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Weighted Tree Automata and Quantitative Logics with a Focus on Ambiguity

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Abstract

We define a framework for adding quantitative properties to monadic second order logic for trees and relate various restrictions of this quantitative logic to subclasses of weighted tree automata. These subclasses are defined by the level of ambiguity allowed in the automata. This yields a generalization of the results by Kreutzer and Riveros, who defined an analogous framework to provide quantitative properties for monadic second order logic for words and proved various fragments of that logic to correspond to subclasses of weighted word automata, characterized by ambiguity.

Along the way we also prove that a finitely ambiguous weighted tree automaton can be decomposed into unambiguous ones and define and analyze polynomial ambiguity for tree automata.

Contents

1 Introduction

A *finite automaton* (FA) is an elementary, very simple model for systems with state transitions. As a fundamental operating principle, an automaton reads a given sequence of events and changes its state according to these events. This concept is rather universal and thus has many applications. Some are very basic, like the modeling of a binary adder, a vending machine or an elevator, but automata can also be used as models for compilers, speech and image recognition software or parsers in general.

At times a simple succession of events is not expressive enough to describe a given problem. A structure commonly used as an extension to sequences is that of a *tree*. Here, events are organized in a parent-child relation. Automata operating on trees, called *finite tree automata* (FTA), have uses including the evaluation of search trees employed in various search algorithms or the analysis of syntax trees as part of a compiler.

There may arise situations in which not only the outcome of a computation done by an automaton is of interest, but also how this computation was done. Certain costs generated, time needed or multiplicity being inherent to our problem may be of importance. An example is the implementation of a natural language parser. As human language is not always unambiguous, a given sentence may have more than one meaning. A "good" language parser should be able to find all different meanings of a sentence and count them or, if possible, assign to each of them a weight describing the likelihood that this is the intended meaning in the specific context. This leads to the concepts of *weighted automata* (WA) and *weighted tree automata* (WTA), which to a given input also assign a value or weight. The concept of weighted automata has first been investigated by Schützenberger [18] and a lot of further research on the subject has been done since then, cf. [17, 16, 2, 7].

The notion of ambiguity inherent to natural language processing is interesting in itself, as the sheer fact of having multiple possible computations for certain input may cause the use of a particular automaton to be ineffective. An automaton is said to be *deterministic* if there is no ambiguity in the basic state transitions, i.e. if given the automaton's current state and an input event there is no more than one possible state for the automaton to change into. It seems desirable for an automaton to work in such a way, but it is also restrictive.

While the classification of automata into deterministic ones and nondeterministic ones is the most prominent, it is possible to distinguish finer nuances of nondeterminism, depending on the number of possible different computations for a given input. Regarding ambiguity, the most commonly distinguished classes of automata are deterministic (DFA), unambiguous (*unamb*-FA), finitely ambiguous (*fin*-FA), polynomially ambiguous (*poly*-FA) and exponentially ambiguous or simply nondeterministic (NFA) finite automata.

Ambiguity of finite automata has already been studied numerous times. For example, [21, 19, 1] present criteria for and algorithms to determine the ambiguity of automata and [12] investigates the decidability of the equivalence problem for finitely ambiguous finance automata.

For finite automata without weights it can be shown, using the powerset construction [4, Theorem 1.1.9], that deterministic automata and nondeterministic automata are equally expressive. The same is not true for weighted automata. It is shown in [14] that the inclusions DWA $\subsetneq$ *unamb*-WA $\subsetneq$ *fin*-WA are strict and in [13] it is shown that the inclusion *fin*-WA $\subsetneq$ *poly*-WA is strict.

Logics are an essential tool to accurately describe problems in theoretical science. In computer science an example for this is *model checking*, the exhaustive and automatic checking of whether a given model for a system meets a desired specification. A typical application of this would be to verify whether a given hardware circuit operates as intended or whether a given software program can produce an infinite loop. Formulating the specification as a logic formula and the system model as an abstract structure, the task of model checking boils down to checking whether the structure satisfies the formula. While formulae are often an accurate and uncomplicated way to formulate these problems, the direct analysis of them can result difficult.

As a remedy, automata come into play. While automata are not always the best means to directly formalize a problem, their structural simplicity allows them to be examined more easily than logic formulas. To make use of the advantages of both logics and automata at the same time, a link between them is needed. Such a link has been established by Büchi's and Elgot's fundamental theorems [3, 9], showing the equal expressive power of finite automata and *monadic second order logic* (MSO-logic). This result has since spawned many extensions to other structures. For example, in [20, 5] it is shown that the same holds true for finite tree automata, in [6] a weighted logic is introduced and shown be just as expressive as weighted automata and in [8] a weighted logic for trees does the same for weighted tree automata.

As ambiguity is an interesting property of automata, the question arises whether logics used to describe these automata preserve the aspect of ambiguity in some form. In the unweighted case we cannot expect this, as deterministic and nondeterministic automata are equally expressive. In the case of weighted automata, however, this question has recently been answered positively by Kreutzer and Riveros [15].

As weighted tree automata are a generalization of weighted automata, the same must be true for them to at least some extent. The objective of this work is to generalize [15] to weighted tree automata and investigate in detail how different degrees of ambiguity translate into logic formulae.

2 Preliminaries

The following introductory definitions are taken from [8] in large parts.

2.1 Tree Languages and Tree Automata

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. A *ranked alphabet* is a pair (Γ, rk_Γ) where Γ is a finite set and $rk_\Gamma \colon \Gamma \to \mathbb{N}$. For every $m \geq 0$ we define $\Gamma^{(m)} := rk_\Gamma^{-1}(m)$ as the set of all symbols of rank m. In the sequel we will abuse notation and denote (Γ, rk_Γ) simply by Γ if rk_Γ is known from the context or not of importance. Furthermore the rank $rk(\Gamma)$ of Γ is defined as $\max\{rk_\Gamma(a) \mid a \in \Gamma\}$. The set of (*finite, labeled and ordered*) Γ*-trees*, denoted by T_Γ, is the smallest subset T of $(\Gamma \cup \{(,)\} \cup \{,\})^*$ such that if $a \in \Gamma^{(m)}$ with $m \geq 0$ and $s_1, \ldots, s_m \in T$, then $a(s_1, \ldots, s_m) \in T$. In case $m = 0$, we identify $a()$ with a. Clearly $T_\Gamma = \emptyset$ iff $\Gamma^{(0)} = \emptyset$. Since we are not interested in the case that $T_\Gamma = \emptyset$, we assume that $\Gamma^{(0)} \neq \emptyset$ for every ranked alphabet Γ considered.

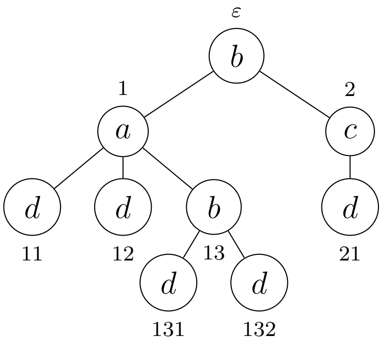
We define the set of *positions in a tree* by means of the mapping $\mathrm{pos} \colon T_\Gamma \to \mathcal{P}(\mathbb{N}^*)$ inductively as follows: (i) if $t \in \Gamma^{(0)}$, then $\mathrm{pos}(t) = \{\varepsilon\}$, and (ii) if $t = a(s_1, \ldots, s_m)$ where $a \in \Gamma^{(m)}$, $m \geq 1$ and $s_1, \ldots, s_m \in T_\Gamma$, then $\mathrm{pos}(t) = \{\varepsilon\} \cup \{iv \mid 1 \leq i \leq m,\ v \in \mathrm{pos}(s_i)\}$. Note that $\mathrm{pos}(t)$ is partially ordered by the prefix relation $\leq_p$ and totally ordered with respect to the lexicographic ordering $\leq_l$. Alluding to the graph structure induced by a tree, we also refer to the elements of $\mathrm{pos}(t)$ as *nodes*, to ε as the *root* of t and to prefix-maximal nodes as *leaves*. Two positions $w_1, w_2 \in \mathrm{pos}(t)$ for which neither $w_1 \leq_p w_2$ nor $w_2 \leq_p w_1$ are called *prefix-independent*.

Now let $t, s \in T_\Gamma$, $w \in \mathrm{pos}(t)$ and $t = a(s_1, \ldots, s_m)$ for some $a \in \Gamma^{(m)}$ with $m \geq 0$ and $s_1, \ldots, s_m \in T_\Gamma$. The *label of t at w* and the *subtree of t at w*, denoted by $t(w)$ and $t|_w$, respectively, are defined inductively as follows: $t(\varepsilon) = a$ and $t|_\varepsilon = t$, and if $w = iv$ and $1 \leq i \leq m$, then $t(w) = s_i(v)$ and $t|_w = s_i|_v$. The *substitution of s into w*, denoted by $t\langle s \to w\rangle$, is defined inductively as $t\langle s \to w\rangle = s$ if $w = \varepsilon$, and if $w = iv$ with $1 \leq i \leq m$ then $t\langle s \to w\rangle = a(s_1, \ldots, s_{i-1}, s_i\langle s \to v\rangle, s_{i+1}, \ldots, s_m)$. To illustrate these concepts we look at an example.

Example 2.1. Assume $\Gamma = \{a, b, c, d\}$ with $rk_\Gamma(a) = 3$, $rk_\Gamma(b) = 2$, $rk_\Gamma(c) = 1$ and $rk_\Gamma(d) = 0$. Then an example tree is:

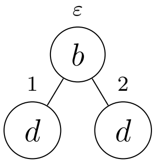
$$t := b(a(d, d, b(d, d)), c(d))$$

$$\text{with } \mathrm{pos}(t) = \{\varepsilon, 1, 11, 12, 13, 131, 132, 2, 21\}.$$



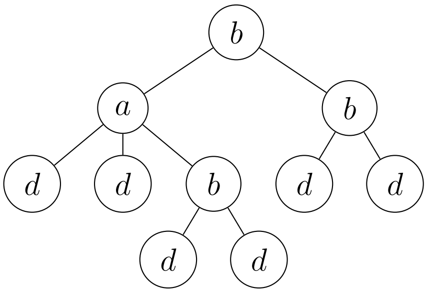
As shown in above diagram, the positions in pos(t) describe the "path" we have to take from the root in order to get to the according node in the graph. The subtree $t|_{13}$ of t at position 13 is:

$$t|_{13} = b(d, d) \text{ with } \mathrm{pos}(t|_{13}) = \{\varepsilon, 1, 2\}.$$



To illustrate substitution, we substitute the subtree $t|_{13}$ into position 2 of t:

$$t\langle t|_{13} \to 2\rangle = b(a(d, d, b(d, d)), b(d, d))$$



Next we recall basic background on bottom-up finite state tree automata, also see [10, 11]. A *bottom-up finite state tree automaton* (for short: BU-FTA) is a tuple $\mathcal{A} = (Q, \Gamma, \delta, F)$ where Q is a finite set (of states), Γ is a ranked alphabet (of input symbols), $\delta \subseteq \bigcup_{m=0}^{rk(\Gamma)} Q^m \times \Gamma^{(m)} \times Q$ (the valid transitions) and $F \subseteq Q$ (the final states). We set $\Delta_{\mathcal{A}} := \bigcup_{m=0}^{rk(\Gamma)} Q^m \times \Gamma^{(m)} \times Q$. A tuple $(\vec{p}, a, q) \in \Delta_{\mathcal{A}}$ is called a *transition* and $(\vec{p}, a, q)$ is called *valid* if $(\vec{p}, a, q) \in \delta$. A BU-FTA is called *deterministic* if for every $m \geq 0$, $a \in \Gamma^{(m)}$ and $\vec{p} \in Q^m$ there exists at most one $q \in Q$ such that $(\vec{p}, a, q) \in \delta$.

Now we define the run semantics of a BU-FTA $\mathcal{A}$. Let $t \in T_\Gamma$. A *quasi-run of $\mathcal{A}$ on t* is a mapping $r\colon \mathrm{pos}(t) \to Q$. If $w \in \mathrm{pos}(t)$ and $t(w) = a \in \Gamma^{(m)}$ for some $m \geq 0$, then

$$\mathbb{t}_r(w) := (r(w1), \ldots, r(wm), a, r(w))$$

is called the *transition with footpoint w* or *transition at w*. A quasi-run r is called *valid* or simply a *run* if for every $w \in \mathrm{pos}(t)$ the transition $\mathbb{t}_r(w)$ is valid and a run is called *accepting* if $r(\varepsilon) \in F$. A run r is also called a *q-run* if $r(\varepsilon) = q$.

Let $\mathrm{Run}_{\mathcal{A}}(t)$, $\mathrm{Run}_{\mathcal{A},q}(t)$, $\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)$ denote the sets of all runs of $\mathcal{A}$ on t, all q-runs of $\mathcal{A}$ on t and all accepting runs of $\mathcal{A}$ on t, respectively.

The *tree language accepted by $\mathcal{A}$* is the set $\mathcal{L}(\mathcal{A}) := \{t \in T_\Gamma \mid \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t) \neq \emptyset\}$. A tree language $L \subseteq T_\Gamma$ is called *(deterministically) recognizable* if there is a (deterministic) bottom-up finite state tree automaton $\mathcal{A}$ such that $L = \mathcal{L}(\mathcal{A})$. By applying the usual power set construction, one obtains that every recognizable tree language is also deterministically recognizable (cf. [20, Theorem 1]). It is well known that the class of recognizable tree languages is closed under the boolean operations (i.e. union, intersection and complement; cf. [20, Theorem 2]).

Next we briefly recall MSO-logic on trees and Büchi's Theorem for trees, namely that MSO-definable tree languages are exactly the recognizable tree languages [20, 5]. Let Γ be a ranked alphabet. The *set* $\mathrm{MSO}(\Gamma)$ *of all formulas of MSO-logic over* Γ is defined as the smallest set F such that

(1) F contains all *atomic formulas* $\mathrm{label}_a(x)$, $\mathrm{edge}_i(x, y)$ and $x \in X$ and

(2) if $\varphi, \psi \in F$, then also $\varphi \vee \psi$, $\varphi \wedge \psi$, $\neg\varphi$, $\exists x.\varphi$, $\exists X.\varphi$, $\forall x.\varphi$, $\forall X.\varphi \in F$,

where $a \in \Gamma$, x, y are first order variables, $1 \leq i \leq rk(\Gamma)$, and X is a second order variable. The set of free variables of φ is denoted by $\text{Free}(\varphi)$.

Let $\mathcal{V}$ be a finite set of first order and second order variables. The ranked alphabet $\Gamma_{\mathcal{V}} = (\Gamma \times \{0,1\}^{\mathcal{V}}, rk)$ is defined by $rk((a,f)) = rk_{\Gamma}(a)$ for every $f \in \{0,1\}^{\mathcal{V}}$. For a symbol $(a,f) \in \Gamma_{\mathcal{V}}$ we denote a by $(a,f)_1$ and f by $(a,f)_2$. A $\Gamma_{\mathcal{V}}$-tree s is *valid* if for every first order variable $x \in \mathcal{V}$, there is exactly one $w \in \text{pos}(s)$ such that $(s(w)_2)(x) = 1$. The subset $T_{\Gamma_{\mathcal{V}}}$ containing all valid trees is denoted by $T^v_{\Gamma_{\mathcal{V}}}$. We put $\Gamma_{\varphi} = \Gamma_{\text{Free}(\varphi)}$.

Every valid $\Gamma_{\mathcal{V}}$-tree s corresponds to a pair (t, ρ) where $t \in T_{\Gamma}$ and ρ is a $(\mathcal{V}, t)$-*assignment*; such an assignment is a function which maps first order variables in $\mathcal{V}$ to elements of $\text{pos}(t)$ and second order variables in $\mathcal{V}$ to subsets of $\text{pos}(t)$. More precisely, we say that s *and* (t, ρ) *correspond to each other* if $\text{pos}(t) = \text{pos}(s)$, t is obtained from s by replacing $s(w)$ by $s(w)_1$ for every $w \in \text{pos}(t)$, and for every first order variable x, second order variable X, and $w \in \text{pos}(s)$, we have that $(s(w)_2)(x) = 1$ iff $\rho(x) = w$, and $(s(w)_2)(X) = 1$ iff $w \in \rho(X)$. In the sequel we will identify a valid $\Gamma_{\mathcal{V}}$-tree with the corresponding pair (t, ρ).

Let s be an arbitrary $\Gamma_{\mathcal{V}}$-tree, x be a first order variable and $w \in \text{pos}(s)$. Then $s[x \to w]$ is the $\Gamma_{\mathcal{V} \cup \{x\}}$-labeled tree obtained from s by putting $(s[x \to w](v)_2)(x) = 1$ iff $v = w$. Similarly, if X is a second order variable and $I \subseteq \text{pos}(s)$, then $s[X \to I]$ is the $\Gamma_{\mathcal{V} \cup \{X\}}$-tree obtained from s by putting $(x[X \to I](v)_2)(X) = 1$ iff $v \in I$. If here $s = (t, \rho)$, we also write $s[x \to w] = (t, \rho[x \to w])$ and $s[X \to I] = (t, \rho[X \to I])$.

Let φ be a formula in $\text{MSO}(\Gamma)$ and $s = (t, \rho)$ be a valid $\Gamma_{\mathcal{V}}$-tree such that $\text{Free}(\varphi) \subseteq \mathcal{V}$. Then the relation "$(t, \rho)$ satisfies φ", denoted by $(t, \rho) \models \varphi$, is defined as usual, i.e.

$$\begin{aligned}
(t,\rho) \models \text{label}_a(x) \quad &:\Leftrightarrow \quad t(\rho(x)) = a \\
(t,\rho) \models \text{edge}_i(x,y) \quad &:\Leftrightarrow \quad \rho(y) = \rho(x)i \\
(t,\rho) \models x \in X \quad &:\Leftrightarrow \quad \rho(x) \in \rho(X) \\
(t,\rho) \models \varphi \wedge \psi \quad &:\Leftrightarrow \quad (t,\rho) \models \varphi \wedge (t,\rho) \models \psi
\end{aligned}$$

$$
\begin{aligned}
(t,\rho) \models \neg\varphi \quad &:\Leftrightarrow \quad \neg((t,\rho) \models \varphi)\\
(t,\rho) \models \exists x.\varphi \quad &:\Leftrightarrow \quad \exists w \in \mathrm{pos}(t) : (t, \rho[x \to w]) \models \varphi\\
(t,\rho) \models \exists X.\varphi \quad &:\Leftrightarrow \quad \exists I \subseteq \mathrm{pos}(t) : (t, \rho[X \to I]) \models \varphi.
\end{aligned}
$$

The remaining cases follow from the above by using double negation. We will usually also assume that $\mathcal{V}$ does not contain any bound variables of φ. We let

$$\mathcal{L}_{\mathcal{V}}(\varphi) := \{(t,\rho) \in T^{v}_{\Gamma_{\mathcal{V}}} \mid (t,\rho) \models \varphi\}$$

and we will simply write $\mathcal{L}(\varphi)$ instead of $\mathcal{L}_{\mathrm{Free}(\varphi)}(\varphi)$. Now we recall the equivalence between recognizable tree languages and MSO-definable tree languages; c.f. [20, Theorems 14 and 17], [5, Theorems 3.7 and 3.9], or [11, Proposition 12.2]: Every tree language $\mathcal{L}_{\mathcal{V}}(\varphi)$ is (deterministically) recognizable over $\Gamma_{\mathcal{V}}$. Conversely, for every recognizable tree language L over Γ, there is an MSO-sentence φ such that $L = \mathcal{L}(\varphi)$. It follows from this, but can also easily be shown directly, that the set $T^{v}_{\Gamma_{\mathcal{V}}}$ is recognizable.

2.2 Tree Series and Weighted Tree Automata

A *semiring* is an algebraic structure $(K, \oplus, \odot, 0, 1)$ with operations sum $\oplus$ and product $\odot$ and constants 0 and 1 such that $(K, \oplus, 0)$ is a commutative monoid and $(K, \odot, 1)$ is a monoid, multiplication distributes over addition, and $k \odot 0 = 0 \odot k = 0$ for every $k \in K$. Whenever the operations and constants of a semiring are clear from the context, we abbreviate $(K, \oplus, \odot, 0, 1)$ by K. The semiring K is *commutative* if $\odot$ is commutative. Important examples of semirings are

- the *boolean semiring* $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$ with disjunction $\vee$ and conjunction $\wedge$

- the *semiring of natural numbers* $(\mathbb{N}, +, \cdot, 0, 1)$, abbreviated by $\mathbb{N}$, with the usual addition and multiplication

- the *tropical semiring* $\mathrm{Trop} := (\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ where the sum and the product operations are min and $+$, respectively, extended to $\mathbb{N} \cup \{\infty\}$ in the usual way.

A *(formal) tree series* is a mapping $S \colon T_\Gamma \to K$. The *support of* S is the set $\operatorname{supp}(S) = \{t \in T_\Gamma \mid S(t) \neq 0\}$. The set of all tree series (over Γ and K) is denoted by $K\langle\langle T_\Gamma \rangle\rangle$. For two tree series $S, T \in K\langle\langle T_\Gamma \rangle\rangle$ and $k \in K$, the sum $S \oplus T$, the *Hadamard product* $S \odot T$, and the product $k \odot S$ are each defined pointwise for every $t \in T_\Gamma$ as follows: $(S \oplus T)(t) = S(t) \oplus T(t)$, $(S \odot T)(t) = S(t) \odot T(t)$, and $(k \odot S)(t) = k \odot S(t)$.

For every $L \subseteq T_\Gamma$, the *characteristic tree series* $\mathbb{1}_L \colon T_\Gamma \to K$ is defined for every $t \in T_\Gamma$ by $\mathbb{1}_L(t) = 1$ if $t \in L$ and $\mathbb{1}_L(t) = 0$ otherwise.

We now introduce weighted bottom-up finite state tree automata and their behavior. Let $(K, \oplus, \odot, 0, 1)$ be a commutative semiring. A *weighted bottom-up finite state tree automaton (short: WTA) over* K *and* Γ is a tuple $\mathcal{A} = (Q, \Gamma, \mu, \alpha)$ where Q is a finite set (of states), Γ is a ranked alphabet (of input symbols), $\mu \colon \bigcup_{m=0}^{rk(\Gamma)} Q^m \times \Gamma^{(m)} \times Q \to K$ (the weight function) and $\alpha \colon Q \to K$ (the function of final weights). We define $\Delta_\mathcal{A} := \bigcup_{m=0}^{rk(\Gamma)} Q^m \times \Gamma^{(m)} \times Q$ as before. A transition $(\vec{p}, a, q) \in \Delta_\mathcal{A}$ is called *valid* if $\mu(\vec{p}, a, q) \neq 0$. A state $q \in Q$ is called *final* if $\alpha(q) \neq 0$. $\mathcal{A}$ is called *deterministic* if for every $m \geq 0$, $a \in \Gamma^{(m)}$ and $\vec{p} \in Q^m$ there exists at most one $q \in Q$ such that $\mu(\vec{p}, a, q) \neq 0$.

Now let $t \in T_\Gamma$. A *quasi-run of* $\mathcal{A}$ *on* t is a mapping $r \colon pos(t) \to Q$. Let $w \in \operatorname{pos}(t)$, then we define $\mathbb{t}_r(w)$ as before and call r *valid* or simply a *run* if for every $w \in \operatorname{pos}(t)$ the transition $\mathbb{t}_r(w)$ is valid. We call a run r *accepting* if $r(\varepsilon)$ is final. If $r(\varepsilon) = q$ then a run r is also called a *q-run*. By $\operatorname{Run}_\mathcal{A}(t)$, $\operatorname{Run}_{\mathcal{A},q}(t)$, $\operatorname{Run}_{\mathcal{A},\mathbb{F}}(t)$ we denote the sets of all runs of $\mathcal{A}$ on t, all q-runs of $\mathcal{A}$ on t and accepting runs of $\mathcal{A}$ on t, respectively.

Let $r \in \operatorname{Run}_\mathcal{A}(t)$, then the *weight of* r is defined by

$$\operatorname{wt}_\mathcal{A}(t, r) := \bigodot_{w \in \operatorname{pos}(t)} \mu(\mathbb{t}_r(w)).$$

The *tree series accepted by* $\mathcal{A}$, denoted by $[\![\mathcal{A}]\!] \in K\langle\langle T_\Gamma \rangle\rangle$, is the tree series defined for every $t \in T_\Gamma$ by

$$[\![\mathcal{A}]\!](t) := \bigoplus_{r \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)} \mathrm{wt}_{\mathcal{A}}(t,r) \odot \alpha(r(\varepsilon)).$$

For $t \in T_\Gamma$, $w \in \mathrm{pos}(t)$, $r \in \mathrm{Run}_{\mathcal{A}}(t)$ and $r_w \in \mathrm{Run}_{\mathcal{A},r(w)}(t|_w)$ we define $r\langle r_w \to w\rangle \in \mathrm{Run}_{\mathcal{A}}(t)$ by

$$r\langle r_w \to w\rangle(\hat{w}) := \begin{cases} r_w(v) & \text{if } w \leq_p \hat{w} \text{ with } \hat{w} = wv \\ r(\hat{w}) & \text{otherwise.} \end{cases}$$

It is easy to see that this is well defined and that given $w \neq \varepsilon$ we have $r\langle r_w \to w\rangle \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)$ iff $r \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)$.

The automaton $\mathcal{A}$ is called trim if

(i) for every $q \in Q$ there exist $t \in T_\Gamma$ and $r \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)$ such that $r(w) = q$ for some $w \in \mathrm{pos}(t)$ and

(ii) for every valid $d \in \Delta_{\mathcal{A}}$ there exist $t \in T_\Gamma$ and $r \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)$ such that $\mathbb{t}_r(w) = d$ for some $w \in \mathrm{pos}(t)$.

The trim part of $\mathcal{A}$ is the automaton obtained by removing all states $q \in Q$ which do not satisfy (i) and setting $\mu(d) = 0$ for all $d \in \Delta_{\mathcal{A}}$ which do not satisfy (ii). This process obviously has no influence on $[\![\mathcal{A}]\!]$.

Now we define the ambiguity of $\mathcal{A}$. We say that $\mathcal{A}$ is

- *deterministic* if for every $m \geq 0$, $a \in \Gamma^{(m)}$ and $\vec{p} \in Q^m$ there exists at most one $q \in Q$ such that $\mu(\vec{p}, a, q) \neq 0$.

- *unambiguous* if $|\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)| \leq 1$ for all $t \in T_\Gamma$.

- *finitely ambiguous* or *m-ambiguous* if $|\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)| \leq m$ for all $t \in T_\Gamma$ and a fixed constant $m \in \mathbb{N}$.

- *(k-)polynomially ambiguous* if $|\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)| \leq p(|\mathrm{pos}(t)|)$ for some polynomial p (of degree k).
- *exponentially ambiguous* in any other case. This naming is justified as an automaton with m states cannot have more than m^n runs on a tree with n nodes

Regarding the definition of polynomial ambiguity, our definition is one of two possibilities to generalize the concept of polynomial ambiguity for automata on words. There, an automaton is said to be polynomially ambiguous if the number of runs on each word is bounded polynomially in the length of the word. In order to have polynomial ambiguity of tree automata be a generalization of the one for word automata, we can either define it with the number of nodes in the tree, as we have done, or with the depth of a tree. Here for $t \in T_\Gamma$ the *depth of* t, denoted by $\mathrm{depth}(t)$, is defined as

$$\mathrm{depth}(t) := 1 + \max_{w \in \mathrm{pos}(t)} |w|.$$

However, the polynomiality using the depth is far more restrictive (see Example 8.1) and, as far as we know, does not possess a characteristic translation into logic formulas as the polynomiality using the number of nodes does.

3 Quantitative Logics for Trees

3.1 Quantitative Monadic Second Order Logic

We now want to define a quantitative logic for trees similar to the ones suggested by Droste and Gastin for words [6] and by Droste and Vogler for trees [8]. We are going to divide our syntax into two levels. The lower, *Boolean level*, will consist of full MSO formulas, without the restrictions on quantors needed in [6, 8]. This level is basically used to access the characteristic functions of regular tree languages. The semiring $(K, \oplus, \odot, 0, 1)$ comes into play in the second level, the *semiring level*, where we will use the operations $\odot$ and $\oplus$ to add and multiply our formulas.

Definition 3.1 (Syntax of QMSO(Γ))**.** The *set* QMSO(Γ) *of all formulas of Quantitative MSO-logic over* K *and* Γ is defined as the smallest set F such that
(1) F contains all $k \in K$ and all $\varphi \in$ MSO(Γ) and
(2) if $\theta, \tau \in F$ then also $\theta \oplus \tau$, $\theta \odot \tau$, $\Sigma x.\theta$, $\Pi x.\theta$, $\Sigma X.\theta \in F$
where x is a first order variable and X is a second order variable.

The operators Σx and ΣX are referred to as first order sum quantification and second order sum quantification, respectively, and Πx is referred to as (first order) product quantification. They are somewhat related to the notions of $\exists X$, $\exists x$, $\forall x$ in [8]. Accordingly, we do not need the operator ΠX (i.e. $\forall X$) for our results. Moreover, the operators Σx, ΣX and Πx also bind the variables x and X, respectively, so that $x \notin$ Free$(\Sigma x.\theta)$ and the same for ΣX and Πx. Other than that the notion of free variables of QMSO(Γ) formulas is the same as for regular MSO(Γ) formulas.

We now come to the semantics of QMSO(Γ). Similar to regular MSO formulas, we take a finite set of first order and second order variables $\mathcal{V}$ and a valid $\Gamma_{\mathcal{V}}$-tree $s = (t, \rho)$. For a formula $\theta \in$ QMSO(Γ) we will define the value $[\![\theta]\!](t, \rho)$ inductively under the assumption that Free$(\theta) \subseteq \mathcal{V}$.

Definition 3.2 (Semantics of QMSO(Γ))**.** Let (t, ρ) be a valid $\Gamma_{\mathcal{V}}$-tree and $\theta \in$

QMSO(Γ) with Free$(\theta) \subseteq \mathcal{V}$. If $\theta = \varphi \in$ MSO(Γ) we set

$$[\![\theta]\!](t,\rho) \;:=\; \begin{cases} 1 & \text{if } (t,\rho) \models \varphi \\ 0 & \text{otherwise.} \end{cases}$$

Otherwise depending on the structure of θ we define

$$\begin{aligned}
[\![k]\!](t,\rho) &:= k \\
[\![\theta_1 \oplus \theta_2]\!](t,\rho) &:= [\![\theta_1]\!](t,\rho) \oplus [\![\theta_2]\!](t,\rho) \\
[\![\theta_1 \odot \theta_2]\!](t,\rho) &:= [\![\theta_1]\!](t,\rho) \odot [\![\theta_2]\!](t,\rho) \\
[\![\Sigma x.\tau]\!](t,\rho) &:= \bigoplus_{w \in \mathrm{pos}(t)} [\![\tau]\!](t,\rho[x \to w]) \\
[\![\Pi x.\tau]\!](t,\rho) &:= \bigodot_{w \in \mathrm{pos}(t)} [\![\tau]\!](t,\rho[x \to w]) \\
[\![\Sigma X.\tau]\!](t,\rho) &:= \bigoplus_{I \subseteq \mathrm{pos}(t)} [\![\tau]\!](t,\rho[X \to I])
\end{aligned}$$

where $k \in K$ and $\theta_1, \theta_2, \tau \in$ QMSO(Γ).

Example 3.3. We consider the semiring $(\mathbb{N}, +, \cdot, 0, 1)$ and the alphabet $\Gamma = \{a, b\}$ where $rk_\Gamma(a) = 2$ and $rk_\Gamma(b) = 0$. We want to construct a formula which for every $t \in T_\Gamma$ outputs the amount of a's taking two b's as child nodes. This is achieved by the following formula.

$$\sum x.\Big(\mathrm{label}_a(x) \wedge \exists y.\big(\mathrm{edge}_1(x,y) \wedge \mathrm{label}_b(y)\big) \wedge \exists y.\big(\mathrm{edge}_2(x,y) \wedge \mathrm{label}_b(y)\big)\Big)$$

Here $\sum$ denotes the addition $+$ in $\mathbb{N}$.

