathrm{SO}(n) := \mathrm{SO}(n,\mathbb{R})$ $\mathrm{SO}(n,\mathbb{K}) := \mathrm{O}(n,\mathbb{K}) \cap \mathrm{SL}(n,\mathbb{K})$
eq. definition:	$\mathrm{SO}(n,\mathbb{K}) = \{Q \in \mathrm{O}(n,\mathbb{K}) \mid \det Q = 1\}$
eq. definition:	the connected component of the unity in $\mathrm{O}(n,\mathbb{K})$
remark:	$\mathrm{SO}(n)$ is the maximal compact subgroup of $\mathrm{GL}^+(n,\mathbb{R})$

structure:	Connected $\mathbb{K}$-Lie group, non-compact if $n \geq 2$ and $\mathbb{K} = \mathbb{C}$, compact otherwise, not simply connected if and only if $n \geq 2$
dimension:	$\frac{1}{2}n(n-1)$ ($\mathbb{K}$-dimension)
Lie algebra:	for $\mathbb{K} = \mathbb{R}$: the Lie algebra of anti-symmetric $n \times n$-matrices, i. e. the anti-symmetric $(2,0)$-tensors, or the set of bivectors in the clifford algebra of $\mathbb{R}^n$.
relation:	$\mathfrak{so}(n, \mathbb{C}) = \mathfrak{so}(n, \mathbb{R}) \otimes \mathbb{C}$
univ. coverings:	$\mathrm{SU}(2) \to \mathrm{SO}(3)$, $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$
$\mathrm{O}(p,q)$	orthogonal group, pseudo-orthogonal group
definition:	all $Q \in \mathrm{GL}(p+q, \mathbb{R})$ leaving invariant the quadratic form $x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$
dimension:	$\frac{1}{2}n(n-1)$
$\mathrm{SO}(p,q)$	special orthogonal group
definition:	$\mathrm{SO}(p,q) := \{Q \in \mathrm{O}(p,q) \mid \det(Q) > 0\}$
$\mathrm{O}_t(p,q)$, $\mathrm{O}_s(p,q)$	time-/space-orientation preserving pseudo-orthogonal groups
definition:	Write each $Q \in \mathrm{O}(p,q)$ $$\begin{pmatrix} X & B \\ C & Y \end{pmatrix},$$ where $X \in \mathrm{Mat}_{p\times p}(\mathbb{R})$ and $Y \in \mathrm{Mat}_{q\times q}(\mathbb{R})$. Then we set $\mathrm{O}_t(p,q) := \{Q \in O(p,q) \mid \det(X) > 0\}$ and $\mathrm{O}_s(p,q) := \{Q \in O(p,q) \mid \det(Y) > 0\}$. Cf. [Bau81, pp. 45].
$\mathrm{SO}^+(p,q)$	space- and time-orientation preserving orthogonal group
definition:	$\mathrm{SO}(p,q) := \mathrm{O}_t(p,q) \cap \mathrm{O}_s(p,q)$
$\mathrm{U}(n)$	unitary group
definition:	$\mathrm{U}(n) := \{Q \in \mathrm{GL}(n, \mathbb{C}) \mid Q^\dagger Q = QQ^\dagger = \mathbb{1}\}$ where $Q^\dagger$ denotes the conjugate transpose
eq. definition:	all $Q \in \mathrm{GL}(n, \mathbb{C})$ such that $\forall x, y \in \mathbb{C}^n \colon \langle Qx, Qy\rangle = \langle x, y\rangle$, where $\langle \cdot, \cdot \rangle$ is the standard hermitian inner product $\sum_i x_i \overline{y_i}$.
structure:	compact, connected, not simply connected *real* Lie group (it is not a complex Lie group)
dimension:	n^2
Lie algebra:	$\mathfrak{u}(n) = \{Q \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid Q^\dagger = -Q\}$, all skew-Hermitian complex $n \times n$-matrices
$\mathrm{SU}(n)$	special unitary group
definition:	$\mathrm{SU}(n) := \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$
eq. definition:	$\mathrm{SU}(n) = \{Q \in \mathrm{U}(n) \mid \det Q = 1\}$
structure:	compact, connected, simply connected *real* Lie group (it is not a complex Lie group)
dimension:	$n^2 - 1$ (real dimension)
Lie algebra:	$\mathfrak{su}(n) = \{Q \in \mathrm{Mat}_{n\times n}(\mathbb{C}) \mid \mathrm{tr}(Q) = 0 \wedge Q^\dagger = -Q\}$ (all traceless anti-Hermitian complex $n \times n$-matrices)
remark:	$\mathrm{SU}(2)$ is isomorphic to the unit quaternions and diffeomorphic to S^3

