

Hyperreflexivity of the bounded n -cocycles spaces of Banach algebras

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ABSTRACT

The concept of hyperreflexivity has previously been defined for subspaces of $B(X, Y)$, where X and Y are Banach spaces. We extend this concept to the subspaces of $B^n(X, Y)$, the space of bounded n -linear maps from $X \times \cdots \times X = X^{(n)}$ into Y , for any $n \in \mathbb{N}$. If A is a Banach algebra and X a Banach A -bimodule, we obtain sufficient conditions under which $\mathcal{Z}^n(A, X)$, the space of all bounded n -cocycles from A into X , is hyperreflexive. To do so, we define two notions related to a Banach algebra: The strong property (\mathbb{B}) and bounded local units (b.l.u.). We show that there are sufficiently many Banach algebras which have both properties. We will prove that all C^* -algebras and group algebras have the strong property (\mathbb{B}) . We also prove that finite CSL algebras and finite nest algebras have this property. We further show that for an arbitrary Banach algebra A and each $n \geq 2$, $M_n(A)$ has the strong property (\mathbb{B}) whenever it is equipped with a Banach algebra norm. In particular, this implies that all Banach algebras are embedded into a Banach algebra with the strong property (\mathbb{B}) . With regard to bounded local units, we show that all C^* -algebras and many group algebras have b.l.u. We investigate the hereditary properties of both notions to construct more example of Banach algebras with these properties. We apply our approach and show that the bounded n -cocycle spaces related to Banach algebras with the strong property (\mathbb{B}) and b.l.u. are hyperreflexive provided that the space of the corresponding $n + 1$ -coboundaries are closed. This includes nuclear C^* -algebras, many group algebras, matrix spaces of certain Banach algebras and finite CSL and nest algebras. We finish the thesis with introducing *the hyperreflexivity constant*. We make our results more precise with finding an upper bound for the hyperreflexivity constant of the bounded n -cocycle spaces.

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CHAPTER 1

INTRODUCTION

In this chapter we give a brief history on reflexivity and hyperreflexivity of linear space of (bounded) operators. As it will be presented, both notions of reflexivity and hyperreflexivity have been defined in different ways on linear maps between Banach spaces. We present all such approaches and give details of how the various notions of reflexivity (hyperreflexivity) are connected. The concept of reflexivity has its root in the problem of invariant subspaces. Some information on the invariant subspace problem and its connection with reflexivity is provided. In the final section, it is briefly described how we approach to the problem of the hyperreflexivity of the bounded n -cocycle spaces in the thesis.

As the name suggests, the notion of hyperreflexivity is a strengthening of the concept of reflexivity. Hence we first present a general history of reflexivity in order to have a better idea of hyperreflexivity.

1.1 Reflexivity

The concept of reflexivity for algebras of bounded operators on Banach spaces has its origin in operator theory.

Definition 1.1.1. Let X be a Banach space, and let $A \subseteq B(X)$ be an algebra of bounded operators on X with unit (i.e. $id_X \in A$).

(1) $Lat A$ denotes the set of all closed subspaces of X invariant under A , i.e., for every

$T \in A$ and $I \in LatA$ we have $T(I) \subseteq I$.

(2) algebra generated by $LatA$ is defined to be the set of all $T \in B(X)$ such that $T(I) \subseteq I$ for each $I \in LatA$. This is denoted by $algLatA$.

(3) A is said to be reflexive if $algLatA = A$.

This definition which is historically the start of the topic of reflexivity was first defined by D. Sarason [56]. During the past decades, the problem of reflexivity has widely been studied by various authors especially for the operator algebras, i.e., the case where $X = H$ is a Hilbert space. In particular, two important classes of operator algebras which are known to be reflexive are:

(i) CSL algebras: Arveson proved in [5] that algebras generated by commutative subspace lattices or briefly CSL algebras are reflexive.

(ii) von Neumann algebras: It follows easily from the double-commutant Theorem that every von Neumann algebra is reflexive.

The concept of reflexivity in the sense of Definition 1.1.1 is closely related to the well-known invariant subspace problem:

“Whether a bounded operator $T \in B(X)$ has a non-trivial invariant subspace?”

Invariant subspace problem and reflexivity. Invariant subspace problem is probably among the most important problems in functional analysis. It is known that a large number of operators on Hilbert spaces have non-trivial invariant subspaces. Let H be a Hilbert space:

(i) If H is finite dimensional with $dimH > 1$, then each operator T on H is a matrix which is known to have at least a non-zero eigenvector v . If we let $L = \{\alpha v : \alpha \in \mathbb{C}\}$, then L is a non-trivial invariant subspace of T .

(ii) If H is not separable, then for each bounded operator T on H , the following is a non-trivial invariant subspace of H :

$$L = \overline{\text{span}}\{T^n(x) : n \geq 0\},$$

where x is a non-zero vector and $\overline{\text{span}}$ stands for the closed linear span.

(iii) If H is an arbitrary Hilbert space, then it is shown in [44] that each normal operator

on H has a non-trivial invariant subspace.

On the other hand, it was shown by Charles Reed that there is a bounded operator on the space l^1 without a non-trivial invariant subspace [46].

Definition 1.1.2. Let X and Y be two Banach spaces. The weak operator topology in $B(X, Y)$ is the topology defined by the basic set of neighborhoods

$$N(T; A, B, \epsilon) = \{R \mid R \in B(X, Y), |y^*((T - R)x)| < \epsilon, y^* \in B, x \in A\}$$

where A and B are arbitrary finite subsets of X and Y^* respectively and $\epsilon > 0$ is arbitrary.

The invariant subspace problem and reflexivity are connected in the following way: Let X be a Banach space with $\dim X > 1$. Let $T \in B(X)$ and define

$$E = \overline{\text{alg}\{id_X, T\}}^{w.o.t},$$

where id_X is the identity operator and $w.o.t$ stands for the weak operator topology. If T does not have a non-trivial invariant subspace, then E is not reflexive. The reason is that clearly

$$\text{Lat}T = \text{Lat}E = \{0, X\}.$$

Hence

$$\text{algLat}E = B(X).$$

On the other hand, it follows easily that $T \in E'$, the commutant of E . However, $T \notin B(X)' = \mathbb{C}id_X$ since it has no non-trivial invariant subspace and $\dim X > 1$. Consequently, $\text{algLat}E \neq E$ and E is not reflexive.

Generalization of reflexivity. The concept of reflexivity was generalized by D.R. Larson both algebraically and topologically to the subspaces of $B(X, Y)$ for Banach spaces X and Y [32].

Definition 1.1.3. Let X and Y be Banach spaces. Let $L(X, Y)$ be the space of all linear operators from X to Y and $B(X, Y)$ be the subspace of $L(X, Y)$ consisting of all bounded

operators. Suppose that \mathcal{S} is a linear subspace of $L(X, Y)$. For each $x \in X$, we define

$$\mathcal{S}(x) = \{S(x) : S \in \mathcal{S}\},$$

and we let $[\mathcal{S}(x)]$ to be the norm-closure of $\mathcal{S}(x)$. Put

$$ref_a(\mathcal{S}) = \{T \in L(X, Y) : T(x) \in \mathcal{S}(x), \text{ for each } x \in X\}.$$

If $\mathcal{S} \subseteq B(X, Y)$, put

$$ref(\mathcal{S}) = \{T \in B(X, Y) : T(x) \in [\mathcal{S}(x)], \text{ for each } x \in X\}.$$

(1) If $\mathcal{S} \subseteq L(X, Y)$, then \mathcal{S} is algebraically reflexive if $\mathcal{S} = ref_a(\mathcal{S})$.

(2) If $\mathcal{S} \subseteq B(X, Y)$, then \mathcal{S} is (topologically) reflexive if $\mathcal{S} = ref(\mathcal{S})$.

Definition 1.1.4. Let X and Y be two Banach spaces. The strong topology in $B(X, Y)$ is the topology defined by the basic set of neighborhoods

$$N(T; A, \epsilon) = \{R \mid R \in B(X, Y), \|(T - R)x\| < \epsilon, x \in A\}$$

where A is an arbitrary finite subset of X and $\epsilon > 0$ is arbitrary.

Remark 1.1.5. It is shown in [18, Corollary VI.5] that a convex subset in $B(X, Y)$ has the same closure in the weak and strong operator topology. On the other hand, it is easy to check that if \mathcal{S} is a subspace of $B(X, Y)$, then $ref(\mathcal{S})$ is closed in the strong operator topology. Consequently, the initial condition for \mathcal{S} to be reflexive is that it should be closed in the strong operator topology and hence in the weak operator topology.

Lemma 1.1.6. *Let \mathcal{S} be a unital subalgebra of $B(X)$. Then \mathcal{S} is reflexive in the sense of Definition 1.1.1 if and only if it is (topologically) reflexive in the sense of Definition 1.1.3.*

Proof. It suffices to show that

$$algLat\mathcal{S} = ref(\mathcal{S}).$$

First let $T \in B(X)$ with the property that for each $x \in X$, there is a sequence $\{T_{n,x}\}$ in \mathcal{S} with

$$T(x) = \lim_{n \rightarrow \infty} T_{n,x}(x).$$

Hence if $I \in \text{Lat}\mathcal{S}$, then for each $x \in I$ we have

$$T(x) = \lim_{n \rightarrow \infty} T_{n,x}(x) \in I.$$

This implies that $T(I) \subseteq I$, and so

$$T \in \text{algLat}\mathcal{S}.$$

Therefore

$$\text{ref}(\mathcal{S}) \subseteq \text{algLat}\mathcal{S}. \tag{1.1.1}$$

Let $T \in \text{algLat}\mathcal{S}$ and pick $v_0 \in X$. Define

$$E = \overline{\{Sv_0 : S \in \mathcal{S}\}}.$$

Then $E \in \text{Lat}\mathcal{S}$. So $TE \subseteq E$. Since \mathcal{S} is unital, $v_0 \in E$. Hence $Tv_0 \in E$. So that there exists a sequence $\{S_{n,v_0}\}$ in \mathcal{S} such that

$$Tv_0 = \lim_{n \rightarrow \infty} S_{n,v_0}v_0.$$

Consequently, $T \in \text{ref}(\mathcal{S})$. □

To generalize the notion of reflexivity, Larson was partly motivated to study the local behavior of derivations from a Banach algebra A into a Banach A -module X . Let $Z^1(A, X)$ be the space of all derivations from A into X . A “*local derivation*” from A into X is an element of $\text{ref}_a(Z^1(A, X))$. A natural question one could consider is for which Banach algebra A and Banach A -bimodule X , local derivations from A into X are derivations? This is equivalent to asking whether $Z^1(A, X)$ is algebraically reflexive? One can also ask the topological version of this question. Let $\mathcal{Z}^1(A, X)$ be the space of bounded derivations from A into X .

“When $\mathcal{Z}(A, X)$ is topologically reflexive?”

If the answer is positive, then every so-called approximate local derivation from A into X would be a derivation.

In the last two decades, the question of algebraic and topological reflexivity of the derivation spaces has received considerable attention from various researchers and some very interesting results have been obtained. In [31], R.D. Kadison showed that bounded local derivations from a von Neumann algebra into any of its dual bimodules are derivations. Kadison’s result was generalized later by showing that the space of bounded derivations from a C^* -algebra into any of its bimodules is both algebraically and topologically reflexive [29, 50]. On the other hand, it was shown in [33] that every local derivation on $B(X)$, for a Banach space X , is a derivation. In [52], E. Samei showed that the space of bounded derivations from $L^1(G)$ into any Banach $L^1(G)$ -bimodule is reflexive provided that G has an open subgroup of polynomial growth. This class includes IN-groups, maximally almost periodic groups, and totally disconnected groups.

1.1.1 A general view of reflexivity

The notion of reflexivity (algebraic reflexivity) as we presented in the preceding section is defined for the subspaces of $B(X, Y)$ ($L(X, Y)$). In 1994, Don Hadwin introduced the concept of E -reflexivity for arbitrary vector spaces [23]. His work is interesting in various aspects:

- (1) It generalizes the concept of reflexivity to arbitrary vector spaces.
- (2) When dealing with the spaces $L(X, Y)$ and $B(X, Y)$, we can define both notions of algebraic and topological reflexivity in terms of E -reflexivity.
- (3) We recall that a Banach space X is called (classically) reflexive if the map $\theta : X \rightarrow X^{**}$ is surjective where

$$\theta(x)(\varphi) = \varphi(x), \quad (\varphi \in X^*, x \in X).$$

In his paper, Don Hadwin rephrased classical reflexivity in terms of E -reflexivity. Hence his work unifies different versions of reflexivity that were already defined.

In this section, we give definition of E -reflexivity and show how this notion coincide with other notions of reflexivity. The reference for our results is [23].

Let X be a vector space and suppose that Y is a set of linear maps from X to \mathbb{C} that separates the points of X . For $A \subseteq X$, we define

$$A^\perp = \{f \in Y : f|_A = 0\}.$$

For $B \subseteq Y$, define

$$B_\perp = \cap \{\ker f : f \in B\}.$$

Suppose that $\emptyset \neq E \subseteq Y$ is closed under scalar multiplication and that $E_\perp = \{0\}$. We call (X, Y, E) a reflexivity triple. Let \mathcal{S} be a subset of X . We define E -reflexive part of \mathcal{S} to be

$$ref_E(\mathcal{S}) = (S^\perp \cap E)_\perp.$$

Hence $x \in ref_E(\mathcal{S})$ if and only if for each $f \in E$ with $f|_{\mathcal{S}} = 0$, we have $f(x) = 0$.

Definition 1.1.7. Let (X, Y, E) be a reflexivity triple. A subset \mathcal{S} of X is called E -reflexive if $ref_E(\mathcal{S}) = \mathcal{S}$.

E -Reflexivity and classical reflexivity:

Suppose that Y is a Banach space, and let $X = Y^*$. Then Y can be viewed as a set of linear maps on X that separates the points of X . Let $E = Y$. If $\mathcal{S} \subseteq X$, then

$$ref_E(\mathcal{S}) = (S^\perp \cap E)_\perp = (S^\perp)_\perp = \overline{\text{span}}^{w^*}(\mathcal{S}). \quad (1.1.2)$$

Here $\overline{\text{span}}^{w^*}(\mathcal{S})$ stands for the weak*-closure of the linear span of \mathcal{S} . The last equation in (1.1.2) is proven in [23, Proposition 1.1]

Lemma 1.1.8. *Y is classically reflexive if and only if every norm closed linear subspace of X is E -reflexive.*

Proof. Assume that Y is classically reflexive, i.e., $Y^{**} = Y$. Then the weak*-topology on $X = Y^*$ is exactly the weak topology induced by Y^{**} . We know that in every normed space, the norm-closure and the weak closure of a convex subset coincide. Hence on account of (1.1.2), for every normed closed linear subspace \mathcal{S} of X , we have

$$\text{ref}_E(\mathcal{S}) = \overline{\mathcal{S}}^{w^*} = \overline{\mathcal{S}}^w = \overline{\mathcal{S}}^{\|\cdot\|} = \mathcal{S}.$$

Conversely, let \mathcal{S} be a closed subspace of X . Then by the assumption, \mathcal{S} is E -reflexive. Hence, on account of (1.1.2), we have

$$\overline{\mathcal{S}}^{w^*} = \overline{\mathcal{S}}^w = \mathcal{S}. \tag{1.1.3}$$

On the other hand, we know that ([47, Theorem 3.10])

$$Y = \{y^{**} \in Y^{**} : y^{**} \text{ is } w^*\text{-continuous on } Y^*\}. \tag{1.1.4}$$

Now pick $y^{**} \in Y^{**}$. It is trivial that y^{**} is w -continuous. Hence $\ker y^{**}$ is w -closed. If we apply (1.1.3), we infer that $\ker y^{**}$ is w^* -closed. It means that y^{**} is w^* -continuous. Consequently, $y^{**} \in Y$ by (1.1.4). Therefore, $Y^{**} = Y$ and Y is (classically) reflexive. \square

E -reflexivity and algebraic reflexivity

Let V and W be two Banach spaces. A rank one-tensor is of the form $x \otimes \alpha$ for $x \in V$ and $\alpha \in W'$, where W' denotes the set of all linear functionals on W . Every rank-one tensor acts as a functional on $L(V, W)$ by

$$(x \otimes \alpha)(T) = \alpha(T(x)) \quad (T \in L(V, W)).$$

Lemma 1.1.9. *Let E be the set of all rank-one tensors, and let $Y = \text{span}E$. If \mathcal{S} is a linear subspace of $L(V, W)$, then $\text{ref}_E(\mathcal{S}) = \text{ref}_a(\mathcal{S})$. In particular, \mathcal{S} is algebraically reflexive if and only if it is E -reflexive.*

Proof. Let $T \in \text{ref}_a(\mathcal{S})$ and suppose that $x \in V$ and $\alpha \in W'$ are chosen such that

$$(x \otimes \alpha)(S) = \alpha(S(x)) = 0 \quad \forall S \in \mathcal{S}.$$

Let $S_x \in \mathcal{S}$ be such that $T(x) = S_x(x)$. Then $(x \otimes \alpha)(T) = \alpha(T(x)) = \alpha(S_x(x)) = 0$. Hence $T \in \text{ref}_E(\mathcal{S})$.

Conversely, suppose that $T \in \text{ref}_E(\mathcal{S})$, and let $x \in V$. By the assumption, for each $\alpha \in W'$ with the property that

$$(x \otimes \alpha)(S) = \alpha(S(x)) = 0 \quad \forall S \in \mathcal{S},$$

We should have

$$(x \otimes \alpha)(T) = \alpha(T(x)) = 0.$$

This implies that $T(x) \in \mathcal{S}(x)$. Hence $T \in \text{ref}_a(\mathcal{S})$. □

***E*-reflexivity and topological reflexivity:**

Let V and W be two Banach spaces. Define

$$E = \{x \otimes \alpha : x \in V, \alpha \in W^*\},$$

where W^* denotes the space of bounded linear functionals on W . Let $Y = B(V, W)^*$.

Lemma 1.1.10. *Let \mathcal{S} be a subspace of $B(V, W)$. Then $\text{ref}_E(\mathcal{S}) = \text{ref}(\mathcal{S})$. In particular, \mathcal{S} is topologically reflexive if and only if it is *E*-reflexive.*

Proof. Let $T \in \text{ref}(\mathcal{S})$. Suppose that $x \in V$ and $\alpha \in W^*$ are chosen such that

$$(x \otimes \alpha)(S) = \alpha(S(x)) = 0 \quad (S \in \mathcal{S}).$$

If $\{S_{n,x}\}$ is a sequence in \mathcal{S} with $T(x) = \lim_{n \rightarrow \infty} S_{n,x}(x)$, then

$$(x \otimes \alpha)(T) = \lim_{n \rightarrow \infty} \alpha(S_{n,x}(x)) = 0.$$

So $T \in \text{ref}_E(\mathcal{S})$.

Conversely, let $T \in \text{ref}_E(\mathcal{S})$. Then for each $x \in V$ and $\alpha \in W^*$ with $x \otimes \alpha|_{\mathcal{S}} = 0$, we have $(x \otimes \alpha)(T) = 0$. Equivalently, we can say that for each $x \in V$ if $\alpha \in W^*$ is such that

$$\alpha(S(x)) = 0 \quad \forall S \in \mathcal{S},$$

then $\alpha(T(x)) = 0$. By the Hahn-Banach theorem, this implies that $T(x) \in \overline{\mathcal{S}(x)}$. Hence $T \in \text{ref}(\mathcal{S})$. \square

1.2 Hyperreflexivity

As mentioned before, the concept of hyperreflexivity is a strengthening of reflexivity. This concept was first introduced in [6] by W. B. Arveson for operator algebras for which it was named “the distance formula problem”. Let H be a Hilbert space and \mathcal{S} a closed unital subalgebra of $B(H)$. Let $d(\cdot, \mathcal{S})$ be the quotient norm on $B(H)/\mathcal{S}$ defined by

$$d(T, \mathcal{S}) = \inf\{\|T - S\| : S \in \mathcal{S}\}.$$

We can also define the following seminorm on $B(H)/\mathcal{S}$,

$$\beta(T, \mathcal{S}) = \sup\{\|P^\perp T P\| : P \in \text{Lat}\mathcal{S}\}. \quad (1.2.1)$$

Here, we identify each element $I \in \text{Lat}(\mathcal{S})$ with its orthogonal projection $P : H \rightarrow I$. It is clear that

$$\beta(T, \mathcal{S}) \leq d(T, \mathcal{S}), \quad T \in B(H).$$

Moreover it is easy to check that $T \in \text{algLat}\mathcal{S}$ if $P^\perp T P = 0$. Hence β defines a norm on $B(H)/\mathcal{S}$ if and only if \mathcal{S} is reflexive.

Definition 1.2.1. A closed unital subalgebra \mathcal{S} of $B(H)$ is called **hyperreflexive** (or C-hyperreflexive) if there is a constant $C > 0$ such that

$$\text{dist}(T, \mathcal{S}) \leq C\beta(T, \mathcal{S}), \quad (T \in B(H)).$$

The smallest such constant is called the hyperreflexivity constant of \mathcal{S} .

There are many results on the hyperreflexivity of the operator algebras. For example, the following two are among the most well-known ones.

(i) Arveson proved in [6] that every nest algebra has a distance formula, and so, it is hyperreflexive. His result provides a very rare example of hyperreflexive operator algebras for which the hyperreflexivity constant is 1. Hence norms d and β coincide in this case.

(ii) There are various results on the hyperreflexivity of different classes of von Neumann algebras mainly due to the works of E. Christensen (see [11] and [12]). In particular, he proved in [12] that every injective von Neumann algebra is hyperreflexive and that its hyperreflexivity constant is less than 4. In [22], Giol provided a unified argument under which many known results on the hyperreflexivity of von Neumann algebras were reobtained.

As the concept of reflexivity was generalized from the operator algebras to the subspaces of $B(X, Y)$, the concept of hyperreflexivity was also generalized in the same manner. Let X and Y be two Banach spaces, and let \mathcal{S} be a closed linear subspace of $B(X, Y)$. For every $T \in B(X, Y)$, we define

$$\text{dist}(T, \mathcal{S}) = \inf_{S \in \mathcal{S}} \|T - S\|,$$

and

$$\text{dist}_r(T, \mathcal{S}) = \sup_{\|x\| \leq 1} \inf_{S \in \mathcal{S}} \|T(x) - S(x)\|.$$

It is clear that

$$\text{dist}_r(T, \mathcal{S}) \leq \text{dist}(T, \mathcal{S}).$$

In general, dist_r defines a seminorm on $B(X, Y)/\mathcal{S}$. Moreover, it gives a norm if and only if \mathcal{S} is reflexive.

Definition 1.2.2. A closed subspace \mathcal{S} of $B(X, Y)$ is called hyperreflexive if there is $C > 0$ such that

$$\text{dist}(T, \mathcal{S}) \leq C \text{dist}_r(T, \mathcal{S}) \quad (T \in B(X, Y)). \quad (1.2.2)$$

In other words, \mathcal{S} is hyperreflexive if $\text{dist}(\cdot, \mathcal{S})$ and $\text{dist}_r(\cdot, \mathcal{S})$ define equivalent norms on $B(X, Y)/\mathcal{S}$. We will show in Proposition 1.2.7 that this coincides with Definition 1.2.1 when $X = Y = H$ is a Hilbert space and \mathcal{S} is a closed unital subalgebra of $B(H)$. It is easy to check that the closed subspace \mathcal{S} is reflexive if

$$\text{dist}_r(T, \mathcal{S}) = 0 \Rightarrow \text{dist}(T, \mathcal{S}) = 0.$$

Hence according to (1.2.2), being hyperreflexive is stronger than being reflexive. K. Davidson and S. Power showed in [16] that the converse might not be true, i.e. not every reflexive operator algebra is hyperreflexive. However, in the special case when a reflexive subspace of $B(X, Y)$ is finite dimensional (for possibly infinite dimensional Banach spaces X and Y), then it has to be hyperreflexive [38].

In recent years, several authors have also considered the hyperreflexivity of the derivation spaces. In [58], V. Shulman showed that $\mathcal{Z}^1(A, A)$, the space of bounded derivations from a C^* -algebra A into itself is hyperreflexive if the second Hochschild cohomology group vanishes ($\mathcal{H}^2(A, A) = 0$). It is also shown in [3] that $\mathcal{Z}^1(L^1(G), L^1(G))$ is hyperreflexive for each amenable group in [SIN]. In [52], E. Samei showed that $\mathcal{Z}^1(L^1(G), X^*)$ is hyperreflexive if G is a locally compact amenable group with an open subgroup which has polynomial growth and X is an essential Banach $L^1(G)$ -bimodule. It was later proven in [4] that $\mathcal{Z}^1(L^1(G), L^1(G))$ is hyperreflexive for each locally compact group with an open subgroup which has polynomial growth, thus eliminating the assumption of amenability. More results on reflexivity and hyperreflexivity can be found in [9, 14, 24, 25, 26, 27, 28, 29, 31, 33, 50, 51, 57, 58].

1.2.1 A general view of hyperreflexivity

The definition of hyperreflexivity as we presented in Section 1.1 is valid for the subspaces of $B(V, W)$. In Section 1.1.1, we demonstrated how Hadwin generalized the concept of reflexivity. In his paper, he also defined the concept of E -hyperreflexivity which generalizes hyperreflexivity [23].

Let X be a Banach space and Y a subspace of X^* separating the points of X . Suppose

that E is a nonempty subset of Y which is closed under scalar multiplication such that $E_{\perp} = \{0\}$. Let

$$\tilde{E} = \{e \in E : \|e\| = 1\}.$$

For a subspace \mathcal{S} of X , we define

$$d_Y(x, \mathcal{S}) = \sup\{|f(x)| : f \in \mathcal{S}^{\perp}, \|f\| = 1\},$$

and

$$d_E(x, \mathcal{S}) = \sup\{|f(x)| : f \in \mathcal{S}^{\perp} \cap \tilde{E}\}.$$

Obviously we have

$$d_E(x, \mathcal{S}) \leq d_Y(x, \mathcal{S}).$$

Definition 1.2.3. A subspace \mathcal{S} of X is said to be E -hyperreflexive if there is a constant $C > 0$ such that

$$d_Y(x, \mathcal{S}) \leq C d_E(x, \mathcal{S}), \quad x \in X.$$

Remark 1.2.4. If \mathcal{S} is a closed subspace, then being E -hyperreflexive is stronger than being E -reflexive, i.e., if \mathcal{S} is E -hyperreflexive, then it is E -reflexive in the sense of Definition 1.1.7. To see this, note that we have

$$\begin{aligned} d_E(x, \mathcal{S}) = 0 &\Leftrightarrow f(x) = 0 \quad \forall f \in \tilde{E} \cap \mathcal{S}^{\perp} \\ &\Leftrightarrow f(x) = 0 \quad \forall f \in E \cap \mathcal{S}^{\perp} \\ &\Leftrightarrow x \in (E \cap \mathcal{S}^{\perp})_{\perp} = \text{ref}_E(\mathcal{S}). \end{aligned}$$

On the other hand,

$$\begin{aligned} d_Y(x, \mathcal{S}) = 0 &\Leftrightarrow f(x) = 0 \quad \forall f \in \mathcal{S}^{\perp} \\ &\Leftrightarrow x \in \mathcal{S}. \end{aligned}$$

E -hyperreflexivity and hyperreflexivity

In this Section, we show that Hadwin's approach toward E -hyperreflexivity generalized the hyperreflexivity in the sense of Definition 1.2.1.

Proposition 1.2.5. *Let V and W be two Banach spaces. Let $X = B(V, W)$, $Y = B(V, W)^*$ and let*

$$E = \{x \otimes \alpha : x \in V, \alpha \in W^*\}.$$

For $T \in B(V, W)$ and a subspace $\mathcal{S} \subseteq B(V, W)$, we have:

(i) $d_Y(T, \mathcal{S}) = \text{dist}(T, \mathcal{S})$.

(ii) $d_E(T, \mathcal{S}) = \text{dist}_r(T, \mathcal{S})$.

In particular, for any closed subspace of $B(V, W)$, being hyperreflexive is equivalent to being E -hyperreflexive.

Proof. (i) Let $f \in \mathcal{S}^\perp$ with $\|f\| = 1$. Then for each $T \in B(V, W)$, we have

$$|f(T)| = |f(T) - f(S)| \leq \|T - S\| \quad (\forall S \in \mathcal{S}).$$

Therefore

$$d_Y(T, \mathcal{S}) \leq \text{dist}(T, \mathcal{S}).$$

To prove the other way around, let $T \in B(V, W)$. If $T \in \overline{\mathcal{S}}$, trivially we have

$$\text{dist}(T, \mathcal{S}) \leq d_Y(T, \mathcal{S}).$$

If $T \notin \overline{\mathcal{S}}$, then by the Hahn-Banach theorem, there exists $f \in X^*$ such that

$$\|f\| = 1, \quad f|_{\mathcal{S}} = 0, \quad f(T) = \|T\|.$$

This implies that

$$\text{dist}(T, \mathcal{S}) \leq d_Y(T, \mathcal{S}).$$

(ii) Let $x \in V$ and $\alpha \in W^*$ be such that

$$\|x \otimes \alpha\| = \|\alpha\| \|x\| = 1, \quad \alpha(S(x)) = 0 \quad \forall S \in \mathcal{S}.$$

Then, for every $S \in \mathcal{S}$, we have

$$\begin{aligned} \|(x \otimes \alpha)(T)\| &= |\alpha(T(x))| \\ &= |\alpha(T(x) - S(x))| \\ &\leq \|T(x) - S(x)\| \|\alpha\| \\ &= \|T(x\|\alpha\|) - S(x\|\alpha\|)\|. \end{aligned}$$

Hence

$$|(x \otimes \alpha)(T)| \leq \inf_{S \in \mathcal{S}} \|T(x\|\alpha\|) - S(x\|\alpha\|)\|.$$

This implies that

$$d_E(T, \mathcal{S}) \leq \text{dist}_r(T, \mathcal{S}).$$

To prove the converse, it suffices to show that for each $x_1 \in V$ with $\|x_1\| = 1$, there are $x_0 \in V$ and $\alpha_0 \in W^*$ such that

$$\|\alpha_0\|\|x_0\| = 1, \quad \alpha_0(S(x_0)) = 0 \quad \forall S \in \mathcal{S}$$

and

$$\inf_{S \in \mathcal{S}} \|T(x_1) - S(x_1)\| \leq |\alpha_0(T(x_0))|.$$

Now if $T(x_1) \in \overline{\mathcal{S}(x_1)}$, then clearly for every α_0 and x_0 with $\|\alpha_0\|\|x_0\| = 1$, the inequality holds. Otherwise, by the Hahn-Banach theorem, there is $\alpha_0 \in W^*$ such that $\|\alpha_0\| = 1$, $\alpha_0(\mathcal{S}(x_1)) = 0$ and $\alpha_0(T(x_1)) = \|T(x_1)\|$ and $\|\alpha_0\| = 1$. Hence it suffices to let $x_0 = x_1$. \square

1.2.2 Two definitions of hyperreflexivity for the operator algebras coincide

In this section, we show that Definition 1.2.1 and Definition 1.2.2 are equivalent. Suppose that X is a Banach space. Let \mathcal{S} be a closed unital subalgebra of $B(X)$. Define

$$\beta(T, \mathcal{S}) = \sup\{\|\pi_M T|_M\| : M \in \text{Lat}\mathcal{S}\}, \tag{1.2.3}$$

where $\pi_M : X \rightarrow \frac{X}{M}$ is the quotient map.