Example 3.4. We consider the field of real numbers $(\mathbb{R}, +, \cdot, 0, 1)$ and the alphabet $\Gamma = \{a, b, c\}$ where $rk_\Gamma(a) = 2$, $rk_\Gamma(b) = 0$ and $rk_\Gamma(c) = 1$.

Assume that given a tree $t \in T_\Gamma$ we want to travel down the tree from the root to a leaf. When we are at a node labeled c there is only one way to go down, when we are at a node labeled b we are finished. In the case of being at a node labeled a we have two possible choices and decide a direction randomly with probability $\frac{1}{2}$ for each possibility.

Given this setting we want to construct a formula which outputs for a given tree t the expected number of c's visited when proceeding as described above. Recall that given a probability measure $\mathbb{P}$ on a finite set Ω the expected value of a random variable $f\colon \Omega \to \mathbb{R}$ is given by

$$\mathbb{E}(f) = \sum_{\omega \in \Omega} f(\omega)\mathbb{P}(\omega).$$

A set of positions $I \subseteq \mathrm{pos}(t)$ visited when traveling down a tree is characterized by (1) containing a leaf, (2) the parent node of every node in I is also in I and (3) no node has two child nodes in I. Such a set is also called a *branch* of the tree. For a second order variable X we therefore define the formula

$$\begin{aligned}\mathrm{branch}(X) := \exists x.\Big(&x \in X \wedge \forall y.\neg(\mathrm{edge}_1(x,y) \vee \mathrm{edge}_2(x,y))\Big)\\ &\wedge \forall x.\forall y.\Big(x \in X \wedge (\mathrm{edge}_1(y,x) \vee \mathrm{edge}_2(y,x))\Big) \to y \in X\\ &\wedge \forall x.\forall y.\forall z.(\mathrm{edge}_1(x,y) \wedge \mathrm{edge}_2(x,z)) \to \neg(y \in X \wedge z \in X).\end{aligned}$$

The probability of a branch I to be traveled is given by $\mathbb{P}(I) = (\frac{1}{2})^{n(a)}$ where $n(a)$ is the amount of nodes labeled a in I. The function value $f(I)$ of I is then the amount of nodes in I labeled c. Applying the formula for the expected value we can hence define our formula as

$$\begin{aligned}&\sum X.\Big(\mathrm{branch}(X) \cdot \sum x.(x \in X \wedge \mathrm{label}_c(x))\\ &\cdot \prod x.(\frac{1}{2} \cdot x \in X \wedge \mathrm{label}_a(x) + \neg(x \in X \wedge \mathrm{label}_a(x)))\Big)\end{aligned}$$

where $\sum$ and $\prod$ are the addition $+$ and multiplication $\cdot$ in $\mathbb{R}$, respectively.

We want to define another operator $(\cdot)^{\rightarrow}$ which, as we will see later, does not increase the expressiveness of QMSO(Γ), but yields a nice characterization for deterministic WTA. For $t \in T_\Gamma$ and $\theta \in \mathrm{QMSO}(\Gamma)$ with $\mathrm{Free}(\theta) = \emptyset$ we define

$$[\![\theta^{\rightarrow}]\!](t) \quad := \quad \bigodot_{w \in \mathrm{pos}(t)} [\![\theta]\!](t|_w),$$

that is, we multiply over the weights of all subtrees of t.

3.2 Fragments of QMSO(Γ)

As done in [15] we want to study various fragments of $\mathrm{QMSO}(\Gamma)$ by restricting the use of certain quantors. For any subset $\mathrm{Op} \subseteq \{\oplus, \odot, \Sigma_x, \Pi_x, \Sigma_X, ^{\rightarrow}\}$ of operators in the semiring level we denote by $\mathrm{QMSO}_\Gamma(\mathrm{Op})$ the restriction of $\mathrm{QMSO}(\Gamma)$ to the operators in Op. For example, $\mathrm{QMSO}_\Gamma(\Sigma_X, \Sigma_x, \Pi_x, \oplus, \odot)$ denotes the full logic. We will simply write $\mathrm{QMSO}(\mathrm{Op})$ if it is clear from context what the underlying alphabet Γ is.

This, however, is often not restrictive enough. For instance, in [8] the $\forall x.\varphi$-operator had to be restricted to be only used on so-called *recognizable step functions* φ, otherwise the resulting formulas could define tree series not definable by a weighted automaton. It is expectable that we will need a similar mechanism. We therefore also consider fragments obtained by restricting the alternation and nesting of the semiring level operators, using an intuitive *quantifier pattern.* Such a pattern is a word over $\{\Sigma_X^n, \Sigma_x^n, \Pi_x^n \mid n \in \mathbb{N}_0 \cup \{\infty\}\}$, where the index $(\cdot)^n$ specifies the (maximum) number of nested quantifiers in a block. For example, the fragment $\mathrm{QMSO}(\Sigma_X^\infty \Sigma_x^\infty \Pi_x^1, \oplus, \odot)$ contains all $\mathrm{QMSO}(\Gamma)$ formulas with any number of second order sum quantifiers followed by any number of first order sum quantifiers followed by at most one non-nested product quantification. As we often do not distinguish between first order and second order sum quantification, we denote by $\Sigma_{X,x}^n$ the quantifier pattern allowing the use of n nested sum quantifiers of any type. We write Σ_x short for Σ_x^∞ and the same for Σ_X and Π_x.

The reason $(\cdot)^{\rightarrow}$ was left out in the quantifier patterns is that for the whole paper we will assume this operator to not be nested when it occurs.

Finally we want to define a restriction on the operators $\oplus$ and $\odot$. Given an operator $\star \in \{\oplus, \odot\}$ and any set Op of operators in the semiring level, we define the fragment QMSO(Op, $\star_b$) as the set of all formulas $\theta \in$ QMSO(Op, $\star$) such that for all subformulas $\tau_1 \star \tau_2$ in θ we have that $\tau_1, \tau_2 \in$ QMSO($\oplus, \odot$), i.e. τ_1 and τ_2 contain no quantifier of any kind. The b stems from the notion that we restrict the operators to a "base level" of the semiring level.

As usual, we say that a tree series $S \in K\langle\langle T_\Gamma \rangle\rangle$ is definable by a QMSO(Op) formula over K and Γ if there exists a formula $\theta \in$ QMSO(Op) such that $S(t) = [\![\theta]\!](t)$ for all $t \in T_\Gamma$.

4 General Weighted Tree Automata and the Fragment $\mathrm{QMSO}(\Sigma^{\infty}_{X,x}\Pi^{1}_{x}, \oplus, \odot)$

The first fragment we have a look at is $\mathrm{QMSO}(\Sigma^{\infty}_{X,x}\Pi^{1}_{x}, \oplus, \odot)$ which, as we will prove in this section, describes exactly the tree series that can also be defined by weighted bottom-up finite state tree automata. This result is in fact not new, as it has already been shown in [8], but we will prove it using our framework nevertheless. All proofs of this section, with the exception of Proposition 4.7, are straight-forward adaptations of the ones used in Section 4 of [15] or [8] for the corresponding statements. The main theorem we want to prove is the following.

Theorem 4.1. *Let* $(K, \oplus, \odot, 0, 1)$ *be a commutative semiring and* (Γ, rk_Γ) *a ranked alphabet. A tree series* $S \in K\langle\langle T_\Gamma\rangle\rangle$ *is definable by a weighted bottom-up finite state tree automaton over* K *and* Γ *if, and only if,* S *is definable by a formula in* $\mathrm{QMSO}_\Gamma(\Sigma^{\infty}_{X,x}\Pi^{1}_{x}, \oplus, \odot)$.

Proof. ($\Rightarrow$) Let $\mathcal{A} = (Q, \Gamma, \mu, \alpha)$ be WTA that defines the tree series $S \in K\langle\langle T_\Gamma\rangle\rangle$. We will define a formula $\theta_{\mathcal{A}} \in \mathrm{QMSO}_\Gamma(\Sigma^{\infty}_{X,x}\Pi^{1}_{x}, \oplus, \odot)$ such that $[\![\mathcal{A}]\!](t) = [\![\theta_{\mathcal{A}}]\!](t)$ for all $t \in T_\Gamma$. The idea is to do the following:

1. Guess a run using second-order sum quantification.
2. Check whether this run is valid using an MSO-formula.
3. Aggregate the cost of this run using the Π-operator.

For a first-order variable x and second-order variables $X_1, \dots, X_n$ we define the abbreviations

$$\mathrm{root}(x) := \forall y. \left(\bigwedge_{i=1}^{rk(\Gamma)} \neg \mathrm{edge}_i(y, x) \right)$$

$$\mathrm{partition}(X_1, \dots, X_n) := \forall x. \bigvee_{i=1}^{n} \left(x \in X_i \wedge \bigwedge_{j \neq i} \neg(x \in X_j) \right).$$

Both formulas are obviously MSO-formulas. The formula $\mathrm{root}(x)$ is true iff x is the root of the given tree and $\mathrm{partition}(X_1, \dots, X_n)$ is true iff $\{X_1, \dots, X_n\}$ forms a partition of set of positions in the given tree.

Now let $D := \{(\vec{q}, a, q) \in \Delta_{\mathcal{A}} \mid \mu(\vec{q}, a, q) \neq 0\}$ be the set of all valid transitions and $D_F := \{(\vec{q}, a, q) \in D \mid \alpha(q) \neq 0\}$ be the set of all valid final transitions. Furthermore, for $(\vec{q}, a, q) \in D$ let $X_{(\vec{q},a,q)}$ be a second order variable and for $n := |D|$ let $v\colon \{X_{(\vec{q},a,q)} \mid (\vec{q}, a, q) \in D\} \to \{1, \ldots, n\}$ be a bijection. We write $X_{(\vec{q},a,q)}$ for $X_{v((\vec{q},a,q))}$ and $\bar{X}$ for $(X_1, \ldots, X_n)$. Using this notation we define the formula $\mathrm{matched}(\bar{X})$ to check whether the guessed partition is well matched.

$$\mathrm{matched}(\bar{X}) := \bigwedge_{(\vec{q},a,q)\in D} \forall x. \Big((x \in X_{(\vec{q},a,q)}) \to \mathrm{label}_a(x)\Big) \wedge \tag{1}$$

$$\bigwedge_{\substack{(\vec{q},a,q)\in D\\ \vec{q}=(q_1,\ldots,q_m)}} \forall x. \left((x \in X_{(\vec{q},a,q)}) \to \exists y_1 \ldots \exists y_m. \left((\bigwedge_{i=1}^{m} \mathrm{edge}_i(x, y_i)) \wedge \bigvee_{\substack{(\vec{p}_1,a_1,q_1)\in D\\ \cdots\\ (\vec{p}_m,a_m,q_m)\in D}} (\bigwedge_{i=1}^{m} (y_i \in X_{\vec{p}_i,a_i,q_i})) \right) \right) \tag{2}$$

Part (1) verifies that the labeling of the run is consistent with the letters in the tree and Part (2) verifies that the transitions used are well matched. With this in hand we define the MSO-formula $\mathrm{valid}_{\mathcal{A}}(\bar{X})$ that checks if $\bar{X}$ encodes a valid run of $\mathcal{A}$.

$$\mathrm{valid}_{\mathcal{A}}(\bar{X}) :=$$
$$\mathrm{partition}(\bar{X}) \wedge \mathrm{matched}(\bar{X}) \wedge \exists x. \left(\mathrm{root}(x) \wedge \bigvee_{(\vec{q},a,q)\in D_F} (x \in X_{(\vec{q},a,q)}) \right)$$

That is, we verify that $\bar{X}$ is a partition, that this partition is well matched and that the state at the root is a final state.

Next we define the formulas that aggregate the cost of the transitions and the weight of the final state.

$$\text{transition}(x, \bar{X}) := \bigoplus_{(\vec{q},a,q)\in D} \Big((x \in X_{(\vec{q},a,q)}) \odot \mu(\vec{q}, a, q)\Big)$$

$$\text{final}(\bar{X}) := \bigoplus_{(\vec{q},a,q)\in D_F} \Big((\exists x.\text{root}(x) \wedge x \in X_{(\vec{q},a,q)}) \odot \alpha(q)\Big)$$

With this in hand we can define the formula $\theta_{\mathcal{A}}$ as

$$\theta_{\mathcal{A}} := \Sigma\bar{X}\Big(\text{valid}_{\mathcal{A}}(\bar{X}) \odot (\Pi x.\text{transition}(x, \bar{X})) \odot \text{final}(\bar{X})\Big)$$

and it is easy to see that $\theta_{\mathcal{A}}$ and $\mathcal{A}$ define the same tree series.

($\Leftarrow$) We prove this direction by structural induction. Given a formula $\theta \in \text{QMSO}_\Gamma(\Sigma^\infty_{X,x}\Pi^1_x, \oplus, \odot)$ we show how to construct an automaton $\mathcal{A}_\theta$ such that θ and $\mathcal{A}_\theta$ define the same tree series. The first step is a simple application of Büchi's Theorem.

Proposition 4.2. *For every formula $\varphi \in \text{MSO}(\Gamma)$and every finite set of first and second order variables $\mathcal{V} \supseteq \text{Free}(\varphi)$ there is a deterministic WTA $\mathcal{A}_\varphi$ such that for every tree $t \in T_\Gamma$ and every $(\mathcal{V}, t)$-assignment ρ we have*

$$[\![\varphi]\!](t, \rho) = [\![\mathcal{A}_\varphi]\!](t, \rho).$$

Proof. Take $\varphi \in \text{MSO}(\Gamma)$ and $\mathcal{V} \subseteq \text{Free}(\varphi)$. By Büchi's Theorem there is a BU-FTA $\mathcal{A} = (Q, \Gamma_{\mathcal{V}}, \delta, F)$ such that

$$(t, \rho) \models \varphi \text{ iff } (t, \rho) \in \mathcal{L}(\mathcal{A}).$$

As every recognizable tree language is also deterministically recognizable (see Section 2.1), we can assume without loss of generality that $\mathcal{A}$ is deterministic.

We define the automaton $\mathcal{A}_\varphi = (Q, \Gamma_{\mathcal{V}}, \mu, \alpha)$ for $d \in \Delta_{\mathcal{A}}$ and $q \in Q$ via

$$\mu(d) := \begin{cases} 1 & \text{if } d \in \delta \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha(q) := \begin{cases} 1 & \text{if } q \in F \\ 0 & \text{otherwise.} \end{cases}$$

For every $(t, \rho) \in \mathcal{L}(\mathcal{A})$ there is now exactly one run of $\mathcal{A}_\varphi$ on t and the weight of this run is 1, all $(t, \rho) \notin \mathcal{L}(\mathcal{A})$ do not possess any run in $\mathcal{A}_\varphi$. $\mathcal{A}_\varphi$ is obviously also deterministic, as $\mathcal{A}$ was, and therefore $\mathcal{A}_\varphi$ satisfies $[\![\varphi]\!](t, \rho) = [\![\mathcal{A}_\varphi]\!](t, \rho)$ for every tree $t \in T_\Gamma$ and $(\mathcal{V}, t)$-assignment ρ. □

We now come to the semiring level, or more precisely to the fragment $\mathrm{QMSO}_\Gamma(\oplus, \odot)$. We prove this special case, because the automata constructed for this fragment can be chosen deterministic, as we will see.

Proposition 4.3. *For every formula $\theta \in \mathrm{QMSO}_\Gamma(\oplus, \odot)$ and every finite set of first and second order variables $\mathcal{V} \supseteq \mathrm{Free}(\theta)$ there is a deterministic WTA $\mathcal{A}_\theta$ such that for every tree $t \in T_\Gamma$ and every $(\mathcal{V}, t)$-assignment ρ we have*

$$[\![\theta]\!](t, \rho) = [\![\mathcal{A}_\theta]\!](t, \rho).$$

Proof. Let $\theta \in \mathrm{QMSO}_\Gamma(\oplus, \odot)$ and $\mathcal{V} \supseteq \mathrm{Free}(\theta)$. The formula θ consists of MSO formulas and semiring elements separated by $\oplus$, $\odot$ and parentheses. We take these MSO formulas and semiring elements and by distributing multiplication over addition and using the commutativity of $\odot$ rewrite θ into the form

$$\theta = \bigoplus_{i=1}^{n} \left(k_i \odot \bigodot_{j=1}^{n_i} \theta_i^j \right)$$

where $n \in \mathbb{N}$, $n_i \in \mathbb{N}$ and $k_i \in K$ for $i \in \{1, \ldots, n\}$ and $\theta_i^j \in \mathrm{MSO}(\Gamma)$ for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n_i\}$. For $t \in T_\Gamma$ and a $(\mathcal{V}, t)$-assignment ρ we see that $[\![\bigodot_{j=1}^{n_i} \theta_i^j]\!](t, \rho) = 1$ iff $(t, \rho) \models \theta_i^j$ for all $j \in \{1, \ldots, n_i\}$ and 0 otherwise, so

$[\![\bigodot_{j=1}^{n_i} \theta_i^j]\!] = [\![\bigwedge_{j=1}^{n_i} \theta_i^j]\!]$. Hence, by Büchi's Theorem we can find a deterministic BU-FTA $\mathcal{A}_i = (Q_i, \Gamma_{\mathcal{V}}, \delta_i, F_i)$ such that for $\theta_i := \bigwedge_{j=1}^{n_i} \theta_i^j$ we have

$$(t, \rho) \models \theta_i \text{ iff } (t, \rho) \in \mathcal{L}(\mathcal{A}_i)$$

for every $t \in T_\Gamma$ and $(\mathcal{V}, \rho)$-assignment ρ. Now consider the WTA $\mathcal{A}_\theta = (Q, \Gamma_{\mathcal{V}}, \mu, \alpha)$ where

$$Q := \bigtimes_{i=1}^{n} (Q_i \cup \{\emptyset\})$$
$$\alpha(q^1, \ldots, q^n) := \bigoplus_{\substack{i=1\\ q^i \in F_i}}^{n} k_i$$

$$\mu((q_1^1, \ldots, q_1^n), \ldots, (q_m^1, \ldots, q_m^n), a, (q^1, \ldots, q^n)) :=$$
$$\begin{cases} 1 & \text{if } \forall i \in \{1, \ldots, n\} : \Big((q_1^i, \ldots, q_m^i, a, q^i) \in \delta_i \\ & \vee (q^i = \emptyset \wedge \neg \exists q \in Q_i : ((q_1^i, \ldots, q_m^i, a, q) \in \delta_i) \Big) \\ 0 & \text{otherwise.} \end{cases}$$

The automaton $\mathcal{A}_\theta$ runs all automata $\mathcal{A}_i$ in parallel. Coordinate $i \in \{1, \ldots, n\}$ behaves exactly like automaton $\mathcal{A}_i$, except for the fact that if for some $m \in \{1, \ldots, rk(\Gamma)\}$, $a \in \Gamma^{(m)}$ and given $q^1, \ldots, q^m \in Q_i$ there is no transition $(q_1, \ldots, q_m, a, q) \in \delta_i$, then, and only then, we can switch into state $\emptyset$, enabling the other coordinates to continue their runs, but making sure that the run of this coordinate i will not influence the weight of the whole run. It is easy to see that $\mathcal{A}_\theta$ is deterministic and that it defines the same tree series as θ. □

Next we turn to the proof for the fragment $\text{QMSO}_\Gamma(\Pi_x^1, \oplus, \odot)$. More precisely, the next proof shows the recognizability of the fragment $\text{QMSO}_\Gamma(\Pi_x^1, \oplus_b, \odot_b)$.

Proposition 4.4. *For every formula* $\theta \in \text{QMSO}_\Gamma(\oplus, \odot)$ *and every finite set of* *first and second order variables* $\mathcal{V} \supseteq \mathrm{Free}(\Pi x.\theta)$ *there is an unambiguous WTA* $\mathcal{A}_\Pi$ *such that for every tree* $t \in T_\Gamma$ *and every* $(\mathcal{V}, t)$*-assignment* ρ *we have*

$$[\![\Pi x.\theta]\!](t, \rho) = [\![\mathcal{A}_\Pi]\!](t, \rho).$$

Proof. Let $\mathcal{A} = (Q, \Gamma_{\mathcal{V}\cup\{x\}}, \mu, \alpha)$ be the deterministic WTA of Proposition 4.3 that defines the tree series $[\![\theta]\!]$. Recall that the weights of all transitions in $\mathcal{A}$ are either 0 or 1. Assuming that $x \notin \mathcal{V}$, there is an obvious bijection between the sets $\Gamma_{\mathcal{V}\cup\{x\}}$ and $\Gamma_\mathcal{V} \times \{0, 1\}$. We abuse notation and identify $\Gamma_{\mathcal{V}\cup\{x\}} = \Gamma_\mathcal{V} \times \{0, 1\}$ in the sequel. Without loss of generality we assume $\mathcal{A}$ to be complete, that is, for all $\vec{q} \in Q^m$ and $a \in \Gamma_{\mathcal{V}\cup\{x\}}$ there is some $q \in Q$ with $\mu(\vec{q}, a, q) = 1$. We can enforce this by simply adding a dummy state if $\mathcal{A}$ does not already have this property. We define the automaton $\mathcal{A}_\Pi = (Q \times \mathcal{P}(Q \times Q), \Gamma_\mathcal{V}, \mu_\Pi, \alpha_\Pi)$ with the help of some abbreviations: given $q_f \in Q$, $m \in \{1, \dots, rk(\Gamma)\}$, $a \in \Gamma^{(m)}_\mathcal{V}$, $\vec{q} = (q_1, \dots, q_m) \in Q^m$ and $\vec{R} = (R_1, \dots, R_m) \in (\mathcal{P}(Q \times Q))^m$ we set

$$\begin{aligned} f_1(a, \vec{q}, \vec{R}) &:= \{(p, q_f') \in Q \times Q \mid \exists i \in \{1, \dots, m\}\ \exists (p_i, q_f') \in R_i : \\ &\qquad \mu(q_1, \dots, q_{i-1}, p_i, q_{i+1}, \dots, q_m, (a, 0), p) = 1\} \\ f_2(a, \vec{q}, q_f) &:= \{(p, q_f) \mid p \in Q \text{ and } \mu(q_1, \dots, q_m, (a, 1), p) = 1\} \end{aligned}$$

$$\alpha_\Pi(q, R) := \begin{cases} 1 & \text{if } R \subseteq \{(p, p) \mid p \in Q\} \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_\Pi((q_1, R_1), \dots, (q_m, R_m), a, (q_0, R_0)) :=$$
$$\begin{cases} k & \text{if for some } q_f \in Q \text{ we have } k = \alpha(q_f),\ \mu(q_1, \dots, q_m, (a, 0), q_0) = 1 \text{ and} \\ & R_0 = f_1(a, \vec{q}, \vec{R}) \cup f_2(a, \vec{q}, q_f) \text{ where } \vec{q} = (q_1, \dots, q_m) \text{ and } \vec{R} = (R_1, \dots, R_m) \\ 0 & \text{otherwise.} \end{cases}$$

In the first coordinate the automaton executes $\mathcal{A}$ on (t, ρ) as if it had not yet read x. Then at each position $w \in \mathrm{pos}(t)$ we guess one state q_f. It is the state we guess the automaton $\mathcal{A}$ would produce at the root if we ran $\mathcal{A}$ on $(t, \rho[x \to w])$. We therefore set the weight of this transition as the final weight $\alpha(q_f)$ of q_f. The second coordinate is then used to check whether we guessed correctly. We add the singleton $f_2(a, \vec{q}, q_f)$ with $a = (t, \rho)(w)$ into the set of the second coordinate. The pair (p, q_f) inside this singleton consists of the state p the automaton $\mathcal{A}$ would have at position w if we ran it on $(t, \rho[x \to w])$ and the state q_f, which we save to check correctness later. We then use the second coordinate to see what the automaton $\mathcal{A}$ will do with the state p, which is exactly what the function f_1 does. Assuming for some $i \in \mathbb{N}$ we have $(p_i, q'_f) \in R_i$, we interpret this as the fact that we guessed q'_f as one final state earlier and the automaton $\mathcal{A}$ would now be in state p_i having done so. Then $f_1(a, \vec{q}, \vec{R})$ will contain the pair (p, q'_f) where p is the state $\mathcal{A}$ would then change into under this assumption. If we guessed correctly, all those pairs in the second coordinate should be of the form (q'_f, q'_f) at the root, which is exactly the condition we impose on states to be final.

We show that for every $t \in T_\Gamma$ and $(\mathcal{V}, t)$-assignment ρ such that for every $w \in \mathrm{pos}(t)$ we have $[\![\theta]\!](t, \rho[x \to w]) \neq 0$ there is exactly one run $r \in \mathrm{Run}_{\mathcal{A}_\Pi, \mathbb{F}}(t, \rho)$ and this run satisfies $\mu_\Pi(\mathbb{t}_r(w)) = [\![\theta]\!](t, \rho[x \to w])$ for all $w \in \mathrm{pos}(t)$. In particular, $\mathcal{A}_\Pi$ is unambiguous.

Assume we have $r \in \mathrm{Run}_{\mathcal{A}_\Pi, \mathbb{F}}(t, \rho)$, $w \in \mathrm{pos}(t)$ and $t(w) \in \Gamma^{(m)}$ for some $m \in \mathbb{N}$. Then given $(q_i, R_i) := r(wi)$ for $i \in \{1, \ldots, m\}$ we know that for some $q_f \in Q$ we must have $r(w) = (q_0, f_1((t, \rho)(w), \vec{q}, \vec{R}) \cup f_2((t, \rho)(w), \vec{q}, q_f))$ where $\vec{q} = (q_1, \ldots, q_m)$, $\vec{R} = (R_1, \ldots, R_m)$ and q_0 is uniquely determined due to $\mathcal{A}$ being deterministic. This $q_f =: q_f(r, w)$ hence is characteristic for r at w, such that two runs r_1 and r_2 are equal iff $q_f(r_1, w) = q_f(r_2, w)$ for all $w \in \mathrm{pos}(t)$. Furthermore, we see that $\mu_\Pi(\mathbb{t}_r(w)) = \alpha(q_f(r, w))$ for all $w \in \mathrm{pos}(t)$.

Now, assuming that for all $w \in \mathrm{pos}(t)$ we have $[\![\theta]\!](t, \rho[x \to w]) \neq 0$, we find exactly one run $r' \in \mathrm{Run}_{\mathcal{A}, \mathbb{F}}(t, \rho[x \to w])$ and for this run we have $\alpha(r'(\varepsilon)) = [\![\theta]\!](t, \rho[x \to w])$ by construction of $\mathcal{A}$. We set $q_f[x \to w] := r'(\varepsilon)$ as the final state of this unique run. If we can show that for every run $r \in \mathrm{Run}_{\mathcal{A}_\Pi, \mathbb{F}}(t, \rho)$ we have $q_f(r, w) = q_f[x \to w]$, we can easily infer from the observations just made, that r has to be uniquely determined and that $\mathrm{wt}((t, \rho), r) = [\![\Pi x.\theta]\!](t, \rho)$.

Take $r \in \mathrm{Run}_{\mathcal{A}_\Pi,\mathbb{F}}(t, \rho)$ and let $(q_\varepsilon, R_\varepsilon) := r(\varepsilon)$ be the state at the root. We show that for all $w \in \mathrm{pos}(t)$ we have $(q_f[x \to w], q_f(r, w)) \in R_\varepsilon$ and due to $(q_\varepsilon, R_\varepsilon)$ being final, we must then have $q[x \to w] = q_f(r, w)$. Fix $w \in \mathrm{pos}(t)$, $t(w) \in \Gamma^{(m)}$ for $m \in \mathbb{N}$, $(q_i, R_i) := r(wi)$ for $i \in \{1, \dots, m\}$, $(q_0, R_0) := r(w)$ and take $r' \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t, \rho[x \to w])$. It is easy to see that the projection of r on the first coordinate is the quasi-run in $\mathcal{A}$ we would get if we did not assign any position of t to x. Hence, we have $q_i = r'(wi)$ for all $i \in \{1, \dots, m\}$ so by definition of $\mathcal{A}_\Pi$ we have $(r'(w), q_f(r, w)) \in R_0$ via f_2. Now assume that for some position $w' \in \mathrm{pos}(t)$ and $j \in \mathbb{N}$ we have $w'j \leq_p w$. We write $(q', R') := r(w')$ and $(q'_j, R'_j) := r(w'j)$. If we have $(r'(w'j), p) \in R'_j$ for some $p \in Q$, then we also have $(r'(w'), p) \in R'$ via f_1. Therefore, we must have $(r'(\varepsilon), q_f(r, w)) = (q_f[x \to w], q_f(r, w)) \in R_\varepsilon$, which was to show.

It is also easy to see now that when defining a quasi-run r by choosing $q_f(r, w)$ as $q_f[x \to w]$ for all $w \in \mathrm{pos}(t)$ we obtain a (valid) run on (t, ρ), so we have shown that $\mathcal{A}_\Pi$ defines the same tree series as $\Pi x.\theta$. Also note that from what we have shown, it easily follows that if for some $w \in \mathrm{pos}(t)$ we have $[\![\theta]\!](t, \rho[x \to w]) = 0$, then there exists no (valid) run of $\mathcal{A}_\Pi$ on (t, ρ), such that $\mathcal{A}_\Pi$ indeed is unambiguous. □

We now come to the proof for the operators $\oplus$ and $\odot$.

Proposition 4.5. *Let $\tau_1, \tau_2 \in \mathrm{QMSO}(\Gamma)$ and let $\mathcal{V} \supseteq \mathrm{Free}(\tau_1) \cup \mathrm{Free}(\tau_2)$ be a finite set of first and second order variables such that there are WTA $\mathcal{A}_1$ and $\mathcal{A}_2$ over $\Gamma_\mathcal{V}$ defining the same tree series as τ_1 and τ_2, respectively. Then there is a WTA $\mathcal{A}_\oplus$ such that for every tree $t \in T_\Gamma$ and every $(\mathcal{V}, t)$-assignment ρ we have*

$$[\![\tau_1 \oplus \tau_2]\!](t, \rho) = [\![\mathcal{A}_\oplus]\!](t, \rho).$$

Proof. Let $\mathcal{A}_1 = (Q_1, \Gamma_\mathcal{V}, \mu_1, \alpha_1)$ and $\mathcal{A}_2 = (Q_2, \Gamma_\mathcal{V}, \mu_2, \alpha_2)$ be automata defining the formulas τ_1 and τ_2. Without loss of generality we assume $Q_1 \cap Q_2 = \emptyset$, then we define the automaton $\mathcal{A}_\oplus = (Q_1 \cup Q_2, \Gamma_\mathcal{V}, \mu_\oplus, \alpha_\oplus)$ for $d \in \Delta_{\mathcal{A}_\oplus}$ and

$q \in Q_1 \cup Q_2$ as

$$\mu_\oplus(d) := \begin{cases} \mu_1(d) & \text{if } d \in \Delta_{\mathcal{A}_1} \\ \mu_2(d) & \text{if } d \in \Delta_{\mathcal{A}_2} \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha_\oplus(q) := \begin{cases} \alpha_1(q) & \text{if } q \in Q_1 \\ \alpha_2(q) & \text{if } q \in Q_2 \end{cases}$$

and it is easy to see that $\mathcal{A}_\oplus$ defines the same tree series as $\tau_1 \oplus \tau_2$, as for each valid $\Gamma_\mathcal{V}$-tree t the set of runs of $\mathcal{A}_\oplus$ on t is the union of runs of $\mathcal{A}_1$ and $\mathcal{A}_2$ on t. □

Proposition 4.6. *Let $\tau_1, \tau_2 \in \mathrm{QMSO}(\Gamma)$ and let $\mathcal{V} \supseteq \mathrm{Free}(\tau_1) \cup \mathrm{Free}(\tau_2)$ be a finite set of first and second order variables such that there are WTA $\mathcal{A}_1$ and $\mathcal{A}_2$ over $\Gamma_\mathcal{V}$ defining the same tree series as τ_1 and τ_2, respectively. Then there is a WTA $\mathcal{A}_\odot$ such that for every tree $t \in T_\Gamma$ and every $(\mathcal{V}, t)$-assignment ρ we have*

$$[\![\tau_1 \odot \tau_2]\!](t, \rho) = [\![\mathcal{A}_\odot]\!](t, \rho).$$

Proof. Let $\mathcal{A}_1 = (Q_1, \Gamma_\mathcal{V}, \mu_1, \alpha_1)$ and $\mathcal{A}_2 = (Q_2, \Gamma_\mathcal{V}, \mu_2, \alpha_2)$ be automata defining the formulas τ_1 and τ_2. We define the automaton $\mathcal{A}_\odot = (Q_1 \times Q_2, \Gamma_\mathcal{V}, \mu_\odot, \alpha_\odot)$ for $((p_1^1, p_1^2), \ldots, (p_m^1, p_m^2), a, (q^1, q^2)) \in \Delta_{\mathcal{A}_\odot}$ and $(q_1, q_2) \in Q_1 \times Q_2$ as

$$\mu_\odot((p_1^1, p_1^2), \ldots, (p_m^1, p_m^2), a, (q^1, q^2)) := \mu_1(p_1^1, \ldots, p_m^1, a, q^1) \odot \mu_2(p_1^2, \ldots, p_m^2, a, q^2)$$
$$\alpha_\odot(q) := \alpha_1(q_1) \odot \alpha_2(q_2)$$

and it is easy to see that $\mathcal{A}_\odot$ defines the same tree series as $\tau_1 \odot \tau_2$, as for each valid $\Gamma_\mathcal{V}$-tree t the set of runs of $\mathcal{A}_\oplus$ on t are the pairs of runs of $\mathcal{A}_1$ and $\mathcal{A}_2$ on t and the weight of each such run is the product of the weights of the runs in each coordinate, since K is commutative. □

Now only the translations of the first and second order sum operators into automata are left. We turn to the second order sum operator first.

