Asymptotic and numerical solutions of trapped Rossby waves in high-latitude shear flows with boundaries

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Abstract

We consider the amplitudes of coastally trapped Rossby waves in a high-latitude shear flow on a modified β-plane, where also the effect of the sphericity of the earth (δ-effect) is taken into account. We present a particular analytical solution and also asymptotic and numerical solutions. We find that the asymptotic WKB solutions are accurate compared to the numerical results. We show that the δ-effect is most important for shorter waves and leads to an enhanced selection of trapped Rossby wave modes.

Zusammenfassung

Wir betrachten die Amplituden von küstennah gefangenen Rossby-Wellen in einer Scherströmung hoher Breiten. Die Rechnungen werden auf einer modifizierten β-Ebene durchgeführt, die auch die Spherizität der Erde berücksichtigt (δ-Effekt). Wir zeigen eine spezielle analytische Lösung und auch asymptotische und numerische Lösungen. Die asymptotischen WKB-Lösungen erweisen sich als genau, verglichen mit den numerischen Resultaten. Der δ-Effekt wirkt sich a stärksten bei den sehr langen und den kurzen Wellen aus und führt zu einer stärkeren Selektion von Moden gefangener Rossby-Wellen.

1 Introduction

Rossby waveguides play an important role in atmospheric and oceanic wave dynamics since in waveguides wavepackets and the corresponding wave activity can propagate far distances along the zonal direction without approaching a critical line (cf. Hoskins and Ambrizzi 1993, Chang and Philander 1989). In a previous paper, Harlander and Metz (1998) discussed the possibility of a Rossby waveguide in a high-latitude shear flow on a modified β-plane. The authors showed by applying the WKB-method that wavepackets can be trapped between a reflecting boundary and a turning latitude for Rossby wavepackets. In the present short note we follow Schönfeldt (1999) and compute single modes of trapped Rossby waves with constant zonal wavenumber and we ask how such modes can be influenced by the sphericity of the earth. Therefore we solve the corresponding amplitude eigenvalue problem with different methods. The eigenfunctions determine the amplitudes of the trapped waves and the eigenvalues the zonal wavenumbers. In contrast to Schönfeldt (1999) we compare WKB solutions to numerical solutions.

In section 2 we briefly recapitulate the basic equations used by Harlander and Metz (1998) and, furthermore, describe the WKB-amplitude equation and the eigencondition for trapped modal waves. In section 3 we show an analytic solution computed by Schönfeld (1999), asymptotic WKB solutions and numerical solutions for trapped large-scale waves.

2 WKB solution of the quasi-geostrophic vorticity equation

We are concerned with high-latitude southern hemisphere flow on a modified β-plane where the β-term alone may not be sufficient to provide an adequate representation of the effects of sphericity of the earth. Therefore we retain the second term in the Taylor-series expansion of the Coriolis parameter on this β-plane and write:

$$f \approx -f_0 + \beta_0 L y + \frac{1}{2} f_0 \frac{L^2}{a^2} y^2 \quad , \tag{1}$$

where $f_0 = 2\Omega \sin \mid \phi_0 \mid$, $\beta_0 = 2\Omega / a \cos \phi_0$, $L = 1000km$ ist the length scale and y the dimensionless meridional coordinate, ϕ_0 is the tangential latitude of the β-plane and a the earth radius.

Assuming a (geostrophic) zonally symmetric basic flow $\bar{U}$ that depends only on the y-coordinate, the linearized (dimensionless) quasi-geostrophic potential vorticity equation for the perturbation streamfunction ψ_0 on the southern hemisphere is written as

$$\left(\frac{\partial}{\partial t} + \bar{U}\frac{\partial}{\partial x}\right)(\nabla^2 \psi_0 - F\psi_0) + B_1 \frac{\partial \psi_0}{\partial x} = 0 \quad , \tag{2}$$

with

$$B_1 = F\bar{U} + \bar{\beta} + \bar{\delta} y - \frac{\partial^2 \bar{U}}{\partial y^2} \quad , \tag{3}$$