Remark 1.2.6. Let $X = H$ be a Hilbert space. Suppose that M is a closed subspace of H , and let P denote the orthogonal projection onto M . Then it is easy to check that

$$\|\pi_M T|_M\| = \|(1 - P)TP\|, \quad (T \in B(H)).$$

In particular, definition of β given by (1.2.3) coincides with that of (1.2.1).

Proposition 1.2.7. *Let X be a Banach space. Suppose that \mathcal{S} is a closed unital subalgebra of $B(X)$, and*

$$E = \{x \otimes \alpha : x \in X, \alpha \in X^*\}.$$

Then

$$d_E(T, \mathcal{S}) = \beta(T, \mathcal{S}) \quad (T \in B(X)).$$

In particular, Definition 1.2.1 and Definition 1.2.2 coincide when X is a Hilbert space and \mathcal{S} is a unital operator algebra.

Proof. First we show that

$$d_E(T, \mathcal{S}) \leq \beta(T, \mathcal{S}). \tag{1.2.4}$$

Equivalently, we need to show that for each $x \otimes \alpha \in E$ with $\|\alpha\| = \|x\| = 1$ and

$$(x \otimes \alpha)|_{\mathcal{S}} = 0, \tag{1.2.5}$$

there is $M \in Lat\mathcal{S}$ such that

$$|(x \otimes \alpha)(T)| \leq \|\pi_M T|_M\|.$$

To this end, we let $M = \mathcal{S}(x)$. Then $M \in Lat\mathcal{S}$. Moreover, for every $m \in M$, according to (1.2.5), $\alpha(m) = 0$. So

$$\begin{aligned} |(x \otimes \alpha)(T)| &= |\alpha(T(x))| \\ &= |\alpha(T(x) - m)| \\ &\leq \|T(x) - m\|. \end{aligned}$$

Consequently we can write (note that $x \in M$ since \mathcal{S} is unital)

$$\begin{aligned} |(x \otimes \alpha)(T)| &\leq \inf_{m \in M} \|T|_M(x) - m\| \\ &= \|\pi_M T|_M(x)\| \\ &\leq \|\pi_M T|_M\|. \end{aligned}$$

Next we show that

$$\beta(T, \mathcal{S}) \leq d_E(T, \mathcal{S}). \quad (1.2.6)$$

To do so, we need to show that for each $M \in \text{Lat}\mathcal{S}$ and each $\epsilon > 0$, there are $\alpha_\epsilon \in X^*$ and $x_\epsilon \in X$ with $\|\alpha_\epsilon\|, \|x_\epsilon\| = 1$ and

$$(x_\epsilon \otimes \alpha_\epsilon)|_{\mathcal{S}} = \alpha_\epsilon(\mathcal{S}(x_\epsilon)) = 0$$

such that

$$\|\pi_M T|_M\| \leq |\alpha_\epsilon(T(x_\epsilon))| + \epsilon. \quad (1.2.7)$$

Note that

$$\|\pi_M T|_M\| = \sup\{\|\pi_M T|_M(x)\| : x \in M, T(x) \notin M \text{ and } \|x\| = 1\}.$$

Hence for $\epsilon > 0$, there is $x_\epsilon \in M$ with $T|_M(x_\epsilon) \notin M$ such that

$$\|\pi_M T|_M\| \leq \|\pi_M T|_M(x_\epsilon)\| + \epsilon. \quad (1.2.8)$$

Since $T|_M(x_\epsilon) \notin M$, the Hahn-Banach Theorem implies that there is $\alpha_\epsilon \in V^*$ with $\|\alpha_\epsilon\| = 1$ and $\alpha_\epsilon|_M = 0$ such that

$$\alpha_\epsilon(T(x_\epsilon)) = \|T|_M(x_\epsilon)\|. \quad (1.2.9)$$

Now by applying (1.2.9),(1.2.8) and using the fact that $\|\pi_M T|_M(x_\epsilon)\| \leq \|T|_M(x_\epsilon)\|$, we obtain (1.2.7). Note that since $M \in \text{Lat}\mathcal{S}$ and $x_\epsilon \in M$, we get

$$\alpha_\epsilon(\mathcal{S}(x_\epsilon)) = 0,$$

as desired. Now (1.2.4) and (1.2.6) imply that

$$d_E(T, \mathcal{S}) = \beta(T, \mathcal{S}).$$

The final result follows by combining the preceding results together with Proposition 1.2.5. □

1.2.3 Our approach to hyperreflexivity

Hyperreflexivity is a powerful tool that allows us to measure the “global” distance of an element to a linear space using its “local” distance. One important subspace related to a given Banach algebra is the space of bounded derivations whose hyperreflexivity for various cases have been studied extensively. The counterparts of bounded derivations in higher dimensions are bounded n -cocycles which play a fundamental role in the cohomology of Banach algebras. Our main goal in this thesis is to extend the concept of hyperreflexivity to these higher cocycles. For Banach spaces X and Y , we first define hyperreflexivity for subspaces of $B^n(X, Y)$. Then we focus on $\mathcal{Z}^n(A, X)$, the space of bounded n -cocycles from a Banach algebra A into a Banach A -bimodule X , and pose the question when it can be hyperreflexive. Our investigation leads us to find sufficient conditions under which $\mathcal{Z}^n(A, X)$ becomes hyperreflexive. Hence answering this question improves our knowledge of the cohomology of Banach algebras. We demonstrate that for a large classes of Banach algebras, including nuclear C^* -algebra, group algebras of amenable groups with open subgroups of polynomial growth, finite CSL and finite nest algebras and matrix spaces of some Banach algebras, these sufficient conditions hold which give evidence that our conditions are satisfactory. For the case when $n = 1$, our results include and, at the same time, generalize all the ones already obtained in the literature pointed out in the section 1.2.

As it is customary, we first present in Chapter 2 some backgrounds of notions that will be needed in the following chapters.

A tool that plays a key role in our discussion is a property related to a Banach algebra which we call it *the strong property* (\mathbb{B}) (Definition 3.1.1). In [2], without defining this property explicitly, it is shown that all C^* -algebras and group algebras have this property. We devote Chapter 3 to the strong property (\mathbb{B}) . We study the hereditary properties of Banach algebras with the strong property (\mathbb{B}) which enables us to construct other algebras with this property from the known ones (Section 3.2).

We will show in Chapter 5 that a fundamental fact towards obtaining the hyperreflex-

ivity of the bounded n -cocycle spaces of a Banach algebra, is to see when the unitization of a Banach algebra possesses the strong property (\mathbb{B}) . This may not be easily seen even if the algebra itself has the strong property (\mathbb{B}) ! That is why in Chapter 4, we introduce the notion of *bounded local units* or briefly *b.l.u.* for a Banach algebra which roughly speaking forces it to have bounded approximate identities consisting of elements that act as local units on a dense subset (see Definition 4.1.1 and Remark 4.1.2). We show in Theorem 4.1.3 that existence of b.l.u. allows us to carry the strong property (\mathbb{B}) from a Banach algebra to its unitization. We also study the hereditary property of algebras having b.l.u. We would like to point out that one advantage of investigating the hereditary properties of both notions of b.l.u. and having the strong property (\mathbb{B}) is that this process might be easier than investigating the hereditary properties of hyperreflexivity of bounded n -cocycles directly. This also provides us with more examples of bounded n -cocycle spaces which are hyperreflexive.

One fundamental result of this thesis is presented in Chapter 5.2. We show that if A is a Banach algebra for which its unitization, i.e., $A^\# = A \oplus \mathbb{C}$, has the strong property (\mathbb{B}) and X a Banach A -bimodule such that $\mathcal{H}^{n+1}(A, X)$ is a Banach space, then $\mathcal{Z}^n(A, X)$ is hyperreflexive (Theorem 5.2.4).

Section 5.3 is devoted to present examples of Banach algebras with hyperreflexive spaces of bounded n -cocycles. We first show that C^* -algebras and group algebras of groups with open subgroups of polynomial growth and some of their ideals have both the strong property (\mathbb{B}) and b.l.u. Using the criterion we obtained in Theorem 5.2.4 (pointed out in the preceding paragraph), we then show that $\mathcal{Z}^n(A, X)$ is hyperreflexive, for all $n \in \mathbb{N}$, for various cases such as when

- (i) A is a nuclear C^* -algebra and X a dual Banach A -bimodule,
- (ii) A is a von Neumann algebra of types I, II_∞ or III and $X = A$ or $X = B(\mathcal{H}) \supseteq A$ for a Hilbert space \mathcal{H} ,
- (iii) $A = I(H^\perp) \triangleleft L^1(G)$ and X a dual Banach $I(H^\perp)$ -bimodule. Here G is a locally compact amenable group with an open subgroup of polynomial growth, and H is a normal closed subgroup of G .

We also show that one could drop the amenability assumption in (iii) in some cases (Theorem 5.3.3). Finally, we show that similar results hold for quotients and tensor products of such algebras.

In Chapter 6, we introduce the notion of “*a constant for the strong property (\mathbb{B})* ”. We show that we can come with a constant for the strong property (\mathbb{B}) for all Banach algebras that we already showed to have this property. We also prove that although it is not true that every Banach algebra has the strong property (\mathbb{B}) , one can construct Banach algebras with the strong property (\mathbb{B}) related to any arbitrary Banach algebra. More precisely, for a Banach algebra A and $n \geq 2$, we show that if we equip $M_n(A)$, the space of matrices with entries in A , with an appropriate Banach algebra norm, then $M_n(A)$ has the strong property (\mathbb{B}) with a constant. This implies, in particular, that every Banach algebra is isometrically embedded into a Banach algebra with the strong property (\mathbb{B}) . We also prove that finite nest algebras on any Hilbert space and finite CSL algebras on separable Hilbert spaces have the strong property (\mathbb{B}) with a constant.

As mentioned before, our works in Chapters 3, 4, 5 shows that the strong property (\mathbb{B}) paves the way to solve the problem of hyperreflexivity of the bounded n -cocycle spaces for various Banach algebras. In Chapter 7, we are interested to have further information on the hyperreflexivity of such spaces. Roughly speaking, by “further information” we mean to find a constant which is called *the hyperreflexivity constant*. This constant in some sense, measures “the distance” of the comparable norms that appear in the hyperreflexivity context. Our analysis shows that existence of a constant for the strong property (\mathbb{B}) enables us to deal with this problem. We use our results in Chapter 6 and 7 to find an upper bound for the hyperreflexivity constant of the bounded n -cocycle spaces related to Banach algebras discussed in the preceding chapters.

CHAPTER 2

PRELIMINARIES

The present chapter contains the background necessary for this thesis. We introduce notations and tools which will be used in the next chapters. In Section 2.1, we review the definition of Banach spaces, Banach algebras and Banach modules. Some properties of locally compact groups and some related Banach algebras including group algebras is provided in Section 2.2. We then define Hochschild cohomology groups, amenable Banach algebras and amenable groups in Section 2.3. Some basic theorems on such groups and Banach algebras is presented. In Section 2.4, we introduce certain operator algebras called CSL and nest algebras. Some results on their Hochschild cohomology groups which will be needed in Chapter 7 is provided.

2.1 Banach spaces, Banach algebras and Banach modules

Definition 2.1.1. (1) Let X be a complex vector space. A norm on X is a function $\|\cdot\| : X \rightarrow \mathbb{R}$ with the following properties: For all $x, y \in X$ and $\alpha \in \mathbb{C}$.

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\alpha x\| = |\alpha| \|x\|$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

In this case, we call the pair $(X, \|\cdot\|)$ a normed space.

(2) A sequence $\{x_n\}$ in the normed space $(X, \|\cdot\|)$ is called a Cauchy sequence if for each

$\epsilon > 0$, there is $N \in \mathbb{N}$ such that for each $m, n \geq N$, we have

$$\|x_n - x_m\| < \epsilon.$$

(3) A normed space $(X, \|\cdot\|)$ is called a Banach space if each Cauchy sequence converges in this space.

Remark 2.1.2. Let $(X, \|\cdot\|)$ be a normed space and M a closed subspace of X . The quotient space $\frac{X}{M}$ becomes a normed space with respect to the following norm known as quotient norm:

$$\|x + M\|_q = \inf\{\|x - y\| : y \in M\}.$$

Let X and Y be two vector spaces. Let $L(X, Y)$ denotes the set of all linear maps from X to Y . $L(X, Y)$ becomes a vector space with respect to the pointwise addition and scalar multiplication. We let $L^n(X, Y) = L(X^{(n)}, Y)$ denote the space of all n -linear maps from $X^{(n)} = \underbrace{X \times \dots \times X}_{n \text{ times}}$ into Y .

If X and Y are normed spaces, then for $T \in L^n(X, Y)$ we define

$$\|T\| = \sup\{\|T(x_1, \dots, x_n)\| : \|x_i\| \leq 1, 1 \leq i \leq n\}.$$

In this case, we define $B^n(X, Y)$ to be the subspace of all bounded n -linear maps in $L^n(X, Y)$. That is

$$B^n(X, Y) = \{T \in L^n(x, Y) : \|T\| < \infty\}.$$

We let $B(X, Y) = B^1(X, Y)$ and $B(X) = B(X, X)$.

Theorem 2.1.3 (The open mapping theorem). *Let X and Y be two Banach spaces. If $T : X \rightarrow Y$ is a bounded surjective linear map, then there is a constant $C > 0$ such that for each $y \in Y$, there exist $x \in X$ with $y = T(x)$ and $\|x\| \leq C\|T(x)\|$.*

Definition 2.1.4. It is proven in [7, Lemma VI.10] that the following defines a norm on the tensor product $X \otimes Y$:

$$\|u\| = \inf\left\{\sum_{i=1}^n \|x_i\| \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i\right\}.$$

The completion of the normed space $(X \otimes Y, \|\cdot\|)$ is called the *projective tensor product* of X and Y and is denoted by $X \hat{\otimes} Y$.

Remark 2.1.5. It is shown in [7, Proposition VI.12] that elements of $X \hat{\otimes} Y$ are of the form $u = \sum_{n=1}^{\infty} x_n \otimes y_n$, where $\sum_{n=1}^{\infty} \|x_n\| \|y_n\| < \infty$. The norm on $X \hat{\otimes} Y$ is given by

$$\|u\| = \inf \left\{ \sum_{n=1}^{\infty} \|x_n\| \|y_n\| : u = \sum_{n=1}^{\infty} x_n \otimes y_n \right\}.$$

Definition 2.1.6. (1) An algebra over \mathbb{C} is a complex vector space A together with a map $A \times A \rightarrow A$, called product or multiplication and written $(a, b) \rightarrow ab$, which is bilinear, i.e., it satisfies

$$a(b + c) = ab + ac, \quad (a + b)c = ac + bc,$$

as well as

$$\lambda(ab) = (\lambda a)b = a(\lambda b) \quad (a, b, c \in A \text{ and } \lambda \in \mathbb{C}).$$

(2) A Banach algebra is an algebra A over complex numbers together with a norm $\|\cdot\|$ such that the underlying normed space is a Banach space and the inequality

$$\|ab\| \leq \|a\| \|b\|$$

holds for all $a, b \in A$.

Remark 2.1.7. (1) Let X be a Banach space. Then $B(X)$ together with the operator norm and composition as multiplication is a Banach algebra.

(2) Let X and Y be Banach algebras. Then $X \hat{\otimes} Y$ together with the projective norm and the multiplication

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 x_2 \otimes y_1 y_2$$

becomes a Banach algebra. (See [7, Propositions VI.17 and VI.18] for proof.)

(3) Let Y be a Banach algebra and X a closed ideal of Y . Then the quotient space together with the quotient norm and the multiplication

$$(a + X)(b + X) = ab + X,$$

becomes a Banach algebra. (See [10, Theorem 4.2] for proof.)

Definition 2.1.8. Let A be a Banach algebra.

(1) A unit for A is an element $1_A \in A$ such that

$$1_A a = a 1_A = a \quad (\forall a \in A).$$

It is easy to check that if A has a unit, then its unit is unique.

(2) A left approximate identity for A is a net $\{\rho_i\}_{i \in I} \subseteq A$ such that for all $a \in A$, we have

$$\lim_{i \in I} \|\rho_i a - a\| = 0.$$

Right approximate identity is defined similarly. A net which is both a left and a right approximate identity is called an approximate identity. Note that (left, right) approximate identity for a Banach algebra might not be unique.

Definition 2.1.9. Let A be an algebra over \mathbb{C} .

(1) An involution is a map $A \rightarrow A$, denoted by $a \rightarrow a^*$ such that for all $a, b \in A$ and $\lambda \in \mathbb{C}$ we have

$$(a + b)^* = a^* + b^*, \quad (\lambda a)^* = \bar{\lambda} a^*, \quad (ab)^* = b^* a^*, \quad \text{and} \quad (a^*)^* = a.$$

(2) A Banach $*$ -algebra is a Banach algebra together with an involution such that for every $a \in A$ we have

$$\|a^*\| = \|a\|.$$

(3) A Banach $*$ -algebra is called a C^* -algebra if

$$\|a^* a\| = \|a\|^2$$

holds for every $a \in A$.

Definition 2.1.10. (1) Let A be an algebra and E a vector space. We call E a left A -module if there is a bilinear map $A \times E \rightarrow E$, denoted by $(a, e) \rightarrow a \cdot e$ such that

$$(ab) \cdot e = a \cdot (b \cdot e) \quad (a, b \in A \quad e \in E).$$

Similarly, we can define right A -module.

(2) E is an A -bimodule if it is both a left and a right A -module and

$$a \cdot (e \cdot b) = (a \cdot e) \cdot b \quad (a, b \in A, e \in E).$$

(3) Let A be a Banach algebra and suppose that E is a left A -module which is a Banach space as well. We say that E is a left Banach A -module if

$$\|a \cdot e\| \leq \|a\| \|e\|, \quad (a \in A, e \in E).$$

Similarly, we can define right Banach A -bimodule.

(4) E is a Banach A -bimodule if it is an A -bimodule, a left Banach A -module and a right Banach A -module.

Remark 2.1.11. Let A be a Banach algebra and E a Banach A -bimodule. Then E^* , the dual space of E , becomes a Banach A -bimodule with the module action defined by

$$a \cdot \varphi(e) = \varphi(e \cdot a), \quad \varphi \cdot a(e) = \varphi(a \cdot e) \quad (a \in A, e \in E, \varphi \in E^*).$$

2.2 Locally compact groups and some related Banach algebras

Definition 2.2.1. (1) A topological group is a group G together with a topology on the set G such that the group multiplication and inversion

$$G \times G \rightarrow G, \quad G \rightarrow G$$

$$(x, y) \mapsto xy, \quad x \mapsto x^{-1}$$

are both continuous maps.

(2) A topological group is called a locally compact group if it is Hausdorff and locally compact.

Definition 2.2.2. Let G be a locally compact group, and let \mathcal{B} be the σ -algebra of Borel sets on G . Then

- (1) A measure $\mu : \mathcal{A} \rightarrow [0, \infty]$ on the measurable space (G, \mathcal{B}) is called a Borel measure.
- (2) A Borel measure μ is called locally finite if every point in G possesses a neighborhood U with $\mu(U) < \infty$.
- (3) A locally finite Borel measure μ on \mathcal{B} is called outer Radon measure or briefly a Radon measure if
 - (i) $\mu(A) = \inf\{\mu(U) : A \subseteq U, U \text{ is open}\}$, for every $A \in \mathcal{B}$.
 - (ii) $\mu(A) = \sup\{\mu(F) : F \subseteq A, F \text{ is compact}\}$, for every $A \in \mathcal{B}$ which is open or $\mu(A) < \infty$.
- (4) μ is called left invariant if $\mu(xA) = \mu(A)$ holds for every $A \in \mathcal{B}$ and $x \in G$.

Theorem 2.2.3. *Let G be a locally compact group. There is a non-zero, left invariant Radon measure on G . It is uniquely determined up to positive multipliers. Every such a measure is called a left Haar measure.*

(See [17, Theorem 1.3.4]) for the proof.)

Definition 2.2.4. Let G be a locally compact group. Suppose that \mathcal{B} is the Borel σ -algebra on G and λ is a fixed left Haar measure of G .

(i) The group algebra of G which is denoted by $L^1(G)$ is defined as follows:

$$L^1(G) = L^1(G, \mathcal{B}, \lambda) = \{f : G \rightarrow \mathbb{C} : f \text{ is } \lambda\text{-measurable and } \|f\|_1 = \int_G |f(x)| d\lambda x < \infty\}.$$

For simplicity, we use “ dx ” to denote “ $d\lambda x$ ” in the integration.

(ii) We also define the following Banach algebra related to G

$$L^\infty(G) = L^\infty(G, \mathcal{B}, \lambda) = \{f : G \rightarrow \mathbb{C} : f \text{ is } \lambda\text{-measurable and } \|f\|_\infty < \infty\}$$

where $\|f\|_\infty = \inf\{N : \lambda\{x : |f(x)| > N\} = 0\}$.

Theorem 2.2.5. *Let G be a locally compact group. Then $L^1(G)$ is a Banach algebra with respect to $\|\cdot\|_1$ and the multiplication (known as the convolution) that is defined by*

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy \quad (f, g \in L^1(G), x \in G).$$

(See [17, Theorem 1.6.2] for proof.)

Definition 2.2.6. Let G be a locally compact group, and let E be a closed subspace of $L^\infty(G)$ containing the constant function.

- (1) A mean on E is a functional $m \in E^*$ such that $m(1) = \|m\| = 1$.
- (2) G is called amenable if there is a left invariant mean on $L^\infty(G)$ i.e., a mean such that

$$m(\delta_g * \varphi) = m(\varphi) \quad (g \in G, \varphi \in L^\infty(G)).$$

Here δ_g is the Dirac measure at g and

$$(\delta_g * \varphi)(t) = \varphi(g^{-1}t), \quad \text{locally almost everywhere.}$$

2.3 Amenable Banach algebras and Hochschild cohomology

Definition 2.3.1. Let A be a Banach algebra, and let X be a Banach A -bimodule.

- (1) An operator $D \in L(A, X)$ is a derivation if for all $a, b \in A$, we have

$$D(ab) = aD(b) + D(a)b.$$

We let $Z^1(A, X)$ and $\mathcal{Z}^1(A, X)$ to be the linear spaces of derivations and bounded derivations from A into X , respectively.

- (2) For each $x \in X$, the operator $ad_x \in B(A, X)$ defined by

$$ad_x(a) = a \cdot x - x \cdot a$$

is a bounded derivation which is called an inner derivation.

- (3) A is called amenable if every bounded derivation from A into any dual A -bimodule is an inner derivation.

Definition 2.3.2. Let A be a Banach algebra. A bounded net $(m_\alpha)_\alpha$ in $A \hat{\otimes} A$ is called a *bounded approximate diagonal* for A if

$$a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0 \quad \text{and} \quad a \Delta_A m_\alpha \rightarrow a \quad (a \in A)$$

where $\Delta_A : A \hat{\otimes} A \rightarrow A$ is the multiplication mapping defined by

$$\Delta_A(a \otimes b) = ab.$$

Theorem 2.3.3. *A Banach algebra A is amenable if and only if there is a bounded approximate diagonal for A .*

(See [48, Theorem 2.2.4] for proof).

Definition 2.3.4. Let A be a Banach algebra and X a Banach A -bimodule.

(1) For $n \in \mathbb{N}$ and $T \in L^n(A, X)$, define

$$\begin{aligned} \delta^n T : (a_1, \dots, a_{n+1}) &\mapsto a_1 T(a_2, \dots, a_n) \\ &+ \sum_{j=1}^n (-1)^j T(a_1, \dots, a_j a_{j+1}, \dots, a_{n+1}) \\ &+ (-1)^{n+1} T(a_1, \dots, a_n) a_{n+1}. \end{aligned}$$

It is clear that δ^n is a linear map from $L^n(A, X)$ into $L^{n+1}(A, X)$; these maps are the *connecting maps*. Moreover, it can be shown that $\delta^{n+1} \circ \delta^n = 0$ for every $n \in \mathbb{N}$. The elements of $\ker \delta^n$ are the *n -cocycles*; we denote this linear space with $Z^n(A, X)$.

(2) If we replace $L^n(A, X)$ with $B^n(A, X)$ in the above, we will have the ‘Banach’ version of the connecting maps; we denote them with the same notation δ^n . In this case, δ^n is a bounded linear map from $B^n(A, X)$ into $B^{n+1}(A, X)$; these maps are the *bounded connecting maps* or *n -coboundary operators*. The elements of $\ker \delta^n$ are the *bounded n -cocycles*; we denote this linear space by $\mathcal{Z}^n(A, X)$. It is easy to check that $Z^1(A, X)$ and $\mathcal{Z}^1(A, X)$ coincide with our previous definition of these notations.

(3) The sequence

$$\{0\} \rightarrow X \xrightarrow{\delta_0} B(A, X) \xrightarrow{\delta^1} B^2(A, X) \xrightarrow{\delta^2} \dots \xrightarrow{\delta^{n-1}} B^n(A, X) \xrightarrow{\delta^n} B^{n+1}(A, X) \xrightarrow{\delta^{n+1}} \dots$$

is called the *Hochschild cochain complex*. Here $\delta_0 : X \rightarrow B(A, X)$ is defined by,

$$\delta_0(x)(a) = a \cdot x - x \cdot a.$$

Definition 2.3.5. Let A be a Banach algebra, and let X be a Banach A -bimodule. For $n \in \mathbb{N}$, let

$$\mathcal{H}^n(A, X) = \frac{\ker \delta^n}{\text{im} \delta^{n-1}}.$$

$\mathcal{H}^n(A, X)$ is called the n^{th} Hochschild cohomology group of A with coefficients in X .

Remark 2.3.6. Let A be a Banach algebra, and let X be a Banach A -bimodule. It is shown in [15, Section 2.8] that for $n \in \mathbb{N}$, the Banach space $B^n(A, X)$ turns into a Banach A -bimodule by the actions defined through:

$$\begin{aligned} (a \star T)(a_1, \dots, a_n) &= aT(a_1, \dots, a_n); \\ (T \star a)(a_1, \dots, a_n) &= T(aa_1, \dots, a_n) \\ &+ \sum_{j=1}^n (-1)^j T(a, a_1, \dots, a_j a_{j+1}, \dots, a_n) \\ &+ (-1)^{n+1} T(a, a_1, \dots, a_{n-1}) a_n. \end{aligned}$$

In particular, when $n = 1$, $B(A, X)$ becomes a Banach A -bimodule with respect to the products

$$(a \star T)(b) = aT(b) \quad , \quad (T \star a)(b) = T(ab) - T(a)b.$$

Remark 2.3.7. Let $\Lambda_n: B^{n+1}(A, X) \rightarrow B^n(A, B(A, X))$ be the identification given by

$$(\Lambda_n(T)(a_1, \dots, a_n))(a_{n+1}) = T(a_1, \dots, a_{n+1}).$$

Then Λ_n is an A -bimodule isometric isomorphism. If we denote the connecting maps for the complex $B^n(A, (B(A, X), \star))$ by Δ^n , then it is shown in [15] that

$$\Lambda_{n+1} \circ \delta^{n+1} = \Delta^n \circ \Lambda_n. \tag{2.3.1}$$

The well-known Theorem of Johnson makes the connection between amenability of groups and Banach algebras.

Theorem 2.3.8. (Johnson's theorem) *For a locally compact group G , the following are equivalent:*

- (i) G is an amenable group.
- (ii) $L^1(G)$ is an amenable Banach algebra.

(See [48, Theorem 2.1.8] for proof.)

By definition, the amenability of a Banach algebra is equivalent to the fact that its first Hochschild cohomology groups with coefficients in a dual Banach bimodule vanishes. The following theorem shows that, it is actually the case for the Hochschild cohomology groups of all orders.

Theorem 2.3.9. *For a Banach algebra A the following are equivalent:*

(i) A is amenable.

(ii) $\mathcal{H}^n(A, X^*) = \{0\}$ for each Banach A -bimodule X and for all $n \in \mathbb{N}$.

(See [48, Theorem 2.4.7]).

2.4 CSL and nest algebras

CSL and nest algebras will be discussed in Chapter 6.

Definition 2.4.1. Let H be a Hilbert space. Let $\mathcal{L} = \{M_i\}_{i \in I}$ be a family of closed subspaces of H .

(i) \mathcal{L} is called a subspace lattice or SL if it is closed under intersection and closed linear span. A subspace lattice is said to be *commutative* if the corresponding orthogonal projections onto its subspaces commute. A commutative subspace lattice is briefly denoted by CSL. The CSL algebra generated by a CSL $\mathcal{L} = \{M_i\}_{i \in I}$, is the subalgebra of $B(H)$ consisting of all bounded linear maps leaving each M_i invariant.

(ii) A CSL is called a nest if it is totally ordered under inclusion. A nest algebra is a CSL algebra corresponding to a nest.

(iii) A finite CSL (respectively, nest) algebra is a CSL (respectively, nest) algebra whose corresponding subspace lattice is finite.

Remark 2.4.2. Since every closed subspace of a Hilbert space can be identified with its range projection, a CSL $\mathcal{L} = \{M_i\}_{i \in I}$ can be thought as a family of projections $\mathcal{L} = \{P_i\}_{i \in I}$ on a Hilbert space H . Both identifications are used interchangeably. In this case, the

corresponding CSL algebra is given by,

$$\text{alg}\mathcal{L} = \{T \in B(H) : P_i^\perp T P_i = 0, i \in I\}.$$

The next theorem gives some information on some cohomology groups of nest algebras.

Theorem 2.4.3. *Let $N \subseteq B(H)$ be a nest algebra. Then:*

(i) $\mathcal{H}^n(N, B(H)) = 0$, for all $n \geq 1$.