remark:	unlike in case of SO(n) and O(n), SU(n) is not a connection component of U(n). Even the dimensions differ by the dimension of the kernel of the determinant map on U(n).
relation:	There is the following splitting short exact sequence of Lie groups: $$1 \to \mathrm{SU}(n) \to \mathrm{U}(n) \stackrel{\det}{\to} \mathrm{U}(1) \to 1\,.$$
$\mathcal{L}$	(full) Lorentz group
definition:	$\mathcal{L} := \mathrm{O}(1,3)$
structure:	non-compact real Lie group with four connection components.
dimension:	6
Lie algebra:	$\mathfrak{sl}(2,\mathbb{C})$
$\mathcal{L}_+$	proper Lorenz group
definition:	$\mathcal{L}_+ = \mathrm{SO}(1,3)$
eq. definition:	$\mathcal{L}_+ = \{Q \in \mathcal{L} \mid \det Q = 1\}$
remark:	$\mathcal{L}_+$ contains such transformation without space and without time reversal and such transformations with both, space and time reversal.
structure:	non-compact real Lie group with two connected components (subgroup of $\mathcal{L}$).
dimension:	6
$\mathcal{L}^\uparrow$, $\mathrm{O}^+(1,3)$	orthochroneuos Lorentz group
definition:	$\mathcal{L}^\uparrow := \{Q \in \mathcal{L} \mid Q^0_0 \geq 1\}$
alt. notation:	$\mathrm{O}^+(1,3)$, $\mathrm{O}^\uparrow(1,3)$
remark:	leaves invariant the time direction.
relation:	$\mathcal{L} = \mathcal{L}^\uparrow \cup T\mathcal{L}^\uparrow$, where T is the time reversal operator, $x^0 \mapsto -x^0$, $x^i \mapsto x^i$, $i = 1,2,3$.
structure:	non-compact real Lie group with two connected components (subgroup of $\mathcal{L}$).
dimension:	6
$\mathcal{L}^\uparrow_+$, $\mathrm{SO}^+(1,3)$	proper orthochroneuos Lorentz group, restricted Lorentz group
definition:	$\mathcal{L}^\uparrow_+ := \mathcal{L}_+ \cap \mathcal{L}^\uparrow$
alt. notation:	$\mathrm{SO}^+(1,3)$, $\mathrm{SO}^\uparrow(1,3)$
remark:	no time and no space reversals.
relation:	$\mathcal{L}^\uparrow = \mathcal{L}^\uparrow_+ \cup P\mathcal{L}^\uparrow_+$, where P is the space reversal operator, $x^0 \mapsto x^0$, $x^i \mapsto -x^i$, $i = 1,2,3$.
relation:	$\mathcal{L}_+ = \mathcal{L}^\uparrow_+ \cup PT\mathcal{L}^\uparrow_+$
relation:	$\mathcal{L} = \mathcal{L}^\uparrow_+ \cup P\mathcal{L}^\uparrow_+ \cup T\mathcal{L}^\uparrow_+ \cup PT\mathcal{L}^\uparrow_+$
structure:	connected, non-compact, not simply connected real Lie group
dimension:	6
univ. covering:	$\mathrm{SL}(2,\mathbb{C})$ as real Lie group; the covering map $\mathrm{SL}(2,\mathbb{C}) \to \mathcal{L}^\uparrow_+$ restricts to the universal covering map $\mathrm{SU}(2) \to \mathrm{SO}(3)$.
Lie algebra:	$\mathfrak{sl}(2,\mathbb{C}) =: \mathfrak{lo}$