where $\epsilon_R = \frac{U}{f_0 L}$ is the Rossby number, $U = 1m/s$, ψ_0 is the $\mathcal{O}(\epsilon_R^0)$ streamfunction, $\bar{\beta} = \frac{\beta}{\epsilon_R} = \mathcal{O}(1)$, $\bar{\delta} = \frac{\delta}{\epsilon_R} = \mathcal{O}(1)$, $\beta = \frac{\beta_0 L}{f_0}$, $\delta = \frac{L^2}{a^2}$. All the other symbols have their conventional meanings. Note that we have included in (2) the effects of an upper free surface ("Froude"-parameter $F = f_0^2 L^2 / gD = \mathcal{O}(1)$), where D is the vertical scale height. Except for the δ-term which results from the second derivative of the Coriolis parameter with respect to latitude (3) is of standard form (for more details see Harlander and Metz, 1998).

We restrict our discussion to the case where $\bar{U}$ and B_1 vary in such a way that we may apply the WKB ansatz

$$\psi_0 = (\tilde{\psi}_0(Y) + \epsilon \tilde{\psi}_1(Y) + \cdots) \exp(i\theta(Y)/\epsilon) \tag{4}$$

to the quasi-geostrophic potential vorticity equation (QGV). Here $Y = \epsilon y$ is a slow variable, introduced usually in the WKB context with $\epsilon = 0.1$. Following e.g. Yang (1991) we may write the dispersion relation of a quasi-geostrophic Rossby wave packet:

$$\sigma = \bar{U} m - \frac{m}{K^2} B_1 \quad , \tag{5}$$

where $K^2 = m^2 + n(Y)^2 + F$ is the total wave number.

2.1 The amplitude equation

Using (5) we can compute the local meridional wavenumber

$$n(Y)^2 = \left(\frac{\partial \theta}{\partial Y}\right)^2 = -\frac{mB_1}{\sigma - \bar{U}m} + m^2 + F =: q(Y) \quad . \tag{6}$$

This equation is called Eikonal-equation. From the order ϵ^1 problem of the WKB approximation we find the equation for the amplitude $\tilde{\psi}_0$ of the wavepacket

$$n(Y)' \tilde{\psi}_0(Y) + 2n(Y) \tilde{\psi}_0(Y)' = 0 \quad , \tag{7}$$

Table 1: **Parameter setting for the two experiments described in the text.**

	U	U_0	F	β	δ
E1	eq.(15)	0.5	1	5	1
E3	eq.(16)	1	1	5	1

with the solution

$$\tilde{\psi}_0 = \frac{c}{n(Y)^{\frac{1}{2}}} \quad , \tag{8}$$

where c is a constant of integration. Sometimes (7) is called transport equation.

The WKB solution is not valid near the turning points Y_T where $q(Y) = 0$. In our case we have a single zero at the turning point with $q'(Y_T) > 0$. Using a linear approximation for $q(Y)$ near the turning point it can be shown (e.g. Holmes 1995) that

$$\tilde{\psi}_0 = \begin{cases} \frac{1}{|q(Y)|^{1/4}}(2a_R \cos(\frac{1}{\epsilon}\theta(Y) - \frac{\Pi}{4}) + b_R \cos(\frac{1}{\epsilon}\theta(Y) + \frac{\Pi}{4}) & : \quad Y < Y_T \\ \frac{1}{q(Y)^{1/4}}(a_R \exp(-\frac{1}{\epsilon}\kappa(Y)) + b_R \exp(\frac{1}{\epsilon}\kappa(Y)) & : \quad Y_T < Y \end{cases} \tag{9}$$

where

$$\theta(Y) = \int_Y^{Y_T} \mid q(s) \mid^{1/2} ds \tag{10}$$

and

$$\kappa(Y) = \int_{Y_T}^{Y} q(s)^{1/2} ds \quad . \tag{11}$$

2.2 Trapped modal waves

The continous spectrum of trapped local wavepackets allows trapped modal waves along the coast for some specific zonal wavenumbers (eigenvalues) m. Hence the boundary conditions for the amplitude at and far away from the coast are

$$\tilde{\psi}(Y_B) = \tilde{\psi}(\infty) = 0 \quad , \tag{12}$$

and this can be used as eigencondition. To satisfy the boundary conditions we see from (9) that b_R must be zero and

$$\frac{1}{\epsilon}\theta(Y) - \frac{\pi}{4} = \frac{\pi}{2}(2n-1) \quad \text{with} \quad n = 1, 2, \cdots, \infty \quad . \tag{13}$$