(ii) $\mathcal{H}^n(N, N) = 0$, for all $n \geq 1$.

Proof. Parts (i) and (ii) are proven in [13, Theorem 2.1] and [13, Theorem 2.3], respectively. □

There are some similar results on some cohomology groups of general CSL algebras. For a CSL algebra $\mathcal{A} \subseteq B(H)$, we define $\mathcal{E}(\mathcal{A})$ to be the following subalgebra of $B(\mathbb{C} \oplus H)$,

$$\mathcal{E}(\mathcal{A}) = \left\{ \begin{pmatrix} z & u \\ 0 & a \end{pmatrix} \in B(\mathbb{C} \oplus H) : z \in \mathbb{C}, u \in H^*, a \in \mathcal{A} \right\}.$$

Theorem 2.4.4. *Let \mathcal{L} be a finite CSL and $\mathcal{A} = \text{alg}\mathcal{L}$. Then for each $n \in \mathbb{N}$,*

$$\mathcal{H}^n(\mathcal{E}(\mathcal{A}), B(\mathbb{C} \oplus H)) = 0,$$

and

$$\mathcal{H}^n(\mathcal{E}(\mathcal{A}), \mathcal{E}(\mathcal{A})) = 0.$$

(For a proof of this theorem, see [42, Lemma 5]).

CHAPTER 3

BANACH ALGEBRAS HAVING THE STRONG PROPERTY (\mathbb{B})

We recall from [1, Definition 2.2] that a Banach algebra A has the property (\mathbb{B}) if, for any Banach space X , every continuous bilinear map $\varphi : A \times A \rightarrow X$ that preserves zero products, i.e., with the property that

$$a, b \in A, ab = 0 \text{ implies } \varphi(a, b) = 0$$

is of the form of

$$\varphi(ab, c) = \varphi(a, bc) \quad (a, b, c \in A).$$

Banach algebras with the property (\mathbb{B}) were mainly defined, in order to unify the problem of disjointness preserving linear maps on different classes of Banach algebras. If A and B are Banach algebras, then a linear mapping $T : A \rightarrow B$ is said to be disjointness preserving if for all $a, b \in A$ with $ab = 0$, we have $T(a)T(b) = 0$. If $T : A \rightarrow B$ is a bounded disjointness preserving linear map, then the bilinear map $\varphi : A \times A \rightarrow B$ defined by

$$\varphi(a, b) = T(a)T(b)$$

is a bounded bilinear map that preserves zero products. Standard examples of bounded disjointness preserving maps are weighted composition operators. Let X and Y be locally compact Hausdorff spaces. Then the operator $T : C_0(X) \rightarrow C_0(Y)$ defined by

$$Tf = h \cdot f \circ \phi$$

is called a weighted composition operator where $\phi : Y \rightarrow X$ is a homeomorphism and $h : Y \rightarrow \mathbb{T}$ is a continuous function.

It was shown in [1] that the property (\mathbb{B}) is very useful in studying local homomorphisms and local derivations. Moreover, in [52] the class of Banach algebras with the property (\mathbb{B}) plays an important role in studying the n -reflexivity of the bounded n -cocycles of some classes of group algebras.

The notion of zero products preserving (bi-)linear maps was generalized in [2] to the notion of approximately zero products preserving (bi-)linear maps. We will show later in chapter 5 that these latter maps play an important role in the problem of the hyperreflexivity of the bounded n -cocycle spaces as maps preserving zero products play in that of reflexivity. That is why we introduce the strong property (\mathbb{B}) in this chapter.

3.1 General definition of the strong property (\mathbb{B})

In order to investigate the hyperreflexivity of the spaces of bounded n -cocycles, we first need to generalize the concept of having the property (\mathbb{B}) .

Definition 3.1.1. We say that a Banach algebra A has the strong property (\mathbb{B}) if for each $K > 0$ there is a continuous function $(L_{A,K} =) L_K : [0, \infty) \rightarrow [0, \infty)$ with $L_K(0) = 0$ such that for any Banach space X and every continuous bilinear map $\varphi : A \times A \rightarrow X$ with $\|\varphi\| \leq K$ and each $0 \leq \alpha < K$ satisfying

$$ab = 0 \Rightarrow \|\varphi(a, b)\| \leq \alpha \|a\| \|b\|,$$

we would have

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq L_K(\alpha) \|a\| \|b\| \|c\| \quad (a, b, c \in A).$$

We call L_K a **function associated to** A and K . It follows routinely from the fact that $L_K(0) = 0$ if $\varphi : A \times A \rightarrow X$ satisfies

$$ab = 0 \Rightarrow \varphi(a, b) = 0,$$

then

$$\varphi(ab, c) - \varphi(a, bc) = 0 \quad (a, b, c \in A).$$

Therefore, having the strong property (\mathbb{B}) is in fact the generalization of having the property (\mathbb{B}) .

In Definition 3.1.1, we presented the concept of the strong property (\mathbb{B}) in its most general form. Remark 3.1.2 and Lemma 3.1.3 below show that in order for a Banach algebra to have the strong property (\mathbb{B}) , we need to handle much fewer bounded bilinear maps.

Remark 3.1.2. In Definition 3.1.1, we need to only investigate the existence of the function L_1 (i.e. when $K = 1$) since for every $K > 0$, if we define $L_K : [0, \infty) \rightarrow [0, \infty)$ by $L_K(\alpha) = KL_1(\alpha/K)$, then it is straightforward to check that L_K satisfies the assumption of Definition 3.1.1.

Lemma 3.1.3. *Let A be a Banach algebra. Then A has the strong property (\mathbb{B}) if and only if for every continuous bilinear map $\varphi : A \times A \rightarrow \mathbb{C}$ with $\|\varphi\| \leq 1$ and each $0 \leq \alpha < 1$ satisfying*

$$ab = 0 \Rightarrow \|\varphi(a, b)\| \leq \alpha\|a\|\|b\|,$$

we would have

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq L_1(\alpha)\|a\|\|b\|\|c\| \quad (a, b, c \in A).$$

Proof. Let X be an arbitrary Banach space and $\varphi : A \times A \rightarrow X$ a linear map with $\|\varphi\| \leq 1$ and $0 \leq \alpha < 1$ with

$$ab = 0 \Rightarrow \|\varphi(a, b)\| \leq \alpha\|a\|\|b\|.$$

Let L be the function associated to \mathbb{C} and 1 and fix $a_0, b_0, c_0 \in A$. We show that

$$\|\varphi(a_0b_0, c_0) - \varphi(a_0, b_0c_0)\| \leq L(\alpha)\|a_0\|\|b_0\|\|c_0\|.$$

Using the Hahn-Banach theorem, we can find a linear map $T : X \rightarrow \mathbb{C}$, with $\|T\| = 1$ and

$$T(\varphi(a_0b_0, c_0) - \varphi(a_0, b_0c_0)) = \|\varphi(a_0b_0, c_0) - \varphi(a_0, b_0c_0)\|. \quad (3.1.1)$$

Now consider the bilinear map $T \circ \varphi : A \times A \rightarrow \mathbb{C}$. It is easy to see that $\|T \circ \varphi\| \leq 1$ and

$$ab = 0 \Rightarrow \|T \circ \varphi(a, b)\| \leq \alpha \|a\| \|b\|.$$

So by the assumption, for $a_0, b_0, c_0 \in A$ we have

$$|T \circ \varphi(a_0b_0, c_0) - T \circ \varphi(a_0, b_0c_0)| \leq L(\alpha) \|a_0\| \|b_0\| \|c_0\|,$$

or by (3.1.1),

$$\|\varphi(a_0b_0, c_0) - \varphi(a_0, b_0c_0)\| \leq L(\alpha) \|a_0\| \|b_0\| \|c_0\|.$$

Since $a_0, b_0, c_0 \in A$ are arbitrary, the proof is complete. \square

We keep the general format of the definition of the strong property (\mathbb{B}) since it is usually more convenient.

Remark 3.1.4. In [2] without defining this property explicitly, it is proven that all group algebras and C^* -algebras have the strong property (\mathbb{B}) . In Chapter 7 where we try to find a *constant for the strong property (\mathbb{B})* (See Definition 6.1.1), we will present an alternative way of showing that group algebras and C^* -algebras have the strong property (\mathbb{B}) . In Chapter 6, we will also construct other examples of Banach algebras with this property. These algebras are of the form of matrices over a given Banach algebra or certain operator algebras.

3.2 Hereditary properties of Banach algebras with the strong property (\mathbb{B})

One way to construct new examples of Banach algebras with the strong property (\mathbb{B}) is to investigate how this property relates to Banach algebras associated to a given Banach algebra with the strong property (\mathbb{B}) . In this section we aim to look into such cases. In

other word, we investigate the hereditary properties of the strong property (\mathbb{B}) .

We start with the following proposition that deals with ideals of a Banach algebra with the strong property (\mathbb{B}) .

Proposition 3.2.1. *Let A be a Banach algebra having the strong property (\mathbb{B}) . Suppose that I is a closed ideal of A such that it has a bounded approximate identity in A . Then I has the strong property (\mathbb{B}) .*

Proof. Let X be a Banach space and $K > 0$. Suppose that $\varphi : I \times I \rightarrow X$ is a bounded bilinear map with $\|\varphi\| \leq K$ and $0 \leq \alpha < K$ is such that

$$uv = 0 \Rightarrow \|\varphi(u, v)\| \leq \alpha\|u\|\|v\|.$$

Fix $u, v \in I$ with $\|u\|, \|v\| \leq 1$. Define $\psi_{u,v} : A \times A \rightarrow X$ with

$$\psi_{u,v}(a, b) = \varphi(ua, bv).$$

Obviously $\|\psi_{u,v}\| \leq \|\varphi\| \leq K$. If $ab = 0$, then $(ua)(bv) = 0$. Hence

$$\begin{aligned} \|\psi_{u,v}(a, b)\| &= \|\varphi(ua, bv)\| \\ &\leq \alpha\|ua\|\|bv\| \\ &\leq \alpha\|a\|\|b\| \end{aligned}$$

Therefore, by Definition 3.1.1, there is a continuous function $L_K : [0, \infty) \rightarrow [0, \infty)$ with $L_K(0) = 0$ such that

$$\|\psi_{u,v}(ab, c) - \psi_{u,v}(a, bc)\| \leq L_K(\alpha)\|a\|\|b\|\|c\|$$

or equivalently,

$$\|\varphi(uab, cv) - \varphi(ua, bcv)\| \leq L_K(\alpha)\|a\|\|b\|\|c\|. \quad (3.2.1)$$

Now suppose that the bounded approximate identity of I in A has a bound M . Then using (3.2.1), we have

$$\|\varphi(ub, v) - \varphi(u, bv)\| \leq M^2 L_K(\alpha)\|b\| \quad (u, v \in I, b \in A).$$

In particular, if $u, v, w \in I$ are arbitrary, then

$$\|\varphi(uw, v) - \varphi(u, wv)\| \leq M^2 L_K(\alpha) \|u\| \|v\| \|w\|$$

Hence I satisfies in the strong property (\mathbb{B}) with $L_{I,K} = M^2 L_K$. \square

Proposition 3.2.2. *Let A be a Banach algebra having the strong property (\mathbb{B}) and suppose that B is a Banach algebra and $\Phi : A \rightarrow B$ is a bounded surjective homomorphism. Then B has the strong property (\mathbb{B}) . In particular, if I is a closed ideal of A , then A/I has the strong property (\mathbb{B}) .*

Proof. Let X be a Banach space and suppose that $\varphi : B \times B \rightarrow X$ is a bounded bilinear map with the property that $\|\varphi\| \leq K$, and let $0 \leq \alpha < K$ with

$$ab = 0 \Rightarrow \|\varphi(a, b)\| \leq \alpha \|a\| \|b\|.$$

We define $\psi : A \times A \rightarrow X$ with

$$\psi(a, b) = \varphi(\Phi(a), \Phi(b)).$$

It is easy to check that $\|\psi\| \leq \|\Phi\|^2 \|\varphi\| \leq \|\Phi\|^2 K$. Suppose that $ab = 0$. Then

$$\begin{aligned} \|\psi(a, b)\| &= \|\varphi(\Phi(a), \Phi(b))\| \\ &\leq \alpha \|\Phi(a)\| \|\Phi(b)\| \\ &\leq \alpha \|\Phi\|^2 \|a\| \|b\|. \end{aligned}$$

This implies that

$$\|\psi(ab, c) - \psi(a, bc)\| \leq L_{K'}(\alpha) \|a\| \|b\| \|c\| \quad \forall a, b, c \in A, \quad (3.2.2)$$

where $L_{K'}$ is the function associated to A and $K' = \|\Phi\|^2 K$. Using the open mapping theorem, there is $C > 0$ such that for each $b_1, b_2, b_3 \in B$ there are $a_1, a_2, a_3 \in A$ with

$$b_1 = \Phi(a_1), \quad \|a_1\| < C \|b_1\|.$$

$$b_2 = \Phi(a_2), \quad \|a_2\| < C \|b_2\|.$$

$$b_3 = \Phi(a_3), \quad \|a_3\| < C\|b_3\|.$$

Now by (3.2.2), we can write

$$\begin{aligned} \|\varphi(b_1 b_2, b_3) - \varphi(b_1, b_2 b_3)\| &\leq L_{K'}(\alpha) \|a_1\| \|a_2\| \|a_3\| \\ &\leq C^3 L_{K'}(\alpha) \|b_1\| \|b_2\| \|b_3\|. \end{aligned}$$

Now it suffices to define $L_{B,K} : [0, \infty) \rightarrow [0, \infty)$ with $L_{B,K} = C^3 L_{K'}$. \square

Although it is not trivial that the strong property (\mathbb{B}) is flexible with respect to the equivalent norms, one implication of Proposition 3.2.2 is that the strong property (\mathbb{B}) is independent of the complete norm of the Banach algebra.

Corollary 3.2.3. *Let A be a Banach algebra having the strong property (\mathbb{B}) with respect to the norm $\|\cdot\|$. Then A has the strong property (\mathbb{B}) with respect to all norms which are equivalent to $\|\cdot\|$.*

A possible way to construct a Banach algebra related to an infinite family of Banach algebras is to define their l^1 -sum. Let I be an index set. If $(A_i, \|\cdot\|_i)_{i \in I}$ is a family of Banach algebras, then we define

$$l^1(I, A_i) = \{(a_i)_{i \in I} : \|(a_i)_{i \in I}\|_1 := \sum_{i \in I} \|a_i\|_i < \infty\}.$$

$l^1(I, A_i)$ becomes a Banach algebra with “pointwise” addition and multiplication, and scalar multiplication defined by $\lambda(a_i) = (\lambda a_i)$.

Proposition 3.2.4. *Let $(A_i)_{i \in \mathbb{N}}$ be a family of Banach algebras having the strong property (\mathbb{B}) . Let $K > 0$ and, for each $i \in \mathbb{N}$, let L_i be a function associated to A_i and K in Definition (3.1.1). Suppose that there is a continuous function $L : [0, \infty) \rightarrow [0, \infty)$ with $L(0) = 0$ such that*

$$\sup_{i \in \mathbb{N}} L_i(\alpha) \leq L(\alpha) \quad (0 \leq \alpha < K).$$

Then $A = l^1(\mathbb{N}, A_i)$ has the strong property (\mathbb{B}) . In particular, if $\{L_i, i \in \mathbb{N}\}$ is a finite set, then A has the strong property (\mathbb{B}) .

Proof. By Remark 3.1.2, it suffices to find a function associated to A and $K = 1$ satisfying in Definition 3.1.1. Define the continuous functions $L_A : [0, \infty) \rightarrow [0, \infty)$ with

$$L_A(\alpha) = \alpha + L(\alpha) \quad (\alpha \geq 0),$$

For each $i \in \mathbb{N}$, let l_i be the natural embedding of A_i into A which is an isometry. Let A_F denote the family of all elements $a = (a_i)_{i \in \mathbb{N}}$ which are zero except for a finite number of elements of \mathbb{N} . Note that A_F is dense in A and for each $a \in A_F$, there is $n \in \mathbb{N}$ such that $a = \sum_{i=1}^n l_i(a)$. Suppose that X is a Banach space, $\varphi : A \times A \rightarrow X$ is a bounded bilinear map with $\|\varphi\| \leq 1$, and $0 \leq \alpha < 1$ is such that

$$ab = 0 \Rightarrow \|\varphi(a, b)\| \leq \alpha \|a\| \|b\|.$$

For each $i \in I$, let $\varphi_i : A_i \times A_i \rightarrow X$ be the bounded linear map defined by

$$\varphi_i(a_i, b_i) = \varphi(l_i(a_i), l_i(b_i)).$$

Note that $\|\varphi_i\| \leq \|\varphi\| \leq 1$. If $a_i b_i = 0$, obviously we have

$$\|\varphi_i(a_i, b_i)\| \leq \alpha \|a_i\| \|b_i\|.$$

By the hypothesis, this implies that

$$\|\varphi_i(a_i b_i, c_i) - \varphi_i(a_i, b_i c_i)\| \leq L_i(\alpha) \|a_i\| \|b_i\| \|c_i\|.$$

Now let $a, b, c \in A_F$. Then

$$\begin{aligned} \|\varphi(ab, c) - \varphi(a, bc)\| &= \left\| \sum_{i=1}^n \sum_{j=1}^n \varphi(l_i(a_i b_i), l_j(c_j)) - \varphi(l_i(a_i), l_j(b_j c_j)) \right\| \\ &\leq \sum_{i=1}^n \sum_{j=1}^n \|\varphi(l_i(a_i b_i), l_j(c_j)) - \varphi(l_i(a_i), l_j(b_j c_j))\| \\ &\leq L_A(\alpha) \left(\sum_{i=1}^n \|a_i\| \right) \left(\sum_{i=1}^n \|b_i\| \right) \left(\sum_{i=1}^n \|c_i\| \right) \quad (*) \\ &= L_A(\alpha) \|a\| \|b\| \|c\|, \end{aligned}$$

where the inequality (*) follows from the following argument:

If $i = j$, we have

$$\begin{aligned}\|\varphi(l_i(a_i b_i), l_i(c_i)) - \varphi(l_i(a_i), l_i(b_i c_i))\| &\leq L_i(\alpha) \|a_i\| \|b_i\| \|c_i\| \\ &\leq L_A(\alpha) \|a_i\| \|b_i\| \|c_i\|.\end{aligned}$$

If $i \neq j$, then

$$l_i(a_i b_i) l_j(c_j) = l_i(a_i) l_j(b_j c_j) = 0.$$

Therefore

$$\begin{aligned}\|\varphi(l_i(a_i b_i), l_j(c_j)) - \varphi(l_i(a_i), l_j(b_j c_j))\| &\leq \alpha \|a_i\| \|b_i\| \|c_j\| + \alpha \|a_i\| \|b_j\| \|c_j\| \\ &\leq L_A(\alpha) (\|a_i\| \|b_i\| \|c_j\| + \|a_i\| \|b_j\| \|c_j\|)\end{aligned}$$

Hence

$$\begin{aligned}\sum_{i=1}^n \sum_{j=1}^n \|\varphi(l_i(a_i b_i), l_j(c_j)) - \varphi(l_i(a_i), l_j(b_j c_j))\| &\leq \sum_{i=1}^n \sum_{j \neq i}^n [L_A(\alpha) (\|a_i\| \|b_i\| \|c_j\| + \|a_i\| \|b_j\| \|c_j\|) \\ &\quad + L_A(\alpha) \|a_i\| \|b_i\| \|c_i\|] \\ &\leq L_A(\alpha) \left(\sum_{i=1}^n \|a_i\| \right) \left(\sum_{i=1}^n \|b_i\| \right) \left(\sum_{i=1}^n \|c_i\| \right) \\ &= L_{A,1}(\alpha) \|a\| \|b\| \|c\|.\end{aligned}$$

Since A_F is dense in A and φ is continuous, We have

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq L_{A,1}(\alpha) \|a\| \|b\| \|c\| \quad (\forall a, b, c \in A).$$

Hence it follows again from Remark 3.1.2 that A has the strong property (\mathbb{B}) . \square

A standard and useful way to relate two arbitrary Banach algebras is to consider their (projective) tensor product. The next theorem shows that performing the projective tensor product, allows us to obtain Banach algebras with the strong property (\mathbb{B}) from the known ones.

Proposition 3.2.5. *Let A and B be two Banach algebras having the strong property (\mathbb{B}) . Then the projective tensor product $A \hat{\otimes} B$ has the strong property (\mathbb{B}) .*

Proof. Let X be a Banach space and $K > 0$. Suppose that $\varphi : (A \hat{\otimes} B) \times (A \hat{\otimes} B) \rightarrow X$ is a continuous bilinear map with $\|\varphi\| \leq K$ and $0 \leq \alpha < K$ satisfying

$$xy = 0 \Rightarrow \|\varphi(x, y)\| \leq \alpha \|x\| \|y\|.$$

Fix $u, v \in B$ with $\|u\|, \|v\| \leq 1$ and define $\varphi_{u,v} : A \times A \rightarrow X$ with

$$\varphi_{u,v}(a, b) = \varphi(a \otimes u, b \otimes v).$$

It is easy to check that $\|\varphi_{u,v}\| \leq \|\varphi\| \leq K$. Moreover if $ab = 0$, then

$$\|\varphi_{u,v}(a, b)\| \leq \alpha \|a\| \|b\|.$$

Hence, by the hypothesis, we get

$$\|\varphi_{u,v}(ab, c) - \varphi_{u,v}(a, bc)\| \leq L_{A,K}(\alpha) \|a\| \|b\| \|c\| \quad (\forall a, b, c \in A).$$

This implies that for all $u, v \in B$,

$$\|\varphi(ab \otimes u, c \otimes v) - \varphi(a \otimes u, bc \otimes v)\| \leq L_{A,K}(\alpha) \|a\| \|b\| \|c\| \|u\| \|v\| \quad (\forall a, b, c \in A, u, v \in B). \quad (3.2.3)$$

Similarly we can show that

$$\|\varphi(a \otimes uv, b \otimes w) - \varphi(a \otimes u, b \otimes vw)\| \leq L_{B,K}(\alpha) \|a\| \|b\| \|u\| \|v\| \|w\| \quad (\forall a, b \in A, u, v, w \in B). \quad (3.2.4)$$

Using inequalities (3.2.3) and (3.2.4) we can write,

$$\begin{aligned} \|\varphi((a \otimes u)(b \otimes v), c \otimes w) - \varphi((a \otimes u), (b \otimes v)(c \otimes w))\| &= \|\varphi(ab \otimes uv, c \otimes w) - \varphi(a \otimes uv, bc \otimes w) \\ &+ \varphi(a \otimes uv, bc \otimes w) - \varphi(a \otimes u, bc \otimes vw)\| \\ &\leq \|\varphi(ab \otimes uv, c \otimes w) - \varphi(a \otimes uv, bc \otimes w)\| \\ &+ \|\varphi(a \otimes uv, bc \otimes w) - \varphi(a \otimes u, bc \otimes vw)\| \\ &\leq (L_{A,K}(\alpha) + L_{B,K}(\alpha)) \|a\| \|b\| \|c\| \|u\| \|v\| \|w\| \end{aligned}$$

Now let $x, y, z \in A \otimes B$ and consider the following representations

$$x = \sum_{i=1}^{n_1} a_i \otimes u_i, \quad y = \sum_{j=1}^{n_2} b_j \otimes v_j, \quad z = \sum_{k=1}^{n_3} c_k \otimes w_k.$$

Then

$$\begin{aligned}
\|\varphi(xy, z) - \varphi(x, yz)\| &= \left\| \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} (\varphi((a_i \otimes u_i)(b_j \otimes v_j), c_k \otimes w_k) - \varphi(a_i \otimes u_i, (b_j \otimes v_j)(c_k \otimes w_k))) \right\| \\
&\leq (L_{A,K}(\alpha) + L_{B,K}(\alpha)) \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \sum_{k=1}^{n_3} \|a_i\| \|u_i\| \|b_j\| \|v_j\| \|c_k\| \|w_k\| \\
&= (L_{A,K}(\alpha) + L_{B,K}(\alpha)) \left(\sum_{i=1}^{n_1} \|a_i\| \|u_i\| \right) \left(\sum_{j=1}^{n_2} \|b_j\| \|v_j\| \right) \left(\sum_{k=1}^{n_3} \|c_k\| \|w_k\| \right)
\end{aligned}$$

Since this is true for all representations of x, y, z , we can deduce that

$$\|\varphi(xy, z) - \varphi(x, yz)\| \leq (L_{A,K}(\alpha) + L_{B,K}(\alpha)) \|x\| \|y\| \|z\| \quad (\forall x, y, z \in A \otimes B).$$

Finally since φ is continuous and $\overline{A \otimes B} = A \hat{\otimes} B$, we get

$$\|\varphi(xy, z) - \varphi(x, yz)\| \leq (L_{A,K}(\alpha) + L_{B,K}(\alpha)) \|x\| \|y\| \|z\| \quad (\forall x, y, z \in A \hat{\otimes} B).$$

The proof is complete if we define $L_{A \hat{\otimes} B, K} = L_{A, K} + L_{B, K}$. □

3.3 An example of Banach algebras without the strong property (\mathbb{B})

When dealing with a new definition or property, a natural question is that whether there is an example that does not satisfy in the assumptions of the definition. The strong property (\mathbb{B}) is a special property in the sense that, one possibly expects that there should exist many Banach algebras without the strong property (\mathbb{B}) . Although this might be true, it is not easy to find such a counterexample. However, in this section we combine some known results to give an example of a Banach algebra without the (strong) property (\mathbb{B}) .

Definition 3.3.1. Let \mathbb{D} be the open unit disk on the complex plane. The disk algebra is defined with,

$$A(\mathbb{D}) = \{f \in C(\overline{\mathbb{D}}) : f \text{ is holomorphic on } \mathbb{D}\}.$$

Hence a function f lies in $A(\mathbb{D})$ if it is continuous on the closed unit disk and holomorphic on the open unit disk.

We will use the next theorems to prove that $A(\mathbb{D})$ does not have the (strong) property (\mathbb{B}) . The following result on the reflexivity of bounded derivations is proven in [52, Theorem 2.5].

Theorem 3.3.2. *Let A be a Banach algebra with local units having the property (\mathbb{B}) . Then for any Banach A -bimodule X , $\mathcal{Z}^1(A, X)$ is reflexive.*

The next Theorem that provides an example of a non-reflexive space of bounded derivations is proven in [54, Theorem 3.1].

Theorem 3.3.3. *Let Ω be an open connected subset of the plane, and let A be a Banach algebra of analytic functions on Ω . Then there is a bounded local derivation from A into A^* which is not a derivation. Moreover, $\mathcal{Z}^1(A, A^*)$ is not reflexive.*

Note that $A(\mathbb{D})$ is an example of a Banach algebra of analytic functions.

Remark 3.3.4. $A(\mathbb{D})$ is a unital Banach algebra. So it trivially has local units. Hence if it also has the property (\mathbb{B}) , then according to Theorem 3.3.2, $\mathcal{Z}^1(A, A^*)$ has to be reflexive which contradicts Theorem 3.3.3. Consequently, $A(\mathbb{D})$ does not have the property (\mathbb{B}) . So it does not have the strong property (\mathbb{B}) either.

We proved in Section 3.2 that various Banach algebras related to a given Banach algebra with the strong property (\mathbb{B}) inherit this property. In Chapter 6, we will use Remark 3.3.4 to prove something in the other way around. Actually, we show that subalgebras of a Banach algebra with the strong property (\mathbb{B}) might not have the same property.

CHAPTER 4

BANACH ALGEBRAS WITH BOUNDED LOCAL UNITS

A Banach algebra A is said to be unital if there exist $1_A \in A$ such that

$$1_A a = a 1_A = a, \quad \forall a \in A.$$

Existence of a unit for a Banach algebra is an extra assumption and it might not exist. However, in many cases, a bounded approximate identity works as effectively as a unit. There are also far more Banach algebras with bounded approximate identities rather than those with a unit.

Among the Banach algebras that fit in our framework with regard to the problem of hyperreflexivity, unital Banach algebras are possibly the best. But we do not want to confine ourselves only to this class of Banach algebras. However we can not simply replace units with bounded approximate identities in our direction yet. That is why we define the notion of bounded local units which is a concept between a unit and a bounded approximate identity. In fact, we explain below, in Remark 4.1.2, the existence of a unit implies existence of bounded local units and existence of bounded local units implies existence of bounded approximate identity. We present examples to show that neither of the converse cases hold true.

4.1 General definition of bounded local units

We recall that the unitization of A is $A^\sharp := A \oplus \mathbb{C}$ with multiplication

$$(a, \lambda)(b, \mu) = (ab + a\mu + b\lambda, \lambda\mu) \quad (a, b \in A, \lambda, \mu \in \mathbb{C}),$$

and norm

$$\|(a, \lambda)\| = \|a\| + |\lambda| \quad (a \in A, \lambda \in \mathbb{C}).$$

Thus A^\sharp is a unital Banach algebra with unit $(0, 1)$ which is denoted by 1 if there is no case of ambiguity. Also A is a closed two-sided ideal of A^\sharp with the codimension 1. We will show in Theorem 5.2.4 that in order to prove the hyperreflexivity of the bounded n -cocycle spaces related to a Banach algebra A , we need the unitization of A to have the strong property (\mathbb{B}) . On the other hand, it follows from Proposition 3.2.1 that if A^\sharp has the strong property (\mathbb{B}) , then so does A . Hence the natural question is that whether the converse is true i.e., if A has the strong property (\mathbb{B}) , can we deduce that A^\sharp has the same property? In this section, even though we can not answer this question in general, we present sufficient conditions on A for which this phenomenon occurs. As we will see, our algebra needs to have sufficiently many local units which are uniformly bounded.

Definition 4.1.1. Let A be a Banach algebra. We say that A has *bounded local units* or in brief *b.l.u* if there are dense subsets A_l and A_r of A and $M > 0$ such that for every $a \in A_l$ (resp. $b \in A_r$) there is $c \in A$ (resp. $d \in A$) with $\|c\| \leq M$ (resp. $\|d\| \leq M$) satisfying

$$ca = a \quad (bd = b).$$

Remark 4.1.2. The concept of bounded local units is something strictly between the notion of a unit and a bounded approximate identity, as we explain below:

(i) Definition 4.1.1 clearly shows that a unital Banach algebra has b.l.u. On the other hand, $C_0(\mathbb{R})$ is an example of a Banach algebra which has b.l.u. but it is not unital. Note that $C_0(\mathbb{R}) = \overline{C_c(\mathbb{R})}$. Urysohn's Lemma shows that elements of $C_c(\mathbb{R})$ have local units with bound 1. Hence $C_0(\mathbb{R})$ has b.l.u. Note that $C_0(\mathbb{R})$ is not unital since \mathbb{R} is not compact.