Proposition 4.7. *Let $\theta \in \mathrm{QMSO}(\Gamma)$ and let $\mathcal{V} \supseteq \mathrm{Free}(\Sigma X.\theta)$ be a finite set of first and second order variables, where X is a second order variable, such that there is a WTA $\mathcal{A}$ over $\Gamma_{\mathcal{V}\cup\{X\}}$ defining the same tree series as θ. Then there is a WTA $\mathcal{A}_X$ such that for every tree $t \in T_\Gamma$ and every $(\mathcal{V},t)$-assignment ρ we have*

$$[\![\Sigma X.\theta]\!](t,\rho) = [\![\mathcal{A}_X]\!](t,\rho).$$

Proof. Let $\mathcal{A} = (Q, \Gamma_{\mathcal{V}\cup\{X\}}, \mu, \alpha)$ be a WTA defining the same tree series as θ. Again we assume $X \notin \mathcal{V}$ and write $\Gamma_{\mathcal{V}\cup\{X\}}$ as $\Gamma_{\mathcal{V}} \times \{0,1\}$. We define the automaton $\mathcal{A}_X = (Q\times\{0,1\}, \Gamma_{\mathcal{V}}, \mu_X, \alpha_X)$ for $\left(\binom{p_1}{k_1},\dots,\binom{p_m}{k_m},a,\binom{q}{k}\right) \in \Delta_{\mathcal{A}_X}$ as

$$\mu_X\left(\binom{p_1}{k_1},\dots,\binom{p_m}{k_m},a,\binom{q}{k}\right) := \mu(p_1,\dots,p_m,(a,k),q)$$
$$\alpha_X\binom{q}{k} := \alpha(q)$$

To show that $\mathcal{A}_X$ defines the same tree series as $\Sigma X.\theta$, take $t \in T_\Gamma$ and a $(\mathcal{V},t)$-assignment ρ. First note that

$$\bigoplus_{r\in \mathrm{Run}_{\mathcal{A}_X,\mathbb{F}}(t,\rho)} \mathrm{wt}_{\mathcal{A}_X}((t,\rho),r)\odot\alpha(r(\varepsilon)) = \bigoplus_{r\in(Q\times\{0,1\})^{\mathrm{pos}(t)}} \mathrm{wt}_{\mathcal{A}_X}((t,\rho),r)\odot\alpha(r(\varepsilon))$$

where $(Q\times\{0,1\})^{\mathrm{pos}(t)}$ is the set of all functions $\mathrm{pos}(t)\to Q\times\{0,1\}$, i.e. the set of all quasi-runs, as quasi-runs that are not runs will not influence the sum. For $w\in\mathrm{pos}(t)$ set $a(w) := (t,\rho)(w)$ and $m(w)\in\mathbb{N}$ such that $a(w)\in\Gamma_{\mathcal{V}}^{(m(w))}$, i.e. $a(w)$ is the label at w of the $\Gamma_{\mathcal{V}}$-tree corresponding to the pair (t,ρ). Furthermore, let $\{0,1\}^{\mathrm{pos}(t)}$ be the set of mappings $\mathrm{pos}(t)\to\{0,1\}$ and for $\sigma\in\{0,1\}^{\mathrm{pos}(t)}$ set $I(\sigma) = \{w\in\mathrm{pos}(t)\mid\sigma(w)=1\}$, then we have

$$\begin{aligned}
&[\![\mathcal{A}_X]\!](t,\rho)\\
&= \bigoplus_{r\in(Q\times\{0,1\})^{\mathrm{pos}(t)}} \mathrm{wt}_{\mathcal{A}_X}((t,\rho),r)\odot\alpha_X(r(\varepsilon))\\
&= \bigoplus_{r\in(Q\times\{0,1\})^{\mathrm{pos}(t)}} \alpha_X(r(\varepsilon))\odot\bigodot_{w\in\mathrm{pos}(t)}\mu_X(\mathbb{t}_r(w))
\end{aligned}$$

$$
\begin{aligned}
&= \bigoplus_{\sigma \in \{0,1\}^{\mathrm{pos}(t)}} \bigoplus_{r \in Q^{\mathrm{pos}(t)}} \alpha(r(\varepsilon)) \odot \bigodot_{w \in \mathrm{pos}(t)} \mu(r(w1), \ldots, r(wm(w)), (a(w), \sigma(w)), r(w)) \\
&= \bigoplus_{\sigma \in \{0,1\}^{\mathrm{pos}(t)}} \bigoplus_{r \in Q^{\mathrm{pos}(t)}} \alpha(r(\varepsilon)) \odot \mathrm{wt}_{\mathcal{A}}((t, \rho[X \to I(\sigma)]), r) \\
&= \bigoplus_{\sigma \in \{0,1\}^{\mathrm{pos}(t)}} [\![\mathcal{A}]\!](t, \rho[X \to I(\sigma)]) \\
&= \bigoplus_{I \subseteq \mathrm{pos}(t)} [\![\theta]\!](t, \rho[X \to I]) \\
&= [\![\Sigma X.\theta]\!](t, \rho)
\end{aligned}
$$

which is what we wanted to show. □

Now lastly we come to the first order sum quantifier. Its construction is similar to the one of the second order sum quantifier.

Proposition 4.8. *Let $\theta \in \mathrm{QMSO}(\Gamma)$ and let $\mathcal{V} \supseteq \mathrm{Free}(\Sigma x.\theta)$ be a finite set of first and second order variables, where x is a first order variable, such that there is a WTA $\mathcal{A}$ over $\Gamma_{\mathcal{V} \cup \{x\}}$ defining the same tree series as θ. Then there is a WTA $\mathcal{A}_x$ such that for every tree $t \in T_\Gamma$ and every $(\mathcal{V}, t)$-assignment ρ we have*

$$[\![\Sigma x.\theta]\!](t, \rho) = [\![\mathcal{A}_x]\!](t, \rho).$$

Proof. Let $\mathcal{A} = (Q, \Gamma_{\mathcal{V} \cup \{x\}}, \mu, \alpha)$ be a WTA defining the same tree series as θ. Again we assume $x \notin \mathcal{V}$ and write $\Gamma_{\mathcal{V} \cup \{x\}}$ as $\Gamma_{\mathcal{V}} \times \{0,1\}$. We define the automaton $\mathcal{A}_x = (Q \times \{0,1\}, \Gamma_{\mathcal{V}}, \mu_x, \alpha_x)$ for $\left(\binom{p_1}{k_1}, \ldots, \binom{p_m}{k_m}, a, \binom{q}{k}\right) \in \Delta_{\mathcal{A}_x}$ as:

$$
\mu_x\left(\binom{p_1}{k_1}, \ldots, \binom{p_m}{k_m}, a, \binom{q}{k}\right) :=
$$

$$
\begin{cases}
\mu(p_1, \ldots, p_m, (a,0), q) & \text{if } k = 0 \wedge k_1 = \ldots = k_m = 0 \\
\mu(p_1, \ldots, p_m, (a,1), q) & \text{if } k = 1 \wedge k_1 = \ldots = k_m = 0 \\
\mu(p_1, \ldots, p_m, (a,0), q) & \text{if } k = 1 \wedge \exists! i \in \{1, \ldots, m\} : k_i = 1 \\
0 & \text{otherwise}
\end{cases}
$$

$$\alpha_x\left(\begin{smallmatrix} q \\ k \end{smallmatrix}\right) := \begin{cases} \alpha(q) & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

That is, from a bottom-up perspective we can run $\mathcal{A}_x$ on a tree $t \in T_{\Gamma_\mathcal{V}}$ as if x had not been read yet, then guess randomly a position for x and from then on only allow transitions that behave as if x was not set on these positions. Finally we require x to be guessed at at least one position, as only states whose second coordinate is 1 are final. For every $w \in \mathrm{pos}(t)$ every run of $\mathcal{A}$ on $t[x \to w]$ corresponds to exactly one run of $\mathcal{A}_x$ on t, given by setting the second coordinate of this run to 1 on all positions $v \leq_p w$ and to 0 otherwise. The weights of these runs are the same, so it is clear that $\mathcal{A}_x$ defines the same tree series as $\Sigma x.\theta$ does. □

All of the above proofs show that for all formulas $\theta \in \mathrm{QMSO}_\Gamma(\Sigma^{\infty}_{X,x}\Pi^{1}_{x}, \oplus, \odot)$, the tree series $[\![\theta]\!]$ is also definable by a WTA $\mathcal{A}$ and we conclude the proof of Theorem 4.1. □

5 Determinism and the Fragment $\mathrm{QMSO}(^{\rightarrow}, \oplus_b, \odot)$

We now come to the tree series definable by deterministic WTA. The idea and proof of this section is a straight-forward adaptation of the idea and proof used in Theorem 5.1 of [15], where the theorem is proven for deterministic weighted automata on words.

Theorem 5.1. *Let $(K, \oplus, \odot, 0, 1)$ be a commutative semiring and (Γ, rk_Γ) a ranked alphabet. A tree series $S \in K\langle\langle T_\Gamma\rangle\rangle$ is definable by a deterministic weighted bottom-up finite state tree automaton over K and Γ if, and only if, S is definable by a formula in $\mathrm{QMSO}_\Gamma(^{\rightarrow}, \oplus_b, \odot)$.*

Proof. ($\Rightarrow$) Let $\mathcal{A} = (Q, \Gamma, \mu, \alpha)$ be a deterministic WTA such that $[\![\mathcal{A}]\!] = S$. We show how to construct a formula $\theta_\mathcal{A} \in \mathrm{QMSO}_\Gamma(^{\rightarrow}, \oplus_b, \odot)$ such that $[\![\theta_\mathcal{A}]\!] = [\![\mathcal{A}]\!]$. The characteristic feature of deterministic WTA we will use is that, given a tree $t \in T_\Gamma$, we know that there is at most one (not necessarily accepting) run of $\mathcal{A}$ on t. Therefore, to know which state a run on t has at a position $w \in \mathrm{pos}(t)$, it suffices to know the subtree $t|_w$ of t at w and look at the run of $\mathcal{A}$ on this subtree.

As we did in the proof of Theorem 4.1, we define D as the set of valid transitions in $\Delta_\mathcal{A}$ and let $\bar{X} = (X_1, \ldots, X_n)$ be an enumeration of the set $\{X_{(\vec{q},a,q)} \mid (\vec{q}, a, q) \in D\}$. We use a formula similar to $\mathrm{valid}_\mathcal{A}(\bar{X})$, but this time do not check whether the state at the root is final or not. We reuse the formulas $\mathrm{partition}(\bar{X})$ and $\mathrm{matched}(\bar{X})$ and define

$$\text{det-valid}_\mathcal{A}(\bar{X}) := \mathrm{partition}(\bar{X}) \wedge \mathrm{matched}(\bar{X})$$

$$\text{det-transition} := \bigoplus_{(\vec{q},a,q)\in D} \big(\exists \bar{X}.\text{det-valid}_\mathcal{A}(\bar{X}) \wedge \big(\exists x.\mathrm{root}(x) \wedge x \in X_{(\vec{q},a,q)}\big)\big)$$
$$\odot\ \mu(\vec{q}, a, q)$$

$$\text{det-final} := \bigoplus_{q \in Q} \left(\exists \bar{X}.\text{det-valid}_{\mathcal{A}}(\bar{X}) \wedge \left(\exists x.\text{root}(x) \wedge \bigvee_{(\vec{q},a,q) \in D} x \in X_{(\vec{q},a,q)} \right)\right)$$
$$\odot\, \alpha(q)$$

and with this we can define $\theta_{\mathcal{A}}$ as

$$\theta_{\mathcal{A}} := (\text{det-transition})^{\rightarrow} \odot \text{det-final}.$$

Clearly, for $t \in T_\Gamma$, $w \in \text{pos}(t)$ and the unique run $r \in \text{Run}_{\mathcal{A},\mathbb{F}}(t)$ we have

$$[\![\text{det-transition}]\!](t|_w) = \mu(\mathfrak{t}_r(w))$$

so we have

$$\begin{aligned} [\![(\text{det-transition})^{\rightarrow} \odot \text{det-final}]\!](t) &= \text{wt}_{\mathcal{A}}(t,r) \odot \alpha(r(\varepsilon)) \\ &= [\![\mathcal{A}]\!](t) \end{aligned}$$

such that indeed $[\![\theta_{\mathcal{A}}]\!] = [\![\mathcal{A}]\!]$.

($\Leftarrow$) For $\tau \in \text{QMSO}_\Gamma(^{\rightarrow}, \oplus_b, \odot)$ we show how to construct a deterministic WTA $\mathcal{A}_\tau$ such that $[\![\mathcal{A}_\tau]\!] = [\![\tau]\!]$. In Proposition 4.3 we have shown that for every formula in $\text{QMSO}_\Gamma(\oplus, \odot)$ we can find a deterministic WTA defining the same tree series as this formula and the construction in the proof of Proposition 4.6 for $\odot$ preserves determinism. Therefore the only thing left to show is that given a formula $\theta \in \text{QMSO}_\Gamma(\oplus, \odot)$ we can construct a deterministic WTA $\mathcal{A}_{\rightarrow}$ such that $[\![\mathcal{A}_{\rightarrow}]\!] = [\![(\theta)^{\rightarrow}]\!]$.

Let $\mathcal{A} = (Q, \Gamma, \mu, \alpha)$ be the deterministic automaton we can find by Proposition 4.3 such that $[\![\mathcal{A}]\!] = [\![\theta]\!]$. Recall that the weights of all transitions in $\mathcal{A}$ are either 1 or 0. We define the automaton $\mathcal{A}_{\rightarrow} = (Q, \Gamma, \mu_{\rightarrow}, \alpha_{\rightarrow})$ for

$(p_1, \ldots, p_m, a, q) \in \Delta_{\mathcal{A}_\rightarrow}$ as

$$\mu_\rightarrow(p_1, \ldots, p_m, a, q) := \begin{cases} \alpha(q) & \text{if } \mu(p_1, \ldots, p_m, a, q) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha_\rightarrow(q) := 1.$$

For $t \in T_\Gamma$, $w \in \mathrm{pos}(t)$, $r \in \mathrm{Run}_{\mathcal{A}_\rightarrow, \mathbb{F}}(t)$ and $r_w \in \mathrm{Run}_{\mathcal{A}, \mathbb{F}}(t|_w)$ we clearly have

$$\begin{aligned} \mu(\mathfrak{t}_r(w)) &= \alpha(r_w(\varepsilon)) \\ &= [\![\theta]\!](t|_w) \end{aligned}$$

and as $\mathcal{A}_\rightarrow$ is deterministic due to the fact that the valid transitions in $\Delta_{\mathcal{A}_\rightarrow}$ form a subset of the valid transitions in $\Delta_\mathcal{A}$, we have

$$\begin{aligned} [\![\mathcal{A}_\rightarrow]\!](t) &= \bigodot_{w \in \mathrm{pos}(t)} [\![\theta]\!](t|_w) \\ &= [\![(\theta)^\rightarrow]\!](t) \end{aligned}$$

which is what we wanted to show. □

6 Unambiguity and the Fragment $\text{QMSO}(\Pi^1_x, \oplus_b, \odot_b)$

We now come to the tree series definable by unambiguous WTA. The idea and proof of this section is a straight-forward adaptation of the idea and proof used in Theorem 5.2 of [15], where the theorem is proven for unambiguous weighted automata on words.

Theorem 6.1. *Let* $(K, \oplus, \odot, 0, 1)$ *be a commutative semiring and* (Γ, rk_Γ) *a ranked alphabet. A tree series* $S \in K\langle\langle T_\Gamma \rangle\rangle$ *is definable by an unambiguous weighted bottom-up finite state tree automaton over* K *and* Γ *if, and only if,* S *is definable by a formula in* $\text{QMSO}_\Gamma(\Pi^1_x, \oplus_b, \odot_b)$.

Proof. ($\Rightarrow$) Let $\mathcal{A} = (Q, \Gamma, \mu, \alpha)$ be an unambiguous WTA such that $[\![\mathcal{A}]\!] = S$. We show how to construct a formula $\theta_\mathcal{A} \in \text{QMSO}_\Gamma(\Pi^1_x, \oplus_b, \odot_b)$ such that $[\![\theta_\mathcal{A}]\!] = [\![\mathcal{A}]\!]$. As for every tree $t \in T_\Gamma$ there is at most one run of $\mathcal{A}$ on t, we don't have to sum over all runs. Instead, we know that if we guess a run using the existential quantifier, this will always produce the same unique run.

As we did in the proof of Theorem 4.1, we define D as the set of valid transitions in $\Delta_\mathcal{A}$ and let $\bar{X} = (X_1, \ldots, X_n)$ be an enumeration of the set $\{X_{(\vec{q},a,q)} \mid (\vec{q}, a, q) \in D\}$. We also reuse the formula $\text{valid}_\mathcal{A}(\bar{X})$ and define

$$\text{unamb-transition}(x) := \bigoplus_{(\vec{q},a,q)\in D} \big(\exists \bar{X}.\text{valid}_\mathcal{A}(\bar{X}) \wedge x \in X_{(\vec{q},a,q)}\big) \odot \mu(\vec{q}, a, q)$$

$$\text{unamb-final} := \bigoplus_{(\vec{q},a,q)\in D} \big(\exists \bar{X}.\text{valid}_\mathcal{A}(\bar{X}) \wedge \exists x.\text{root}(x) \wedge x \in X_{(\vec{q},a,q)}\big) \odot \alpha(q)$$

In the first formula we guess the unique run, if it exists, using the existential operator and take the weight at position x. In the second formula we find the final weight of the state at the root of this run. It is easy to see that by defining the formula $\theta'_\mathcal{A}$ as

$$\theta'_\mathcal{A} := (\Pi x.\text{unamb-transition}(x)) \odot \text{unamb-final}$$

we have $[\![\mathcal{A}]\!] = [\![\theta'_{\mathcal{A}}]\!]$. However, this formula is not in $\mathrm{QMSO}_\Gamma(\Pi_x^1, \oplus_b, \odot_b)$ yet. We therefore consider the following: for $\tau_1, \tau_2 \in \mathrm{QMSO}_\Gamma(\oplus, \odot)$, the formula

$$(\Pi x.\tau_1) \odot \tau_2$$

can also be written as

$$\Pi x. \left((\tau_1 \odot \tau_2 \odot \mathrm{root}(x)) \oplus (\tau_1 \odot \neg\mathrm{root}(x))\right)$$

after relabeling x in τ_2, if it is used in the formula. Of course, the commutativity of $\odot$ is crucial here. Latter formula obviously is in $\mathrm{QMSO}_\Gamma(\Pi_x^1, \oplus_b, \odot_b)$ and as $\theta'_{\mathcal{A}}$ has above form, we obtain $\theta_{\mathcal{A}}$ as needed by applying this idea to $\theta'_{\mathcal{A}}$.

($\Leftarrow$) This direction has been proven in Proposition 4.4. □

7 Finite Ambiguity and the Fragment $\mathrm{QMSO}(\Pi_x^1, \oplus, \odot_b)$

We now come to the tree series definable by finitely ambiguous WTA. The main idea here is the same as in [15, Theorem 5.3], just that we have to prove an equivalent version of the Lemma A.7 in [15] for the case of trees.

Theorem 7.1. *Let $(K, \oplus, \odot, 0, 1)$ be a commutative semiring and (Γ, rk_Γ) a ranked alphabet. A tree series $S \in K\langle\langle T_\Gamma \rangle\rangle$ is definable by a finitely ambiguous weighted bottom-up finite state tree automaton over K and Γ if, and only if, S is definable by a formula in $\mathrm{QMSO}_\Gamma(\Pi_x^1, \oplus, \odot_b)$.*

Proof. ($\Leftarrow$) Take $\theta \in \mathrm{QMSO}_\Gamma(\Pi_x^1, \oplus, \odot_b)$. Obviously θ is a finite sum of formulas in $\mathrm{QMSO}_\Gamma(\Pi_x^1, \oplus_b, \odot_b)$, that is

$$\theta = \bigoplus_{i=1}^{n} \theta_i$$

for some $n \in \mathbb{N}$ and $\theta_1, \ldots, \theta_n \in \mathrm{QMSO}_\Gamma(\Pi_x^1, \oplus_b, \odot_b)$. By Theorem 6.1 there are unambiguous WTA $\mathcal{A}_1, \ldots, \mathcal{A}_n$ such that $[\![\mathcal{A}_i]\!] = [\![\theta_i]\!]$ for all $i \in \{1, \ldots, n\}$ and by the proof of Proposition 4.5 the automaton we can construct for the sum $\theta = \bigoplus_{i=1}^n \theta_i$ is then n-ambiguous.

($\Rightarrow$) For this direction we use Lemma 7.2 below, which we yet have to prove. Given the lemma, for a finitely ambiguous WTA $\mathcal{A}$ we can find $n \in \mathbb{N}$ and unambiguous WTA $\mathcal{A}_1, \ldots, \mathcal{A}_n$ such that for all $t \in T_\Gamma$ we have

$$[\![\mathcal{A}]\!](t) = \bigoplus_{i=1}^{n} [\![\mathcal{A}_i]\!](t).$$

By Theorem 6.1 for $i \in \{1, \ldots, n\}$ we can find $\theta_i \in \mathrm{QMSO}_\Gamma(\Pi_x^1, \oplus_b, \odot_b)$ such that $[\![\mathcal{A}_i]\!] = [\![\theta_i]\!]$ so for $\theta := \bigoplus_{i=1}^n \theta_i \in \mathrm{QMSO}_\Gamma(\Pi_x^1, \oplus, \odot_b)$ we have $[\![\theta]\!] = [\![\mathcal{A}]\!]$. This was to show. □

The rest of the section is dedicated to prove the lemma used in above proof.

Lemma 7.2. *Let $\mathcal{A} = (A, \Gamma, \mu, \alpha)$ be a finitely ambiguous weighted bottom-up finite state tree automaton. Then there exist finitely many unambiguous weighted bottom-up finite state tree automata $\mathcal{A}_1, \ldots, \mathcal{A}_n$ satisfying*

$$[\![\mathcal{A}]\!] = [\![\mathcal{A}_1]\!] \oplus \ldots \oplus [\![\mathcal{A}_n]\!].$$

The proof roughly follows the construction applied in [14, Section 4]. Let $\mathcal{A}' = (A, \Gamma, \mu', \alpha')$ be a finitely ambiguous WTA. Without loss of generality, $\mathcal{A}'$ is assumed to be trim. We will prove the lemma by constructing a finite set of unambiguous automata such that every accepting run in $\mathcal{A}'$ will correspond to an accepting run in one of the new automata in a 1-to-1 manner. Note first, that an accepting run is characterized only by all its transitions and the final state having non-zero weights. Therefore, for the construction we consider the "booleanized" automaton $\mathcal{A} = (A, \Gamma, \mu, \alpha)$ over $\mathbb{B}$ where for $d \in \Delta_{\mathcal{A}}$ and $q \in A$

$$\mu(d) := \begin{cases} 1 & \text{if } \mu'(d) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\alpha(q) := \begin{cases} 1 & \text{if } \alpha'(q) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\mathcal{A}$ is an automaton over the boolean semiring $\mathbb{B}$ having the same accepting runs as $\mathcal{A}'$. Now, remember that there is an obvious 1-to-1 correspondence between the WTA over the boolean semiring and the standard BU-FTA. In the following, we will therefore not make a strict distinction between the two.

We will now use the power set construction to obtain a deterministic WTA $\mathcal{B} = (B, \Gamma, \nu, \beta)$ over $\mathbb{B}$ having the same support as $\mathcal{A}$. As we are going to need its properties in detail, we will recapitulate the construction, but refer to [4, Theorem 1.1.9] for a proof of correctness. Let $\mathcal{B}' = (B', \Gamma, \nu', \beta')$ be the WTA defined by $B' := \mathcal{P}(A)$,

$$\nu'(P_1, \ldots, P_m, a, Q) = 1 :\Leftrightarrow$$
$$Q = \{q \in A \,|\, \exists p_1 \in P_1 \ldots \exists p_m \in P_m : \mu(p_1, \ldots, p_m, a, q) = 1\}$$

and

$$\beta'(Q) = 1 :\Leftrightarrow \exists q \in Q : \alpha(q) = 1.$$

Then $\mathcal{B}$ is defined as the trim part of $\mathcal{B}'$.

With the help of $\mathcal{B}$, we now define the *Schützenberger covering* $\mathcal{S} = (S, \Gamma, \zeta, \omega)$ of $\mathcal{A}$ over $\mathbb{B}$. Let the tensor product $\mathcal{A} \boxdot \mathcal{B} = (A \times B, \Gamma, \mu \boxdot \nu, \alpha \boxdot \beta)$ of $\mathcal{A}$ and $\mathcal{B}$ be the automaton defined by

$$\mu \boxdot \nu((p_1, P_1), \ldots, (p_m, P_m), a, (q, Q)) = 1 :\Leftrightarrow$$
$$\mu(p_1, \ldots, p_m, a, q) = 1 \wedge \nu(P_1, \ldots, P_m, a, Q) = 1$$

and

$$\alpha \boxdot \beta(q, Q) = 1 :\Leftrightarrow \alpha(q) = 1 \wedge \beta(Q) = 1.$$

Then $\mathcal{S}$ is defined as the trim part of $\mathcal{A} \boxdot \mathcal{B}$.

Now let $(q, Q) \in S$. The *past of* (q, Q), denoted by $\mathrm{Past}_{\mathcal{S}}(q, Q)$, is defined as

$$\mathrm{Past}_{\mathcal{S}}(q, Q) := \{t \in T_\Gamma \,|\, \mathrm{Run}_{\mathcal{S},(q,Q)}(t) \neq \emptyset\}.$$

Proposition 7.3. *Let $\mathcal{A}, \mathcal{B}, \mathcal{S}$ be as above, then the following holds:*

(i) *For all states (q, Q) in $\mathcal{S}$ we have $q \in Q$.*

(ii) *We have a canonical bijection between the accepting runs in $\mathcal{S}$ and $\mathcal{A}$.*

(iii) *For every pair of states* $(q_1, Q), (q_2, Q)$ *in* $\mathcal{S}$ *we have* $\text{Past}_{\mathcal{S}}(q_1, Q) = \text{Past}_{\mathcal{S}}(q_2, Q)$.

Proof. (i) Let $t \in T_\Gamma$ and $r \in \text{Run}_{\mathcal{A}\boxdot\mathcal{B},\mathbb{F}}(t)$. For the leaves $w \in \text{pos}(t)$ with $a := t(w) \in \Gamma^{(0)}$ and $(q, Q) := r(w)$ we have $\mu(a, q) = 1$ and $\nu(a, Q) = 1 \Leftrightarrow Q = \{\hat{q} \in A \mid \mu(a, \hat{q}) = 1\}$ by definition of $\mathcal{B}$ and $\mathcal{A} \boxdot \mathcal{B}$, so obviously $q \in Q$.

For $w \in \text{pos}(t)$ with $t(w) \in \Gamma^{(m)}$, $m \geq 1$ we assume by induction that for $(p_i, P_i) := r(wi)$ $(1 \leq i \leq m)$ we have $p_i \in P_i$ and obtain that for $(q, Q) := r(w)$ we have $\mu(p_1, \ldots, p_m, a, q) = 1$ and $\nu(P_1, \ldots, P_m, a, Q) = 1 \Leftrightarrow Q = \{\hat{q} \in A \mid \exists \hat{p}_1 \in P_1 \ldots \exists \hat{p}_m \in P_m : \mu(\hat{p}_1, \ldots, \hat{p}_m, a, \hat{q}) = 1\}$. Trivially, $q \in Q$ must hold. By trimness, this proves (i).

(ii) Let $t \in T_\Gamma$. Define $\Psi_1 \colon \text{Run}_{\mathcal{S},\mathbb{F}}(t) \to \text{Run}_{\mathcal{A},\mathbb{F}}(t)$ by projection on the first coordinate, i.e. if $r(w) = (q, Q)$ then $\Psi_1(r)(w) = q$. Similarly, let $\Psi_2 \colon \text{Run}_{\mathcal{S},\mathbb{F}}(t) \to \text{Run}_{\mathcal{B},\mathbb{F}}(t)$ be the projection on the second coordinate. By definition of $\mathcal{A} \boxdot \mathcal{B}$, Ψ_1 and Ψ_2 are well defined. As $\mathcal{B}$ is deterministic ($\to$ unambiguous), we get $|\text{Run}_{\mathcal{B},\mathbb{F}}(t)| \leq 1$ and it is therefore easy to see that for every two $r_1, r_2 \in \text{Run}_{\mathcal{S},\mathbb{F}}(t)$ we have $\Psi_2(r_1) = \Psi_2(r_2)$. It follows that Ψ_1 is injective. For surjectivity of Ψ_1 take $r_{\mathcal{A}} \in \text{Run}_{\mathcal{A},\mathbb{F}}(t)$ and the unique run $r_{\mathcal{B}} \in \text{Run}_{\mathcal{B},\mathbb{F}}(t)$. It is easy to see that $r(w) := (r_{\mathcal{A}}(w), r_{\mathcal{B}}(w)) \in \text{Run}_{\mathcal{A}\boxdot\mathcal{B},\mathbb{F}}(t)$ and therefore $r \in \text{Run}_{\mathcal{S},\mathbb{F}}(t)$, $\Psi_1(r) = r_{\mathcal{A}}$.