If, for a chosen n, a k can be found from the continous spectrum of trapped wave packets so that

$$(\frac{\pi}{2}(2n-1) + \frac{\pi}{4})\epsilon = \int_{Y_B}^{Y_T} \mid q(s) \mid^{1/2} ds \quad , \tag{14}$$

then a trapped modal wave exists with zonal wave number k and mode $n-1$. (The mode defines the number of zeros between the boundary and the turning latitude.)

3 Results

Here we focus on the Rossby wave propagation properties in sheared high-latitude westerly ocean currents where β becomes smaller than in middle latitudes and the second derivative of the Coriolis parameter becomes more important. The current is assumed to be i) bounded at its southern side by a rigid east-west oriented boundary and ii) situated in the southern hemisphere.

In the experiments performed two different basic flows are used, i) a linear flow profile

$$\bar{U}(Y) = \frac{U_0}{\epsilon}Y + 1.02 \tag{15}$$

and ii) a jet-like basic flow profile

$$\bar{U}(Y) = U_0 \exp\left\{-\left(\frac{Y}{\epsilon}\right)^2\right\} \quad , \tag{16}$$

where U_0 is constant.

The order of magnitude of the different effects considered are

$$\bar{U} \sim \mathcal{O}(1) \quad , \quad F \sim \mathcal{O}(1) \quad , \quad \bar{\beta} \sim \mathcal{O}(1) \quad , \quad \bar{\delta} \sim \mathcal{O}(1) \quad . \tag{17}$$

It is worth to mention that the specific setting of U_0, F, $\bar{\beta}$ and $\bar{\delta}$ of the experiments performed (cf. Table 1) is consistent with these order of magnitudes. As shown in Tab. 1 we assumed that the β-effect is half as strong than in middle latitudes and that the δ-effect is five times smaller than the β-effect. The use of $U_0 = 0.5$ in experiment E1 corresponds to a velocity gradient of $1ms^{-1}/2000km$ and $U_0 = 1$ in experiment E3 corresponds to a maximal jet velocity of $1ms^{-1}$ 2000km north of the "coast".

3.1 Trapped modal waves

3.1.1 Analytical solution

In the following we present the analytic solution given in Schönfeldt (1999) but include the δ-effect. Let us use $\psi_0 = \tilde{\psi}(y)\exp i(mx - \sigma t)$ to transform (2) to

$$\epsilon^2\frac{d^2\tilde{\psi}}{dY^2} - q(Y)\tilde{\psi} = 0 \quad , \tag{18}$$

where $q(Y)$ is given by (6). For the linear basic flow an analytic solution of (18) can be found if $\sigma - m\bar{U}(Y_B) = 0$, i.e. if the boundary is a critical line. Here we assume that $\sigma = \bar{U}(Y_B) = 0$ and therefore

$$q(Y) = m^2 - \left(\frac{\bar{\beta} - \delta_\epsilon\bar{Y}}{U_\epsilon(\bar{Y} + Y)} + \frac{\delta_\epsilon}{U_\epsilon}\right) \quad , \tag{19}$$

where $\bar{Y} = \epsilon/U_0$, $U_\epsilon = \bar{Y}^{-1}$, $\delta_\epsilon = \bar{\delta}/\epsilon$. Using the ansatz of Schönfeldt (1999)

$$\tilde{\psi} = \zeta\hat{\psi}(\tilde{m}\zeta)\exp(-\tilde{m}\zeta) \quad , \tag{20}$$

where $\zeta = Y + \bar{Y}$ and $\tilde{m} = (\frac{m^2}{\epsilon^2} - \frac{\delta_\epsilon}{\epsilon^2 U_\epsilon})^{1/2}$ and substituting (20) in (18) we obtain an equation for $\hat{\psi}$

$$\zeta\tilde{m}\hat{\psi}'' + 2(1 - \zeta\tilde{m})\hat{\psi}' + \left(\frac{\bar{\beta} - \delta_\epsilon\bar{Y}}{\epsilon^2 U_\epsilon\tilde{m}} - 2\right)\hat{\psi} = 0 \quad . \tag{21}$$