(ii) The terminology bounded local units has been inspired by the concept of bounded approximate units. We recall that a Banach algebra A has *bounded approximate units* or in brief *b.a.u.* if there is a bounded subset U of A such that for every $\epsilon > 0$ and $a \in A$, there is $e \in U$ such that $\|ae - a\| + \|ea - a\| < \epsilon$. It is clear from Definition 4.1.1 that

if A has b.l.u., then it has b.a.u. (simply put $e = c + d - dc$ to get $ea = a$ and $be = b$). Moreover, it is proven in [15, Corollary 2.9.15] that a Banach algebra has b.a.u if and only if it has a bounded approximate identity. Consequently, the existence of b.l.u. implies existence of bounded approximate identity. However the converse may not be true! One can construct radical Banach algebras with b.a.u. (see [49, Section 4] or [45]). It is shown in [15, Corollary 1.5.3] that such algebras can never have any local units.

Hence we are looking for Banach algebras that have bounded approximate units which also act as local units for some dense subsets.

The next theorem constructs a bridge and makes a connection between a Banach algebra and its unitization when we are concerned about having the strong property (\mathbb{B}) . It demonstrates that the existence of a b.l.u. is of great importance to make such a connection.

Theorem 4.1.3. *Let A be a Banach algebra with b.l.u. and having the strong property (\mathbb{B}) . Then A^\sharp , the unitization of A , has the strong property (\mathbb{B}) .*

Proof. Let X be a Banach space and $K > 0$. Suppose that $\varphi : A^\sharp \times A^\sharp \rightarrow X$ is a bounded bilinear map with $\|\varphi\| \leq K$, and let $0 \leq \alpha < K$ satisfying

$$a, b \in A^\sharp, \quad ab = 0 \Rightarrow \|\varphi(a, b)\| \leq \alpha \|a\| \|b\|. \quad (4.1.1)$$

In particular, this holds for each $a, b \in A$ with $ab = 0$. Hence, by hypothesis,

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq L_{A,K}(\alpha) \|a\| \|b\| \|c\| \quad (\forall a, b, c \in A). \quad (4.1.2)$$

Suppose that $a \in A^\sharp$, $c \in A$, $b \in A_l$ (see Definition 4.1.1) and let $e \in A$ with $\|e\| \leq M$ be such that $eb = b$. So

$$a(e - 1)bc = 0.$$

Therefore, by (4.1.1), we can write

$$\begin{aligned} \|\varphi(ae, bc) - \varphi(a, bc)\| &= \|\varphi(a(e - 1), bc)\| \\ &\leq \alpha \|ae - a\| \|bc\| \\ &\leq \alpha(M + 1) \|a\| \|b\| \|c\|. \end{aligned}$$

Now if we define $L' : [0, \infty) \rightarrow [0, \infty)$ with

$$L'(\alpha) = ML_{A,K}(\alpha) + (M + 1)\alpha,$$

then, by applying (4.1.2), we get

$$\begin{aligned} \|\varphi(ab, c) - \varphi(a, bc)\| &= \|\varphi(aeb, c) - \varphi(ae, bc) + \varphi(ae, bc) - \varphi(a, bc)\| \\ &\leq \|\varphi(aeb, c) - \varphi(ae, bc)\| + \|\varphi(ae, bc) - \varphi(a, bc)\| \\ &\leq L_{A,K}(\alpha)\|ae\|\|b\|\|c\| + \alpha(M + 1)\|a\|\|b\|\|c\| \\ &\leq L'(\alpha)\|a\|\|b\|\|c\| \end{aligned}$$

Since φ is continuous and $\overline{A_l} = A$, we conclude that

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq L'(\alpha)\|a\|\|b\|\|c\| \quad (\forall a \in A^\sharp, \forall b, c \in A). \quad (4.1.3)$$

Now suppose that $a, c \in A^\sharp$, $b \in A_r$ (see Definition 4.1.1) and let $e \in A$ with $\|e\| \leq M$ be such that $be = b$. Then

$$ab(1 - e)c = 0.$$

Define the continuous function $L_{A^\sharp, K} : [0, \infty) \rightarrow [0, \infty)$ by

$$L_{A^\sharp, K}(\alpha) = ML'(\alpha) + \alpha(M + 1).$$

Using (4.1.3) and (4.1.1), we can write

$$\begin{aligned} \|\varphi(a, bc) - \varphi(ab, c)\| &= \|\varphi(a, bec) - \varphi(ab, ec) + \varphi(ab, ec - c)\| \\ &\leq \|\varphi(a, bec) - \varphi(ab, ec)\| + \|\varphi(ab, ec - c)\| \\ &\leq L'(\alpha)\|a\|\|b\|\|ec\| + \alpha\|a\|\|b\|\|ec - c\| \\ &\leq L_{A^\sharp, K}(\alpha)\|a\|\|b\|\|c\|. \end{aligned}$$

Using the continuity of φ and the fact that A_r is dense in A , we deduce that

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq L_{A^\sharp, K}(\alpha)\|a\|\|b\|\|c\| \quad (\forall a, c \in A^\sharp, \forall b \in A). \quad (4.1.4)$$

Finally suppose that $a, c \in A^\sharp$, $b \in A$ and $\lambda \in \mathbb{C}$. Then

$$\begin{aligned} \|\varphi(a(b + \lambda), c) - \varphi(a, (b + \lambda)c)\| &= \|\varphi(ab, c) - \varphi(a, bc)\| \\ &\leq L_{A^\sharp, K}(\alpha)\|a\|\|b\|\|c\| \\ &\leq L_{A^\sharp, K}(\alpha)\|a\|\|b + \lambda\|\|c\|, \end{aligned}$$

where the first inequality follows from (4.1.4). This completes the proof. \square

We explained in Remark 4.1.2 that a Banach algebra A has b.a.u. if and only if it has a bounded approximate identity. Hence if A has bounded local units, then it must have a bounded approximate identity. We finish this section with the following proposition which shows when the converse holds. This is practical because it shows the connection between having bounded local units and bounded approximate identities.

Proposition 4.1.4. *Let A be a Banach algebra. Then A has b.l.u. if and only if it has a bounded approximate identity and dense subsets A_l and A_r of A such that for every $a \in A_l$ (resp. $b \in A_r$) there is $c \in A$ (resp. $d \in A$) satisfying $ca = a$ and $bd = b$.*

Proof. “ \implies ” Clear.

“ \impliedby ” Let $a \in A_l$ and $c \in A$ such that $ca = a$. Define

$$\text{Ann}_l(a) = \{x \in A : xa = 0\}.$$

Clearly $\text{Ann}_l(a)$ is a closed left ideal of A and for every $x \in A$,

$$xc + \text{Ann}_l(a) = x + \text{Ann}_l(a).$$

Now let $\{e_i\}$ to be an approximate identity of A bounded by a constant M . Then for each i ,

$$e_i c + \text{Ann}_l(a) = e_i + \text{Ann}_l(a),$$

and so,

$$\|c + \text{Ann}_l(a)\| = \lim_{i \rightarrow \infty} \|e_i c + \text{Ann}_l(a)\| = \lim_{i \rightarrow \infty} \|e_i + \text{Ann}_l(a)\| \leq M.$$

Therefore there is $x \in \text{Ann}_l(a)$ such that $\|c + x\| < M + 1$. Since $(c + x)a = ca + xa = a$, we conclude that the elements of A_l can have local units which are uniformly bounded by $M + 1$. Similarly, we can show this holds for A_r . Hence, by Definition 4.1.1, A has bounded local units. □

4.2 Examples of Banach algebras with bounded local units

In this section we provide examples of possibly non-unital Banach algebras which have b.l.u. Our examples contain all C^* -algebras and large classes of group algebras.

Proposition 4.2.1. *Suppose that A is a C^* -algebra. Then A has b.l.u.*

Proof. It is clear that commutative C^* -algebras have b.l.u. (use either Proposition 4.1.4 or a direct construction using Urysohn's lemma). Now suppose that A is a $*$ -subalgebra of $B(H)$ for some Hilbert space H . For every $a \in A$, we can write the polar decomposition $a = U|a|$, where $U \in B(H)$ is a partial isometry and $|a|$ is the positive part of a . Since $C^*(|a|)$, the commutative C^* -algebra generated by $|a|$, lies in A , a can be approximated by elements in A having right local units in $C^*(|a|) \subseteq A$. Similarly we can show that A has a dense subset whose elements have left local units. Therefore A has b.l.u. by Proposition 4.1.4. □

Proposition 4.2.2. *Let G be a locally compact group with an open subgroup of polynomial growth. Then $L^1(G)$ has b.l.u.*

Proof. It is shown in [52, Lemma 3.1] that if G is a locally compact group with an open subgroup of polynomial growth, then $L^1(G)$ has bounded approximate identities $\{\varphi_i\}_{i \in I}$ and $\{\psi_i\}_{i \in I}$ such that for each $i \in I$,

$$\varphi_i * \psi_i = \psi_i * \varphi_i = \varphi_i.$$

Hence if we define

$$L^1(G)_l = \{\varphi_i * f : f \in L^1(G), i \in I\}$$

and

$$L^1(G)_r = \{f * \varphi_i : f \in L^1(G), i \in I\},$$

then $L^1(G)_l$ and $L^1(G)_r$ satisfies the assumption of Definition 4.1.1. Hence $L^1(G)$ has b.l.u. □

4.3 Hereditary properties of Banach algebras with bounded local units

In this section, we investigate the hereditary property of Banach algebras with b.l.u. This will be useful due to Theorem 4.1.3 and Theorem 5.2.4.

Proposition 4.3.1. *Let A be a Banach algebra with b.l.u. Suppose that B is a Banach algebra and $\Phi : A \rightarrow B$ a continuous algebraic homomorphism with dense range. Then B has b.l.u. In particular, if I is a closed ideal of A , then A/I has a b.l.u.*

Proof. Let A_l (resp. A_r) be the set of all elements in A with left (resp. right) local unit as in Definition 4.1.1. Let $M > 0$ be a bound for all local units. Define $B_l = \Phi(A_l)$ and $B_r = \Phi(A_r)$. Since Φ is continuous and A_l and A_r are dense in A , we have

$$B = \overline{\Phi(A_l)} = \overline{B_l}, \quad B = \overline{\Phi(A_r)} = \overline{B_r}.$$

Let $b \in B_l$, and let $a \in A_l$ be such that $b = \Phi(a)$. By the hypothesis, there is $e \in A$ with $ea = a$ and $\|e\| \leq M$. Then

$$b = \Phi(ea) = \Phi(e)b \quad \text{and} \quad \|\Phi(e)\| \leq M\|\Phi\|.$$

A similar argument can be applied to B_r so that B has b.l.u. □

Proposition 4.3.2. *Let A and B be Banach algebras with b.l.u. Then $A \hat{\otimes} B$ has b.l.u.*

Proof. Let A_r (resp. B_r) be the set of all elements in A (resp. B) with right local units and those local units are bounded by M and N , respectively. Define $(A\hat{\otimes}B)_r$ to be

$$(A\hat{\otimes}B)_r := \{u(a \otimes b) : u \in A\hat{\otimes}B, a \in A_r, b \in B_r\}.$$

We show that $(A\hat{\otimes}B)_r$ satisfies the assumption of Definition 4.1.1. For every $u \in A\hat{\otimes}B$, $a \in A_r$ and $b \in B_r$, there are $c \in A$ and $e \in B$ with $\|c\| \leq M$ and $\|e\| \leq N$ such that

$$ac = c \quad \text{and} \quad be = b.$$

So

$$u(a \otimes b)(c \otimes e) = u(a \otimes b).$$

Hence it remains to show that $(A\hat{\otimes}B)_r$ is dense in $A\hat{\otimes}B$. Let $\{a_i\}$ and $\{b_i\}$ be bounded approximate identities for A and B , respectively. Since A_r (resp. B_r) is dense in A (resp. B), we can assume that $\{a_i\} \subset A_r$ and $\{b_j\} \subset B_r$. Hence for every $u \in A\hat{\otimes}B$,

$$u = \lim_{(i,j) \rightarrow \infty} u(a_i \otimes b_j) \in \overline{(A\hat{\otimes}B)_r}.$$

Similarly, we can show that $(A\hat{\otimes}B)_l$ defined by

$$(A\hat{\otimes}B)_l := \{(a \otimes b)u : u \in A\hat{\otimes}B, a \in A_l, b \in B_l\}.$$

satisfies the assumption of Definition 4.1.1. Hence $A\hat{\otimes}B$ has b.l.u. □

Let Ω be a locally compact Hausdorff space and A a Banach algebra. We let $C_0(\Omega, A)$ denote the space of all continuous functions $f : \Omega \rightarrow A$ vanishing at infinity. $C_0(\Omega, A)$ together with the canonical sup-norm and pointwise operations becomes a Banach algebra.

Proposition 4.3.3. *Let Ω be a locally compact Hausdorff space and A a Banach algebra with b.l.u. Then $C_0(\Omega, A)$ has a b.l.u.*

Proof. We know that

$$C_0(\Omega) \check{\otimes} A \cong C_0(\Omega, A)$$

where $\check{\otimes}$ stands for the injective tensor product. Let $i : C_0(\Omega) \hat{\otimes} A \rightarrow C_0(\Omega) \check{\otimes} A$ be the canonical map. i is a homomorphism with dense range. By Proposition 4.2.1, $C_0(\Omega)$ has

b.l.u. Hence Proposition 4.3.2 shows that $C_0(\Omega) \hat{\otimes} A$ has a b.l.u. The result now follows from Proposition 4.3.1. \square

CHAPTER 5

HYPERREFLEXIVITY OF THE BOUNDED n -COCYCLE SPACES OF BANACH ALGEBRAS

In the present chapter, we first generalize the notion of hyperreflexivity to the subspaces of bounded n -linear maps. Then we show that having the strong property (\mathbb{B}) and bounded local units can be fundamental in handling the problem of the hyperreflexivity of the bounded n -cocycle spaces. We are inspired by the idea used in [52], where it is shown that having the property (\mathbb{B}) and local units can conveniently solve the problem of the reflexivity of the bounded n -cocycle spaces. We use our results from Chapters 3 and 4 to give examples of Banach algebras whose bounded n -cocycle spaces are hyperreflexive. Further results will be provided in Chapter 7.

5.1 Generalizing the notion of hyperreflexivity

As it was mentioned before, the notion of hyperreflexivity is defined for the linear subspaces of $B(X, Y)$. We extend this notion to the linear subspaces of $B^n(X, Y)$. We do this in a natural way, as follows.

Definition 5.1.1. Let X and Y be Banach spaces, and let \mathfrak{S} be a closed subspace of $B^n(X, Y)$. For every $T \in B^n(X, Y)$, we define

$$\text{dist}(T, \mathfrak{S}) = \inf_{S \in \mathfrak{S}} \|T - S\|$$

and

$$\text{dist}_r(T, \mathfrak{S}) = \sup_{\|x_i\| \leq 1} \inf_{S \in \mathfrak{S}} \|T(x_1, \dots, x_n) - S(x_1, \dots, x_n)\|.$$

It is clear that for all $T \in B^n(X, Y)$,

$$\text{dist}_r(T, \mathfrak{S}) \leq \text{dist}(T, \mathfrak{S}).$$

We define \mathfrak{S} to be *(n-)reflexive* if for every $T \in B^n(X, Y)$, $\text{dist}_r(T, \mathfrak{S}) = 0$ implies that $\text{dist}(T, \mathfrak{S}) = 0$. We say that \mathfrak{S} is *hyperreflexive* if there exist some $C > 0$ such that for all $T \in B^n(X, Y)$,

$$\text{dist}(T, \mathfrak{S}) \leq C \text{dist}_r(T, \mathfrak{S}).$$

Remark 5.1.2. In [52], the concept of reflexivity for linear subspace of n -linear maps was introduced. It is straightforward to verify that dist_r defines a seminorm on the quotient space $B^n(X, Y)/\mathfrak{S}$ given by

$$\|T + \mathfrak{S}\|_r = \text{dist}_r(T, \mathfrak{S}).$$

Now it follows easily from the definition that \mathfrak{S} is reflexive if and only if $\|\cdot\|_r$ is a norm on $B^n(X, Y)/\mathfrak{S}$. On the other hand, \mathfrak{S} is hyperreflexive if and only if $\|\cdot\|_r$ is equivalent to the dist norm on $B^n(X, Y)/\mathfrak{S}$ which is nothing but the quotient norm on $B^n(X, Y)/\mathfrak{S}$.

5.2 hyperreflexivity of bounded n -cocycle spaces

In this section, we show how one can apply having the strong property (\mathbb{B}) to deduce that certain spaces of bounded n -cocycles are hyperreflexive. The first such relation is presented in Theorem 5.2.2. But first we need the following proposition.

Proposition 5.2.1. *Let A be a unital Banach algebra having the strong property (\mathbb{B}) . Suppose that $K > 0$. Then:*

(i) *There is a continuous function $M_K : [0, \infty) \rightarrow [0, \infty)$ with $M_K(0) = 0$ such that for every right Banach A -module X and a bounded operator $D : A \rightarrow X$ with $\|D\| \leq K$ and each $0 \leq \alpha < K$ satisfying*

$$ab = 0 \Rightarrow \|D(a)b\| \leq \alpha \|b\| \|a\|$$

we have

$$\|D(ab)c - D(a)bc\| \leq M_K(\alpha)\|a\|\|b\|\|c\| \quad (\forall a, b, c \in A).$$

(ii) There is a continuous function $N_K : [0, \infty) \rightarrow [0, \infty)$ with $N_K(0) = 0$ such that for every Banach A -bimodule X and a bounded operator $D : A \rightarrow X$ with $\|D\| \leq K$ and each $0 \leq \beta < K$ satisfying

$$ab = bc = 0 \Rightarrow \|aD(b)c\| \leq \beta\|a\|\|b\|\|c\|$$

we have

$$\|d[D(acb) - aD(cb) - D(ac)b + aD(c)b]e\| \leq N_K(\beta)\|a\|\|b\|\|c\|\|d\|\|e\| \quad (\forall a, b, c, d, e \in A).$$

Proof. (i) We define $\varphi : A \times A \rightarrow X$ with $\varphi(a, b) = D(a)b$. Note that $\|\varphi\| \leq K$. Moreover, if $ab = 0$, then

$$\|\varphi(a, b)\| = \|D(a)b\| \leq \alpha\|a\|\|b\|.$$

Since A has the strong property (\mathbb{B}) , there is a continuous function $L_K : [0, \infty) \rightarrow [0, \infty)$ with $L_K(0) = 0$ such that

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq L_K(\alpha)\|a\|\|b\|\|c\|.$$

or equivalently,

$$\|D(ab)c - D(a)bc\| \leq L_K(\alpha)\|a\|\|b\|\|c\|.$$

So let $M_K = L_K$.

(ii) Fix $a_2, b_2 \in A$ with $a_2b_2 = 0$ and $\|a_2\| = \|b_2\| = 1$. Define $\varphi : A \times A \rightarrow X$ with

$$\varphi(a, b) = aD(ba_2)b_2.$$

Note that $\|\varphi\| \leq K$ and if $ab = 0$, then $a(ba_2) = (ba_2)b_2 = 0$. Hence

$$\|\varphi(a, b)\| = \|aD(ba_2)b_2\| \leq \beta\|a\|\|ba_2\|\|b_2\| \leq \beta\|a\|\|b\|.$$

Since A has the strong property (\mathbb{B}) , there is a continuous function $L_K : [0, \infty) \rightarrow [0, \infty)$ with $L_K(0) = 0$ such that

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq L_K(\beta)\|a\|\|b\|\|c\|.$$

Hence

$$\|abD(ca_2)b_2 - aD(bca_2)b_2\| \leq L_K(\beta)\|a\|\|b\|\|c\|. \quad (5.2.1)$$

Now fix $a, c, d \in A$ with $\|a\| = \|c\| = \|d\| = 1$. Define $\psi : A \times A \rightarrow X$ with

$$\psi(f, b) = daD(cf)b - dD(acf)b.$$

Obviously $\|\psi\| \leq 2K$ and if $fb = 0$, then by (5.2.1)

$$\|\psi(f, b)\| \leq L_K(\beta)\|f\|\|b\|.$$

Let $K' = \max\{L_K(\beta) + 2K : 0 \leq \beta \leq K\}$. Then there is a continuous function $L_{K'} : [0, \infty) \rightarrow [0, \infty)$ with $L_{K'}(0) = 0$ such that

$$\|\psi(fb, e) - \psi(f, be)\| \leq L_{K'}(L_K(\beta))\|f\|\|b\|\|e\|,$$

or equivalently,

$$\|daD(cfb)e - dD(acfb)e - daD(cf)be + dD(acf)be\| \leq L_{K'}(L_K(\beta))\|f\|\|b\|\|e\|.$$

By putting $f = 1$, we get

$$\|d[D(acb) - aD(cb) - D(ac)b + aD(c)b]e\| \leq L_{K'}(L_K(\beta))\|a\|\|b\|\|c\|\|d\|\|e\| \quad (\forall a, b, c, d, e \in A).$$

The final result follows if we put $N_K = L_{K'} \circ L_K$.

□

Now we generalize the result of Proposition 5.2.1 (ii) to higher dimensions.

Theorem 5.2.2. *Let A be a unital Banach algebra with unit 1 having the strong property (\mathbb{B}) . Suppose that X is a unital Banach A -bimodule, $n \in \mathbb{N}$, $K > 0$, $T \in B^n(A, X)$ with $\|T\| \leq K$ and let $0 \leq \gamma < K$ satisfying*

$$a_0a_1 = a_1a_2 = \cdots = a_na_{n+1} = 0 \Rightarrow \|a_0T(a_1, \dots, a_n)a_{n+1}\| \leq \gamma\|a_0\| \cdots \|a_{n+1}\|.$$

Also $T(a_1, \dots, a_n) = 0$ if for some $1 \leq i \leq n$, $a_i = 1$. Then there exists a continuous function $L_{n,K} : [0, \infty) \rightarrow [0, \infty)$ with $L_{n,K}(0) = 0$, depending only on A , n and K , such that

$$\|\delta^n(T)\| \leq L_{n,K}(\gamma).$$

Proof. We prove the statement by induction on n . For $n = 1$, the result follows from Proposition 5.2.1(ii) together with the fact that X is unital and $T(1) = 0$.

Now suppose that the result is true for $n \in \mathbb{N}$. We prove it for $n + 1$. Consider $T \in B^{n+1}(A, X)$ with $\|T\| \leq K$ and $0 \leq \gamma < K$ satisfying

$$a_0 a_1 = a_1 a_2 = \cdots = a_{n+1} a_{n+2} = 0 \Rightarrow \|a_0 T(a_1, \dots, a_{n+1}) a_{n+2}\| \leq \gamma \|a_0\| \cdots \|a_{n+2}\|.$$

Also $T(a_1, \dots, a_{n+1}) = 0$ if for some $1 \leq i \leq n + 1$, $a_i = 1$. Take $a_i \in A$, $i = 0, \dots, n + 1$ with $a_0 a_1 = a_1 a_2 = \cdots = a_n a_{n+1} = 0$. We first show that there is a continuous function $N_K : [0, \infty) \rightarrow [0, \infty)$ with $N_K(0) = 0$, depending only on A and K , such that

$$\|a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1}\| \leq N_K(\gamma) \|a_0\| \cdots \|a_{n+1}\| \quad (5.2.2)$$

where the action \star is defined in Remark 2.3.6. First suppose that $\|a_0\| = \cdots = \|a_{n+1}\| = 1$, and let

$$S = a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1}.$$

For every $b, c \in A$ with $bc = 0$, we have

$$\begin{aligned} S(b)c &= [a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1}](b)c \\ &= a_0 \Lambda_n(T)(a_1, \dots, a_n)(a_{n+1}b)c - a_0 \Lambda_n(T)(a_1, \dots, a_n)(a_{n+1})bc \\ &= a_0 T(a_1, \dots, a_n, a_{n+1}b)c. \end{aligned}$$

But $a_0 a_1 = \cdots = a_n(a_{n+1}b) = (a_{n+1}b)c = 0$. Thus, by our hypothesis

$$\|a_0 T(a_1, \dots, a_n, a_{n+1}b)c\| \leq \gamma \|a_0\| \cdots \|a_{n+1}b\| \|c\| \leq \gamma \|b\| \|c\|,$$

implying that $\|S(b)c\| \leq \gamma \|b\| \|c\|$. Hence, by Proposition 5.2.1(i), there exist a continuous function $N'_K : [0, \infty) \rightarrow [0, \infty)$ with $N'_K(0) = 0$, depending only on A and K , such that

$$\|S(bc) - S(b)c\| \leq N'_K(\gamma) \|b\| \|c\| \quad (\forall b, c \in A). \quad (5.2.3)$$

On the other hand,

$$\begin{aligned} S(1) &= (a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1})(1) \\ &= a_0 \Lambda_n(T)(a_1, \dots, a_n)(a_{n+1}1) - a_0 \Lambda_n(T)(a_1, \dots, a_n)(a_{n+1})1 \\ &= 0. \end{aligned}$$

Putting $b = 1$ in (5.2.3) and $N_K = \|1\|N'_K$, we get

$$\|S(c)\| \leq N_K(\gamma)\|c\| \quad (c \in A),$$

or equivalently,

$$\|S\| = \|a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1}\| \leq N_K(\gamma). \quad (5.2.4)$$

Now consider the general case. If for some $0 \leq i \leq n+1$, $a_i = 0$, then we clearly have

$$\|a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1}\| \leq N_K(\gamma)\|a_0\| \dots \|a_{n+1}\|.$$

Now suppose that for all $0 \leq i \leq n+1$, $a_i \neq 0$. Then

$$\frac{a_0}{\|a_0\|} \frac{a_1}{\|a_1\|} = \dots = \frac{a_{n+1}}{\|a_{n+1}\|} \frac{a_{n+2}}{\|a_{n+2}\|} = 0,$$

and so, by (5.2.4),

$$\left\| \frac{a_0}{\|a_0\|} \star \Lambda_n(T)\left(\frac{a_1}{\|a_1\|}, \dots, \frac{a_n}{\|a_n\|}\right) \star \frac{a_{n+1}}{\|a_{n+1}\|} \right\| \leq N_K(\gamma)$$

implying that (5.2.2) holds.

Now let $B_A(A, X)$ denote the space of all (bounded) right A -module morphisms from A into X and suppose that $q : B(A, X) \rightarrow \frac{B(A, X)}{B_A(A, X)}$ is the natural quotient mapping. It is straightforward to verify that $\frac{B(A, X)}{B_A(A, X)}$ is a unital Banach A -bimodule and q is an A -bimodule morphism. Thus, by (5.2.2),

$$\begin{aligned} \|a_0 \star q(\Lambda_n(T)(a_1, \dots, a_n)) \star a_{n+1}\| &= \|q(a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1})\| \\ &\leq \|q\| \|a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1}\| \\ &\leq N_K(\gamma)\|a_0\| \dots \|a_{n+1}\|. \end{aligned}$$

Moreover if for some i , $1 \leq i \leq n$, $a_i = 1$, then for every $a \in A$,

$$\Lambda_n(T)(a_1, \dots, a_n)(a) = T(a_1, \dots, a_n, a) = 0.$$

This shows that $q \circ \Lambda_n(T)(a_1, \dots, a_n) = 0$ if for some $1 \leq i \leq n$, $a_i = 1$. Let

$$K' = \sup\{N_K(\gamma) + K : 0 \leq \gamma \leq K\}.$$

Then, by the assumption of the induction, there is a continuous function $L_{K'} : [0, \infty) \rightarrow [0, \infty)$ with $L_{K'}(0) = 0$, depending only on A , n and K , such that for all $a_0, \dots, a_{n+1} \in A$,

$$\|\Delta_q^n(q \circ \Lambda_n(T))(a_1, \dots, a_{n+1})\| \leq L_{K'}(N_K(\gamma))\|a_1\| \cdots \|a_{n+1}\|, \quad (5.2.5)$$

where $\Delta_q^n : B^n(A, \frac{B(A, X)}{B_A(A, X)}) \rightarrow B^{n+1}(A, \frac{B(A, X)}{B_A(A, X)})$ is the corresponding connecting map defined in Definition 2.3.4. On the other hand, since q is a Banach A -bimodule morphism, it is easy to check that for all $a_0, \dots, a_{n+1} \in A$,

$$\begin{aligned} \Delta_q^n(q \circ \Lambda_n(T))(a_1, \dots, a_{n+1}) &= q(\Delta^n(\Lambda_n(T))(a_1, \dots, a_{n+1})) \\ &= q(\Lambda_{n+1}(\delta^{n+1}(T))(a_1, \dots, a_{n+1})) \end{aligned}$$

where the last equality follows from (2.3.1). Hence, by (5.2.5),

$$\|q(\Lambda_{n+1}(\delta^{n+1}(T))(a_1, \dots, a_{n+1}))\| \leq L_{K'}(N_K(\gamma))\|a_1\| \cdots \|a_{n+1}\|,$$

implying that for $S = \Lambda_{n+1}(\delta^{n+1}(T))(a_1, \dots, a_{n+1})$,

$$\|\text{dist}(S, B_A(A, X))\| \leq L_{K'}(N_K(\gamma))\|a_1\| \cdots \|a_{n+1}\|.$$

So for every $a \in A$, we have

$$\|S(a) - S(1)a\| \leq [2L_{K'}(N_K(\gamma))\|a_1\| \cdots \|a_{n+1}\|]\|a\|. \quad (5.2.6)$$

On the other hand,

$$\begin{aligned} S(1) &= \Lambda_{n+1}(\delta^{n+1}(T))(a_1, \dots, a_{n+1})(1) \\ &= \delta^{n+1}(T)(a_1, \dots, a_{n+1}, 1) \\ &= a_1 T(a_2, \dots, a_{n+1}, 1) + \sum_{j=0}^{n-1} (-1)^j T(a_1, \dots, a_j a_{j+1}, \dots, a_{n+1}, 1) + (-1)^n T(a_1, \dots, a_{n+1} 1) \\ &+ (-1)^{n+1} T(a_1, \dots, a_{n+1}) 1 \\ &= 0. \end{aligned}$$

Therefore by putting $a = a_{n+2}$ in (5.2.6), we have

$$\begin{aligned} \|\delta^{n+1}(T)(a_1, \dots, a_{n+2})\| &= \|\Lambda_{n+1}(\delta^{n+1}(T))(a_1, \dots, a_{n+1})(a_{n+2})\| \\ &= \|S(a_{n+2})\| \\ &\leq 2L_{K'}(N_K(\gamma))\|a_1\| \cdots \|a_{n+1}\|\|a_{n+2}\|. \end{aligned}$$

Hence our proof is complete if we define $L_{n+1,K} : [0, \infty) \rightarrow [0, \infty)$ by

$$L_{n+1,K} = 2L_{K'} \circ N_K.$$

□

Even though Theorem 5.2.2 represents a nice formula, it heavily depends on the fact that the Banach algebra is unital. A possible way to extend and apply this theorem is to consider the unitization of the given Banach algebra. Although we do not know whether the unitization of a Banach algebra with the strong property (\mathbb{B}) inherits this property, the existence of b.l.u. for a Banach algebra makes this happen by Theorem 4.1.3. Hence for many Banach algebras, passing to the unitization is a good method to eliminate the assumption of being unital.

