The Geometry and Topology of Twisted Quiver Varieties

A Thesis Submitted to the

College of Graduate and Postdoctoral Studies

in Partial Fulfillment of the Requirements

for the degree of Master of Science

in the Department of Mathematics and Statistics

University of Saskatchewan

Saskatoon

By

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Abstract

Quivers have a rich history of being used to construct algebraic varieties via their representations in the category of vector spaces. It is also natural to consider quiver representations in a larger category, namely that of vector bundles on some complex variety equipped with a fixed locally free sheaf that twists the morphisms.

For A-type quivers, such representations can be identified with the critical points of a Morse-Bott function on the moduli space of twisted Higgs bundles. Hence these "twisted quiver varieties" can be used to extract topological information about the Higgs bundle moduli space. We find a formula for the dimension of the moduli space of twisted representations of A-type quivers and geometric descriptions when each node of the quiver is represented by a line bundle. We then specialize to the so-called "argyle quivers", studied using Bradlow-Daskaloploulous stability parameters and pullback diagrams. Next we focus on the Riemann sphere $\mathbb{P}^1$ and obtain explicit expressions for the twisted quiver varieties as well as a stratification of these spaces via collisions of invariant zeroes of polynomials. We apply these results to some low-rank Higgs bundle moduli spaces.

We then study representations of cyclic quivers, which can be viewed as corresponding to certain deformations of the Hitchin representations in non-abelian Hodge theory. When all of the ranks are 1, we describe the moduli spaces as subvarieties of the Hitchin system. We also draw out descriptions of the twisted quiver varieties for when the underlying curve is $\mathbb{P}^1$ and extend this to some other labellings of the quiver.

We close with a discussion of possible applications of these ideas to hyperpolygon spaces as well as possible directions that use the motivic approach to moduli theory.

Declaration

The content of this thesis is, to the best of my knowledge, original except where attributed to others within the bibliography.

Portions of Chapters 3 and 4 appear as a joint work with Steven Rayan in *Twisted argyle quivers and Higgs bundles.* Bull. Sci. Math. 146 (2018), 1–32.

Acknowledgements

My first thanks must go to Steve Rayan, my advisor and friend, for guiding me through this strange and wonderful world of mathematics and academic communities. We used up quite a bit of coffee and chalk, and he has been a constant inspiration.

I am indebted to many people at the mathematics department at the University of Saskatchewan, namely the remainder of my graduate committee Jacek Szmigielski and Ebrahim Samei, as well as to George Patrick for giving me the opportunity to be involved in research while I was an undergraduate. I'd like to thank Diego Barba for being a great person to share an office with (by often sharing snacks) and certainly Kyla Denton and Jessica Parker for helping me take care of paperwork and for helping me grow my collection of office plants. Thanks to Chris Mahadeo and Ale Malusà for being good friends and listening to different versions of the same talk mulitiple times. From my highschool days, I thank Michael Gange for making math interesting to me in the first place.

I am hugely grateful to my family and friends for helping to keep me sane. Thanks especially to my parents Paula and Darren Sundbo for raising, loving, and encouraging me, to Brenda and Dwayne Mintzler for working hard to memorize some mathematical terms, and to Kennedi Sundbo for being a distraction.

On the monetary side of things, I am happy to thank the University of Saskatchewan for a Graduate Teaching Fellowship as well as NSERC for an Alexander Graham Bell Canada Graduate Scholarship - Master's.

Finally, this would have been impossible without Kelsey, not only for copy editing and coining the term "argyle quivers". You light up my life.

To Mom and Dad.

Contents

1 INTRODUCTION

Quivers are simple objects which have proven to be powerful tools for studying mathematical entities and the maps between them, particularly when it comes to representation theory and geometry. Quiver varieties are formed by labelling the nodes of a directed graph with nonnegative integers and then considering linear representations up to isomorphism. Broadly speaking, quiver varieties have applications to toric geometry, vertex algebras, noncommutative geometry, integrable systems, and gauge theories. Our focus, which is on representations in a broader scope of categories, can be viewed as an extension from geometry over a *point* to geometry over a *manifold*, reflecting the philosophy of the Grothendieck school: mathematics via the study of families.

One of the main goals of this thesis is to extract topological information about the moduli space of twisted Higgs bundles on $\mathbb{P}^1$. In this case, thanks to the Birkhoff-Grothendieck Theorem, the geometry of the fixed points is easier to get a handle on than when the genus of the underlying curve is strictly positive. Understanding these moduli spaces is a meaningful pursuit as Higgs bundles have proven themselves to be potent problem-solving instruments. For instance, they play an essential role in Ngô's celebrated proof of the Fundamental Lemma (an important result in the geometric Langlands program) ([55, 56]) and also exemplify the intersection between geometry and physics through their connection to hyperkähler geometry, mirror symmetry, and string theory (see for instance [40]).

These fixed points of the moduli space which we study can be naturally identified with moduli spaces of quiver representations in a richer category than the usual category of vector spaces. To be more specific, we study the category $\mathrm{Bun}(X, L)$ whose objects are holomorphic vector bundles on a fixed complex projective variety X and whose morphisms are maps between them graded by exterior powers of L: that is, $\mathrm{Mor}^k(U, V) = \mathrm{Hom}(U, V \otimes \Lambda^k L)$. Such representations of A-type quivers in a twisted category of bundles have acquired the name "holomorphic chains" and, in the particular case of the quiver A_2, "holomorphic triples" ([26, 11]). When the quiver is not specifically A-type, the nomenclature "quiver bundle" has been used. Given their importance to Higgs bundles, various aspects of the topology, geometry, and homological algebra of holomorphic chains and quiver bundles have been explored in recent years (see for example [26, 1, 11, 27, 61, 58, 21, 49, 62, 60, 50, 28]). Generalizing what is known about quivers to this more general framework is an interesting problem and reflects our "point versus manifold" comment from earlier. This can be viewed as part of a larger program of generalizations involving quiver representations and linear algebra. Let k be a field, V a vector space over k, and X a variety over k:

$$\begin{tikzcd}
\begin{array}{c}\text{Linear algebra with}\\ \text{a single vector space } V\\ \text{and a map } V \to V\end{array} \arrow[r, "\text{``manifold-ification''}"] \arrow[d, "\text{``quiver-ification''}"'] & \begin{array}{c}\text{Higgs bundles with a}\\ \text{single bundle } E \text{ and a map}\\ \Phi : E \to E \text{ (possibly twisted)}\end{array} \arrow[d, "\text{``quiver-ification''}"'] \\
\begin{array}{c}\text{Linear algebra with multiple}\\ \text{vector spaces and maps between}\\ \text{them (quiver representations)}\end{array} \arrow[r, "\text{``manifold-ification''}"] & \begin{array}{c}\text{Quiver representations in the}\\ \text{(twisted) category of bundles}\\ \text{on } X \text{ (quiver bundles)}\end{array}
\end{tikzcd}$$

This approach also provides possible ways of constructing new geometric structures. Typically, geometry arises from operations on vector spaces such as tensor products and direct sums as well as from quotients by the action of a group. One can also solve polynomial equations (giving rise to varieties) or differential equations on a vector space V. Higgs bundles arise as solutions to differential equations when V is finite dimensional, but in particular they do *not* arise as V/G for V finite dimensional. The geometry of linear algebra with multiple vector spaces can be viewed as V/G for $V = \bigoplus_{i,j} \mathrm{Hom}(V_i, V_j)$ and G reductive. Quiver representations in the twisted category cannot be seen this way, but on $\mathbb{P}^1$ at least we can construct their geometry via V/G for $V = \bigoplus_{i,j} \mathrm{Hom}(E_i, E_j \otimes L)$ for non-reductive G.

We begin in Chapter 2 by giving an overview of some of the facts used in the rest of the thesis. This includes an introduction to vector bundles, Higgs bundles and their stability and localizations, fine and coarse moduli spaces, and quivers.

In Chapter 3 we study twisted representations of A-type quivers for any genus of underlying curve and any twisting bundle L. Our first result is a calculation of the generic dimension of the moduli space using spectral sequences and hypercohomology for the differential induced by the morphisms themselves in concert with the Čech differential. Then we begin our journey through the geometry of these representations by studying the simplest case, the quiver A_n with all nodes labelled with $r_i = 1$. We obtain descriptions of the $SL(n, \mathbb{C})$, $PGL(n, \mathbb{C})$ moduli spaces in terms of finite covers of symmetric products. From these results (in concert with [32]) we can describe the $GL(n, \mathbb{C})$ moduli space, at least topologically. Such calculations appear in [35] for $n = 2$ and in [26] (via the methods of pullback diagrams, cf. Section 3.9) for $n = 3$. We follow up by specializing to a particular configuration for which we are also able to obtain concrete results for the geometry of the moduli space:

Definition 1.0.1. A labelled A-type quiver of the form

$$\bullet_{1,d_1} \longrightarrow \bullet_{r_2,d_2} \longrightarrow \bullet_{1,d_3} \longrightarrow \cdots \longrightarrow \bullet_{r_{n-1},d_{n-1}} \longrightarrow \bullet_{1,d_n}$$

is called an *argyle quiver*.

Our first result concerning this shape of quiver is Theorem 3.3.1, which generalizes the relationship between holomorphic triples and "stable pairs" [64] to general argyle quivers in the form of a generalized pullback diagram. As part of this result, we show that the Hitchin stability condition always has a corresponding Bradlow-Daskalopoulos stability parameter (cf. [10]) and vice-versa, a correspondence that boils down to the fact that a particular linear system always has a unique solution. From this result the Betti numbers of the moduli space could conceivably be calculated via variation of stability parameters and flips and flops (as is done for stable pairs in [64]).

Chapter 4 concerns our focus on representations with underlying bundle $\mathbb{P}^1$. In this case the non-reductive contributions from the automorphism group act in a particular way, namely via Euclidean reduction on spaces of polynomials. Our first result in this direction is Theorem 4.1.1, which involves argyle quivers of length $n = 3$, which we also refer to as type $(1, k, 1)$ to reflect the prescribed ranks of the three nodes. Here we remove the "collision locus" where zeroes of maps in a representation that are invariant under the action of automorphisms become coincident. We fix the holomorphic type of the central bundle and identify the projective closure of the collision-free subvariety of the moduli space with a product of flag varieties. From this, we are able to state Theorem 4.1.2, which describes how to compute the moduli space associated to an arbitrary argyle quiver from $(1, k, 1)$ pieces. Next, we give a number of examples of how strata are glued together by identifying $(1, k, 1)$ quiver varieties for different holomorphic types as collision varieties of one another.

We conclude the chapter with applications to the topology of twisted Higgs bundle moduli spaces on $\mathbb{P}^1$ with a complete account of the rational Betti numbers in rank 2 and any twisting line bundle $L = \mathcal{O}(t)$, as well as examples in rank 3. For ordinary Higgs bundles of rank $r = 2$ on a Riemann surface of genus $g \geq 2$, these were calculated by Hitchin in [35]. The rank 3 and rank 4 cases were computed in [25] and [21], respectively. In the parabolic Higgs setting on punctured Riemann surfaces, rational Poincaré series in low rank were computed in [9] and [19]. All of these calculations are largely Morse-theoretic, although [21] uses moduli stacks and motivic zeta functions. In the genus 0 setting, there are now general results on Donaldson-Thomas invariants due to Mozgovoy in [49], obtained by plethystic counting techniques, from which the Betti numbers can be extracted. In comparison, the $\mathbb{C}^*$-localization tends to becomes unmanageable outside of low rank due to the number of types of fixed points. That being said, bearing with it can reap rewards such as information on the stratification of the moduli space as organized by the Morse flow, as well as insight into the structure of the cohomology ring (not to mention an abundance of finer information, such as Verlinde formulae [2, 31], although this requires a much deeper analysis of the fixed-point geometry). We also mention an application of facts about Higgs bundle moduli spaces to the stability theory of quiver representations.

In Chapter 5, we shift focus from A-type quivers to cyclic quivers. These have a close relationship with cyclic Higgs bundles, which are beginning to attract attention (see, for example, [14]), and with Hitchin representations. Their moduli spaces can also be viewed as generalizations of the Hitchin section. For type $(1, \ldots, 1)$ cyclic quivers we can give set-theoretic descriptions of the $SL(n, \mathbb{C})$ and $PGL(n, \mathbb{C})$ moduli spaces for any genus in terms of divisors (Theorem 5.1.1). In the following sections we describe how this specializes to give the moduli spaces for $\mathbb{P}^1$ in terms of (almost) finite-to-one coverings of the Hitchin base. This "almost" comes from the geometry of the nilpotent cone, as one might guess. We describe how this finite-to-one covering degenerates at this fibre in terms of the $\mathbb{C}^*$ flows. We also endeavor to expand this approach to other types of cyclic quivers starting with type $(k, 1)$, which we can decompose as k different type $(1, 1)$ cyclic quiver varieties (with reductions coming from the automorphism groups) as stated in Theorem 5.3.1. These moduli spaces, arising as quotients, display interesting geometries and are reminescent of weighted projective spaces. We remark on the obstructions that prevent us from progressing further.

Finally, Chapter 6 explores some future directions that work on twisted quiver varieties could take. We introduce hyperpolygon spaces which arise as Nakajima quiver varieties in the category of vector spaces and explain how realizing the correct notion of Nakajima's construction ([52, 53, 54]) in the twisted category could lead to a compactification of this space (and certainly other new geometries). The approach to calculating the topological information of the moduli space via motives and Grothendieck ring of varieties is also detailed, following [21]. We suggest generalizations of these ideas to any twisting bundle L and curves of any genus g.

2 PRELIMINARIES

2.1 Some Geometry

We begin with a brief tour of some important elements of algebraic geometry. Vector bundles over Riemann surfaces pervade the thesis; recall that a *compact Riemann surface* (or sometimes *curve*) X is a 1-dimensional smooth, compact, connected, complex manifold. A *complex vector bundle* $E \to X$ of rank r is a smoothly varying family of r-dimensional complex vector spaces which is locally trivial. This idea of "smoothly varying" manifests as the transition functions (which tell us how the local trivializations fit together) being smooth. We will work exclusively with *holomorphic vector bundles*, which have holomorphic transition functions (this is clearly a stronger condition). In the special case $r = 1$, E is called a *line bundle*.

A *section* of a vector bundle E is a map $s : X \to E$ such that $\pi \circ s = \mathrm{id}_X$, where $\pi : E \to X$ is the projection map. The space of holomorphic sections of E is identified throughout as the zeroth sheaf cohomology group $H^0(X, E)$. The *degree* of a holomorphic line bundle $L \to X$ is the number of times that a generic holomorphic section of L vanishes, while the degree of a rank r holomorphic vector bundle $E \to X$ is given by the degree of its determinant line bundle $\det(E) = \Lambda^r E$. The space of line bundles of a given degree d on a Riemann surface X of genus g is isomorphic to a complex torus $\mathbb{C}^g / \mathbb{Z}^{2g}$, which we denote by $\mathrm{Jac}^d(X)$. The space of all line bundles on X is $\mathrm{Pic}(X) = \bigcup_{d \in \mathbb{Z}} \mathrm{Jac}^d(X)$, which is a group with the operation of tensor multiplication. The identity of $\mathrm{Pic}(X)$ is the trivial line bundle $X \times \mathbb{C}$, often denoted $\mathcal{O}_X$ (or simply $\mathcal{O}$). We note that for vector bundles E_1 and E_2 we have

$$\deg(E_1 \otimes E_2) = \mathrm{rk}(E_2)\deg(E_1) + \mathrm{rk}(E_1)\deg(E_2)$$

and

$$\deg(E_1 \oplus E_2) = \deg(E_1) + \deg(E_2).$$

On Riemann surfaces, the cotangent bundle T^*X (the dual of the familiar tangent bundle TX) is known as the *canonical line bundle*, and is denoted by ω_X or K_X. On a Riemann surface of genus g, the canonical line bundle has degree $2g-2$.

The following two theorems are fundamental:

Theorem 2.1.1. (The Riemann-Roch Theorem). *If E is a vector bundle on a compact Riemann surface X of genus g, then*[1]

$$h^0(X,E) - h^1(X,E) = deg(E) + (1-g)rk(E).$$

Theorem 2.1.2. (Serre Duality). *If L is a line bundle on a compact Riemann surface X, then*

$$H^1(X,L) \cong H^0(X, L^* \otimes \omega_X)^*.$$

The Riemann sphere $\mathbb{CP}^1$ (henceforth $\mathbb{P}^1$) is the unique Riemann surface of genus 0 and enjoys some properties which lead to us giving it special attention. Firstly $\mathrm{Jac}^d(\mathbb{P}^1) \cong \{\mathrm{pt}\}$, which is to say that there is a unique holomorphic line bundle of degree d on $\mathbb{P}^1$, up to isomorphism. We denote it by $\mathcal{O}(d)$. The next theorem is also important to us and so we prove it for completeness, following the presentation of [38].

Theorem 2.1.3. (The Birkhoff-Grothendieck Theorem). *If $E \to \mathbb{P}^1$ is a rank r holomorphic vector bundle of degree d, then there is a unique length r integer partition of d, $(a_1 \geq a_2 \geq \cdots \geq a_r)$ such that*

$$E \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r).$$

Proof. We will proceed by induction. Certainly this holds for rank 1 by the above comment, so suppose that E is a rank r holomorphic vector bundle. Consider the vector bundle $E \otimes \mathcal{O}(n)$. We will first show that for large enough n, this bundle will have holomorphic sections. We have a short exact sequence of sheaves[2]

$$0 \longrightarrow E \otimes \mathcal{O}(n-1) \xrightarrow{s_p} E \otimes \mathcal{O}(n) \longrightarrow \mathcal{P} \longrightarrow 0 \tag{2.1}$$

[1] Throughout the thesis, we adopt the notation $h^p(X,E) = \dim H^p(X,E)$.

[2] In the exact sequences throughout this proof, we simply write F for the sheaf of sections of a vector bundle F, which should formally be written as $\mathcal{O}(F)$.

(where s_p is the section of $\mathcal{O}(1)$ which vanishes at $p \in \mathbb{P}^1$ and $\mathcal{P}$ is the quotient sheaf) which gives us the injection

$$0 \longrightarrow H^0(\mathbb{P}^1, E \otimes \mathcal{O}(n-1)) \xrightarrow{s_p} H^0(\mathbb{P}^1, E \otimes \mathcal{O}(n)) \tag{2.2}$$

and thus $h^0(\mathbb{P}^1, E \otimes \mathcal{O}(n-1)) \leq h^0(\mathbb{P}^1, E \otimes \mathcal{O}(n))$. Invoking Riemann-Roch, we see

$$\begin{aligned} h^0(\mathbb{P}^1, E \otimes \mathcal{O}(n)) &= h^1(\mathbb{P}^1, E \otimes \mathcal{O}(n)) + \deg(E \otimes \mathcal{O}(n)) + \operatorname{rk}(E \otimes \mathcal{O}(n))(1-g) \\ &\geq d + r(1+n) \end{aligned}$$

and thus for n large enough, $h^0(\mathbb{P}^1, E \otimes \mathcal{O}(n)) > 0$. Considering 2.2 again, suppose that $H^0(\mathbb{P}^1, E \otimes \mathcal{O}(n-1))$ and $H^0(\mathbb{P}^1, E \otimes \mathcal{O}(n))$ have the same dimension. This would mean that s_p is an isomorphism for all $p \in \mathbb{P}^1$, and hence all sections of $E \otimes \mathcal{O}(n)$ vanish at p for all $p \in \mathbb{P}^1$. This is a contradiction, so we have

$$h^0(\mathbb{P}^1, E \otimes \mathcal{O}(n-1)) < h^0(\mathbb{P}^1, E \otimes \mathcal{O}(n)).$$

In particular, there exists an integer n such that $h^0(\mathbb{P}^1, E \otimes \mathcal{O}(n-1)) = 0$ and $h^0(\mathbb{P}^1, E \otimes \mathcal{O}(n)) \neq 0$. In this case, the long exact sequence induced by 2.1 becomes

$$0 \longrightarrow 0 \longrightarrow H^0(\mathbb{P}^1, E \otimes \mathcal{O}(n)) \longrightarrow H^0(\mathbb{P}^1, \mathcal{Q}) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}(n-1)) \longrightarrow \dots$$

If s is a nontrivial section of $E \otimes \mathcal{O}(n)$, then the map to $H^0(\mathbb{P}^1, \mathcal{P})$ is given by evaluation at p. This map is injective by exactness of the sequence, and thus $s(p) \neq 0$ for all $p \in \mathbb{P}^1$. That is, s is a nonvanishing section of $E \otimes \mathcal{O}(n)$, so s defines an inclusion of the trivial bundle $\mathcal{O}$ into $E \otimes \mathcal{O}(n)$ by $\mathbb{P}^1 \times \mathbb{C} \to E \otimes \mathcal{O}(n)$, $(z, \lambda) \mapsto \lambda s(z)$. This tells us that we have an exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow E \otimes \mathcal{O}(n) \xrightarrow{\alpha} \mathcal{Q} \longrightarrow 0 \tag{2.3}$$

where $\mathcal{Q}$ is the quotient sheaf. We would like to show that this exact sequence splits: that is, that $E \otimes \mathcal{O}(n) \cong \mathcal{O} \oplus \mathcal{Q}$. For this to be the case, there must be a homomorphism $\beta : \mathcal{Q} \to E \otimes \mathcal{O}(n)$ such that $\alpha \circ \beta = \mathrm{id}_{\mathcal{Q}}$. To show that such a homomorphism exists, consider the short exact sequence obtained by tensoring 2.3 with $\mathcal{Q}^*$:

$$0 \longrightarrow \mathcal{O} \otimes \mathcal{Q}^* \longrightarrow E \otimes \mathcal{O}(n) \otimes \mathcal{Q}^* \longrightarrow \mathcal{Q} \otimes \mathcal{Q}^* \longrightarrow 0$$

which gives the long exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^1, \mathcal{Q}^*) \longrightarrow H^0(\mathbb{P}^1, E \otimes \mathcal{O}(n) \otimes \mathcal{Q}^*) \longrightarrow H^0(\mathbb{P}^1, \mathcal{Q} \otimes \mathcal{Q}^*) \longrightarrow H^1(\mathbb{P}^1, \mathcal{Q}^*) \longrightarrow \dots$$

Clearly, there exists a nonvanishing section of $\mathcal{Q} \otimes \mathcal{Q}^* \cong \mathrm{Hom}(\mathcal{Q}, \mathcal{Q})$ given by the identity

$$\mathrm{id}_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{Q}.$$

If we can show that $\mathrm{id}_{\mathcal{Q}}$ maps to zero in $H^1(\mathbb{P}^1, \mathcal{Q}^*)$, then by exactness we will have an element of $H^0(\mathbb{P}^1, E \otimes \mathcal{O}(n) \otimes \mathcal{Q}^*)$, which is what we desire. To this end, note that by our inductive hypothesis, $\mathcal{Q}$ splits into a direct sum of line bundles

$$\mathcal{Q} \cong \mathcal{O}(b_1) \oplus \dots \oplus \mathcal{O}(b_m)$$

Define $\mathcal{Q}(-1) = \mathcal{O}(b_1 - 1) \oplus \dots \oplus \mathcal{O}(b_m - 1)$, and consider the short exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow E \otimes \mathcal{O}(n-1) \longrightarrow \mathcal{Q}(-1) \longrightarrow 0$$

which gives rise to

$$0 \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}(-1)) \longrightarrow H^0(\mathbb{P}^1, E \otimes \mathcal{O}(n-1)) \longrightarrow H^0(\mathbb{P}^1, \mathcal{Q}(-1)) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}(-1)) \longrightarrow \dots \tag{2.4}$$

Now, $H^0(\mathbb{P}^1, \mathcal{O}(-1))$ is trivial since $\mathcal{O}(-1)$ has negative degree, and we have chosen n so that $H^0(\mathbb{P}^1, E \otimes \mathcal{O}(n-1)) = 0$ as well. So by applying Riemann-Roch to $\mathcal{O}(-1)$, we have

$$\begin{aligned} h^1(\mathbb{P}^1, \mathcal{O}(-1)) &= h^0(\mathbb{P}^1, \mathcal{O}(-1)) - \deg(\mathcal{O}(-1)) - (1-g) \\ &= 0 - (-1) - (1-0) \\ &= 0 \end{aligned}$$

so that 2.4 implies

$$H^0(\mathbb{P}^1, \mathcal{Q}(-1)) = \bigoplus_i H^0(\mathbb{P}^1, \mathcal{O}(b_i - 1)) = 0$$

and it follow that $b_i - 1$ must be negative for all i, and thus $b_i \leq 0$ for all i. Next, applying Riemann-Roch to $\mathcal{O}(b_i)$ gives us

$$\begin{aligned} h^1(\mathbb{P}^1, \mathcal{O}(-b_i)) &= h^0(\mathbb{P}^1, \mathcal{O}(-b_i)) - \deg(\mathcal{O}(-b_i)) - (1-g) \\ &= (-b_i + 1) - (-b_i) - (1-0) \\ &= 0 \end{aligned}$$

from which we see that

$$H^1(\mathbb{P}^1, Q^*) = \bigoplus_i H^1(\mathbb{P}^1, \mathcal{O}(b_i)) = 0$$

Hence all sections of $\mathrm{Hom}(\mathcal{Q}, \mathcal{Q})$ are mapped to 0, in particular id_Q, so id_Q lifts to a section of $\mathrm{Hom}(\mathcal{Q}, E \otimes \mathcal{O}(n))$ as desired. This means that $E \otimes \mathcal{O}(n)$ splits as $\mathcal{O} \oplus \mathcal{Q}$. That is,

$$E \otimes \mathcal{O}(n) \cong \mathcal{O} \oplus \mathcal{O}(b_1) \oplus \ldots \oplus \mathcal{O}(b_m)$$

which implies

$$E \cong \mathcal{O}(-n) \oplus \mathcal{O}(b_1 - n) \oplus \ldots \oplus \mathcal{O}(b_m - n)$$

and the proof is complete.

$\square$

2.2 Moduli

If we want to organize equivalence classes of objects, we use a *moduli space*. This can be thought of as a geometric solution to a classification problem. Here we will formalize this idea using the theory of categories after defining a few terms, starting with schemes:

Definition 2.2.1. An *affine scheme* is a locally ringed space $(\mathcal{S}, \mathcal{O}_\mathcal{S})$ which is isomorphic to the spectrum $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ of some commutative ring R. A *scheme* is a locally ringed space $(\mathcal{S}, \mathcal{O}_\mathcal{S})$ for which there is an open cover $\{U_\alpha\}$ of $\mathcal{S}$ such that the restriction $(U_\alpha, \mathcal{O}_\mathcal{S}\big|_{U_\alpha})$ is isomorphic to an affine scheme.

We will abuse notation and write $\mathcal{S}$ for the scheme $(\mathcal{S}, \mathcal{O}_\mathcal{S})$.

Definition 2.2.2. A *morphism* between two schemes $\mathcal{S}$ and $\mathcal{T}$ is a pair $(\psi, \psi^\#)$ of maps

$$\psi : \mathcal{S} \to \mathcal{T}$$

$$\psi^\# : \mathcal{O}_\mathcal{T} \to \psi_* \mathcal{O}_\mathcal{S}$$

such that ψ is continuous and for any point $s \in \mathcal{S}$, any neighbourhood U of $\psi(s) \in \mathcal{T}$, and any $f \in \mathcal{O}_\mathcal{T}$, f vanishes at $\psi(s)$ if and only if $\psi^\#$ vanishes at s.

With these definitions, we can consider the category of schemes, denoted **Sch**. Schemes are going to be our candidates for moduli spaces[3].