This equation can be solved by an infinite power series expansion yielding the eigenfunctions

$$\hat{\psi}_n(\zeta\tilde{m}) = \sum_{j=0}^{n} \binom{n}{j} \frac{1}{(j+1)!}(-2\zeta\tilde{m})^j \quad , \tag{22}$$

and the corresponding eigenvalues

$$m_n = \left\{ (\frac{\bar{\beta} - \delta_\epsilon \bar{Y}}{U_0 2(n+1)})^2 + \frac{\delta_\epsilon}{U_\epsilon} \right\}^{1/2} \tag{23}$$

for all $n = 0, 1, 2, \cdots, \infty$. Equations (22) and (23) are the solutions of Schönfeldt (1999) (extended by the δ-term), where also a brief discussion of the particular properties of such waves is given. Using the parameters given in Tab. 1 we find $m_0 = 3.31$, $m_1 = 2.06$, $m_2 = 1.73$, $m_3 = 1.6$ with $\bar{\delta} = 1$ and $m_0 = 5$, $m_1 = 2.5$, $m_2 = 1.67$, $m_3 = 1.25$ with $\bar{\delta} = 0$, i.e. trapped long and short waves are strongly modified by the sphericity of the earth, whereas waves with a wavelength of about $3000km$ show no significant modification. However, this result is not independent of the geometrie of the modified β-channel, e.g. short waves are not modified by the δ-effect if the line $y = 0$ corresponds with the coast line.

3.1.2 WKB and numerical solutions

Unfortunately, a WKB solution of the situation considered in the previous subsection can not be expected to be accurate since the coastal boundary is a critical line. To be more precise, we can not compute the zonal wavenumbers via (14) with a sufficient accuracy since $q(s)$ has a singularity at the boundary. However, we compare WKB-solutions to solutions obtained numerically by a forth-order Runge-Kutta method.

The Figs. 1 and 2 show the amplitudes of different modes of stationary coastally trapped Rossby waves, with and without the $\delta-$effect, for the linear basic flow profile and the jet-like profile, respectively. It is obvious that both basic flows used do not violate the WKB assumptions and therefore the WKB solutions are accurate, especially for the jet-like basic flow. Near the turning latitude, however, the WKB solution (9) is not valid. As can be expected from Fig. 1 of Harlander and Metz (1998), the δ-effect shifts the turning latitude and therefore the maximal amplitude of the trapped waves closer to the boundary.

A trapped Rossby wave with mode 2 or larger does not exist for the jet-like basic flow, and even mode 1 occurs only if the δ-effect is neglected. Therefore, the δ-effect enhances not only the selection of trapped wavepackets but also reduces the number of modes of trapped Rossby waves.

Finally, we belief that (together with the results given in Harlander and Metz (1998)) we have shown a rather complete picture of possible large-scale trapped waves in bounded zonal high-latitude flows and the importance of the sphericity of the earth on the wave characteristics.

Acknowledgments
The author thanks Dr. H.-J. Schönfeldt for many fruitful discussions and for proofreading the manuscript.

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Figure 1: Normed amplitudes of trapped stationary Rossby waves of E1. $m_1^{num} \approx 1.971$, $m_1^{WKB} \approx 2.02$ (a), $\bar{\delta} = 0$, $m_1^{num} \approx 2.241$, $m_2^{WKB} \approx 2.324$ (b), $m_2^{num} \approx 1.697$, $m_2^{WKB} \approx 1.714$ (c), $\bar{\delta} = 0$, $m_2^{num} \approx 1.547$, $m_2^{WKB} \approx 1.586$ (d).