(iii) As the problem is symmetric, it suffices to show that $\text{Past}_{\mathcal{S}}(q_1, Q) \subseteq \text{Past}_{\mathcal{S}}(q_2, Q)$. By (i) we know that $q_1 \in Q$ and $q_2 \in Q$. Let $t \in \text{Past}_{\mathcal{S}}(q_1, Q)$ and $r_1 \in \text{Run}_{\mathcal{S},(q_1,Q)}(t)$. We define $r_2 \in \text{Run}_{\mathcal{S},(q_2,Q)}(t)$ inductively starting from the root, beginning with $r_2(\varepsilon) := (q_2, Q)$. Now assume r_2 is defined on $w \in \text{pos}(t)$ with $r_2(w) = (f_2, F)$, $r_1(w) = (f_1, F)$ and $t(w) =: a \in \Gamma^{(m)}$. Let $(p_i, P_i) := r_1(wi)$ $(1 \leq i \leq m)$. By induction we assume $f_2 \in F$ and by definition of $\mathcal{B}$ we have $F = \{f \in A \mid \exists \hat{p}_1 \in P_1 \ldots \exists \hat{p}_m \in P_m : \mu(\hat{p}_1, \ldots, \hat{p}_m, a, f) = 1\}$. This implies that we can find $\hat{p}_1 \in P_1, \ldots, \hat{p}_m \in P_m$ such that $\mu(\hat{p}_1, \ldots, \hat{p}_m, a, f_2) = 1$. We define $r_2(wi) := (\hat{p}_i, P_i)$. Now $\zeta(\mathfrak{t}_{r_2}(w)) = 1$ and, as $\hat{p}_i \in P_i$, the prerequisite for the next induction step is fulfilled and we obtain r_2 as needed.

□

Definition 7.4. Let $a \in \Gamma^{(m)}$. Two valid transitions

$$((p_1^j, P_1), \dots, (p_m^j, P_m), a, (q, Q)) \in \Delta_{\mathcal{S}} \ (j = 1, 2)$$

are said to *compete*. Competing is an equivalence relation on the valid transitions and the equivalence class $\mathbb{T}$ of a valid transition is called a *competing set* if it contains at least 2 different transitions. A transition belonging to a competing set is called a *competing transition*.

Note that every transition $((p_1, P_1), \dots, (p_m, P_m), a, (q, Q))$ competes with itself by definition, but we will only refer to it as a competing transition if it actually belongs to a competing set.

We are now going to show that for each competing set there must exist a constant $\chi \in \mathbb{N}$ such that for every tree $t \in T_\Gamma$ and every run $r \in \mathrm{Run}_{\mathcal{S},\mathbb{F}}(t)$ the number of footpoints of transitions of this given competing set is bounded by χ.

Proposition 7.5. *Let $\mathbb{T}$ be a competing set. Then there exists a constant $\chi = \chi(\mathbb{T}) \in \mathbb{N}$ satisfying*

$$\forall t \in T_\Gamma \ \forall r \in \mathrm{Run}_{\mathcal{S},\mathbb{F}}(t) : |\{w \in \mathrm{pos}(t) \mid \mathbb{t}_r(w) \in \mathbb{T}\}| \leq \chi$$

Proof. We prove the statement by contradiction. Let $\mathbb{T}$ be a competing set such that for every $n \in \mathbb{N}$ we have a tree $t_n \in T_\Gamma$ and a run $r_n \in \mathrm{Run}_{\mathcal{S},\mathbb{F}}(t)$ such that for $V_n := \{w \in \mathrm{pos}(t_n) \mid \mathbb{t}_{r_n}(w) \in \mathbb{T}\}$ we have $|V_n| \geq n$. Let $v \in V_n$ and $((p_1, P_1), \dots, (p_m, P_m), a, (q, Q))$ be the transition $\mathbb{t}_r(v)$ at v and fix a transition $((\hat{p}_1, P_1), \dots, (\hat{p}_m, P_m), a, (q, Q)) \in \mathbb{T}$ different from the former. By definition of a run and Proposition 7.3 (iii) we have $t_n|_{vi} \in \mathrm{Past}_{\mathcal{S}}(p_i, P_i) = \mathrm{Past}_{\mathcal{S}}(\hat{p}_i, P_i)$ for all $i \in \{1, \dots, m\}$. Thus, we find $\hat{r}_i \in \mathrm{Run}_{\mathcal{S},(\hat{p}_i,P_i)}(t_n|_{vi})$ which means that

$$r_v(w) := \begin{cases} \hat{r}_i(u) & \text{if } vi \leq_p w \text{ for some } i \in \{1, \dots, m\} \text{ and } w = viu \\ r_n(w) & \text{otherwise} \end{cases}$$

is an accepting run for t_n. r_v differs strictly from r_n but the difference is restricted to the subtree at v. Thus, for $v_1, v_2 \in V_n$ with $v_1 \neq v_2$, two runs r_{v_1} and r_{v_2} constructed in this way will differ either at both v_1 and v_2, if v_1 and v_2 are prefix-independent, or they will differ at least at the prefix-smaller position. This means there exists a different run $r_v \in \mathrm{Run}_{\mathcal{S},\mathbb{F}}(t_n)$ for each $v \in V_n$, i.e. $|\mathrm{Run}_{\mathcal{S},\mathbb{F}}(t_n)| \geq |V_n| \geq n$. For $n \to \infty$ this contradicts the finite ambiguity of $\mathcal{S}$. □

As both Γ and S are finite, there is only a finite amount of different competing sets $\mathbb{T}$ and we can take the maximum of all $\chi(\mathbb{T})$ to obtain a global constant χ.

To make proper use of Proposition 7.5 we need to decompose our automaton $\mathcal{S}$ for the first time. Let F be the set of final states in $\mathcal{S}$. Then for each $f \in F$ we let $\mathcal{S}_f$ be the automaton behaving exactly like $\mathcal{S}$ with the exception that f is the only final state. Obviously, each accepting run r in $\mathcal{S}$ will now be a run in $\mathcal{S}_{r(\varepsilon)}$ and only in $\mathcal{S}_{r(\varepsilon)}$. To prove Lemma 7.2 it therefore suffices to deal with the $\mathcal{S}_f$ separately and then take the sum of those. For sake of notation, we simply assume $\mathcal{S}$ to have only one final state (q_f, Q_f).

We will now show that every accepting run r is characterized uniquely by the order in which the competing transitions are visited, from a bottom-up point of view, assuming the transitions are ordered using the lexicographical order of their footpoints.

Proposition 7.6. *Let $t \in T_\Gamma$ and $r_1, r_2 \in \mathrm{Run}_{\mathcal{S},\mathbb{F}}(t)$ such that $r_1 \neq r_2$. Then there exists a competing set $\mathbb{T}$ satisfying the following. Let $W^j := \{w \in \mathrm{pos}(t_n) \,|\, \mathbb{t}_{r_j}(w) \in \mathbb{T}\}$ and write $W^j = \{w^j_1, \ldots, w^j_{n_j}\}$ with $w^j_1 \leq_l \ldots \leq_l w^j_{n_j}$ for $j = 1, 2$. Then for some $1 \leq k \leq \min\{n_1, n_2\}$ we have $\mathbb{t}_{r_1}(w^1_k) \neq \mathbb{t}_{r_2}(w^2_k)$.*

Proof. Take $t \in T_\Gamma$ and $r_1, r_2 \in \mathrm{Run}_{\mathcal{S},\mathbb{F}}(t)$ with $r_1 \neq r_2$. Let w be the lexicographically smallest position in $\mathrm{pos}(t)$ such that $r_1(w) \neq r_2(w)$. By assumption on $\mathcal{S}$ we have $r_1(\varepsilon) = r_2(\varepsilon) = (q_f, Q_f)$, so $\varepsilon \neq w = w'i'$ for some $i' \in \mathbb{N}$. Let $t(w') =: a \in \Gamma^{(m)}$. By minimality of w we have $r_1(w') = r_2(w') =: (q, Q)$.

Now let $(p_i^j, P_i) := r_j(w'i)$ for $1 \le i \le m$, $j = 1, 2$. Remember that $\mathcal{S}$ is "deterministic" in the second coordinate which means that both runs are identical there. It is now easy to see that w' is the footpoint of the transition $((p_1^1, P_1), \dots, (p_m^1, P_m), a, (q, Q))$ in r_1 and $((p_1^2, P_1), \dots, (p_m^2, P_m), a, (q, Q))$ in r_2, belonging to the same competing set $\mathbb{T}$ and being strictly different as $p_{i'}^1 \neq p_{i'}^2$. Again by minimality of w, we get that r_1 and r_2 are identical on all positions $\hat{w} \le_l w'$. In particular, all footpoints $\hat{w}_1, \dots, \hat{w}_n \in \operatorname{pos}(t)$ of transitions in r_1 belonging to $\mathbb{T}$, i.e. with $\mathbb{t}_{r_1}(\hat{w}_i) \in \mathbb{T}$, such that $\hat{w}_i \le_l w'$, are also footpoints of transitions from $\mathbb{T}$ in r_2 and vice versa. □

We are now going to use this characterization to define finitely many automata such that each accepting run in $\mathcal{S}$ will correspond to an accepting run in exactly one of the newly constructed automata. The idea is to make the automata remember which ones of the competing transitions have been used in which order. By Proposition 7.5 we only have to care about remembering finitely many transitions and by Proposition 7.6 this will cause different accepting runs for a tree to be excepted in different automata.

Let $\mathbb{T}_1, \dots, \mathbb{T}_k$ be an enumeration of the competing sets in $\mathcal{S}$. For each $\xi \in \bigtimes_{l=1}^{k} \bigcup_{j=0}^{\chi} (\mathbb{T}_l)^j$ take $\mathcal{S}^\xi := (S \times \bigtimes_{l=1}^{k} \bigcup_{j=0}^{\chi} (\mathbb{T}_l)^j, \Gamma, \eta, \omega^\xi)$ defined by

$$\eta\left(\begin{pmatrix} {\scriptstyle (p_1,P_1)} \\ e_1^1 \\ \vdots \\ e_1^k \end{pmatrix}, \dots, \begin{pmatrix} {\scriptstyle (p_m,P_m)} \\ e_m^1 \\ \vdots \\ e_m^k \end{pmatrix}, a, \begin{pmatrix} {\scriptstyle (q,Q)} \\ e^1 \\ \vdots \\ e^k \end{pmatrix}\right) = 1 :\Leftrightarrow$$

$$\zeta((p_1, P_1), \dots, (p_m, P_m), a, (q, Q)) = 1 \wedge$$

$$\forall l \in \{1, \dots, k\} \begin{cases} e^l = d e_1^l \cdots e_m^l & \text{if } d := ((p_1, P_1), \dots, (p_m, P_m), a, (q, Q)) \in \mathbb{T}_l \\ e^l = e_1^l \cdots e_m^l & \text{otherwise} \end{cases}$$

and

$$\omega^\xi \begin{pmatrix} {\scriptstyle (q,Q)} \\ e^1 \\ \vdots \\ e^k \end{pmatrix} = 1 \quad :\Leftrightarrow \quad (q, Q) = (q_f, Q_f) \wedge \begin{pmatrix} e^1 \\ \vdots \\ e^k \end{pmatrix} = \xi$$

It is easy to see by construction, that the automata gain no "new" accepting runs, as they are in fact "deterministic" in the $\bigtimes_{l=1}^{k} \bigcup_{j=0}^{\chi} (\mathbb{T}_l)^j$ coordinate, and that this second coordinate effectively saves the order of all competing transitions for each competing set in lexicographical order. By Proposition 7.5 we can execute every accepting run in one of the automata and by Proposition 7.6 different accepting runs for the same tree end up in different automata, making the automata unambiguous. Finally, we "redecorate" these automata with our original weights, i.e.

$$\begin{aligned} &\eta\left(\begin{pmatrix} {\scriptstyle (p_1,P_1)} \\ e_1^1 \\ \vdots \\ e_1^k \end{pmatrix}, \dots, \begin{pmatrix} {\scriptstyle (p_m,P_m)} \\ e_m^1 \\ \vdots \\ e_m^k \end{pmatrix}, a, \begin{pmatrix} {\scriptstyle (q,Q)} \\ e^1 \\ \vdots \\ e^k \end{pmatrix}\right) &&= 1 \\ \rightsquigarrow \quad &\eta'\left(\begin{pmatrix} {\scriptstyle (p_1,P_1)} \\ e_1^1 \\ \vdots \\ e_1^k \end{pmatrix} \cdots \begin{pmatrix} {\scriptstyle (p_m,P_m)} \\ e_m^1 \\ \vdots \\ e_m^k \end{pmatrix}, a, \begin{pmatrix} {\scriptstyle (q,Q)} \\ e^1 \\ \vdots \\ e^k \end{pmatrix}\right) &&:= \mu'(p_1, \dots, p_m, a, q) \end{aligned}$$

and the same for the final states. The result is a set of unambiguous WTA whose sum equals our initial automaton. This concludes the proof of Lemma 7.2. □

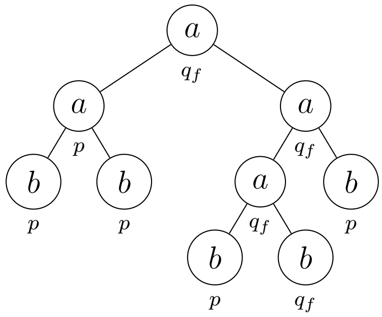
8 Polynomial Ambiguity and the Fragment $\mathrm{QMSO}(\Sigma_x^k\Pi_x^1, \oplus, \odot_b)$

We now come to the tree series definable by polynomially ambiguous WTA. Given a polynomially ambiguous WTA $\mathcal{A}$ we can define the function $\mathbb{r}_{\mathcal{A}} \colon \mathbb{N} \to \mathbb{N}$ that counts the maximum number of possible runs given trees with a limited number of nodes, i.e. $\mathbb{r}_{\mathcal{A}}(n) = \max\{|\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)| \,|\, t \in T_\Gamma, |\mathrm{pos}(t)| \leq n\}$. We then define the *degree of ambiguity of* $\mathcal{A}$ by

$$\mathrm{degree}(\mathcal{A}) \;:=\; \min\{k \in \mathbb{N} \,|\, \mathbb{r}_{\mathcal{A}} \in \mathcal{O}(n^k)\}.$$

This is well defined if $\mathcal{A}$ is polynomially ambiguous. We illustrate this by giving an example for a simple polynomially ambiguous automaton.

Example 8.1. We consider the alphabet $\Gamma = \{a, b\}$ where $rk_\Gamma(a) = 2$ and $rk_\Gamma(b) = 0$. We construct an automaton $\mathcal{A} = (Q, \Gamma, \mu, \alpha)$ over the tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ which to a tree $t \in T_\Gamma$ assigns the minimum amount of a's we have to visit to reach any leaf b starting from the root. For this, we let $Q = \{p, q_f\}$, where p will serve as a "filler state" and q_f as a "counting state". Given the tree t, we want that for every leaf b in t there is exactly one run of $\mathcal{A}$ on t, given by mapping all nodes between this leaf and the root to q_f and all other nodes to p. The following figure gives an example of how such a run should look like.



We therefore define

$$\begin{aligned} \mu(b, q_f) &:= 0 \\ \mu(b, p) &:= 0 \end{aligned} \qquad \begin{aligned} \mu(p, p, a, p) &:= 0 \\ \mu(q_f, p, a, q_f) &:= 1 \\ \mu(p, q_f, a, q_f) &:= 1 \end{aligned}$$

and μ is ∞ for all other transitions. For the final weights we set $\alpha(q_f) := 0$ and $\alpha(p) := \infty$. In other words, we can "enter" into the tree at the leaves with both states p and q_f without cost. The transitions at the letter a then serve to "forward q_f upwards", but cannot "create q_f out of nothing". Note here, that the transition (q_f, q_f, a, q_f) is not valid, so we can enter at no more than one leaf with the state q_f to get a run. Finally, the root has to be mapped to q_f due to $\alpha(p) := \infty$, which in turn forces us to enter at least one leaf with state the q_f.

For $w \in \operatorname{pos}(t)$ with $t(w) = b$, the run r associated to w by $r(w) = q_f$ will have weight $|w|$, hence the weight of t will be the minimum over all $|v|$ for $v \in \operatorname{pos}(t)$ with $t(v) = b$. As every run of $\mathcal{A}$ on t corresponds to exactly one leaf of t and a tree trivially has no more leaves than nodes, $\mathcal{A}$ is 1-polynomially ambiguous. As we can construct trees with as many leaves as we want, $\mathcal{A}$ is also not finitely ambiguous.

This example can also be used to show that polynomiality defined using the depth of the tree is not equivalent to our definition. For $n \in \mathbb{N}$ we let $t_n \in T_\Gamma$ be the "largest" tree of depth n possible, i.e. the full binary tree of depth n with 2^{n-1} leaves. Formally, we set $t_1 := b()$ and $t_{n+1} := a(t_n, t_n)$ for $n \in \mathbb{N}$. Then indeed $\operatorname{depth}(t_n) = n$ and t_n has exactly 2^{n-1} leaves, so there are 2^{n-1} different runs of $\mathcal{A}$ on t_n for every $n \in \mathbb{N}$. So if we would regard n, the depth of the tree, to be its "size", $\mathcal{A}$ would in fact not be polynomially ambiguous.

A formula describing $\mathcal{A}$ is found in

$$\min x. \sum y. \Big(\operatorname{label}_b(x) + \min \Big\{1 + (\operatorname{label}_a(y) \wedge y \leq_p x), \neg(\operatorname{label}_a(y) \wedge y \leq_p x)\Big\}\Big)$$

where $\sum$ is the addition $+$ in $\mathbb{N}$. A definition of the prefix relation $\leq_p$ for formulas is given in Definition 8.23.

Theorem 8.2. *Let $(K, \oplus, \odot, 0, 1)$ be a commutative semiring and (Γ, rk_Γ) a ranked alphabet. A tree series $S \in K\langle\langle T_\Gamma\rangle\rangle$ is definable by a polynomially ambiguous weighted bottom-up finite state tree automaton of degree k over K and Γ if, and only if, S is definable by a formula in $\mathrm{QMSO}_\Gamma(\Sigma_x^k\Pi_x^1, \oplus, \odot_b)$.*

Proof. ($\Leftarrow$) This direction can be proven with the idea used in the proof of Theorem 6.2 of [15] for weighted automata over words. Take $k \in \mathbb{N}$ and $\theta \in \mathrm{QMSO}_\Gamma(\Sigma_x^k\Pi_x^1, \oplus, \odot_b)$. Due to the fact that for $\tau_1, \tau_2 \in \mathrm{QMSO}(\Gamma)$ we can rewrite $\Sigma x.(\tau_1 \oplus \tau_2)$ into $\Sigma x.\tau_1 \oplus \Sigma x.\tau_2$, we can assume that θ is a sum of formulas in $\mathrm{QMSO}_\Gamma(\Sigma_x^k\Pi_x^1, \oplus_b, \odot_b)$, that is

$$\theta = \bigoplus_{i=1}^{n} \theta_i$$

for some $n \in \mathbb{N}$ and $\theta_i \in \mathrm{QMSO}_\Gamma(\Sigma_x^k\Pi_x^1, \oplus_b, \odot_b)$ for $i \in \{1, \ldots, n\}$. It then suffices to show that for all $i \in \{1, \ldots, n\}$ we can find a polynomially ambiguous WTA $\mathcal{A}_i$ of degree k such that $[\![\mathcal{A}_i]\!] = [\![\theta_i]\!]$. We prove this by induction over k. For $k = 0$ this is clear due to Theorem 6.1. For $k > 0$ we consider the proof of Proposition 4.8 in more detail, where we have shown how to construct an automaton for the first order sum operator.

By induction we assume that for $\tau \in \mathrm{QMSO}_\Gamma(\Sigma_x^{k-1}\Pi_x^1, \oplus_b, \odot_b)$ we can find a $k-1$-polynomially ambiguous WTA $\mathcal{A}$ such that $[\![\tau]\!] = [\![\mathcal{A}]\!]$. Now let $\mathcal{A}_x$ be the automaton constructed from $\mathcal{A}$ as done in the proof of Proposition 4.8 such that $[\![\mathcal{A}_x]\!] = [\![\Sigma x.\tau]\!]$ and let p be a polynomial of degree $k-1$ such that for all $t \in T_\Gamma$ we have

$$|\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)| \leq p(|\mathrm{pos}(t)|).$$

Now let $t \in T_\Gamma$. As we have seen in the construction of $\mathcal{A}_x$, every run of $\mathcal{A}_x$ on t corresponds to exactly one run of $\mathcal{A}$ on $t[x \to w]$ for some $w \in \mathrm{pos}(t)$. In particular, we have

$$|\mathrm{Run}_{\mathcal{A}_x,\mathbb{F}}(t)| \leq \sum_{w \in \mathrm{pos}(t)} |\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t[x \to w])|$$

$$\begin{aligned} &\leq \sum_{w \in \mathrm{pos}(t)} p(|\mathrm{pos}(t)|) \\ &= |\mathrm{pos}(t)| \cdot p(|\mathrm{pos}(t)|) \end{aligned}$$

so that $\mathcal{A}_x$ is k-polynomially ambiguous, as $|\mathrm{pos}(t)| \cdot p(|\mathrm{pos}(t)|)$ is a polynomial of degree k in $|\mathrm{pos}(t)|$.

($\Rightarrow$) The proof for this direction takes more effort and we will therefore divide it into five smaller parts. First, we will make some definitions and observations applicable to polynomially ambiguous WTA in general. Secondly, we will show that we can represent any polynomially ambiguous WTA as a sum of polynomially ambiguous WTA which are in a standard form we yet have to define. Thirdly, we will analyze this standard form and fourthly, we will prove some purely logic-related statements we need in order to utilize the properties we have found the standard form to possess. Finally, we will combine all of this to conclude the proof of Theorem 8.2.

8.1 General Definitions and Observations

For the rest of this section we will assume, without loss of generality, that all WTA, which are not the result of an explicit construction, are trim. We begin by introducing a more elaborate concept for runs. For now let $\mathcal{A} = (Q, \Gamma, \mu, \gamma)$ be a polynomially ambiguous WTA.

Definition 8.3 ($\mathrm{Run}_{\mathcal{A}}(t; \vec{w}, \vec{q})$, $\mathrm{Run}_{\mathcal{A}}(t; \vec{w}, \vec{d})$)**.** Let $t \in T_\Gamma$, $\vec{w} = (w_1, \ldots, w_n) \in \mathrm{pos}(t)^n$, $\vec{q} = (q_1, \ldots, q_n) \in Q^n$ and $\vec{d} = (d_1, \ldots, d_n) \in \Delta_{\mathcal{A}}^n$, then

$$\mathrm{Run}_{\mathcal{A}}(t; \vec{w}, \vec{q}) := \{r \in \mathrm{Run}_{\mathcal{A}}(t) \,|\, r(w_i) = q_i \text{ for all } i = 1, \ldots, n\}$$

and

$$\mathrm{Run}_{\mathcal{A}}(t; \vec{w}, \vec{d}) := \{r \in \mathrm{Run}_{\mathcal{A}}(t) \,|\, \mathbb{t}_r(w_i) = d_i \text{ for all } i = 1, \ldots, n\}.$$

The sets $\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t;\vec{w},\vec{q})$, $\mathrm{Run}_{\mathcal{A},q}(t;\vec{w},\vec{q})$, $\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t;\vec{w},\vec{d})$ and $\mathrm{Run}_{\mathcal{A},q}(t;\vec{w},\vec{d})$ for $q \in Q$ are defined in a similar manner to the above and $\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)$ and $\mathrm{Run}_{\mathcal{A},q}(t)$. We also need the notion of partial runs as defined in [19]. For $t \in T_\Gamma$, a tuple $\vec{w} = (w_1, \dots, w_n) \in \mathrm{pos}(t)^n$ of pairwise prefix-independent positions and $q_1, \dots, q_n \in Q$ a map

$$r\colon \mathrm{pos}(t) \setminus \left(\bigcup_{i=1}^{n} w_i \mathrm{pos}(t|_{w_i}) \right) \cup \{w_1, \dots, w_n\} \to Q$$

is called a *partial run of $\mathcal{A}$ on t relative to $q_1, \dots, q_n$ at $w_1, \dots, w_n$* if for all $w \in \mathrm{pos}(t) \setminus (\bigcup_{i=1}^{n} w_i \mathrm{pos}(t|_{w_i}))$ the transition $\mathbb{t}_r(w)$ is valid and $r(w_i) = q_i$ for $i \in \{1, \dots, n\}$. We denote the set of all such runs by $\mathrm{Run}^{\partial}_{\mathcal{A}}(t;\vec{w},\vec{q})$ and the sets $\mathrm{Run}^{\partial}_{\mathcal{A},q}(t;\vec{w},\vec{q})$ for $q \in Q$ and $\mathrm{Run}^{\partial}_{\mathcal{A},\mathbb{F}}(t;\vec{w},\vec{q})$ are defined analogously to the previous cases.

Definition 8.4 ($\preccurlyeq$, $\mathfrak{C}$, $\mathfrak{Q}$)**.** We define a relation $\preccurlyeq$ on Q by

$$q_1 \preccurlyeq q_2 \quad :\Leftrightarrow \quad \exists t \in T_\Gamma \ \exists w \in \mathrm{pos}(t) : \mathrm{Run}_{\mathcal{A},q_1}(t; w, q_2) \neq \emptyset.$$

This relation is reflexive and transitive. For transitivity take $q_1 \preccurlyeq q_2$ and $q_2 \preccurlyeq q_3$ and trees $t_1, t_2 \in T_\Gamma$, positions $w_2 \in \mathrm{pos}(t_1)$, $w_3 \in \mathrm{pos}(t_2)$ and runs $r_i \in \mathrm{Run}_{\mathcal{A},q_i}(t_i; w_{i+1}, q_{i+1})$ for $i = 1, 2$ as in the definition of $\preccurlyeq$. Then $r_1\langle r_2 \to w_2\rangle \in \mathrm{Run}_{\mathcal{A},q_1}(t_1\langle t_2 \to w_2\rangle; w_3, q_3)$, i.e. $q_1 \preccurlyeq q_3$. Intuitively, $q_1 \preccurlyeq q_2$ means that there is a "path" from q_1 down to q_2, cf. [21]. This gives rise to a relation $\approx$ on Q defined by

$$q_1 \approx q_2 \quad :\Leftrightarrow \quad q_1 \preccurlyeq q_2 \wedge q_2 \preccurlyeq q_1.$$

This is an equivalence relation inducing equivalence classes $[q]_\approx \in Q_{/\approx}$. One may think of the classes as strongly connected components of states. We set $\mathfrak{C}(q) := [q]_\approx$ and $\mathfrak{Q} := Q_{/\approx}$ and refer to $\mathfrak{C}(q)$ as the *component of* q and to $\mathfrak{Q}$ as the *components of* Q. Then again, $\preccurlyeq$ induces a partial order $\preccurlyeq$ on $\mathfrak{Q}$, defined by

$$\mathfrak{C}(q_1) \preccurlyeq \mathfrak{C}(q_2) \quad :\Leftrightarrow \quad q_1 \preccurlyeq q_2.$$

We will use this relation to derive various structural properties of our automaton. We also need the notion of a *bridge*, similar to the one used in [21].

Definition 8.5 (Bridge)**.** A valid transition $\mathfrak{b} = (p_1, \dots, p_m, a, q) \in \Delta_{\mathcal{A}}$ is called a *bridge out of* $\mathfrak{C}(q)$ if $\mathfrak{C}(p_i) \neq \mathfrak{C}(q)$ for all $i \in \{1, \dots, m\}$. Notice that all valid transitions of the form (a, q) with $a \in \Gamma^{(0)}$ and $q \in Q$ are bridges.

Definition 8.6 ($\mathcal{F}_p$)**.** For every $p \in Q$ we define the WTA $\mathcal{F}_p = (Q, \Gamma, \mu, \gamma_p)$ where for $q \in Q$ we define γ_p as

$$\gamma_p(q) \quad := \quad \begin{cases} 1 & \text{if } q = p \\ 0 & \text{otherwise.} \end{cases}$$

The intuition is that for $t \in T_\Gamma$ the accepting runs of the automaton $\mathcal{F}_p$ on t are exactly the p-runs of $\mathcal{A}$ on t, i.e. the ones that "begin" with p at the root. Though we have defined $\mathcal{F}_p$ specifically for our automaton $\mathcal{A}$, the construction is applicable in an obvious way to arbitrary WTA and depending on context, we will change the underlying automaton used for $\mathcal{F}_p$ in the following considerations. Now for some properties of $\mathcal{F}_p$.

Proposition 8.7.

(i) $\mathrm{Run}_{\mathcal{F}_p,\mathbb{F}}(t) = \mathrm{Run}_{\mathcal{A},p}(t)$ *for all* $p \in Q$ *and* $t \in T_\Gamma$*. In particular* $\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t) = \bigcup_{p \in F} \mathrm{Run}_{\mathcal{F}_p,\mathbb{F}}(t)$*, where* F *is the set of final states of* $\mathcal{A}$*.*

(ii) $\mathcal{F}_p$ *is polynomially ambiguous for every* $p \in Q$*.*

(iii) *If* $p_1 \preccurlyeq p_2$ *then* $\mathrm{degree}(\mathcal{F}_{p_1}) \geq \mathrm{degree}(\mathcal{F}_{p_2})$*.*

Proof. (i) Clear.

(ii) By trimness we can find a run using p, i.e. there is a tree $t \in T_\Gamma$ with a run $r_t \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)$ and $w \in \mathrm{pos}(t)$ such that $r_t(w) = p$. Now for all $s \in T_\Gamma$ and $r_s \in \mathrm{Run}_{\mathcal{F}_p,\mathbb{F}}(s)$ we have $r_t\langle r_s \to w\rangle \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t\langle s \to w\rangle)$ and as $\mathcal{A}$ is polynomially ambiguous this means that $\mathcal{F}_p$ must be polynomially ambiguous.

(iii) For $p_1 = p_2$ this is clear, otherwise from the fact that $p_1 \preccurlyeq p_2$ we can find $t \in T_\Gamma$ and $w \in \mathrm{pos}(t)$ such that some $r_t \in \mathrm{Run}_{\mathcal{A},p_1}(t; w, p_2)$ exists. Then for any tree $s \in T_\Gamma$ and $r_s \in \mathrm{Run}_{\mathcal{F}_{p_2},\mathbb{F}}(s)$ we get $r_t\langle r_s \to w\rangle \in \mathrm{Run}_{\mathcal{F}_{p_1},\mathbb{F}}(t\langle s \to w\rangle)$ so that $\mathrm{degree}(\mathcal{F}_{p_2}) \leq \mathrm{degree}(\mathcal{F}_{p_1})$.

□

Definition 8.8 ($\mathrm{degree}_\mathcal{A}(p)$)**.** For $p \in Q$ we define $\mathrm{degree}_\mathcal{A}(p) := \mathrm{degree}(\mathcal{F}_p)$ which is well defined by Proposition 8.7 (ii). Furthermore we define

$$\mathrm{degree}_\mathcal{A}(\mathfrak{C}(p)) := \mathrm{degree}_\mathcal{A}(p)$$

which is now well defined by Proposition 8.7 (iii). If it is clear from context about which automaton we are talking, we will simply write $\mathrm{degree}(p)$ and $\mathrm{degree}(\mathfrak{C}(p))$.

We now show some properties which are characteristic for polynomially ambiguous WTA. The first three points deal with restrictions the polynomial ambiguity imposes on the automaton. The last point shows that polynomially ambiguous WTA, which are not also finitely ambiguous, possess at least a lower linear bound on their ambiguity. The ideas for the following proof are the same as the ones applied by Seidl and Weber in [21, 19].