For any collection of objects A, we say that a *family* of objects of A parametrized by $\mathcal{S}$ is a collection $\{E_s\}_{s\in\mathcal{S}}$ of objects of A indexed by the points of $\mathcal{S}$. If we have an equivalence relation $\sim$ on A, then two families E and E' are *equivalent* if $E_s \sim E'_s$ for all $s \in \mathcal{S}$. We often ask that these families satisfy a certain algebraic condition, which will be specified in each case. Intuitively, this condition should make the families vary over $\mathcal{S}$ in some nice way. Given such a condition P, we say that families E which satisfy it are of *P-type*.

The information $(A, P, \sim)$ defines a *moduli problem*. A *moduli functor* for this moduli problem is a functor from the category of schemes to the category of sets

$$\begin{aligned}\mathrm{Mod}(A,P,\sim):\ \mathbf{Sch} &\longrightarrow \mathbf{Set}\\ \mathcal{S} &\longmapsto \{\text{equivalence classes of } P\text{-type families}\\ &\qquad\quad \text{of objects of } A \text{ parametrized by } \mathcal{S}\}.\end{aligned}$$

The ideal "solution" to a moduli problem is the following:

Definition 2.2.3. A scheme $\mathcal{M} \in \mathbf{Sch}$ is a *fine moduli space* for $\mathrm{Mod}(A, P, \sim)$ if there exists a universal family $\mathbb{U}$ over $\mathcal{M}$.

Of course, this begs the question of what a universal family is. Loosely, a universal family should encode information about all other families parametrized by all other schemes:

Definition 2.2.4. A family $\mathbb{U} \xrightarrow{\pi} \mathcal{M}$ is *universal* if, for any family of objects E parametrized by any scheme $\mathcal{S}$, there exists a unique map $f_E : \mathcal{S} \to \mathcal{M}$ such that $f_E^*\mathbb{U} = E$.

Recall the definition of a pullback bundle $f_E^*\mathbb{U} = \{(u, s) \in \mathbb{U} \times \mathcal{S} : f_E(s) = \pi(u)\}$. In a diagram, this looks like

$$\begin{tikzcd} E \arrow[d] & \mathbb{U} \arrow[d, "\pi"] \\ \mathcal{S} \arrow[r, dotted, "\exists! f_E"'] & \mathcal{M} \end{tikzcd}$$

[3]There is a fully faithful functor $t : \mathbf{Var}(k) \to \mathbf{Sch}$ from the category of algebraic varieties over an algebraically closed field k to the category of schemes. That is, we can think of varieties as a specific kind of scheme. Indeed, after this section, all the moduli spaces we encounter or construct are considered as varieties.

It turns out that the geometry and topology of $\mathcal{M}$, as well as the vector bundle structure of $\mathbb{U}$, is all encoded in the universal property. That is, there exists an identification

$$\{\text{morphisms } f : \mathcal{S} \to \mathcal{M}\} \longleftrightarrow \left\{ \begin{matrix} \text{equivalence classes of } P\text{-type families} \\ \text{of objects of } A \text{ parametrized by } \mathcal{S} \end{matrix} \right\}$$

This can be restated as an isomorphism of functors

$$\mathrm{Hom}_{\mathbf{Sch}}(-, \mathcal{M}) \cong \mathrm{Mod}(A, P, \sim)$$

where $\mathrm{Hom}_{\mathbf{Sch}}(-, \mathcal{M}) : \mathbf{Sch}^{\mathrm{opp}} \to \mathbf{Set}$ is the contravariant functor (sometimes referred to as the *functor of points*) defined as follows:

- for $\mathcal{S} \in \mathbf{Sch}$, $\mathcal{S} \mapsto \mathrm{Hom}_{\mathbf{Sch}}(\mathcal{S}, \mathcal{M})$
- for a map $f : \mathcal{S} \to \mathcal{T}$, define $\mathrm{Hom}_{\mathbf{Sch}}(g, \mathcal{M}) : \mathrm{Hom}_{\mathbf{Sch}}(\mathcal{T}, \mathcal{M}) \to \mathrm{Hom}_{\mathbf{Sch}}(\mathcal{S}, \mathcal{M})$ by $g \mapsto g \circ f$ for all $g : \mathcal{T} \to \mathcal{M}$.

A functor which is isomorphic to a functor of the form $\mathrm{Hom}(-, \mathcal{S})$ is called *representable*. In fact, the moduli functor $\mathrm{Mod}(A, P, \sim)$ has a fine moduli space if and only if it is representable.

Example 2.2.1. An example of a fine moduli space is the Grassmannian variety $\mathrm{Gr}(k, n)$ which parametrizes k-dimensional complex subspaces of $\mathbb{C}^n$. This space corresponds to the moduli problem where A is the collection of k-dimensional complex subspaces of $\mathbb{C}^n$, $\sim$ is trivial, and a family $E \to \mathcal{S}$ is of P-type if it is a vector subbundle of $\mathcal{S} \times \mathbb{C}^n$.

The universal family $\mathbb{U}$ over $\mathrm{Gr}(k, n)$ is the tautological bundle, meaning that its fibre at a point that corresponds to $V \subset \mathbb{C}^n$ is the subspace V itself. Since the families E and $\mathbb{U}$ can both be viewed as lying in trivial bundles, we have the following picture:

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[dr] & \mathbb{C}^n \arrow[r] \arrow[d] & \mathbb{C}^n \arrow[dr] & \mathbb{U} \arrow[l, hook'] \arrow[d, "\pi"] \\
& \mathcal{S} \arrow[rr, dotted, "\exists! f_E"'] & & \mathrm{Gr}(k, n)
\end{tikzcd}$$

The fact that the structure of the moduli space $\mathcal{M}$ can be realized from the universal property is a consequence of the Yoneda Lemma ([46]), which says that if $\mathbf{C}$ is a locally small category, $\mathcal{S} \in \mathbf{C}$, and $F : \mathbf{C}^{\text{opp}} \to \mathbf{Set}$, then there is an equivalence between the set of natural transformations $\text{Hom}_{\mathbf{C}}(-, \mathcal{S}) \to F$ and the elements of $F(\mathcal{S})$. In our context, one consequence of this is that the natural isomorphism

$$\eta : \text{Mod}(A, P, \sim) \to \text{Hom}_{\mathbf{Sch}}(-, \mathcal{M})$$

is the same a point in

$$\text{Mod}(A, P, \sim)(\mathcal{M}) = \{\text{equivalence classes of } P\text{-type families of objects of } A \text{ parametrized by } \mathcal{M}\}.$$

Specifically, the isomorphism η corresponds to the universal family $\mathbb{U}$!

The Yoneda Lemma has as a corollary that $\text{Hom}_{\mathbf{C}}(-, \mathcal{S})$ is naturally isomorphic to $\text{Hom}_{\mathbf{C}}(-, \mathcal{S}')$ if and only if $\mathcal{S} \cong \mathcal{S}'$. This tells us that if a fine moduli space exists, it is unique. There is, however, a serious hurdle; fine moduli spaces often do not exist! One problem is that the objects in question may have "too many" automorphisms.

Example 2.2.2. Consider a family of objects E parametrized by $[0, 1]$ such that each E_x is isomorphic, then identify E_0 with E_1 via a nontrivial automorphism. We now have a family E parametrized by S^1 which is not trivial. Each point of the circle carries an isomorphic object, so each point should be mapped to the same point of the fine moduli space $\mathcal{M}$. However, the constant map $S^1 \mapsto \{m\} \in \mathcal{M}$ classifies the trivial family $E \times S^1$. So we see that there must be more families than there are maps, and so no fine moduli space exists.

Our focus will be constructing *coarse* moduli spaces:

Definition 2.2.5. A scheme $\mathcal{M} \in \mathbf{Sch}$ is a *coarse moduli space* for $\text{Mod}(A, P, \sim)$ if there exists a natural transformation (not necessarily an isomorphism) $\eta : \text{Mod}(A, P, \sim) \to \text{Hom}_{\mathbf{Sch}}(-, \mathcal{M})$ which:

- induces a bijection of sets $\text{Mod}(A, P, \sim)(\{pt\}) \cong \text{Hom}_{\mathbf{Sch}}(\{pt\}, \mathcal{M}) \cong \mathcal{M}$
- is initial among all such natural transformations.

That is, the points of a coarse moduli space $\mathcal{M}$ are in bijection with the equivalence classes which we are trying to classify, and $\mathrm{Hom}_{\mathbf{Sch}}(-, \mathcal{M})$ is the functor of points which is "closest" to $\mathrm{Mod}(A, P, \sim)$. From now on, we use the terms "moduli space" and "coarse moduli space" interchangeably.

We also ask that our coarse moduli spaces be separable. This leads to the Geometric Invariant Theory of [51] and the idea of stable and unstable points, the unstable points of the moduli space being the ones which we ignore to ensure that our space is Hausdorff. We will explore this more directly in our specific cases.

2.3 (Twisted) Higgs bundles

2.3.1 Definitions

In 1987, the notion of a Higgs bundle was introduced by Hitchin in [35]. Hitchin was considering solutions to the self-dual Yang-Mills equations on a Riemann surface X of genus $g \geq 2$:

$$\begin{aligned} F(\nabla) + \Phi \wedge \Phi^* &= 0 \\ \partial_{\nabla}^{0,1} \Phi &= 0 \end{aligned} \tag{2.5}$$

where ∇ is a unitary connection on a Hermitian vector bundle E on X, $F(\nabla)$ is the curvature of ∇, Φ is an ω_X-twisted endomorphism of E, and Φ^* is its Hermitian adjoint. In terms of intuition, the second equation is asking only that Φ be holomorphic with respect to the holomorphic structure on E induced by ∇, while the first is asking if we can perturb a connection ∇ by a linear operator Φ so that it is flat. A related notion is that of a Higgs bundle: The Hitchin-Kobayashi correspondence ([35]; for more generality, see [47]) allows us to move between solutions (∇, Φ) to these equations and stable Higgs bundles.

Definition 2.3.1. A *Higgs bundle* on X is a pair (E, Φ) consisting of a holomorphic vector bundle $E \to X$ and a holomorphic map $\Phi : E \to E \otimes \omega_X$.

The study of Higgs bundles has proved to be very fruitful indeed for mathematics as a whole. We will be considering a slight generalization, which allows one to easily consider Higgs bundles on surfaces of any genus $g \geq 0$.

Definition 2.3.2. Given a holomorphic line bundle L on X, an *L-twisted Higgs bundle* on X is a pair (E, Φ) consisting of a holomorphic vector bundle $E \to X$ and a holomorphic map $\Phi : E \to E \otimes L$.

Definition 2.3.3. We say that two (twisted) Higgs bundles (E, Φ) and (E', Φ') on X are *equivalent* if E and E' are isomorphic as vector bundles and $\Phi = \Psi\Phi'\Psi^{-1}$ for some $\Psi \in H^0(X, \mathrm{Aut}(E))$. This is equivalent to asking that the diagram

$$\begin{tikzcd}
E \arrow[r, "\Psi"] \arrow[d, "\Phi"'] & E' \arrow[d, "\Phi'"] \\
E \otimes L \arrow[r, "\Psi \otimes 1_L"] & E' \otimes L
\end{tikzcd}$$

commutes for some isomorphism Ψ.

These twisted objects have attracted attention for several years ([22, 34, 59]) one of the reasons for which being that traditional Higgs bundles are always unstable on the Riemann sphere, but there are a cornucopia of stable twisted Higgs bundles.

We have again mentioned the idea of *stability*. The situation is that the coarse moduli space of all (twisted) Higgs bundles is ill-behaved (it is non-Hausdorff). This problem arises often in moduli problems, and the solution is to throw away the so-called unstable objects. In the context of (twisted) Higgs bundles we have the following stability condition.

Definition 2.3.4. Let (E, Φ) be an L-twisted Higgs bundle. The *slope* of E is defined as

$$\mu(E) = \frac{\deg(E)}{\mathrm{rank}(E)}.$$

A subbundle F of E is *Φ-invariant* if $\Phi(F) \subseteq F \otimes L$. We say that (E, Φ) is *semistable* if $\mu(F) \leq \mu(E)$ for all nonzero, proper, Φ-invariant subbundles F of E. If this inequality is strict for all such F, then we say that (E, Φ) is *stable*, and if this inequality fails for some F then we say that (E, Φ) is *unstable*.

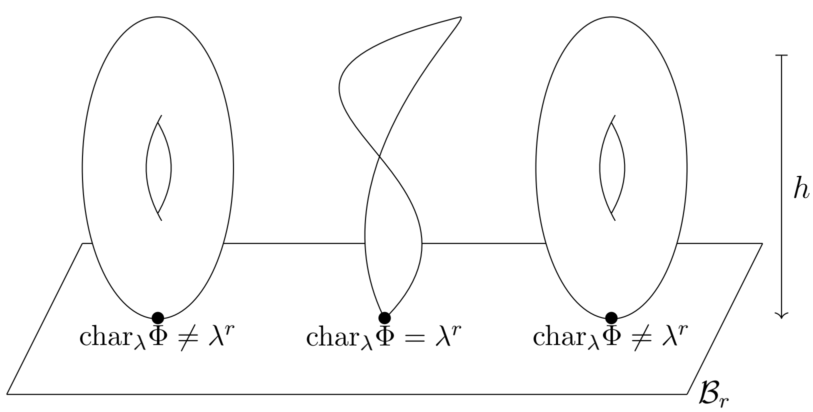
We will sometimes denote $\mu(E)$ by μ_{tot}. We also assume $r = \mathrm{rank}(E)$ and $d = \deg(E)$ are coprime throughout so that the semistable objects are exactly the stable objects.

The relationship between the equations in 2.5 (often called the *Hitchin equations*) and Higgs bundles is that solutions (∇, Φ) to the Hitchin equations are stable Higgs bundles. This is the simplest form Hitchin-Kobayashi correspondence.

In terms of moduli functors, let $A_{X,L}(r,d)$ be the collection of stable L-twisted Higgs bundles on X of degree d and rank r and let $\sim$ be the equivalence for Higgs bundles defined above. Given a line bundle $L \to X$ and a scheme $\mathcal{S}$, define a line bundle $L^{\mathcal{S}}$ on $X \times \mathcal{S}$ by $L^{\mathcal{S}}\big|_{X\times\{s\}} \cong L$. A family of Higgs bundles parametrized by $\mathcal{S}$, which is on its own simply a choice of an arbitrary Higgs bundle for each point of $\mathcal{S}$, is said to be of P-type if it can be written as $(E^{\mathcal{S}}, \Phi^{\mathcal{S}})$, where $E^{\mathcal{S}}$ is a holomorphic vector bundle over $X \times \mathcal{S}$ and $\Phi^{\mathcal{S}}$ is a holomorphic section of $\mathrm{End}(E^{\mathcal{S}}) \otimes L^{\mathcal{S}}$. So there is a Higgs bundle moduli functor

$$\mathrm{Mod}(A_{X,L}(r,d), P, \sim) : \ \mathbf{Sch} \to \mathbf{Set}$$

for which a coarse moduli space exists, due to Nitsure in [57]. We denote this coarse moduli space of stable L-twisted Higgs bundles of rank r and degree d on X by $\mathcal{M}_{X,L}(r,d)$. There is a popular and very useful characterization of this space known as the *Hitchin fibration*:



The projection h is known as the *Hitchin map* and is the map that sends (E, Φ) to the characteristic polynomial $\mathrm{char}_\lambda\Phi$ of Φ (equivalently, the set of eigenvalues of Φ). The base $\mathcal{B}_r = \bigoplus_{i=1}^r H^0(X, L^{\otimes i})$ is known as the *Hitchin base*. This fibration is well-known in the case that $L = \omega_X$, but exists for any L.

Technically, Definition 2.3.2 defines L-twisted $GL(r,\mathbb{C})$-Higgs bundles. While there is a notion of G-Higgs bundles for any complex reductive Lie group G (and corresponding moduli spaces denoted $\mathcal{M}^{G}_{X,L}(r,d)$), we will only touch on a few cases and so the following definitions will suffice.

Definition 2.3.5. Given a holomorphic line bundle L on X, an *L-twisted $SL(r,\mathbb{C})$-Higgs bundle with determinant*[4] P on X is a pair (E,Φ) consisting of a rank r holomorphic vector bundle $E \to X$ with $\det(E) = P$ and a holomorphic map $\Phi : E \to E \otimes L$ with $\mathrm{tr}(\Phi) = 0$.

Definition 2.3.6. Given a holomorphic line bundle L on X, an *L-twisted $PGL(r,\mathbb{C})$-Higgs bundle* on X is an equivalence class $[(E,\Phi)]$ of L-twisted $SL(r,\mathbb{C})$-Higgs bundles where (E,Φ) and (E',Φ') are equivalent if $E \cong E' \otimes M$ for some line bundle M over X and $\Phi = \Phi' \otimes 1_M$.

In the second definition, note that the condition $E \cong E' \otimes M$ forces $M^r = \mathcal{O}_X$, and hence M lies in the group of order-r roots of unity in the divisor group of X. It follows that an alternative description of $\mathcal{M}^{PGL(n,\mathbb{C})}_{X,L}(r,d)$ is as the quotient of $\mathcal{M}^{SL(n,\mathbb{C})}_{X,L}(Q)$ by this group, acting by tensor product.

Remark 2.3.1. When the underlying curve is $X = \mathbb{P}^1$, the cohomologies of the $GL(r,\mathbb{C})$, $SL(r,\mathbb{C})$, and $PGL(r,\mathbb{C})$ moduli spaces coincide, so we will not make a distinction between them in our calculations.

2.3.2 Fixed point loci of the moduli space

Calculating the Betti numbers of the moduli space $\mathcal{M}_{X,L}(r,d)$ motivates much of what is to come. Here we use Morse-Bott theory to explain how we can determine such information using moduli of spaces holomorphic chains. These ideas appear in Hitchin's original paper [35].

According to Morse-Bott theory, we can find the Poincaré polynomial of $\mathcal{M}_{X,L}(r,d)$ via the formula

$$\mathcal{P}_x(\mathcal{M}_{X,L}(r,d)) = \sum_{\mathcal{N}} x^{\beta(\mathcal{N})}\mathcal{P}_x(\mathcal{N}), \tag{2.6}$$

where $\mathcal{N}$ denotes a connected component of the critical set of a Morse-Bott function and $\beta(\mathcal{N})$ is the Morse index of any point in $\mathcal{N}$. These indices can be computed algebraically as dimensions of weight spaces or by using differential topology ([60]).

[4] This is a small abuse of notation, as $SL(r,\mathbb{C})$-Higgs bundle should really only mean the case of fixed determinant $\mathcal{O}_X$.

There is a natural height function on $\mathcal{M}_{X,L}(r,d)$, which turns out to be a perfect Morse-Bott function, given by the L^2 norm

$$f((E,\Phi)) = \|\phi\|^2 = 2i\int_X \mathrm{tr}(\Phi\Phi^*)dx \wedge dy.$$

Taking a more algebraic point of view, the critical subvarieties of this function are the fixed "points" of the group action of (the compact part of) $\mathbb{C}^*$ acting on $\mathcal{M}_{X,L}(r,d)$ by

$$\theta \cdot (E,\Phi) = (E, e^{i\theta}\Phi).$$

The connected components of the fixed point set are the objects $\mathcal{N}$ in Equation 2.6. The connection to the differential approach of Morse theory is that the function f is a moment map for the action of the compact part $S^1 \subset \mathbb{C}^*$.

If (E,Φ) is fixed, then Φ and $e^{i\theta}\Phi$ have the same characteristic polynomials, and in particular the same determinant,

$$\det\Phi = (e^{i\theta})^r \det\Phi$$

and hence $\det\Phi(1-e^{ir\theta})$ for all θ, implying that $\det\Phi = 0$. We can apply this approach to all the terms in the characteristic polynomial, and so conclude that $\mathrm{char}_\lambda\Phi = \lambda^r$. Thus Φ is nilpotent of order r and all the fixed points lie in the nilpotent cone $h^{-1}(0)$ of $\mathcal{M}_{X,L}(r,d)$.

To take a closer look at which objects will specifically be fixed, note that (E,Φ) fixed implies that there exists a family of automorphisms $\Psi_\theta \in H^0(X, \mathrm{Aut}(E))$ such that $e^{i\theta}\Phi = \Psi_\theta\Phi\Psi_\theta^{-1}$ for all θ. Differentiating yields

$$\begin{aligned} ie^{i\theta}\Phi &= \frac{d\Psi_\theta}{d\theta}\Phi\Psi_\theta^{-1} + \Psi_\theta\Phi\frac{d\Psi_\theta^{-1}}{d\theta} \\ &= \frac{d\Psi_\theta}{d\theta}\Phi\Psi_\theta^{-1} - \Psi_\theta\Phi\Psi_\theta^{-1}\frac{d\Psi_\theta}{d\theta}\Psi_\theta^{-1}. \end{aligned} \tag{2.7}$$

The 1-parameter family $\{\Psi_\theta\}$ is generated by the endomorphism $\Theta := \left.\frac{d\Psi_\theta}{d\theta}\right|_{\theta=0}$. Thus, taking $\theta \to 0$ in Equation 2.7 gives

$$\begin{aligned} i\Phi &= \Theta\Phi - \Phi\Theta \\ &= [\Theta,\Phi]. \end{aligned}$$

As an endomorphism, Θ itself has eigenvalues and eigenspaces, say $(\lambda_1, U_1), \ldots, (\lambda_m, U_m)$, $1 \leq m \leq r$. Now consider

$$\begin{aligned} i\Phi U_j &= [\Theta, \Phi] U_j \\ &= \Theta\Phi U_j - \Phi\Theta U_j \\ &= \Theta\Phi U_j - \lambda_j \Phi U_j. \end{aligned}$$

this means that $(\lambda_j + i)\Phi U_j = \Theta\Phi U_j$, which further implies that ΦU_j is contained in the eigenspace of Θ for $\lambda_j + i$, and in particular $\Lambda_j + i$ is an eigenvalue and so must be equal to some other λ_k. We can do this for any (λ_j, U_j) and can re-order the eigenvalues and eigenspaces of Θ as $(\lambda_1, U_1), (\lambda_1 + i, U_2), \ldots, (\lambda_1 + (m-1)i, U_m)$. Using the notation $\phi_j = \Phi\big|_{U_j}$, we have

$$\begin{aligned} \phi_1 &: U_1 \to U_2 \otimes L \\ \phi_2 &: U_2 \to U_3 \otimes L \\ &\vdots \\ \phi_{m-1} &: U_{m-1} \to U_m \otimes L \\ \phi_m &: U_m \to 0. \end{aligned}$$

There is only one such sequence of U_i since if there were multiple, each would be Φ-invariant and E could be presented as a direct sum of Φ-invariant subbundles.

In general, (E, Φ) is a fixed point if $E = \bigoplus_{i=1}^{m} U_i$ and $\Phi : E \to E \otimes L$ has the property that $\Phi : U_i \to U_{i+1} \otimes L$. Such Higgs bundles are sometimes called *holomorphic chains*.

The fixed point subvarieties $\mathcal{N}$ are the moduli spaces of holomorphic chains with fixed ranks and degrees $((r_i, d_i))_{i=1,\ldots,m}$. Holomorphic chains can be viewed as representations of A-type quivers in a twisted category, as introduced in the following section.

2.4 Quivers

Definition 2.4.1. A *quiver* is a finite directed graph. Specifically, a quiver is a quadruple (V, A, s, t) consisting of two finite sets V (the set of *vertices* or *nodes*) and A (the set of *arrows*) and two maps $s, t : A \to V$ which assign to each arrow its *source* and *target*, respectively.

Often, one studies quivers which are labelled in some way.

Definition 2.4.2. A *representation* of a quiver Q in the category $\mathcal{C}$ is the assignment of an object of $\mathcal{C}$ to each of the vertices of Q, possibly subject to some labelling, and a morphism of $\mathcal{C}$ to each of the arrows.

The space of all stable representations of Q in $\mathcal{C}$ is denoted by $\mathrm{Rep}(Q)$.

Definition 2.4.3. Two representations $((E_i)_{i\in V}, (f_j)_{j\in A})$ and $((F_i)_{i\in V}, (g_j)_{j\in A})$ of a quiver Q in the category $\mathcal{C}$ are said to be *equivalent* if there exists a family of isomorphisms $(h_i : E_i \to F_i)_{i\in V}$ of objects of $\mathcal{C}$ such that

$$\begin{tikzcd} E_{s(j)} \arrow[r, "u_{s(j)}"] \arrow[d, "f_j"'] & F_{s(j)} \arrow[d, "g_j"] \\ E_{t(j)} \arrow[r, "u_{t(j)}"] & F_{t(j)} \end{tikzcd}$$

commutes for every $j \in A$.

One is often interested in the moduli space of such representations, which is sometimes called a *quiver variety*, although we will not carefully construct the moduli functor here as we did above.

Moduli spaces of representations of quivers Q which have each node labelled by (r_i, d_i), with $r_i \in \mathbb{Z}_{>0}$ and $d_i \in \mathbb{Z}$, will be our objects of study. Such a quiver Q has a *dual quiver* Q^* which is constructed by reversing the direction of each arrow and changing the labelling of each node to $(r_i, -d_i)$. The moduli spaces of representations of Q is isomorphic to that of Q^*, a fact we will make several uses of.

By and large, the study of quiver varieties has focused on representations of quivers in the category of vector spaces and has created a vast theory (see for example [41]). Representations in certain twisted categories have been used to study Higgs bundles and that is where we set our sights in this thesis. To be specific, we study representations in the twisted category $\mathrm{Bun}(X, L)$, whose objects are holomorphic vector bundles on X and whose morphisms are vector bundle morphisms twisted by L. Given a quiver Q with nodes labelled by (r_i, d_i), a representation amounts to a choice of vector bundle U_i of rank r_i and degree d_i to each node

$\bullet_{r_i,d_i}$ and a choice of map $\phi_{ij} : U_i \to U_j \otimes L$ to each arrow $\bullet_{r_i,d_i} \longrightarrow \bullet_{r_j,d_j}$. We denote the moduli space of representations of Q in this category by $\mathcal{M}_{X,L}(Q)$.

Definition 2.4.4. An *A-type quiver* is a quiver of the form

$$Q = \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet.$$

Denote by A_n the A-type quiver with n nodes. It is an easy but crucial observation that representations of A-type quivers in $\mathrm{Bun}(X, L)$ are holomorphic chains.

There is an alternative method of constructing varieties from quivers due to Nakajima which we introduce in Chapter 6.

3 A-TYPE QUIVERS

3.1 Deformation theory

Here we calculate the dimension of the moduli space $\mathcal{M}_{X,L}(Q)$ introduced in the previous chapter, when Q is the quiver A_n and L is any holomorphic line bundle on a Riemann surface X of any genus g. For this, we will need deformation theory.

3.1.1 Background on hypercohomology

We make use of some tools from deformation theory to understand the local behaviour of twisted Higgs bundles and to calculate the dimension of the moduli space of type A quiver varieties. For us, it suffices to focus (admittedly narrowly) on hypercohomology groups. This background follows [30] while the subsequent section is inspired by the deformation theory of Higgs bundles which appears, for example, in [57].

Consider an object O and its class $[O]$ in the moduli space $\mathcal{M}$ of objects of the same type as O. The idea is that the tangent space $T_{[O]}\mathcal{M}$ is equal to the first hypercohomology group $\mathbb{H}^1(O)$.