Let A be a Banach algebra, and let X be a Banach A -bimodule. We extend X to a Banach A^\sharp -bimodule by defining

$$1 \cdot x = x \cdot 1 = x.$$

Let $\sigma : B^n(A, X) \rightarrow B^n(A^\sharp, X)$ be the linear isometry defined by

$$\sigma(T)(a_1 + \lambda_1, \dots, a_n + \lambda_n) = T(a_1, \dots, a_n) \quad (5.2.7)$$

where $a_i \in A$ and $\lambda_i \in \mathbb{C}$. It is to check that if $\delta^{\sharp n} : B^n(A^\sharp, X) \rightarrow B^{n+1}(A^\sharp, X)$ is the corresponding connecting map, then we have

$$\delta^{\sharp n}(\sigma(T))(a_1 + \lambda_1, \dots, a_{n+1} + \lambda_{n+1}) = \delta^n(T)(a_1, \dots, a_{n+1})$$

So $\sigma(T)$ is an n -cocycle if and only if T is an n -cocycle.

Lemma 5.2.3. *Let A be a Banach algebra. Let X be a Banach A -bimodule, and let $T \in B^n(A, X)$. Then for every $a_i \in A^\sharp$, $i = 0, \dots, n+1$ with $a_0 a_1 = \dots = a_n a_{n+1} = 0$, we have*

$$\|a_0 \sigma(T)(a_1, \dots, a_n) a_{n+1}\| \leq \text{dist}_r(T, \mathcal{Z}^n(A, X)) \|a_0\| \cdots \|a_{n+1}\|$$

where $\sigma : B^n(A, X) \rightarrow B^n(A^\sharp, X)$ is the linear map defines in (5.2.7).

Proof. Let $a_i = b_i + \lambda_i \in A^\sharp$, $0 \leq i \leq n+1$ with $b_i \in A$ and $\lambda_i \in \mathbb{C}$ such that $a_0 a_1 = \cdots = a_n a_{n+1} = 0$. Then for $D \in \mathcal{Z}^n(A, X)$, $\sigma(D) \in \mathcal{Z}^n(A^\sharp, X)$, and so,

$$a_0 \sigma(D)(a_1, \dots, a_n) a_{n+1} = 0.$$

Thus

$$\begin{aligned} \|a_0 \sigma(T)(a_1, \dots, a_n) a_{n+1}\| &= \|a_0 [\sigma(T)(a_1, \dots, a_n) - \sigma(D)(a_1, \dots, a_n)] a_{n+1}\| \\ &= \|a_0 \sigma(T - D)(a_1, \dots, a_n) a_{n+1}\| \\ &\leq \|a_0\| \| (T - D)(b_1, \dots, b_n) \| \|a_{n+1}\|. \end{aligned}$$

Since $D \in \mathcal{Z}^n(A, X)$ was arbitrary, we have

$$\begin{aligned} \|a_0 \sigma(T)(a_1, \dots, a_n) a_{n+1}\| &\leq \inf_{D \in \mathcal{Z}^n(A, X)} \|T(b_1, \dots, b_n) - D(b_1, \dots, b_n)\| \|a_0\| \|a_{n+1}\| \\ &\leq \text{dist}_r(T, \mathcal{Z}^n(A, X)) \|b_1\| \cdots \|b_n\| \|a_0\| \|a_{n+1}\| \\ &\leq \text{dist}_r(T, \mathcal{Z}^n(A, X)) \|a_0\| \cdots \|a_{n+1}\|. \end{aligned}$$

□

We are now ready to present the main result of this section.

Theorem 5.2.4. *Let A be a Banach algebra for which its unitization has the strong property (\mathbb{B}) . Let $n \in \mathbb{N}$, and let X be a Banach A -bimodule such that $\mathcal{H}^{n+1}(A, X)$ is a Banach space. Then $\mathcal{Z}^n(A, X)$ is hyperreflexive.*

Proof. We recall from [52] that $\mathcal{Z}^n(A, X)$ is reflexive. Hence according to Remark 5.1.2, dist_r defines a norm and it suffices to show that norms given by dist and dist_r on $B^n(A, X)/\mathcal{Z}^n(A, X)$ are equivalent. Suppose otherwise, i.e., there is a sequence $\{T_m\} \subseteq B^n(A, X)$ such that for all $m \in \mathbb{N}$,

$$\gamma_m := \text{dist}_r(T_m, \mathcal{Z}^n(A, X)) < \frac{1}{m+1} \tag{5.2.8}$$

but

$$\text{dist}(T_m, \mathcal{Z}^n(A, X)) = \frac{1}{2}. \tag{5.2.9}$$

We can replace T_m by $T_m + D_m$ for some suitable $D_m \in \mathcal{Z}^n(A, X)$ to assume that $\|T_m\| \leq 1$. By Lemma 5.2.3, for every $a_0, \dots, a_{n+1} \in A^\sharp$ with $a_0 a_1 = \dots = a_n a_{n+1} = 0$,

$$\|a_0 \sigma(T_m)(a_1, \dots, a_n) a_{n+1}\| \leq \gamma_m \|a_0\| \dots \|a_{n+1}\|.$$

Hence, by our hypothesis and Theorem 5.2.2, there exists a continuous function $L : [0, \infty) \rightarrow [0, \infty)$ with $L(0) = 0$ such that

$$\|\delta^{\sharp n}(\sigma(T_m))\| \leq L(\gamma_m),$$

where $\delta^{\sharp n} : B^n(A^\sharp, X) \rightarrow B^{n+1}(A^\sharp, X)$ is the corresponding connecting map. On the other hand, it is clear that

$$\|\delta^n(T_m)\| \leq \|\delta^{\sharp n}(\sigma(T_m))\|.$$

Therefore

$$\|\delta^n(T_m)\| \leq L(\gamma_m).$$

In particular, from (5.2.8), we have

$$\lim_{m \rightarrow \infty} \|\delta^n(T_m)\| = 0. \quad (5.2.10)$$

However, $\mathcal{H}^{n+1}(A, X)$ is a Banach space which implies that $\text{Im} \delta^n$ is closed. Hence, by the open mapping theorem, there is a constant $C > 0$ such that for each $T \in B^n(A, X)$,

$$\text{dist}(T, \mathcal{Z}^n(A, X)) \leq C \|\delta^n(T)\|.$$

In particular, for all $m \in \mathbb{N}$,

$$\text{dist}(T_m, \mathcal{Z}^n(A, X)) \leq C \|\delta^n(T_m)\|.$$

Hence it follows from (5.2.10) that

$$\lim_{m \rightarrow \infty} \text{dist}(T_m, \mathcal{Z}^n(A, X)) = 0,$$

which is a contradiction to our assumption (5.2.9). Thus $\mathcal{Z}^n(A, X)$ is hyperreflexive. \square

As it is described below, the result of Theorem 5.2.4 does not depend on the choice of the complete norm on the unitization of A .

Remark 5.2.5. Let A be a closed subalgebra of a unital Banach algebra B with unit e such that $e \notin A$. Then the (closed) subalgebra in B generated by A and e is nothing but $A \oplus \mathbb{C}e$. Now it is clear that the mapping

$$\Lambda : A^\# \rightarrow A \oplus \mathbb{C}e \quad a + \lambda \mapsto a + \lambda e,$$

is a bounded linear bijection. Hence, by the Inverse Mapping Theorem, it is a bounded algebra isomorphism. Therefore $A^\#$ can be embedded isomorphically into B and can be viewed as a closed subalgebra with an equivalent norm. According to Corollary 3.2.3 $A^\#$ has the strong property (\mathbb{B}) if and only if $A \oplus \mathbb{C}e \subseteq B$ has the strong property (\mathbb{B}) . This implies that having the strong property (\mathbb{B}) is independent of the choice of adding a unit to our Banach algebra and hence, the choice of the unitization norm in Theorem 5.2.4 was just out of convenience.

5.3 Examples of Banach algebras with hyperreflexive bounded n -cocycles spaces

In this section, we put together our results we presented in the previous sections to give examples of classes of Banach algebras for which certain bounded n -cocycle spaces are hyperreflexive.

Our first result is on C^* -algebras. In [58], Shulman proved that for a C^* -algebra A , $\mathcal{Z}^1(A, A)$ is hyperreflexive provided that $\mathcal{H}^2(A, A) = 0$. In the following theorem, we extend Shulman's result to larger classes of bounded n -cocycles.

Theorem 5.3.1. *Suppose that A is a C^* -algebra, $n \in \mathbb{N}$, and X is a Banach A -bimodule for which $\mathcal{H}^{n+1}(A, X)$ is a Banach space. Then $\mathcal{Z}^n(A, X)$ is hyperreflexive. In particular, this is true in either of the following cases:*

- (i) A is amenable and X is a dual Banach A -bimodule;
- (ii) $A = X$ and A is an injective von Neumann algebra.

Proof. By Remark 3.1.4, every C^* -algebra has the strong property (\mathbb{B}) . The result now follows from Proposition 4.2.1, Theorem 4.1.3, and Theorem 5.2.4. We also note that for

every $n \in \mathbb{N}$, $\mathcal{H}^n(A, X) = 0$ when A is an injective von Neumann algebra and $A = X$ or A is amenable and X is a dual Banach A -bimodule. \square

We now turn our attention to group algebras associated to locally compact groups. The hyperreflexivity of the derivation space from groups algebras have been studied in [3], [4], and [52]. In particular, it is shown in [52, Theorem 4.3] that if G is a locally compact amenable group with an open subgroup of polynomial growth, and X a Banach A -bimodule, then $\mathcal{Z}^1(L^1(G), X^*)$ is hyperreflexive. In [4, Theorem 4.5], the preceding result was generalized for $X = C_0(G)$ by removing the amenability condition. In the following theorem, we generalized [52, Theorem 4.3] to the space of bounded n -cocycles from $L^1(G)$ and certain ideas associated to it.

For a normal closed subgroup H of G , we let $T_H : C_c(G) \rightarrow C_c(G/H)$ be operator defined by

$$T_H(f)(xH) = \int_H f(xh)dh \quad (f \in C_c(G))$$

where dh is the Haar measure on H . It is well-known that T_H extended to a continuous algebra homomorphism from $L^1(G)$ onto $L^1(G/H)$. We denote $I(H^\perp) = \ker T_H$.

Theorem 5.3.2. *Let G be a locally compact amenable group with an open subgroup of polynomial growth, and let H be a normal closed subgroup of G . Then:*

- (1) $I(H^\perp)$ has both b.l.u. and the strong property (\mathbb{B}) ;
- (2) For every $n \in \mathbb{N}$ and Banach $I(H^\perp)$ -bimodule X , $\mathcal{Z}^n(I(H^\perp), X^*)$ is hyperreflexive.

Proof. (1) By Remark 3.1.4, every group algebra has the strong property (\mathbb{B}) . Also Proposition 4.2.2 shows that $L^1(G)$ has b.l.u. Now let H be a normal closed subgroup. Since G is amenable, it is known that $I(H^\perp)$ has a bounded approximate identity. Hence it has the strong property (\mathbb{B}) from Proposition 3.2.1. On the other hand, it is shown in [35, Theorem 2] (see also [35, Lemma 3]) that $I(H^\perp)$ has a dense linear space such that each element has a local unit. Hence it follows from Proposition 4.1.4 that $I(H^\perp)$ has b.l.u.

(2) Since $L^1(G)$ is an amenable Banach algebra and $I(H^\perp)$ has a bounded approximate

identity, $I(H^\perp)$ is amenable so that its n^{th} Hochschild cohomology with coefficient in dual modules vanishes. Thus the result follows from part(1), Theorem 4.1.3 and Theorem 5.2.4. \square

Theorem 5.3.3. *Let G be a locally compact group with an open subgroup of polynomial growth. Then $\mathcal{Z}^1(L^1(G), (L^1(G))^{(n)})$ is hyperreflexive for $n = 0$ and for each odd $n \in \mathbb{N}$ where $(L^1(G))^{(n)}$ stands for the n^{th} dual space of $L^1(G)$. In particular, $\mathcal{Z}^1(L^1(G), L^\infty(G))$ is hyperreflexive.*

Proof. It was shown in the proof of Theorem 5.3.2 that if G is a locally compact group with an open subgroup of polynomial growth, then $L^1(G)$ has both the strong property (\mathbb{B}) and b.l.u. It is also shown in [4, Theorem 2.5] that $\mathcal{H}^2(L^1(G), L^1(G))$ is a Banach space. Moreover [43, Theorem 3.3] shows that $\mathcal{H}^2(L^1(G), (L^1(G))^{(n)})$ is a Banach space for each locally compact group G and each odd $n \in \mathbb{N}$. Hence the result follows from Theorem 4.1.3 and Theorem 5.2.4. \square

Remark 5.3.4. This is proven in [52] that each of the following locally compact groups has an open subgroup of polynomial growth:

- (i) G is a group of polynomial growth.
- (ii) G is an IN -group.
- (iii) G is a maximally almost periodic group.
- (iv) G is a totally disconnected group.

We finish this section by presenting the following two propositions which enable us to construct more examples of Banach algebras with hypereflexive bounded n -cocycle spaces from the known ones. As it is with the general approach of this thesis, we need to consider Banach algebras which inherit both the strong property (\mathbb{B}) and b.l.u. It is shown in Chapters 3 and 4 that quotients and tensor products behave well with this regards.

Proposition 5.3.5. *Let A be a Banach algebra with b.l.u. which has the strong property (\mathbb{B}) . Suppose that I is a closed ideal of A and X is a Banach A/I -module such that $\mathcal{H}^{n+1}(A/I, X)$ is a Banach space. Then $\mathcal{Z}^n(A/I, X)$ is hyperreflexive.*

Proof. By Proposition 4.3.1, A/I has a b.l.u. Also Corollary 3.2.2 shows that A/I has the strong property (\mathbb{B}) . Thus the result now follows from Theorems 4.1.3 and 5.2.4. \square

Proposition 5.3.6. *Let A and B be Banach algebras having b.l.u. and the strong property (\mathbb{B}) . Let X be a Banach $A\hat{\otimes}B$ -module such that $\mathcal{H}^{n+1}(A\hat{\otimes}B, X)$ is a Banach space. Then $\mathcal{Z}^n(A\hat{\otimes}B, X)$ is hyperreflexive.*

Proof. By Proposition 4.3.2, $A\hat{\otimes}B$ has a b.l.u. Also Proposition 3.2.5 shows that $A\hat{\otimes}B$ has the strong property (\mathbb{B}) . The result now follows from Theorems 4.1.3 and 5.2.4. \square

CHAPTER 6

A CONSTANT FOR THE STRONG PROPERTY (\mathbb{B})

In the present chapter, we provide an important tool which will be used in the following chapter to obtain upper bounds for the *hyperreflexivity constant* of the bounded n -cocycle spaces of Banach algebras. Our approach towards the problem of hyperreflexivity shows that a possible way to find such a constant is to show a Banach algebra has the strong property (\mathbb{B}) with a “special associated function” (see Definition 6.1.1 below).

We show that for many Banach algebras which we have shown to have the strong property (\mathbb{B}) , we can choose a linear function to represent this associated function, so that we can associate a constant. This includes group algebras, C^* -algebras, finite CSL and nest algebras. We also prove that for any arbitrary Banach algebra, there are related Banach algebras which have the strong property (\mathbb{B}) with a constant.

6.1 General definition

A Banach algebra is said to have the strong property (\mathbb{B}) with a constant if its associated function is a line as described below.

Definition 6.1.1. We say that a Banach algebra A has the strong property (\mathbb{B}) with a constant $r > 0$ if for each Banach space X and every bounded bilinear map $\varphi : A \times A \rightarrow X$ with the property that

$$a, b \in A \quad ab = 0 \Rightarrow \|\varphi(a, b)\| \leq \alpha \|a\| \|b\|,$$

We can infer that

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq r\alpha \|a\| \|b\| \|c\| \quad (\forall a, b, c \in A).$$

Remark 6.1.2. Definition 6.1.1 of a Banach algebra with the strong property (\mathbb{B}) is equivalent to the following definition:

A Banach algebra A has the strong property (\mathbb{B}) with a constant $r > 0$ if for every bounded bilinear map $\varphi : A \times A \rightarrow \mathbb{C}$, we have

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq r\alpha(\varphi)\|a\|\|b\|\|c\|, \quad (\forall a, b, c \in A)$$

where

$$\alpha(\varphi) = \sup\{\|\varphi(a, b)\| : a, b \in A, \|a\|, \|b\| \leq 1, ab = 0\}.$$

We will use this alternative definition when it is more convenient.

We will see later in Chapter 7 that existence of a constant for the strong property (\mathbb{B}) is fundamental in finding an upper bound for the hyperreflexivity constant of the bounded n -cocycle spaces.

6.2 Fourier algebra of the unit circle

As it was mentioned above, in order to achieve our goal in finding an upper bound for the hyperreflexivity constant of the bounded n -cocycle spaces of C^* -algebras and group algebras, we need to find a constant for the strong property (\mathbb{B}) for such Banach algebras. In the present section we aim to find such a constant for the Fourier algebra of the unit circle. This result is shown to be fundamental to find a constant for the strong property (\mathbb{B}) of C^* -algebras and group algebras. We start with the following essential lemma. Let \mathbb{T} denote the unit circle in \mathbb{C} , i.e.

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

Here we identify \mathbb{T} with $\frac{\mathbb{R}}{\mathbb{Z}} \cong [-\pi, \pi]$. In this case $s = t$ if $s \equiv t \pmod{2\pi\mathbb{Z}}$. For every $f \in L^1(\mathbb{T})$, the Fourier transform on f , denoted by \hat{f} , is defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int} dt, \quad (n \in \mathbb{Z}).$$

The Fourier algebra of the unit circle is defined as follows

$$A(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \|f\|_{A(\mathbb{T})} = \sum_{n \in \mathbb{Z}} |\hat{f}(n)| < \infty\}.$$

It is well-known that $A(\mathbb{T}) \subseteq C(\mathbb{T})$, the space of continuous functions on \mathbb{T} . Also $A(\mathbb{T})$ with the pointwise addition and multiplication and $\|\cdot\|_{A(\mathbb{T})}$, is a Banach algebra.

Lemma 6.2.1. *Let X be a Banach space and $F : A(\mathbb{T}) \rightarrow X$ a linear map with $\|F\| \leq 1$. Suppose that $0 \leq \alpha \leq 1$ is such that for each $\varphi, \psi \in A(\mathbb{T})$ with $\text{supp } \varphi \cap \text{supp } \psi = \emptyset$, we have*

$$\|F(\varphi * \check{\psi})\| \leq \alpha \|\varphi\| \|\psi\|.$$

Let $f \in A(\mathbb{T})$ be given by $f(s) = e^{is} - 1$. Then

$$\|F(f)\| \leq 12\sqrt{\pi(1 + \sqrt{2})}\sqrt{\alpha}.$$

Proof. Let $0 < \epsilon < 3$. Define

$$W_\epsilon = \{x \in \mathbb{T} : \|f - R_x f\|_{A(\mathbb{T})} < \epsilon\},$$

where $(R_x f)(s) = f(s + x)$. Note that for $s \in \mathbb{T}$

$$(f - R_x f)(s) = e^{is}(1 - e^{ix}).$$

Hence if we define $e_1(s) = e^{is}$, then

$$\|f - R_x f\|_{A(\mathbb{T})} = \|e_1\| \|1 - e^{ix}\| = |1 - e^{ix}|.$$

So

$$W_\epsilon = \{x \in \mathbb{T} : |1 - e^{ix}| < \epsilon\}.$$

We show that for each $0 < \delta < \epsilon$, $[-(\epsilon - \delta), (\epsilon - \delta)] \subseteq W_\epsilon$. Define $g : [-\pi, \pi] \rightarrow \mathbb{T}$ by $g(s) = 1 - e^{is}$. Let $0 < x < \pi$. Applying vector-valued mean value theorem to the function $g|_{[0,x]}$, we find $0 < c < x$ with

$$|g(x)| = |g(x) - g(0)| \leq |g'(c)||x| \leq |x|.$$

If $-\pi < x < 0$, we use the same argument on the interval $[x, 0]$. For $x = 0$, the inequality trivially holds. So for each $0 < \delta < \epsilon$ and for all $x \in [-(\epsilon - \delta), (\epsilon - \delta)]$ we get

$$|e^{ix} - 1| = |g(x)| \leq |x| < \epsilon.$$

It means that $[-(\epsilon - \delta), (\epsilon - \delta)] \subseteq W_\epsilon$. Define

$$V_{\epsilon, \delta} = \left[\frac{-(\epsilon - \delta)}{3}, \frac{(\epsilon - \delta)}{3} \right], \quad U_{\epsilon, \delta} = \left[\frac{-(\epsilon - \delta)}{6}, \frac{(\epsilon - \delta)}{6} \right].$$

Then $V_{\epsilon, \delta} + V_{\epsilon, \delta} + V_{\epsilon, \delta} \subseteq W_\epsilon$ and $U_{\epsilon, \delta} + U_{\epsilon, \delta} = V_{\epsilon, \delta}$. Now put

$$u = \frac{1}{\lambda(U_{\epsilon, \delta})^2} 1_{U_{\epsilon, \delta}} * 1_{U_{\epsilon, \delta}}$$

and

$$v = f\left(\frac{1}{\lambda(V_{\epsilon, \delta})} 1_{V_{\epsilon, \delta} + V_{\epsilon, \delta}} * 1_{V_{\epsilon, \delta}}\right). \quad (6.2.1)$$

Obviously, $1_{U_{\epsilon, \delta}} \in L^2(\mathbb{T})$. Since $A(\mathbb{T}) = L^2(\mathbb{T}) * L^2(\mathbb{T})$, we have $u \in A(\mathbb{T}) \subseteq C(\mathbb{T}) \subseteq L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$. It is easy to check that $\|1_{U_{\epsilon, \delta}}\|_2 = \sqrt{\lambda(U_{\epsilon, \delta})}$. By definition of the Fourier norm,

$$\begin{aligned} \|u\|_{A(\mathbb{T})} &\leq \frac{1}{\lambda(U_{\epsilon, \delta})^2} \|1_{U_{\epsilon, \delta}}\|_2 \|1_{U_{\epsilon, \delta}}\|_2 \\ &= \frac{1}{\lambda(U_{\epsilon, \delta})} = \frac{6\pi}{\epsilon - \delta}. \end{aligned} \quad (6.2.2)$$

Since $1_{U_{\epsilon, \delta}} \in L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$ and $L^2(\mathbb{T})$ is $L^1(\mathbb{T})$ -module with respect to the convolution,

$$\|u\|_2 \leq \frac{1}{\lambda(U_{\epsilon, \delta})^2} \|1_{U_{\epsilon, \delta}}\|_1 \|1_{U_{\epsilon, \delta}}\|_2. \quad (6.2.3)$$

It is easy to check that $\|1_{U_{\epsilon, \delta}}\|_1 = \lambda(U_{\epsilon, \delta})$. So by (6.2.3),

$$\|u\|_2 \leq \frac{\lambda(U_{\epsilon, \delta})^{\frac{3}{2}}}{\lambda(U_{\epsilon, \delta})^2} = \frac{1}{\lambda(U_{\epsilon, \delta})^{\frac{1}{2}}} = \sqrt{\frac{6\pi}{\epsilon - \delta}}. \quad (6.2.4)$$

We show that $\text{supp } u \subseteq U_{\epsilon, \delta} + U_{\epsilon, \delta}$. Let $x \in [-\pi, \pi]$. Then

$$\begin{aligned} (1_{U_{\epsilon, \delta}} * 1_{U_{\epsilon, \delta}})(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} 1_{U_{\epsilon, \delta}}(y) 1_{U_{\epsilon, \delta}}(x - y) dy \\ &= \frac{1}{2\pi} \int_{U_{\epsilon, \delta}} 1_{U_{\epsilon, \delta}}(x - y) dy. \end{aligned}$$

So for x to be in $\text{supp } u$, there should exist $y \in \text{supp } 1_{U_{\epsilon,\delta}} = U_{\epsilon,\delta}$ such that $x - y \in \text{supp } 1_{U_{\epsilon,\delta}} = U_{\epsilon,\delta}$. So $x \in U_{\epsilon,\delta} + U_{\epsilon,\delta}$.

We also have

$$\|u\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) dx = \frac{1}{\lambda(U_{\epsilon,\delta})^2} \|1_{U_{\epsilon,\delta}}\|_1 \|1_{U_{\epsilon,\delta}}\|_1 = 1. \quad (6.2.5)$$

Next we prove some properties related to v defined in (6.2.1).

First of all, note that $1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}}, 1_{V_{\epsilon,\delta}} \in L^2(\mathbb{T})$. So $1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}} * 1_{V_{\epsilon,\delta}} \in A(\mathbb{T})$ which implies that $v \in A(\mathbb{T})$. Also

$$\begin{aligned} \|v\|_{A(\mathbb{T})} &\leq \|f\|_{A(\mathbb{T})} \left\| \frac{1}{\lambda(V_{\epsilon,\delta})} 1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}} * 1_{V_{\epsilon,\delta}} \right\|_{A(\mathbb{T})} \\ &\leq \frac{1}{\lambda(V_{\epsilon,\delta})} \|f\|_{A(\mathbb{T})} \|1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}}\|_2 \|1_{V_{\epsilon,\delta}}\|_2. \end{aligned}$$

Obviously, $\|1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}}\|_2 = \sqrt{\lambda(V_{\epsilon,\delta} + V_{\epsilon,\delta})}$ and $\|1_{V_{\epsilon,\delta}}\|_2 = \sqrt{\lambda(V_{\epsilon,\delta})}$. So

$$\begin{aligned} \|v\|_{A(\mathbb{T})} &\leq \|f\|_{A(\mathbb{T})} \left(\frac{\lambda(V_{\epsilon,\delta} + V_{\epsilon,\delta})}{\lambda(V_{\epsilon,\delta})} \right)^{\frac{1}{2}} \\ &= 2 \left(\frac{\frac{4(\epsilon-\delta)}{6\pi}}{\frac{2(\epsilon-\delta)}{6\pi}} \right)^{\frac{1}{2}} = 2\sqrt{2}. \end{aligned} \quad (6.2.6)$$

Using (6.2.6), we can write

$$\begin{aligned} \|f - v\|_{A(\mathbb{T})} &\leq \|f\|_{A(\mathbb{T})} + \|v\|_{A(\mathbb{T})} \\ &\leq 2(1 + \sqrt{2}). \end{aligned} \quad (6.2.7)$$

Similar to what we proved for u , we have

$$\text{supp } v \subseteq \text{supp } (1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}} * 1_{V_{\epsilon,\delta}}) \subseteq V_{\epsilon,\delta} + V_{\epsilon,\delta} + V_{\epsilon,\delta} \subseteq W_{\epsilon}.$$

We now show that for each $x \in V_{\epsilon,\delta}$, $f(x) = v(x)$. To see this, take $x \in V_{\epsilon,\delta}$. Then

$$\begin{aligned} (1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}} * 1_{V_{\epsilon,\delta}})(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} 1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}}(x-w) 1_{V_{\epsilon,\delta}}(w) dw \\ &= \frac{1}{2\pi} \int_{V_{\epsilon,\delta}} 1_{V_{\epsilon,\delta}+V_{\epsilon,\delta}}(x-w) dw \\ &= \lambda(V_{\epsilon,\delta}) \end{aligned}$$

Hence $f(x) = v(x)$. This implies that

$$\text{supp}(f - v) \subseteq V_{\epsilon, \delta}^c. \quad (6.2.8)$$

We show that $\|v\|_2 \leq 2\epsilon\sqrt{\frac{\epsilon-\delta}{3}}$. Let $x \in W_\epsilon$. Then

$$\begin{aligned} |f(x)| &= |f(0) - R_x f(0)| \\ &\leq \|f - R_x f\|_\infty \\ &\leq \|f - R_x f\|_{A(\mathbb{T})} \\ &< \epsilon. \end{aligned}$$

Since $\text{supp } v \subseteq W_\epsilon$, we get

$$\begin{aligned} \|v\|_2^2 &= \frac{1}{2\pi} \int_{W_\epsilon} |f(t)|^2 \left| \frac{1}{\lambda(V_\epsilon)} 1_{V_{\epsilon, \delta} + V_{\epsilon, \delta}} * 1_{V_{\epsilon, \delta}}(t) \right|^2 dt \\ &\leq \epsilon^2 \frac{1}{\lambda(V_{\epsilon, \delta})^2} \|1_{V_{\epsilon, \delta} + V_{\epsilon, \delta}} * 1_{V_{\epsilon, \delta}}\|_2^2 \\ &\leq \epsilon^2 \frac{1}{\lambda(V_{\epsilon, \delta})^2} \|1_{V_{\epsilon, \delta} + V_{\epsilon, \delta}}\|_2^2 \|1_{V_{\epsilon, \delta}}\|_1^2 \\ &= \epsilon^2 \frac{1}{\lambda(V_{\epsilon, \delta})^2} \lambda(V_\epsilon + V_\epsilon) \lambda(V_{\epsilon, \delta})^2 \\ &= \epsilon^2 \frac{4(\epsilon - \delta)}{6\pi}. \end{aligned}$$

This implies that

$$\|v\|_2 \leq 2\epsilon\sqrt{\frac{\epsilon - \delta}{6\pi}}. \quad (6.2.9)$$

We now show that $\|f - f * \check{u}\|_{A(\mathbb{T})} \leq \epsilon$. We can write $f * \check{u}$ as a Bochner integral

$$f * \check{u} = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) R_x f dx.$$

By (6.2.5), $\frac{1}{2\pi} \int_{-\pi}^{\pi} u(x) dx = 1$. Therefore

$$\begin{aligned} \|f - f * \check{u}\|_{A(\mathbb{T})} &= \frac{1}{2\pi} \left\| \int_{-\pi}^{\pi} (f - R_x f) u(x) dx \right\|_{A(\mathbb{T})} \\ &\leq \frac{1}{2\pi} \int_{U_{\epsilon, \delta} + U_{\epsilon, \delta}} \|(f - R_x f)\|_{A(\mathbb{T})} |u(x)| dx \\ &< \epsilon, \end{aligned}$$

where the last inequality follows from the fact that $U_{\epsilon, \delta} + U_{\epsilon, \delta} \subseteq W_\epsilon$ and $\|u\|_1 = 1$. On the other hand, using (6.2.4) and (6.2.9), we get

$$\begin{aligned} \|v * \check{u}\|_{A(\mathbb{T})} &\leq \|u\|_2 \|v\|_2 \\ &\leq \sqrt{\frac{6\pi}{\epsilon - \delta}} 2\epsilon \sqrt{\frac{\epsilon - \delta}{6\pi}} = 2\epsilon. \end{aligned}$$

So if we put $a = (f - v) * \check{u}$, then

$$\begin{aligned} \|f - a\|_{A(\mathbb{T})} &\leq \|f - f * \check{u}\|_{A(\mathbb{T})} + \|v * \check{u}\|_{A(\mathbb{T})} \\ &< \epsilon + 2\epsilon = 3\epsilon. \end{aligned} \tag{6.2.10}$$

Now we can write

$$\begin{aligned} \|F(f)\| &= \|F(f - a + a)\| \\ &\leq \|F(f - a)\| + \|F(a)\|. \end{aligned}$$

Since $a = (f - v) * \check{u}$ and by (6.2.8), $\text{supp}(f - v) \cap \text{supp} \check{u} \subseteq V_{\epsilon, \delta}^c \cap V_{\epsilon, \delta} = \emptyset$, we have (by hypothesis)

$$\|F(a)\| \leq \alpha \|f - v\|_{A(\mathbb{T})} \|u\|_{A(\mathbb{T})}.$$

Hence

$$\|F(f)\| \leq \|f - a\|_{A(\mathbb{T})} + \alpha \|f - v\|_{A(\mathbb{T})} \|u\|_{A(\mathbb{T})}.$$

Using (6.2.2), (6.2.7) and (6.2.10), we get

$$\|F(f)\| \leq 3\epsilon + \alpha 2(1 + \sqrt{2}) \frac{6\pi}{\epsilon - \delta} \quad (0 < \epsilon < 3 \quad 0 < \delta < \epsilon).$$

Letting $\delta \rightarrow 0$, $A = 3$ and $B = 12\pi(1 + \sqrt{2})$, we have

$$\|F(f)\| \leq \inf \left\{ A\epsilon + \frac{\alpha B}{\epsilon}, \quad 0 < \epsilon < 3 \right\}. \tag{6.2.11}$$

Define $k : (0, 3) \rightarrow R^+$ by $k(\epsilon) = A\epsilon + \frac{\alpha B}{\epsilon}$. Then

$$k'(\epsilon) = A - \frac{\alpha B}{\epsilon^2} = 0 \Rightarrow \epsilon = \sqrt{\frac{\alpha B}{A}}.$$

Note that for each $0 \leq \alpha \leq 1$ we have $\sqrt{\frac{\alpha B}{A}} < 3$. So by (6.2.11) we can write

$$\|F(f)\| \leq k\left(\sqrt{\frac{\alpha B}{A}}\right) = 2\sqrt{AB\alpha} = 12\sqrt{\pi(1 + \sqrt{2})}\sqrt{\alpha}.$$

□

We are now ready to prove the main result of this section which was partly inspired by [2, Lemma 3.1] and its proof.