Proposition 8.9. *Let $\mathcal{A} = (Q, \Gamma, \mu, \gamma)$ be a polynomially ambiguous WTA.*

(i) *For $t \in T_\Gamma$, $w \in \mathrm{pos}(t)$ and $q \in Q$ we have $|\mathrm{Run}^\partial_{\mathcal{A},q}(t; w, q)| \leq 1$.*

(ii) *For $t \in T_\Gamma$, $w \in \mathrm{pos}(t)$, $q \in Q$ and $p \in \mathfrak{C}(q)$ we have $|\mathrm{Run}^\partial_{\mathcal{A},q}(t; w, p)| \leq 1$.*

(iii) *Let $d = (p_1, \dots, p_m, a, q) \in \Delta_\mathcal{A}$ be a valid transition. If $p_i \in \mathfrak{C}(q)$ for some $i \in \{1, \dots, m\}$, then $\mathcal{F}_{p_j}$ is unambiguous for $j \in \{1, \dots, m\}$, $j \neq i$.*

(iv) *If $\mathrm{degree}(\mathcal{A}) > 0$ then there exists a sequence of trees $(t_n)_{n\in\mathbb{N}}$ with $|\mathrm{pos}(t)| \leq C \cdot n$ and $|\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t_n)| \geq n$, i.e. we have a lower linear bound on the ambiguity of $\mathcal{A}$.*

Proof. (i) Assume we have $q \in Q$, $t \in T_\Gamma$ and $w \in \mathrm{pos}(t)$ such that $r_1, r_2 \in \mathrm{Run}^\partial_{\mathcal{A},q}(t; w, q)$ with $r_1 \neq r_2$. We can then "concatenate" these partial runs repeatedly arbitrarily mixing them to get runs on trees growing like n in size but having at least 2^n partial runs. Formally let $t_1 := t$ and $t_n := t_{n-1}\langle t \to w^{n-1}\rangle$ for $n > 1$. For $n > 1$ and a word $x = x'l \in \{1,2\}^n$ where $l \in \{1,2\}$ we set $r_x := r_{x'}\langle r_l \to w^{n-1}\rangle \in \mathrm{Run}^{\partial}_{\mathcal{A},q}(t_n; w^n, q)$. In conclusion we have a sequence of trees $(t_n)_{n\in\mathbb{N}}$ with $|\mathrm{pos}(t_n)| \leq n \cdot C$ where $C = |\mathrm{pos}(t)|$ and $|\mathrm{Run}^{\partial}_{\mathcal{A},q}(t_n; w^n, q)| \geq 2^n$. By trimness we can find some $s \in T_\Gamma$ and $r_s \in \mathrm{Run}_{\mathcal{A},q}(s)$. By joining $t'_n := t_n\langle s \to w^n\rangle$ and $r'_x := r_x\langle r_s \to w^n\rangle$ we get $|\mathrm{pos}(t'_n)| \leq n \cdot C + |\mathrm{pos}(s)|$ and $|\mathrm{Run}_{\mathcal{A},q}(t_n)| \geq 2^n$ which clearly is a contradiction to the polynomial ambiguity of $\mathcal{F}_q$, i.e. Proposition 8.7 (ii).

(ii) Assume we have $q \in Q$, $p \in \mathfrak{C}(q)$, $t \in T_\Gamma$ and $w \in \mathrm{pos}(t)$ such that there exist $r_1, r_2 \in \mathrm{Run}^{\partial}_{\mathcal{A},q}(t; w, p)$ with $r_1 \neq r_2$. As $p \preccurlyeq q$ we can find $s \in T_\Gamma$, $v \in \mathrm{pos}(s)$ and $r_s \in \mathrm{Run}_{\mathcal{A},p}(s; v, q)$. Then considering $t' := t\langle s \to w\rangle$ and the runs $r'_1 := r_1\langle r_s \to w\rangle$ and $r'_2 := r_2\langle r_s \to w\rangle$ we easily see that $|\mathrm{Run}^{\partial}_{\mathcal{A},q}(t'; wv, q)| > 1$ which is a contradiction to (i).

(iii) Assume we have a valid transition $d = (p_1, \ldots, p_m, a, q) \in \Delta_{\mathcal{A}}$ such that $p_i \in \mathfrak{C}(q)$ for $i \in \{1, \ldots, m\}$ and $\mathcal{F}_{p_j}$ is not unambiguous for some $j \in \{1, \ldots, m\}$ with $j \neq i$. Then we can find $s \in T_\Gamma$ with $r_1, r_2 \in \mathrm{Run}_{\mathcal{A},p_j}(t)$ such that $r_1 \neq r_2$. By trimness we can find a run that uses d, i.e. there are $t_d \in T_\Gamma$ and $r_d \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t_d)$ such that for some $w_d \in \mathrm{pos}(t_d)$ we have $\mathbb{t}_{r_d}(w_d) = d$. We consider the subtree $t := t_d|_{w_d}$ with the run $r \in \mathrm{Run}_{\mathcal{A},q}(t)$ defined by $r(w) := r_d(w_d w)$, i.e. $\mathbb{t}_r(\varepsilon) = d$. Then the tree $t' := t\langle s \to j\rangle$ with runs $r'_1 := r\langle r_1 \to j\rangle$ and $r'_2 := r\langle r_2 \to j\rangle$ clearly shows that $|\mathrm{Run}^{\partial}_{\mathcal{A},q}(t'; i, p_i)| > 1$ which is a contradiction to (ii).

(iv) This has been proven in [19], so we just sum up the argumentation. We refrain from repeating the exact formulations here, as it would require introducing a lot of definitions not used elsewhere in this paper. As degree$(\mathcal{A}) > 0$, $\mathcal{A}$ is not finitely ambiguous. By [19, Prop. 2.5] this means that $\mathcal{A}$ must satisfy at least one of three properties (T1.1), (T1.2) or (T2) which in our notation look like the following.

(T1) $\exists j \in \{1, \dots, rk(\Gamma)\}$ $\exists p, q, q_j \in Q : p \approx q_j \approx q$ such that

(T1.1) There exist two different valid transitions $(q_1^{(i)}, \dots, q_{j-1}^{(i)}, q_j, q_{j+1}^{(i)}, \dots, q_m^{(i)}, a, q) \in \Delta_{\mathcal{A}}$, $i = 1, 2$, and trees $t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_m \in T_\Gamma$ with $\mathrm{Run}_{\mathcal{A}, q_{j'}^{(i)}}(t_{j'}) \neq \emptyset$ for $i = 1, 2$ and all $j' \neq j$.

(T1.2) There exists a valid transition $(q_1, \dots, q_{j-1}, q_j, q_{j+1}, \dots, q_m, a, q) \in \Delta_{\mathcal{A}}$ and a tree $t \in T_\Gamma$ with $|\mathrm{Run}_{\mathcal{A}, q_{j'}}(t)| > 1$ for some $j' \neq j$.

(T2) There exist states $p, q \in Q$ with $p \neq q$ such that for some $t \in T_\Gamma$ and $w \in \mathrm{pos}(t)$ all of the sets $\mathrm{Run}^{\partial}_{\mathcal{A},p}(t; w, p)$, $\mathrm{Run}^{\partial}_{\mathcal{A},p}(t; w, q)$ and $\mathrm{Run}^{\partial}_{\mathcal{A},q}(t; w, q)$ are non-empty.

(T1.1) is basically a negation of (ii) and (T1.2) a negation of (iii). Therefore, in our case (T2) must hold. As a consequence we can find $t \in T_\Gamma$, $w \in \mathrm{pos}(t)$ and $p, q \in Q$ with $p \neq q$ for which there are partial runs $r^p_p \in \mathrm{Run}^{\partial}_{\mathcal{A},p}(t; w, p)$, $r^p_q \in \mathrm{Run}^{\partial}_{\mathcal{A},p}(t; w, q)$ and $r^q_q \in \mathrm{Run}^{\partial}_{\mathcal{A},q}(t; w, q)$.

Conceptually, we can now "concatenate" these partial runs in a fashion $r^p_p \cdots r^p_p r^p_q r^q_q \cdots r^q_q$. This creates partial runs on trees growing like n in size and having at least n partial runs. We set $t_1 := t$, $t_{n+1} := t_n\langle t \to w^n\rangle$ and $r^1_1 := r^p_q$. For $n \geq 1$ and $i \in \{1, \dots, n\}$ we set $r^i_{n+1} := r^i_n\langle r^q_q \to w^n\rangle$ (adding r^q_q at the bottom) and $r^{n+1}_{n+1} := r^p_p\langle r^n_n \to w\rangle$ (to get the run with r^p_q at the bottom). Then for $n \in \mathbb{N}$ and $1 \leq i < j \leq n$ we have $r^i_n \in \mathrm{Run}^{\partial}_{\mathcal{A},p}(t_n; w^n, q)$ and $r^i_n \neq r^j_n$ such that $|\mathrm{pos}(t_n)| \leq n \cdot |\mathrm{pos}(t)|$ and $|\mathrm{Run}^{\partial}_{\mathcal{A},p}(t; w^n, q)| \geq n$. By extending the trees t_n with some fixed tree $\hat{s} \in T_\Gamma$ and run $\hat{r} \in \mathrm{Run}^{\partial}_{\mathcal{A},\mathbb{F}}(\hat{s}; \hat{v}, p)$ for $\hat{v} \in \mathrm{pos}(\hat{s})$ at the top and $\check{s} \in T_\Gamma$ and run $\check{r} \in \mathrm{Run}_{\mathcal{A},q}(\check{s})$ at the bottom we obtain trees as needed. □

8.2 Decomposition into a Sum of Standardized Automata

We can now define what we want to understand by a standardized WTA.

Definition 8.10 (Polynomial Standard Form). We call a (polynomially ambiguous) WTA $\mathcal{A} = (Q, \Gamma, \mu, \gamma)$ *standardized* or say it to be in *polynomial standard form* if

(i) $\mathcal{A}$ is polynomially ambiguous, trim and possesses only one final state $q_f \in Q$ and

(ii) for every $p \in Q$ with $\text{degree}_{\mathcal{A}}(p) > 0$ there is exactly one bridge out of $\mathfrak{C}(p)$ and every accepting run r uses this bridge exactly once. Formally

$$\{d \in \Delta_{\mathcal{A}} \mid d \text{ is a bridge out of } \mathfrak{C}(p)\} = \{\mathfrak{b}(p)\}$$

for some $\mathfrak{b}(p) \in \Delta_{\mathcal{A}}$ and

$$\forall t \in T_\Gamma \; \forall r \in \text{Run}_{\mathcal{A},\mathbb{F}}(t) : |\{w \in \text{pos}(t) \mid \mathfrak{t}_r(w) = \mathfrak{b}(p)\}| = 1.$$

The fundamental concept of standardized WTA is close to the notion of *chain NFAs* as introduced in [21].

Lemma 8.11. *Let $\mathcal{A} = (Q, \Gamma, \mu, \gamma)$ be a polynomially ambiguous WTA, then there exist $n \in \mathbb{N}$ and WTA $\mathcal{A}_1, \ldots, \mathcal{A}_n$ in polynomial standard form such that* $\text{degree}(\mathcal{A}_i) \leq \text{degree}(\mathcal{A})$ *for all $i \in \{1, \ldots, n\}$ and*

$$[\![\mathcal{A}]\!] = \bigoplus_{i=1}^{n} [\![\mathcal{A}_i]\!].$$

The rest of this subsection is dedicated to the proof of this lemma. We begin with a first elementary simplification. Let $\mathcal{A} = (Q, \Gamma, \mu, \gamma)$ be a polynomially ambiguous WTA and F be the final states of $\mathcal{A}$. For $q \in F$ take the automaton $\mathcal{A}_q = (Q, \Gamma, \mu, \gamma_q)$ where $\gamma_q(q) := \gamma(q)$ and $\gamma_q(p) := 0$ if $p \neq q$. Clearly, we have

$$[\![A]\!] = \bigoplus_{q \in F} [\![A_q]\!].$$

Therefore, without loss of generality, it suffices to prove Lemma 8.11 under the assumption that $\mathcal{A}$ possesses only one final state q_f. In the next step we construct a WTA $\mathcal{A}'$, accepting the same tree series as $\mathcal{A}$, in which for every component $\mathfrak{c}' \in \mathfrak{Q}'$ and every run r' there is at most one footpoint of a bridge out of $\mathfrak{c}'$ in r'. The idea is to make several copies of the states of $\mathcal{A}$ and adapt μ accordingly.

Set $N := |Q| + 1$, $U := \{1, \dots, rk(\Gamma)\}$ and $\mathfrak{U} := \bigcup_{i=0}^{N} U^i$. Then we consider the WTA $\mathcal{A}' = (Q', \Gamma, \mu', \gamma')$ where $Q' := Q \times \mathfrak{U}$ and μ' and γ' are defined by

$$\mu'((p_1, u_1), \dots, (p_m, u_m), a, (q, u)) :=$$
$$\begin{cases} \mu(p_1, \dots, p_m, a, q) & \text{if } (p_1, \dots, p_m, a, q) \text{ is not a bridge and } u = u_1 = \ldots = u_m \\ & \text{or if } (p_1, \dots, p_m, a, q) \text{ is a bridge and } u_i = ui \text{ for all} \\ & i \in \{1, \dots, m\} \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma'(q, u) \quad := \quad \begin{cases} \gamma(q) & \text{if } u = \varepsilon \\ 0 & \text{otherwise.} \end{cases}$$

The automaton $\mathcal{A}'$ behaves like $\mathcal{A}$ in the Q-coordinate and, using a top-down perspective, upon encountering a bridge saves the direction it took passing this bridge accumulating these directions into a word. Every state $(p, u) \in Q'$ can be interpreted as a copy of the state $p \in Q$. For $(p, u) \in Q'$ we denote the component of (p, u) in terms of $\preccurlyeq$ by $\mathfrak{C}'(p, u)$ and the components of Q' by $\mathfrak{Q}'$.

Proposition 8.12.

(i) *$\mathcal{A}'$ is polynomially ambiguous with* $\text{degree}(\mathcal{A}') = \text{degree}(\mathcal{A})$ *and* $[\![\mathcal{A}']\!] = [\![\mathcal{A}]\!]$.

(ii) *For all $(p, u) \in Q'$ we have* $\text{degree}_{\mathcal{A}'}(p, u) \leq \text{degree}_{\mathcal{A}}(p)$.

(iii) *States of $\mathcal{A}'$ with non-zero degree being in $\preccurlyeq$-relation are always at positions being in in $\leq_p$-relation: For all $t \in T_\Gamma$, $r' \in \text{Run}_{\mathcal{A}',\mathbb{F}}(t)$ and $w_1, w_2 \in \text{pos}(t)$ such that $r'(w_1) \preccurlyeq r'(w_2)$,* $\text{degree}_{\mathcal{A}'}(r'(w_1)) > 0$ *and* $\text{degree}_{\mathcal{A}'}(r'(w_2)) > 0$ *we have $w_1 \leq_p w_2$ or $w_2 \leq_p w_1$.*

Proof. (i) Let $t \in T_\Gamma$ and $\pi\colon \text{Run}_{\mathcal{A}',\mathbb{F}}(t) \to \text{Run}_{\mathcal{A},\mathbb{F}}(t)$ be the projection on the Q-coordinate. By definition of μ' the well definition of π is clear. We will show that π is a bijection and that the weights of the runs are preserved. Take

$r \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)$ and define $r' \in \mathrm{Run}_{\mathcal{A}',\mathbb{F}}(t)$ inductively starting from the root with $r'(\varepsilon) := (q_f, \varepsilon)$, which is the only final state. Then assume that r' is defined at position $w \in \mathrm{pos}(t)$ with $r'(w) = (q, u)$ and that $\mathbb{t}_r(w) = (p_1, \dots, p_m, a, q) =: d$.

1) If d is not a bridge, we set $d' := ((p_1, u), \dots, (p_m, u), a, (q, u))$ and define r' at position w using this transition. By construction of μ' this is the only possible extension of r' such that $\mathbb{t}_{r'}(w)$ is valid. We also see that $\mu(d) = \mu'(d')$.

2) If d is a bridge we set $d' := ((p_1, u1), \dots, (p_m, um), a, (q, u))$. If $u \notin U^N$ we have $d' \in \Delta_{\mathcal{A}'}$ and can define r' at position w using this transition. This then again is the only possible extension of r' such that $\mathbb{t}_{r'}(w)$ is valid and we have $\mu(d) = \mu'(d')$ in this case.

3) Now if d was a bridge and $u \in U^N$ at the same time, we can find N pairwise distinct positions $w_1 \leq_p \dots \leq_p w_N = w$ such that $\mathbb{t}_r(w_i)$ is a bridge for all $i \in \{1, \dots, N\}$. But then also $\mathfrak{C}(r(w_1)), \dots, \mathfrak{C}(r(w_N))$ are pairwise distinct such that $|Q| \geq |\mathfrak{Q}| \geq N = |Q| + 1$ which is a contradiction. Therefore this case can not arise.

We now have $\pi(r') = r$. The uniqueness of the construction of r' implies injectivity of π and the fact that we can always construct r' as shown above shows surjectivity. This bijection yields the polynomial ambiguity of $\mathcal{A}'$ with the same degree as $\mathcal{A}$ and together with $\mu(\mathbb{t}_r(w)) = \mu'(\mathbb{t}_{r'}(w))$ for all $w \in \mathrm{pos}(t)$ and $\gamma(q_f) = \gamma'(q_f, \varepsilon)$ we have $[\![\mathcal{A}]\!] = [\![\mathcal{A}']\!]$.

(ii) Take $(p, u) \in Q'$, then with the same construction as in (i) we can show that for $t \in T_\Gamma$ the projection on the Q-coordinate $\pi\colon \mathrm{Run}_{\mathcal{A}',(p,u)}(t) \to \mathrm{Run}_{\mathcal{A},p}(t)$ is injective such that $|\mathrm{Run}_{\mathcal{A}',(p,u)}(t)| \leq |\mathrm{Run}_{\mathcal{A},p}(t)|$ and therefore $\mathrm{degree}_{\mathcal{A}'}(p, u) \leq \mathrm{degree}_{\mathcal{A}}(p)$.

(iii) Assume we have $t \in T_\Gamma$, $r' \in \mathrm{Run}_{\mathcal{A}',\mathbb{F}}(t)$ and prefix-independent $w_1, w_2 \in \mathrm{pos}(t)$ such that $r'(w_1) \preccurlyeq r'(w_2)$, $\mathrm{degree}_{\mathcal{A}'}(r'(w_1)) > 0$ and $\mathrm{degree}_{\mathcal{A}'}(r'(w_2)) > 0$. Now let v be the largest common prefix of w_1 and w_2, i.e. $w_1 = viw_1'$ and $w_2 = vjw_2'$ with $i \neq j$. Take π as in (i) and consider $r := \pi(r') \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)$. We have $\mathrm{degree}_{\mathcal{A}}(r(vi)) \geq \mathrm{degree}_{\mathcal{A}}(r(w_1)) \geq \mathrm{degree}_{\mathcal{A}'}(r'(w_1)) > 0$ and similarly

$\mathrm{degree}_{\mathcal{A}}(r(vj)) > 0$ which by Proposition 8.9 (iii) implies that $d := \mathbb{t}_r(v)$ is a bridge. By what we have shown in (i) this then means that for some $u, u_1, u_2 \in \mathfrak{U}$ the state $r'(w_1)$ is of the form (p_1, uiu_1) and $r'(w_2)$ is of the form (p_2, uju_2). But due to $r'(w_1) \preccurlyeq r'(w_2)$ and the construction of μ we also must have that uiu_1 is a prefix of uju_2, which is impossible due to $i \neq j$. This is a contradiction, so w_1 and w_2 as chosen cannot exist. □

The next proposition shows, simply put, that components of non-zero degree always form a "straight line" in a run and that no components of non-zero degree can "branch off" of these lines.

Proposition 8.13. *Let* $t \in T_\Gamma$, $r' \in \mathrm{Run}_{\mathcal{A}',\mathbb{F}}(t)$, $q \in Q'$ *with* $\mathrm{degree}_{\mathcal{A}'}(q) > 0$ *and define* $\mathfrak{W} := \{w \in \mathrm{pos}(t) \,|\, r'(w) \in \mathfrak{C}'(q)\}$, *then:*

(i) $\mathfrak{W} = \emptyset$ *or* $\mathfrak{W} = \{v, vi_1, vi_1i_2, \ldots, vi_1 \cdots i_n\}$ *for some* $v \in \mathrm{pos}(t)$, $n \in \mathbb{N}_0$ *and* $i_1, \ldots, i_n \in \mathbb{N}$.

(ii) *If* $w \in \mathrm{pos}(t)$ *with* $v \leq_p w$ *for some* $v \in \mathfrak{W}$ *and* $\mathrm{degree}_{\mathcal{A}'}(r'(w)) > 0$, *then either* $w \in \mathfrak{W}$ *or* $v \leq_p w$ *for all all* $v \in \mathfrak{W}$.

(iii) *The run* r' *uses at most one bridge out of* $\mathfrak{C}'(q)$, *that is for*

$$\mathfrak{V} := \{w \in \mathrm{pos}(t) \,|\, \mathbb{t}_{r'}(w) \textit{ is a bridge out of } \mathfrak{C}'(q)\}$$

we have $|\mathfrak{V}| \leq 1$ *and* $|\mathfrak{V}| = 1$ *iff* $\mathfrak{W} \neq \emptyset$.

Proof. (i) We assume $\mathfrak{W} \neq \emptyset$. By 8.12 (iii) $\mathfrak{W}$ is a $\leq_p$-totally ordered set. Now if $w' \in \mathrm{pos}(t)$ such that $w_1 \leq_p w' \leq_p w_2$ for some $w_1, w_2 \in \mathfrak{W}$ we have $r'(w_1) \preccurlyeq r'(w') \preccurlyeq r'(w_2)$ such that $r'(w') \in \mathfrak{C}'(q)$ and $w' \in \mathfrak{W}$. Hence, $\mathfrak{W} = \{v, vi_1, vi_1i_2, \ldots, vi_1 \cdots i_n\}$ for some $v \in \mathrm{pos}(t)$, $n \in \mathbb{N}_0$ and $i_1, \ldots, i_n \in \mathbb{N}$.

(ii) Take $w \in \mathrm{pos}(t)$ with $v \leq_p w$ for some $v \in \mathfrak{W}$ and $\mathrm{degree}_{\mathcal{A}'}(r'(w)) > 0$ and write $\mathfrak{W} = \{v_1, \ldots, v_k\}$ with $v_1 \leq_p \ldots \leq_p v_k$. Take the largest $l \in \{1, \ldots, k\}$ such that $v_l \leq_p w$. If $l = k$ we are finished, otherwise write $v_{l+1} = v_l j$ and take $d := \mathbb{t}_{r'}(v_l) = (p_1, \ldots, p_m, a, p)$. This is not a bridge as $\mathfrak{C}'(p) = \mathfrak{C}'(p_j)$, so by

Proposition 8.9 (iii) $\text{degree}_{\mathcal{A}'}(p_i) = 0$ for all $i \neq j$. In particular, $p_i \not\preccurlyeq r'(w)$ for $i \neq j$ (Proposition 8.7 (iii)) and so $v_l i \not\leq_p w$ for $i \neq j$ and as $v_l j = v_{l+1} \not\leq_p w$ we must have $w = v_l$, i.e. $w \in \mathfrak{W}$.

(iii) Let $\mathfrak{V} := \{w \in \text{pos}(t) \mid \mathfrak{t}_{r'}(w) \text{ is a bridge out of } \mathfrak{C}'(q)\}$. It is easy to see that $\mathfrak{V} \subseteq \mathfrak{W}$, so if $\mathfrak{W} = \emptyset$ then also $\mathfrak{V} = \emptyset$. Assume $\mathfrak{W} \neq \emptyset$, let $v \in \mathfrak{W}$ and take $d := \mathfrak{t}_{r'}(v)$. If v is the maximal element in $\mathfrak{W}$, d must clearly be a bridge. Otherwise $vj \in \mathfrak{W}$ for some $j \in \mathbb{N}$, due to the structure of $\mathfrak{W}$ we proved in (i), so that d cannot be a bridge, in particular $|\mathfrak{V}| \leq 1$. □

In conclusion, we now have an automaton that defines the same tree series $[\![\mathcal{A}]\!]$ as $\mathcal{A}$ does and is polynomially ambiguous with the same degree as $\mathcal{A}$, but is simpler in structure, as every run uses at most one bridge out of each component of $\mathfrak{Q}'$ of non-trivial degree. Therefore without loss of generality, we assume $\mathcal{A}$ to have had this property from the beginning. We will continue to denote the only final state of $\mathcal{A}$ by q_f.

We now come to the final construction needed to prove Lemma 8.11. We assume $\text{degree}(\mathcal{A}) > 0$ as for finitely ambiguous WTA the lemma obviously holds true. Let $\mathfrak{c}_1, \ldots, \mathfrak{c}_n \in \mathfrak{Q}$ be an enumeration of all components of Q of non-trivial degree and for $i \in \{1, \ldots, n\}$ let $\mathfrak{b}^{(i)}_1, \ldots, \mathfrak{b}^{(i)}_{k_i} \in \Delta_{\mathcal{A}}$ be an enumeration of all bridges out of $\mathfrak{c}_i$ and set $J := \bigtimes_{i=1}^n \{1, \ldots, k_i\}$. Then for $x = (x_1, \ldots, x_n) \in J$ we define the automaton $\mathcal{A}_x = (Q, \Gamma, \mu_x, \gamma)$ by

$$\mu_x(d) \quad := \quad \begin{cases} \mu(d) & \text{if } d \neq \mathfrak{b}^{(i)}_j \text{ for all } i \in \{1, \ldots, n\} \text{ and } j \in \{1, \ldots, k_i\} \\ & \text{or if } d = \mathfrak{b}^{(i)}_{x_i} \text{ for some } i \in \{1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

for $d \in \Delta_{\mathcal{A}}$. That is, for every component of Q of non-trivial degree we remove all but one bridge out of this component. By assumption on $\mathcal{A}$, i.e. Proposition 8.13 (iii), for any $t \in T_\Gamma$ and $r \in \text{Run}_{\mathcal{A},\mathbb{F}}(t)$ the run r is also an accepting run for t in one of the $\mathcal{A}_x$, but r might still be an accepting run in more than one of the

$\mathcal{A}_x$. We can resolve this by taking only a subset of the $\mathcal{A}_x$.

Let $\mathfrak{B} := \{b_j^{(i)} \mid i \in \{1, \ldots, n\},\ j \in \{1, \ldots, k_i\}\}$. For $t \in T_\Gamma$ and $r \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)$ define $\mathfrak{B}(r) := \{\mathfrak{b} \in \mathfrak{B} \mid \exists w \in \mathrm{pos}(t) : \mathbb{t}_r(w) = \mathfrak{b}\}$ as the set of all bridges in $\mathfrak{B}$ used by r.

Proposition 8.14.

(i) *For every $x = (x_1, \ldots, x_n) \in J$ and $r_1, r_2 \in \mathrm{Run}_{\mathcal{A}_x,\mathbb{F}}(t)$ we have that $\mathfrak{B}(r_1) = \mathfrak{B}(r_2)$. The definition $\mathfrak{B}(x) := \mathfrak{B}(r_1)$ is therefore well-defined.*

(ii) *For every $t \in T_\Gamma$ and $r \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)$ there exists some $x \in J$ such that $r \in \mathrm{Run}_{\mathcal{A}_x,\mathbb{F}}(t)$.*

Proof. The idea is very simple here. In every run the root has to be mapped to q_f. Hence given $x \in J$ every run of $\mathcal{A}_x$ uses the one bridge out of the component of q_f it possesses. The "child states" of non-zero degree of this transition therefore occur in every run as well and so do the bridges leaving their components. Iterating this argument we get that every run of $\mathcal{A}_x$ uses the same set of bridges in $\mathfrak{B}$. A formal proof follows.

Take $t \in T_\Gamma$ and $r_1, r_2 \in \mathrm{Run}_{\mathcal{A}_x,\mathbb{F}}(t)$. We show $\mathfrak{B}(r_1) \subseteq \mathfrak{B}(r_2)$ and consider $B := \mathfrak{B}(r_1) \setminus \mathfrak{B}(r_2)$. If $B = \emptyset$ there is nothing to show, otherwise we have $V := \{v \in \mathrm{pos}(t) \mid r_1(v) \in B\} \neq \emptyset$. Now select some prefix-minimal $v \in V$, let $\mathfrak{W}_1 := \{w \in \mathrm{pos}(t) \mid r_1(w) \in \mathfrak{C}(r_1(v))\}$ and let v_0 be the prefix-smallest element of $\mathfrak{W}_1$, which exists due to Proposition 8.13 (i). If $v_0 = \varepsilon$ then from $r(\varepsilon) = q_f$ we get that $\mathfrak{b} := \mathbb{t}_{r_1}(v)$ is a bridge out of $\mathfrak{C}(q_f)$. As $r_2(\varepsilon) = q_f$ as well and there is only one (in $\mathcal{A}_x$ valid) bridge out of $\mathfrak{C}(q_f)$, we get that $\mathfrak{b} \in \mathfrak{B}(r_2)$, so $\mathfrak{b} \notin B$ and $v \notin V$, which is a contradiction.

We conclude that $\varepsilon \neq v_0 = w_1 l$ for some $l \in \mathbb{N}$ and $w_1 \in \mathrm{pos}(t)$. As $\mathrm{degree}_{\mathcal{A}}(r_1(v_0)) > 0$ the transition $\mathfrak{b} := \mathbb{t}_{r_1}(w_1)$ must be a bridge, i.e. $\mathfrak{b} \in \mathfrak{B}$. As we chose v to be minimal, we have $w_1 \notin V$ and therefore $\mathfrak{b} \notin B$, so $\mathfrak{b} \in \mathfrak{B}(r_2)$. Hence, there is some $w_2 \in \mathrm{pos}(t)$ with $\mathbb{t}_{r_2}(w_2) = \mathfrak{b}$. But then $r_2(w_2 l) = r_1(w_1 l) \in \mathfrak{C}(r_1(v))$, so $\mathfrak{W}_2 := \{w \in \mathrm{pos}(t) \mid r_2(w) \in \mathfrak{C}(r_1(v))\} \neq \emptyset$. In particular, for some $w \in \mathrm{pos}(t)$ the transition $\mathbb{t}_{r_2}(w)$ is a bridge out of $\mathfrak{C}(r_1(v))$

and as there is only one such bridge, $\mathfrak{t}_{r_2}(w) = \mathfrak{t}_{r_1}(v)$ must hold. In particular, $\mathfrak{t}_{r_1}(v) \in \mathfrak{B}(r_2)$, so $\mathfrak{b} \notin B$ and $v \notin V$. It follows, that $V = B = \emptyset$ and $\mathfrak{B}(r_1) = \mathfrak{B}(r_2)$.

(ii) By Proposition 8.13 (iii) we have that for all $i \in \{1, \ldots, n\}$ there can be at most one $w \in \mathrm{pos}(t)$ such that $r(w)$ is a bridge out of $\mathfrak{c}_i$. In particular, for every $i \in \{1, \ldots, n\}$ there exists at most one $j \in \{1, \ldots, k_i\}$ such that $\mathfrak{b}_j^{(i)} \in \mathfrak{B}(r)$. For $i \in \{1, \ldots, n\}$ we set $x_i = j$ if $\mathfrak{b}_j^{(i)} \in \mathfrak{B}(r)$ for some $j \in \{1, \ldots, k_i\}$ and if such j does not exist we choose x_i arbitrary. Clearly we have $r \in \mathrm{Run}_{\mathcal{A}_x, \mathbb{F}}(t)$. □

We now have an equivalence relation on J induced by $x \approx y$ iff $\mathfrak{B}(x) = \mathfrak{B}(y)$ for $x, y \in J$. We select a representative of every equivalence class and obtain a set $\{x_1, \ldots, x_k\} \subseteq J$. For this selection we can prove the following proposition, which in essence concludes the proof of Lemma 8.11.

Proposition 8.15.

(i) $[\![\mathcal{A}]\!] = \bigoplus_{i=1}^{k} [\![\mathcal{A}_{x_i}]\!]$.