Standard cohomologies arise from single differentials: suppose that O is equipped with two differentials δ and δ'. We now have bidegree cochains $C^{\bullet,\bullet}(O)$ associated to O such that

$$\begin{array}{ccc} C^{p,q}(O) & \xrightarrow{\delta} & C^{p+1,q}(O) \\ {\scriptstyle\delta'}\downarrow & & \downarrow{\scriptstyle\delta'} \\ C^{p,q+1}(O) & \xrightarrow{\delta} & C^{p+1,q+1}(O) \end{array}$$

commutes and there exists a left exact sequence[5]

$$0 \longrightarrow \mathcal{E}_2^{0,1}(O) \longrightarrow \mathbb{H}^1(O) \longrightarrow \mathcal{E}_2^{1,0}(O) \longrightarrow \mathcal{E}_2^{0,2}(O) \longrightarrow \mathbb{H}^2(O). \tag{3.1}$$

The spaces $\mathcal{E}_2^{\bullet,\bullet}(O)$ are defined by first setting

$$\begin{aligned}\mathcal{E}_1^{p,q} &= H_\delta^p(C^{\bullet,\bullet}(O)) \\ &= \frac{\ker\big(C^{p,q}(O) \xrightarrow{\delta} C^{p+1,q}(O)\big)}{\operatorname{im}\big(C^{p-1,q}(O) \xrightarrow{\delta} C^{p,q}(O)\big)}\end{aligned}$$

and then

$$\begin{aligned}\mathcal{E}_2^{p,q}(O) &= H_{\delta'}^q(\mathcal{E}_1^{\bullet,\bullet}(O)) \\ &= \frac{\ker\big(\mathcal{E}_1^{p,q}(O) \xrightarrow{\delta'} \mathcal{E}_1^{p,q+1}(O)\big)}{\operatorname{im}\big(\mathcal{E}_1^{p,q-1}(O) \xrightarrow{\delta'} \mathcal{E}_1^{p,q}(O)\big)}.\end{aligned}$$

If $\mathbb{H}^2 = 0$, then it is said that the deformations are *unobstructed*, and when this is the case it is clear that

$$T_{[O]}\mathcal{M} = \mathbb{H}^1(O) = \mathcal{E}_2^{0,1}(O) \oplus \mathcal{E}_2^{1,0}(O),$$

since a short exact sequence of vectors spaces is split.

3.1.2 Dimension of the moduli space

We now calculate the dimension of the tangent space $T_{\mathcal{C}}\mathcal{M}_{X,L}(Q)$ at a point $\mathcal{C}$ of the moduli space and the expected dimension of the moduli space $\mathcal{M}_{X,L}(Q)$. We note that our result in this section can be deduced from more general arguments in [61]. In parallel, parabolic versions of this result can be found in [9, 19].

Given a choice of $\mathcal{C} = (U_1, \ldots, U_n; \phi_1, \ldots, \phi_{n-1})$, we use the usual differential δ to define vector spaces $V^{p,q}(\mathcal{C})$ by

$$V^{p,q}(\mathcal{C}) = H^p\left(\left(\bigoplus_{i=1}^{n-q} U_i^* \otimes U_{i+q}\right) \otimes \Lambda^q L\right)$$

[5] This is an example of a *five-term exact sequence*, a type of exact sequence which arises from the study of spectral sequences.

(for the remainder of this section we will suppress $\mathcal{C}$). It is worth noting that $(\phi_1, \dots \phi_{n-1}) \in V^{0,1}$. We can also define a differential[6] $\delta_\Phi : V^{p,q} \to V^{p,q+1}$ by the following:

$$\delta_\Phi(\psi_1, \dots, \psi_{n-q}) =$$
$$(\psi_2\phi_1 - \phi_{1+q}\psi_1,\ \psi_3\phi_2 - \phi_{2+q}\psi_2, \dots, \psi_{n-q}\phi_{n-(q+1)} - \phi_{n-1}\psi_{n-(q+1)}).$$

The map δ_Φ is named for its dependence on the total map $\Phi := \bigoplus_{i=1}^{n-1} \phi_i$. Now we have given $\bigoplus_{p,q} V^{p,q}$ the structure of a bi-graded Lie algebra, with $\delta_\Phi(-)$ being the Lie bracket. The hypercohomology $\mathbb{H}^1$ that we are looking for fits into an exact sequence as in Equation 3.1 with

$$\mathcal{E}_2^{p,q} = \frac{\ker\big(V^{p,q} \xrightarrow{\delta_\Phi} V^{p,(q+1)}\big)}{\operatorname{im}\big(V^{p,(q-1)} \xrightarrow{\delta_\Phi} V^{p,q}\big)}.$$

Proposition 3.1.1. *If X is a Riemann surface and L is a line bundle, then the deformations of $\mathcal{M}_{X,L}(Q)$ are unobstructed.*

Proof. By the usual filtration, the space $\mathbb{H}^2$ consists of contributions from three spaces (which we will show are all trivial): $\mathcal{E}_2^{0,2}$, $\mathcal{E}_2^{2,0}$, and $\mathcal{E}_2^{1,1}$.

To begin, note that L is a line bundle on a projective algebraic curve so $\Lambda^2 L = 0$. Therefore $\mathcal{E}_2^{0,2}$, whose numerator consists of the kernel of

$$H^0((U_1^*U_3 \oplus \dots \oplus U_{n-2}^*U_n) \otimes \Lambda^2 L)$$

under δ_Φ, is zero.

To see that $\mathcal{E}_2^{2,0}$ is also zero, note that its numerator is the kernel of

$$H^2(\mathrm{End}U_1 \oplus \dots \oplus \mathrm{End}U_n)$$

under δ_Φ, which is trivial on a smooth, compact curve.

Thirdly, we must deal with

$$\begin{aligned}\mathcal{E}_2^{1,1} &= \frac{\ker\big(H^1((U_1^*U_2 \oplus \dots \oplus U_{n-1}^*U_n) \otimes L) \xrightarrow{\delta_\Phi} H^1((U_1^*U_3 \oplus \dots \oplus U_{n-2}^*U_n) \otimes \Lambda^2 L)\big)}{\operatorname{im}\big(H^1\big(\mathrm{End}U_1 \oplus \dots \oplus \mathrm{End}U_n\big) \xrightarrow{\delta_\Phi} H^1((U_1^*U_2 \oplus \dots \oplus U_{n-1}^*U_n) \otimes L)\big)} \\ &= \frac{H^1((U_1^*U_2 \oplus \dots \oplus U_{n-1}^*U_n) \otimes L)}{\operatorname{im}\big(H^1\big(\mathrm{End}U_1 \oplus \dots \oplus \mathrm{End}U_n\big) \xrightarrow{\delta_\Phi} H^1((U_1^*U_2 \oplus \dots \oplus U_{n-1}^*U_n) \otimes L)\big)}\end{aligned}$$

[6] This definition is inspired by the deformation theory of Higgs bundles, which have a differential

$$[-, \Phi] : C^{p,q}(E, \Phi) \to C^{p,q+1}(E, \Phi)$$

on Čech cochains $C^{p,q}(E, \Phi) = \check{C}^p(\mathrm{End}(E) \otimes \Lambda^q L)$.

If we can show that the map in the denominator is surjective, then we will have shown that $\mathcal{E}_2^{1,1}$ is trivial. To do this, consider the Serre-dual map

$$H^0((U_1^*U_2 \oplus \ldots \oplus U_{n-1}^*U_n)^* \otimes L^* \otimes \omega_X) \xrightarrow{\delta_\Phi^*} H^0((\mathrm{End}U_1 \oplus \ldots \oplus \mathrm{End}U_n)^* \otimes \omega_X)$$

where ω_X is the canonical line bundle on X. This map is equivalent to

$$H^0(U_1U_2^* \oplus \ldots \oplus U_{n-1}U_n^*) \xrightarrow{\delta_\Phi^*} H^0((\mathrm{End}U_1 \oplus \ldots \oplus \mathrm{End}U_n) \otimes L)$$

The map δ_Φ^* is injective if and only if δ_Φ is surjective, and vice versa. We can calculate

$$\begin{aligned}&\delta_\Phi^*(\eta_1, \ldots, \eta_{n-1}) = \\ &\qquad (\phi_1^*\eta_1^*,\ \phi_2^*\eta_2^* - \eta_1^*\phi_1^*, \ldots, \phi_{n-1}^*\eta_{n-1}^* - \eta_{n-2}^*\phi_{n-2}^*, -\eta_{n-1}^*\phi_{n-1}^*)\end{aligned}$$

from which we can see that the kernel of δ_Φ^* is trivial, since $\phi_1^*\eta_1^* = 0$ will imply $\eta_1 = 0$, so $\phi_2^*\eta_2^* - \eta_1^*\phi_1^* = 0$ implies $\phi_2^*\eta_2^* = 0$ and so on. Thus δ_Φ^* is injective, δ_Φ is surjective, and the image of $H^1(\mathrm{End}U_1 \oplus \ldots \oplus \mathrm{End}U_n)$ is $H^1((U_1^*U_2 \oplus \ldots \oplus U_{n-1}^*U_n) \otimes L)$, which tells us that $\mathcal{E}_2^{1,1} = 0$, as required.

$\square$

Remark 3.1.1. Recall that these finding mean that we can write $\mathbb{H}^1 = \mathcal{E}_2^{0,1} \oplus \mathcal{E}_2^{1,0}$ which allows for a natural interpretation of $\mathbb{H}^1$ as a direct sum of deformations of Φ which respect $(U_1, \ldots, U_n)$ and deformations of $(U_1, \ldots, U_n)$ which respect Φ ($\mathcal{E}_2^{0,1}$ and $\mathcal{E}_2^{1,0}$ respectively).

This proposition also gives us an inroad to calculating the expected dimension of $\mathcal{M}_{X,L}(Q)$. In particular,

$$\dim\mathcal{M}_{X,L}(Q) = e_2^{0,1} + e_2^{1,0}$$

where

$$e_2^{0,1} = \dim\left(\frac{H^0\big((U_1^*U_2 \oplus \ldots \oplus U_{n-1}^*U_n) \otimes L\big)}{\mathrm{im}\big(H^0\big(\mathrm{End}U_1 \oplus \ldots \oplus \mathrm{End}U_n\big) \xrightarrow{\delta_\Phi} H^0\big((U_1^*U_2 \oplus \ldots \oplus U_{n-1}^*U_n) \otimes L\big)\big)}\right) \tag{3.2}$$

and

$$e_2^{1,0} = \dim\Big(\ker\big(H^1\big(\mathrm{End}U_1 \oplus \ldots \oplus \mathrm{End}U_n\big) \xrightarrow{\delta_\Phi} H^1\big((U_1^*U_2 \oplus \ldots \oplus U_{n-1}^*U_n) \otimes L\big)\Big). \tag{3.3}$$

Theorem 3.1.1. *Given a quiver*

$$Q = \bullet_{r_1,d_1} \longrightarrow \bullet_{r_2,d_2} \longrightarrow \cdots \longrightarrow \bullet_{r_n,d_n},$$

the dimension of the moduli space of representations in the category of L-twisted vector bundles (with L of degree t) over a Riemann surface X of genus g is

$$\sum_{i=1}^{n-1} \big(r_i d_{i+1} - r_{i+1} d_i + r_i r_{i+1} t\big) + (1-g)\left(\sum_{i=1}^{n-1} r_i r_{i+1} - \sum_{i=1}^{n} r_i^2\right) + \min_{1 \le i \le n}\{h^0(\mathrm{End}U_i)\}.$$

Proof. We have from Proposition 3.1.1 that $\dim\mathcal{M}_{X,L}(Q) = e_2^{0,1} + e_2^{1,0}$, where $e_2^{0,1}$ and $e_2^{1,0}$ are given by equations (3.2) and (3.3), respectively. We also showed in the proof that the map

$$H^1\big(\mathrm{End}U_1 \oplus \ldots \oplus \mathrm{End}U_n\big) \xrightarrow{\delta_\Phi} H^1\big((U_1^* U_2 \oplus \ldots \oplus U_{n-1}^* U_n) \otimes L\big)$$

is surjective, so we can say

$$e_2^{1,0} = h^1\big(\mathrm{End}U_1 \oplus \ldots \oplus \mathrm{End}U_n\big) - h^1\big((U_1^* U_2 \oplus \ldots \oplus U_{n-1}^* U_n) \otimes L\big)$$

and a similar argument will allow us to analyze $e_2^{0,1}$. We would like to say that

$$H^0\big(\mathrm{End}U_1 \oplus \ldots \oplus \mathrm{End}U_n\big) \xrightarrow{\delta_\Phi} H^0\big((U_1^* U_2 \oplus \ldots \oplus U_{n-1}^* U_n) \otimes L\big)$$

is injective, but this is not quite true. By inspecting the map

$$\delta_\Phi(\psi_1, \ldots, \psi_n) =$$
$$(\psi_2\phi_1 - \phi_1\psi_1,\ \psi_3\phi_2 - \phi_2\psi_2, \ldots, \psi_n\phi_{n-1} - \phi_{n-1}\psi_{n-1})$$

it can be seen that it is injective only if we except one of the terms ψ_i. Ignoring an arbitrary ψ_i would result in a map that was injective but may not have an image of the same dimension as the full δ_Φ map. We must ignore a ψ_i coming from $H^0(\mathrm{End}U_i)$ of minimal dimension. If there are more than one such $H^0(\mathrm{End}U_i)$ having the minimal dimension, then it will not matter which we remove as the resulting dimensions will be the same. That is, if $H^0(\mathrm{End}U_j)$ is of minimal dimension among the $H^0(\mathrm{End}U_i)$, then we can think of δ_Φ as being $H^0(\mathrm{End}U_j)$-far away from being injective. This tells us that

$$e_2^{0,1} = h^0\big((U_1^* U_2 \oplus \ldots \oplus U_{n-1}^* U_n) \otimes L\big) - h^0\big(\mathrm{End}U_1 \oplus \ldots \oplus \mathrm{End}U_n\big) + \min_{1 \le i \le n}\{h^0(\mathrm{End}U_i)\}.$$

Now we can apply Riemann-Roch to $e_2^{0,1}+e_2^{1,0}$ to obtain

$$
\begin{aligned}
&e_2^{0,1}+e_2^{1,0}\\
&\quad= h^1\big(\mathrm{End}U_1\oplus\ldots\oplus\mathrm{End}U_n\big)-h^1\big((U_1^*U_2\oplus\ldots\oplus U_{n-1}^*U_n)\otimes L\big)\\
&\qquad+h^0\big((U_1^*U_2\oplus\ldots\oplus U_{n-1}^*U_n)\otimes L\big)-h^0\big(\mathrm{End}U_1\oplus\ldots\oplus\mathrm{End}U_n\big)\\
&\qquad+\min_{1\le i\le n}\{h^0(\mathrm{End}U_i)\}\\
&\quad=\deg\big((U_1^*U_2\oplus\ldots\oplus U_{n-1}^*U_n)\otimes L\big)+\mathrm{rank}\big((U_1^*U_2\oplus\ldots\oplus U_{n-1}^*U_n)\otimes L\big)(1-g)\\
&\qquad-\deg\big(\mathrm{End}U_1\oplus\ldots\oplus\mathrm{End}U_n\big)-\mathrm{rank}\big(\mathrm{End}U_1\oplus\ldots\oplus\mathrm{End}U_n\big)(1-g)\\
&\qquad+\min_{1\le i\le n}\{h^0(\mathrm{End}U_i)\}\\
&\quad=\sum_{i=1}^{n-1}\deg(U_i^*U_{i+1}L)+(1-g)\sum_{i=1}^{n-1}\mathrm{rank}(U_i^*U_{i+1}L)\\
&\qquad-\sum_{i=1}^{n}\deg(\mathrm{End}U_i)-(1-g)\Big(\sum_{i=1}^{n}\mathrm{rank}(\mathrm{End}U_i)\Big)+\min_{1\le i\le n}\{h^0(\mathrm{End}U_i)\}\\
&\quad=\sum_{i=1}^{n-1}\deg(U_i^*U_{i+1}L)+(1-g)\Big(\sum_{i=1}^{n-1}r_ir_{i+1}-\sum_{i=1}^{n}r_i^2\Big)+\min_{1\le i\le n}\{h^0(\mathrm{End}U_i)\}
\end{aligned}
\tag{3.4}
$$

It remains to calculate $\deg(U_i^*U_{i+1}L)$. Note that the following calculation also serves to demonstrate that the dimension of the moduli space only depends on the degrees and ranks of the U_i, not on their specific structures (how they may split, etc.). We decompose[7] the determinant of $U_i^*U_{i+1}L$ as follows:

$$
\begin{aligned}
\det(U_i^*U_{i+1}L)&=\det(U_i^*)^{\otimes r_{i+1}}\otimes\det(U_{i+1}L)^{\otimes r_i}\\
&=\det(U_i^*)^{\otimes r_{i+1}}\otimes\det(U_{i+1})^{\otimes r_i}\otimes\det(L)^{\otimes r_ir_{i+1}}
\end{aligned}
$$

[7]The determinant of a product of three vector bundles decomposes in a unique way just as one would expect. If V_1, V_2, and V_3 are vector bundles with ranks r_1, r_2, and r_3 respectively, then

$$
\begin{aligned}
\det(V_1V_2V_3)&=\det(V_1V_2)^{r_3}\det(V_3)^{r_2r_3}\\
&=\det(V_1)^{r_2r_3}\det(V_2)^{r_1r_3}\det(V_3)^{r_1r_2}\\
&=\det(V_1)^{r_2r_3}\det(V_2V_3)^{r_1}\\
&=\det(V_1V_2V_3)
\end{aligned}
$$

thus

$$\begin{aligned}\deg(U_i^* U_{i+1} L) &= \deg(\det(U_i^* U_{i+1} L)) \\ &= r_{i+1}\deg(U_i^*) + r_i\deg(U_{i+1}) + r_i r_{i+1}\deg(L) \\ &= r_i d_{i+1} - r_{i+1} d_i + r_i r_{r+1} t.\end{aligned}$$

This calculation along with equation (3.4) gives the result. □

3.2 Type $(1, \ldots, 1)$ quivers

In this section we consider the moduli space of representations of A-type quivers in the twisted category of bundles $\mathrm{Bun}(X, L)$, for X of arbitrary genus, L of degree t, and quivers labelled as such:

$$Q = \bullet_{1,d_1} \longrightarrow \bullet_{1,d_2} \longrightarrow \cdots \longrightarrow \bullet_{1,d_n}.$$

As mentioned previously, a point in the representation space amounts to the assignment of a holomorphic line bundle U_i of degree d_i to each vertex, and a map $\phi_i : U_i \to U_{i+1} \otimes L$ to each arrow. Note that for each $1 < m \leq n$, the subbundle $U_m \oplus U_{m+1} \oplus \ldots \oplus U_n$ is ϕ-invariant, and so by stability we can calculate

$$\begin{aligned}\frac{\sum_{i=m}^n d_i}{n-(m-1)} &< \frac{d}{n} \\ \frac{d - \sum_{i=1}^{m-1} d_i}{n-(m-1)} &< \frac{d}{n} \\ nd - n\sum_{i=m}^{n} d_i &< (n-(m-1))d \\ \frac{d(m-1)}{n} &< \sum_{i=1}^{m-1} d_i\end{aligned} \tag{3.5}$$

Consider the subbundle $U_1 \oplus \ldots \oplus U_{m-1}$, which has slope $\sum_{i=1}^{m-1} d_i/m-1$. By the above calculation,

$$\frac{\sum_{i=1}^{m-1} d_i}{m-1} > \frac{d}{n} = \mu_{tot},$$

and thus a stable representation cannot have $U_1 \oplus \ldots \oplus U_{m-1}$ being ϕ-invariant. In particular, this means that ϕ_{m-1} cannot be zero and so we can say that ϕ_i is nonzero for all $0 \leq i \leq n-1$.

We know that $\phi_i : U_i \to U_{i+1} \otimes L$, which is to say that $\phi_i \in H^0(X, U_i^* U_{i+1} L)$. The bundle $U_i^* U_{i+1} L$ can have nonzero sections only if $\deg(U_i^* U_{i+1} L) = d_{i+1} - d_i + t > 0$.

We begin with a concrete geometric description of the $SL(n,\mathbb{C})$ and $PGL(n,\mathbb{C})$ moduli spaces, which we then use to describe the $GL(n,\mathbb{C})$ case.

Theorem 3.2.1. *Let X be a Riemann surface of genus g, L a holomorphic line bundle of degree t on X, and Q be the quiver*

$$\bullet_{1,d_1} \longrightarrow \bullet_{1,d_2} \longrightarrow \cdots \longrightarrow \bullet_{1,d_n}.$$

Then the $SL(n,\mathbb{C})$ and $PGL(n,\mathbb{C})$ moduli spaces of representations of Q in $Bun(X,L)$ are

$$\mathcal{M}_{X,L}^{SL(n,\mathbb{C})}(Q) \cong \widetilde{\left(\prod_{i=1}^{n-1} Sym^{d_{i+1}-d_i+t}(X)\right)}$$

and

$$\mathcal{M}_{X,L}^{PGL(n,\mathbb{C})}(Q) \cong \prod_{i=1}^{n-1} Sym^{d_{i+1}-d_i+t}(X)$$

respectively, where the superscript $\sim$ is used to denote an n^{2g}-sheeted covering.

Proof. Let us first consider the moduli space of L-twisted $SL(n,\mathbb{C})$-Higgs bundles. That is, we are asking that a representation

$$U_1 \xrightarrow{\phi_1} U_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_{n-1}} U_n$$

have $\det\left(\bigoplus_{i=1}^n U_i\right) = P$ for some fixed $P \in \mathrm{Jac}^d(X)$. Since all the U_i are line bundles, we have $\bigotimes_{i=1}^n U_i = P$ which tells us that one of the U_i depends on the others; say

$$U_n = U_1^* U_2^* \dots U_{n-1}^* P$$

Recall that $\phi_i \in H^0(X, U_i^* U_{i+1} L) \setminus \{0\}$, so $\deg(\phi_i) = d_{i+1} - d_i + t$. In addition, since we are modding out by the action of $\mathbb{C}^*$ on this space, the information here amounts to a choice of $U_i^* U_{i+1} L$ and of projective class of ϕ_i, which we will denote by $[\phi_i]$. By the divisor correspondence, the information $(U_i^* U_{i+1} L, [\phi_i])$ is a point in the symmetric product of X with itself $d_{i+1} - d_i + t$ times. That is,

$$(U_i^* U_{i+1} L, [\phi_i]) \in \mathrm{Sym}^{d_{i+1}-d_i+t}(X) = \left(\prod_{i=1}^{d_{i+1}-d_i+t} X\right) \Big/ \Sigma_{d_{i+1}-d_i+t} \tag{3.6}$$

where $\Sigma_{d_{i+1}-d_i+t}$ is the symmetric group on $d_{i+1}-d_i+t$ elements. So, the information

$$\left(\left(U_1^*U_2L,[\phi_1]\right),\ldots,\left(U_{n-1}^*U_nL,[\phi_{n-1}]\right)\right)$$

lies in $\mathrm{Sym}^{d_2-d_1+t}(X)\times\ldots\times\mathrm{Sym}^{d_n-d_{n-1}+t}(X)$. This is not the moduli space we are seeking, since we want the U_i, not these $U_i^*U_{i+1}L$. If we are given such a point in $\mathrm{Sym}^{d_2-d_1+t}(X)\times\ldots\times\mathrm{Sym}^{d_n-d_{n-1}+t}(X)$, we can attempt to recover the U_i. Write $V_i=U_i^*U_{i+1}L$. Then,

$$\begin{aligned}
&V_1V_2^2V_3^3\ldots V_{n-2}^{n-2}V_{n-1}^*P\left(L^*\right)^{-1+\sum_{i=1}^{n-2}i}\\
&\qquad=(U_1^*U_2L)(U_2^*U_3L)^2\ldots(U_{n-2}^*U_{n-1}L)^{n-2}(U_{n-1}^*U_nL)^*P\left(L^*\right)^{-1+\sum_{i=1}^{n-2}i}\\
&\qquad=(U_1^*U_2L)(U_2^*U_3L)^2\ldots(U_{n-2}^*U_{n-1}L)^{n-2}(U_{n-1}^*U_1^*U_2^*\ldots U_{n-1}^*PL)^*P\left(L^*\right)^{-1+\sum_{i=1}^{n-2}i}\\
&\qquad=(U_1^*U_2)(U_2^*U_3)^2\ldots(U_{n-2}^*U_{n-1})^{n-2}(U_1U_2\ldots U_{n-2}U_{n-1}^2)\\
&\qquad=U_{n-1}^n
\end{aligned}$$

Thus, a point in $\mathrm{Sym}^{d_2-d_1+t}(X)\times\ldots\times\mathrm{Sym}^{d_n-d_{n-1}+t}(X)$ fixes the n-th power of U_{n-1}. Accounting for torsion in the Jacobian, we know that U_{n-1}^n has n^{2g} distinct roots[8]. We can choose one of these, and this fixes all the other U_i. This tells us that $\mathcal{M}_{X,L}^{SL(n,\mathbb{C})}(Q)$ is in fact an n^{2g}-fold covering of $\prod_{i=1}^{n-1}\mathrm{Sym}^{d_{i+1}-d_i+t}(X)$, which we denote by $^{\sim}\left(\prod_{i=1}^{n-1}\mathrm{Sym}^{d_{i+1}-d_i+t}(X)\right)$.

Define the n-th roots of unity

$$\mathrm{Jac}^0(X)[n]:=\{J\in\mathrm{Jac}^0(X)|J^n=\mathcal{O}_X\},$$

which is a finite subgroup of $\mathrm{Jac}^0(X)$. This group acts on the covering $^{\sim}\left(\prod_{i=1}^{n-1}\mathrm{Sym}^{d_{i+1}-d_i+t}(X)\right)$ in the following way:

$$J\cdot(U_1,\ldots,U_n,[\phi_1],\ldots,[\phi_{n-1}])=(J\otimes U_1,\ldots,J\otimes U_n,[\phi_1],\ldots,[\phi_{n-1}]).$$

This is a true action because the maps $[\phi_i]$ are unaffected:

$$H^0\left(X,(JU_i)^*JU_{i+1}L\right)=H^0\left(X,U_i^*U_{i+1}L\right),$$

and we still have the same fixed determinant:

$$\det\left(\bigoplus_{i=1}^n JU_i\right)=J^n\det\left(\bigoplus_{i=1}^n U_i\right)=P.$$

[8]This is a generalization of Atiyah's work in [4].

The orbits of this action are precisely points $(U_1, \ldots, U_n, [\phi_1], \ldots, [\phi_{n-1}])$ arising from different choices of the root of U_{n-1}^n, since if R is an n-th root of U_{n-1}^n then the others arise by tensoring R with the n-th roots of unity in $\mathrm{Jac}^0(X)$. That is, the orbit of R under this action is $\{R \otimes \mathcal{O}_X, R \otimes J_1, \ldots, R \otimes J_{n-1}\}$, where $\{\mathcal{O}_X, J_1, \ldots, J_{n-1}\}$ are the elements of $\mathrm{Jac}^0(X)[n]$. By definition we have

$$\mathcal{M}_{X,L}^{PGL(n,\mathbb{C})}(Q) = \frac{\mathcal{M}_{X,L}^{SL(n,\mathbb{C})}(Q)}{\mathrm{Jac}^0(X)[n]}$$

and the preceding analysis has thus shown that

$$\mathcal{M}_{X,L}^{PGL(n,\mathbb{C})}(Q) \cong \prod_{i=1}^{n-1} \mathrm{Sym}^{d_{i+1}-d_i+t}(X).$$

Note that $\mathcal{M}_{X,L}^{PGL(n,\mathbb{C})}(Q)$ is not smooth, due to its construction as a quotient by a finite group[9]. □

Corollary 3.2.1. *The $GL(n,\mathbb{C})$ moduli space is, cohomologically,*

$$Jac^0(X) \times \prod_{i=1}^{n-1} Sym^{d_{i+1}-d_i+t}(X).$$

Proof. It is shown in [32] that the mixed Hodge polynomials of $\mathcal{M}_{X,L}^{GL(n,\mathbb{C})}(Q)$ and $\mathcal{M}_{X,L}^{PGL(n,\mathbb{C})}(Q)$ satisfy

$$H(\mathcal{M}_{X,L}^{GL(n,\mathbb{C})}(Q), x, y, t) = (1 + xyt)^{2g} H(\mathcal{M}_{X,L}^{PGL(n,\mathbb{C})}(Q), x, y, t).$$

This gives us the above topological description of $\mathcal{M}_{X,L}^{GL(n,\mathbb{C})}(Q)$. □

The fact that the n-th symmetric product of the projective line is projective n-space leads to the following additional corollary:

Corollary 3.2.2. *Let Q be the quiver*

$$\bullet_{1,d_1} \longrightarrow \bullet_{1,d_2} \longrightarrow \cdots \longrightarrow \bullet_{1,d_n}.$$

Then the moduli space of representations of Q in $Bun(\mathbb{P}^1, \mathcal{O}(t))$ is

$$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q) = \mathbb{P}^{d_2-d_1+t} \times \ldots \times \mathbb{P}^{d_n-d_{n-1}+t}.$$

[9]That is, $\mathcal{M}_{X,L}^{PGL(n,\mathbb{C})}(Q)$ is an *orbifold.*

This can also be derived directly from the methods of Section 4.1.

