Adding Threshold Concepts to the Description Logic $\mathcal{EL}$

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To Margot

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Contents

Chapter 1

Introduction

Description Logics (DLs) [BCM$^+$03] are a family of logic-based knowledge representation formalisms, which can be used to represent the conceptual knowledge of an application domain in a structured and formally well-understood way. They allow their users to define the important notions of the domain as concepts by stating necessary and sufficient conditions for an individual to belong to the concept. These conditions can be atomic properties required for the individual (expressed by concept names) or properties that refer to relationships with other individuals and their properties (expressed as role restrictions). The expressivity of a particular DL is determined by what sort of properties can be required and how they can be combined.

The DL $\mathcal{EL}$, in which concepts can be built using concept names as well as the concept constructors conjunction ($\sqcap$), existential restriction ($\exists r.C$), and the top concept ($\top$), has drawn considerable attention in the last decade since, on the one hand, important inference problems such as the subsumption problem are polynomial in $\mathcal{EL}$, even with respect to expressive terminological axioms [Bra04]. On the other hand, though quite inexpressive, $\mathcal{EL}$ can be used to define biomedical ontologies, such as the large medical ontology SNOMED CT.[1] In $\mathcal{EL}$ we can, for example, define the concept of a *happy man* as a male human that is healthy and handsome, has a rich and intelligent wife, a son and a daughter, and a friend:

$$\begin{aligned} &\mathsf{Human} \sqcap \mathsf{Male} \sqcap \mathsf{Healthy} \sqcap \mathsf{Handsome} \sqcap \\ &\exists \mathsf{spouse}.(\mathsf{Rich} \sqcap \mathsf{Intelligent} \sqcap \mathsf{Female}) \sqcap \\ &\exists \mathsf{child}.\mathsf{Male} \sqcap \exists \mathsf{child}.\mathsf{Female} \sqcap \exists \mathsf{friend}.\top \end{aligned} \tag{1.1}$$

For an individual to belong to this concept, all the stated properties need to be satisfied. However, maybe we would still want to call a man happy if most, though not all, of the properties hold. It might be sufficient to have just a daughter without a son, or a wife that is only intelligent but not rich, or maybe an intelligent and rich spouse of a different gender. But still, not too many of the properties should be violated.

In this thesis, we introduce a DL extending $\mathcal{EL}$ that allows us to define concepts in such an approximate way. The main idea is to use a *graded membership function* m, which instead of a Boolean membership value 0 or 1 yields a membership degree from the interval $[0, 1]$. We can then require a happy man to belong to the $\mathcal{EL}$ concept (1.1) with degree at least .8. More generally, if C is an $\mathcal{EL}$ concept, then the *threshold concept* $C_{\geq t}$ for $t \in [0, 1]$ collects all the individuals that belong to C with degree at least t. In

[1] see http://www.ihtsdo.org/snomed-ct/

addition to such upper threshold concepts, we will also consider lower threshold concepts $C_{\leq t}$ and allow the use of strict inequalities in both. For example, an unhappy man could be required to belong to the $\mathcal{EL}$ concept (1.1) with a degree less than .2. Using these constructors and defining their underlying semantics based on a graded membership function, we define the family of DLs $\tau\mathcal{EL}(m)$ where m is a parameter of the logic representing the chosen function.

We then go further and define a particular membership degree function *deg*. Its definition is a natural extension of the homomorphism characterization of crisp membership in $\mathcal{EL}$. Basically, an individual is punished (in the sense that its membership degree is lowered) for each missing property in a uniform way. For instance, suppose that some individual d belongs to the sets corresponding to Human and Healthy under some interpretation, but does not belong to the ones corresponding to Handsome and Male. Then, regarding the concept description $\mathsf{Human} \sqcap \mathsf{Male} \sqcap \mathsf{Healthy} \sqcap \mathsf{Handsome}$, the computation of *deg* will punish d for the two missing properties, and give the value $deg(d, \mathsf{Human} \sqcap \mathsf{Male} \sqcap \mathsf{Healthy} \sqcap \mathsf{Handsome}) = 1/2$ as the degree of membership of d in that concept (see Chapter 4 for the precise details).

From a technical point of view, this function is akin to the similarity measures for $\mathcal{EL}$ concepts introduced in [LT12, Sun13], though only [Sun13] directly draws its inspirations from the homomorphism characterization of subsumption in $\mathcal{EL}$. The threshold logic $\tau\mathcal{EL}(deg)$ induced by *deg* constitutes the main subject of study in Chapters 5 and 6, where we investigate the complexity of reasoning in $\tau\mathcal{EL}(deg)$ with respect to the empty terminology and to a particular form of acyclic TBoxes.

The last part of the thesis is devoted to better understand the relationship between concept similarity measures and our threshold logic formalism. We will describe a particular form of constructing membership degree functions from concept similarity measures, which then originates a wide family of threshold Description Logics. In this way, we obtain a variety of logics that could be useful in diverse scenarios according to the specific properties of their underlying similarity measures.

The remainder of this introduction is concerned with an overview on related work, and a more detailed summary of the subsequent chapters in this document.

1.1 Related work

We now provide an overview of some of the existing approaches to represent *imprecise* knowledge in Description Logics. We consider the ones that we believe look closest to our work. Nevertheless, there exists a vast number of other proposals. See for example [PZ13] for an extension of $\mathcal{EL}$ with the notion of rough sets, [LS10] for a family of probabilistic DLs, and [LS08] for a survey on managing *uncertainty* and *vagueness* in Description Logics.