Bibliography

[Bau81] BAUM, Helga: *Spin-Strukturen und Dirac-Operatoren über pseudoriemannschen Mannigfaltigkeiten*. Teubner, 1981

[BD85] BRÖCKER, Theodor ; DIECK, Tammo tom: *Representations of Compact Lie Groups*. 1st edition. Springer GTM, 1985

[BGP] BÄR, Christian ; GINOUX, Nicolas ; PFÄFFLE, Frank: *Wave Equations on Lorentzian Manifolds and Quantization*

[BGV92] BERLINE, N. ; GETZLER, E. ; VERGNE, M.: *Heat Kernels and Dirac Operators*. Springer, 1992

[BR96] BRATTELI, Ola ; ROBINSON, Derek W.: *Operator Algebras and Quantum Statistical Mechanics 2*. 2nd edition. Springer, 1996

[BS03] BERNAL, Antonio N. ; SÁNCHEZ, Miguel: On smooth Cauchy hypersurfaces and Geroch's splitting theorem. In: *Comm. Math. Phys.* 243 (2003), S. 461–470

[BS05] BERNAL, Antonio N. ; SÁNCHEZ, Miguel: Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes. In: *Comm. Math. Phys.* 257 (2005), S. 43–50

[Buc62] BUCHDAHL, H. A.: On the compatibility of relativistic wave equations in Riemann spaces. In: *Nouvo Cimento* 25 (1962), S. 486

[Buc82a] BUCHDAHL, H. A.: On the compatibility of relativistic wave equations in Riemann spaces: II. In: *J. Phys. A: Math. Gen.* 15 (1982), S. 1–5

[Buc82b] BUCHDAHL, H. A.: On the compatibility of relativistic wave equations in Riemann spaces: III. In: *J. Phys. A: Math. Gen.* 15 (1982), S. 1057–1062

[CM00] CARMELI, M. ; MALIN, S.: *Theory of Spinors – An Introduction*. 1st edition. World Scientific, 2000

[Cno02] CNOPS, Jan: *An Introduction to Dirac Operators on Manifolds*. 1st edition. Birkhäuser, 2002

[Dim82] DIMOCK, J.: Dirac Quantum Fields on a Manifold. In: *Trans. AMS* 269 (1982), S. 133–147

[Dir36] DIRAC, P. A. M.: Relativistic Wave Equations. In: *Proc. Roy. Soc. A* 155 (1936), S. 447–459

[FH91] FULTON, William ; HARRIS, Joe: *Representation Theory, A First Course*. 1st. Springer GTM, 1991

[FP39] Fierz, M. ; Pauli, W.: On Relativistic Wave Equations for Particles of Arbitrary Spin in an Electromagnetic Field. In: *Proc. Roy. Soc. A* 173 (1939), Nr. 953, S. 211–232

[Fri97] Friedrich, Thomas: *Dirac-Operatoren in der Riemannschen Geometrie*. 1st. Vieweg, 1997

[FV02] Fewster, Christopher F. ; Verch, Rainer: A Quantum Weak Energy Inequality for Dirac Fields in Curved Spacetime. In: *Comm. Math. Phys.* 225 (2002), S. 331–359

[Ger68] Geroch, Robert: Spinor Structure of Space-Times in General Relativity I. In: *J. Math. Phys.* 9 (1968), Nr. 11, S. 1739–1744