We also take this opportunity to test our dimension formula from Theorem 3.1.1 against the expression for the moduli space in Corollary 3.2.1. The dimension of $\mathrm{Sym}^j(X)$ is j since X is a curve, and so the dimension of

$$\mathrm{Jac}^0(X) \times \mathrm{Sym}^{d_2-d_1+t}(X) \times \ldots \times \mathrm{Sym}^{d_n-d_{n-1}+t}(X)$$

is easily seen to be

$$g + \sum_{i=1}^{n-1}(d_{i+1} - d_i + t) = g + d_n - d_1 + (n-1)t.$$

On the other hand, putting $r_i = 1$ for all i in the dimension formula yields

$$\begin{aligned}
&\sum_{i=1}^{n-1}\left(r_i d_{i+1} - r_{i+1}d_i + r_i r_{i+1} t\right) + (1-g)\left(\sum_{i=1}^{n-1} r_i r_{i+1} - \sum_{i=1}^{n} r_i^2\right) + \min_{1\le i\le n}\{h^0(\mathrm{End}U_i)\} \\
&\quad = \sum_{i=1}^{n-1}\left(d_{i+1} - d_i + t\right) + (1-g)\left((n-1) - n\right) + 1 \\
&\quad = g + d_n - d_1 + (n-1)t.
\end{aligned}$$

3.3 Pullback diagrams and stability of argyle quivers

Definition 3.3.1. A labelled A-type quiver of the form

$$\bullet_{1,d_1} \longrightarrow \bullet_{r_2,d_2} \longrightarrow \bullet_{1,d_3} \longrightarrow \cdots \longrightarrow \bullet_{r_{n-1},d_{n-1}} \longrightarrow \bullet_{1,d_n}$$

is called an *argyle quiver*.

In this section we expand on the work of Thaddues in [64] and Gothen in [26], allowing us to make some deductions about the moduli space of twisted representations with fixed determinant of argyle quivers over a curve X of genus g. In the sequel we specialize to $\mathbb{P}^1$, where our results about argyle quivers are very concrete.

3.3.1 Stable tuples

Given an argyle quiver Q of length $n = 2q+1$, we will need to consider a space of *stable* $4q$-*tuples*, analagous to the stable pairs studied by Thaddeus. Stability for these tuples depends on $2q$ parameters, which we denote by $\sigma = (\sigma_1, \ldots, \sigma_{2q}) \in \mathbb{R}^{2q}$. Accordingly, we will define a space

$$\mathcal{R}_X^\sigma(k_1, \ldots, k_{2q}; e_1, \ldots, e_{2q})$$

which parametrizes stable tuples of the form $\{(V_1, \ldots, V_{2q}; \phi_1, \ldots, \phi_{2q})\}$ where V_i is a vector bundle of rank k_i and degree e_i (with $k_i = 1$ whenever i is even), $\phi_i \in H^0(X, V_i)$ for i odd, and $\phi_i \in H^0(X, V_{i-1}^* V_i)$ for i even.

The stability condition arising from the choice of σ follows from the well known α-stability condition on the space of holomorphic chains (equivalently, the moduli space of representations of the A-type quiver Q). This space is

$$\mathcal{M}_{X,L}^\alpha(r_1, \ldots, r_n; d_1, \ldots, d_n) = \{(U_1, \ldots, U_n; \phi_1, \ldots, \phi_{n-1})\}/\sim$$

with $\mathrm{rk}(U_i) = r_i$, $\deg(U_i) = d_i$, and $\phi_i \in H^0(X, U_i^* U_{i+1} L)$. The α-slope of a holomorphic chain $\mathcal{C} = (U_1, \ldots, U_n; \phi_1, \ldots, \phi_{n-1})$ depends on the $2q$-tuple $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{R}^{2q}$, and is defined as

$$\mu_\alpha(\mathcal{C}) = \frac{\sum_{i=1}^n d_i + \sum_{i=1}^{n-1} \alpha_i r_{i+1}}{\sum_{i=1}^n r_i} \tag{3.7}$$

We say that a holomorphic chain $\mathcal{C} \in \mathcal{M}_X^\alpha(r_1, \ldots, r_n; d_1, \ldots, d_n)$ is α-stable if $\mu_\alpha(\mathcal{C}') < \mu_\alpha(\mathcal{C})$ for each proper, $(\phi_1 \oplus \ldots \oplus \phi_{n-1})$-invariant subchain $\mathcal{C}' \subset \mathcal{C}$. Now we will play with this a little bit; recall the usual slope $\mu(\mathcal{C}) = \frac{d}{r}$ and set

$$\alpha_i = \frac{r}{r_{i+1}}\big(\sigma_i - \frac{1}{n-1}\mu(\mathcal{C})\big)$$

for all $i = 1, \ldots, n-1$. Now the expression $\mu_{\vec{\alpha}}(\mathcal{C}') < \mu_{\vec{\alpha}}(\mathcal{C})$ becomes

$$\frac{d' + \sum_{i=1}^{n-1} r'_{i+1} \frac{r}{r_{i+1}}\big(\sigma_i - \frac{1}{n-1}\mu(\mathcal{C})\big)}{r'} < \frac{d + \sum_{i=1}^{n-1} r_{i+1} \frac{r}{r_{i+1}}\big(\sigma_i - \frac{1}{n-1}\mu(\mathcal{C})\big)}{r}$$

$$\mu(\mathcal{C}') + \sum_{i=1}^{n-1} \frac{r}{r'} \frac{r'_{i+1}}{r_{i+1}}\big(\sigma_i - \frac{1}{n-1}\mu(\mathcal{C})\big) < \mu(\mathcal{C}) + \sum_{i=1}^{n-1}\big(\sigma_i - \frac{1}{n-1}\mu(\mathcal{C})\big)$$

$$\mu(\mathcal{C}') < \sum_{i=1}^{n-1} \frac{r}{r'} \frac{r'_{i+1}}{r_{i+1}}\big(\frac{1}{n-1}\mu(\mathcal{C}) - \sigma_i\big) + \sum_{i=1}^{n-1} \sigma_i$$

$$d' < \frac{d}{n-1} \sum_{i=1}^{n-1} \frac{r'_{i+1}}{r_{i+1}} + \sum_{i=1}^{n-1} \sigma_i \left(r' - r\frac{r'_{i+1}}{r_{i+1}}\right) \tag{3.8}$$

This is the σ-stability condition for holomorphic chains of length n. To specialize to $\mathcal{R}_X^\sigma(k_1, \ldots, k_{2q}; e_1, \ldots, e_{2q})$, we need to focus on chains of the form

$$\mathcal{C} = (\mathcal{O}, U_2, \ldots, U_n; \phi_1, \ldots, \phi_{n-1}),$$

where U_i is a line bundle for each i odd. This can be viewed as a $4q$-tuple in $\mathcal{R}_X^\sigma(k_1, \ldots, k_{2q}; e_1, \ldots, e_{2q})$ for $k = r - 1$, $k_i = r_{i+1}$, and $e_i = d_{i+1}$. Since we have set $d_1 = d_1' = 0$, we have $d = e$ and can write (3.8) as

$$e' < \frac{e}{2q} \sum_{i=1}^{2q} \frac{k_i'}{k_i} + \sum_{i=1}^{2q} \sigma_i \left(r' - (k+1) \frac{k_i'}{k_i} \right)$$

Now, this expression still depends explicitly on r_1', which is certainly strange if we are trying to look at this as a stability condition for a $4q$-tuple. To remedy this, we note that if $\phi_1 \in H^0(X, U_2') \setminus \{0\}$, then it is clear that $r_1' = 1$. Conversely, if $\phi_1 \notin H^0(X, U_2') \setminus \{0\}$, then we must have $r_1' = 0$. Hence, the stability condition on $\mathcal{R}_X^\sigma(k_1, \ldots, k_{2q}; e_1, \ldots, e_{2q})$ settles nicely into two cases:

Definition 3.3.2. A $4q$-tuple $(V_1, \ldots, V_{2q}; \phi_1, \ldots, \phi_{2q})$ with $\mathrm{rk}(V_i) = r_i$ and $\deg(V_i) = e_i$ is *stable* if for every sub-$4q$-tuple $(V_1', \ldots, V_{2q}'; \phi_1', \ldots, \phi_{2q}')$ of $(V_1, \ldots, V_{2q}; \phi_1, \ldots, \phi_{2q})$ where we denote $\mathrm{rk}(V_i') = k_i'$ and $\deg(V_i') = e_i'$, we have

$$e' < \frac{e}{n-1} \sum_{i=1}^{2q} \frac{k_i'}{k_i} + \sum_{i=1}^{2q} \sigma_i \left(1 + k' - (k+1) \frac{k_i'}{k_i} \right) \quad \text{if} \quad \phi_1 \in H^0(X, V_1') \setminus \{0\}$$

$$e' < \frac{e}{n-1} \sum_{i=1}^{2q} \frac{k_i'}{k_i} + \sum_{i=1}^{2q} \sigma_i \left(k' - (k+1) \frac{k_i'}{k_i} \right) \qquad \text{if} \quad \phi_1 \notin H^0(X, V_1') \setminus \{0\}$$

3.3.2 Pullback diagrams

The connection between twisted representations of an argyle quiver on the Riemann surface X and $4q$-tuples is captured by the following result:

Theorem 3.3.1. *For a labelled argyle quiver Q of length $n = 2q + 1$*

$$\bullet_{1,d_1} \longrightarrow \bullet_{r_2,d_2} \longrightarrow \bullet_{1,d_3} \longrightarrow \cdots \longrightarrow \bullet_{r_{n-1},d_{n-1}} \longrightarrow \bullet_{1,d_n}$$

there exists a unique $\sigma \in \mathbb{R}^{2q}$ and $b_i \in \mathbb{Z}$ such that the moduli space of representations of Q in the twisted category of holomorphic vector bundles with fixed determinant P is given by the pullback diagram

$$\begin{tikzcd}
\mathcal{M}_{X,L}^{SL(r,\mathbb{C})}(Q) \arrow[r, "g"] \arrow[d, "\pi"'] & \prod_{i=1,\mathrm{odd}}^{n} \mathrm{Jac}^{d_i}(X) \arrow[d, "\pi'"'] \\
\mathcal{R}_X^{\sigma}(k_1, \dots, k_{2q}; e_1, \dots, e_{2q}) \arrow[r, "h"] & \prod_{i=1,\mathrm{odd}}^{n} \mathrm{Jac}^{b_i}(X)
\end{tikzcd}$$

with maps described as follows:

$$\begin{aligned}
&\pi : (U_1, \dots, U_n; \phi_1, \dots, \phi_{n-1}) \\
&\qquad \mapsto (U_1^* U_2 L, U_1^* U_3 L^2, U_3^* U_4 L, U_3^* U_5 L^2, \dots, U_{n-2}^* U_n L^2; \phi_1, \dots, \phi_{n-1}) \\
&g : (U_1, \dots, U_n; \phi_1, \dots, \phi_{n-1}) \mapsto (U_1, U_3, \dots, U_n) \\
&h : (V_1, \dots, V_{2q}; \phi_1, \dots, \phi_{n-1}) \\
&\qquad \mapsto \left(\bigotimes_{i=1,\mathrm{odd}}^{2q-1} \det(V_i), \bigotimes_{i=1,\mathrm{odd}}^{2q-3} \det(V_i) \otimes \det(V_{2q-1} V_{2q}^*), \dots, \bigotimes_{i=1,\mathrm{odd}}^{2q-1} \det(V_i V_{i+1}^*) \right) \\
&\pi' : (U_1, U_3, \dots, U_n) \\
&\qquad \mapsto \Big(P L^{\sum_{i=2,\mathrm{even}}^{2q} r_i} (U_1^*)^{r_2+1} (U_3^*)^{r_4+1} \dots U_{2q+1}^*, \\
&\qquad\quad P L^{\sum_{i=2,\mathrm{even}}^{2q-2} r_i - r_{2q}} (U_1^*)^{r_2+1} \dots U_{2q-1}^* (U_{2q+1}^*)^{r_{2q-1}+1}, \dots \\
&\qquad\quad \dots, P L^{-\sum_{i=2,\mathrm{even}}^{2q} r_i} U_1^* (U_3^*)^{r_2+1} \dots (U_{2q+1}^*)^{r_{2q-1}+1} \Big)
\end{aligned}$$

Moreover, the maps π and π' are finite-to-one covering maps.

Remark 3.3.1. One way to interpret Theorem 3.3.1 is that the map h generalizes the determinant map of vector bundles to tuples; the determinant of a $4q$-tuple (which contains $2q$ bundles) is a tuple of $q + 1$ determinants. Therefore the fibres of h are the generalization of moduli spaces of bundles of fixed determinant.

Proof. To show that this diagram commutes, we will consider $h \circ \pi(U_1, \dots, U_n; \phi_1, \dots, \phi_{n-1})$.

Recalling that $P = U_1 \det(U_2) U_3 \det(U_4) U_5 \dots U_{2q+1}$, the first term is

$$\begin{aligned}\bigotimes_{i=1,\text{odd}}^{2q-1} \det(U_i^* U_{i+1} L) &= \bigotimes_{i=1,\text{odd}}^{2q-1} ((U_i^*)^{r_{i+1}} \det(U_{i+1}) L^{r_{i+1}}) \\ &= L^{\sum_{i=1,odd}^{2q-1}(r_i+1)} (U_1^*)^{r_2} \det(U_2) (U_3^*)^{r_4} \det(U_4) \dots (U_{2q-1}^*)^{r_{2q}} \det(U_{2q}) \\ &= L^{\sum_{i=2,even}^{2q} r_i} P (U_1^*)^{r_2+1} (U_3^*)^{r_4+1} \dots (U_{2q-1}^*)^{r_{2q}+1} U_{2q+1}^*\end{aligned}$$

which is exactly the first term of $\pi' \circ g(U_1, \dots, U_n; \phi_1, \dots, \phi_{n-1})$. The other terms are similar. The unique b_i extolled in the statement of the theorem are nothing but the degrees of these line bundles.

To see that π is a r^{2g}-fold covering map, write

$$(V_1, \dots, V_{2q}) = (U_1^* U_2 L, U_1^* U_3 L^2, U_3^* U_4 L, U_3^* U_5 L^2, \dots, U_{n-2}^* U_n L^2)$$

and then note

$$\det(V_1) = (U_1^*)^{r_2} L^{r_2} \left(P^* U_1^* U_3^* \det(U_4)^* U_5^* \dots U_{2q+1}^* \right)$$

We can say that π is a finite covering for the following reasons. For i odd, $\det V_i = (U_i^*)^{r_{i+1}} \det(U_{i+1}) L^{r_{i+1}}$ and so $\det(U_{i+1})^* = (U_i^*)^{r_{i+1}} L^{r_{i+1}} \det(V_i)$. In addition, $\det(V_{i+1}) = U_i^* U_{i+2} L^2$ and so $U_{i+2}^* = U_{i-1}^* U_i L^2 \det(V_{i+1})^*$. In particular, this tells us that $\det(U_{i+1})^*$ and U_{i+2}^* can be written in terms of U_i^* and some other known quantities. By doing this for all odd i from 3 to $2q-1$, we can write $(U_1^*)^{1+r_2+1+\dots+r_{2q}+1} = (U_i^*)^r$ in known terms. Then, accounting for torsion in the Jacobian, π is an r^{2g}-fold covering map. A similar approach shows that π' is a finite-to-one covering map.

Now it remains to show that there exist unique $(\sigma_1, \dots, \sigma_{2q})$ for which the above holds. We begin by defining the line bundles $U_j' = \phi_{j-1}(U_{j-1})$ and $U_j'' = \phi_j^{-1}(U_{j+1})$ for all j even. For any line subbundle U_j''' of U_j which is *not* equal to either U_j' or U_j'', we can define a subrepresentation

$$(0, \dots, U_j''', \dots, 0; 0, \dots, 0)$$

of

$$(U_1, \dots, U_n; \phi_1, \dots, \phi_{n-1}) \in \mathcal{M}_{X,L}^{SL(r,\mathbb{C})}(Q).$$

It is clear that stability implies $\deg(U_j''') < \frac{d}{r}$. Now, such subrepresentations are in one-to-one correspondence with sub-$4q$-tuples

$$(0, \dots, U_{j-1}^* U_j''' L \dots, 0; 0, \dots, 0)$$

of

$$(V_1, \dots, V_{2q}; \phi_1, \dots, \phi_{2q}) \in \mathcal{R}_X^{\sigma}(k_1, \dots, k_{2q}; e_1, \dots, e_{2q}).$$

By definition, such a $4q$-tuple is stable if and only if

$$\begin{aligned} e' = \deg(U_{j-1}^* U_j''' L) &< \frac{e}{2q} \sum_{i=1}^{2q} \frac{k_i'}{k_i} + \sum_{i=1}^{2q} \sigma_i \left(k' - (k+1) \frac{k_i'}{k_i} \right) \\ &= \frac{e}{2qk_{j-1}} + \sum_{i=1}^{2q} \sigma_i \left(1 - (k+1) \frac{k_i'}{k_i} \right) \\ &= \frac{e}{2qr_j} + \sigma_{j-1} \left(1 - \frac{r}{r_j} \right) + \sum_{i=1, i \neq j-1}^{2q} \sigma_i \end{aligned}$$

where

$$e = \sum_{i=1}^{2q+1} d_i + \sum_{i=1,\mathrm{odd}}^{2q-1} \left((r_{i+1}+2)t - (r_{i+1}+1) d_i \right).$$

Since we also know that

$$\deg(U_{j-1}^* U_j''' L) = -d_{j-1} + \deg(U_j''') + t < -d_{j-1} + \frac{d}{r} + t$$

we see that equivalence of stability in $\mathcal{M}_{X,L}^{SL(r,\mathbb{C})}(Q)$ and in $\mathcal{R}_X^{\sigma}(k_1, \dots, k_{2q}; e_1, \dots, e_{2q})$ boils down to the equation

$$-d_{j-1} + \frac{d}{r} + t = \frac{e}{2qr_j} + \sigma_{j-1} \left(1 - \frac{r}{r_j} \right) + \sum_{i=1, i \neq j-1}^{2q} \sigma_i$$

which allows us to deduce

$$\sigma_{j-1} \left(1 - \frac{r}{r_j} \right) + \sum_{i=1, i \neq j-1}^{2q} \sigma_i = -d_{j-1} + \frac{d}{r} + t - \frac{e}{2qr_j} \tag{3.9}$$

for all j even.

Considering the subrepresentation $(0, \dots, U_j''', \dots, 0; 0, \dots, 0)$ again, we note that is also in correspondence with sub-$4q$-tuples

$$(0, \dots 0, U_{j-1}^* U_j''' L, U_{j-1}^* U_{j+1} L^2, 0 \dots, 0; 0, \dots, 0)$$

of

$$(V_1, \dots, V_{2q}; \phi_1, \dots, \phi_{2q}) \in \mathcal{R}_X^{\sigma}(k_1, \dots, k_{2q}; e_1, \dots, e_{2q}),$$

for which the stability condition is

$$
\begin{aligned}
e' &< \frac{e}{2q}\sum_{i=1}^{2q}\frac{k_i'}{k_i} + \sum_{i=1}^{2q}\sigma_i\left(k' - (k+1)\frac{k_i'}{k_i}\right)\\
&= \frac{e}{2q}\left(\frac{1}{k_{j-1}} + \frac{1}{k_j}\right) + \sum_{i=1}^{2q}\sigma_i\left(2 - (k+1)\frac{k_i'}{k_i}\right)\\
&= \frac{e}{2q}\left(\frac{1}{r_j} + \frac{1}{r_{j+1}}\right) + \sigma_{j-1}\left(2 - \frac{r}{r_j}\right) + \sigma_j\left(2 - \frac{r}{r_{j+1}}\right) + \sum_{i=1, i\neq j-1,j}^{2q} 2\sigma_i
\end{aligned}
$$

where e is as above and e' is

$$
\begin{aligned}
e' &= \deg(U_{j-1}^* U_j''' L) + \deg(U_{j-1}^* U_{j+1} L)\\
&= -2d_{j-1} + d_j''' + d_{j+1} + 3t\\
&< -2d_{j-1} + \frac{d}{r} + d_{j+1} + 3t
\end{aligned}
$$

Hence, we set

$$
-2d_{j-1} + \frac{d}{r} + d_{j+1} + 3t = \frac{e}{2q}\left(\frac{1}{r_j} + \frac{1}{r_{j+1}}\right) + \sigma_{j-1}\left(2 - \frac{r}{r_j}\right) + \sigma_j\left(2 - \frac{r}{r_{j+1}}\right) + \sum_{i=1, i\neq j-1,j}^{2q} 2\sigma_i
$$

from which we can calculate

$$
\begin{aligned}
\sigma_{j-1}\left(2 - \frac{r}{r_j}\right) + \sigma_j\left(2 - r\right) + \sum_{i=1, i\neq j-1,j}^{2q} 2\sigma_i&\\
= -2d_{j-1} + \frac{d}{r} + d_{j+1} + 3t - \frac{e}{2q}\left(\frac{1}{r_j} + \frac{1}{r_{j+1}}\right)&
\end{aligned}
\tag{3.10}
$$

for all j even.

It remains to show that the system of equations defined by (3.9) and (3.10) has a unique solution. The associated $2q \times 2q$ matrix is

$$
\Sigma_q = \begin{pmatrix}
1 - \frac{r}{r_2} & 1 & 1 & 1 & \cdots & 1\\
1 & 1 & 1 - \frac{r}{r_4} & 1 & & \vdots\\
\vdots & & & & & \\
2 - \frac{r}{r_2} & 2 - r & 2 & 2 & \cdots & \\
2 & 2 & 2 - \frac{r}{r_4} & 2 - r & & \\
\vdots & & & & & \\
2 & 2 & 2 & 2 & \cdots & 2 - r
\end{pmatrix}
$$

which can be transformed to

$$\Sigma_q' = \begin{pmatrix} 1-\frac{r}{r_2} & 1 & 1 & \cdots & 1 \\ 1 & 1-r & 1 & & \vdots \\ 1 & 1 & 1-\frac{r}{r_4} & & \\ \vdots & & & \ddots & \\ 1 & 1 & 1 & \cdots & 1-r \end{pmatrix}$$

via elementary row operations. The determinant of Σ_q' can be calculated via the matrix determinant lemma, which states that for an invertible $n \times n$ matrix A and column vectors u and v,

$$\det(A + uv^T) = (1 + v^T A^{-1} u)\det(A).$$

By factoring Σ_q' as

$$\Sigma_q' = A + uv^T = \begin{pmatrix} -\frac{r}{r_2} & 0 & 0 & \cdots & 0 \\ 0 & -r & 0 & & \vdots \\ 0 & 0 & -\frac{r}{r_4} & & \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & -r \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}$$

we can calculate

$$\begin{aligned} (1 + v^T A^{-1} u) &= \left(1 + \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} -\frac{r_2}{r} & 0 & 0 & \cdots & 0 \\ 0 & -\frac{1}{r} & 0 & & \vdots \\ 0 & 0 & -\frac{r_4}{r} & & \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & -\frac{1}{r} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right) \\ &= \left(1 - \frac{r_2}{r} - \frac{1}{r} - \frac{r_4}{r} - \ldots - \frac{1}{r} \right) \end{aligned}$$

as well as

$$\det(A) = \left(-\frac{r}{r_2}\right)(-r)\left(-\frac{r}{r_4}\right)\ldots(-r).$$

From these we have

$$\begin{aligned}\det(\Sigma'_q) &= \left(1 - \frac{r_2}{r} - \frac{1}{r} - \frac{r_4}{r} - \ldots - \frac{1}{r}\right)\left(-\frac{r}{r_2}\right)(-r)\left(-\frac{r}{r_4}\right)\ldots(-r) \\ &= (r - r_2 - 1 - r_4 - \ldots - 1)\,\frac{(-r)^{2q-1}}{r_2 r_4 \ldots r_{2q-1}} \\ &= (r - q - r_2 - r_4 - \ldots - r_{2q-1})\,\frac{(-r)^{2q-1}}{r_2 r_4 \ldots r_{2q-1}}.\end{aligned}$$

Since $r = 1 + (q + r_2 + r_4 + \ldots + r_{2q-1})$, the determinant is always nonzero, and the proof is complete. □

Remark 3.3.2. The reason for restricting ourselves to argyle quivers in the theorem is that our analysis of σ depends explicitly on the fact that every second bundle is a line bundle[10]. We do not expect such a clean formulation of the pullback property in the non-argyle case.

Remark 3.3.3. When the genus of X is 0, the image of h is only a point, meaning that there is no useful fibration structure coming from h. However, $\mathcal{M}_{X,L}^{SL(n,\mathbb{C})}(Q)$ is still a finite-to-one cover of $\mathcal{R}_X^{\sigma}(k_i; e_i)$. When $g = 1$, the Jacobians and the elliptic curve X itself can be identified and $\mathcal{M}_{X,L}^{SL(n,\mathbb{C})}(Q)$ fibres over a Cartesian product of the elliptic curve with itself some number of times. In this case, one can view the pullback procedure as expressing the data of a representation of Q, which consists of bundles and twisted maps, in terms of simpler data on X. This data is a tuple of points, after fixing the determinant of the representation (by picking a fibre of h) and up to some choice of roots of unity (the map π). In some sense, this picture is reminiscent of the spectral viewpoint and the Hitchin fibration for Higgs bundles, which transforms the data of a Higgs bundle on a Riemann surface X to a point on the Jacobian of another Riemann surface, the so-called "spectral curve" of the Higgs bundle (see, for example, [36, 7]). In the pullback diagram for tuples, we see products of Jacobians rather than a single Jacobian.

Remark 3.3.4. We also stress the general utility of the pullback diagram. In [64] (cf. also [25]), a special case of Theorem 3.3.1 is used to obtain an exact geometric identification of the moduli space of stable pairs (a single bundle with a single map). This is achieved by variation of stability, wherein the stability parameter is initialized at an extreme value and

[10]This comes up in how we define U_j''' in the second batch of sub-$4q$-tuples.

then the desired moduli space is constructed in steps by flips and flops as the parameter crosses certain walls. In principle, the same procedure can be applied for tuples associated to the more general argyle quivers above but this would involve quite a number of birational transformations.

4 A-TYPE QUIVERS ON $\mathbb{P}^1$ AND HIGGS BUNDLES

4.1 Argyle quiver bundles

In this chapter, we seek explicit identifications of moduli spaces of twisted representations of argyle quivers when X is $\mathbb{P}^1$, the most concrete setting. We start with the case where the length of the quiver is $n = 3$ and work from there.