(ii) *For $i \in \{1, \ldots, k\}$ we have* $\mathrm{degree}(\mathcal{A}_{x_i}) \leq \mathrm{degree}(\mathcal{A})$ *such that $\mathcal{A}_{x_i}$ is polynomially ambiguous and* $\max_{i \leq k} \mathrm{degree}(\mathcal{A}_{x_i}) = \mathrm{degree}(\mathcal{A})$.

Now for $i \in \{1, \ldots, k\}$ let $\mathcal{A}_i = (Q_i, \Gamma, \mu_i, \gamma)$ be the trim part of $\mathcal{A}_{x_i}$. Note that trimming has no influence on properties (i) and (ii). For $p \in Q_i$ we denote the component of p by $\mathfrak{C}_i(p) \in \mathfrak{Q}_i$, where $\mathfrak{Q}_i$ denotes the components of Q_i.

(iii) *For $i \in \{1, \ldots, k\}$ and every $p \in Q_i$ with* $\mathrm{degree}_{\mathcal{A}_i}(p) > 0$ *there is exactly one bridge out of $\mathfrak{C}_i(p)$ and every accepting run r uses this bridge exactly once. Formally*

$$\{d \in \Delta_{\mathcal{A}_i} \mid d \textit{ is a bridge out of } \mathfrak{C}_i(p)\} = \{\mathfrak{b}(p)\}$$

for some $\mathfrak{b}(p) \in \Delta_{\mathcal{A}_i}$ and

$$\forall t \in T_\Gamma \ \forall r \in \mathrm{Run}_{\mathcal{A}_i, \mathbb{F}}(t) : |\{w \in \mathrm{pos}(t) \mid \mathfrak{t}_r(w) = \mathfrak{b}(p)\}| = 1.$$

Proof. (i) Take $t \in T_\Gamma$ and $r \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)$, then by Proposition 8.14 (ii) we have that $r \in \mathrm{Run}_{\mathcal{A}_x,\mathbb{F}}(t)$ for some $x \in J$. By construction we have $x \approx x_i$ for some $i \in \{1,\dots,k\}$, such that $\mathfrak{B}(r) = \mathfrak{B}(x) = \mathfrak{B}(x_i)$. This means $r \in \mathrm{Run}_{\mathcal{A}_{x_i},\mathbb{F}}(t)$, yielding $\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t) = \bigcup_{i=1}^{k} \mathrm{Run}_{\mathcal{A}_{x_i},\mathbb{F}}(t)$. Now assume we have $r \in \mathrm{Run}_{\mathcal{A}_{x_i},\mathbb{F}}(t) \cap \mathrm{Run}_{\mathcal{A}_{x_j},\mathbb{F}}(t)$ for $1 \leq i < j \leq k$, then $\mathfrak{B}(r) = \mathfrak{B}(x_i) \neq \mathfrak{B}(x_j) = \mathfrak{B}(r)$ which is a contradiction, hence $\mathrm{Run}_{\mathcal{A}_{x_i},\mathbb{F}}(t) \cap \mathrm{Run}_{\mathcal{A}_{x_j},\mathbb{F}}(t) = \emptyset$, i.e. we have a partition of the accepting runs in $\mathcal{A}$.

(ii) $\mathrm{degree}(\mathcal{A}_{x_i}) \leq \mathrm{degree}(\mathcal{A})$ is clear as for every $t \in T_\Gamma$ and $i \in \{1,\dots,k\}$ we have $\mathrm{Run}_{\mathcal{A}_{x_i},\mathbb{F}}(t) \subseteq \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)$. The second property is clear by $|\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)| = \sum_{i=1}^{k} |\mathrm{Run}_{\mathcal{A}_{x_i},\mathbb{F}}(t)|$ for every $t \in T_\Gamma$ and the definition of the function degree.

(iii) Let $i \in \{1,\dots,k\}$, then for all $p \in Q_i$ we have $\mathrm{degree}_{\mathcal{A}_i}(p) \leq \mathrm{degree}_{\mathcal{A}_{x_i}}(p) \leq \mathrm{degree}_{\mathcal{A}}(p)$. We prove that if $\mathrm{degree}_{\mathcal{A}_i}(p) > 0$ then $\mathfrak{C}_i(p) = \mathfrak{C}(p)$. Let $t \in T_\Gamma$ such that we find $w \in \mathrm{pos}(t)$ and $r \in \mathrm{Run}_{\mathcal{A}_i,\mathbb{F}}(t; w, p) \subseteq \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t; w, p)$.

Now take $q \in \mathfrak{C}(p)$, then we find $s_1 \in T_\Gamma$ with $w_q \in \mathrm{pos}(s_1)$ and $r_1 \in \mathrm{Run}_{\mathcal{A},p}(s_1; w_q, q)$ and $s_2 \in T_\Gamma$ with $w_p \in \mathrm{pos}(s_2)$ and $r_2 \in \mathrm{Run}_{\mathcal{A},q}(s_2; w_p, p)$. Then for $s := s_1\langle s_2 \to w_q\rangle$ and $r_s := r_1\langle r_2 \to w_q\rangle$ we have $r_s \in \mathrm{Run}_{\mathcal{A},p}(s; w_q, w_q w_p, q, p)$. That is, we have a tree s with a run in $\mathcal{A}$ that goes from p to q to p again.

By Proposition 8.13 (ii) we know, as $\mathrm{degree}_{\mathcal{A}}(p) > 0$, that all footpoints v of bridges out of components of Q of non-trivial degree fulfill $w_q w_p \leq_p v$. In other words, there are no bridges "between" the two p's. Hence, by considering the tree $t' := t\langle s \to w\rangle\langle t|_w \to w w_q w_p\rangle$, which inserts s into t at position w, and the run r' "glued" accordingly, we see $r' \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t')$ and $\mathfrak{B}(r') = \mathfrak{B}(r)$ which means $r' \in \mathrm{Run}_{\mathcal{A}_{x_i},\mathbb{F}}(t') = \mathrm{Run}_{\mathcal{A}_i,\mathbb{F}}(t')$. As q is used by r' we have $q \in Q_i$ and $q \in \mathfrak{C}_i(p)$.

Now if $\mathrm{degree}_{\mathcal{A}_i}(p) > 0$ and $\mathfrak{b} \in \Delta_{\mathcal{A}_i}$ is a bridge out of $\mathfrak{C}_i(p)$ then due to $\mathrm{degree}_{\mathcal{A}}(p) > 0$ and $\mathfrak{C}_i(p) = \mathfrak{C}(p)$ we have that $\mathfrak{b}$ is a bridge out of $\mathfrak{C}(p)$ as well and by construction of $\mathcal{A}_{x_i}$ this bridge $\mathfrak{b} = \mathfrak{b}(p)$ is unique. As $p \in Q_i$ by trimness we can find some $t \in T_\Gamma$, $r \in \mathrm{Run}_{\mathcal{A}_i,\mathbb{F}}$ and $w \in \mathrm{pos}(t)$ with $r(w) = p$. For this run we clearly have $\mathfrak{b}(p) \in \mathfrak{B}(r) = \mathfrak{B}(x_i)$ so by Proposition 8.14 (i) every valid run in $\mathcal{A}_{x_i}$ uses $\mathfrak{b}(p)$, hence so does $\mathcal{A}_i$, as trimming does not influence runs. □

The automata $\mathcal{A}_i$ are all in polynomial standard form and have a degree less than or equal to the degree of $\mathcal{A}$. As their sum equals $\mathcal{A}$, we have proven Lemma 8.11.

8.3 Analysis of the Polynomial Standard Form

From now on, let $\mathcal{A} = (Q, \Gamma, \mu, \gamma)$ be a WTA in polynomial standard form. In this subsection we will show that there exist degree($\mathcal{A}$) many bridges in $\mathcal{A}$, such that given any tree, the number of runs on that tree is bounded universally if we fix the position of these bridges. The bound does not depend on the given tree. This property gives a rather intuitive understanding of what polynomial ambiguity means: if our automaton has degree n, then fixing the positions of n predetermined transitions will determine every run up to a constant number of possibilities.

Definition 8.16 (Λ, rk_Λ, Top)**.** Fix $p \in Q$ with $\mathrm{degree}_{\mathcal{A}}(p) > 0$. As there is exactly one bridge $\mathfrak{b} \in \Delta_{\mathcal{A}}$ out of $\mathfrak{C}(p)$ we define $\mathfrak{b}(\mathfrak{C}(p)) := \mathfrak{b}$ and $\mathfrak{b}(p) := \mathfrak{b}$ as this bridge. We set $\Lambda := \{\mathfrak{b}(q) \,|\, q \in Q,\ \mathrm{degree}_{\mathcal{A}}(q) > 0\}$ and for $\mathfrak{b}(p) = (p_1, \ldots, p_m, a, p_0)$ define the rank of $\mathfrak{b}(p)$, denoted by $rk_\Lambda(\mathfrak{b}(p))$, as $rk_\Lambda(\mathfrak{b}(p)) := |\{i \in \{1, \ldots, m\} \,|\, \mathrm{degree}_{\mathcal{A}}(p_i) > 0\}|$, the amount of p_i of non-trivial degree. We also extend the relation $\preccurlyeq$ to Λ, that is

$$\mathfrak{b}(p_1) \preccurlyeq \mathfrak{b}(p_2) :\Leftrightarrow p_1 \preccurlyeq p_2.$$

For for $\mathfrak{N} \subseteq \Lambda$ we define the set Top($\mathfrak{N}$) as the set of all minimal elements in $\mathfrak{N}$, that is

$$\mathrm{Top}(\mathfrak{N}) := \{d \in \mathfrak{N} \,|\, \forall e \in \mathfrak{N} : (e \preccurlyeq d \longrightarrow e \approx d)\}.$$

Proposition 8.17.

(i) *Let* $\mathfrak{N} \subseteq \Lambda$. *If* $\mathrm{Top}(\mathfrak{N}) = \{\mathfrak{b}_1, \dots, \mathfrak{b}_n\}$ *and* $w_1, \dots, w_n$ *are not pairwise prefix-independent, then*

$$\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t; w_1, \dots, w_n, \mathfrak{b}_1, \dots, \mathfrak{b}_n) = \emptyset.$$

(ii) *For* $\mathfrak{b}_1, \mathfrak{b}_2 \in \Lambda$ *with* $\mathfrak{b}_1 \preccurlyeq \mathfrak{b}_2$, $t \in T_\Gamma$, $w_1, w_2 \in \mathrm{pos}(t)$ *we have*

$$\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t; w_1, w_2, \mathfrak{b}_1, \mathfrak{b}_2) \neq \emptyset \longrightarrow w_1 \leq_p w_2.$$

Proof. (i) If this does not hold, it is an obvious contradiction to the minimality of the elements in $\mathrm{Top}(\mathfrak{N})$.

(ii) Take $\mathfrak{b}_1, \mathfrak{b}_2 \in \Lambda$ with $\mathfrak{b}_1 \preccurlyeq \mathfrak{b}_2$, $t \in T_\Gamma$, $w_1, w_2 \in \mathrm{pos}(t)$. Assume that there exists some $r \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t; w_1, w_2, \mathfrak{b}_1, \mathfrak{b}_2)$. Let $p_1 := r(w_1)$ and $p_2 := r(w_2)$. Due to $\mathfrak{b}(p_1) = \mathfrak{b}_1$, $\mathfrak{b}(p_1) = \mathfrak{b}_2$ and $\mathfrak{b}_1 \preccurlyeq \mathfrak{b}_2$ we have that $p_1 \preccurlyeq p_2$.

If $w_1 \leq_p w_2$ there is nothing to show. If $w_2 \leq_p w_1$, then clearly $p_2 \preccurlyeq p_1$ and so $p_1 \approx p_2$. As $\mathcal{A}$ is standardized this implies $\mathfrak{b}_1 = \mathfrak{b}_2$ and $w_1 = w_2$. In particular, $w_1 \leq_p w_2$.

Now assume that neither $w_1 \leq_p w_2$ nor $w_2 \leq_p w_1$, i.e. w_1 and w_2 are prefix-independent. Due to $p_1 \preccurlyeq p_2$ we can find some $s \in T_\Gamma$ and $v \in \mathrm{pos}(s)$ such that $r_s \in \mathrm{Run}_{\mathcal{A},p_1}(s; v, p_2)$ exists. Then for $t' := t\langle s \to w_1 \rangle$ we have $r' := r\langle r_s \to w_1 \rangle \in \mathrm{Run}_{\mathcal{A},\mathbb{F}}(t'; w_1 v, w_2, p_2, p_2)$. As we have that $w_1 v$ and w_2 are prefix-independent, there must be at least 2 bridges out of $\mathfrak{C}(p_2)$ in r', i.e. $|\{w \in \mathrm{pos}(t) \mid \mathbb{t}_{r'}(w) = \mathfrak{b}(p_2)\}| \geq 2$. This is a contradiction to the assumption that $\mathcal{A}$ is standardized. □

Before we get to some deeper results, we need one more construction. For $p \in Q$ the automata $\mathcal{F}_p$ accept subtrees with runs that from a top-down point of view "begin" with p. But we also need an automaton for the "upper part" of the run, ending in p, partial runs ending in p, so to speak.

Definition 8.18 ($\mathcal{H}_p^q$)**.** For $p, q \in Q$ we define the automaton $\mathcal{H}_p^q = (Q \times$

$\{0,1\}, \Gamma_\perp, \mu_p^q, \alpha_p^q)$ where $\Gamma_\perp := \Gamma \cup \{\perp\}$ with $\perp \notin \Gamma$ and $rk_{\Gamma_\perp}(\perp) := 0$. The rank of all other letters is preserved and μ_p^q and α_p^q are defined for $(\binom{p_1}{k_1}, \ldots, \binom{p_m}{k_m}, a, \binom{p_0}{k_0}) \in \Delta_{\mathcal{H}_p^q}$ as:

$$\mu_p^q\left(\binom{p_1}{k_1}, \ldots, \binom{p_m}{k_m}, a, \binom{p_0}{k_0}\right) :=$$

$$\begin{cases} 1 & \text{if } a = \perp \wedge \binom{p_0}{k_0} = \binom{p}{1} \\ \mu(p_1, \ldots, p_m, a, p_0) & \text{if } a \in \Gamma \wedge k_0 = 0 \wedge k_1 = \ldots = k_m = 0 \\ & \text{or if } a \in \Gamma \wedge k_0 = 1 \wedge \exists! i \in \{1, \ldots, m\} : k_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha_p^q\binom{p_0}{k_0} := \begin{cases} 1 & \text{if } \binom{p_0}{k_0} = \binom{q}{1} \\ 0 & \text{otherwise.} \end{cases}$$

From a bottom-up point of view, the automaton emulates $\mathcal{A}$ in the first coordinate of the states, deterministically remembers every occurrence of $\perp$ in the second and forces the runs to take value p at the leaves labeled $\perp$.

Proposition 8.19. *Let $p, q \in Q$, then*

(i) *If $t \in T_{\Gamma_\perp}$ and $\text{Run}_{\mathcal{H}_p^q,\mathbb{F}}(t) \neq \emptyset$, then $\exists! w \in \text{pos}(t) : t(w) = \perp$.*

(ii) *For $t \in T_\Gamma$ and $w \in \text{pos}(t)$ we can identify $\text{Run}_{\mathcal{H}_p^q,\mathbb{F}}(t\langle \perp \to w \rangle)$ with $\text{Run}^\partial_{\mathcal{A},q}(t; w, p)$.*

(iii) *$\mathcal{H}_p^q$ is polynomially ambiguous.*

(iv) *$\mathcal{H}_q^q$ is unambiguous.*

(v) *If $p \preccurlyeq p'$ for $p' \in Q$ then $\text{degree}(\mathcal{H}_p^q) \leq \text{degree}(\mathcal{H}_{p'}^q)$.*

Proof. (i) Take $t \in T_{\Gamma_\perp}$ and $r \in \text{Run}_{\mathcal{A}_p^q,\mathbb{F}}(t)$. If $t(w) \neq \perp$ for all $w \in \text{pos}(t)$ we easily see that for all $w \in \text{pos}(t)$ we have $r(w) = \binom{p(w)}{0}$ for some $p(w) \in Q$. In particular, $r(\varepsilon) = \binom{p(\varepsilon)}{0}$ for some $p(\varepsilon) \in Q$ and r is not accepting. If $w_1, w_2 \in \text{pos}(t)$

with $w_1 \neq w_2$ and $t(w_1) = t(w_2) = \bot$ then by definition of μ for $w \in \mathrm{pos}(t)$ with $w \leq_p w_1$ or $w \leq_p w_2$ we have $r(w) = \binom{p(w)}{1}$ for some $p(w) \in Q$. As $\bot$ can only occur at leaves, w_1 and w_2 must be prefix-independent. Let v be the largest common prefix of w_1 and w_2, i.e. $w_1 = viw_1'$, $w_2 = vjw_2'$ with $i \neq j$, then $\mathfrak{t}_r(v)$ cannot be a valid transition, as the second coordinate of both $r(vi)$ and $r(vj)$ is 1.

(ii) Take $t \in T_\Gamma$, $w \in \mathrm{pos}(t)$ and let $\pi\colon \mathrm{Run}_{\mathcal{H}_p^q,\mathbb{F}}(t\langle\bot \to w\rangle) \to \mathrm{Run}^\partial_{\mathcal{A},q}(t; w, p)$ be defined by projection on the Q-coordinate. This is well defined as $\mathrm{pos}(t\langle\bot \to w\rangle) = (\mathrm{pos}(t) \setminus w\mathrm{pos}(t|_w)) \cup \{w\}$ and by definition of μ_p^q. It is also injective as $\mathcal{H}_p^q$ is "deterministic" in the second coordinate of the states. For surjectivity take $r \in \mathrm{Run}^\partial_{\mathcal{A},q}(t; w, p)$ and for $v \in \mathrm{pos}(t\langle\bot \to w\rangle)$ define

$$r'(v) := \begin{cases} \binom{r(v)}{1} & \text{if } v \leq_p w \\[2em] \binom{r(v)}{0} & \text{otherwise.} \end{cases}$$

then $r' \in \mathrm{Run}_{\mathcal{H}_p^q,\mathbb{F}}(t\langle\bot \to w\rangle)$ and $\pi(r') = r$. Since the runs correspond to each other by removing or supplying the second coordinate, we can identify r' with r.

(iii) Clear by taking some fixed $t \in T_\Gamma$ and $r \in \mathrm{Run}_{\mathcal{A},p}(t)$ and joining r into the runs of $\mathcal{H}_p^q$.

(iv) Clear with (ii) and Proposition 8.9(i).

(v) Same procedure as Proposition 8.7(iii). □

Now we come to the main result of this subsection.

Lemma 8.20. *Let $p \in Q$ with $l := \mathrm{degree}_{\mathcal{A}}(p) \geq 0$.*

(I) *There exists a set $\mathfrak{N}(p) = \{\mathfrak{b}_1, \dots, \mathfrak{b}_l\} \subseteq \Lambda$ fulfilling the following properties:*

(i) *There is a constant $C > 0$ such that for all $t \in T_\Gamma$ and $w_1, \dots, w_l \in$*

$\mathrm{pos}(t)$ *we have*

$$|\mathrm{Run}_{\mathcal{A},p}(t; w_1, \dots, w_l, \mathfrak{b}_1, \dots, \mathfrak{b}_l)| \leq C.$$

(ii) *Assume* $\mathfrak{b}_i = (p_1^{(i)}, \dots, p_{m_i}^{(i)}, a^{(i)}, q^{(i)})$ *for all* $i \in \{1, \dots, l\}$ *and without loss of generality* $\mathrm{Top}(\mathfrak{N}(p)) = \{\mathfrak{b}_1, \dots, \mathfrak{b}_n\}$ *for some* $n \in \mathbb{N}$. *Then there is a constant* $\hat{C} > 0$ *such that for all* $t \in T_\Gamma$ *and pairwise prefix-independent* $w_1, \dots, w_n \in \mathrm{pos}(t)$ *we have*

$$|\mathrm{Run}^{\partial}_{\mathcal{A},p}(t; w_1, \dots, w_n, q^{(1)}, \dots, q^{(n)})| \leq \hat{C}$$

and

(iii)

$$n + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \mathrm{degree}_{\mathcal{A}}(p_j^{(i)}) = l.$$

(II) *Furthermore there exists a sequence of trees* $(t_n)_{n\in\mathbb{N}}$ *in* T_Γ *and a constant* $\check{C} > 0$ *such that for all* $n \in \mathbb{N}$*:*

- $|\mathrm{pos}(t_n)| \leq \check{C} \cdot n$ *and*
- $|\mathrm{Run}_{\mathcal{A},p}(t_n)| \geq n^l$.

That is, we want to prove that if $\mathcal{F}_p$ is of degree l, then for all trees the runs of $\mathcal{F}_p$ on those trees are determined up to a constant C by fixing the location of l bridges. Furthermore, the degree of $\mathcal{F}_p$ is not only an upper bound on the amount of runs for a given tree, but also a lower bound.

Proof. Let $p \in Q$. If $\mathrm{degree}_{\mathcal{A}}(p) = 0$, i.e. $\mathcal{F}_p$ is finitely ambiguous, the proposition is not more than the definition of finite ambiguity. We therefore only need to consider the case $\mathrm{degree}_{\mathcal{A}}(p) > 0$ in greater depth. In this case we always have the (one) bridge $\mathfrak{b}(p) = (p_1, \dots, p_m, a, p_0)$ out of $\mathfrak{C}(p)$. We prove the statement by induction: we assume it is true for $p' \in Q$ with $p \preccurlyeq p'$ and $p \not\approx p'$ and then prove it for p. Set $k_i := \mathrm{degree}_{\mathcal{A}}(p_i)$ for $i \in \{1, \dots, m\}$ and $k := \sum_{i=1}^{m} k_i$.

Step 1: We show $k \leq \mathrm{degree}_{\mathcal{A}}(p) \leq k+1$.
For $k \leq \mathrm{degree}_{\mathcal{A}}(p)$ take $t \in T_\Gamma$ such that $r \in \mathrm{Run}_{\mathcal{A},p}(t)$ exists and let $w \in \mathrm{pos}(t)$ be the position where $\mathbb{t}_r(w) = \mathfrak{b}(p)$. Furthermore for $i \in \{1, \dots, m\}$ take tree sequences $(t_n^{(i)})_{n\in\mathbb{N}}$ with $|\mathrm{pos}(t_n^{(i)})| \leq C_i n$ and $|\mathrm{Run}_{\mathcal{A},p_i}(t_n^{(i)})| \geq n^{k_i}$. By assumption such sequences exist and when considering the tree sequence defined by $s_n := t\langle t_n^{(1)} \to w1\rangle \dots \langle t_n^{(m)} \to wm\rangle$ we see

$$\mathrm{pos}(s_n)| \leq |\mathrm{pos}(t)| + \sum_{i=1}^{m} |\mathrm{pos}(t_n^{(i)})| \leq C_0 + \sum_{i=1}^{m} C_i n \leq Cn$$

for $C_0 := |\mathrm{pos}(t)|$ and $C := \sum_{i=0}^{m} C_i$ and

$$\mathrm{Run}_{\mathcal{A},p}(s_n)| \geq \prod_{i=1}^{m} |\mathrm{Run}_{\mathcal{A},p_i}(t_n^{(i)})| \geq \prod_{i=1}^{m} n^{k_i} = n^k.$$

This clearly shows $\mathrm{degree}_{\mathcal{A}}(p) \geq k$. The sequence $(s_n)_{n\in\mathbb{N}}$ also fulfills (II) for p if $\mathrm{degree}_{\mathcal{A}}(p) = k$.

For $\mathrm{degree}_{\mathcal{A}}(p) \leq k+1$ we consider the sets $\mathfrak{N}(p_i) = \{\mathfrak{b}_1^{(i)}, \dots, \mathfrak{b}_{k_i}^{(i)}\}$ for $i \in \{1, \dots, m\}$. Now take $t \in T_\Gamma$, $w_0, w_1^{(1)}, w_2^{(1)}, \dots, w_{k_m}^{(m)} \in \mathrm{pos}(t)$ and consider $r \in \mathrm{Run}_{\mathcal{A},p}(t; w_0, w_1^{(1)}, \dots, w_{k_m}^{(m)}, \mathfrak{b}(p), \mathfrak{b}_1^{(1)}, \dots, \mathfrak{b}_{k_m}^{(m)})$. By Proposition 8.9 (ii) we get that r is uniquely determined on all positions v "above" w_0, i.e. when $\neg(w_0 \leq_p v)$. Now take $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, k_i\}$, then $r(w_0 i) = p_i$ by choice of r. Due to Proposition 8.17 (ii) and the fact that $\mathfrak{b}(p_i) \preccurlyeq \mathfrak{b}_j^{(i)}$ this means that we have $w_0 i \leq_p w_j^{(i)}$ such that $w_j^{(i)} = w_0 i v_j^{(i)}$ for some $v_j^{(i)} \in \mathrm{pos}(t|_{w_0 i})$. Combining this with the assumption that $|\mathrm{Run}_{\mathcal{A},p_i}(t|_{w_0 i}; v_1^{(i)}, \dots, v_{k_i}^{(i)}, \mathfrak{b}_1^{(1)}, \dots \mathfrak{b}_{k_i}^{(i)})| \leq C_i$ for some constants $C_i > 0$ we get

$$|\mathrm{Run}_{\mathcal{A},p}(t; w_0, w_1^{(1)}, \dots, w_{k_m}^{(m)}, \mathfrak{b}(p), \mathfrak{b}_1^{(1)}, \dots, \mathfrak{b}_{k_m}^{(m)})| \leq \prod_{i=1}^{m} C_i.$$

Finally, with $C := \prod_{i=1}^{m} C_i$,

$$|\mathrm{Run}_{\mathcal{A},p}(t)|$$

$$\begin{aligned}&= \sum_{w_0, w_1^{(1)}, \dots, w_{k_m}^{(m)} \in \mathrm{pos}(t)} |\mathrm{Run}_{\mathcal{A},p}(t; w_0, w_1^{(1)}, \dots, w_{k_m}^{(m)}, \mathfrak{b}(p), \mathfrak{b}_1^{(1)}, \dots, \mathfrak{b}_{k_m}^{(m)})| \\ &\leq C|\mathrm{pos}(t)|^{k+1}\end{aligned}$$

gives us $\mathrm{degree}_{\mathcal{A}}(p) \leq k+1$. Also, if $\mathrm{degree}_{\mathcal{A}}(p) = k+1$, the set $\mathfrak{N}(p) := \{\mathfrak{b}(p)\} \cup \bigcup_{i=1}^m \mathfrak{N}(p_i)$ fulfills requirement (i) of our proposition. As obviously $\mathrm{Top}(\mathfrak{N}(p)) = \mathfrak{b}(p)$, we get (ii) by using Proposition 8.9 (ii) again and (iii) by definition of k which means in this case we have proven (I) already.

Step 2: (I) for $\mathrm{degree}_{\mathcal{A}}(p) = k$ and (II) for $\mathrm{degree}_{\mathcal{A}}(p) = k+1$
Now we know that we only need to consider the cases $\mathrm{degree}_{\mathcal{A}}(p) = k$ and $\mathrm{degree}_{\mathcal{A}}(p) = k+1$. Furthermore, if $\mathrm{degree}_{\mathcal{A}}(p) = k$, we have (II) already, and if $\mathrm{degree}_{\mathcal{A}}(p) = k+1$, we have (I) already. We consider the remaining cases.

We use a recursive method to find a certain state $p' \in Q$. For the start we set $p' := p$. By Proposition 8.19 (iv) the automaton $\mathcal{H}_p^p$ is unambiguous. Now assume $\mathfrak{b}(p') = (p'_1, \dots, p'_{m'}, a', p'_0)$. As long as $\mathcal{H}_{p'}^p$ is finitely ambiguous and $rk_\Lambda(\mathfrak{b}(p')) = 1$, we set $p' := p'_i$ for the one $i \in \{1, \dots, m'\}$ with $\mathrm{degree}_{\mathcal{A}}(p'_i) > 0$. We stop this procedure once either $\mathcal{H}_{p'}^p$ is not finitely ambiguous anymore or $rk_\Lambda(\mathfrak{b}(p')) \neq 1$. We consider four different cases which can occur after stopping.

Case 1: $\mathcal{H}_{p'}^p$ finitely ambiguous and $rk_\Lambda(\mathfrak{b}(p')) = 0$
Take $t \in T_\Gamma$, $w \in \mathrm{pos}(t)$ and consider the set $\mathrm{Run}_{\mathcal{A},p}(t; w, \mathfrak{b}(p'))$. Assume $\mathfrak{b}(p') = (p'_1, \dots, p'_{m'}, a', p'_0)$, then for $i \in \{1, \dots, m'\}$ we have $|\mathrm{Run}_{\mathcal{A},p_i}(t|_{wi})| \leq C_i$ for $C_i > 0$ not depending on t. As we assume $\mathcal{H}_{p'}^p$ to be finitely ambiguous and by Proposition 8.19 (v), we have a constant $C_0 > 0$ such that

$$|\mathrm{Run}^{\partial}_{\mathcal{A},p}(t; w, p'_0)| = |\mathrm{Run}_{\mathcal{H}_{p'_0}^p, \mathbb{F}}(t\langle \perp \to w\rangle)| \leq C_0.$$

This means

$$|\mathrm{Run}_{\mathcal{A},p}(t;w,\mathfrak{b}(p'))| \le |\mathrm{Run}^{\partial}_{\mathcal{A},p}(t;w,p'_0)| \cdot \prod_{i=1}^{m'} |\mathrm{Run}_{\mathcal{A},p_i}(t|_{wi})| \le \prod_{i=0}^{m'} C_i$$

and, with $C := \prod_{i=0}^{m'} C_i$, we get

$$|\mathrm{Run}_{\mathcal{A},p}(t)| = \sum_{w\in\mathrm{pos}(t)} |\mathrm{Run}_{\mathcal{A},p}(t;w,\mathfrak{b}(p'))| \le C \cdot |\mathrm{pos}(t)|$$

which means

$$0 < \mathrm{degree}_{\mathcal{A}}(p) \le 1$$

so $\mathrm{degree}_{\mathcal{A}}(p) = 1$. As we have seen the set $\mathfrak{N}(p) := \{\mathfrak{b}(p')\}$ now fulfills (i), (ii) and (iii), so we have proven (I) in this case. Moreover we have proven property (II) in this case in Proposition 8.9 (iv).