Theorem 6.2.2. *Let $\phi : A(\mathbb{T}) \times A(\mathbb{T}) \rightarrow \mathbb{C}$ be a continuous bilinear map satisfying the property*

$$f, g \in A(\mathbb{T}), \text{ supp } f \cap \text{ supp } g = \emptyset \Rightarrow |\phi(f, g)| \leq \alpha \|f\| \|g\| \quad (6.2.12)$$

for some $\alpha \geq 0$. Then

$$|\phi(fg, h) - \phi(f, gh)| \leq 288\pi(1 + \sqrt{2})\alpha \|f\| \|g\| \|h\| \quad (6.2.13)$$

for all $f, g, h \in A(\mathbb{T})$.

Proof. First assume that $0 \leq \alpha < 1$ and $\|\phi\| \leq 1$. The map ϕ gives rise to a continuous linear operator Φ on the projective tensor product $A(\mathbb{T}) \hat{\otimes} A(\mathbb{T}) (= A(\mathbb{T} \times \mathbb{T}))$ defined through

$$\Phi(f \otimes g) = \phi(f, g) \quad (f, g \in A(\mathbb{T})). \quad (6.2.14)$$

We define $N : A(\mathbb{T}) \rightarrow A(\mathbb{T} \times \mathbb{T})$ with

$$Nk(s, t) = k(s - t) \quad (k \in A(\mathbb{T}), s, t \in \mathbb{T}).$$

Pick $f, h \in A(\mathbb{T})$ with $\|f\|, \|h\| \leq 1$ and define $N_{f,h} : A(\mathbb{T}) \rightarrow A(\mathbb{T} \times \mathbb{T})$ with

$$N_{f,h}k = Nk(f \otimes e_1 h)$$

where $e_1 \in A(\mathbb{T})$ is given by $e_1(s) = e^{is}$. Then it is easy to check that

$$N_{f,h}(e_1 - 1) = fe_1 \otimes h - f \otimes e_1 h. \quad (6.2.15)$$

Note that for $\psi, \varphi \in A(\mathbb{T})$, we have the Bochner integral equality

$$N(\varphi * \check{\psi}) = \int_{\mathbb{T}} R_x \varphi \otimes R_x \psi dx.$$

Hence

$$N_{f,h}(\varphi * \check{\psi}) = \int_{\mathbb{T}} (R_x \varphi) f \otimes (R_x \psi) e_1 h dx. \quad (6.2.16)$$

If $\text{supp } \varphi \cap \text{supp } \psi = \emptyset$, then we have

$$\text{supp } ((R_x \varphi) f) \cap \text{supp } ((R_x \psi) e_1 h) = \emptyset.$$

Hence using (6.2.16) we get

$$\begin{aligned} |\Phi \circ N_{f,h}(\varphi * \check{\psi})| &\leq \int_{\mathbb{T}} \|\Phi((R_x \varphi) f \otimes (R_x \psi) e_1 h)\| dx \\ &\leq \int_{\mathbb{T}} \|\phi((R_x \varphi) f, (R_x \psi) e_1 h)\| dx \quad (\text{by (6.2.12)}) \\ &\leq \int_{\mathbb{T}} \alpha \|\phi(R_x \varphi) f\| \|(R_x \psi) e_1\| dx \\ &\leq \alpha \|\varphi\| \|\psi\|. \end{aligned}$$

Hence by Lemma 6.2.1, we should have

$$|(\Phi \circ N_{f,h})(e_1 - 1)| \leq 12\sqrt{\pi(1 + \sqrt{2})}\sqrt{\alpha},$$

which by (6.2.15), it implies that

$$\begin{aligned} |\varphi(fe_1, h) - \varphi(f, e_1 h)| &= |\Phi(fe_1 \otimes h - f \otimes e_1 h)| \\ &\leq 12\sqrt{\pi(1 + \sqrt{2})}\sqrt{\alpha}. \end{aligned} \quad (6.2.17)$$

Now we show that

$$|\phi(fe_n, h) - \phi(f, e_n h)| \leq 12\sqrt{\pi(1 + \sqrt{2})}\sqrt{\alpha} \|f\| \|h\| \quad (6.2.18)$$

for all $f, h \in A(\mathbb{T})$, where e_n denotes the function in $A(\mathbb{T})$ defined by

$$e_n(s) = e^{ins} \quad (s \in \mathbb{R}, n \in \mathbb{Z}).$$

For $a \in A(\mathbb{T})$, let $a_n \in A(\mathbb{T})$ be the function defined by

$$a_n(x) = a(nx).$$

Note that $e_n = (e_1)_n$. Define $\tau : A(\mathbb{T}) \times A(\mathbb{T}) \rightarrow \mathbb{C}$ by

$$\tau(a, b) = \phi(fa_n, hb_n) \quad (a, b \in A(\mathbb{T})).$$

Note that if $a \in A(\mathbb{T})$, then $a(s) = \sum_{k=-\infty}^{+\infty} \hat{a}(k)e^{iks}$, hence $a(ns) = \sum_{k=-\infty}^{+\infty} \hat{a}(k)e^{ikns}$ and so $a_n \in A(\mathbb{T})$ with

$$\|a_n\| \leq \sum_{k=-\infty}^{+\infty} |\hat{a}(k)| = \|a\|.$$

Moreover, if $a, b \in A(\mathbb{T})$ are such that $\text{supp } a \cap \text{supp } b = \emptyset$, then it is easily seen that $\text{supp } fa_n \cap \text{supp } hb_n = \emptyset$. So

$$\begin{aligned} |\tau(a, b)| &\leq \|\phi(fa_n, hb_n)\| \\ &\leq \alpha \|fa_n\| \|hb_n\| \\ &\leq \alpha \|a\| \|b\|. \end{aligned}$$

From (6.2.17), we deduce that

$$|\tau(e_1, 1) - \tau(1, e_1)| \leq 12\sqrt{\pi(1 + \sqrt{2})}\sqrt{\alpha}. \quad (6.2.19)$$

On the other hand, we have

$$\tau(e_1, 1) = \phi(fe_n, h), \quad \tau(1, e_1) = \phi(f, e_nh)$$

which, together with (6.2.19), gives (6.2.18).

Now let $g \in A(\mathbb{T})$. Since $g = \sum_{k=-\infty}^{+\infty} \hat{g}(k)e_k$, by applying (6.2.18) we get

$$\begin{aligned} |\phi(fg, h) - \phi(f, gh)| &= \left| \phi\left(\sum_{k=-\infty}^{+\infty} \hat{g}(k)fe_k, h\right) - \phi\left(f, \sum_{k=-\infty}^{+\infty} \hat{g}(k)e_k h\right) \right| \\ &\leq \sum_{k=-\infty}^{+\infty} |\hat{g}(k)| |\phi(fe_k, h) - \phi(f, e_k h)| \\ &\leq \sum_{k=-\infty}^{+\infty} |\hat{g}(k)| 12\sqrt{\pi(1 + \sqrt{2})}\sqrt{\alpha} \\ &= 12\sqrt{\pi(1 + \sqrt{2})}\sqrt{\alpha} \|g\|. \end{aligned}$$

Therefore if $f, h \in A(\mathbb{T})$ are arbitrary elements, we get

$$|\phi(fg, h) - \phi(f, gh)| \leq 12\sqrt{\pi(1 + \sqrt{2})}\sqrt{\alpha}\|f\|\|g\|\|h\|. \quad (6.2.20)$$

Next, let $m : A(\mathbb{T} \times \mathbb{T}) \rightarrow A(\mathbb{T})$ be the multiplication map which maps every elementary tensor $f \otimes g \in A(\mathbb{T} \times \mathbb{T})$ to $fg \in A(\mathbb{T})$. It follows from (6.2.20) that for $u = \sum_{i=1}^{\infty} f_i \otimes g_i \in A(\mathbb{T} \times \mathbb{T})$ we can write

$$\begin{aligned} |\Phi(u) - \phi(1, m(u))| &= |\Phi(\sum_{i=1}^{\infty} f_i \otimes g_i - \sum_{i=1}^{\infty} 1 \otimes f_i g_i)| \\ &\leq 12\sqrt{\pi(1 + \sqrt{2})}\sqrt{\alpha} \sum_{i=1}^{\infty} \|f_i\|\|g_i\|, \end{aligned}$$

In particular, for every $u \in I := \ker m$,

$$|\Phi(u)| \leq 12\sqrt{\pi(1 + \sqrt{2})}\sqrt{\alpha}\|u\|,$$

implying that

$$\|\Phi|_I\| \leq 12\sqrt{\pi(1 + \sqrt{2})}\sqrt{\alpha}. \quad (6.2.21)$$

Now consider the general case. Let $\phi : A(\mathbb{T}) \times A(\mathbb{T}) \rightarrow \mathbb{C}$ be a continuous bilinear map satisfying (6.2.12) for some $\alpha > 0$. Without loss of generality, we can assume that $\Phi|_I \neq 0$. Let $\Phi_0 \in I^*$ with $\Phi_0 = \frac{\Phi|_I}{\|\Phi|_I\|}$. Then $\|\Phi_0\| = 1$. By the Hahn-Banach Theorem, Φ_0 can be extended to $\Psi \in A(\mathbb{T} \times \mathbb{T})^*$ with $\|\Psi\| = 1$. For $f, g \in A(\mathbb{T})$ with $\text{supp } f \cap \text{supp } g = \emptyset$ we have

$$\begin{aligned} |\Psi(f \otimes g)| &= |\Phi_0(f \otimes g)| \\ &= \frac{1}{\|\Phi|_I\|} |\Phi(f \otimes g)| \\ &\leq \frac{\alpha}{\|\Phi|_I\|} \|f\|\|g\|. \end{aligned}$$

Put $\alpha_0 = \frac{\alpha}{\|\Phi|_I\|}$. Then $\|\Psi\| = 1$ and $0 \leq \alpha_0 \leq 1$ (We can assume $\alpha \leq \|\Phi|_I\|$, otherwise the statement is trivial). By the first part and (6.2.21),

$$\begin{aligned} 1 = \|\Phi_0\| &= \|\Psi|_I\| \\ &\leq 12\sqrt{\pi(1 + \sqrt{2})}\sqrt{\alpha_0} \\ &= 12\sqrt{\pi(1 + \sqrt{2})}\sqrt{\frac{\alpha}{\|\Phi|_I\|}}. \end{aligned}$$

This implies that

$$\|\Phi|_I\| \leq 144\pi(1 + \sqrt{2})\alpha.$$

In particular, for every $u \in I$

$$|\Phi(u)| \leq 144\pi(1 + \sqrt{2})\alpha\|u\|.$$

Finally for $f, g, h \in A(\mathbb{T})$, it is clear that $fg \otimes h - f \otimes gh \in I$. So we can write

$$\begin{aligned} |\phi(fg, h) - \phi(f, gh)| &= |\Phi(fg \otimes h - f \otimes gh)| \\ &\leq 144\pi(1 + \sqrt{2})\alpha\|fg \otimes h - f \otimes gh\| \\ &\leq 288\pi(1 + \sqrt{2})\alpha\|f\|\|g\|\|h\|. \end{aligned}$$

□

6.3 Group algebras and C*-algebras

In this section, we use the result of Section 6.2 to obtain a constant for the strong property (B) of C*-algebras and group algebras. The approach we use in this section is entirely adopted from [2] with a slight modification using our result from the preceding section. The idea is based on passing to the multiplier algebra of the given Banach algebra and then considering some special elements of the multiplier algebra called the doubly power bounded elements. We highlight that the approach in [2] does not give a constant for the strong property (B), whereas our modification does.

We first present some definitions which are required in the discussion.

Definition 6.3.1. Let A be a Banach algebra. We say that A is left faithful if

$$\{a \in A : aA = \{0\}\} = \{0\}.$$

A is said to be right faithful if

$$\{a \in A : Aa = \{0\}\} = \{0\}.$$

Definition 6.3.2. A multiplier on a Banach algebra A is a pair (L, R) , where $L, R : A \rightarrow A$ are linear maps such that, for all $a, b \in A$, the following identities hold

$$L(ab) = L(a)b, \quad R(ab) = aR(b) \quad \text{and} \quad aL(b) = R(a)b.$$

The set of all multipliers on A is denoted by $\mathcal{M}(A)$. It turns out that every multiplier on a right and left faithful Banach algebra A consists of continuous linear operators on A so that $\mathcal{M}(A)$ becomes a unital closed subalgebra of $B(A) \times B(A)^{op}$ called the multiplier algebra of A . Here we write $B(A)^{op}$ for the opposite algebra to $B(A)$ and we take the norm on $B(A) \times B(A)^{op}$ to be given by

$$\|(S, T)\| = \max\{\|S\|, \|T\|\}$$

for all $S, T \in B(A)$. Moreover, A is canonically embedded into $\mathcal{M}(A)$ by

$$a \rightarrow (L_a, R_a)$$

where $L_a(b) = ab = R_b(a)$ for all $a, b \in A$ and the embedding of A into $\mathcal{M}(A)$ is continuous with

$$\|(L_a, R_a)\| \leq \|a\|, \quad (a \in A).$$

If, in addition A has a contractive approximate identity, then we can identify A isometrically with its image via this embedding.

Definition 6.3.3. Let A be a left and right faithful Banach algebra. An invertible element $\mu \in \mathcal{M}(A)$ is called doubly power bounded if

$$\sup_{k \in \mathbb{Z}} \|\mu^k\| < \infty.$$

The following lemma shows how doubly power bounded elements fit in the definition of the strong property (\mathbb{B}) . This is a modification of [2][Lemma 3.2]

Lemma 6.3.4. *Let A be a Banach algebra, and let $\varphi : A \times A \rightarrow X$ be a continuous bilinear map into a Banach space X satisfying the property*

$$a, b \in A, ab = 0 \Rightarrow \|\varphi(a, b)\| \leq \alpha \|a\| \|b\| \tag{6.3.1}$$

for some $\alpha \geq 0$. If $\mu \in \mathcal{M}(A)$ is a doubly power bounded element with

$$M = \sup_{k \in \mathbb{Z}} \|\mu^k\|, \quad (6.3.2)$$

then

$$\|\varphi(a\mu, b) - \varphi(a, \mu b)\| \leq 288\pi(1 + \sqrt{2})M^2\alpha\|a\|\|b\| \quad (6.3.3)$$

for all $a, b \in A$.

Proof. Pick $a, b \in A$ and let $\mu \in \mathcal{M}(A)$ satisfying in (6.3.2). We define a continuous linear operator

$$T_\mu : A(\mathbb{T}) \rightarrow \mathcal{M}(A), \quad T_\mu(f) = \sum_{k=-\infty}^{+\infty} \hat{f}(k)\mu^k \quad (f \in A(\mathbb{T})).$$

T_μ is well-defined since

$$\begin{aligned} \|T_\mu(f)\| &\leq \sum_{k=-\infty}^{+\infty} |\hat{f}(k)|\|\mu^k\| \\ &\leq M\|f\| \quad (f \in A(\mathbb{T})). \end{aligned} \quad (6.3.4)$$

Moreover, T_μ is an algebraic homomorphism as it is shown below

$$\begin{aligned} T_\mu(fg) &= \sum_{j=-\infty}^{+\infty} (\widehat{fg})(j)\mu^j \\ &= \sum_{j=-\infty}^{+\infty} (\hat{f} * \hat{g})(j)\mu^j \\ &= \sum_{j=-\infty}^{+\infty} \left(\sum_{k=-\infty}^{+\infty} \hat{f}(k)\hat{g}(j-k) \right) \mu^j \\ &= \left(\sum_{k=-\infty}^{+\infty} \hat{f}(k)\mu^k \right) \left(\sum_{k=-\infty}^{+\infty} \hat{g}(k)\mu^k \right) \\ &= T_\mu(f)T_\mu(g) \quad (f, g \in A(\mathbb{T})). \end{aligned} \quad (6.3.5)$$

Now we define the continuous bilinear map

$$\psi : A(\mathbb{T}) \times A(\mathbb{T}) \rightarrow X, \quad \psi(f, g) = \varphi(aT_\mu(f), T_\mu(g)b) \quad (f, g \in A(\mathbb{T})). \quad (6.3.6)$$

Suppose that $f, g \in A(\mathbb{T})$ are such that

$$\text{supp } f \cap \text{supp } g = \emptyset.$$

Then

$$(aT_\mu(f))(T_\mu(g)b) = aT_\mu(fg)b = 0.$$

Hence we can write

$$\begin{aligned} \|\psi(f, g)\| &\leq \alpha \|aT_\mu(f)\| \|T_\mu(g)b\| \\ &\leq \alpha M^2 \|a\| \|b\| \|f\| \|g\|. \end{aligned} \tag{6.3.7}$$

Using Theorem 6.2.2, taking into account of (6.3.6) and (6.3.7) we can deduce that

$$\begin{aligned} \|\varphi(a\mu, b) - \varphi(a, \mu b)\| &= \|\psi(e_1, 1) - \psi(1, e_1)\| \\ &\leq 288\pi(1 + \sqrt{2})M^2\alpha \|a\| \|b\| \end{aligned}$$

where $e_1 \in A(\mathbb{T})$ is given by $e_1(s) = e^{is}$. This completes the proof. \square

Group algebras and C^* -algebras are among the Banach algebras that fit in the framework of Lemma 6.3.4.

6.3.1 Group algebras.

For a locally compact group G , we write $M(G)$ for the linear space of all complex, regular Borel measures on G . It is well-known that $M(G)$ is a Banach algebra with respect to the convolution product and the total variation as the norm. Also, $L^1(G)$ can be seen as the two-sided closed ideal of $M(G)$ consisting of all measures in $M(G)$ which are absolutely continuous with respect to λ , a fixed Haar measure on G . In fact, by Wendels theorem ([15, Theorem 3.3.40]), $M(G)$ is nothing but the multiplier algebra of $L^1(G)$. By [15, Theorem 3.3.15], the Banach space $C_0(G)$ is a Banach $M(G)$ -bimodule and $M(G)$ with respect to the convolution product is the dual of $C_0(G)$ as a Banach $M(G)$ -bimodule. For every $t \in G$ we denote by δ_t the unit point mass measure at t . It is important to notice

that the convolution product in $L^1(G)$ can be expressed in the following way:

$$\begin{aligned} f * g &= \int_G f(t)(\delta_t * g)d\lambda(t) \\ &= \int_G (f * \delta_t)g(t)d\lambda(t) \quad (f, g \in L^1(G)) \end{aligned} \tag{6.3.8}$$

where the expressions on the right are considered as (Bochner) integrals of measurable $L^1(G)$ -valued functions of t : see [39, Table 1 on page 144 and Appendix 1.9.16]. The following Theorem and its proof is a modification of [2, Theorem 3.4].

Theorem 6.3.5. *Let G be a locally compact group and let*

$$\varphi : L^1(G) \times L^1(G) \rightarrow X$$

be a continuous bilinear map on a Banach space X satisfying the property

$$f, g \in A, f * g = 0 \Rightarrow \|\varphi(f, g)\| \leq \alpha \|f\|_1 \|g\|_1 \tag{6.3.9}$$

for some $\alpha \geq 0$. Then

$$\|\varphi(f * g, h) - \varphi(f, g * h)\| \leq 288\pi(1 + \sqrt{2})\alpha \|f\|_1 \|g\|_1 \|h\|_1$$

for all $f, g, h \in L^1(G)$.

Proof. For every $t \in G$, the unit point mass measure δ_t at t clearly satisfies in

$$\|\delta_t^k\| = \|\delta_{t^k}\| = 1$$

for each $k \in \mathbb{Z}$. Therefore (6.3.3) in Lemma 6.3.4 implies that

$$\|\varphi(f * \delta_t, h) - \varphi(f, \delta_t * h)\| \leq 288\pi(1 + \sqrt{2})\alpha \|f\|_1 \|h\|_1 \tag{6.3.10}$$

for all $f, h \in L^1(G)$, $t \in G$. We now pick $f, g, h \in L^1(G)$ and multiply (6.3.10) by $|g(t)|$, with $t \in G$, and get

$$\|\varphi(f * g(t)\delta_t, h) - \varphi(f, g(t)\delta_t * h)\| \leq 288\pi(1 + \sqrt{2})\alpha \|f\|_1 \|h\|_1 |g(t)|. \tag{6.3.11}$$

If we integrate (6.3.11), we arrive at

$$\begin{aligned} \int_G \|\varphi(f * g(t)\delta_t, h) - \varphi(f, g(t)\delta_t * h)\| d\lambda(t) &\leq 288\pi(1 + \sqrt{2})\alpha \|f\|_1 \|h\|_1 \int_G |g(t)| d\lambda(t) \\ &= 288\pi(1 + \sqrt{2})\alpha \|f\|_1 \|h\|_1 \|g\|_1. \end{aligned} \quad (6.3.12)$$

Note that

$$\begin{aligned} \int_G \varphi(f * g(t)\delta_t, h) d\lambda(t) &= \varphi\left(\int_G f * g(t)\delta_t d\lambda(t), h\right) \\ &= \varphi(f * g, h), \end{aligned} \quad (6.3.13)$$

and

$$\begin{aligned} \int_G \varphi(f, g(t)\delta_t * h) d\lambda(t) &= \varphi\left(f, \int_G g(t)\delta_t * h d\lambda(t)\right) \\ &= \varphi(f, g * h). \end{aligned} \quad (6.3.14)$$

Since

$$\begin{aligned} &\left\| \int_G (\varphi(f * g(t)\delta_t, h) - \varphi(f, g(t)\delta_t * h)) d\lambda(t) \right\| \\ &\leq \int_G \|\varphi(f * g(t)\delta_t, h) - \varphi(f, g(t)\delta_t * h)\| d\lambda(t), \end{aligned}$$

combining (6.3.12), (6.3.13) and (6.3.14) imply that

$$\|\varphi(f * g, h) - \varphi(f, g * h)\| \leq 288\pi(1 + \sqrt{2})\alpha \|f\|_1 \|g\|_1 \|h\|_1,$$

as desired. \square

6.3.2 C^* -algebras.

Let A be a C^* -algebra. It is well-known that the multiplier algebra of A , $\mathcal{M}(A)$ becomes a unital C^* -algebra (see [15, Proposition 3.2.39]).

Theorem 6.3.6. *Let A be a C^* -algebra, and let $\varphi : A \times A \rightarrow X$ be a continuous bilinear map on a Banach space X satisfying the property*

$$a, b \in A, ab = 0 \Rightarrow \|\varphi(a, b)\| \leq \alpha \|a\| \|b\| \quad (6.3.15)$$

for some $\alpha \geq 0$. Then

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq 288\pi(1 + \sqrt{2})\alpha\|a\|\|b\|\|c\|$$

for all $a, b, c \in A$.

Proof. Recall that the unitary elements of $\mathcal{M}(A)$ are those $u \in \mathcal{M}(A)$ such that

$$uu^* = u^*u = 1,$$

which clearly entails that $\|u^k\| = 1$ for each $k \in \mathbb{Z}$. Hence unitary elements are doubly power bounded and consequently Lemma 6.3.4 gives that

$$\|\varphi(au, c) - \varphi(a, uc)\| \leq 288\pi(1 + \sqrt{2})\alpha\|a\|\|c\|, \quad (6.3.16)$$

for each $a, c \in A$, and a unitary $u \in \mathcal{M}(A)$. It is clear that (6.3.16) still holds true in the case when u lies in the convex hull of the set of the unitary elements of $\mathcal{M}(A)$. Since $\mathcal{M}(A)$ is a C^* -algebra, by the Russo-Dye Theorem [15, Theorem 3.2.18] this convex hull is norm-dense in the closed unit ball of $\mathcal{M}(A)$. Consequently (6.3.16) also holds for each u in the closed unit ball of $\mathcal{M}(A)$. In particular, we get

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq 288\pi(1 + \sqrt{2})\alpha\|a\|\|b\|\|c\|$$

for all $a, b, c \in A$. This completes the proof. \square

6.4 Banach algebras' matrix spaces

It was pointed out in Remark 3.3.4 that there is a Banach algebra without the strong property (\mathbb{B}) . Indeed, for majority of Banach algebras, it is not known whether or not they have the strong property (\mathbb{B}) . So we could ask if there is a way that an arbitrary Banach algebra relates to the strong property (\mathbb{B}) . In this section, we show that there is such a way. In fact, we prove that if \mathcal{A} is an arbitrary Banach algebra such that $M_n(\mathcal{A})$ is a Banach algebra for some $n \geq 2$ satisfying an identity (inequalities (6.4.1)), then $M_n(\mathcal{A})$ has the strong property (\mathbb{B}) and contains \mathcal{A} as a closed subalgebra in a natural way. This

result, in particular, shows that subalgebras might not inherit the strong property (\mathbb{B}) . In this section, we show that matrix spaces related to an arbitrary Banach algebra have the strong property (\mathbb{B}) . The following Lemma is fundamental to prove such a fact. It shows that idempotents fit nicely in the strong property (\mathbb{B}) .

Lemma 6.4.1. *Let \mathcal{A} be a unital Banach algebra. For $n \in \mathbb{N}$, suppose that $(M_n(\mathcal{A}), \|\cdot\|)$ is a Banach algebra with the property that for each $[a_{ij}] \in M_n(\mathcal{A})$*

$$\|a_{ij}\| \leq \|[a_{ij}]\| \leq \sum_{i,j} \|a_{ij}\| = \|[a_{ij}]\|_s. \quad (6.4.1)$$

Let X be a Banach space and $\varphi : M_n(\mathcal{A}) \times M_n(\mathcal{A}) \rightarrow X$ a bounded bilinear map with the property that

$$A, B \in M_n(\mathcal{A}), \quad AB = 0 \Rightarrow \|\varphi(A, B)\| \leq \alpha \|A\| \|B\|,$$

for some $\alpha > 0$. Then for all $A, B \in M_n(\mathcal{A})$ and every idempotent $P \in M_n(\mathcal{A})$

$$\|\varphi(AP, B) - \varphi(A, PB)\| \leq 2\|P\|(n + \|P\|)\alpha \|A\| \|B\|.$$

Proof. Let I be the identity matrix in $M_n(\mathcal{A})$. We have $AP(I - P)B = 0$. So by the assumption

$$\begin{aligned} \|\varphi(AP, (I - P)B)\| &\leq \alpha \|A\| \|B\| \|P\| (\|I\| + \|P\|) \\ &\leq \alpha \|P\| (n + \|P\|) \|A\| \|B\|. \end{aligned}$$

So

$$\|\varphi(AP, B) - \varphi(AP, PB)\| \leq \alpha \|P\| (n + \|P\|) \|A\| \|B\|.$$

Hence we can write

$$\begin{aligned} \|\varphi(AP, B) - \varphi(A, PB)\| &= \|\varphi(AP + A(I - P), PB) - \varphi(AP, B)\| \\ &\leq \|\varphi(AP, PB) - \varphi(AP, B)\| + \|\varphi(A(I - P), PB)\| \\ &\leq 2\alpha \|P\| (n + \|P\|) \|A\| \|B\|. \end{aligned}$$

□

From now on, we always assume that $(M_n(\mathcal{A}), \|\cdot\|)$ is a Banach algebra satisfying in (6.4.1) given in Lemma 6.4.1.