[Ger70a] Geroch, Robert: Domain of Dependence. In: *J. Math. Phys.* 11 (1970), Nr. 2, S. 437–449

[Ger70b] Geroch, Robert: Spinor Structure of Space-Times in General Relativity II. In: *J. Math. Phys.* 11 (1970), Nr. 1, S. 343–348

[GHL04] Gallot, S. ; Hulin, D. ; Lafontaine, J.: *Riemannian Geometry*. Third edition. Springer, 2004

[Goo55] Good, R. H.: Properties of the Dirac Matrices. In: *Rev. Mod. Phys.* 27 (1955), S. 187–211

[Hal03] Hall, Brian C.: *Lie Groups, Lie Algebras, and Representations*. Springer GTM, 2003

[Hum72] Humpfreys, James E.: *Introduction to Lie Algebras and Representation Theory*. 1st. Springer GTM, 1972

[HV00] Hurley, Donald J. ; Vandyck, Michael A.: *Geometry, Spinors and Applications*. 1st. Springer Praxis, 2000

[Ill93] Illge, Reinhard: Massive Fields of Arbitrary Spin in Curved Space-Times. In: *Comm. Math. Phys.* 158 (1993), S. 433–457

[Ill96] Illge, Reinhard: Energy-momentum tensor for massive fields of arbitrary spin. In: *Class. Quantum Grav.* 13 (1996), S. 1499–1507

[IS99] Illge, Reinhard ; Schimming, Rainer: Consistent Field Equations for Higher Spin on Curved Spacetimes. In: *Ann. Phys.* 8 (1999), Nr. 4, S. 319–329

[Lan02] Lang, Serge: *Algebra*. 3rd Edition. Springer GTM, 2002

[Lee03] Lee, John M.: *Introduction to Smooth Manifolds*. Springer, 2003

[LM89] Lawson, H. B. ; Michelsohn, Marie-Louise: *Spin Geometry*. 1st Edition. Princeton University Press, 1989

[Mes58] Messiah, A.: *Quantum Mechanics*. Two Volumes Bound as One. Dover, 1958

[Mil74] Milnor, John W.: *Characteristic Classes*. Princeton Univ. Press., 1974

[Mur90] Murphy, Gerard J.: *C^*-Algebras and Operator Theory*. 1st edutuib. Academic Press, 1990

[O'N83] O'Neill, Barrett: *Semi-Riemannian Geometry*. 1st edition. Academic Press, 1983

[Pau36] Pauli, Wolfgang: Contributions mathématiques à la théorie des matrices de Dirac. In: *Ann. de l'I. H. P.* 6 (1936), S. 109–136

[Roe98] Roe, John: *Elliptic operators, topology and asymptotic methods*. 2nd. Chapman&Hall/CRC, 1998

[Rol03] Rollnik, Horst: *Quantentheorie 2*. 1st. Springer, 2003

[Ros02] Rossmann, Wulf: *Lie Groups, An Introduction Through Linear Groups*. 1st. Oxford University Press, 2002

[Rud] Rudolph, Gerd: Lecture Notes Quantenmechanik 1, Prof. Gerd Rudolph, Universität Leipzig.

[Str02] Straumann, Norbert: *Quantenmechanik: ein Grundkurs über nichtrelativistische Quantentheorie*. 1st. Springer, 2002

[SU01] Sexl, Roman U. ; Urbantke, Helmuth K.: *Relativity, Groups, Particles*. 1st. Springer, 1992/2001

[Wal84] Wald, Robert M.: *General Relativity*. 1st. The University of Chicago Press, 1984

[Wal94] Wald, Robert M.: *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*. 1st. The University of Chicago Press, 1994

[Wün85] Wünsch, Volkmar: Cauchy's Problem and Huygen's Principle for Relativistic Higher Spin Wave Equations in an Arbitrary Curved Space-Time. In: *Gen. Rel. Grav.* 17 (1985), S. 15–38