Case 2: $\mathcal{H}^p_{p'}$ finitely ambiguous and $rk_\Lambda(\mathfrak{b}(p')) > 1$

Assume $\mathfrak{b}(p') = (p'_1, \dots, p'_{m'}, a', p'_0)$, set $k'_i := \mathrm{degree}_{\mathcal{A}}(p'_i)$ and take $j_1 \neq j_2$ with $k'_{j_1} > 0$ and $k'_{j_2} > 0$. We set $k' := \sum_{i=1}^{m'} k'_i$. If $p = p'$ we have $k = k'$ trivially. Otherwise $k' \le \mathrm{degree}_{\mathcal{A}}(p')$ due to Step 1 and $\mathrm{degree}_{\mathcal{A}}(p') \le k$ due to Proposition 8.7 (iii), so $k' \le k$. We write $\mathfrak{N}(p'_i) = \{\mathfrak{b}^{(i)}_1, \dots, \mathfrak{b}^{(i)}_{k'_i}\}$ for $i \in \{1, \dots, m'\}$. Now Take $t \in T_\Gamma$, $w_1, w_2 \in \mathrm{pos}(t)$, consider $r \in \mathrm{Run}_{\mathcal{A},p}(t; w_1, w_2, \mathfrak{b}^{(j_1)}_1, \mathfrak{b}^{(j_2)}_1)$ and take $w \in \mathrm{pos}(t)$ with $\mathbb{t}_r(w) = \mathfrak{b}(p')$ which always exists by Proposition 8.15 (iii). From Proposition 8.17 (ii) we get that $w \le_p w_1$ and $w \le_p w_2$. Furthermore we have $wj_1 \le_p w_1$ and $wj_2 \le w_2$ and due to $j_1 \neq j_2$ this means w is the largest common prefix of w_1 and w_2. In particular, $w = w(w_1, w_2)$ is a function of the positions of $\mathfrak{b}^{(j_1)}_1$ and $\mathfrak{b}^{(j_2)}_1$. As it was in Case 1 we can find a constant $C_0 > 0$ and constants $C_1, \dots, C_{m'} > 0$ such that

$$\begin{aligned}&|\mathrm{Run}_{\mathcal{A},p}(t)|\\ &= \sum_{w^{(1)}_1\in\mathrm{pos}(t)} \cdots \sum_{w^{(m')}_{k'_{m'}}\in\mathrm{pos}(t)} |\mathrm{Run}_{\mathcal{A},p}(t; w^{(1)}_1, \dots, w^{(m')}_{k'_{m'}}, \mathfrak{b}^{(1)}_1, \dots, \mathfrak{b}^{(m')}_{k_{m'}})|\end{aligned}$$

$$= \sum_{w_1^{(1)} \in \mathrm{pos}(t|_{w'1})} \cdots \sum_{w^{(m')}_{k'_{m'}} \in \mathrm{pos}(t|_{w'm'})} |\mathrm{Run}^{\partial}_{\mathcal{A},p}(t; w', p'_0)| \\ \cdot \prod_{i=1}^{m'} |\mathrm{Run}_{\mathcal{A},p_i}(t|_{w'i}; w_1^{(i)}, \dots, w^{(i)}_{k'_i}, \mathfrak{b}_1^{(i)}, \dots, \mathfrak{b}^{(i)}_{k'_i})|$$

with $w' = w(w_1^{(j_1)}, w_1^{(j_2)})$

$$\leq \sum_{w_1^{(1)} \in \mathrm{pos}(t)} \cdots \sum_{w^{(m')}_{k'_{m'}} \in \mathrm{pos}(t)} \prod_{i=0}^{m'} C_i = C|\mathrm{pos}(t)|^{k'} \qquad \text{with } C := \prod_{i=0}^{m'} C_i$$

so $k \leq \mathrm{degree}_{\mathcal{A}}(p) \leq k'$, i.e. $\mathrm{degree}_{\mathcal{A}}(p) = k = k'$. We also see that in this case the set $\mathfrak{N}(p) := \bigcup_{i=1}^{m'} \mathfrak{N}(p'_i)$ satisfies (i). For (ii) and (iii) we notice that $\mathrm{Top}(\mathfrak{N}(p)) = \bigcup_{i=1}^{m'} \mathrm{Top}(\mathfrak{N}(p'_i))$. Write $\mathrm{Top}(\mathfrak{N}(p'_i)) = \{\mathfrak{b}_1^{(i)}, \dots, \mathfrak{b}_{n_i}^{(i)}\}$ and let $q_1^{(i)}, \dots, q_{n_i}^{(i)}$ be the states associated to these bridges as done in (ii). Let $t \in T_\Gamma$, $w_1^{(1)}, \dots, w_{n_{m'}}^{(m')} \in \mathrm{pos}(t)$ be pairwise prefix-independent and
$r \in \mathrm{Run}^{\partial}_{\mathcal{A},p}(t; w_1^{(1)}, \dots, w_{n_{m'}}^{(m')}, q_1^{(1)}, \dots, q_{n_{m'}}^{(m')})$. With the same reasoning as earlier, we find that the position $w \in \mathrm{pos}(t)$ with $\mathbb{t}_r(w) = \mathfrak{b}(p')$ is a function of the positions of $q_1^{(j_1)}$ and $q_1^{(j_2)}$. We abbreviate this position by w' again and if r as chosen exists can write $w_n^{(i)} = w'iv_n^{(i)}$ for $i \in \{1, \dots, m'\}$, $n \in \{1, \dots, n_i\}$ and $v_n^{(i)} \in \mathrm{pos}(t|_{w'i})$. Then for some constants $C'_i > 0$ we have

$$|\mathrm{Run}^{\partial}_{\mathcal{A},p}(t; w_1^{(1)}, \dots, w_{n_{m'}}^{(m')}, q_1^{(1)}, \dots, q_{n_{m'}}^{(m')})| \\ = |\mathrm{Run}^{\partial}_{\mathcal{A},p}(t; w', p'_0)| \cdot \prod_{i=1}^{m'} |\mathrm{Run}^{\partial}_{\mathcal{A},p_i}(t|_{w'i}; v_1^{(i)}, \dots, v_{n_i}^{(i)}, q_1^{(i)}, \dots, q_{n_i}^{(i)})| \\ \leq C_0 \cdot \prod_{i=1}^{m'} C'_i \qquad \text{with } w' = w(w_1^{(j_1)}, w_1^{(j_2)})$$

so we have (ii). That (iii) also holds is clear by induction and the definition of $k' = k$, therefore we have (I) in this case.

Case 3: $\mathcal{H}^p_{p'}$ not finitely ambiguous and $k' := \mathrm{degree}_{\mathcal{A}}(p') < k$
As $\mathcal{H}^p_{p'}$ is not finitely ambiguous, we know that $p \neq p'$. Therefore right before coming to p' we considered some $q' \in Q$ with $\mathcal{H}^p_{q'}$ finitely ambiguous and

$rk_\Lambda(\mathfrak{b}(q')) = 1$. Let $\mathfrak{b}(q') = (q'_1, \dots, q'_{m'}, a', q'_0)$ and take $j \in \{1, \dots, m'\}$ with $q'_j = p'$. For $i \neq j$ we have $\text{degree}_\mathcal{A}(q'_i) = 0$, so there are constants $C_i > 0$ with $|\text{Run}_{\mathcal{A},q'_i}(t)| \leq C_i$ for every $t \in T_\Gamma$. As $\mathcal{H}^p_{q'}$ is finitely ambiguous there is a constant $C_0 > 0$ such that for every $t \in T_\Gamma$ and $w \in \text{pos}(t)$ we have $|\text{Run}^\partial_{\mathcal{A},p}(t; w, q'_0)| \leq C_0$. For p' we write $\mathfrak{N}(p') = \{\mathfrak{b}_1, \dots, \mathfrak{b}_{k'}\}$ and see that for some constant $C_j > 0$ and $t \in T_\Gamma$ we have

$$\begin{aligned}
&|\text{Run}_{\mathcal{A},p}(t)| \\
&= \sum_{w_0 \in \text{pos}(t)} \cdots \sum_{w_{k'} \in \text{pos}(t)} |\text{Run}_{\mathcal{A},p}(t; w_0, w_1, \dots, w_{k'}, \mathfrak{b}(q'), \mathfrak{b}_1, \dots, \mathfrak{b}_{k'})| \\
&= \sum_{w_0 \in \text{pos}(t)} \sum_{w_1 \in \text{pos}(t|_{w_0 j})} \cdots \sum_{w_{k'} \in \text{pos}(t|_{w_0 j})} \Bigg(|\text{Run}^\partial_{\mathcal{A},p}(t; w_0, q'_0)| \cdot \prod_{\substack{i=1 \\ i \neq j}}^{m'} |\text{Run}_{\mathcal{A},q'_i}(t|_{w_0 i})| \\
&\qquad \cdot |\text{Run}_{\mathcal{A},p}(t|_{w_0 j}; w_1, \dots, w_{k'}, \mathfrak{b}_1, \dots, \mathfrak{b}_{k'})| \Bigg) \\
&\leq |\text{pos}(t)|^{k'+1} \prod_{i=0}^{m'} C_i
\end{aligned}$$

and this means $\text{degree}_\mathcal{A}(p) \leq k' + 1 \leq k$. We see that in this case the set $\mathfrak{N}(p) := \mathfrak{N}(p') \cup \{\mathfrak{b}(q')\}$ fulfills (i). As then $\text{Top}(\mathfrak{N}(p)) = \{\mathfrak{b}(q')\}$ we get (ii) from the fact that $|\text{Run}^\partial_{\mathcal{A},p}(t; w, q'_0)| \leq C_0$ for all $t \in T_\Gamma$ and every $w \in \text{pos}(t)$. We get (iii) from $k' = \text{degree}_\mathcal{A}(p') = k - 1$. Hence, we have (I) for this case.

Case 4: $\mathcal{H}^p_{p'}$ not finitely ambiguous and $\text{degree}_\mathcal{A}(p') = k$

As $\mathcal{H}^p_{p'}$ is not finitely ambiguous, we have $p \neq p'$ and by Proposition 8.9 (iv) we can find a sequence of trees $(t_n)_{n\in\mathbb{N}} \subseteq T_{\Gamma_\perp}$ with $|\text{pos}(t_n)| \leq C_1 \cdot n$ and $|\text{Run}_{\mathcal{H}^p_{p'},\mathbb{F}}(t_n)| \geq n$ for some constant $C_1 > 0$ and all $n \in \mathbb{N}$. By induction we can also find a sequence of trees $(s_n)_{n\in\mathbb{N}}$ such that $|\text{pos}(s_n)| \leq C_2 \cdot n$ and $|\text{Run}_{\mathcal{A},p'}(s_n)| \geq n^k$ for some constant $C_2 > 0$ and $n \in \mathbb{N}$. We set $w_n \in \text{pos}(t_n)$ as the unique position for which $t_n(w_n) = \perp$ and define $t'_n := t_n\langle s_n \to w_n\rangle$. Then with the help of Proposition 8.19 (ii) we see that for $n \in \mathbb{N}$

$$\begin{aligned}|\mathrm{Run}_{\mathcal{A},p}(t'_n)| &\geq |\mathrm{Run}^{\partial}_{\mathcal{A},p}(t'_n; w_n, p')| \cdot |\mathrm{Run}_{\mathcal{A},p'}(t'_n|_{w_n})| \\ &= |\mathrm{Run}_{\mathcal{H}^p_{p'},\mathbb{F}}(t_n)| \cdot |\mathrm{Run}_{\mathcal{A},p'}(s_n)| \\ &\geq n^{k+1}\end{aligned}$$

and

$$|\mathrm{pos}(t'_n)| \leq |\mathrm{pos}(t_n)| + |\mathrm{pos}(s_n)| \leq (C_1 + C_2)n$$

so $\mathrm{degree}_{\mathcal{A}}(p) \geq k+1$. We clearly have also proven (II) in this case.

Step 3: Conclusion

First notice, that the case analysis in Step 2 is exhaustive: if $\mathcal{H}^p_{p'}$ is finitely ambiguous, then we continue the procedure if $rk_\Lambda(\mathfrak{b}(p')) = 1$ and otherwise the only cases left are $rk_\Lambda(\mathfrak{b}(p')) = 0$ and $rk_\Lambda(\mathfrak{b}(p')) > 1$, which we both covered. If $\mathcal{H}^p_{p'}$ is not finitely ambiguous, then we know that $p' \neq p$, so that $\mathrm{degree}_{\mathcal{A}}(p') \leq k$ simply due to the way our recursion works in the very first step. We have covered both the cases $\mathrm{degree}_{\mathcal{A}}(p') = k$ and $\mathrm{degree}_{\mathcal{A}}(p') < k$.

Now by Step 1 we know that either $\mathrm{degree}_{\mathcal{A}}(p) = k$ or $\mathrm{degree}_{\mathcal{A}}(p) = k+1$.

If $\mathrm{degree}_{\mathcal{A}}(p) = k$ then also by Step 1 we have property (II). In this case, only the cases 1, 2 and 3 of Step 2 are possible. In each of these cases, we have property (I).

If $\mathrm{degree}_{\mathcal{A}}(p) = k+1$, we have property (I) by Step 1. Furthermore, for $\mathrm{degree}_{\mathcal{A}}(p) = k+1$ only the cases 1 and 4 of Step 2 are possible. In both of these cases we have proven property (II). In conclusion, (I) and (II) hold in every possible case. □

While the preceding lemma is interesting as a whole, we will only need point (I) in the sequel. To use this property, we define another automaton very similar to $\mathcal{H}^q_p$.

Definition 8.21 ($\mathcal{G}_p$, Γ_l)**.** First take $p \in Q$ and we write $\mathrm{Top}(\mathfrak{N}(p)) =$

$\{\mathfrak{b}_1, \dots, \mathfrak{b}_l\}$ with $\mathfrak{b}_i = (p_1^{(i)}, \dots, p_{m_i}^{(i)}, a^{(i)}, q^{(i)})$ for $i \in \{1, \dots, l\}$. Then let $\mathcal{G}_p = (Q \times \{0,1\}^l, \Gamma_l, \nu_p, \beta_p)$ be the automaton defined in the following way:

$$\Gamma_l := \Gamma \cup \{\bot_1, \dots, \bot_l\}$$

with $\bot_i \notin \Gamma$ for $i \in \{1, \dots, l\}$ and

$$rk_{\Gamma_l}(a) := \begin{cases} rk_\Gamma(a) & \text{if } a \in \Gamma \\ 0 & \text{otherwise.} \end{cases}$$

For $m \in \mathbb{N}$, $a \in \Gamma_l^{(m)}$, $p_0, p_1, \dots, p_m \in Q$ and $k_0, k_1, \dots, k_m \in \{0,1\}^l$ with $k_i = (k_i^{(1)}, \dots, k_i^{(l)})$ for $i \in \{0, \dots, l\}$ we define

$$\nu_p\left(\left(\begin{smallmatrix} p_1 \\ k_1 \end{smallmatrix}\right), \dots, \left(\begin{smallmatrix} p_m \\ k_m \end{smallmatrix}\right), a, \left(\begin{smallmatrix} p_0 \\ k_0 \end{smallmatrix}\right)\right) :=$$

$$\begin{cases} 1 & \text{if } a = \bot_i \text{ for some } i \in \{1, \dots, l\} \text{ and } p_0 = q^{(i)} \text{ and} \\ & k_0^{(i)} = 1 \text{ and } k_0^{(j)} = 0 \text{ for all } j \in \{1, \dots, l\} \text{ with } j \neq i \\ \mu(p_1, \dots, p_m, a, p_0) & \text{if } a \in \Gamma \text{ and for all } i \in \{1, \dots, l\} \\ & \text{either } k_0^{(i)} = k_1^{(i)} = \ldots = k_m^{(i)} = 0 \\ & \text{or } k_0^{(i)} = 1 \wedge \exists! j \in \{1, \dots, m\} : k_j^{(i)} = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta_p\left(\begin{smallmatrix} p_0 \\ k_0 \end{smallmatrix}\right) := \begin{cases} 1 & \text{if } p_0 = p \text{ and } k_0^{(1)} = \ldots = k_0^{(l)} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 8.22. *Let $p \in Q$ and $l := |\mathrm{Top}(\mathfrak{N}(p))|$.*

(i) *If $t \in T_{\Gamma_l}$ and $\mathrm{Run}_{\mathcal{G}_p,\mathbb{F}}(t) \neq \emptyset$, then $\forall i \in \{1, \dots, l\}\ \exists! w_i \in \mathrm{pos}(t) : t(w_i) = \bot_i$.*

(ii) *For $t \in T_\Gamma$ and pairwise prefix-independent $w_1, \dots, w_l \in \mathrm{pos}(t)$ we can identify $\mathrm{Run}_{\mathcal{G}_p,\mathbb{F}}(t\langle\bot_1 \to w_1\rangle \dots \langle\bot_l \to w_l\rangle)$ with $\mathrm{Run}^\partial_{\mathcal{A},p}(t; w_1, \dots, w_l,$*

$q^{(1)}, \ldots, q^{(l)})$.

(iii) *$\mathcal{G}_p$ is finitely ambiguous.*

Proof. (i) Same as Proposition 8.19 (i)

(ii) Same as Proposition 8.19 (ii).

(iii) Combine (ii) and property (ii) of Lemma 8.20. □

8.4 Two Transformations on Logic Formulas

What we essentially want to do is, instead of letting the automaton $\mathcal{A}$ run directly on a given tree, to cut this tree into several pieces at prefix-independent positions and run automata $\mathcal{F}_p$ on the resulting subtrees and an automaton $\mathcal{G}_p$ on the upper part. However, even if we find logic formulas to describe the automata $\mathcal{F}_p$ and $\mathcal{G}_p$, we have no way of using them. A formula always evaluates a whole tree and there is no elementary method to tell a formula to only evaluate a subtree, for example. To remedy this, we define two mappings $\mathfrak{G}$ and $\mathfrak{F}$ which will effectively turn a given formula into one evaluating only a part of a tree. First, we define some abbreviations:

Definition 8.23 (Basic Abbreviations)**.**

$$x \leq_p y := \forall X.\Bigg(\Bigg(y \in X \wedge \forall z.\Big(\big(\bigvee_{i=1}^{rk(\Gamma)} (\exists z'.(\text{edge}_i(z,z') \wedge z' \in X))\big) \to z \in X\Big)\Bigg) \to x \in X\Bigg)$$

$$x = z := z \leq_p x \wedge x \leq_p z$$

$$x > z := z \leq_p x \wedge \neg(z = x)$$

$$x >_i z := \exists y.(\text{edge}_i(z,y) \wedge y \leq_p x)$$

where x, y, z, y' are first order variables, X a second order variable and $i \in \{1, \ldots, rk(\Gamma)\}$. The first formula is taken from [4] and is an MSO-formulation of the prefix-relation: $x \leq_p y$ iff x is an element of every prefix-closed set X which contains y. Here we call a set $X \subseteq \mathbb{N}^*$ prefix-closed if $z' \in X$ and $z \leq_p z'$ implies $z \in X$.

Now we define the two transformations. As indicated by the naming, the transformation $\mathfrak{G}$ is linked to the automata $\mathcal{G}_p$ and $\mathfrak{F}$ to the automata $\mathcal{F}_p$.

Definition 8.24 **($\mathfrak{G}_l^z$, $\mathfrak{F}_i^z$).** Let $l \in \mathbb{N}$ and $\theta \in \mathrm{QMSO}_{\Gamma_l}(\Pi_x, \oplus, \odot)$. Then for a tuple $z = (z_1, \ldots, z_l)$ of first order variables not used in θ we define $\mathfrak{G}_l^z(\theta)$ by induction. The idea is that the z_j stand for positions we imagine to substitute $\perp_j$ into. The existence quantifier is then restricted to only find positions not "below" any of the z_j and the product quantifier effectively only multiplies over all positions not "below" any of the z_j. The definition is as follows:

$$\begin{aligned}
\mathfrak{G}_l^z(\mathrm{label}_a(x)) &:= \mathrm{label}_a(x) & \mathfrak{G}_l^z(\varphi \wedge \psi) &:= \mathfrak{G}_l^z(\varphi) \wedge \mathfrak{G}_l^z(\psi)\\
\mathfrak{G}_l^z(\mathrm{edge}_i(x,y)) &:= \mathrm{edge}_i(x,y) & \mathfrak{G}_l^z(\varphi \vee \psi) &:= \mathfrak{G}_l^z(\varphi) \vee \mathfrak{G}_l^z(\psi)\\
\mathfrak{G}_l^z(x \in X) &:= x \in X & \mathfrak{G}_l^z(\tau_1 \oplus \tau_2) &:= \mathfrak{G}_l^z(\tau_1) \oplus \mathfrak{G}_l^z(\tau_2)\\
\mathfrak{G}_l^z(\neg\varphi) &:= \neg\mathfrak{G}_l^z(\varphi) & \mathfrak{G}_l^z(\tau_1 \odot \tau_2) &:= \mathfrak{G}_l^z(\tau_1) \odot \mathfrak{G}_l^z(\tau_2)\\
\mathfrak{G}_l^z(k) &:= k
\end{aligned}$$

$$\begin{aligned}
\mathfrak{G}_l^z(\mathrm{label}_{\perp_j}(x)) &:= (x = z_j)\\
\mathfrak{G}_l^z(\exists x.\varphi) &:= \exists x.(\mathfrak{G}_l^z(\varphi) \wedge \neg(\bigvee_{k=1}^{l} x > z_k))\\
\mathfrak{G}_l^z(\exists X.\varphi) &:= \exists X.(\mathfrak{G}_l^z(\varphi) \wedge \neg\exists x.(x \in X \wedge (\bigvee_{k=1}^{l} x > z_k)))\\
\mathfrak{G}_l^z(\Pi x.\tau) &:= \Pi x.((\mathfrak{G}_l^z(\tau) \odot \neg(\bigvee_{k=1}^{l} x > z_k)) \oplus \bigvee_{k=1}^{l} x > z_k)
\end{aligned}$$

where $j \in \{1, \ldots, l\}$, $i \in \{1, \ldots, rk(\Gamma_l)\}$, $k \in K$, $a \in \Gamma$, $\tau, \tau_1, \tau_2 \in \mathrm{QMSO}_{\Gamma_l}(\Pi_x, \oplus, \odot)$ and $\varphi, \psi \in \mathrm{MSO}(\Gamma_l)$.

Let $i \in \{1, \ldots, rk(\Gamma)\}$ and $\theta \in \mathrm{QMSO}_\Gamma(\Sigma_x^k\Pi_x^1, \oplus, \odot)$. Then for a first order variable z not used in θ we define $\mathfrak{F}_i^z(\theta)$ by induction. The variable z stands for a position we imagine to substitute some $\bot_j$ into. Existence, sum and product quantifiers are then restricted to only consider the subtree at "zi". The definition is as follows:

$$\begin{aligned}
\mathfrak{F}_i^z(\mathrm{label}_a(x)) &:= \mathrm{label}_a(x) & \mathfrak{F}_i^z(\varphi \wedge \psi) &:= \mathfrak{F}_i^z(\varphi) \wedge \mathfrak{F}_i^z(\psi)\\
\mathfrak{F}_i^z(\mathrm{edge}_j(x,y)) &:= \mathrm{edge}_j(x,y) & \mathfrak{F}_i^z(\varphi \vee \psi) &:= \mathfrak{F}_i^z(\varphi) \vee \mathfrak{F}_i^z(\psi)\\
\mathfrak{F}_i^z(x \in X) &:= x \in X & \mathfrak{F}_i^z(\tau_1 \oplus \tau_2) &:= \mathfrak{F}_i^z(\tau_1) \oplus \mathfrak{F}_i^z(\tau_2)\\
\mathfrak{F}_i^z(\neg\varphi) &:= \neg\mathfrak{F}_i^z(\varphi) & \mathfrak{F}_i^z(\tau_1 \odot \tau_2) &:= \mathfrak{F}_i^z(\tau_1) \odot \mathfrak{F}_i^z(\tau_2)\\
\mathfrak{F}_i^z(k) &:= k
\end{aligned}$$

$$\begin{aligned}
\mathfrak{F}_i^z(\exists x.\varphi) &:= \exists x.(\mathfrak{F}_i^z(\varphi) \wedge x >_i z)\\
\mathfrak{F}_i^z(\exists X.\varphi) &:= \exists X.(\mathfrak{F}_i^z(\varphi) \wedge \neg\exists x.(x \in X \wedge \neg(x >_i z)))\\
\mathfrak{F}_i^z(\Sigma x.\tau) &:= \Sigma x.(\mathfrak{F}_i^z(\tau) \odot x >_i z)\\
\mathfrak{F}_i^z(\Pi x.\tau) &:= \Pi x.((\mathfrak{F}_i^z(\tau) \odot x >_i z) \oplus \neg(x >_i z))
\end{aligned}$$

where $j \in \{1, \ldots, rk(\Gamma)\}$, $k \in K$, $a \in \Gamma$, $\tau, \tau_1, \tau_2 \in \mathrm{QMSO}_\Gamma(\Sigma_x\Pi_x^1, \oplus, \odot)$ and $\varphi, \psi \in \mathrm{MSO}(\Gamma)$.

As we can always rewrite $\forall x.\varphi$ and $\forall X.\varphi$ to $\neg\exists x.\neg\varphi$ and $\neg\exists X.\neg\varphi$, respectively, we do not need to define these cases explicitly.

Proposition 8.25. *Let $t \in T_\Gamma$ and $l \in \mathbb{N}$, then*

(i) *For $w_1, \ldots, w_l \in \mathrm{pos}(t)$ pairwise prefix-independent, $\theta \in \mathrm{QMSO}_{\Gamma_l}(\Pi_x^1, \oplus, \odot)$, a finite set of first and second order variables $\mathcal{V} \supseteq \mathrm{Free}(\theta)$, a $(\mathcal{V}, t\langle\bot_1 \to w_1\rangle \ldots \langle\bot_l \to w_l\rangle)$-assignment $\rho_\bot$, a tuple of first order variables $z = (z_1, \ldots, z_l)$ not in $\mathcal{V}$ and not occurring in θ and the $(\mathcal{V}, t)$-assignment ρ defined by $\rho(x) := \rho_\bot(x)$ where $x \in \mathcal{V}$ is a first or second order variable we have*

$$[\![\theta]\!](t\langle\bot_1 \to w_1\rangle \ldots \langle\bot_l \to w_l\rangle, \rho_\bot) = [\![\mathfrak{G}_l^z(\theta)]\!](t, \rho[z_1 \to w_1] \ldots [z_l \to w_l]).$$

(ii) *For $w \in \mathrm{pos}(t)$, $i \in \{1, \ldots, rk_\Gamma(t(w))\}$, $\theta \in \mathrm{QMSO}_\Gamma(\Sigma_x\Pi^1_x, \oplus, \odot)$, a finite set of first and second order variables $\mathcal{V} \supseteq \mathrm{Free}(\theta)$, a $(\mathcal{V}, t|_{wi})$-assignment ρ', a first order variable z not in $\mathcal{V}$ and not occurring in θ and the $(\mathcal{V}, t)$-assignment ρ defined by $\rho(x) := wi\rho'(x)$ where $x \in \mathcal{V}$ is a first or second order variable we have*

$$[\![\theta]\!](t|_{wi}, \rho') = [\![\mathfrak{F}^z_i(\theta)]\!](t, \rho[z \to w]).$$

Proof. (i) We prove the statement inductively and take t, $w_1, \ldots, w_l$, θ, $\mathcal{V}$, z, $\rho_\perp$ and ρ as in the proposition, set $t_\perp := t\langle \perp_1 \to w_1\rangle \ldots \langle \perp_l \to w_l\rangle$ and abbreviate $\rho[z_1 \to w_1] \ldots [z_l \to w_l]$ to $\rho[z \to w]$. We start by proving the case $\theta = \varphi \in \mathrm{MSO}(\Gamma_l)$ and show

$$(t_\perp, \rho_\perp) \models \varphi \Leftrightarrow (t, \rho[z \to w]) \models \mathfrak{G}^z_l(\varphi).$$

For the atomic formulas $\mathrm{edge}_i(x, y)$, $x \in X$ and $\mathrm{label}_a(x)$ with $a \in \Gamma$ this is easily verified, as $\mathfrak{G}^z_l(\varphi) = \varphi$ in those cases. For $i \in \{1, \ldots, l\}$ and $\varphi = \mathrm{label}_{\perp_i}(x)$ we see that

$$\begin{aligned}(t_\perp, \rho_\perp) \models \mathrm{label}_{\perp_i}(x) &\Leftrightarrow t_\perp(\rho_\perp(x)) = \perp_i\\ &\Leftrightarrow \rho_\perp(x) = w_i\\ &\Leftrightarrow \rho[z \to w](x) = w_i\\ &\Leftrightarrow (t, \rho[z \to w]) \models x = z_i\end{aligned}$$

as $\rho[z \to w](z_i) = w_i$ by definition. For $\varphi = \psi_1 \vee \psi_2$, $\varphi = \psi_1 \wedge \psi_2$ and $\varphi = \neg\psi$ we get the statement from $\mathfrak{G}^z_l(\psi_1 \vee \psi_2) = \mathfrak{G}^z_l(\psi_1) \vee \mathfrak{G}^z_l(\psi_2)$, $\mathfrak{G}^z_l(\psi_1 \wedge \psi_2) = \mathfrak{G}^z_l(\psi_1) \wedge \mathfrak{G}^z_l(\psi_2)$ and $\mathfrak{G}^z_l(\neg\psi) = \neg\mathfrak{G}^z_l(\psi)$. Now for the case $\varphi = \exists x.\psi$ we have

$$\begin{aligned}&(t_\perp, \rho_\perp) \models \exists x.\psi\\ &\Leftrightarrow \exists v \in \mathrm{pos}(t_\perp) : (t_\perp, \rho_\perp[x \to v]) \models \psi\\ &\Leftrightarrow \exists v \in \mathrm{pos}(t_\perp) : (t, \rho[x \to v][z \to w]) \models \mathfrak{G}^z_l(\psi) \qquad \text{(by induction)}\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \exists v \in \mathrm{pos}(t) : \neg \bigvee_{k=1}^{l} (w_k \leq_p v \wedge v \neq w_k) \wedge (t, \rho[x \to v][z \to w]) \models \mathfrak{G}_l^z(\psi)\\
&\Leftrightarrow \exists v \in \mathrm{pos}(t) : (t, \rho[x \to v][z \to w]) \models (\mathfrak{G}_l^z(\psi) \wedge \neg(\bigvee_{k=1}^{l} x > z_k))\\
&\Leftrightarrow (t, \rho[z \to w]) \models \exists x.(\mathfrak{G}_l^z(\psi) \wedge \neg(\bigvee_{k=1}^{l} x > z_k))\\
&\Leftrightarrow (t, \rho[z \to w]) \models \mathfrak{G}_l^z(\exists x.\psi)
\end{aligned}$$

and for $\varphi = \exists X.\psi$ we have

$$\begin{aligned}
&(t_\perp, \rho) \models \exists X.\psi\\
&\Leftrightarrow \exists V \subseteq \mathrm{pos}(t_\perp) : (t_\perp, \rho_\perp[X \to V]) \models \psi\\
&\Leftrightarrow \exists V \subseteq \mathrm{pos}(t_\perp) : (t, \rho[X \to V][z \to w]) \models \mathfrak{G}_l^z(\psi) \qquad \text{(by induction)}\\
&\Leftrightarrow \exists V \subseteq \mathrm{pos}(t) : \neg\exists v \in \mathrm{pos}(t) : (v \in V \wedge \bigvee_{k=1}^{l} (w_k \leq_p v \wedge v \neq w_k))\\
&\qquad\qquad \wedge (t, \rho[X \to V][z \to w]) \models \mathfrak{G}_l^z(\psi)\\
&\Leftrightarrow \exists V \subseteq \mathrm{pos}(t) : (t, \rho[X \to V][z \to w]) \models (\mathfrak{G}_l^z(\psi) \wedge \neg\exists x.(x \in X \wedge \bigvee_{k=1}^{l} x > z_k))\\
&\Leftrightarrow (t, \rho[z \to w]) \models \exists X.(\mathfrak{G}_l^z(\psi) \wedge \neg\exists x.(x \in X \wedge \bigvee_{k=1}^{l} x > z_k))\\
&\Leftrightarrow (t, \rho[z \to w]) \models \mathfrak{G}_l^z(\exists X.\psi)
\end{aligned}$$

which proves that for MSO-formulas we have

$$[\![\varphi]\!](t_\perp, \rho_\perp) = [\![\mathfrak{G}_l^z(\varphi)]\!](t, \rho[z \to w]).$$