4.1.1 Type $(1, k, 1)$ quivers

We begin with the quiver

$$Q = \bullet_{1,d_1} \longrightarrow \bullet_{k,d_2} \longrightarrow \bullet_{1,d_3}$$

and put $r = k + 2$ and $d = d_1 + d_2 + d_3$. A representation of Q is a tuple of the form $(U_1, U_2, U_3; \phi_1, \phi_2)$ in which $U_1 \cong \mathcal{O}(d_1)$ and $U_3 \cong \mathcal{O}(d_3)$ since $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$. In addition, U_2 splits as

$$\mathcal{O}(a_1)^{\oplus s_1} \oplus \cdots \oplus \mathcal{O}(a_m)^{\oplus s_m}$$

for some $a_i \in \mathbb{Z}$ and some $s_i > 0$, where $\sum_{i=1}^m s_i a_i = d_2$ and $k = \sum_{i=1}^m s_i$. We always sort the a_i's as $a_1 > a_2 > \ldots > a_m$. With this information in hand, we can rewrite the representation

$$\mathcal{O}(d_1) \xrightarrow{\Xi} U_2 \xrightarrow{\Phi} \mathcal{O}(d)$$

as

$$
\begin{tikzcd}
 & \mathcal{O}(a_1) \arrow[dddr, "\phi_1^1"] & \\
 & \oplus \\ \vdots \\ \oplus & \\
 & \mathcal{O}(a_1) \arrow[dr, "\phi_1^{s_1}"] & \\
\mathcal{O}(d_1) \arrow[uuur, "\xi_1^1"] \arrow[ur, "\xi_1^{s_1}"] \arrow[dr, "\xi_2^1"] \arrow[dddr, "\xi_m^{s_m}"'] & \oplus & \mathcal{O}(d_3) \\
 & \mathcal{O}(a_2) \arrow[ur, "\phi_2^1"'] & \\
 & \oplus \\ \vdots \\ \oplus & \\
 & \mathcal{O}(a_m) \arrow[uuur, "\phi_m^{s_m}"'] &
\end{tikzcd}
$$

In this diagram $\phi_i^j \in H^0(\mathbb{P}^1, \mathcal{O}(d - a_i + t))$ and $\xi_i^j \in H^0(\mathbb{P}^1, \mathcal{O}(d_3 - a_i + t))$. This picture is acted upon by elements of $\mathrm{Aut}(U_1) \times \mathrm{Aut}(U_2) \times \mathrm{Aut}(U_3)$. From this group, there are degree 0 maps between each pair of nodes of equal degree, as well as degree $a_i - a_j$ maps from $\mathcal{O}(a_i)$ to $\mathcal{O}(a_j)$ for all $i < j$. If we must be very specific, we write ψ_{kl}^{ij} for the map from the k-th a_i node to the l-th a_j node. Most of the time when considering such maps, it is not important which of the $\mathcal{O}(a_i)$ nodes we consider, so we simply write ϕ_i and ψ_{ij}.

Next we will consider which values of $d_1, a_1, \ldots, a_k, d_3$ are allowable under the standard slope-stability conditions. Since we have already sorted the a_i as $a_1 > a_2 > \ldots > a_m$, it suffices to impose the following:

$$
\begin{aligned}
d_3 &< \mu_{\text{tot}} \\
\frac{d_3 + a_1}{2} &< \mu_{\text{tot}} \\
&\vdots \\
\frac{d_3 + s_1 a_1}{1 + s_1} &< \mu_{tot} \\
\frac{d_3 + s_1 a_1 + a_2}{2 + s_1} &< \mu_{\text{tot}} \\
&\vdots \\
\frac{d_3 + \sum_{i=1}^{m} s_i a_i}{k + 1} &< \mu_{\text{tot}}
\end{aligned}
$$

Recall that $\phi_i \in H^0(\mathbb{P}^1, \mathcal{O}(d_2 - a_i + t)) \setminus 0 \cong \mathbb{C}^{d_2 - a_i + t + 1} \setminus 0$. Define i' so that $a_{i'+1} < \mu_{\text{tot}} < a_{i'}$ (allowing the cases $i' = 0$ and $i' = m$). This will allow us to say something about the ϕ

and ξ maps. We can see that for any $i > i'$, $\mathcal{O}(a_i)$ can be allowed to be invariant without stability issues, meaning any of the ϕ_i^j can be allowed to be zero. On the other hand, for $i > i'$, none of ξ_i^j can be allowed to be zero. If one is zero, then the subbundle consisting of all the nodes except a single $\mathcal{O}(a_i)$ would be invariant, but this is not stable since $a_i < \mu_{\text{tot}}$. In a similar way, for any $i \leq i'$, ϕ_i^j cannot be zero, but ξ_i^j can. The final restriction to note is that while we allow any of $\xi_1^1, \ldots, \xi_{i'}^{s_{i'}}, \phi_{i'+1}^1, \ldots, \phi_m^{s_m}$ to vanish, they cannot all be zero concurrently as that would imply that the representation could be presented as a direct sum of two stable representations.

We will reduce the amount of freedom that some of the ϕ_i and ξ_i have by letting them be acted upon by some of the ψ_{pq}. In other words, we construct the moduli variety by performing *reduction in stages*. We are performing a geometric-invariant-theoretic (GIT) reduction using the Φ-stability condition, but note that we are quotienting by a *non-reductive* group. In general, an element $\Psi \in \mathrm{Aut}(U_i)$ is an invertible matrix-valued polynomial (in the affine parameter $z \in \mathbb{P}^1$) whose degree 0 piece is an element of $\mathrm{GL}(r_i, \mathbb{C})$. The diagonal terms in particular comprise the usual maximal torus in $\mathrm{GL}(r_i, \mathbb{C})$. The off-diagonal terms, which are all zero to one side of the diagonal by degree considerations, measure the non-reductiveness of the group. Fortunately, the off-diagonal terms act on the polynomials ϕ_i in the representation in a predictable way: they reduce the degree of ϕ_i or ξ_i in accordance with the Euclidean algorithm.

To be precise, consider $\psi_{ij} : \mathcal{O}(a_i) \to \mathcal{O}(a_j)$ where $a_i \neq a_j$ and $i, j \leq i'$. We send $\phi_i \mapsto \phi_i + \phi_j \psi_{ij} := \phi_i'$. We know that $\psi_{ij} \in H^0(\mathbb{P}^1, \mathcal{O}(a_j - a_i)) \cong \mathbb{C}^{a_j - a_i + 1}$, so one can see that we can use the $a_j - a_i + 1$ degrees of freedom of ψ_{ij} to kill off some of the freedom of ϕ_i. In particular, the dimension of the space that parametrizes ϕ_i' will be $d_2 - a_i + t + 1 - (a_j - a_i + 1) = d_2 - a_j + t$. To be more precise, if

$$\begin{aligned}\phi_i' &= \phi_i + \phi_j \psi_{ij} \\ &= (A_p z^p + \ldots + A_0) + (B_q z^q + \ldots + B_0)(C_r z^r + \ldots + C_0)\end{aligned}$$

then we set $C_r = \frac{-A_p}{B_q}$ so that $C_r B_q = -A_p$, as well as $C_{r-1} = \frac{-1}{B_q}(A_{p-1} + C_r B_{q-1})$ so that $C_r B_{q-1} + C_{r-1} B_q = -A_{p-1}$, etc. In general, we set

$$C_{r-i} = \frac{-1}{B_q}(A_{p-i} + \sum_{j=0}^{i-1} C_{r-j} B_{m-i+j})$$

for $i = 1, \dots, r$.

An additional property of this action is that the size of the automorphism group is not constant; it changes in accordance with divisor equivalences. This is best explained from the point of view of the spectral correspondence, in which we appeal to the identification of these quiver representations with twisted Higgs bundles. As previously mentioned, the spectral correspondence [36, 7] is a bijection between Higgs bundles of fixed generic characteristic polynomial on a curve and line bundles supported on another curve. This additional curve is called a *spectral curve*, as its points are precisely the spectrum of the Higgs fields on one side of the correspondence. The spectral curve, $\widetilde{X}$, is a finite-to-one cover of the original curve ($\mathbb{P}^1$ in this case), branched over a finite number of points where the characteristic polynomial develops eigenvalues with multiplicity. The spectral line bundles record the eigenspaces of the Higgs fields.

Most importantly, the spectral correspondence respects isomorphism classes. If two Higgs bundles (E, Φ) and (E', Φ') are isomorphic, then their spectral line bundles L and L' are isomorphic, and vice-versa. If the genus of $\widetilde{X}$ is g, then the Jacobian of $\widetilde{X}$ is a g-dimensional complex torus modelled on the symmetric product $\mathrm{Sym}^g(\widetilde{X})$. It fails to be globally isomorphic to $\mathrm{Sym}^g(\widetilde{X})$ because of special divisors. Specifically, if the degree of the covering map is r, then we have an induced surjection $\mathrm{Sym}^g(\widetilde{X}) \to \mathbb{P}^r$. Preimages of points in $\mathbb{P}^r$ with a repeated coordinate induce extra automorphisms of the corresponding divisors in $\mathrm{Sym}^g(\widetilde{X})$. The quotient of $\mathrm{Sym}^g(\widetilde{X})$ by these automorphisms results in $\mathrm{Jac}(\widetilde{X})$. The classical example is the Jacobian of the genus 2 hyperelliptic curve. The covering map is a degree 2 map $f : \widetilde{X} \to \mathbb{P}^1$, and its fibres form a $\mathbb{P}^1$ of linearly-equivalent divisors. The Jacobian is obtained by blowing down the "canonical series" (the preimage of this $\mathbb{P}^1$ under $\mathrm{Sym}^2(\widetilde{X}) \to \mathbb{P}^2$) in $\mathrm{Sym}^2(\widetilde{X})$. In higher genus and for higher degrees of the covering map, these equivalences are more numerous and complicated.

For us, these repeated coordinates in $\mathbb{P}^r$ correspond to coincidences of invariant zeroes of polynomials in the Higgs fields determined by the representation of the quiver, meaning zeroes of ϕ_i's that are preserved by the action of automorphisms. Suppose that we fix the splitting type $\mathbf{a} = (a_1, \dots, a_m; s_1, \dots, s_m)$ of U_2 in our quiver. This is tantamount to adding $2m$ labels to the central node that fix U_2. The resulting moduli space, which we denote

$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q,\mathbf{a})$, keeps track of ϕ_i data without any contribution from vector bundle moduli. We will excise any representations with collisions of invariant zeroes. We denote the removal of the "collision manifold" (i.e. keeping the regular part) by a superscript Δ.

Theorem 4.1.1. *Let Q be a quiver of type $(1,k,1)$ and let $\mathbf{a}$ be the splitting type of U_2. The projective closure of $\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}(Q,\mathbf{a})$ is*

$$\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(Q,\mathbf{a}) \cong \mathbb{P}^q \times \prod_{j=1}^{i'} \mathrm{Gr}\Big(s_j, d_3 - a_j + t + 1 - \sum_{k=1}^{j-1} s_k(a_k - a_j + 1)\Big)$$
$$\times \prod_{j=i'+1}^{m} \mathrm{Gr}\Big(s_j, a_j - d_1 + t + 1 - \sum_{k=j}^{m-1} s_k(a_k - a_j + 1)\Big)$$

where

$$q = \sum_{j=1}^{i'} s_j(d_3 - a_j + t + 1) + \sum_{j=i'+1}^{m} s_j(a_j - d_1 + t + 1) - 1 - \sum_{j=1}^{i'} \sum_{k=i'+1}^{m} s_j s_k(a_j - a_k + 1).$$

Proof. We can act on all ϕ_i for $i \le i'$ by all maps ψ_{ij} that go from $\mathcal{O}(a_i)$ to nodes of higher degree by the Euclidean algorithm, and similarly on all ξ_i for $i' < i$ by maps ψ_{ij} that go to $\mathcal{O}(a_j)$ from nodes of lower degree. It is important to note that if the power of ψ_{ij} would reduce the amount of freedom of one of these maps (which are not allowed to be zero by stability) to zero, then the representation is not stable. Lastly, ψ_{ij} for $j \le i' < i$ each reduce the freedom of one of $\xi_1^1, \ldots, \xi_{i'}^{s_{i'}}, \phi_{i'+1}^1, \ldots, \phi_m^{s_m}$. We know that not all of these can simultaneously vanish, so they contribute a single projective space to the moduli variety.

Now, after "using up" the power of the ψ_{ij} between nodes of different degree and accounting for the data contributed by $\xi_1^1, \ldots, \xi_{i'}^{s_{i'}}, \phi_{i'+1}^1, \ldots, \phi_m^{s_m}$, we can split up and rewrite the remaining information as

$$\begin{array}{c}\mathcal{O}\\ \oplus\\ \vdots\\ \oplus\\ \mathcal{O}\end{array} \begin{array}{c} \xrightarrow{\phi_1^1} \\ \\ \xrightarrow{\phi_1^{s_1}} \end{array} \mathcal{O}(d_3 - a_1) \qquad \cdots \qquad \begin{array}{c}\mathcal{O}\\ \oplus\\ \vdots\\ \oplus\\ \mathcal{O}\end{array} \begin{array}{c} \xrightarrow{\phi_{i'}^1} \\ \\ \xrightarrow{\phi_{i'}^{s_{i'}}} \end{array} \mathcal{O}(d_3 - a_{i'})$$

and

$$
\begin{tikzcd}[row sep=tiny]
 & \mathcal{O} & & & \mathcal{O} \\
 & \oplus & & & \oplus \\
\mathcal{O}(d_1-a_1) \arrow[uur, "\xi^1_{i'+1}"] \arrow[ddr, "\xi^{s_{i'+1}}_{i'+1}"'] & \vdots & \cdots & \mathcal{O}(d_1-a_m) \arrow[uur, "\xi^1_m"] \arrow[ddr, "\xi^{s_m}_m"'] & \vdots \\
 & \oplus & & & \oplus \\
 & \mathcal{O} & & & \mathcal{O}
\end{tikzcd}
$$

Write $\Phi_i := \phi_i^{1\prime} \oplus \ldots \oplus \phi_i^{s_i\prime}$. We claim that the induced map of sections for each of these is, in fact, injective. If $\tilde{\Phi}_1 : \mathbb{C}^{s_1} \to \mathbb{C}^{d_3-a_1}$ is not injective, then there exists some nontrivial kernel A which is generated by some subbundle B of $\mathcal{O} \oplus \ldots \oplus \mathcal{O}$. We can say $\mathrm{rank}B < s$ and also note that B must have sections since A is nontrivial. If $\mathrm{rank}B = 1$, the only degree of B that allows B to have sections is zero, in which case B is destabilizing. If $\mathrm{rank}B \geq 2$, it is possible that $\deg B \leq -1$ and B can have sections and may not be destabilizing. However, B must have some subbundle with non-negative degree, which would be destabilizing. Thus, $\tilde{\Phi}_1$ is injective and contributes $\mathrm{Gr}(s_i, d_3 - a_1 + t + 1)$ to the moduli space. The same argument holds for any $\tilde{\Phi}_i : \mathbb{C}^{s_i} \to \mathbb{C}^{d_3-a_i+t-\sum_{j=1}^{i-1} s_j(a_j-a_i+1)}$ once noting that the reductions done above can be done in such a way that each ϕ_i' induces a map from $\mathbb{C}$ into the subspace $\mathbb{C}^{d_3-a_i+t-\sum_{j=1}^{i-1} s_j(a_j-a_i+1)}$ of $\mathbb{C}^{d_3-a_i+t+1}$, which corresponds to the space of degree $d_3 - a + t - 2s$ polynomials. That is, each of the reduced ϕ_i' maps into the 'same' $\mathbb{C}^{d_3-a_i+t-\sum_{j=1}^{i-1} s_j(a_j-a_i+1)}$. Moreover, the equality of the moduli spaces of a quiver and its dual allows us to state a similar result for $\Xi_i = \xi_i^{1\prime} \oplus \ldots \oplus \xi_i^{s_i\prime}$. In particular, it contributes $\mathrm{Gr}\Big(s_j, a_j - d_1 + t + 1 - \sum_{k=j}^{m-1} s_k(a_k - a_j + 1)\Big)$ to the moduli space. $\square$

In the sequel, we reintegrate the collision manifold by identifying it with a twisted $(1, k, 1)$ quiver variety for a different splitting type, leading to a stratification of $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q, \mathbf{a})$ by the algebraic type of U_2.

The above description of the action of the automorphisms also allows us to calculate the moduli space of representations of type $(k, 1)$ quivers (and thus, type $(1, k)$ quivers). In this case, stability imposes a straightforward condition: none of the maps ϕ_i are allowed to be zero. This leads to the following corollary.

Corollary 4.1.1. *Let Q be the quiver*

$$\bullet_{k,d_1} \longrightarrow \bullet_{1,d_2}$$

and $\mathbf{a} = (a_1, \ldots, a_m; s_1, \ldots, s_m)$ *be the splitting type of* U_1*. The projective closure of* $\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}(Q, \mathbf{a})$ *is*

$$\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(Q, \mathbf{a}) \cong \prod_{i=1}^{m} \mathrm{Gr}\Big(s_i, d_2 - a_i + t + 1 - \sum_{j=1}^{i-1} s_j(a_j - a_i + 1)\Big).$$

4.1.2 General argyle quivers

The structure of an argyle quiver allows us to calculate the moduli space as a product of appropriately adjusted $(1, k, 1)$ quiver varieties.

Theorem 4.1.2. *Given an argyle quiver* Q *with* $\mathbf{a}_i$ *being the splitting type of* U_i*, the projective closure of the regular part of the moduli space of representations of* Q *in the category of* $\mathcal{O}(t)$*-twisted holomorphic vector bundles over* $\mathbb{P}^1$ *is*

$$\begin{aligned}\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(Q, \mathbf{a}_2, \mathbf{a}_4, \ldots, \mathbf{a}_{n-1}) = \overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(\bullet_{1,d_1} \longrightarrow \bullet_{r_2,d_2} \longrightarrow \bullet_{1,d_3}, \mathbf{a}_2) \times \ldots \\ \cdots \times \overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(\bullet_{1,d_{n-2}} \longrightarrow \bullet_{r_{n-1},d_{n-1}} \longrightarrow \bullet_{1,d_n}, \mathbf{a_{n-1}})\end{aligned}$$

where

$$\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(\bullet_{1,d_i} \longrightarrow \bullet_{r_{i+1},d_{i+1}} \longrightarrow \bullet_{1,d_{i+2}}, \mathbf{a}_{i+1})$$

is the projective closure of the moduli space of the quiver

$$\bullet_{1,d_i} \longrightarrow \bullet_{r_{i+1},d_{i+1}} \longrightarrow \bullet_{1,d_{i+2}}$$

with splitting type of U_i *given by* $\mathbf{a}_i$*, with stability condition induced by* Q*.*

Proof. Given a general argyle quiver

$$Q = \bullet_{1,d_1} \longrightarrow \bullet_{r_2,d_2} \longrightarrow \bullet_{1,d_3} \longrightarrow \cdots \longrightarrow \bullet_{r_{n-1},d_{n-1}} \longrightarrow \bullet_{1,d_n}$$

we can write a representation $(U_1, \ldots, U_n; \phi_1, \ldots, \phi_{n-1})$ as

$$\begin{tikzcd}
 & \mathcal{O}(a_1) \arrow[ddr, "\phi_1"] & & \mathcal{O}(b_1) \arrow[ddr, "\chi_1"] & & & & & \\
 & \oplus & & \oplus & & & & \arrow[dr] & \\
\mathcal{O}(d_1) \arrow[uur, "\xi_1"] \arrow[ddr, "\xi_{m_a}"'] & \vdots & \mathcal{O}(d_2) \arrow[uur, "\zeta_1"] \arrow[ddr, "\zeta_{m_b}"'] & \vdots & \mathcal{O}(d_4) \arrow[ur] \arrow[dr] & & \cdots & & \mathcal{O}(d_n) \\
 & \oplus & & \oplus & & & & \arrow[ur] & \\
 & \mathcal{O}(a_{m_a}) \arrow[uur, "\phi_{m_a}"'] & & \mathcal{O}(b_{m_b}) \arrow[uur, "\chi_{m_b}"'] & & & & &
\end{tikzcd}$$

The conditions on the degrees of the nodes that allow stability are akin to those shown for the $(1, k, 1)$ case, although there are many more. From this picture, it is clear that whether some ζ_j are allowed to be zero or not, they do not effect the behaviour of the ϕ_i in terms of stability, and vice versa. The same is not true of ϕ_i and ξ_i, as we have seen. This suggests that we could consider the moduli space of Q as decomposing as the moduli of the "diamonds". Since the bundles associated to nodes labelled with rank 1 are fixed, this does not account for any information more than once. Thus, to calculate $\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(Q, \mathbf{a}_2, \mathbf{a}_4, \ldots, \mathbf{a}_{n-1})$, we only need to calculate $\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(\bullet_{1,d_i} \longrightarrow \bullet_{r_{i+1},d_{i+1}} \longrightarrow \bullet_{1,d_{i+2}}, \mathbf{a}_{i+1})$ for each of the $(1, k, 1)$ blocks, with the following difference: i' is defined so that $a_{i'+1} < \mu_{\text{tot}} < a_{i'}$, where μ_{tot} is the slope of Q, not only the slope of the particular $(1, k, 1)$ block.

$\square$

We note finally that this result could be made a bit more general. The most important feature of argyle quivers is the regular appearance of nodes labelled with rank 1. We could, for example, calculate the moduli space of representations of quiver

$$\bullet_{1,d_1} \longrightarrow \bullet_{r_2,d_2} \longrightarrow \bullet_{1,d_3} \longrightarrow \bullet_{1,d_4} \longrightarrow \bullet_{r_5,d_5}$$

using the techniques presented in this section, although this quiver is not *strictly* speaking argyle. It is less clear that this loosening of definition is harmless for our results in genus $g \geq 1$.

4.1.3 Stratification of the moduli space by collisions

In the preceding section, we computed the closure of a single stratum of the $(1, k, 1)$ moduli space corresponding to fixing the holomorphic type of the rank k piece and removing collision data. Here we will explore examples of how to glue the strata in some low r and low t cases by realizing one stratum as the "collision submanifold" of a more generic stratum. In a sense, we take a finer look at the invariant theory of the representations by indentifying explicit invariants of the isomorphism class that coordinatize the strata. These invariants take the form of zeroes of certain ϕ_i's, regarded as polynomials over $\mathbb{P}^1$. These descriptions show that there is at least a birational equivalence between $\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(Q, \mathbf{a}_2, \mathbf{a}_4, \ldots, \mathbf{a}_{n-1})$ and

$\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}(Q, \mathbf{a}_2, \mathbf{a}_4, \ldots, \mathbf{a}_{n-1})$.

In the type-change stratification, the largest-dimensional stratum corresponds to representations of generic type, where "generic" means precisely the following:

Definition 4.1.1. Given a bundle U on $\mathbb{P}^1$ of rank r and degree d, its *generic splitting* is the decomposition of U as

$$\mathcal{O}(a+1)^{\oplus s} \oplus \mathcal{O}(a)^{\oplus r-s}$$

such that $s(a+1) + (r-s)a = d$.

The bundle U admits other infinitely many "less generic" splitting types that are related to the one above by adding 1 to the degree of a summand and simultaneously removing 1 from the degree of another summand. As per usual, it is stability that caps the number of splitting types that appear in the moduli space.

Consider the general $(1, k, 1)$ case. For a representation with U_2 of type

$$(a_1, \ldots, a_1; \ldots; a_m, \ldots, a_m),$$

we have

$$\Phi = \begin{pmatrix} 0 & 0 & \cdots & & & & \cdots & 0 \\ \xi_1^1 & 0 & & & & & & \vdots \\ \vdots & \vdots & \ddots & & & & & \\ \xi_1^{s_1} & & & & & & & \\ \xi_2^1 & & & & & & & \\ \vdots & & & & & & & \\ \xi_m^{s_m} & 0 & & & & & & \vdots \\ 0 & \phi_1^1 & \cdots & \phi_1^{s_1} & \phi_2^1 & \cdots & \phi_m^{s_m} & 0 \end{pmatrix}.$$

By observing $\Psi^{-1}\Phi\Psi$, we see that ϕ_i^j will have an invariant zero if and only if $\phi_1^1, \ldots, \phi_1^{s_1}, \ldots, \phi_i^1, \ldots, \phi_i^{s_i}$ have a common zero. As well, ξ_i^j will have an invariant zero if and only if $\xi_j^1, \ldots, \xi_j^{s_j}, \ldots, \xi_m^1, \ldots, \xi_m^{s_m}$ have a common zero. For $1 \leq i \leq j \leq k$, we would like to construct a way to map a representation with U_2 of type $(a_1, \ldots, a_1; \ldots; a_m, \ldots, a_m)$ to a representation with of the same type, except that a term a_i has been replaced with $b_i + 1$ and an a_j has been replaced with $a_j - 1$.

In view of the above description of the invariant zeroes, it is possible to construct a meromorphic automorphism Θ that will create the above transformation when $\phi_1^1, \ldots, \phi_i^{s_i}, \xi_j^1, \ldots \xi_m^{s_m}$ all share a zero, which is precisely when the automorphism has determinant equal to 1 (as opposed to having a determinant which is a meromorphic section of $\mathcal{O}$).

We pose the following algorithm that controls how the holomorphic type of U_2 changes due to a collision of invariant zeroes.

The Type-Change Algorithm: Begin with an empty set $\mathcal{S}$. Given a splitting $S_0 = (a_1, \ldots, a_1; \ldots; a_m, \ldots, a_m)$ of U_2 (where a_i appears s_i times), add S_0 to $\mathcal{S}$. Given $S_p = (b_1, \ldots, b_1; \ldots; b_m, \ldots, b_m)$, choose integers i, j such that $1 \leq i \leq j \leq k$ then construct the sequence S_{p+1} which is identical to S_p except for that a term b_i has been replaced with $b_i + 1$ and a b_j has been replaced with $b_j - 1$. If this S_{p+1} is not in $\mathcal{S}$ and the corresponding representation type is stable, then we glue in the moduli space of representations corresponding to type S_{p+1} in place of the collision locus of type S_p. Then, add S_{p+1} to $\mathcal{S}$ and restart this procedure with S_{p+1}. If S_{p+1} is unstable then add it to $\mathcal{S}$, and if $p > 0$, apply the procedure to S_{p-1}. If $p = 0$, terminate.

The moduli space of representations of the quiver

$$Q = \bullet_{1,d_1} \longrightarrow \bullet_{r_2,d_2} \longrightarrow \bullet_{1,d_3}$$

in the category of $\mathcal{O}(t)$-twisted holomorphic vector bundles over $\mathbb{P}^1$ can then be geometrically realized as the moduli space corresponding to the generic splitting, subject to the type-change algorithm.