We show that if \mathcal{A} is unital, then there is a fixed number N such that each element of $M_n(\mathcal{A})$ is generated by at most N idempotents.

Lemma 6.4.2. *Let \mathcal{A} be a unital Banach algebra with unit e . For $a \in \mathcal{A}$ and $1 \leq i, j \leq n$, define the following matrices in $M_n(\mathcal{A})$:*

$$(A_{ij,a})_{ks} = \begin{cases} a & ks = ij \\ 0 & \text{otherwise} \end{cases} \quad 1 \leq k, s \leq n.$$

Then every matrix $A = [a_{ij}] \in M_n(\mathcal{A})$ can be written in the following form:

$$\begin{aligned} A &= A_{12,e}A_{21,a_{11}} \\ &+ \sum_{j=2}^n A_{j(j-1),e}A_{(j-1)j,a_{jj}} \\ &+ \sum_{i \neq j} A_{ij,a_{ij}}. \end{aligned}$$

Proof. It is easy to check that the following hold for each $a \in \mathcal{A}$,

$$A_{11,a} = A_{12,e}A_{21,a} , \tag{6.4.2}$$

and

$$A_{jj,a} = A_{j(j-1),e}A_{(j-1)j,a} \quad 2 \leq j \leq n. \tag{6.4.3}$$

Now using (6.4.2) and (6.4.3), we have

$$\begin{aligned} A &= \sum_{i,j} A_{ij,a_{ij}} \\ &= A_{12,e}A_{21,a_{11}} \\ &+ \sum_{j=2}^n A_{j(j-1),e}A_{(j-1)j,a_{jj}} \\ &+ \sum_{i \neq j} A_{ij,a_{ij}}. \end{aligned}$$

□

Taking into account Lemma 6.4.2, we prove Lemma 6.4.3 and Lemma 6.4.4 below to proceed to prove the main result of this section which is Theorem 6.4.5.

Lemma 6.4.3. *Let $B_1, \dots, B_k \in M_n(\mathcal{A})$ with $\|B_i\| \leq M$ for each $1 \leq i \leq k$. Suppose that for each $A, C \in M_n(\mathcal{A})$,*

$$\|\varphi(AB_i, C) - \varphi(A, B_iC)\| \leq L_i\alpha\|A\|\|C\|,$$

for some $L_i > 0$. Then

$$(i) \quad \|\varphi(AB_1 \cdots B_k, C) - \varphi(A, B_1 \cdots B_kC)\| \leq M^{k-1}(L_1 + \cdots + L_k)\alpha\|A\|\|B\|.$$

$$(ii) \quad \|\varphi(A(B_1 + \cdots + B_k), C) - \varphi(A, (B_1 + \cdots + B_k)C)\| \leq (L_1 + \cdots + L_k)\alpha\|A\|\|B\|.$$

Proof. (ii) is trivial. We prove (i) for $k = 2$. The general case is obtained by applying induction. We have

$$\|\varphi(AB_1B_2, C) - \varphi(AB_1, B_2C)\| \leq L_2\alpha\|AB_1\|\|C\| \leq ML_2\alpha\|A\|\|C\|. \quad (6.4.4)$$

and

$$\|\varphi(AB_1, B_2C) - \varphi(A, B_1B_2C)\| \leq L_1\|A\|\|B_2C\| \leq ML_1\alpha\|A\|\|C\|. \quad (6.4.5)$$

Hence using (6.4.4) and (6.4.5) we get

$$\begin{aligned} \|\varphi(AB_1B_2, C) - \varphi(A, B_1B_2C)\| &\leq \|\varphi(AB_1, B_2C) - \varphi(AB_1, B_2C)\| \\ &\quad + \|\varphi(AB_1, B_2C) - \varphi(A, B_1B_2C)\| \\ &\leq M(L_1 + L_2)\alpha\|A\|\|C\|. \end{aligned}$$

□

Lemma 6.4.4. *Let $a \in \mathcal{A}$ with $\|a\| \leq 1$. Then for each $i \neq j$, and $A, B \in M_n(\mathcal{A})$,*

$$\|\varphi(AA_{ij,a}, B) - \varphi(A, A_{ij,a}B)\| \leq 2(3n + 5)\alpha\|A\|\|B\|.$$

Proof. For $a, b \in \mathcal{A}$ and $1 \leq i, j, k, l \leq n$, we define the following matrix in $M_n(\mathcal{A})$,

$$(B_{ij,a}^{kl,b})_{ms} = \begin{cases} a & ms = ij \\ b & ms = kl \\ 0 & \text{otherwise} \end{cases} \quad 1 \leq m, s \leq n.$$

For i and j with $i \neq j$, it is easy to check that

$$A_{ij,a} = B_{ij,a}^{jj,e} - A_{jj,e},$$

It is straightforward to see that both matrices in the right hand side of the preceding equality are idempotents. Since $(M_n(\mathcal{A}), \|\cdot\|)$ satisfies in (6.4.1), for each $1 \leq i, j \leq n$ with $i \neq j$, we have

$$\|A_{jj,e}\| \leq \|e\| = 1 \text{ and } \|B_{ij,a}^{jj,e}\| \leq \|a\| + \|e\| \leq 2.$$

Using this fact and Lemma 6.4.3(ii), and Lemma 6.4.1, we have

$$\begin{aligned} \|\varphi(AA_{ij,a}, B) - \varphi(A, A_{ij,a}B)\| &\leq 2[(2n+4) + (n+1)]\alpha\|A\|\|B\| \\ &= 2(3n+5)\alpha\|A\|\|B\|. \end{aligned}$$

□

We can now show that the matrix spaces of a unital Banach algebra have the strong property (\mathbb{B}) with a constant.

Theorem 6.4.5. *Let \mathcal{A} be a unital Banach algebra such that $M_n(\mathcal{A})$ is a Banach algebra satisfying in (6.4.1) for some $n \geq 2$. Then $M_n(\mathcal{A})$ has the strong property (\mathbb{B}) with a constant given by*

$$C_n = (6n^3 + 16n^2 + 10n).$$

Proof. Let X be a Banach space and $\varphi : M_n(\mathcal{A}) \times M_n(\mathcal{A}) \rightarrow X$ a bilinear map (not necessarily bounded!) with the property that

$$AB = 0 \Rightarrow \|\varphi(A, B)\| \leq \alpha\|A\|\|B\|.$$

Let $A, B, C \in M_n(\mathcal{A})$ with $\|A\| = \|[a_{ij}]\| \leq 1$ and $\|B\|, \|C\| \leq 1$. Since $(M_n(\mathcal{A}), \|\cdot\|)$ satisfies in (6.4.1), we have that $\|a_{ij}\| \leq 1$ for each $1 \leq i, j \leq n$. Consequently (again using (6.4.1)), for all $1 \leq i, j, k, s \leq n$, we should have $\|A_{ks, a_{ij}}\| \leq 1$. Now using Lemma 6.4.2, we can write

$$\begin{aligned} \|\varphi(BA, C) - \varphi(B, AC)\| &\leq \|\varphi(BA_{12, e}A_{21, a_{11}}, C) - \varphi(B, A_{12, e}A_{21, a_{11}}C)\| \\ &\quad + \sum_{j=2}^n \|\varphi(BA_{j(j-1), e}A_{(j-1)j, a_{jj}}, C) - \varphi(B, A_{j(j-1), e}A_{(j-1)j, a_{jj}}C)\| \\ &\quad + \sum_{i \neq j} \|\varphi(BA_{ij, a_{ij}}, C) - \varphi(B, A_{ij, a_{ij}}C)\|. \end{aligned}$$

If we apply Lemma 6.4.3.(i) and Lemma 6.4.4 we get

$$\begin{aligned} \|\varphi(BA, C) - \varphi(B, AC)\| &\leq 4(3n + 5)\alpha \\ &\quad + (n - 1)[4(3n + 5)\alpha] \\ &\quad + (n^2 - n)[2(3n + 5)\alpha]. \end{aligned}$$

A simple calculation now shows that the constant in the right hand side is equal to

$$2(3n^3 + 8n^2 + 5n).$$

This completes the proof. □

We can improve the result of Theorem 6.4.5 by eliminating the assumption on the Banach algebra to be unital. As one may expect, we do this with considering the unitization of the given Banach algebra.

Theorem 6.4.6. *Let \mathcal{A} be a Banach algebra. Let \mathcal{A}^\sharp denote the unitization of \mathcal{A} . Suppose that for $n \geq 2$, $(M_n(\mathcal{A}^\sharp), \|\cdot\|)$ is a Banach algebra satisfying (6.4.1). Then $(M_n(\mathcal{A}), \|\cdot\|)$ has the strong property (\mathbb{B}) with a constant bounded by*

$$n^2(6n^3 + 16n^2 + 10n).$$

Proof. According to Theorem 6.4.5, $M_n(\mathcal{A}^\sharp)$ has the strong property (\mathbb{B}) with a constant given by

$$C_n = (6n^3 + 16n^2 + 10n).$$

Since $M_n(\mathcal{A})$ is an ideal of $M_n(\mathcal{A}^\sharp)$, Proposition 3.2.1 implies that it has the strong property (\mathbb{B}) with a constant given by

$$\|I \otimes 1\|_{M_n(\mathcal{A})}^2(6n^3 + 16n^2 + 10n) \leq n^2(6n^3 + 16n^2 + 10n).$$

□

In the following examples, we present various classes of norms on $M_n(\mathcal{A})$ for which Theorem 6.4.5 and Theorem 6.4.6 hold. Thus demonstrating the generality of our results.

Example 6.4.7. Let \mathcal{A} be a Banach algebra. For $n \geq 2$, $M_n(\mathcal{A})$ becomes a Banach algebra with the following norm

$$\|[a_{ij}]\|_s = \sum_{i,j} \|a_{ij}\|.$$

It is clear that $(M_n(\mathcal{A}^\sharp), \|\cdot\|_s)$ satisfies in (6.4.1). Hence, by Theorem 6.4.6, $(M_n(\mathcal{A}), \|\cdot\|_s)$ has the strong property (\mathbb{B}) with a constant given by

$$C_n = n^2(6n^3 + 16n^2 + 10n).$$

If \mathcal{A} is unital, then Theorem 6.4.5 shows that this constant can be reduced to

$$C_n = (6n^3 + 16n^2 + 10n).$$

In some cases, the constants given in Theorem 6.4.5 and Theorem 6.4.6 can be reduced.

Example 6.4.8. Let \mathcal{A} be a unital Banach algebra. For $n \geq 2$, consider the operator norm on M_n and equip $M_n(\mathcal{A}) \cong M_n \otimes \mathcal{A}$ with the projective tensor norm. Then $\|id_{M_n \otimes \mathcal{A}}\| = \|id_{M_n}\| \|id_{\mathcal{A}}\| = 1$. Hence, in this case, proof of Lemma 6.4.1 can be modified to show that for each idempotent $P \in M_n(\mathcal{A})$,

$$\|\varphi(AP, B) - \varphi(A, PB)\| \leq 2\|P\|(1 + \|P\|)\alpha\|A\|\|B\|.$$

Consequently, proof of Theorem 6.4.5 can be modified to show that $M_n(\mathcal{A})$ has the strong property (\mathbb{B}) with a constant given by

$$C_n = 16(n^2 + n).$$

If \mathcal{A} is an arbitrary Banach algebra, then a similar argument as that of Theorem 6.4.6 can be applied to shows that $M_n(\mathcal{A})$ has the strong property (\mathbb{B}) with a constant given by

$$C_n = 16(n^2 + n).$$

Proposition 6.4.9. *Every Banach algebra is isometrically embedded into a Banach algebra which has the strong property (\mathbb{B}) .*

Proof. Let \mathcal{A} be an arbitrary Banach algebra . According to Example 6.4.7, $(M_2(A), \|\cdot\|_s)$ has the strong property (\mathbb{B}) . It is easy to check that the mapping $\theta : A \rightarrow M_2(A)$ given by

$$\theta(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

is an isometric algebraic homomorphism. □

Corollary 6.4.10. *Subalgebras might not inherit the strong property (\mathbb{B}) .*

Proof. According to Corollary 3.3.4, $A(\mathbb{D})$ does not have the strong property (\mathbb{B}) . However, it is a closed subalgebra of $M_2(A(\mathbb{D}))$ which has the strong property (\mathbb{B}) according to Proposition 6.4.9. □

6.5 Finite nest algebras on arbitrary Hilbert spaces

CSL algebras and more specially, nest algebras are important classes of non-self adjoint operator algebras. In particular, when dealing with the problems of the reflexivity and hyperreflexivity of operator algebras, they where amongst the first to be studied. In [5], Arveson showed that every CSL algebra is a reflexive operator algebra. He proved latter in [6] that some special CSL algebras called the nest algebra are hyperreflexive. Actually, his result highlighted nest algebras to be very rare hyperreflexive operator algebras for which *the hyperreflexivity constant* is 1, i.e., for every nest algebra $N \subseteq B(H)$,

$$\text{dist}(T, N) = \text{dist}_r(T, N), \quad T \in B(H).$$

In the present and the following sections, we prove that finite nest algebras on any Hilbert space and finite CSL algebras on separable Hilbert spaces have the strong property (\mathbb{B}) with a constant. To do so, we first show that such operator algebras can always be represented as $n \times n$ -matrices for $n \geq 2$.

Clearly the largest possible nest algebra that can be defined on a Hilbert space H is $B(H)$ which is generated by the trivial nest $\{0, H\}$. A nice property of $B(H)$ is that its elements are linear combinations of at most 10 idempotents.

Theorem 6.5.1. *Let H be a Hilbert space and $T \in B(H)$. Then*

$$T = \sum_{n=1}^{10} \lambda_n P_n$$

where for $1 \leq n \leq 10$, $P_n \in B(H)$ is a projection and $\lambda_n \in \mathbb{C}$ with $|\lambda_n| \leq 2\|T\|$.

Proof. For the proof and more details see [37]. □

The next lemma shows that elements of a finite nest algebra can be represented as matrices. This was pointed out in [13] without proof. But for sake of completion, we present a proof for it. This lemma together with Theorem 6.5.1 will be used later to show that a finite nest algebras is generated by its idempotents.

Lemma 6.5.2. *Let H be a Hilbert space and $0 = e_0 \leq e_1 \leq \dots \leq e_n = 1$ a finite nest on H . If H_j is the range of $e_j - e_{j-1}$, then*

$$H = H_1 \oplus \dots \oplus H_n.$$

Moreover, if N is the corresponding nest algebra, then elements of N are exactly those represented as upper triangular matrices with respect to above decomposition.

Proof. It is easy to check that for each $1 \leq j \leq n$, $e_j - e_{j-1}$ is a projection. Moreover it is clear that,

$$\sum_{i=1}^n e_i - e_{i-1} = 1. \tag{6.5.1}$$

Note that if $i < j$, then

$$(e_j - e_{j-1})(e_i - e_{i-1}) = 0.$$

Consequently, we find that

$$H_j \cap H_i = \text{ran}(e_j - e_{j-1}) \cap \text{ran}(e_i - e_{i-1}) = \{0\}. \quad (6.5.2)$$

(6.5.1) and (6.5.2) imply that

$$H = H_1 \oplus \cdots \oplus H_n.$$

Note that each $a \in B(H)$ is represented as a matrix $[a_{ij}]$ with respect to this decomposition where $a_{ij} : H_j \rightarrow H_i$ maps $h \in H_j$ to the orthogonal projection of ah on H_i . Hence

$$a_{ij} = (e_i - e_{i-1})a(e_j - e_{j-1}).$$

If $a \in N$, then for each $1 \leq j \leq n$ we have $ae_j = e_jae_j$. Hence

$$a_{ij} = (e_i - e_{i-1})(e_jae_j - e_{j-1}ae_{j-1}).$$

This implies that

$$i > j \Rightarrow a_{ij} = 0.$$

Hence a is represented as an upper triangular matrix.

Conversely, suppose that a has an upper triangular representation. Then for $i > j$, we have

$$a_{ij} = (e_i - e_{i-1})a(e_j - e_{j-1}) = 0. \quad (6.5.3)$$

To show $a \in N$, we need to show that

$$(1 - e_j)ae_j = 0, \quad 1 \leq j \leq n.$$

We prove it for $j = 1$; other cases are proven similarly. According to (6.5.3) we can write

$$\begin{aligned} (e_2 - e_1)ae_1 &= 0 \\ (e_3 - e_2)ae_1 &= 0 \\ &\vdots \\ (e_n - e_{n-1})ae_1 &= 0 \end{aligned} \quad (6.5.4)$$

If we add all equations in (6.5.4) we get,

$$(1 - e_1)ae_1 = 0,$$

as desired. □

As it is shown in the next Theorem, representing finite nest algebras as matrices is fundamental in proving that these type of operator algebras have the strong property (\mathbb{B}) .

Theorem 6.5.3. *Let H be a Hilbert space and $0 = e_0 \leq e_1 \leq \dots \leq e_n = 1$ a finite nest on H . If N is the corresponding nest algebra, then N has the strong property (\mathbb{B}) with a constant given by $r_n = 33n + 7n^2$.*

Proof. For $1 \leq j \leq n$, if H_j is the range of $e_j - e_{j-1}$, then according to Lemma 6.5.2

$$H = H_1 \oplus \dots \oplus H_n$$

and for $T \in B(H)$, we have $T \in N$ if and only if its matrix form with respect to above decomposition is upper triangular. Now assume that $\varphi : N \times N \rightarrow \mathbb{C}$ is a bilinear map with the property that

$$a, b \in N \quad ab = 0 \Rightarrow \|\varphi(a, b)\| \leq \alpha \|a\| \|b\|.$$

Using the same argument as we used to prove Lemma 6.4.1, we can check that for any idempotent $P \in N$ and any $a, b \in N$, we have

$$\|\varphi(aP, b) - \varphi(a, Pb)\| \leq 2\|P\|(\|I - P\|)\alpha \|a\| \|b\|. \quad (6.5.5)$$

Note that for $T \in B(H)$, if the matrix form is given by $T = [T_{ij}]$, then

$$\|T_{ij}\| \leq \|T\| \leq \sum_{i,j} \|T_{ij}\|. \quad (6.5.6)$$

Now pick $c = [c_{ij}] \in N$ with $\|c\| \leq 1$. Then

$$c = \sum_{i \leq j} A_{c_{ij}, ij},$$

where $A_{c_{ij},ij} \in B(H)$ is a matrix all entries of which are 0 except the ij^{th} entry which is c_{ij} . If $i < j$, then we can write

$$A_{c_{ij},ij} = B_{c_{ij},ij} - E_{jj},$$

where

$$(B_{c_{ij},ij})_{mn} = \begin{cases} 1 & mn = jj \\ c_{ij} & mn = ij \\ 0 & otherwise \end{cases}$$

and

$$(E_{jj})_{mn} = \begin{cases} 1 & mn = jj \\ 0 & otherwise \end{cases}.$$

Note that $B_{c_{ij},ij}$ and E_{jj} are both idempotents and by (6.5.6) we have $\|B_{c_{ij},ij}\| \leq 2$ and $\|E_{jj}\| \leq 1$. Consequently by applying (6.5.5) we obtain that

$$\|\varphi(aA_{c_{ij},ij}, b) - \varphi(a, A_{c_{ij},ij}b)\| \leq 14\alpha\|a\|\|b\|. \quad (6.5.7)$$

On the other hand, for $1 \leq i \leq n$, we have $c_{ii} \in B(H_i)$. So by Theorem 6.5.1 we have

$$c_{ii} = \sum_{k=1}^{10} \lambda_k^i P_k^i,$$

where $P_k^i \in B(H_i) \subseteq N$ is a projection and

$$|\lambda_k^i| \leq 2\|c_{ii}\| \leq 2\|c\| \leq 2.$$

As a result, we have

$$\begin{aligned} \|\varphi(aA_{c_{ii},ii}, b) - \varphi(a, A_{c_{ii},ii}b)\| &\leq \sum_{k=1}^{10} 2\|\varphi(aA_{P_k^i,ii}, b) - \varphi(a, A_{P_k^i,ii}b)\| \\ &\leq 40\alpha\|a\|\|b\|. \end{aligned} \quad (6.5.8)$$

We have

$$c = \sum_{i=1}^n A_{c_{ii},ii} + \sum_{i < j} A_{c_{ij},ij}.$$

Hence, by applying (6.5.7) and (6.5.8) we can write

$$\begin{aligned}
\|\varphi(ac, b) - \varphi(a, cb)\| &\leq \sum_{i=1}^n \|\varphi(aA_{c_{ii}, ii}, b) - \varphi(a, A_{c_{ii}, ii}b)\| \\
&\quad + \sum_{i < j} \|\varphi(aA_{c_{ij}, ij}, b) - \varphi(a, A_{c_{ij}, ij}b)\| \\
&\leq (40n + 14\frac{n(n-1)}{2})\alpha\|a\|\|b\| \\
&= (33n + 7n^2)\alpha\|a\|\|b\|.
\end{aligned} \tag{6.5.9}$$

Finally applying (6.5.9), for arbitrary $c \in N$ we obtain that

$$\|\varphi(ac, b) - \varphi(a, cb)\| \leq (33n + 7n^2)\alpha\|a\|\|b\|\|c\|,$$

as claimed. □

Remark 6.5.4. The idea that is used to prove Theorem 6.2.11, can be applied to construct more examples of operator algebras with the strong property (\mathbb{B}) . Let H be a Hilbert space. Assume that $H = H_1 \oplus \cdots \oplus H_n$ is any decomposition of H . Let \mathcal{A} be a subalgebra of $B(H)$ having the following properties:

(i) If $a = [a_{ij}] \in \mathcal{A}$, then $A_{a_{ij}, ij} \in \mathcal{A}$ for each $1 \leq i, j \leq n$, where

$$(A_{ij})_{mn} = \begin{cases} a_{ij} & mn = ij \\ 0 & otherwise \end{cases}.$$

(ii) For each $1 \leq i \leq n$, $E_{ii} \in \mathcal{A}$.

(iii) For $1 \leq i \leq n$, if there is $A \in \mathcal{A}$ such that $A_{a_{ii}, ii} \notin \mathbb{C}E_{ii}$, then we require that $B(H_i) \subseteq \mathcal{A}$.

Then \mathcal{A} has the strong property (\mathbb{B}) .

Example 6.5.5. Let H be Hilbert space.

(i) If $H = H_1 \oplus \cdots \oplus H_n$ is any decomposition of H , then subalgebra of $B(H)$ containing all operators with upper triangular representation as well as subalgebra of $B(H)$ containing all operators with lower triangular representation with respect to this decomposition have the strong property (\mathbb{B}) .

(ii) Let $H = H_1 \oplus H_2 \oplus H_3$ be a decomposition of H . It is easy to check that the following is a subalgebra of $B(H)$ satisfying the assumptions of Remark 6.5.4,

$$\mathcal{A} = \left\{ \begin{pmatrix} b_{11} & 0 & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} : b_{ij} \in B(H_j, H_i) \right\}.$$

So \mathcal{A} has the strong property (\mathbb{B}) .

(iii) Consider the following subalgebra of \mathcal{A} defined in (ii).

$$\mathcal{B} = \left\{ \begin{pmatrix} b_{11} & 0 & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} : b_{ii} \in \mathbb{C}1_{H_i}, b_{ij} \in B(H_j, H_i) \right\}.$$

According to Remark 6.5.4, \mathcal{B} has the strong property (\mathbb{B}) .

6.6 Finite CSL algebras on Separable Hilbert spaces

As it was presented in the previous section, the main tool to prove a finite nest algebra has the strong property (\mathbb{B}) is to represent elements of the nest algebra as upper triangular matrices. In the following, we present a similar representation for finite CSL algebras on separable Hilbert spaces. The main reference for this section is [34].

Definition 6.6.1. Let $n \in \mathbb{N}$, and let \preceq be a partial order on $\{1, 2, \dots, n\}$. A subset \mathcal{F} of $\{1, 2, \dots, n\}$ is \preceq -hereditary (or simply *hereditary* if no confusion arises) if

$$i \preceq j \in \mathcal{F} \Rightarrow i \in \mathcal{F}.$$

The class of all hereditary subsets is denoted by $\mathcal{D}_n(\preceq)$.

Remark 6.6.2. It is easy to check that $\mathcal{D}_n(\preceq)$ is closed under intersection and union. Moreover, for $1 \leq j \leq n$, each subset of the form

$$\{i : 1 \leq i \leq n \text{ and } i \preceq j\} \tag{6.6.1}$$

is in $\mathcal{D}_n(\preceq)$. It can be shown that in general, not all hereditary subsets of $\{1, 2, \dots, n\}$ are of the form (6.6.1). However, it is not difficult to check that any hereditary subset has to be a union of subsets of the form (6.6.1).

Theorem 6.6.3. *Let H be a separable Hilbert space. Assume that \mathcal{L} is a finite CSL consisting of n non-zero projections. Then there is a partial order \preceq on the set $\{1, \dots, n\}$ consistent with the natural order, i.e.,*

$$i \preceq j \Rightarrow i \leq j, \quad 1 \leq i, j \leq n.$$

and a decomposition $H = \bigoplus_{i=1}^n H_i$ for the Hilbert space such that

$$\text{alg}\mathcal{L} = \{[T_{ij}] \in B(H) : T_{ij} \in B(H_j, H_i), T_{ij} = 0 \text{ if } i \not\preceq j\}. \quad (6.6.2)$$

So $\text{alg}\mathcal{L}$ consists of all upper triangular matrices with possibly some fixed zeros up the diagonal.

Proof. For a proof to this theorem see [34]. □

Remark 6.6.4. According to Theorem 6.6.3, the “building blocks” of finite CSL’s are:

- (i) an orthogonal decomposition $H = H_1 \oplus \dots \oplus H_n$.
- (ii) a partial order \preceq consistent with the natural order.

Note that in order to construct finite CSL algebras we do not require the partial order to be consistent with the natural order. Actually, if $H = H_1 \oplus \dots \oplus H_n$, and \preceq is a partial order on $\{1, \dots, n\}$ (not necessarily consistent with the natural order), then according to Remark 6.6.2,

$$\mathcal{L} = \{\bigoplus_{i \in \mathcal{F}} H_i : \mathcal{F} \in \mathcal{D}_n(\preceq)\}$$

is a subspace lattice. Also corresponding CSL algebra is given by (6.6.2). To see this, let $T = [T_{ij}] \in B(H)$ with $T_{ij} = 0$ if $i \not\preceq j$. We show that $T \in \text{alg}\mathcal{L}$. According to Remark 6.6.2, it suffices to show that subspaces of the form $M = \bigoplus_{i \preceq j_0} H_i$, for some fixed $1 \leq j_0 \leq n$, are invariant under T . We notice that each operator T_{ij} acts on H_j and maps H_j into H_i . Hence T_{ij} acts non-trivially on M only if $j \preceq j_0$ and $i \preceq j$. In this case, we have

$$T_{ij}(H_j) \subseteq H_i \subseteq M.$$

Hence T leaves M invariant. Consequently, $T \in \text{alg}\mathcal{L}$.

Conversely, let $T = [T_{ij}]$ be an operator on H . Suppose that for some $1 \leq i_0, j_0 \leq n$ with $i_0 \not\leq j_0$, we have $T_{i_0 j_0} \neq 0$. We define

$$M = \bigoplus_{i \preceq j_0} H_i \in \mathcal{L}.$$

Then $H_{j_0} \subseteq M$ and $T_{i_0 j_0}(H_{j_0}) \subseteq H_{i_0} \not\subseteq M$. Hence $T \notin \text{alg}\mathcal{L}$.

Note that a straightforward algorithm can be used to show that this CSL algebra, $\text{alg}\mathcal{L}$, is isomorphic to another CSL algebra generated by another partial order consistent with the natural order.

Example 6.6.5. Let $H = H_1 \oplus H_2 \oplus H_3$. Define the following partial order on $\{1, 2, 3\}$,

$$2 \preceq 1 \preceq 3 \text{ and } n \preceq n \quad (n = 1, 2, 3).$$

If we let

$$\mathcal{L} = \{\bigoplus_{i \in \mathcal{F}} H_i : \mathcal{F} \in \mathcal{D}_3(\preceq)\},$$

then

$$\text{alg}\mathcal{L} = \left\{ \begin{pmatrix} b_{11} & 0 & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{pmatrix} : b_{ij} \in B(H_j, H_i) \right\}.$$

which is exactly algebra constructed in Example 6.5.5 (ii).

Similar to nest algebras, representing separable CSL algebras as matrices paves the way to prove that these operator algebras have the strong property (\mathbb{B}) .

Proposition 6.6.6. *Let \mathcal{L} be a finite CSL consisting of n non-zero projection on a separable Hilbert space H . Then $\text{alg}\mathcal{L}$ has the strong property (\mathbb{B}) with a constant given by*

$$r_n = 33n + 7n^2.$$

Proof. We use the same argument as was used to prove Theorem 6.5.3. Note that according to Theorem 6.6.3, every finite CSL algebra has a representation satisfying in the assumptions of Remark 6.5.4. Note that the constant r_n for the strong property (\mathbb{B}) of

nest algebras was obtained based on the fact that each matrix in N has at most n non-zero entries on and $\frac{n^2-n}{2}$ non-zero entries off the diagonal. Theorem 6.6.3 shows that the same thing holds true for CSL algebras. Hence we can end up with obtaining the constant

$$r_n = 33n + 7n^2$$

for the strong property (\mathbb{B}) of CSL algebras. \square

Remark 6.6.7. The constant r_n given in Proposition 6.6.6 is a universal constant for all finite CSL algebras on separable Hilbert spaces whose lattices have n non-zero elements. This constant can possibly be reduced if we pick certain finite CSL algebras.

For example, let H be a separable Hilbert space with the following decomposition,

$$H = H_1 \oplus \cdots \oplus H_n.$$

Define the partial order \preceq on $\{1, \dots, n\}$ by

$$i \preceq n \text{ and } i \preceq i, \quad 1 \leq i \leq n.$$

By Theorem 6.6.3, these decomposition and partial order give us a finite CSL algebra the $*$ -diagram of which is given by

$$\begin{pmatrix} * & & & * \\ & * & 0 & \vdots \\ 0 & & \ddots & \vdots \\ & & & * \end{pmatrix}.$$

So there are n non-zero entries on and $n - 1$ non-zero entries off the diagonal. Using this fact, one can apply the argument that was used in the proof of Theorem 6.5.3 to obtain $s_n = 54n - 14$ as a constant for the strong property (\mathbb{B}) of this finite CSL algebra.