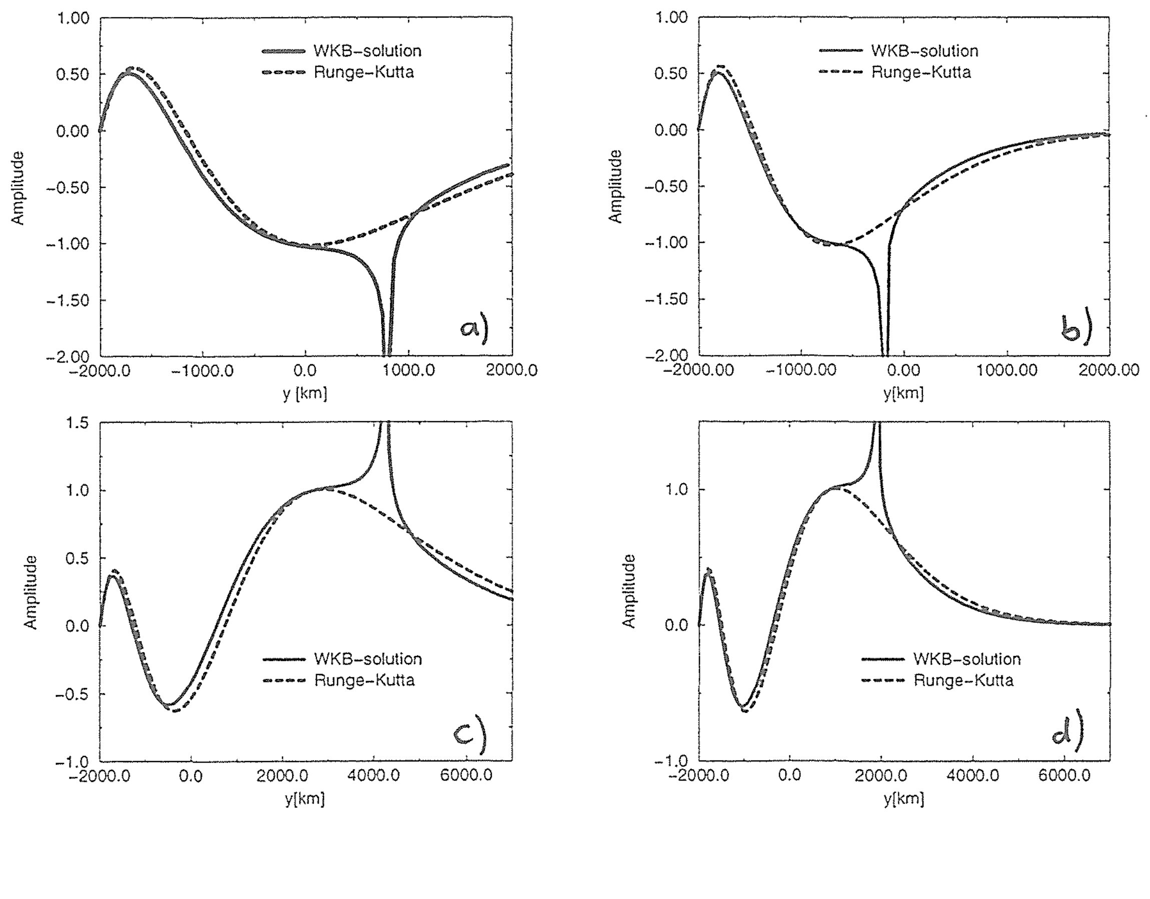

Figure 2: Normed amplitudes of trapped stationary Rossby waves of E3. $m_0^{num} \approx 5.935$, $m_1^{WKB} \approx 5.992$ (a), $\bar{\delta} = 0$, $m_0^{num} \approx 8.814$, $m_2^{WKB} \approx 8.879$ (b), $\bar{\delta} = 0$, $m_1^{num} \approx 4.150$, $m_2^{WKB} \approx 4.175$ (c).