Example 4.1.1. Consider the quiver $Q = \bullet_{1,2} \longrightarrow \bullet_{2,-1} \longrightarrow \bullet_{1,-2}$ with $\mathbf{a} = (0; -1)$ and $t = 5$. A representation of Q looks like

$$\begin{tikzcd} & \mathcal{O} \arrow[dr, "\phi_1"] & \\ \mathcal{O}(2) \arrow[ur, "\xi_1"] \arrow[dr, "\xi_2"'] & \oplus & \mathcal{O}(-2) \\ & \mathcal{O}(-1) \arrow[ur, "\phi_2"'] & \end{tikzcd}$$

Here, $\xi_1 \in H^0(\mathbb{P}^1, \mathcal{O}(3))$, $\xi_2 \in H^0(\mathbb{P}^1, \mathcal{O}(2))$, $\phi_1 \in H^0(\mathbb{P}^1, \mathcal{O}(3))$ and $\phi_2 \in H^0(\mathbb{P}^1, \mathcal{O}(4))$. By stability, ξ_2 and ϕ_1 are not allowed to vanish and they contribute $\mathbb{P}^2$ and $\mathbb{P}^3$ to the moduli space, respectively. Either of ξ_1 or ϕ_2 can be zero, but they cannot vanish concurrently. The automorphism $\psi_{21} : \mathcal{O}(-1) \to \mathcal{O}$, $\psi_{21} \in H^0(\mathbb{P}^1, \mathcal{O}(1))$ acts on either ξ_1 or ϕ_2, reducing the amount of freedom by 2 and so (ξ_1, ϕ_2) contributes $\mathbb{P}^6$. Hence,

$$\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(5)}}(Q, \mathbf{a}) = \mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^6.$$

The only other splitting type of U_2 which corresponds to a stable representation of Q is $\mathbf{b} = (1, -2; 1, 1)$. Such a representation looks like

$$\begin{tikzcd} & \mathcal{O}(1) \arrow[dr, "\phi_1'"] & \\ \mathcal{O}(2) \arrow[ur, "\xi_1'"] \arrow[dr, "\xi_2'"'] & \oplus & \mathcal{O}(-2) \\ & \mathcal{O}(-2) \arrow[ur, "\phi_2'"'] & \end{tikzcd}$$

In a way completely analagous to the above, we have

$$\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(5)}}(Q, \mathbf{b}) = \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^6.$$

We can identify this space with one of the collision manifolds of $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(5)}(Q, \mathbf{a})$; in particular, when ξ_2 and ϕ_1 share a zero z', we can construct the following meromorphic automorphism

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{z-z'} & 0 & 0 \\ 0 & 0 & z - z' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

that acts by conjugation to take a representation with U_2 of type $\mathbf{a}$ to a representation with U_2 of type $\mathbf{b}$. This amounts to a change of basis of the Higgs field. Moreover, in this case we can make a fairly explicit identification of the full moduli space $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(5)}(Q)$: it is $\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^6$ blown down to $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^6$ along the collision locus of ξ_2 and ϕ_1, which lies in $\mathbb{P}^2 \times \mathbb{P}^3$.

Example 4.1.2. For a slightly trickier example, consider $Q = \bullet_{1,2} \longrightarrow \bullet_{2,0} \longrightarrow \bullet_{1,-3}$ with $\mathbf{a} = (0, 0)$ and $t = 6$. A representation of Q looks like

$$\begin{tikzcd} & \mathcal{O} \arrow[dr, "\phi_1"] & \\ \mathcal{O}(2) \arrow[ur, "\xi_1"] \arrow[dr, "\xi_2"'] & \oplus & \mathcal{O}(-3) \\ & \mathcal{O} \arrow[ur, "\phi_2"'] & \end{tikzcd}$$

Here, $\xi_1, \xi_2 \in H^0(\mathbb{P}^1, \mathcal{O}(4))$ and $\phi_1, \phi_2 \in H^0(\mathbb{P}^1, \mathcal{O}(3))$. By stability, neither ϕ_1 nor ϕ_2 can be zero, and ξ_1 and ξ_2 cannot be zero concurrently. Hence,

$$\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(6)}}(Q, \mathbf{a}) = \mathbb{P}^9 \times \mathrm{Gr}(2,4).$$

Once again, there is only one other splitting type of U_2 which corresponds to a stable representation, and in this case it is $\mathbf{b} = (1,-1;1,1)$. Such a representation has moduli space calculated in the same way as the first example:

$$\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(6)}}(Q, \mathbf{b}) = \mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^9.$$

How this space fits into $\mathbb{P}^9 \times \mathrm{Gr}(2,4)$ is not immediately clear. This is due to the fact that these two representations have different "stability types"; the maps that are allowed to be zero and those that are intertwined with each other are different in each of the representation types. In the generic stratum, neither ϕ_1 and ϕ_2 can be zero while ξ_1 and ξ_2 cannot simultaneously be zero. In the less generic stratum, neither ϕ_1 nor ξ_2 can be zero while ϕ_2 and ξ_1 form an analogous pair. The change in stability in crossing from one stratum to the other is reminiscent of a conifold transition in reductive GIT, but where the dimension need not be the same on both sides of the transition.

Example 4.1.3. Finally, we consider an argyle quiver with two $(1,k,1)$ blocks. Let

$$Q = \bullet_{1,0} \longrightarrow \bullet_{2,0} \longrightarrow \bullet_{1,3} \longrightarrow \bullet_{3,-2} \longrightarrow \bullet_{1,-2}$$

and $t = 5$. The generic splittings $(\mathbf{a}_2, \mathbf{a}_4)$ are $\mathbf{a}_2 = (0,0)$ and $\mathbf{a}_4 = (0;-1,-1)$. A representation with these splittings looks like

$$\begin{tikzcd}
 & & & \mathcal{O} \arrow[ddr, "\phi_1"] & \\
 & \mathcal{O} \arrow[dr, "\zeta_1"] & & \oplus & \\
\mathcal{O} \arrow[ur, "\eta_1"] \arrow[dr, "\eta_2"'] & \oplus & \mathcal{O}(3) \arrow[uur, "\xi_1"] \arrow[r, "\xi_2"] \arrow[ddr, "\xi_3"'] & \mathcal{O}(-1) \arrow[r, "\phi_2"] & \mathcal{O}(-2) \\
 & \mathcal{O} \arrow[ur, "\zeta_2"'] & & \oplus & \\
 & & & \mathcal{O}(-1) \arrow[uur, "\phi_3"'] &
\end{tikzcd}$$

Note that the left diamond is certainly not a stable representation of the quiver $\bullet_{1,0} \longrightarrow \bullet_{2,0} \longrightarrow \bullet_{1,3}$, but with stability condition induced by Q, we can calculate

$$\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(5)}}(\bullet_{1,0} \longrightarrow \bullet_{2,0} \longrightarrow \bullet_{1,3}, \mathbf{a}_2) = \mathbb{P}^{11} \times \mathrm{Gr}(2,9).$$

Similarly,

$$\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(5)}}(\bullet_{1,3} \longrightarrow \bullet_{3,-2} \longrightarrow \bullet_{1,-2}, \mathbf{a}_4) = \{pt\} \times \mathbb{P}^3 \times \mathbb{P}^8.$$

Here, t is small enough and the quiver labelling is such that none of the other possible splittings of U_2 or U_4 correspond to stable representations. Thus we can actually say

$$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(5)}(Q) = \mathbb{P}^3 \times \mathbb{P}^8 \times \mathbb{P}^{11} \times \mathrm{Gr}(2,9).$$

This is another opportune time to use our dimension formula from Theorem 3.1.1. Since the dimension of $\mathrm{Gr}(k,n)$ is $k(n-k)$, the dimension of

$$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(5)}(Q) = \mathbb{P}^3 \times \mathbb{P}^8 \times \mathbb{P}^{11} \times \mathrm{Gr}(2,9)$$

is

$$3 + 8 + 11 + 2(9-2) = 36.$$

On the other hand, the dimension formula gives

$$\begin{aligned}
&\sum_{i=1}^{n-1} \big(r_i d_{i+1} - r_{i+1} d_i + r_i r_{i+1} t\big) + (1-g)\left(\sum_{i=1}^{n-1} r_i r_{i+1} - \sum_{i=1}^{n} r_i^2\right) + \min_{1\leq i\leq n}\{h^0(\mathrm{End}U_i)\} \\
&\quad = ((1)(0) - (2)(0) + (1)(2)(5)) + ((2)(3) - (1)(0) + (2)(1)(5)) \\
&\qquad + ((1)(-2) - (3)(3) + (3)(1)(5)) + ((3)(-2) - (1)(-2) + (3)(1)(5)) \\
&\qquad + (1-0)((1)(2) + (1)(2) + (1)(3) + (1)(3) - 1 - 4 - 1 - 9 - 1) + 1 \\
&\quad = 10 + 6 + 10 - 2 - 9 + 15 - 6 + 2 + 15 + 2 + 2 + 3 + 3 - 1 - 4 - 1 - 9 - 1 + 1 \\
&\quad = 36.
\end{aligned}$$

Example 4.1.4. If we consider $t = 6$ with this same quiver Q as in Example 4.1.3, we observe a stratification which is more difficult to categorize. The splitting $\mathbf{a}_2 = (0,0)$ is still the only stable type for U_2, but for U_4 we also have $\mathbf{b}_4 = (1;-1;-2)$ and $\mathbf{c}_4 = (0,0;-2)$ corresponding to stable representations. We calculate

$$\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(6)}}(Q, \mathbf{a}_2, \mathbf{a}_4) = \mathbb{P}^{13} \times \mathrm{Gr}(2,10) \times \mathbb{P}^4 \times \mathbb{P}^{11} \times \mathrm{Gr}(2,3),$$

$$\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(6)}}(Q,\mathbf{a}_2,\mathbf{b}_4) = \mathbb{P}^{13}\times \mathrm{Gr}(2,10)\times\mathbb{P}^1\times\mathbb{P}^3\times\mathbb{P}^{10},$$

and

$$\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(6)}}(Q,\mathbf{a}_2,\mathbf{c}_4) = \mathbb{P}^{13}\times \mathrm{Gr}(2,10)\times\mathbb{P}^1\times\mathbb{P}^8\times\mathrm{Gr}(2,5).$$

It is unclear how to glue these into the collision loci of $\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(6)}}(Q,\mathbf{a}_2,\mathbf{a}_4)$. This is partially due to the conifold-like transition mentioned earlier, and also because $\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(6)}}(Q,\mathbf{a}_2,\mathbf{b}_4)$ can be viewed as lying in a collision locus of $\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(6)}}(Q,\mathbf{a}_2,\mathbf{c}_4)$, but from the point of view of collisions in $\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(6)}}(Q,\mathbf{a}_2,\mathbf{a}_4)$, $\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(6)}}(Q,\mathbf{a}_2,\mathbf{c}_4)$ is a special case of $\overline{\mathcal{M}'^{\Delta}_{\mathbb{P}^1,\mathcal{O}(6)}}(Q,\mathbf{a}_2,\mathbf{b}_4)$.

Remark 4.1.1. We end this section by noting that different twists t certainly have an affect on whether very non-generic splittings correspond to stable representations or not. This suggests that different stability conditions (that is, different stability parameters α c.f. Equation 3.7) would have a similar influence. For certain α, we expect $\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(Q,\mathbf{a}_2,\mathbf{a}_4,\ldots,\mathbf{a}_{n-1})$ and $\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}(Q,\mathbf{a}_2,\mathbf{a}_4,\ldots,\mathbf{a}_{n-1})$ to coincide. This may help us to understand the case $\alpha=0$ since wall-crossing is fairly well-understood (see for example [64, 18]), but we will not approach this here.

4.2 Applications to twisted Higgs bundles and back again

The primary application of twisted quiver representations in a category of bundles is to the topology of Higgs bundle moduli spaces. In this context, the natural application of our results in the preceding sections (which concern representations over the projective line) is to twisted Higgs bundles at genus 0. The dimension over $\mathbb{C}$ of the moduli space $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(r,d)$ is tr^2+1 [57]. As noted earlier, this space comes equipped with a linear algebraic action of $\mathbb{C}^*$ that sends (E,Φ) to $(E,e^{i\theta}\Phi)$. Each fixed point of this action is a holomorphic chain, a representation of the quiver A_n for some n with $1\leq n\leq r$, and with a labelling by pairs of integers r_i,d_i in which $\sum r_i=r$, $\sum d_i=d$, and $r_i>0$ [26, 60]. When $r>1$, there are no fixed points with length $n=1$, as these correspond to stable Higgs bundles with the zero Higgs field which are simply stable bundles on $\mathbb{P}^1$, of which there are none other than line bundles.

The action induces a localization of cohomology to the fixed-point locus (as seen in Section 2.3.2), and the Poincaré series of $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(r,d)$ is the weighted sum of the Poincaré series of the connected components of the fixed-point set:

$$\mathcal{P}_x(\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(r,d)) = \sum_{\mathcal{N}} x^{\beta(\mathcal{N})}\mathcal{P}_x(\mathcal{N}).$$

The initial case of interest is $r = 2$ with any odd d and any $t > 0$. The dimension of $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(2,d)$ is $4t+1$. There is a single quiver that controls the fixed points: A_2— with nodes labelled $1, a$ and $1, d-a$, respectively. This is an argyle quiver of type $(1,1)$, for which the moduli space is relatively simple to compute. For any a, d, t, the moduli space is just $\mathbb{P}^{-2a+d+t}$. Note that there is no collision or type-change behaviour in this case, as both nodes correspond to line bundles and so a and $d-a$ fix the bundles up to isomorphism.

These components of the fixed-point locus are indexed by a and the admissible values of a are determined by stability. If a is too large and positive, the only morphism between the nodes will be the zero map, and a copy of $\mathcal{O}(a)$ will be invariant with slope larger than $d/2$. If a is too negative, the copy of $\mathcal{O}(d-a)$ will be destabilizing. It is possible to enumerate the labelled quivers directly. For instance, for $d = -1$, we have $\left\lfloor \frac{t+1}{2} \right\rfloor$ integers a such that $\mathcal{O}(a) \xrightarrow{\phi} \mathcal{O}(d-a) \to 0$ is stable:

$$\begin{aligned}
\mathcal{O} \to \mathcal{O}(-1) \to 0 \\
\mathcal{O}(1) \to \mathcal{O}(-2) \to 0 \\
\vdots \\
\mathcal{O}(-1+\lfloor (t+1)/2 \rfloor) \to \mathcal{O}(-\lfloor (t+1)/2 \rfloor) \to 0
\end{aligned}$$

For any other odd d, the list will have the same number of entries, but with degrees that have been shifted appropriately. Using the Betti numbers of $\mathbb{P}^{-2a+d+t}$ for each admissible a, the corresponding Morse index from [60], and the localization formula, we arrive at:

Theorem 4.2.1. *For any odd d and any $t > 0$, we have*

$$\mathcal{P}_x(\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(2,d)) \quad = \quad \sum_{k=0}^{t-1} \left(\frac{2k+4-[(2k) \bmod 4]}{4} \right) x^{2k}.$$

The even Betti numbers are $1, 1, 2, 2, 3, 3, 4, 4, \dots$ up to $(t-1)/2, (t+1)/2$ if t is odd or $t/2, t/2$ if t is even. From a combinatorial point of view, these count partitions of even integers into unordered combinations of the numbers 2 and 4, i.e. the "change-making problem". To emphasize this, one can rewrite the series as

$$\mathcal{P}_x(\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(2,d)) = \frac{1}{(1-x^2)(1-x^4)} - \left\{ \frac{(\lfloor t/2 \rfloor + 1)x^{2t}}{1-x^2} + \frac{x^{4\lfloor t/2 \rfloor + 4}}{(1-x^2)(1-x^4)} \right\},$$

which displays more of the structure regarding the generators and relations in the cohomology ring (these results in the $t = 2$ or "co-Higgs bundle" case were found in [59]).

In the rank 3 case, the quiver types are now $(1, 1, 1)$, $(1, 2)$, and $(2, 1)$, all of which are argyle. In this case we must contend with collisions, which makes writing down a general Poincaré series cumbersome. We provide two examples, one without and one with a tractable type change.

Example 4.2.1. For the first example we consider $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(2)}(3, -1)$, seen also in [59]. The complex dimension of the moduli space is 19 in this case. As with $r = 2$, the fixed-point set consists entirely of representations of argyle quivers, with the types being $(1, 1, 1)$, $(2, 1)$, and $(1, 2)$. Stability rapidly eliminates any of type $(1, 2)$. For type $(1, 1, 1)$, there are three degree labellings that produce stable representations:

$$1, 0, -2; \quad 1, -1, -1; \quad \text{and } 0, 0, -1,$$

which have Morse indices of $6, 4$, and 2 respectively. Again, there are no type-changing collisions possible because the bundles are line bundles and are therefore fixed up to isomorphism by these degree labellings. By Theorem 4.1.1, the associated quiver varieties are

$$\mathbb{P}^{-1+0+2} \times \mathbb{P}^{-0-2+2}, \; \mathbb{P}^{-1-1+2} \times \mathbb{P}^{1-1+2}, \quad \text{and } \mathbb{P}^{-0+0+2} \times \mathbb{P}^{0-1+2},$$

respectively. For type $(2, 1)$, the only degree labelling that admits stable representations is $0, -1$, which has Morse index 0 (and so we are at the "bottom" of the moduli space). We can deduce from the arguments leading to Theorem 4.1.1 that the associated quiver variety is just a point. More directly, the representation $\phi : \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(-1) \otimes \mathcal{O}(2)$ is stable if and only if it is surjective, in which case the induced map $\widetilde{\phi}$ between spaces of global sections must have full rank. Acting on this copy of $GL(2, \mathbb{C})$ on the right by automorphisms of $\mathcal{O} \oplus \mathcal{O}$

leaves nothing, save for the identity. Weaving together this information with the localization formula, we obtain

$$\mathcal{P}_x(\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(2)}(3,-1)) = 1 + x^2 + 3x^4 + 4x^6 + 3x^8.$$

As with the $r = 2$ case, the top degree is decidedly less than the actual dimension of the moduli space. This is due to the contribution to the moduli space of the Hitchin base; the space of possible coefficients of the characteristic polynomial of Φ, which itself is topologically trivial. The moduli space itself deformation retracts onto the central fibre over the base.

Example 4.2.2. Finally, we consider $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(6)}(3,-1)$. The basic types are the same ($(1,1,1)$, $(2,1)$, and $(1,2)$) but type-change phenomena occurs. Here, the complex dimension of the moduli space is 55. For type $(2,1)$, the labellings $(0,-1)$, $(1,-2)$, and $(2,-3)$ produce stable representations with Morse indices $0, 4$, and 12, respectively. The variety corresponding to the labelling $(0,-1)$ is $\mathrm{Gr}(2,6)$ and that corresponding to $(1,-2)$ is $\mathbb{P}^3 \times \mathbb{P}^2$. Each of these labellings has only one splitting of the left node that corresponds to stable representations. The same is not true of the labelling $(2,-3)$, where we contend with type-change phenomena. We have both

$$\begin{tikzcd} \mathcal{O}(1) \arrow[dr, "\phi_1"] & \\ \oplus & \mathcal{O}(-3) \\ \mathcal{O}(1) \arrow[ur, "\phi_2"] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathcal{O}(2) \arrow[dr, "\phi_1'"] & \\ \oplus & \mathcal{O}(-3) \\ \mathcal{O} \arrow[ur, "\phi_2'"] & \end{tikzcd}$$

The quiver variety of the first is $\mathrm{Gr}(2,3) \cong \mathbb{P}^2$, and the quiver variety of the second is $\mathbb{P}^1$. The locus of $\mathbb{P}^2$ where ϕ_1 and ϕ_2 share a zero is a copy of $\mathbb{P}^1$. We remove this and paste in the second variety, which is just $\mathbb{P}^1$ again. So in this case we have that the moduli space correponding to this labelling of a type $(2,1)$ quiver is $\mathbb{P}^2$. In addition, we have two stable labellings of the $(1,2)$-type quiver, $(1,-2)$ and $(2,-3)$, with respective Morse indices 4 and 10. The labelling $(1,-2)$ has associated quiver variety $\mathrm{Gr}(2,4)$, and $(2,-3)$ has $\mathbb{P}^2 \times \mathbb{P}^1$. Finally, we have the following allowed labellings for the $(1,1,1)$ quiver type:

$$\begin{gathered} 0,0,-1;\;\; 0,1,-2;\;\; 1,-1,-1;\;\; 0,2,-3;\;\; 1,0,-2;\;\; 2,-2,-1;\;\; 1,1,-3; \\ 2,-1,-2;\;\; 3,-3,-1;\;\; 1,2,-4;\;\; 2,0,-3;\;\; 3,-2,-2;\;\; 2,1,-4;\;\; 3,-1,-3;; \\ 3,0,-4;\;\; 4,-2,-3;\;\; 3,1,-5;\;\; 4,-1,-4;\;\; 4,0,-5;\;\; 5,-1,-5;\;\; \text{and } 5,0,-6. \end{gathered}$$

These have Morse indices

$$10, 12, 12, 14, 14, 14, 16, 16, 16, 18, 18, 18, 20, 20, 22, 22, 24, 24, 26, 28 \text{ and } 30$$

respectively, and associated quiver varieties

$$\begin{gathered}\mathbb{P}^6 \times \mathbb{P}^5,\ \mathbb{P}^7 \times \mathbb{P}^3,\ \mathbb{P}^4 \times \mathbb{P}^6,\ \mathbb{P}^8 \times \mathbb{P}^1,\ \mathbb{P}^5 \times \mathbb{P}^4,\ \mathbb{P}^2 \times \mathbb{P}^7,\ \mathbb{P}^6 \times \mathbb{P}^2,\\ \mathbb{P}^3 \times \mathbb{P}^5,\ \mathbb{P}^8,\ \mathbb{P}^7,\ \mathbb{P}^4 \times \mathbb{P}^3,\ \mathbb{P}^1 \times \mathbb{P}^6,\ \mathbb{P}^5 \times \mathbb{P}^5,\ \mathbb{P}^2 \times \mathbb{P}^4,\\ \mathbb{P}^3 \times \mathbb{P}^2,\ \mathbb{P}^5;\ \mathbb{P}^4,\ \mathbb{P}^1 \times \mathbb{P}^3,\ \mathbb{P}^2 \times \mathbb{P}^1,\ \mathbb{P}^2, \text{ and } \mathbb{P}^1.\end{gathered}$$

We can bring all of this together to calculate

$$\begin{aligned}\mathcal{P}_x(\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(6)}(3,-1)) = 1 &+ x^2 + 3x^4 + 4x^6 + 7x^8 + 9x^{10} + 14x^{12} + 17x^{14} + 24x^{16} + 29x^{18}\\ &+ 38x^{20} + 45x^{22} + 49x^{24} + 49x^{26} + 45x^{28} + 36x^{30} + 21x^{32}.\end{aligned}$$

After $r = 3$, $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(r, d)$ will always contain topological contributions from at least one A-type quiver of non-argyle type. For instance, $r = 4$ contains a $(2, 2)$ quiver variety, which was for some time the obstruction to computing Betti numbers for ordinary Higgs bundles in higher genus before [21]. On $\mathbb{P}^1$, the $(2, 2)$ quiver is less formidable and, with some effort, one can find

$$\begin{aligned}\mathcal{P}_x(\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(2)}(4,-d)) = 1 &+ x^2 + 3x^4 + 5x^6 + 9x^8 + 13x^{10} + 18x^{12}\\ &+ 22x^{14} + 20x^{16} + 10x^{18},\end{aligned}$$

for instance, where d is any integer coprime to 4. We also remark that all of the above calculations agree with the conjectural Poincaré series for these moduli spaces arising from the ADHM recursion formula [48].

We can also use some results about Higgs bundles to obtain results about stability of quiver representations which are not readily available from our approach. In [17], Franco shows that if X is an elliptic curve, then $\mathcal{M}^{GL(r,\mathbb{C})}_{X,\omega_X}(r, d) \cong T^*X$. One of the main tools used is the isomorphism between moduli spaces of vector and line bundles on an elliptic curve: $M_X(r, d) \cong M_X(1, d)$. From the perspective of fixed points, this says something about the possible quiver representations:

Corollary 4.2.1. *Let X be a Riemann surface of genus 0 or 1 and let $L = \mathcal{O}$. Then the only A-type quiver with stable representations is $Q = \bullet_{r,d}$, for which we have $\mathcal{M}^{GL(r,\mathbb{C})}_{X,\mathcal{O}}(Q) \cong M_X(r, d)$.*

Proof. A quiver Q of the form $\bullet_{r,d}$ has stable representations corresponding to the stable vector bundles. Beginning with $g = 1$, we have by definition $\mathcal{M}_{X,\mathcal{O}}^{GL(r,\mathbb{C})}(Q) \cong M_X(r,d) \cong \mathrm{Jac}^d(X)$. We know that

$$\mathcal{P}_x(\mathcal{M}_{X,\mathcal{O}}^{GL(r,\mathbb{C})}(r,d)) \;=\; \sum_{\mathcal{N}} x^{\beta(\mathcal{N})}\mathcal{P}_x(\mathcal{N}),$$

and so $\mathcal{M}_{X,\omega_X}^{GL(r,\mathbb{C})}(r,d) \cong T^*X$ gives $\mathcal{P}_x(\mathcal{M}_{X,\mathcal{O}}^{GL(r,\mathbb{C})}(r,d)) = \mathcal{P}_x(T^*X) = 1 + 2x + x^2$. On the other hand, $\mathcal{P}_x(M_X(r,d)) = \mathcal{P}_x(\mathrm{Jac}^d(X)) = 1 + 2x + x^2$ and $\beta(M_X(r,d)) = 0$. This means there are no other fixed points in the moduli space, and thus no other A-type quivers have stable representations.

If an A-type quiver Q has a stable representation in $\mathrm{Bun}(\mathbb{P}^1, \mathcal{O})$, we can build a corresponding stable representation in $\mathrm{Bun}(X, \mathcal{O})$ where X is an elliptic curve by using bundles on X which split into line bundles and have the same splitting types as the bundles on $\mathbb{P}^1$. If Q is not of the form $\bullet_{r,d}$, this is a contradiction.

□

5 CYCLIC QUIVERS

Definition 5.0.1. A *cyclic quiver* is a quiver of the form

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[lll, bend right] \end{tikzcd}$$

Definition 5.0.2. An *L-twisted cyclic Higgs bundle* on X is an L-twisted Higgs bundle (E, Φ) on X of the form

$$E = U_1 \oplus \cdots \oplus U_n, \qquad \Phi = \begin{pmatrix} 0 & \cdots & & \phi_n \\ \phi_1 & \ddots & & \\ & \ddots & & \\ 0 & & \phi_{n-1} & 0 \end{pmatrix}$$

where U_i are holomorphic line bundles on X and $\phi_i : U_i \to U_{i+1} \otimes L$. Note that the subscript is counted modulo n.

With the experience of the previous chapters, it is clear that we can exploit cyclic quivers to study cyclic Higgs bundles by again considering representations in the L-twisted category of vector bundles $\mathrm{Bun}(X, L)$. Cyclic Higgs bundles were first introduced by Baraglia [5] in a slightly different form, and have attracted attention lately for their role in investigating the Labourie Conjecture ([44, 45]), because their harmonic metric is diagonal ([14]), and because of their close relation to the affine Toda equations ([6]). Moreover, cyclic Higgs bundles arise via the study of certain special representations of the fundamental group of X (thanks to non-abelian Hodge Correspondence: see for example [23]). These so-called Hitchin representations correspond to the Hitchin section, and can be generalized in a certain way to yield cyclic Higgs bundles. Thus, the moduli spaces of cyclic Higgs bundles can be viewed as generalizations of the Hitchin section, which we now recall. Let X be a Riemann surface of genus $g \geq 2$ and consider ω_X-twisted $SL(2,\mathbb{C})$-Higgs bundles of the form

$$E = \omega_X^{\frac{1}{2}} \oplus \omega_X^{-\frac{1}{2}}, \qquad \Phi = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}$$

where $\omega_X^{\frac{1}{2}}$ is a choice of holomorphic square root of ω_X and $q : \omega_X^{-\frac{1}{2}} \to \omega_X^{\frac{1}{2}} \otimes \omega_X$. That is to say q lies in $H^0(X, \omega_X^2)$, the space of holomorphic quadratic differentials. In the $SL(2,\mathbb{C})$ moduli space, this is exactly the Hitchin base $\mathcal{B}_2$. All Higgs bundles of this form are stable, and so we have a map $\iota : \mathcal{B}_2 \to \mathcal{M}_{X,\omega_X}^{SL(2,\mathbb{C})}(2,0)$. The image of this map is called the *Hitchin section*.