For the semiring level we take $k \in K$ and $\tau_1, \tau_2 \in \mathrm{QMSO}_{\Gamma_l}(\Pi_x^1, \oplus, \odot)$. By definition have $\mathfrak{G}_l^z(k) = k$, $\mathfrak{G}_l^z(\tau_1 \odot \tau_2) = \mathfrak{G}_l^z(\tau_1) \odot \mathfrak{G}_l^z(\tau_2)$ and $\mathfrak{G}_l^z(\tau_1 \oplus \tau_2) = \mathfrak{G}_l^z(\tau_1) \oplus \mathfrak{G}_l^z(\tau_2)$ so for $\mathcal{V} \supseteq \mathrm{Free}(\tau_1) \cup \mathrm{Free}(\tau_2)$ and a $(\mathcal{V}, t_\perp)$-assignment ρ we have

$$[\![k]\!](t_\perp, \rho_\perp) = [\![\mathfrak{G}_l^z(k)]\!](t, \rho[z \to w])$$

trivially. Furthermore by induction

$$\begin{aligned}[\![\tau_1 \oplus \tau_2]\!](t_\perp, \rho_\perp) &= [\![\tau_1]\!](t_\perp, \rho_\perp) \oplus [\![\tau_2]\!](t_\perp, \rho_\perp)\\ &= [\![\mathfrak{G}_l^z(\tau_1)]\!](t, \rho[z \to w]) \oplus [\![\mathfrak{G}_l^z(\tau_2)]\!](t, \rho[z \to w])\\ &= [\![\mathfrak{G}_l^z(\tau_1 \oplus \tau_2)]\!](t, \rho[z \to w])\end{aligned}$$

and in the same manner

$$[\![\tau_1 \odot \tau_2]\!](t_\perp, \rho_\perp) = [\![\mathfrak{G}_l^z(\tau_1 \odot \tau_2)]\!](t, \rho[z \to w]).$$

We prove the last case, i.e. $\theta = \Pi x.\tau$ for some $\tau \in \mathrm{QMSO}_{\Gamma_l}(\Pi_x^1, \oplus, \odot)$:

$$\begin{aligned}&[\![\Pi x.\tau]\!](t_\perp, \rho_\perp)\\ &= \bigodot_{v \in \mathrm{pos}(t_\perp)} [\![\tau]\!](t_\perp, \rho_\perp[x \to v])\\ &= \bigodot_{v \in \mathrm{pos}(t_\perp)} [\![\mathfrak{G}_l^z(\tau)]\!](\underbrace{t, \rho[x \to v][z \to w]}_{=:s(v)}) \qquad \text{by induction}\\ &= \bigodot_{v \in \mathrm{pos}(t)} (([\![\mathfrak{G}_l^z(\tau)]\!](s(v)) \odot [\![\neg(\bigvee_{k=1}^{l} x > z_k)]\!](s(v))) \oplus [\![\bigvee_{k=1}^{l} x > z_k]\!](s(v)))\\ &= \bigodot_{v \in \mathrm{pos}(t)} [\![(\mathfrak{G}_l^z(\tau) \odot \neg(\bigvee_{k=1}^{l} x > z_k)) \oplus \bigvee_{k=1}^{l} x > z_k]\!](s(v))\\ &= [\![\Pi x.((\mathfrak{G}_l^z(\tau) \odot \neg(\bigvee_{k=1}^{l} x > z_k)) \oplus \bigvee_{k=1}^{l} x > z_k)]\!](t, \rho[z \to w])\\ &= [\![\mathfrak{G}_l^z(\theta)]\!](t, \rho[z \to w])\end{aligned}$$

(ii) We prove this statement by induction as well. Take t, w, θ, $\mathcal{V}$, z, ρ' and ρ as in the proposition and set $t' = t|_{wi}$. Again we start with the case $\theta = \varphi \in \mathrm{MSO}(\Gamma)$, that is we show

$$(t', \rho') \models \varphi \Leftrightarrow (t, \rho[z \to w]) \models \mathfrak{F}_i^z(\varphi).$$

For $\varphi = \mathrm{label}_a(x)$ we have

$$\begin{aligned}(t', \rho') \models \mathrm{label}_a(x) &\Leftrightarrow t'(\rho'(x)) = a\\ &\Leftrightarrow t(wi\rho'(x)) = a\\ &\Leftrightarrow t(\rho[z \to w](x)) = a\\ &\Leftrightarrow (t, \rho[z \to w]) \models \mathrm{label}_a(x),\end{aligned}$$

for $\varphi = \mathrm{edge}_j(x, y)$ we have

$$\begin{aligned}(t', \rho') \models \mathrm{edge}_j(x, y) &\Leftrightarrow \rho'(y) = \rho'(x)j\\ &\Leftrightarrow wi\rho'(y) = wi\rho'(x)j\\ &\Leftrightarrow \rho[z \to w](y) = \rho[z \to w](x)j\\ &\Leftrightarrow (t, \rho[z \to w]) \models \mathrm{edge}_j(x, y),\end{aligned}$$

and for $\varphi = x \in X$ we have

$$\begin{aligned}(t', \rho') \models x \in X &\Leftrightarrow \rho'(x) \in \rho'(X)\\ &\Leftrightarrow wi\rho'(x) \in wi\rho'(X)\\ &\Leftrightarrow \rho[z \to w](x) \in \rho[z \to w](X)\\ &\Leftrightarrow (t, \rho[z \to w]) \models x \in X.\end{aligned}$$

The cases $\varphi = \neg\psi$, $\varphi = \psi_1 \vee \psi_2$ and $\varphi = \psi_1 \wedge \psi_2$ are easily derived from the definition of $\mathfrak{F}_i^z$. Now assume the cases $\varphi = \exists x.\varphi$:

$$\begin{aligned}&(t', \rho') \models \exists x.\psi\\ &\Leftrightarrow \exists v \in \mathrm{pos}(t') : (t', \rho'[x \to v]) \models \psi\\ &\Leftrightarrow \exists v \in \mathrm{pos}(t') : (t, \rho[x \to wiv][z \to w]) \models \mathfrak{F}_i^z(\psi) \qquad \text{(by induction)}\\ &\Leftrightarrow \exists v \in \mathrm{pos}(t) : wi \leq_p v \wedge (t, \rho[x \to v][z \to w]) \models \mathfrak{F}_i^z(\psi)\\ &\Leftrightarrow \exists v \in \mathrm{pos}(t) : \exists u \in \mathrm{pos}(t) : u = wi \wedge u \leq_p v \wedge (t, \rho[x \to v][z \to w]) \models \mathfrak{F}_i^z(\psi)\\ &\Leftrightarrow \exists v \in \mathrm{pos}(t) : (t, \rho[x \to v][z \to w]) \models (\mathfrak{F}_i^z(\psi) \wedge x >_i z)\end{aligned}$$

$$\Leftrightarrow (t, \rho[z \to w]) \models \mathfrak{F}_i^z(\exists x.\psi)$$

and $\varphi = \exists X.\varphi$:

$$\begin{aligned}
&(t', \rho') \models \exists X.\psi \\
&\Leftrightarrow \exists V \subseteq \text{pos}(t') : (t', \rho'[X \to V]) \models \psi \\
&\Leftrightarrow \exists V \subseteq \text{pos}(t') : (t, \rho[X \to wiV][z \to w]) \models \mathfrak{F}_i^z(\psi) \qquad \text{(by induction)} \\
&\Leftrightarrow \exists V \subseteq \text{pos}(t) : \neg\exists v \in \text{pos}(t) : (v \in V \land \neg(wi \leq_p v)) \\
&\qquad\qquad\qquad \land (t, \rho[X \to V][z \to w]) \models \mathfrak{F}_i^z(\psi) \\
&\Leftrightarrow \exists V \subseteq \text{pos}(t) : (t, \rho[X \to V][z \to w]) \models (\mathfrak{F}_i^z(\psi) \land \neg\exists x.(x \in X \land \neg(x >_i z))) \\
&\Leftrightarrow (t, \rho[z \to w]) \models \mathfrak{F}_i^z(\exists X.\psi)
\end{aligned}$$

so for MSO formulas φ we have

$$[\![\varphi]\!](t', \rho') = [\![\mathfrak{F}_i^z(\varphi)]\!](t, \rho[z \to w]).$$

For the semiring level and $\theta = k$, $\theta = \tau_1 \oplus \tau_2$ and $\theta = \tau_1 \odot \tau_2$ it is again easy to see from the definition of $\mathfrak{F}_i^z$ that the induction holds, that is

$$\begin{aligned}
[\![k]\!](t', \rho') &= [\![\mathfrak{G}_l^z(k)]\!](t, \rho[z \to w]) \\
[\![\tau_1 \oplus \tau_2]\!](t', \rho') &= [\![\mathfrak{G}_l^z(\tau_1 \oplus \tau_2)]\!](t, \rho[z \to w]) \\
[\![\tau_1 \odot \tau_2]\!](t', \rho') &= [\![\mathfrak{G}_l^z(\tau_1 \odot \tau_2)]\!](t, \rho[z \to w]).
\end{aligned}$$

For $\theta = \Sigma x.\tau$ consider

$$\begin{aligned}
&[\![\Sigma x.\tau]\!](t', \rho') \\
&= \bigoplus_{v \in \text{pos}(t')} [\![\tau]\!](t', \rho'[x \to v]) \\
&= \bigoplus_{v \in \text{pos}(t')} [\![\mathfrak{F}_i^z(\tau)]\!](t, \rho[x \to wiv][z \to w]) \qquad \text{(by induction)}
\end{aligned}$$

$$\begin{aligned}
&= \bigoplus_{v\in \mathrm{pos}(t)} [\![\mathfrak{F}_i^z(\tau)]\!](t,\rho[x\to v][z\to w]) \odot [\![x >_i z]\!](t,\rho[x\to v][z\to w])\\
&= \bigoplus_{v\in \mathrm{pos}(t)} [\![\mathfrak{F}_i^z(\tau)\odot x >_i z]\!](t,\rho[x\to v][z\to w])\\
&= [\![\Sigma x.(\mathfrak{F}_i^z(\tau)\odot x >_i z)]\!](t,\rho[z\to w])\\
&= [\![\mathfrak{F}_i^z(\Sigma x.\tau)]\!](t,\rho[z\to w])
\end{aligned}$$

and for $\theta = \Pi x.\tau$

$$\begin{aligned}
&[\![\Pi x.\tau]\!](t',\rho')\\
&= \bigodot_{v\in \mathrm{pos}(t')} [\![\tau]\!](t',\rho'[x\to v])\\
&= \bigodot_{v\in \mathrm{pos}(t')} [\![\mathfrak{F}_i^z(\tau)]\!](t,\rho[x\to wiv][z\to w]) \qquad \text{(by induction)}\\
&= \bigodot_{v\in \mathrm{pos}(t)} (([\![\mathfrak{F}_i^z(\tau)]\!](s(v))\odot [\![x >_i z]\!](s(v)))\oplus [\![\neg(x >_i z)]\!](s(v)))\\
&\qquad \text{with } s(v) := (t,\rho[x\to v][z\to w])\\
&= [\![\Pi.x(\mathfrak{F}_i^z(\tau)\odot x >_i z)\oplus \neg(x >_i z))]\!](t,\rho[z\to w])\\
&= [\![\mathfrak{F}_i^z(\Pi x.\tau)]\!](t,\rho[z\to w]).
\end{aligned}$$

□

8.5 Conclusion and a Corollary

The following proposition brings the results of the preceding subsections together and proves that the automata $\mathcal{F}_p$ can be converted into logic formulas of the desired form. We still assume the automaton $\mathcal{A}$ to be standardized.

Proposition 8.26. *For $p \in Q$ and $k = \mathrm{degree}_{\mathcal{A}}(p)$ there is a formula $\theta \in \mathrm{QMSO}_\Gamma(\Sigma_x^k\Pi_x^1, \oplus, \odot_b)$ such that $[\![\mathcal{F}_p]\!] = [\![\theta]\!]$ and θ can be chosen as a finite sum of formulas in $\mathrm{QMSO}_\Gamma(\Sigma_x^k\Pi_x^1, \oplus_b, \odot_b)$.*

Proof. We prove the theorem by induction. We will assume it is true for $q \in Q$ with $p \preccurlyeq q$ and $\mathfrak{C}(q) \neq \mathfrak{C}(p)$ and from that conclude that it is true for p. If

$\text{degree}_{\mathcal{A}}(p) = 0$ then $\mathcal{F}_p$ is finitely ambiguous so by Theorem 7.1 there is a formula $\theta \in \text{QMSO}_\Gamma(\Pi^1_x, \oplus, \odot_b)$ with $[\![\mathcal{F}_p]\!] = [\![\theta]\!]$. As the theorem's proof shows we can even assume the stronger fact, that θ is a finite sum of formulas in $\text{QMSO}_\Gamma(\Pi^1_x, \oplus_b, \odot_b)$. For $\text{degree}_{\mathcal{A}}(p) > 0$ we consider $\text{Top}(\mathfrak{N}(p)) = \{\mathfrak{b}_1, \ldots, \mathfrak{b}_l\}$ with $\mathfrak{b}_i = (p^{(i)}_1, \ldots, p^{(i)}_{m_i}, a^{(i)}, q^{(i)})$ for $i \in \{1, \ldots, l\}$. By induction we assume that the proposition is true for $p^{(i)}_j$ with $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, m_i\}$, so for $k^i_j := \text{degree}_{\mathcal{A}}(p^{(i)}_j)$ we find $\theta^i_j \in \text{QMSO}_\Gamma(\Sigma^{k^i_j}_x \Pi^1_x, \oplus, \odot_b)$ with $[\![\theta^i_j]\!] = [\![\mathcal{F}_{p^{(i)}_j}]\!]$ such that all θ^i_j are finite sums of formulas in $\text{QMSO}_\Gamma(\Sigma^{k^i_j}_x \Pi^1_x, \oplus_b, \odot_b)$. Furthermore the automaton $\mathcal{G}_p$ we defined earlier is finitely ambiguous by Proposition 8.22 (iii), so we find some $\tau \in \text{QMSO}_{\Gamma_l}(\Pi^1_x, \oplus, \odot_b)$ with $[\![\tau]\!] = [\![\mathcal{G}_p]\!]$ such that τ is a finite sum of formulas in $\text{QMSO}_{\Gamma_l}(\Pi^1_x, \oplus_b, \odot_b)$. For $t \in T_\Gamma$ we have a partition

$$\text{Run}_{\mathcal{A},p}(t) = \bigcup_{w_1,\ldots,w_l \in \text{pos}(t)} \underbrace{\text{Run}_{\mathcal{A},p}(t; w_1, \ldots, w_l, \mathfrak{b}_1, \ldots, \mathfrak{b}_l)}_{=:R(w_1,\ldots,w_l)}.$$

By Proposition 8.17 (i) we only have to consider pairwise prefix-independent positions $w_1, \ldots, w_l$ in the above formula, so we fix $w_1, \ldots, w_l \in \text{pos}(t)$ pairwise prefix-independent and let $z_1, \ldots, z_l$ be first order variables not occurring in τ or θ^i_j for $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, m_i\}$. We define the abbreviations

$$\begin{aligned} t[z \to w] &:= t[z_1 \to w_1] \ldots [z_l \to w_l] \\ \text{bridge}(z) &:= \bigodot_{i=1}^{l} (\mu(\mathfrak{b}_i) \odot \text{label}_{a^{(i)}}(z_i)) \\ \text{indep}(z) &:= (\bigwedge_{i=1}^{l} \bigwedge_{\substack{j=1 \\ j\neq i}}^{l} \neg(z_i \leq_p z_j \vee z_j \leq_p z_i)) \end{aligned}$$

then by Propositions 8.22 (ii), 8.7 (i) and 8.25 we can write

$$\begin{aligned} &\sum_{r \in R(w_1,\ldots,w_l)} \text{wt}_{\mathcal{F}_p}(t, r) \\ &= [\![\tau]\!](t\langle \bot_1 \to w_1\rangle \ldots \langle \bot_l \to w_l\rangle) \odot [\![\text{bridge}(z)]\!](t[z \to w]) \odot \bigodot_{i=1}^{l} \bigodot_{j=1}^{m_i} [\![\theta^i_j]\!](t|_{w_i j}) \end{aligned}$$

$$= [\![\mathfrak{G}^z_l(\tau)]\!](t[z \to w]) \odot [\![\mathrm{bridge}(z)]\!](t[z \to w]) \odot \bigodot_{i=1}^{l} \bigodot_{j=1}^{m_i} [\![\mathfrak{F}^{z_i}_j(\theta^i_j)]\!](t[z_i \to w_i])$$

$$= [\![\mathfrak{G}^z_l(\tau) \odot \mathrm{bridge}(z) \odot \bigodot_{i=1}^{l} \bigodot_{j=1}^{m_i} \mathfrak{F}^{z_i}_j(\theta^i_j)]\!](t[z \to w])$$

so in conclusion we get

$$\begin{aligned}
&[\![\mathcal{F}_p]\!](t)\\
&= \sum_{\substack{w_1,\dots,w_l \in \mathrm{pos}(t)\\ \text{pairwise prefix-independent}}} \; \sum_{r \in R(w_1,\dots,w_l)} \mathrm{wt}_{\mathcal{F}_p}(t, r)\\
&= \sum_{\substack{w_1,\dots,w_l \in \mathrm{pos}(t)\\ \text{pairwise prefix-independent}}} [\![\mathfrak{G}^z_l(\tau) \odot \mathrm{bridge}(z) \odot \bigodot_{i=1}^{l} \bigodot_{j=1}^{m_i} \mathfrak{F}^{z_i}_j(\theta^i_j)]\!](t[z \to w])\\
&= \sum_{w_1,\dots,w_l \in \mathrm{pos}(t)} [\![\mathrm{indep}(z) \odot \mathfrak{G}^z_l(\tau) \odot \mathrm{bridge}(z) \odot \bigodot_{i=1}^{l} \bigodot_{j=1}^{m_i} \mathfrak{F}^{z_i}_j(\theta^i_j)]\!](t[z \to w])\\
&= [\![\Sigma z_1 \dots \Sigma z_l.\mathrm{indep}(z) \odot \mathfrak{G}^z_l(\tau) \odot \mathrm{bridge}(z) \odot \bigodot_{i=1}^{l} \bigodot_{j=1}^{m_i} \mathfrak{F}^{z_i}_j(\theta^i_j)]\!](t)
\end{aligned}$$

Recall that for $\tau_1, \tau_2 \in \mathrm{QMSO}_\Gamma(\oplus, \odot)$ we can always rewrite

$$[\![\tau_1 \odot \Pi x.\tau_2]\!] = [\![\Pi x.((\tau_1 \odot \tau_2 \odot \mathrm{root}(x)) \oplus (\tau_2 \odot \neg\mathrm{root}(x)))]\!]$$

due to the commutativity of $\odot$. Assuming that θ^i_j is a finite sum of formulas in $\mathrm{QMSO}_\Gamma(\Sigma^{k^i_j}_x\Pi^1_x, \oplus_b, \odot_b)$ and the definition of $\mathfrak{F}^{z_i}_j$ for all $i \in \{1, \dots, l\}$ and $j \in \{1, \dots, m_i\}$ we get that $\mathfrak{F}^{z_i}_j(\theta^i_j)$ is a finite sum of formulas of the form

$$\Sigma x_1 \dots \Sigma_{x_{k^i_j}}((\bigodot_{n=1}^{k^i_j} x_n >_i z_i) \odot \Pi y.\tau_2)$$

for some $\tau_2 \in \mathrm{QMSO}_\Gamma(\oplus, \odot)$ by using the distributivity of $\odot$ over $\oplus$. Using above rewriting we see that $\mathfrak{F}^{z_i}_j(\theta^i_j)$ is a sum of formulas in $\mathrm{QMSO}_\Gamma(\Sigma^{k^i_j}_x\Pi^1_x, \oplus_b, \odot_b)$. By using that for $\tau_1, \tau_2 \in \mathrm{QMSO}_\Gamma(\oplus, \odot)$ we can write

$$[\![(\Pi x_1.\tau_1) \odot (\Pi x_2.\tau_2)]\!] = [\![\Pi x.(\tau_1 \odot \tau_2)]\!]$$

after adequate relabeling of variables in τ_1 and τ_2, we can expand

$$\bigodot_{i=1}^{l} \bigodot_{j=1}^{m_i} \mathfrak{F}_j^{z_i}(\theta_j^i)$$

to a sum of formulas in $\mathrm{QMSO}_\Gamma(\Sigma_x^{k-l}\Pi_x^1, \oplus_b, \odot_b)$, as by Lemma 8.20 (I) we have that $\sum_{i=1}^{l}\sum_{j=1}^{m_i} k_j^i = k-l$. Now note that by definition of $\mathfrak{G}_l^z$ the formula $\mathfrak{G}_l^z(\tau)$ is actually a sum of formulas in $\mathrm{QMSO}_\Gamma(\Pi_x^1, \oplus_b, \odot_b)$ so with the same constructions as above we can rewrite

$$\mathrm{indep}(z) \odot \mathfrak{G}_l^z(\tau) \odot \mathrm{bridge}(z)$$

into a sum of formulas in $\mathrm{QMSO}_\Gamma(\Pi_x^1, \oplus_b, \odot_b)$ as well. By applying distributivity once more and the fact that for $\tau_1, \tau_2 \in \mathrm{QMSO}_\Gamma(\Sigma_x\Pi_x^1, \oplus, \odot)$ we have

$$[\![\Sigma x.(\tau_1 \oplus \tau_2)]\!] = [\![\Sigma x.\tau_1 \oplus \Sigma x.\tau_2]\!]$$

we can rewrite

$$\Sigma z_1 \ldots \Sigma z_l.\mathrm{indep}(z) \odot \mathfrak{G}_l^z(\tau) \odot \mathrm{bridge}(z) \odot \bigodot_{i=1}^{l} \bigodot_{j=1}^{m_i} \mathfrak{F}_j^{z_i}(\theta_j^i)$$

into a sum of formulas in $\mathrm{QMSO}_\Gamma(\Sigma_x^k\Pi_x^1, \oplus_b, \odot_b)$, which is what we wanted to show. □

To conclude the proof of Theorem 8.2 note that

$$[\![\mathcal{A}]\!] = \gamma(q_f) \odot [\![\mathcal{F}_{q_f}]\!]$$

so by rewriting $\gamma(q_f)$ into $\Pi x.((\gamma(q_f) \odot \mathrm{root}(x)) \oplus \neg(\mathrm{root}(x))$ and applying distributivity on the formula we can find by Proposition 8.26 we obtain a formula $\theta \in \mathrm{QMSO}_\Gamma(\Sigma_x^k\Pi_x^1, \oplus, \odot_b)$, where $k = \mathrm{degree}(\mathcal{A})$, such that $[\![\mathcal{A}]\!] = [\![\theta]\!]$. For an arbitrary polynomially ambiguous automaton we combine this with Lemma 8.11 to write the automaton as a sum of standardized automata and obtain the result we wanted to show. □

As a corollary of Lemma 8.20 we also get that the ambiguity of a WTA $\mathcal{A}$ is either bounded below and above by a fixed polynomial or has a lower exponential bound. While this is a well known result for word automata [21], we could not find a similar result for tree automata.

Corollary 8.27. *Let* $\mathcal{A} = (Q, \Gamma, \mu, \alpha)$ *be a weighted bottom-up finite state tree automaton. Either* $\mathcal{A}$ *is polynomially ambiguous and* $\mathbb{r}_{\mathcal{A}} \in \Theta(n^k)$ *for* $k := \mathrm{degree}(\mathcal{A})$ *or there exists a sequence of trees* $(t_n)_{n\in\mathbb{N}}$ *in* T_Γ *and a constant* $C > 0$ *such that for all* $n \in \mathbb{N}$

(i) $|\mathrm{pos}(t_n)| \leq C \cdot n$

(ii) $|\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t_n)| \geq 2^n$*.*

Proof. First note that the components of Q can be defined as we did earlier independent from the ambiguity of $\mathcal{A}$. The same is true for the definition of bridges. Now assume a sequence $(t_n)_{n\in\mathbb{N}}$ of trees with above properties does not exist. Then the proof of Proposition 8.9 (ii) shows that for all $t \in T_\Gamma$, all $w \in \mathrm{pos}(t)$, all $q \in Q$ and all $p \in \mathfrak{C}(p)$ we must have

$$|\mathrm{Run}^{\partial}_{\mathcal{A},q}(t; w, p)| \leq 1.$$

Otherwise we could construct such a sequence of trees. We will now prove by induction that for all $q \in Q$ there exists a constant C and an integer $n \in \mathbb{N}$ such that

$$|\mathrm{Run}_{\mathcal{A},q}(t)| \leq C \cdot |\mathrm{pos}(t)|^n$$

for all $t \in T_\Gamma$.

If $\mathfrak{C}(q)$ is maximal, i.e. $\mathfrak{C}(q) \preccurlyeq \mathfrak{C}(p)$ implies $\mathfrak{C}(q) \approx \mathfrak{C}(p)$ for all $p \in Q$, then $|\mathrm{Run}_{\mathcal{A},q}(t)| \leq 1$ as we just found, so $C := 1$ and $n := 0$ fulfill our requirements. For the induction step we assume that our claim is true for all $p \in Q$ with $\mathfrak{C}(q) \preccurlyeq \mathfrak{C}(p)$ and $\mathfrak{C}(q) \not\approx \mathfrak{C}(p)$. Define

$$\mathfrak{B}(q) := \{d \in \Delta_{\mathcal{A}} \mid d \text{ is a bridge out of } \mathfrak{C}(q)\}.$$

If $|\mathrm{Run}_{\mathcal{A},q}(t)| \leq 1$ holds for all $t \in T_\Gamma$ we have nothing to show. Otherwise using the same reasoning as in Proposition 8.9 (iii) we can write

$$\begin{aligned}
&|\mathrm{Run}_{\mathcal{A},q}(t)| \\
&\leq \sum_{\mathfrak{b}\in\mathfrak{B}(q)} \sum_{w\in\mathrm{pos}(t)} |\mathrm{Run}_{\mathcal{A},q}(t; w, \mathfrak{b})| \\
&\leq \sum_{\substack{\mathfrak{b}\in\mathfrak{B}(q)\\ \mathfrak{b}=(p_1,\ldots,p_{m_\mathfrak{b}},a,q')}} \sum_{w\in\mathrm{pos}(t)} \left(\underbrace{|\mathrm{Run}^{\partial}_{\mathcal{A},q}(t; w, q')|}_{\leq 1} \cdot \prod_{i=1}^{m_\mathfrak{b}} \underbrace{|\mathrm{Run}_{\mathcal{A},p_i}(t|_{wi})|}_{\substack{\leq C_{i,\mathfrak{b}}\cdot|\mathrm{pos}(t|_{wi})|^{n_{i,\mathfrak{b}}}\\ \text{for some } n_{i,\mathfrak{b}}\in\mathbb{N} \text{ and } C_{i,\mathfrak{b}}\in\mathbb{R}}} \right) \\
&\leq \sum_{\mathfrak{b}\in\mathfrak{B}(q)} C_\mathfrak{b} \cdot |\mathrm{pos}(t)|^{n_\mathfrak{b}+1} \\
&\leq |\mathfrak{B}(q)| \cdot C \cdot |\mathrm{pos}(t)|^{n+1}
\end{aligned}$$

for $C_\mathfrak{b} := \prod_{i=1}^{m_\mathfrak{b}} C_{i,\mathfrak{b}}$ and $n_\mathfrak{b} := \sum_{i=1}^{m_\mathfrak{b}} n_{i,\mathfrak{b}}$ and C and n as the maxima of the $C_\mathfrak{b}$ and $n_\mathfrak{b}$, respectively. We set

$$F := \{q \in Q \mid \alpha(q) \neq 0\}$$

and obtain

$$\begin{aligned}
|\mathrm{Run}_{\mathcal{A},\mathbb{F}}(t)| &\leq \sum_{q\in F} |\mathrm{Run}_{\mathcal{A},q}(t)| \\
&\leq \sum_{q\in F} C_q \cdot |\mathrm{pos}(t)|^{n_q}
\end{aligned}$$

for some $n_q \in \mathbb{N}$ and $C_q \in \mathbb{R}$ and every $t \in T_\Gamma$. This obviously means that $\mathcal{A}$ is polynomially ambiguous and for $l := \mathrm{degree}(\mathcal{A})$ we have $\mathbb{r}_\mathcal{A} \in \mathcal{O}(n^l)$. Due to Lemma 8.20 (II) we also have $\mathbb{r}_\mathcal{A} \in \Omega(n^l)$ so in conclusion $\mathbb{r}_\mathcal{A} \in \Theta(n^l)$ holds. □

9 Pure Weighted Tree Automata and the Fragment $\mathrm{QMSO}(\Sigma_X^\infty, \oplus_b, \odot_b)$

We now come to a special class of WTA, one that uses only weights 0 and 1 on its transitions. Formally, we call a WTA $\mathcal{A} = (Q, \Gamma, \mu, \alpha)$ *pure*, if for all $d \in \Delta_\mathcal{A}$ we have $\mu(d) \in \{0, 1\}$. These automata are interesting in so far, that we can describe them with formulas not using the product quantifier Π. The idea and proof of this section is a straight-forward adaptation of the corresponding proof of [15, Proposition 6.1], where the case of automata on words is considered.

Theorem 9.1. *Let $(K, \oplus, \odot, 0, 1)$ be a commutative semiring and (Γ, rk_Γ) a ranked alphabet. A tree series $S \in K\langle\langle T_\Gamma \rangle\rangle$ is definable by a pure weighted bottom-up finite state tree automaton over K and Γ if, and only if, S is definable by a formula in $\mathrm{QMSO}_\Gamma(\Sigma_X^\infty, \oplus_b, \odot_b)$.*

Proof. ($\Rightarrow$) Let $\mathcal{A} = (Q, \Gamma, \mu, \alpha)$ be a pure WTA. We use the notation from Theorem 4.1 and define $\theta_\mathcal{A}$ as

$$\theta_\mathcal{A} := \Sigma\bar{X}.\left(\mathrm{valid}_\mathcal{A}(\bar{X}) \odot \mathrm{final}(\bar{X})\right).$$

It is clear that $\theta_\mathcal{A} \in \mathrm{QMSO}_\Gamma(\Sigma_X^\infty, \oplus_b, \odot_b)$ and as the transitions have only weights 0 or 1, it is also clear that $[\![\mathcal{A}]\!] = [\![\theta_\mathcal{A}]\!]$.

($\Leftarrow$) Take $\theta \in \mathrm{QMSO}_\Gamma(\Sigma_X^\infty, \oplus_b, \odot_b)$, that is $\theta = \Sigma X_1 \ldots \Sigma X_n \tau$ for some $n \in \mathbb{N}$ and $\tau \in \mathrm{QMSO}_\Gamma(\oplus, \odot)$. By Proposition 4.3 we can find a pure WTA $\mathcal{A}_0$ over $\Gamma_{\{X_1,\ldots,X_n\}}$ that defines the same tree series as τ. Then assuming by induction that for $i \in \{0, \ldots, n-1\}$ we have a pure WTA $\mathcal{A}_i$ over $\Gamma_{\{X_1,\ldots,X_{n-i}\}}$ such that $\mathcal{A}_i$ defines the same tree series as $\Sigma X_{n+1-i} \ldots \Sigma X_n \tau$, the proof of Proposition 4.7 yields that we can find a pure WTA $\mathcal{A}_{i+1}$ over $\Gamma_{\{X_1,\ldots,X_{n-(i+1)}\}}$ such that $\mathcal{A}_{i+1}$ and $\Sigma X_{n-i} \ldots \Sigma X_n \tau$ define the same tree series. The automaton $\mathcal{A}_n$ constructed this way is then pure and defines the same tree series as θ, which is what was to prove. □

10 Conclusion

We have shown that the correlation between the fragments of a quantative logic and the ambiguity of automata as described by Kreutzer and Riveros [15] holds true for tree automata as well and this extension to tree automata can be done in a rather obvious manner. In more detail, to each class of tree series definable by deterministic, unambiguous, finitely ambiguous, polynomially ambiguous and exponentially ambiguous weighted tree automata we have related a characteristic fragment of our logic.

While under the aspect of commonly distinguished degrees of ambiguity of automata this investigation was exhaustive, there are many more fragments of the logic other than the ones we considered. Whether and how these fragments correspond to more general automata models is an issue for further research. For example, the fragment $\mathrm{QMSO}(\Pi_x, \oplus_b, \odot)$, which for word automata Kreutzer and Riveros showed to correspond to a certain model of two-way weighted automata with pebbles, remains unresolved for the tree case. The findings of Kreutzer and Riveros suggest that, if there exists a translation of this fragment into an automata model, this model is likely a pebble tree walking automaton, which itself is an object of current research.

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