CHAPTER 7

AN UPPER BOUND FOR THE HYPERREFLEXIVITY CONSTANT OF THE BOUNDED n -COCYCLE SPACES OF BANACH ALGEBRAS

In the present chapter, we aim to take the final step towards the hyperreflexivity of the bounded n -cocycle spaces by finding *the hyperreflexivity constant*. Let $\mathcal{S} \subseteq B(X, Y)$ be a closed subspace. So far, the question was that whether *there exist* a constant $C > 0$ such that

$$\text{dist}(T, \mathcal{S}) \leq C \text{dist}_r(T, \mathcal{S}), \quad (T \in B(H)). \quad (7.0.1)$$

Now we ask, if such a constant exists, then what is the smallest value it can attain? We call this the hyperreflexivity constant of \mathcal{S} . Usually it is not easy to find the hyperreflexivity constant. Hence, we are satisfied if we can find an upper bound for this constant.

The problem of finding the hyperreflexivity constant was introduced at the same time when the problem of hyperreflexivity was set for the operator algebras. Possibly, nest algebras have the most elegant hyperreflexivity constant. It is proven in [5] that the hyperreflexivity constant for any nest algebra is 1. Some results on the upper bounds for the hyperreflexivity constant of some classes of von Neumann algebras are provided in [11, 12]. Similar results for certain spaces of matrices can be found in [8]. Although we are usually more interested in the upper bounds for the hyperreflexivity constant, some information in the other way around is provided in [16] where a lower bound for the hyperreflexivity constant of certain matrix spaces and the CSL algebras related to the finite tensor product of some non-trivial nests is found.

We use our approach in the preceding chapter where we found constants for the strong property (\mathbb{B}) of various Banach algebras to obtain upper bounds for the hyperreflexivity constant of the bounded n -cocycle spaces related to these Banach algebras.

7.1 General theory

So far we have introduced many Banach algebras with the strong property (\mathbb{B}) . More so, we could come with a constant for the strong property (\mathbb{B}) of all such Banach algebras. In this section, we show how existence of such a constant can help us to find an upper bound for the hyperreflexivity constant of the bounded n -cocycle spaces. We achieve our goal by modifying our approach in Section 5.2 and its main result. We start with the following proposition which is a modification of Proposition 5.2.1

Proposition 7.1.1. *Let A be a unital Banach algebra having the strong property (\mathbb{B}) with a constant r .*

(i) *For every right Banach A -module X and a bounded operator $D: A \rightarrow X$ and each $\alpha \geq 0$ satisfying*

$$ab = 0 \Rightarrow \|D(a)b\| \leq \alpha \|b\| \|a\|$$

we have

$$\|D(ab)c - D(a)bc\| \leq r\alpha \|a\| \|b\| \|c\| \quad (\forall a, b, c \in A).$$

(ii) *For every right Banach A -module X and a bounded operator $D: A \rightarrow X$ and each $\beta \geq 0$ satisfying*

$$ab = bc = 0 \Rightarrow \|aD(b)c\| \leq \beta \|a\| \|b\| \|c\|$$

we have

$$\|d[D(acb) - aD(cb) - D(ac)b + aD(c)b]e\| \leq r^2\beta \|a\| \|b\| \|c\| \|d\| \|e\| \quad (\forall a, b, c, d, e \in A).$$

Proof. (i) We define $\varphi: A \times A \rightarrow X$ with $\varphi(a, b) = D(a)b$. If $ab = 0$, then

$$\|\varphi(a, b)\| = \|D(a)b\| \leq \alpha \|a\| \|b\|.$$

Therefore by the assumption we get

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq r\alpha\|a\|\|b\|\|c\|,$$

or equivalently,

$$\|D(ab)c - D(a)bc\| \leq r\alpha\|a\|\|b\|\|c\|,$$

as desired.

(ii) Fix $a_2, b_2 \in A$ with $a_2b_2 = 0$ and $\|a_2\| = \|b_2\| = 1$. Define $\varphi : A \times A \rightarrow X$ with

$$\varphi(a, b) = aD(ba_2)b_2.$$

If $ab = 0$, then $a(ba_2) = (ba_2)b_2 = 0$. Hence

$$\|\varphi(a, b)\| = \|aD(ba_2)b_2\| \leq \beta\|a\|\|ba_2\|\|b_2\| \leq \beta\|a\|\|b\|.$$

By the assumption, we have

$$\|\varphi(ab, c) - \varphi(a, bc)\| \leq r\beta\|a\|\|b\|\|c\|,$$

or equivalently,

$$\|abD(ca_2)b_2 - aD(bca_2)b_2\| \leq r\beta\|a\|\|b\|\|c\|. \quad (7.1.1)$$

Now fix $a, c, d \in A$ with $\|a\| = \|c\| = \|d\| = 1$. Define $\psi : A \times A \rightarrow X$ with

$$\psi(f, b) = daD(cf)b - dD(acf)b.$$

Obviously if $fb = 0$, then by (7.1.1),

$$\|\psi(f, b)\| \leq r\beta\|f\|\|b\|.$$

Hence, again by our assumption we deduce that for every $f, b, e \in A$,

$$\begin{aligned} \|\psi(fb, e) - \psi(f, be)\| &\leq rr\beta\|f\|\|b\|\|e\| \\ &= r^2\beta\|f\|\|b\|\|e\|, \end{aligned}$$

or equivalently,

$$\|daD(cfb)e - dD(acfb)e - daD(cf)be + dD(acf)be\| \leq r^2\beta\|f\|\|b\|\|e\|.$$

By putting $f = 1$, we get

$$\|d[D(acb) - aD(cb) - D(ac)b + aD(c)b]e\| \leq r^2\beta\|a\|\|b\|\|c\|\|d\|\|e\| \quad (\forall a, b, c, d, e \in A).$$

□

Proposition 7.1.1 (ii) can be improved to the higher dimensions using the induction as it is demonstrated in the following Theorem. We note that this is a modification of Theorem 5.2.2.

Theorem 7.1.2. *Let A be a unital Banach algebra with unit 1 having the strong property (B) with a constant r . Suppose that X is a unital Banach A -bimodule, $n \in \mathbb{N}$, $T \in B^n(A, X)$ and let $\gamma \geq 0$ satisfying*

$$a_0a_1 = a_1a_2 = \cdots = a_n a_{n+1} = 0 \Rightarrow \|a_0T(a_1, \dots, a_n)a_{n+1}\| \leq \gamma\|a_0\| \cdots \|a_{n+1}\|.$$

Also $T(a_1, \dots, a_n) = 0$ if for some $1 \leq i \leq n$, $a_i = 1$. Then

$$\|\delta^n(T)\| \leq 2^{n-1}r^{n+1}\gamma.$$

Proof. We prove the statement by induction on n . For $n = 1$, the result follows from Proposition 7.1.1(ii) together with the fact that X is unital and $T(1) = 0$.

Now suppose that the result is true for $n \in \mathbb{N}$. We prove it for $n + 1$. Consider $T \in B^{n+1}(A, X)$ and $\gamma \geq 0$ satisfying

$$a_0a_1 = a_1a_2 = \cdots = a_{n+1}a_{n+2} = 0 \Rightarrow \|a_0T(a_1, \dots, a_{n+1})a_{n+2}\| \leq \gamma\|a_0\| \cdots \|a_{n+2}\|.$$

Also $T(a_1, \dots, a_{n+1}) = 0$ if for some $1 \leq i \leq n + 1$, $a_i = 1$. Take $a_i \in A$, $i = 0, \dots, n + 1$ with $a_0a_1 = a_1a_2 = \cdots = a_n a_{n+1} = 0$. We first show that

$$\|a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1}\| \leq r\gamma\|a_0\| \cdots \|a_{n+1}\|. \quad (7.1.2)$$

First suppose that $\|a_0\| = \cdots = \|a_{n+1}\| = 1$, and let

$$S = a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1}.$$

For every $b, c \in A$ with $bc = 0$, we have

$$\begin{aligned} S(b)c &= [a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1}](b)c \\ &= a_0 \Lambda_n(T)(a_1, \dots, a_n)(a_{n+1}b)c - a_0 \Lambda_n(T)(a_1, \dots, a_n)(a_{n+1})bc \\ &= a_0 T(a_1, \dots, a_n, a_{n+1}b)c. \end{aligned}$$

But $a_0 a_1 = \cdots = a_n(a_{n+1}b) = (a_{n+1}b)c = 0$. Thus, by our hypothesis

$$\|a_0 T(a_1, \dots, a_n, a_{n+1}b)c\| \leq \gamma \|a_0\| \cdots \|a_{n+1}b\| \|c\| \leq \gamma \|b\| \|c\|$$

implying that $\|S(b)c\| \leq \gamma \|b\| \|c\|$. Hence, by Proposition 7.1.1(i), we get

$$\|S(bc) - S(b)c\| \leq r\gamma \|b\| \|c\| \quad (\forall b, c \in A). \quad (7.1.3)$$

On the other hand,

$$\begin{aligned} S(1) &= (a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1})(1) \\ &= a_0 \Lambda_n(T)(a_1, \dots, a_n)(a_{n+1}1) - a_0 \Lambda_n(T)(a_1, \dots, a_n)(a_{n+1})1 \\ &= 0. \end{aligned}$$

Putting $b = 1$ in (7.1.3), we can write

$$\|S(c)\| \leq r\gamma \|c\| \quad (c \in A)$$

or equivalently,

$$\|S\| = \|a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1}\| \leq r\gamma. \quad (7.1.4)$$

Now consider the general case. If for some $0 \leq i \leq n+1$, $a_i = 0$, then we clearly have

$$\|a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1}\| \leq r\gamma \|a_0\| \cdots \|a_{n+1}\|.$$

Now suppose that for all $0 \leq i \leq n+1$, $a_i \neq 0$. Then

$$\frac{a_0}{\|a_0\|} \frac{a_1}{\|a_1\|} = \cdots = \frac{a_{n+1}}{\|a_{n+1}\|} \frac{a_{n+2}}{\|a_{n+2}\|} = 0,$$

and so, by (7.1.4),

$$\left\| \frac{a_0}{\|a_0\|} \star \Lambda_n(T) \left(\frac{a_1}{\|a_1\|}, \dots, \frac{a_n}{\|a_n\|} \right) \star \frac{a_{n+1}}{\|a_{n+1}\|} \right\| \leq r\gamma,$$

implying that (7.1.2) holds.

Now let $B_A(A, X)$ denote the space of all left multipliers from A into X and suppose that $q : B(A, X) \rightarrow \frac{B(A, X)}{B_A(A, X)}$ is the natural quotient mapping. It is straightforward to verify that $\frac{B(A, X)}{B_A(A, X)}$ is a unital Banach A -bimodule and q is an A -bimodule morphism. Thus, by (7.1.2),

$$\begin{aligned} \|a_0 \star q(\Lambda_n(T)(a_1, \dots, a_n)) \star a_{n+1}\| &= \|q(a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1})\| \\ &\leq \|q\| \|a_0 \star \Lambda_n(T)(a_1, \dots, a_n) \star a_{n+1}\| \\ &\leq r\gamma \|a_0\| \cdots \|a_{n+1}\|. \end{aligned}$$

Moreover, if for some i , $1 \leq i \leq n$, $a_i = 1$, then for every $a \in A$,

$$\Lambda_n(T)(a_1, \dots, a_n)(a) = T(a_1, \dots, a_n, a) = 0.$$

This shows that $q \circ \Lambda_n(T)(a_1, \dots, a_n) = 0$ if for some $1 \leq i \leq n$, $a_i = 1$. Now using the assumption of the induction, we have

$$\begin{aligned} \|\Delta_q^n(q \circ \Lambda_n(T))(a_1, \dots, a_{n+1})\| &\leq (2^{n-1}r^{n+1})(r\gamma) \|a_1\| \cdots \|a_{n+1}\| \\ &\leq 2^{n-1}r^{n+2}\gamma \|a_1\| \cdots \|a_{n+1}\| \end{aligned} \tag{7.1.5}$$

where $\Delta_q^n : B^n(A, \frac{B(A, X)}{B_A(A, X)}) \rightarrow B^{n+1}(A, \frac{B(A, X)}{B_A(A, X)})$ is the corresponding connecting map in Definition 2.3.4. On the other hand, since q is a Banach A -bimodule morphism, it is easy to check that for all $a_0, \dots, a_{n+1} \in A$,

$$\begin{aligned} \Delta_q^n(q \circ \Lambda_n(T))(a_1, \dots, a_{n+1}) &= q(\Delta^n(\Lambda_n(T))(a_1, \dots, a_{n+1})) \\ &= q(\Lambda_{n+1}(\delta^{n+1}(T))(a_1, \dots, a_{n+1})). \end{aligned}$$

Hence, by (7.1.5)

$$\|q(\Lambda_{n+1}(\delta^{n+1}(T))(a_1, \dots, a_{n+1}))\| \leq 2^{n-1}r^{n+2}\gamma\|a_1\| \cdots \|a_{n+1}\|$$

implying that for $S = \Lambda_{n+1}(\delta^{n+1}(T))(a_1, \dots, a_{n+1})$,

$$\|\text{dist}(S, B_A(A, X))\| \leq 2^{n-1}r^{n+2}\gamma\|a_1\| \cdots \|a_{n+1}\|.$$

So for every $a \in A$, we have

$$\begin{aligned} \|S(a) - S(1)a\| &\leq 2[2^{n-1}r^{n+2}\gamma\|a_1\| \cdots \|a_{n+1}\|\|a\|] \\ &\leq 2^n r^{n+2} \gamma \|a_1\| \cdots \|a_{n+1}\| \|a\|. \end{aligned} \tag{7.1.6}$$

On the other hand,

$$\begin{aligned} S(1) &= \Lambda_{n+1}(\delta^{n+1}(T))(a_1, \dots, a_{n+1})(1) \\ &= \delta^{n+1}(T)(a_1, \dots, a_{n+1}, 1) \\ &= a_1 T(a_2, \dots, a_{n+1}, 1) + \sum_{j=0}^{n-1} (-1)^j T(a_1, \dots, a_j a_{j+1}, \dots, a_{n+1}, 1) + (-1)^n T(a_1, \dots, a_{n+1}, 1) \\ &+ (-1)^{n+1} T(a_1, \dots, a_{n+1})1 \\ &= 0. \end{aligned}$$

Therefore by putting $a = a_{n+2}$ in (7.1.6), we have

$$\begin{aligned} \|\delta^{n+1}(T)(a_1, \dots, a_{n+2})\| &= \|\Lambda_{n+1}(\delta^{n+1}(T))(a_1, \dots, a_{n+1})(a_{n+2})\| \\ &= \|S(a_{n+2})\| \\ &= \|S(a_{n+2}) - S(1)a_{n+2}\| \\ &\leq 2^n r^{n+2} \gamma \|a_1\| \cdots \|a_{n+2}\|. \end{aligned}$$

This completes the proof. □

Now we are ready to give the main result of this chapter.

Theorem 7.1.3. *Let A be a Banach algebra having b.l.u. and the strong property (\mathbb{B}) with a constant r . Let M be a bound for the local units of A . Let $n \in \mathbb{N}$, suppose that X is a Banach A -bimodule such that $\mathcal{H}^{n+1}(A, X)$ is a Banach space. Then for each $T \in B^n(A, X)$, we have*

$$\text{dist}(T, \mathcal{Z}^n(A, X)) \leq C2^{n-1}(M^2r + (M + 1)^2)^{n+1}\text{dist}_r(T, \mathcal{Z}^n(A, X))$$

where C is a constant satisfying

$$\text{dist}(T, \mathcal{Z}^n(A, X)) \leq C\|\delta^n(T)\|, \quad (T \in B^n(Z, X)). \quad (7.1.7)$$

Proof. Let $T \in B^n(A, X)$. By Lemma 5.2.3, for every $a_i \in A^\sharp$, $i = 0, \dots, n + 1$ with $a_0a_1 = \dots = a_na_{n+1} = 0$, we have

$$\|a_0\sigma(T)(a_1, \dots, a_n)a_{n+1}\| \leq \text{dist}_r(T, \mathcal{Z}^n(A, X))\|a_1\| \cdots \|a_{n+1}\|$$

where $\sigma(T) : A^\sharp \rightarrow X$ is defined by

$$\sigma(T)(b_1 + \lambda_1, \dots, b_n + \lambda_n) = T(b_1, \dots, b_n) \quad (b_i \in A, \lambda_i \in \mathbb{C}).$$

On the other hand, if for some $1 \leq i \leq n$, $a_i = 1$, then

$$\sigma(T)(a_1, \dots, a_n) = 0.$$

If we apply Theorem 4.1.3, we find that A^\sharp has the strong property (\mathbb{B}) with a constant given by,

$$M^2r + (M + 1)^2.$$

Hence we can use Theorem 7.1.2 to write

$$\|\delta^{\sharp n}(\sigma(T))\| \leq 2^{n-1}(M^2r + (M + 1)^2)^{n+1}\text{dist}_r(T, \mathcal{Z}^n(A, X)). \quad (7.1.8)$$

Now since $\mathcal{H}^{n+1}(A, X)$ is a Banach space, $\text{Im}\delta^n$ is closed. Hence, by the open mapping theorem, there is a constant $C > 0$ such that for each $T \in B^n(A, X)$,

$$\text{dist}(T, \mathcal{Z}^n(A, X)) \leq C\|\delta^n(T)\|. \quad (7.1.9)$$

It is straightforward to check that

$$\|\delta^n(T)\| \leq \|\delta^{\sharp n}(\sigma(T))\|. \quad (7.1.10)$$

Hence putting (7.1.8), (7.1.9) and (7.1.10) together we get,

$$\text{dist}(T, \mathcal{Z}^n(A, X)) \leq C2^{n-1}(M^2r + (M + 1)^2)^{n+1} \text{dist}_r(T, \mathcal{Z}^n(A, X)), \quad (7.1.11)$$

as desired. □

7.2 Group algebras and C*-algebras

We showed in Section 6.3 that every C*-algebra and group algebra has the strong property (B) with a constant. On account of Theorem 7.1.3, this enables us to obtain an upper bound for the hyperreflexivity constant of the bounded n -cocycle spaces of C*-algebras and group algebras. First we need to introduce the notion of amenability constant.

Definition 7.2.1. Let A be a Banach algebra. The amenability constant of A , which we denote by $AM(A)$, is

$$\inf\{\sup_{\alpha} \|\mu_{\alpha}\| : (\mu_{\alpha})_{\alpha} \text{ is a bounded approximate diagonal for } A\}$$

where we define the infimum of the empty set to be $+\infty$. Hence A is amenable if and only if $AM(A) < \infty$ (see Definition 2.3.2 and Theorem 2.3.3).

Remark 7.2.2. (i) Let G be a locally compact amenable group. Then $AM(L^1(G)) = 1$ (see [59, Corollary 1. 10]).

(ii) Let A be an amenable C*-algebra. Then $AM(A) = 1$.

Remark 7.2.3. Let A be an amenable Banach algebra and suppose that X is a dual Banach A -bimodule. Then $C \leq AM(A)$ where C is the constant given in (7.1.7) (see [3]).

Theorem 7.2.4. *Suppose that A is a C*-algebra or the group algebra of a group with an open subgroup of polynomial growth. Let $n \in \mathbb{N}$, and let X be a Banach A -bimodule such*

that $\mathcal{H}^{n+1}(A, X)$ is a Banach space. Then $\mathcal{Z}^n(A, X)$ is hyperreflexive with a constant bounded by

$$C2^{n-1}(288\pi M^2(1 + \sqrt{2}) + (M + 1)^2)^{n+1}$$

where M is a bound for the local units of A and C is a constant satisfying in (7.1.7). In particular, we have

- (i) If A is a C^* -algebra or the group algebra of a discrete group, then $M = 1$.
- (ii) In the case where A is amenable and X is the dual of a Banach A -bimodule, we can assume that $C = 1$.

Proof. The main statement follows if we combine Corollary 6.3.6 (resp. Corollary 6.3.5) and Theorem 7.1.3. To prove (i), note that the group algebra of a discrete group is unital. Moreover, Proposition 4.2.1 shows that local units of a C^* -algebra are bounded by 1. Finally, (ii) follows if we apply Remark 7.2.2 and Remark 7.2.3. \square

7.3 Banach algebras' matrix spaces

In Section 6.4 we showed that Banach algebras' matrix spaces when equipped with an appropriate Banach algebra norm have the strong property (\mathbb{B}) . Hence according to Theorem 4.1.3 and Theorem 5.2.4, if such a Banach algebra has b.l.u., then we will be able to give some results on the hyperreflexivity of bounded n -cocycles related to it. In the current section we assume that M_m and $M_m(\mathcal{A}) = M_m \otimes \mathcal{A}$ are respectively equipped with the operator norm and the projective tensor norm. Then according to Example 6.4.8, it has the strong property (\mathbb{B}) with a constant given by $16(m^2 + m)$.

Theorem 7.3.1. *Let $n \in \mathbb{N}$ and $m \geq 2$. Suppose that \mathcal{A} is a Banach algebra with b.l.u. whose local units have bound N , and let X be a Banach $M_m(\mathcal{A})$ -bimodule with the property that $\mathcal{H}^{n+1}(M_m(\mathcal{A}), X)$ is a Banach space. Then $\mathcal{Z}^n(M_m(\mathcal{A}), X)$ is hyperreflexive with a constant bounded by*

$$C2^{n-1}(16N^2(m^2 + m) + (N + 1)^2)^{n+1},$$

where C is a constant satisfying in (7.1.7).

Proof. According to Example 6.4.8, $M_m(\mathcal{A})$ has the strong property (\mathbb{B}) with a constant given by $16(m^2 + m)$. Since M_m and \mathcal{A} both have b.l.u., Proposition 4.3.2 shows that $M_m(\mathcal{A})$ has b.l.u. with bound N . The result now follows from Theorem 7.1.3. \square

Since amenable Banach algebras are the most well-known Banach algebras whose Hochschild cohomology groups are Banach space, Theorem 7.3.1 specially gives some results on such Banach algebras.

Corollary 7.3.2. *Let \mathcal{A} be an amenable Banach algebra with b.l.u. Suppose that $n \in \mathbb{N}$ and $m \geq 2$. Then for every Banach $M_m(\mathcal{A})$ -bimodule X , $\mathcal{Z}^n(M_m(\mathcal{A}), X^*)$ is hyperreflexive with a constant bounded by*

$$2^{n-1}(16N^2(m^2 + m) + (N + 1)^2)^{n+1},$$

where N is a bound for local units of \mathcal{A} and C is a constant satisfying in (7.1.7).

Proof. It is known that M_m is an amenable Banach algebra. Hence $M_m(\mathcal{A}) = M_m \otimes \mathcal{A}$ is amenable as well. This implies that for each Banach $M_m(\mathcal{A})$ -bimodule X and for each $n \in \mathbb{N}$, $\mathcal{H}^{n+1}(M_m(\mathcal{A}), X^*) = 0$. The result now follows from Theorem 7.3.1. \square

Example 7.3.3. Let X be a Banach space. Suppose that $m \geq 2$ and equip $M_m(B(X)) \cong B(X^{(m)})$ with the operator norm. Note that $B(X^{(m)})$ is a unital Banach algebra. Moreover, it is proven in [30] that for each $n \in \mathbb{N}$, we have

$$\mathcal{H}^n(B(X^{(m)}), B(X^{(m)})) = \{0\}.$$

Hence, Theorem 7.3.1 implies that $\mathcal{Z}^n(B(X^{(m)}), B(X^{(m)}))$ is hyperreflexive with a constant bounded by

$$C2^{n-1}(16(m^2 + m) + 4)^{n+1},$$

where C is a constant satisfying in (7.1.7).

Example 7.3.4. Let G be a locally compact amenable group with an open subgroup of polynomial growth. Corollary 4.2.2 shows that $L^1(G)$ has b.l.u. Suppose that $n \in \mathbb{N}$ and

$m \geq 2$. Then according to Corollary 7.3.2, for every Banach $M_m(L^1(G))$ -bimodule X , $\mathcal{Z}^n(M_m(L^1(G)), X^*)$ is hyperreflexive with a constant bounded by

$$C2^{n-1}(16N^2(m^2 + m) + (N + 1)^2)^{n+1},$$

where N is a bound for local units of \mathcal{A} and C is a constant satisfying in (7.1.7). If G is discrete, then $N = 1$.

7.4 Finite nest and CSL algebras

We finish this chapter by providing results on the hyperreflexivity constant of bounded n -cocycle spaces related to finite nest and finite CSL algebras. Since there have already been some information on the Hochschild cohomology groups of CSL and nest algebras ([13, 42]), on account of Theorem 7.1.3, we can come with results on the hyperreflexivity of the relevant bounded n -cocycle spaces.

Theorem 7.4.1. *Let A be a finite nest (resp. finite CSL) algebra on a (resp. Separable) Hilbert space generated by m nonzero projections. Let $n \in \mathbb{N}$, and let X be a Banach A -bimodule such that $\mathcal{H}^{n+1}(A, X)$ is a Banach space. Then $\mathcal{Z}^n(A, X)$ is hyperreflexive with a constant bounded by*

$$C2^{n-1}(33m + 7m^2 + 4)^{n+1},$$

where C is a constant satisfying in (7.1.7).

Proof. According to Theorem 6.5.3 (resp. Theorem 6.6.6) a constant for the strong property (\mathbb{B}) of a finite nest (resp. finite CSL) algebra generated by m nonzero projections is given by

$$33m + 7m^2.$$

Now the result follows from Theorem 7.1.3. □

Using the fact that finite nest algebras are unital Banach algebras with the strong property (\mathbb{B}) , we can present the following result.

Corollary 7.4.2. *Let $N \subseteq B(H)$ be a finite nest algebra generated by m elements. Then:*

(i) *For all $n \geq 1$, $\mathcal{Z}^n(N, B(H))$ is hyperreflexive.*

(ii) *For all $n \geq 1$, $\mathcal{Z}^n(N, N)$ is hyperreflexive.*

Moreover, an upper bound for the hyperreflexivity constant of these spaces is given by,

$$C2^{n-1}(33m + 7m^2 + 4)^{n+1},$$

where C is a constant satisfying in (7.1.7).

Proof. According to Theorem 2.4.3,

$$\mathcal{H}^n(N, B(H)) = 0 \text{ and } \mathcal{H}^n(N, N) = 0.$$

The result now follows if we apply Theorem 7.4.1. □

In the preceding corollary, in order to show that $\mathcal{Z}^n(N, N)$ (resp. $\mathcal{Z}^n(N, B(H))$), for a nest algebra N , is hyperreflexive, we used the fact that $\mathcal{H}^n(N, N) = 0$ (resp. $\mathcal{H}^n(N, B(H)) = 0$). In general, the same result might not be true for a (finite) CSL algebra. But according to Theorem 2.4.4, there are some alternatives for certain CSL algebras. This enables us to give the following result on the hyperreflexivity of the bounded n -cocycle spaces of CSL algebras.

Remark 7.4.3. Let \mathcal{A} be a norm closed unital subalgebra of $B(H)$ for some Hilbert space H . Let $\mathcal{E}(\mathcal{A})$ be the following subalgebra of $B(\mathbb{C} \oplus H)$,

$$\mathcal{E}(\mathcal{A}) = \left\{ \begin{pmatrix} z & u \\ 0 & a \end{pmatrix} \in B(\mathbb{C} \oplus H) : z \in \mathbb{C}, u \in H^*, a \in \mathcal{A} \right\}.$$

It is proven in [42] that if \mathcal{L} is a CSL and $\mathcal{A} = \text{alg}\mathcal{L}$, then $\mathcal{E}(\mathcal{A})$ is a CSL algebra and

$$\text{Lat}\mathcal{E}(\mathcal{A}) = \{e_1 \oplus L : L \in \mathcal{L}\} \cup \{0\}$$

where e_1 is the projection of $\mathbb{C} \oplus H$ onto \mathbb{C} . In particular, if \mathcal{L} is finite, then $\text{Lat}\mathcal{E}(\mathcal{A})$ is a finite CSL as well.

Proposition 7.4.4. *Let H be a separable Hilbert space, let $\mathcal{L} \subseteq H$ be a finite CSL generated by m projections, and let $\mathcal{A} = \text{alg}\mathcal{L}$. Then for each $n \in \mathbb{N}$, $\mathcal{Z}^n(\mathcal{E}(\mathcal{A}), B(\mathbb{C} \oplus H))$ and $\mathcal{Z}^n(\mathcal{E}(\mathcal{A}), \mathcal{E}(\mathcal{A}))$ are hyperreflexive. Moreover, an upper bound for the hyperreflexivity constant of these spaces is given by*

$$C2^{n-1}(33(m+1) + 7(m+1)^2 + 4)^{n+1},$$

where C is a constant satisfying in (7.1.7).

Proof. By Remark 7.4.3, $\mathcal{E}(\mathcal{A})$ is a finite CSL algebra on a separable Hilbert space. By Theorem 2.4.4,

$$\mathcal{H}^n(\mathcal{E}(\mathcal{A}), B(\mathbb{C} \oplus H)) = 0,$$

and

$$\mathcal{H}^n(\mathcal{E}(\mathcal{A}), \mathcal{E}(\mathcal{A})) = 0.$$

Moreover Proposition 6.6.6 shows that $\mathcal{E}(\mathcal{A})$ is a unital Banach algebra with the strong property (\mathbb{B}) . On the other hand, $\mathcal{E}(\mathcal{A})$ is a CSL algebra generated by $(m+1)$ non-zero projections. According to Theorem 6.5.3, a constant for the strong property (\mathbb{B}) of $\mathcal{E}(\mathcal{A})$ is given by

$$33(m+1) + 7(m+1)^2.$$

The result now follows from Theorem 7.1.3. □

Proposition 7.4.5. *Let \mathcal{A} be a finite dimensional CSL algebra. Let X be a finite dimensional Banach \mathcal{A} -bimodule. Then for each $n \in \mathbb{N}$, $\mathcal{Z}^n(\mathcal{A}, X)$ is hyperreflexive.*

Proof. By Theorem 6.6.6, \mathcal{A} is a unital Banach algebra with the strong property (\mathbb{B}) . Since \mathcal{A} and X are both finite dimensional, $\mathcal{H}^n(\mathcal{A}, X)$ is a finite dimensional normed algebra and hence it is a Banach space. The result now follows from Theorem 7.1.3. □

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