On the Riemann sphere $\mathbb{P}^1$ there is a clear analogue of the Hitchin section in $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(2)}(2,0)$, which we can come upon by studying representations of the cyclic quiver

$$\begin{tikzcd} \bullet_{1,1} \arrow[r, bend right] & \bullet_{1,-1} \arrow[l, bend right] \end{tikzcd}$$

which amounts to looking at the family of (E, Φ) of the form

$$E = \mathcal{O}(1) \oplus \mathcal{O}(-1), \qquad \Phi = \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}$$

with $q \in H^0(\mathbb{P}^1, \mathcal{O}(2)^2)$. We have again formed a section. However, we will see that in higher ranks (or with different labellings or twisting line bundles) the moduli space of representations of a cyclic quiver is not, in general, a section. It should also be noted that these constructions of the Hitchin section take place with degree $d = 0$ (so the rank and degree are not coprime) but the setup ensures that no representations exist which are semistable but not stable.

Remark 5.0.1. There exists a different generalization of the Hitchin section which is in fact a section of $\mathcal{M}_{X,\omega_X}(r, \mathcal{O}_X)$ (cf. [12, 13, 3, 15], for instance). These sections can be viewed as representations of (appropriately labelled) quivers

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \arrow[r] \arrow[l, bend right] & \bullet \arrow[r] \arrow[l, bend right] \arrow[ll, bend right] & \cdots \arrow[r] & \bullet \arrow[llll, bend right] \arrow[lll, bend right] \arrow[ll, bend right] \end{tikzcd}$$

in which every possible "backwards facing" arrow is turned on.

5.1 Type $(1, \dots, 1)$ cyclic quivers

Inspired by cyclic Higgs bundles, we will begin by considering cyclic quivers whose labellings have $r_i = 1$ for all i.

Proposition 5.1.1. *If*

$$\begin{tikzcd} U_1 \arrow[r, "\phi_1"'] & U_2 \arrow[r, "\phi_2"'] & \cdots \arrow[r, "\phi_{n-1}"'] & U_n \arrow[lll, bend right, "\phi_n"'] \end{tikzcd}$$

is a stable representation of the cyclic quiver

$$\begin{tikzcd} \bullet_{1,d_1} \arrow[r] & \bullet_{1,d_2} \arrow[r] & \cdots \arrow[r] & \bullet_{1,d_n} \arrow[lll, bend right] \end{tikzcd}$$

in Bun(X, L), then exactly one of the maps ϕ_i is allowed the possibility of being identically zero.

Proof. Thoughout the proof, let the indices be counted modulo n.

If ϕ_i and ϕ_j with $i > j$ were both allowed to be zero, then $\bigoplus_{i=1}^n U_i$ could be presented as a direct sum of two Φ-invariant subbundles, both of which have slope less than μ_{tot}. This is a contradiction, and so at most one map can be the zero map.

To show that such a map always exists, suppose that $\phi_i \neq 0$ for all $i = 1, \dots, n$. That is, for each ϕ_i there is at least one subbundle of $\bigoplus_{i=1}^n U_i$ which has slope greater than μ_{tot} and which is Φ-invariant if and only if $\phi_i = 0$. For ϕ_i, such an associated destabilizing subbundle has the form $U_j \oplus U_{j+1} \oplus \cdots \oplus U_i$ for some j. Now for each $i = 1, \dots, n$, define V_i to be the subbundle of $\bigoplus_{i=1}^n U_i$ which has these properties and has the lowest rank:

$$V_i = U_{v(i)} \oplus U_{v(i)+1} \oplus \cdots \oplus U_i,$$

where $v : \{1, \dots, n\} \to \{1, \dots, n\}$.

If each V_i is a line bundle, then $V_i = U_i$, and since $\mu(V_i) > \mu_{tot}$ for all $i = 1, \dots, n$ we have

$$\sum_{i=1}^n \deg(U_i) = \sum_{i=1}^n \mu(V_i) > n\mu_{tot} = \sum_{i=1}^n \deg(U_i),$$

which is a contradiction.

Therefore, we assume that at least one V_i has rank greater than 1. For any such V_i we have by definition that $\mu(U_k \oplus U_{k+1} \oplus \dots U_i) < \mu_{tot}$ for all k such that $v(i) < k \leq i$. This also tells us that $\mu(U_{v(i)} \oplus U_{v(i)+1} \oplus \cdots \oplus U_{k-1}) > \mu_{tot}$. The existence of these subbundles with slope greater than μ_{tot} gives us information about V_{k-1}, namely that $V_{k-1} \subset V_i$ for $v(i) \leq k-1 < i$, and so in fact if $V_j \cap V_i \neq \varnothing$ for any $i \neq j$, one must be contained in the other.

Since all the V_i are proper subbundles of $\bigoplus_{i=1}^n U_i$, there must exist a subset $I \subset \{1, \dots, n\}$ with $|I| > 1$ such that

$$\bigoplus_I V_i = \bigoplus_{i=1}^n U_i \quad \text{and } V_j \cap V_k = 0_X \; \forall\, j \neq k \in I,$$

where 0_X is the zero bundle. Recalling that $\mu(V_i) > \mu_{tot}$ for all $i = 1, \dots, n$, we have a contradiction. Thus, there is exactly one map ϕ_i which is allowed to be the zero map. $\square$

With this result in our pocket, we re-index all representations in this chapter so that ϕ_n is the map which is allowed to be zero. This also provides a restriction regarding which labelled cyclic quivers we should be considering: Q admits stable represention if and only if $t \geq d_i - d_{i+1}$ for all $i = 1, \dots, n-1$.

Now let us consider how the automorphism group acts on a representation. By our earlier definition of equivalence and the structure of a cyclic quiver, we have, for $\psi_i \in \mathrm{Aut}(U_i) \cong \mathbb{C}^*$,

$$\Psi\Phi\Psi^{-1} = \begin{pmatrix} \psi_1 & \cdots & & 0 \\ & \psi_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \psi_n \end{pmatrix} \begin{pmatrix} 0 & \cdots & & \phi_n \\ \phi_1 & \ddots & & \\ & \ddots & & \\ 0 & & \phi_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \psi_1^{-1} & \cdots & & 0 \\ & \psi_2^{-1} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \psi_n^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \cdots & & \psi_1\psi_n^{-1}\phi_n \\ \psi_2\psi_1^{-1}\phi_1 & \ddots & & \\ & \ddots & & \\ 0 & & \psi_n\psi_{n-1}^{-1}\phi_{n-1} & 0 \end{pmatrix}$$

By defining

$$\lambda_1 = \psi_2 \psi_1^{-1}$$
$$\vdots$$
$$\lambda_{n-1} = \psi_n \psi_{n-1}^{-1}$$

we can realize the action of $\Psi \in \bigoplus_{i=1}^{n} \mathrm{Aut}(U_i)$ as the action of $(\mathbb{C}^*)^{n-1}$ on the Φ part of $\mathrm{Rep}(Q)$, given by

$$(\lambda_1, \ldots, \lambda_{n-1}) \cdot \begin{pmatrix} 0 & \cdots & & \phi_n \\ \phi_1 & \ddots & & \\ & \ddots & & \\ 0 & & \phi_{n-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cdots & & (\lambda_1^{-1} \ldots \lambda_{n-1}^{-1})\phi_n \\ \lambda_1\phi_1 & \ddots & & \\ & \ddots & & \\ 0 & & \lambda_{n-1}\phi_{n-1} & 0 \end{pmatrix} \tag{5.1}$$

Now we can think more about the structure of the moduli space itself. By identifying representations of a cyclic quiver Q with cyclic Higgs bundles, we can get an idea of how $\mathcal{M}_{X,L}(Q)$ lies in $\mathcal{M}_{X,L}(r,d)$.

Proposition 5.1.2. *The Hitchin map h maps $\mathcal{M}_{X,L}(Q)$ surjectively onto $H^0(X, L^{\otimes n}) \subset \mathcal{B}_n$.*

Proof. By definition,

$$\begin{aligned} h((E,\Phi)) &= \mathrm{char}_\lambda(\Phi) \\ &= \det \begin{pmatrix} -\lambda & \cdots & & \phi_n \\ \phi_1 & -\lambda & & \\ & \ddots & \ddots & \\ 0 & & \phi_{n-1} & -\lambda \end{pmatrix} \\ &= \pm\lambda^r \pm (\phi_1 \ldots \phi_n). \end{aligned}$$

That is, the Hitchin map sends any cyclic quiver representation to the determinant of Φ. Moreover, all such determinants can be obtained. □

We will now restrict ourselves to studying the $SL(n,\mathbb{C})$ and $PGL(n,\mathbb{C})$ moduli spaces. It is difficult to give a global geometric description of either of these, but we can exploit the Hitchin system to describe their structure in each fibre.

Theorem 5.1.1. *Given a Riemann surface X of genus g, a holomorphic line bundle L of degree t, and a $(1,\ldots,1)$-type cyclic quiver Q, we have the following description of the $SL(n,\mathbb{C})$ and $PGL(n,\mathbb{C})$ moduli spaces, parametrized by $\gamma \in H^0(X, L^{\otimes n}) \subset \mathcal{B}_n$:*

$$\mathcal{M}_{X,L}^{SL(n,\mathbb{C})}(Q)\Big|_{h^{-1}(\gamma)} \cong \left\{ (U_1,\ldots,U_n;[\phi_1],\ldots,[\phi_{n-1}]) \in {}^{\sim}\!\left(\prod_{i=1}^{n-1} Sym^{d_{i+1}-d_i+t}(X)\right) : (\phi_1 \ldots \phi_{n-1}) \subseteq (\gamma) \right\}$$

and

$$\mathcal{M}_{X,L}^{PGL(n,\mathbb{C})}(Q)\Big|_{h^{-1}(\gamma)} \cong \left\{ (U_1,\ldots,U_n;[\phi_1],\ldots,[\phi_{n-1}]) \in \prod_{i=1}^{n-1} Sym^{d_{i+1}-d_i+t}(X) : (\phi_1 \ldots \phi_{n-1}) \subseteq (\gamma) \right\}$$

where $(\phi_1 \ldots \phi_{n-1})$ and (γ) are the divisors defined by the holomorphic sections $\phi_1 \ldots \phi_{n-1}$ and γ respectively.

Proof. The beginning of this proof can be adapted directly from the proof of Theorem 3.2.1. Using the techniques presented there tells us that $\mathcal{M}_{X,L}^{SL(n,\mathbb{C})}(Q)\big|_{h^{-1}(\gamma)}$ is the n^{2g}-sheeted cover ${}^{\sim}\!\left(\prod_{i=1}^{n-1} \mathrm{Sym}^{d_{i+1}-d_i+t}(X)\right)$ of $\prod_{i=1}^{n-1} \mathrm{Sym}^{d_{i+1}-d_i+t}(X)$, subject to some condition based on the fixed determinant, namely $\phi_1 \ldots \phi_n = \gamma$. Not all points of ${}^{\sim}\!\left(\prod_{i=1}^{n-1} \mathrm{Sym}^{d_{i+1}-d_i+t}(X)\right)$ allow for a corresponding $\phi_n \in H^0(X, U_n^* U_1 L)$ to be chosen so that this condition is satisfied. We require that the corresponding divisors satisfy $(\phi_1 \ldots \phi_{n-1}) \subseteq (\gamma)$; this tells us that $(\phi_{n-1} \ldots \phi_1)^{-1}\gamma$ is well-defined and holomorphic and that there is, in fact, a suitable ϕ_n (technically, a suitable projective class $[\phi_n]$). We now have the above set-theoretic description of $\mathcal{M}_{X,L}^{SL(n,\mathbb{C})}(Q)\big|_{h^{-1}(\gamma)}$, and the action of $\mathrm{Jac}^0(X)[n]$ gives us $\mathcal{M}_{X,L}^{PGL(n,\mathbb{C})}(Q)\big|_{h^{-1}(\gamma)}$, just as in Theorem 3.2.1. We note that these descriptions are well-defined since all ϕ_i in the projective class $[\phi_i]$ define the same divisor.

□

We also note that making this fibre-wise description is only legitimate since the automorphism group commutes with shifting between fibres (that is, acting by the automorphism group cannot move us between fibres).

At $\gamma = 0 \in H^0(X, L^{\otimes n})$, we expect

$$\mathcal{M}_{X,L}^{SL(n,\mathbb{C})}(Q)\Big|_{h^{-1}(0)} \cong {}^{\sim}\!\left(\prod_{i=1}^{n-1} \mathrm{Sym}^{d_{i+1}-d_i+t}(X)\right)$$

and

$$\mathcal{M}_{X,L}^{PGL(n,\mathbb{C})}(Q)\Big|_{h^{-1}(0)} \cong \prod_{i=1}^{n-1} \mathrm{Sym}^{d_{i+1}-d_i+t}(X)$$

based on Theorem 3.2.1. This agrees exactly with Theorem 5.1.1, since any divisor $(\phi_1, \dots \phi_{n-1})$ lies inside the "divisor" determined by $\gamma = 0$.

5.2 Type $(1, \dots, 1)$ cyclic quivers on $\mathbb{P}^1$

Working over the Riemann sphere, investigating the moduli space of representations of a type $(1, \dots, 1)$ quiver Q is simplified by the fact that there is only a single line bundle of a given degree. So in this context, all of the moduli information lies in the maps ϕ_i. Using Proposition 5.1, we can write

$$\begin{aligned} \mathrm{Rep}(Q) &\cong \prod_{i=1}^{n-1} \Big(H^0(\mathbb{P}^1, \mathcal{O}(-d_i) \otimes \mathcal{O}(d_{i+1}) \otimes \mathcal{O}(t)) \setminus \{0\} \Big) \times H^0(\mathbb{P}^1, \mathcal{O}(-d_n) \otimes \mathcal{O}(d_1) \otimes \mathcal{O}(t)) \\ &\cong \prod_{i=1}^{n-1} \Big(\mathbb{C}^{d_{i+1} - d_i + t + 1} \setminus \{0\} \Big) \times \mathbb{C}^{d_1 - d_n + t + 1} \end{aligned}$$

as well as

$$\mathcal{M}_{\mathbb{P}^1, \mathcal{O}(t)}(Q) \cong \frac{\prod_{i=1}^{n-1} \Big(\mathbb{C}^{d_{i+1} - d_i + t + 1} \setminus \{0\} \Big) \times \mathbb{C}^{d_1 - d_n + t + 1}}{(\mathbb{C}^*)^{n-1}},$$

where the action of $(\mathbb{C}^*)^{n-1}$ is given by Equation (5.1). This is an interesting quotient which is reminiscent of weighted projective space[11]. For example, if we consider representations of the quiver

$$Q = \begin{tikzcd} \bullet_{1,d_1} \arrow[r, bend right] & \bullet_{1,d_2} \arrow[l, bend right] \end{tikzcd}$$

then the moduli space is

$$\mathcal{M}_{\mathbb{P}^1, \mathcal{O}(t)}(Q) \cong \frac{(\mathbb{C}^{d_2 - d_1 + t} \setminus \{0\}) \times \mathbb{C}^{d_2 - d_1 + t}}{\mathbb{C}^*},$$

[11] Let $\mathbf{b} = (b_0, \dots, b_n)$, $b_i \in \mathbb{N}$ and define the action of $\mathbb{C}^*_{\mathbf{b}}$ on $\mathbb{C}^{n+1} \setminus \{0\}$ as the following action of $\mathbb{C}^*$:

$$\lambda \cdot (x_0, \dots, x_n) = (\lambda^{b_0} x_0, \dots \lambda^{b_n} x_n).$$

We then define **b**-*weighted complex projective space* as

$$\mathbb{P}(b_0, \dots, b_n) = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\mathbb{C}^*_{\mathbf{b}}}.$$

which is a singular analogue of weighted projective space in which we allow for negative weights (the action looks like $\lambda \cdot (\phi_1, \phi_2) = (\lambda\phi_1, \lambda^{-1}\phi_2)$). Higher rank examples can be thought of as products of these spaces, which are somehow intertwined at the part which is acted on by negative weight. For example, recall that in rank 3, $(\mathbb{C}^*)^2$ acts as

$$(\lambda_1, \lambda_2) \cdot (\phi_1, \phi_2, \phi_3) = (\lambda_1\phi_1, \lambda_2\phi_2, \lambda_1^{-1}\lambda_2^{-1}\phi_3).$$

Now we have some idea of what $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q)$ is, but these quotients are unusual. We can say more about the structure of the moduli space by exploiting the Hitchin map $h : \mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q) \twoheadrightarrow H^0(\mathbb{P}^1, \mathcal{O}(nt))$. Choose a generic $\gamma \in H^0(\mathbb{P}^1, \mathcal{O}(nt))$ and consider the restriction $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q)\big|_{h^{-1}(\gamma)}$. Fixing the determinant of Φ amounts to fixing the zeroes of the polynomial $\phi_1, \dots, \phi_n$. Recalling the action of $(\mathbb{C}^*)^{n-1}$ from Equation (5.1), we see that it only acts by scaling. That is, the roots of ϕ_i are fixed for all i. Thus, different distributions of the zeroes of γ into the ϕ_i lead to legitimately different points in the moduli space. This tells us that $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q)\big|_{h^{-1}(\gamma)}$ consists of finitely many distinct points, the number of which is given by the multinomial coefficient

$$\eta(Q) := \binom{nt}{d_2 - d_1 + t, \dots, d_n - d_{n-1} + t, d_1 - d_n + t}.$$

This fails over points of $H^0(\mathbb{P}^1, \mathcal{O}(nt))$ which, interpreted as polynomials, have repeated zeroes. So, we have that $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q)$ is an $\eta(Q)$-sheeted covering of $H^0(\mathbb{P}^1, \mathcal{O}(nt))$ which degenerates over points which have zeroes with multiplicity greater than one. However, this is not quite a full description; we have so far neglected to mention the fibre $h^{-1}(0)$. Here we must always have $\phi_n = 0$ and so by Corollary 3.2.2,

$$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q)\big|_{h^{-1}(0)} \cong \mathbb{P}^{d_2-d_1+t} \times \ldots \times \mathbb{P}^{d_n-d_{n-1}+t}.$$

How this fits into the covering described above can be seen by looking at the $\mathbb{C}^*$ flows in $H^0(\mathbb{P}^1, \mathcal{O}(nt))$. Fix a point $p \in \mathbb{P}^{d_2-d_1+t} \times \ldots \times \mathbb{P}^{d_n-d_{n-1}+t}$, which we know consists only of the information

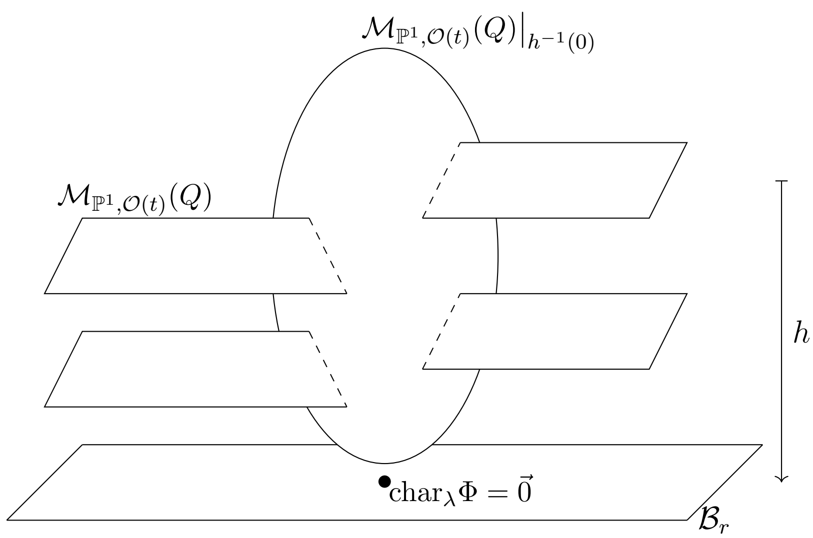
$$\Phi = \begin{pmatrix} 0 & \cdots & & 0 \\ \phi_1 & \ddots & & \\ & \ddots & & \\ 0 & & \phi_{n-1} & 0 \end{pmatrix}.$$

Choose any map $c\phi_n : \mathcal{O}(d_n) \to \mathcal{O}(d_1) \otimes \mathcal{O}(t)$ and we see that the ray given by

$$\begin{pmatrix} 0 & \cdots & & c\phi_n \\ \phi_1 & \ddots & & \\ & \ddots & & \\ 0 & & \phi_{n-1} & 0 \end{pmatrix}$$

goes to p as $c \to 0$. That is, every point of $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q)\big|_{h^{-1}(0)}$ is in the intersection with the cover. This leads the following theorem:

Theorem 5.2.1. *The moduli space of representations of a* $(1,\ldots,1)$*-type cyclic quiver* Q *in the category of* $\mathcal{O}(t)$*-twisted holomorphic vector bundles on* $\mathbb{P}^1$ *is an* $\eta(Q)$*-sheeted covering of* $H^0(\mathbb{P}^1,\mathcal{O}(nt)) \setminus \{0\}$ *which branches over points with roots of multiplicity greater than one, and whose sheets intersect over the point* $0 \in H^0(\mathbb{P}^1,\mathcal{O}(nt))$ *as* $\prod_{i=1}^{n-1} \mathbb{P}^{d_{i+1}-d_i+t}$.



This result also follows from Theorem 5.1.1 since over $\mathbb{P}^1$ the bundles U_i are fixed, $\mathrm{Sym}^d(\mathbb{P}^1) \cong \mathbb{P}^d$, and γ and the ϕ_i can be thought of as polynomials so the divisor condition corresponds to the discussion of the distribution of zeroes above.

Example 5.2.1. Let $X = \mathbb{P}^1$, $L = \mathcal{O}(4)$, and

$$Q = \begin{tikzcd} \bullet_{1,0} \arrow[r, bend right] & \bullet_{1,-1} \arrow[l, bend right] \end{tikzcd}$$

so a representation looks like

$$\begin{tikzcd} \mathcal{O} \arrow[r, bend right, "\phi_1"'] & \mathcal{O}(-1) \arrow[l, bend right, "\phi_2"'] \end{tikzcd}$$

with $\phi_1 \in H^0(\mathbb{P}^1, \mathcal{O}\otimes\mathcal{O}(-1)\otimes\mathcal{O}(4))\setminus\{0\} \cong \mathbb{C}^4\setminus\{0\}$ and $\phi_2 \in H^0(\mathbb{P}^1, \mathcal{O}(1)\otimes\mathcal{O}\otimes\mathcal{O}(4)) \cong \mathbb{C}^6$. Fix a generic point $\gamma \in H^0(\mathbb{P}^1, \mathcal{O}(8))$, say $\gamma = c(z-z_1)\dots(z-z_8)$. We have $\eta(Q) = \binom{8}{3} = 56$ ways to distribute the roots z_i, and using the power of the automorphism group we can put the constant c with ϕ_2:

$$\Phi = \begin{pmatrix} 0 & c\prod_{j\in J}(z-z_j) \\ \prod_{i\in I}(z-z_i) & 0 \end{pmatrix} \text{ where } I\cap J = \varnothing, |I| = 3, |J| = 5.$$

This gives a 56-fold ramified covering of $H^0(\mathbb{P}^1, \mathcal{O}(8)) \setminus \{0\}$. At $\gamma = 0$, we must have $\phi_2 = 0$, and so

$$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(4)}(Q)\big|_{h^{-1}(0)} \cong \mathbb{P}^3.$$

We can reach any point in $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(4)}(Q)\big|_{h^{-1}(0)}$ by choosing a suitable point in the cover and then taking $c \to 0$.

5.3 Type $(k, 1)$ cyclic quivers on $\mathbb{P}^1$

We would like to expand to quivers which have some nodes labelled with higher ranks, and will start by having a look at cyclic quivers of the form

$$Q = \begin{tikzcd} \bullet_{k,d_1} \arrow[r, bend right] & \bullet_{1,d_2} \arrow[l, bend right] \end{tikzcd}$$

where the underlying curve is $\mathbb{P}^1$. We will then restrict focus a little further. Recall the splitting type of a bundle U of rank k over $\mathbb{P}^1$

$$\mathbf{a} = (a_1, \dots, a_m; s_1, \dots, s_m)$$

which defines that U splits as

$$U \cong \mathcal{O}(a_1)^{\oplus s_1} \oplus \dots \oplus \mathcal{O}(a_m)^{\oplus s_m}.$$

We content ourselves with the case that $s_i = 1$ for all $i = 1, \ldots, m$, meaning that $m = k$ and the line bundles are of mutually distinct degrees. We further ask that[12] $\mu_{tot} < a_i$ for all i. The reasons for this are discussed in Remark 5.3.3. For the remainder of this section, we assume that $\mathbf{a}$ has these properties.

With these restrictions in place, we are considering moduli of representations which look like

$$\begin{tikzcd}
\mathcal{O}(a_1) \arrow[dr, bend left, "\phi_1"] & \\
\vdots & \mathcal{O}(d_2) \arrow[ul, bend left, "\phi_2"] \arrow[dl, bend left, "\phi_{2k}"] \\
\mathcal{O}(a_k) \arrow[ur, bend left, "\phi_{2k-1}"] &
\end{tikzcd} \tag{5.2}$$

along with automorphisms $\psi_{ij} : \mathcal{O}(a_i) \to \mathcal{O}(a_j)$ for all $i > j$. Here stability implies that none of $\phi_1, \phi_3, \ldots, \phi_{2k-1}$ can be zero, but any of $\phi_2, \phi_4, \ldots, \phi_{2k}$ can be. This imposes the further condition $-a_i + d_2 + t \geq 0$ for all $i = 1, \ldots, k$.

Proposition 5.3.1.

$$h : \mathcal{M}_{\mathbb{P}^1, \mathcal{O}(t)}(Q; \mathbf{a}) \twoheadrightarrow H^0(\mathbb{P}^1, \mathcal{O}(t)^{\otimes 2})$$

Proof. This follows from the same argument as Propostion 5.1.2. In this case the calculation gives

$$h((E, \Phi)) = \pm \lambda^r \mp \lambda^{r-1}(\phi_1\phi_2 + \phi_3\phi_4 + \cdots + \phi_{2k-1}\phi_{2k}).$$

□

Our strategy to understand $\mathcal{M}_{\mathbb{P}^1, \mathcal{O}(t)}(Q; \mathbf{a})$ is to view a representation such as (5.2) as k separate $(1,1)$ quiver representations by first "using up" the power of the automorphisms ψ_{ij}. To state our result, we need to use the following notation.

Let Q be the type $(1,1)$ cyclic quiver

$$Q = \begin{tikzcd} \bullet_{1,d_1} \arrow[r, bend right] & \bullet_{1,d_2} \arrow[l, bend right] \end{tikzcd}$$

[12]This also covers the case $\mu_{tot} > a_i$ for all i, simply by considering the dual quiver representation, which will have $\mu_{tot} < a_i$ for all i.

oriented with $d_1 > d_2$ so that in a representation, the map $\phi_1 : \mathcal{O}(d_1) \to \mathcal{O}(d_2)$ cannot be zero by stability. Denote by $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q^{-b})$ the moduli space of representations of Q in $\mathrm{Bun}(\mathbb{P}^1, \mathcal{O}(t))$ where ϕ_1 has had its amount of freedom (in terms of complex dimensions) reduced by b.

Example 5.3.1. For example, let $L = \mathcal{O}(4)$ and

$$Q = \begin{tikzcd} \bullet_{1,0} \arrow[r, bend right] & \bullet_{1,-1} \arrow[l, bend right] \end{tikzcd}$$

as in Example 5.2.1. Consider $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q^{-2})$, saying that now $\phi_1 \in \mathbb{C}^2 \setminus \{0\}$ and $\phi_2 \in \mathbb{C}^6$. Now for generic $\gamma \in H^0(\mathbb{P}^1, \mathcal{O}(8))$, reduced to $H^0(\mathbb{P}^1, \mathcal{O}(6))$, we have only to distribute 6 zeroes into the ϕ_1 and ϕ_2. We have $\binom{6}{1} = 6$ ways to do so. At $\gamma = 0$, we have $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q^{-2})\big|_{h-1(0)} \cong \mathbb{P}^1$.

Note that even though we may not have an interpretation of the information of $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q^{-2})$ as an unadjusted $(1,1)$ cyclic quiver variety (although we do in some cases), its structure is easily calculated in a familiar way. Now we can write the moduli space of representations of a $(k,1)$ cyclic quiver in terms of these adjusted moduli spaces of representations of $(1,1)$ cyclic quivers.

Theorem 5.3.1. *Let Q be a type $(k,1)$ cyclic quiver, $\mathbf{a} = (a_1, \ldots, a_k; 1, \ldots, 1)$ be a splitting type, and Q_i be the quivers*

$$\begin{tikzcd} \bullet_{1,a_i} \arrow[r, bend right] & \bullet_{1,d_2} \arrow[l, bend right] \end{tikzcd}$$

Then

$$\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(Q;\mathbf{a}) \cong \mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q_1) \times \prod_{i=2}^{k} \mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}\Big(Q_i^{-\sum_{j=1}^{i-1}(a_j - a_i + 1)}\Big).$$

Proof. Recall the visualization of a representation from Equation 5.2. Due to our assumptions on $\mathbf{a}$, the maps $\phi_1, \phi_3, \ldots, \phi_{2k-1}$ cannot be zero, but any of $\phi_2, \phi_4, \ldots, \phi_{2k}$ can be. The automorphisms ψ_{ij} are not able reduce the amount of freedom of any of the maps $\phi_2, \phi_4, \ldots, \phi_{2k}$. This means that the reductions which take place are exactly the ones that would take place in the A-type $(k,1)$ case with $\mathbf{a}$ as the splitting type. In particular, the map ϕ_{2i-1} has its moduli reduced by $\sum_{j=1}^{m-1}(a_j - a_i + 1)$. Note that ϕ_1 is not reduced at all. Now we have broke our moduli problem into k parts, each of the form $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}\left(Q_i^{-\sum_{j=1}^{i-1}(a_j-a_i+1)}\right)$, and we have our result.

$\square$

Remark 5.3.1. As in Chapter 4, we have only calculated the projective completion of the regular part of the moduli space. Since this approach only allows us to understand specific splittings types $\mathbf{a}$, we will not approach the question of how $\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}(Q)$ is stratified by splitting types.

Corollary 5.3.1. *The moduli space restricted to a fibre,* $\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(t)}}(Q;\mathbf{a})\big|_{h^{-1}(\gamma)}$, *is a*

$$\binom{(r-1)t-\sum_{j=1}^{k-1}(a_j-a_k+1)}{d_2-a_k+t-\sum_{j=1}^{k-1}(a_j-a_k+1)}$$

-to-one covering of

$$\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}(Q_1)\times\prod_{i=2}^{k-1}\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(t)}\left(Q_i^{-\sum_{j=1}^{i-1}(a_j-a_i+1)}\right),$$

except over points $(\phi_1,\phi_2,\dots,\phi_{2k-2})$ *such that* $\phi_1\phi_2+\phi_3\phi_4+\cdots+\phi_{2k-3}\phi_{2k-2}=-\gamma$, *where the sheets intersect as* $\mathbb{P}^{d_2-a_k+t-\sum_{j=1}^{k-1}(a_j-a_k+1)}$.

Proof. If we fix $\gamma=\phi_1\phi_2+\phi_3\phi_4+\cdots+\phi_{2k-1}\phi_{2k}$, it is clear that we have the freedom to choose any $(\phi_1,\phi_2,\dots,\phi_{2k-2})$, which will then place restrictions on ϕ_{2k-1} and ϕ_{2k}. (ϕ_1,ϕ_2) must be chosen before we can reduce the freedom of ϕ_3, and so forth, which is why (ϕ_{2k-1},ϕ_{2k}) is the last to be chosen. Now the problem amounts to distributing the zeroes of $\gamma-\phi_1\phi_2-\phi_3\phi_4-\cdots-\phi_{2k-3}\phi_{2k-2}$ which are not already fixed into ϕ_{2k-1} and ϕ_{2k}. In the case $\phi_1\phi_2+\phi_3\phi_4+\cdots+\phi_{2k-3}\phi_{2k-2}=-\gamma$, we must have $\phi_{2k}=0$, and so the fibre is the projective space $\mathbb{P}^{d_2-a_k+t-\sum_{j=1}^{k-1}(a_j-a_k+1)}$. $\square$

Remark 5.3.2. In contrast to the $(1,\dots,1)$ cyclic case, in the $(k,1)$ cyclic case there is nothing unusual happening to the moduli space at the nilpotent cone $h^{-1}(0)$. There is an analogue of the corresponding $(k,1)$ A-type quiver variety living inside each fibre.

Example 5.3.2. Let $t=5$, Q be the quiver

$$\begin{tikzcd} \bullet_{2,1} \arrow[r, bend right] & \bullet_{1,-2} \arrow[l, bend right] \end{tikzcd}$$

and $\mathbf{a}$ be the splitting type $(1,0;1,1)$. So a representation looks like

$$\begin{tikzcd}
\mathcal{O}(1) \arrow[dr, bend left, "\phi_1"] & \\
& \mathcal{O}(-2) \arrow[ul, bend left, "\phi_2"] \arrow[dl, bend left, "\phi_4"] \\
\mathcal{O} \arrow[uu, dotted, "\psi_{21}"] \arrow[ur, bend left, "\phi_3"] &
\end{tikzcd}$$

where $\phi_1 \in \mathbb{C}^3 \setminus \{0\}$, $\phi_2 \in \mathbb{C}^9$, $\phi_3 \in \mathbb{C}^4 \setminus \{0\}$, and $\phi_1 \in \mathbb{C}^8$. The automorphism $\psi_{21} \in \mathbb{C}^2$ reduces ϕ_3 to $\mathbb{C}^2 \setminus \{0\}$, and

$$\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(5)}}(Q;\mathbf{a}) \cong \mathcal{M}_{\mathbb{P}^1,\mathcal{O}(5)}(Q_1) \times \mathcal{M}_{\mathbb{P}^1,\mathcal{O}(5)}(Q_2^{-2}).$$

Considering $\overline{\mathcal{M}^{\Delta}_{\mathbb{P}^1,\mathcal{O}(5)}}(Q;\mathbf{a})\big|_{h^{-1}(\gamma)}$, we fix $\gamma = c(z-z_1)\dots(z-z_{10}) = \phi_1\phi_2 + \phi_3\phi_4$. We can choose any $(\phi_1,\phi_2) \in \mathcal{M}_{\mathbb{P}^1,\mathcal{O}(5)}(Q_1)$ and then consider $\gamma - \phi_1\phi_2 = \phi_3\phi_4$. We know that the map ϕ_3 is reduced by ϕ_{21}, so we ignore the top 2 degrees of $\gamma - \phi_1\phi_2$. Then the moduli problem amounts to distributing 8 zeroes, 1 into ϕ_3 and 7 into ϕ_4, resulting in an 8-fold covering of $\mathcal{M}_{\mathbb{P}^1,\mathcal{O}(5)}(Q_1)$, except over the points $\gamma = \phi_1\phi_2$, where we have $\mathbb{P}^1$.

This same behaviour is displayed in the intersection with the nilpotent cone, although we can identify the locus

$$\{(\phi_1,\phi_2) \in \mathcal{M}_{\mathbb{P}^1,\mathcal{O}(5)}(Q_1) : \phi_1\phi_2 = 0\} \times \mathbb{P}^1 \cong \mathbb{P}^2 \times \mathbb{P}^1$$

with the moduli space of the similarly labelled (2,1) A-type quiver.

Remark 5.3.3. The condition $\mu_{tot} < a_i$ on the splittings we consider is essentially asking for Proposition 5.1.1 to hold for $(k,1)$ quivers. We put this in place because when the maps ϕ_i which are and are not allowed to be zero by stability are less rigidly structured, the actions of ψ_{ij} become less clear. Without being able to say exactly which maps the automorphisms reduce, our approach of considering k different $(1,1)$ cyclic quiver varieties is less effective. This is also why we impose $s_i = 1$. It is this lack of a clear decomposition of the moduli space into products of varieties which we understand that prevents us from using this procedure to study splittings where some of the line bundles $\mathcal{O}(a_i)$ have the same degree, as well as $(1,k,1)$ and general argyle quivers. In these later cases, one must also contend with the fact that the terms of the fixed characteristic polynomial are, in general, no longer simply products of the maps ϕ_i.

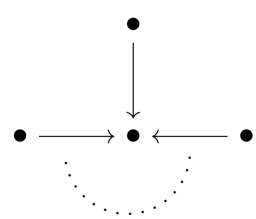
6 FURTHER DIRECTIONS

6.1 Star-shaped quivers and hyperpolygons

Here we will introduce hyperpolygon space using quiver representations. This hyperkähler counterpart to the moduli space of polygons has close ties to the moduli space of parabolic Higgs bundles. We discuss some possible research directions relating to representations in the twisted category $\mathrm{Bun}(X, L)$.

6.1.1 Background

Definition 6.1.1. A *star-shaped quiver* is a quiver of the form



If we consider representations of a star-shaped quiver Q with n outer nodes each labelled 1, and interior node labelled r in the category of vector spaces, we arrive naturally at the moduli space of n-gons in $\mathbb{R}^{r^2-1}$ with edge lengths $\alpha = (\alpha_1, \ldots, \alpha_n)$, denoted $\mathcal{P}_n^r(\alpha)$. This is achieved by way of a symplectic quotient. Each of the n outer nodes is assigned $\mathbb{C}$, the inner node is assigned $\mathbb{C}^r$, and each of the arrows is given a map $x_i \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^r) \cong \mathrm{Mat}_{r\times 1}(\mathbb{C}) \cong \mathbb{C}^r$. Hence, $\mathrm{Rep}(Q) = (\mathbb{C}^r)^{\oplus n} \cong \mathbb{C}^{rn}$. Each of the nodes is acted on by a compact real group ($U(1)$ for the outer nodes and $SU(r)$ for the inner node) and so we have an action of $G = SU(r) \times U(1)^n$ on $\mathrm{Rep}(Q)$, given by $x_i \mapsto hx_ig = hx_ie^{i\theta}$ and so $(x_1, \ldots, x_n) \mapsto (hx_1e^{i\theta_1}, \ldots, hx_ne^{i\theta_n})$, where $h \in SU(r)$ and $e^{i\theta_j} \in U(1)$. The picture now looks like

$$
\begin{tikzcd}
 & \mathbb{C} \arrow[loop above, "U(1)"] \arrow[d, "x_2"'] & \\
\mathbb{C} \arrow[loop above, "U(1)"] \arrow[r, "x_1"] \arrow[rr, bend right, dotted, no head] & \mathbb{C}^r \arrow[loop above, "SU(r)"] & \mathbb{C} \arrow[loop above, "U(1)"] \arrow[l, "x_3"']
\end{tikzcd}
$$

The *polygon space* $\mathcal{P}_n^r(\alpha)$ is defined as the symplectic quotient

$$\mathcal{P}_n^r(\alpha) := \mathrm{Rep}(Q) /\!\!/_{\alpha} G = \mu^{-1}(0_{r\times r}, \alpha)/G$$

where the moment map[13] μ is defined by

$$
\begin{aligned}
\mu : \mathrm{Rep}(Q) &\longrightarrow \mathfrak{g}^* \cong \mathfrak{su}(r)^* \oplus \mathbb{R}^n \\
(x_1, \ldots, x_n) &\longmapsto \big(\sum_{i=1}^{n} (x_i x_i^*)_0, \|x_1\|^2, \ldots, \|x_n\|^2 \big).
\end{aligned}
$$

Indeed $\mathcal{P}_n^r(\alpha)$ is the moduli space of n-gons in $\mathbb{R}^{r^2-1}$ with edge lengths $\alpha = (\alpha_1, \ldots, \alpha_n)$, up to equivalence by the action of $SU(r)$, which acts by rotations and translations. The topology of these spaces for $r = 2$ was studied by Klyachko in [42]. For $r \geq 2$, their topology can be computed using the techniques in [39], thanks to their construction as symplectic quotients.

A natural further question is whether there is a hyperkähler analogue of such spaces. Nakajima ([52, 53, 54]) developed a way to construct hyperkähler varieties from quivers, and in [43] Konno uses such methods to construct the following spaces as well as to compute their Betti numbers. Effectively, we double the arrows.

[13]The components of this moment map have been rescaled for convenience. Strictly speaking, the image of μ does not lie in the designated Lie algebra.

$$
\begin{tikzcd}
 & \mathbb{C} \arrow[d, bend left, "x_2"] & \\
\mathbb{C} \arrow[r, bend left, "x_1"] & \mathbb{C}^r \arrow[l, dotted, bend left, "y_1"] \arrow[u, dotted, bend left, "y_2"] \arrow[r, dotted, bend left, "y_3"] & \mathbb{C} \arrow[l, bend left, "x_3"]
\end{tikzcd}
$$

In this picture, each y_i is interpreted as an element of the cotangent fibre of $\mathrm{Rep}(Q)$ at x_i and in this way the y_i depend on the x_i. We have

$$y_i \in T^*_{x_i}\mathrm{Rep}(Q) \cong \mathrm{Rep}(Q)^* \cong \mathrm{Hom}(\mathbb{C}, \mathbb{C}^r)^* \cong \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}) \cong \mathbb{C}^r.$$

Hence, the information of a Nakajima star-shaped quiver representation is $(x_1, \ldots x_n, y_1, \ldots y_n) \in T^*\mathrm{Rep}(Q)$. We often write $(x_i|y_i)$ for $(x_1, \ldots x_n, y_1, \ldots y_n)$. Still taking $G = SU(r) \times U(1)^n$, we can now define *hyperpolygon space* as the hyperkähler quotient (see [37]) given by

$$\mathcal{X}_n^r(\alpha) := T^*\mathrm{Rep}(Q) /\!\!/\!\!/_\alpha\, G = \frac{\mu_{\mathbb{R}}^{-1}(0_{r\times r}, \alpha) \cap \mu_{\mathbb{C}}^{-1}(0_{r\times r}, 0_{1\times r})}{G}$$

where

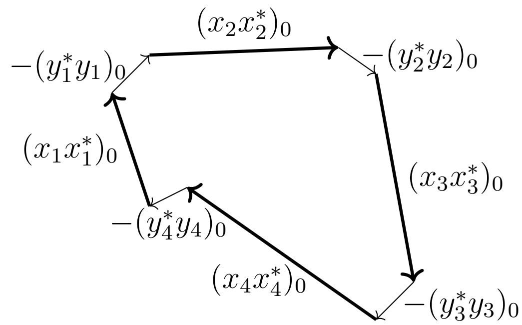
$$\begin{aligned} \mu_{\mathbb{R}} : T^*\mathrm{Rep}(Q) &\longrightarrow \mathfrak{g}^* \cong \mathfrak{su}(r)^* \oplus \mathbb{R}^n \\ (x_i|y_i) &\longmapsto \Big(\sum_{i=1}^n (x_i x_i^*)_0 - (y_i^* y_i)_0, \|x_1\|^2 - \|y_1\|^2, \ldots, \|x_n\|^2 - \|y_n\|^2\Big) \end{aligned}$$

and

$$\begin{aligned} \mu_{\mathbb{C}} : T^*\mathrm{Rep}(Q) &\longrightarrow \mathrm{Lie}(G^{\mathbb{C}}) \cong \mathfrak{sl}(r, \mathbb{C}) \oplus \mathbb{C}^n \\ (x_i|y_i) &\longmapsto \Big(\sum_{i=1}^n (x_i y_i)_0, y_1 x_1, \ldots, y_n x_n\Big). \end{aligned}$$

This is a hyperkähler manifold of complex dimension $2(r-1)(n-r-1)$. An element of $\mathcal{X}_n^r(\alpha)$ has a geometric interpretation as a "corrected" polygon; on their own, the vectors given by the information $(x_i x_i^*)_0$ do not close, but by correcting each one by vectors given by $-(y_i^* y_i)_0$, we indeed have a polygon in $\mathbb{R}^{r^2-1}$ (by the fact that we set $\sum_{i=1}^n (x_i x_i^*)_0 - (y_i^* y_i)_0 =$

0). For example,



Note that $\alpha = (\alpha_1, \dots, \alpha_n)$ no longer refers to the side lengths, but to the differences in length between $(x_ix_i^*)_0$ and $(y_i^*y_i)_0$.

6.1.2 Relationship with Higgs bundles

The moduli space $\mathcal{X}_n^r(\alpha)$ has been identified with a certain open subspace of a parabolic Higgs moduli space, first for rank 2 in [24] and for general rank in [16]. For more detail on parabolic Higgs bundles and the geometry of their moduli spaces, see [9] or [29].

Definition 6.1.2. Let $D = \sum_{i=1}^n p_i$ be a divisor on a Riemann surface X. A *(minimal) parabolic bundle* on X is a holomorphic vector bundle E along with a choice of line bundle and parabolic weights

$$0 \subset L_i \subset E_{p_i}$$
$$1 \geq \beta_2(p_i) > \beta_2(p_i).$$

A *parabolic Higgs bundle* on X is a pair (E, Φ) consisting of a parabolic bundle E and a map $\Phi : E \to E \otimes \omega_X(D)$, which is meromorphic with simple poles at D whose residues are nilpotent with respect to the flags of E_{p_i}.

Fix a divisor $D = \sum_{i=1}^n p_i$ on $\mathbb{P}^1$. Then the moduli space of hyperpolygons $\mathcal{X}_n^r(\alpha)$ is isomorphic to the moduli space of rank r trace-free parabolic Higgs bundles on $\mathbb{P}^1$ (with respect to D) with trivial underlying bundle (that is, $E = \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}$). This correspondence is constructed by taking a point $(x_i|y_i) \in \mathcal{X}_n^r(\alpha)$ and defining a parabolic Higgs field

$$\Phi(z) = \sum_{i=1}^n \frac{x_iy_i}{z - p_i} dz.$$

This also endows $\mathcal{X}_n^r(\alpha)$ with a Hitchin map $h : \mathcal{X}_n^r(\alpha) \to \bigoplus_{i=2}^r H^0(\mathbb{P}^1, \omega_X^{\otimes i}(D))$ defined by $h((x_i|y_i)) = (\mathrm{tr}(\Phi^2), \ldots, \mathrm{tr}(\Phi^r))$. In [16], this is exploited to show that for $r \leq 3$, $\mathcal{X}_n^r(\alpha)$ is a completely integrable system. However, this map is not proper; its fibres are open subvarieties of the fibres of the moduli space of rank r trace-free parabolic Higgs bundles on $\mathbb{P}^1$ (with respect to D) with no extra conditions on the underlying bundle. To be more specific, the fibres of $\mathcal{X}_n^r(\alpha)$ are homeomorphic to $(S^1)^n \setminus V$, where V is some compact subvariety. This is potentially problematic since Nakajima quiver varieties are hyperkähler and thus Calabi-Yau, and the SYZ conjecture [63] says that all Calabi-Yau manifolds are torus fibrations.

6.1.3 Twisted hyperpolygons

Let us now consider two possible ways in which the ideas surrounding hyperpolygon spaces could be extended, using ideas similiar to those explored in the main body of this thesis.

First, consider our "manifold-ification" philosophy from Chapter 1. Here, this amounts to replacing the category in which we are choosing representations and leads to a possible way of compactifying $\mathcal{X}_n^r(\alpha)$. Note that $\mathrm{Hom}(\mathbb{C}, \mathbb{C}^r) \cong \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}^{\oplus r})$. So, choosing representations in the category of holomorphically trivial bundles on $\mathbb{P}^1$ instead of the category of vector spaces does not actually change anything. However, it leads naturally to the idea of letting the rank r bundle assume other splitting types. Leaning on the correspondence with parabolic Higgs bundles, these are exactly the points "missing" from $\mathcal{X}_n^r(\alpha)$. If we can construct parabolic Higgs bundles directly from a star-shaped quiver (perhaps by utilizing the twisted category of bundles considered in the main body of the thesis), this could allow us to compactify the hyperpolygon space.

Such an idea could also help to understand whether there is a McKay-type correspondence between hyperpolygon moduli spaces and some certain subgroups of $SL(r, \mathbb{C})$. The usual McKay corresponence is part of a larger program which gives relationships between gravitational instantons[14], surfaces with Du Val singularities, finite subgroups of $SU(2)$, and appropriately labelled quivers of type ADE ([52]). The Nakajima quiver variety of the D-type

[14]A *gravitational instanton* is a non-compact, complete, hyperkähler 4-manifold which is asymptotically locally Euclidean (ALE), meaning that the hyperkähler metric decays with polynomial order 4 to the Euclidean metric as a radial coordinate tends to infinity.

quiver

$$\begin{tikzcd} & \bullet_1 \arrow[d] & \\ \bullet_1 \arrow[r] & \bullet_2 & \bullet_1 \arrow[l] \\ & \bullet_1 \arrow[u] & \end{tikzcd}$$

which is the hyperpolygon space $\mathcal{X}_4^2(\alpha)$, fits cleanly into this picture, but $\mathcal{X}_n^r(\alpha)$, which is not in a general a 4-manifold, does not.

Alternatively, note that hyperpolygons are analagous to traditional Higgs bundles in that they both require a choice of an element of a space involving specifically a cotangent bundle. Recall that $y_i \in T^*_{x_i}\mathrm{Rep}(Q)$, while a Higgs field Φ is a section of $\mathrm{End}E \otimes \omega_X \cong \mathrm{End}E \otimes T^*X$. Twisted Higgs bundles are obtained by twisting by an arbitrary line bundle. Can we apply the same generalization in the hyperpolygon case? On $\mathrm{Rep}(Q) \cong \mathbb{C}^r$, all bundles of the same rank are isomorphic, but we can expand our scope by letting y_i lie in bundles of different ranks and slightly adjusting the moment map equations.

Let $x_i \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^r) \cong \mathrm{Mat}_{r\times 1}(\mathbb{C})$ as usual and let $y_i \in \mathrm{Mat}_{s\times r}(\mathbb{C})$ for some $s \in \mathbb{N}$. That is, y_i lies in the fibre over x_i of the trivial bundle of rank sr on $\mathrm{Hom}(\mathbb{C}, \mathbb{C}^r)$. Now let

$$\mu_{\mathbb{C}}^s : (x_i|y_i) \longmapsto \Big(\sum_{i=1}^n (x_i A_s y_i)_0, A_s^T y_1 x_1, \ldots, A_s^T y_n x_n\Big)$$

where A_s is the $1 \times s$ matrix

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \end{pmatrix}.$$

This moment map takes values in $\mathfrak{sl}(r, \mathbb{C}) \oplus \mathbb{C}^n$ as before, and we can take the quotient with respect to level sets of $\mu_{\mathbb{R}}$ and $\mu_{\mathbb{C}}^s$ to yield the moduli space $\mathcal{X}_{r,s}^n(\alpha)$ of twisted hyperpolygons. This new space is not in general hyperkähler despite its construction via a hyperkähler-like quotient. As a corollary of the Marsden-Weinstein theorem, the real dimension of this moduli space can be calculated from dimension of the configuration space, the number of linearly independent moment map conditions, and the dimension of the group. We can calculate

$$\begin{aligned} \dim_{\mathbb{R}}(\mathcal{X}_{r,s}^n(\alpha)) &= n(2r + 2rs) - ((r^2 - 1) + n) - (2(r^2) + 2n) - ((r^2 - 1) + n) \\ &= 2rn(s + 1) - 4(n + r^2 - 1). \end{aligned}$$

Thus $\dim_{\mathbb{C}}(\mathcal{X}^n_{r,s}(\alpha)) = rn(s+1) - 2(n + r^2 - 1)$, and setting $s = 1$ recovers the expression for $\dim_{\mathbb{C}}(\mathcal{X}^n_r(\alpha))$.

It now becomes a natural question whether $\mathcal{X}^n_{r,s}(\alpha)$ can be identified with some subvariety of a (possibly twisted) parabolic Higgs moduli space, as is the case for $s = 1$.

6.2 Motivic methods

As previously mentioned, the theory of motives has been used to study the topological invariants of some (ω_X-twisted) Higgs bundle moduli spaces, namely in [21] and [20], building of work concerning the motives of the stack of vector bundles in [8]. The idea at play is that there is a ring homomorphism from (the dimensional completion of) the Grothendieck ring of varieties to $\mathbb{Z}[x,y][[\frac{1}{xy}]]$ given by the E-polynomial (see [33]). By calculating the class of a given variety in terms of simpler ones, we can read off its E-polynomial. The E-polynomial is a fairly fine invariant, and from it both the Betti and Hodge numbers of a variety can be extracted.

For k an arbitrary field, the *Grothendieck ring of varieties* $K_0(\mathrm{Var}_k)$ is the free abelian group on the set of isomorphism classes of varieties over k up to the equivalence $[X] \sim [X \setminus Y] + [Y]$ when Y is a closed subvariety of X. We work in the so-called *dimensional completion* $\hat{K}_0(\mathrm{Var}_k)$ of $K_0(\mathrm{Var}_k)[\frac{1}{[\mathbb{A}^1]}]$, in which the class of the affine line $\mathbb{A}^1$ is invertible.

The *zeta function* of a variety V is given by

$$Z(V,t) = \sum_{i \geq 0} [\mathrm{Sym}^i(V)] t^i \in \hat{K}_0(\mathrm{Var}_k)[[t]]$$

and one can further define, for a curve X of genus g,

$$P(X,t) = \sum_{i=0}^{2g} \mathrm{Sym}^i([X] - [\mathbb{P}^1]) t^i.$$

The goal is then (after translating to the language of quiver representations) to write the classes of the moduli spaces of representations of A-type quivers in $\mathrm{Bun}(X, \omega_X)$ in terms of $[X]$, $Z(X,t)$, and $P(X,t)$. As per Section 2.3.2, this allows for the calculation of the class of the whole space $\mathcal{M}_{X,\omega_X}(r,d)$.

In [21], this is done up to rank 4 (for odd degree) by use of the formula

$$\begin{aligned}\frac{1}{[\mathbb{A}^1]-1}[\mathcal{M}_{X,\omega_X}(Q)] &= [\mathbb{M}^{ss}_{X,\omega_X}(Q)] \\ &= [\mathbb{M}_{X,\omega_X}(Q)] - \sum \left\{\text{HN-strata of } \mathbb{M}_{X,\omega_X}(Q)\right\}\end{aligned}$$

where $\mathbb{M}_{X,\omega_X}(Q)$ is stack of representations of Q in $\mathrm{Bun}(X,\omega_X)$ and $\mathbb{M}^{ss}_{X,\omega_X}(Q)$ is the smooth substack of semi-stable representations. Furthermore, "HN-strata" refers to the Harder-Narasimhan stratification of $\mathbb{M}_{X,\omega_X}(Q)$, the stratification corresponding to different types of canonical destabilizing subrepresentations. There is an issue of convergence of $[\mathbb{M}_{X,\omega_X}(Q)]$ in $\hat{K}_0(\mathrm{Var}_k)$ (that is, it does not always define a class) which is addressed by a truncation procedure in [21] and by wall-crossing methods in [20].

These constructions could be extended to Higgs bundles with different twisting line bundles L, and more specifically to twisted Higgs bundles on curves of genus 0 and 1. It is possible that in the low genus case, the convergence problem is less troublesome, and hence recursive formulas for the classes may be found.

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