1.1.1 Fuzzy DLs

The use of membership degree functions with values in the interval $[0, 1]$ may remind the reader of fuzzy logics. However, there is no strong relationship between this work and the work on fuzzy DLs [BDP15] for two reasons. First, in fuzzy DLs the semantics is extended to fuzzy interpretations where concept and role names are interpreted as fuzzy sets and relations, respectively. Basically, given an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ and a set of truth values $\mathfrak{B}$:

- concept names A are interpreted as fuzzy sets $A^{\mathcal{I}} : \Delta^{\mathcal{I}} \to \mathfrak{B}$, and
- role names r as binary fuzzy relations $r^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to \mathfrak{B}$.

The membership degree of an individual to belong to a complex concept is then computed using fuzzy interpretations of the concept constructors (e.g., conjunction is interpreted using an appropriate triangular norm $\otimes$).

In our setting, we consider crisp interpretations of concept and role names, and directly define membership degrees for complex concepts based on them. Second, we use membership degrees to obtain new concept constructors, but the threshold concepts obtained by applying these constructors are again crisp rather than fuzzy. Additionally, for our threshold logics the membership degree value in a complex concept need not be *systematically* determined by the membership degree values of its parts. Let us illustrate this situation with a simple example. Consider the following two fragments of the concept (1.1):

$$\mathsf{Human} \sqcap \mathsf{Male} \quad \text{and} \quad \mathsf{Human} \sqcap \mathsf{Handsome}$$

For an individual x in an interpretation $\mathcal{I}$ that only belongs to the concept Human, the intuition explained above for *deg* yields the membership degrees:

$$deg(x, \mathsf{Human} \sqcap \mathsf{Male}) = 1/2 \quad \text{and} \quad deg(x, \mathsf{Human} \sqcap \mathsf{Handsome}) = 1/2$$

Now, the relevant aspect is that, when computing $deg(x, \mathsf{Human} \sqcap \mathsf{Male} \sqcap \mathsf{Human} \sqcap \mathsf{Handsome})$, we do not want to count the fact that x is a Human twice, but rather give $1/3$ as the membership degree of x in the composed concept. This intuition will be captured in one of two conditions that membership functions are required to satisfy. Hence, if a specific t-norm $\otimes$ were to be used to interpret conjunction in this particular scenario, it would satisfy $1/2 \otimes 1/2 = 1/3$. However, let us further consider two more concepts:

$$\mathsf{Human} \sqcap \mathsf{Male} \quad \text{and} \quad \mathsf{Handsome} \sqcap \mathsf{Healthy}$$

If in addition, x also belongs to $\mathsf{Healthy}$ we obtain similar values as before:

$$deg(x, \mathsf{Human} \sqcap \mathsf{Male}) = 1/2 \quad \text{and} \quad deg(x, \mathsf{Handsome} \sqcap \mathsf{Healthy}) = 1/2$$

The difference is that the membership degree of x in the concept representing the composition of these two concepts, as already explained above, is $deg(x, \mathsf{Human} \sqcap \mathsf{Male} \sqcap \mathsf{Handsome} \sqcap \mathsf{Healthy}) = 1/2$. Obviously, this is not consistent with the initial definition of $\otimes$ for the pair $(1/2, 1/2)$, as required before.

1.1.2 The logic *sim*-$\mathcal{ALCQO}$

In [LWZ03], the authors introduced the Description Logic *sim*-$\mathcal{ALCQO}$ for expressing "vague" concepts and reasoning about them. This logic is obtained as the result of combining the DL $\mathcal{ALCQO}$ [HS01], and the logic $\mathcal{MS}$ introduced in [WZ03] for reasoning about metric spaces. In particular, the integration of $\mathcal{MS}$ with $\mathcal{ALCQO}$ allows to express concepts of the form $E^{\leq a}C$ (among others). The interpretation of such a concept description collects the set of all individuals x that are similar to at least one instance y of C with degree at most a. Here, x and y are elements of a domain W and the similarity is measured by a *distance function* d, where $<W, d>$ is a *metric space*.

In principle we could try to express a threshold concept $C_{\geq t}$ as the *sim-$\mathcal{ALCQO}$* concept $E^{\leq(1-t)}C$. This is based on the idea that by using a distance function, the closer two individuals are the more similar they are. Therefore, if y is an instance of C and $d(x, y) \leq 1-t$, we could interpret this as that x is an element of $C_{\geq t}$. However, the most important difference to our approach is that [LWZ03] do not fix an specific distance function d, but reason with respect to all possible such functions. This, for example, includes distance functions which do not take into account the conceptual structure of the domain elements in an interpretation to measure the distance between them.

1.1.3 Concept similarity measures

In the last few years, the idea of measuring similarity between concepts described in DLs has received considerable attention. Many concept similarity measures have been proposed to approach problems from a different/new perspective in very dissimilar domains. See for example [BWH05] for an early survey on the topic, and [dSF08, dFE06, EPT15, Sun13] for recently proposed measures and their applications.

One particular application of concept similarity measures to DLs is suggested in [EPT14, EPT15]. Instead of requiring that an individual is an instance of a query concept, the authors only require that it is an instance of a concept that is "similar enough" to the query concept. A somehow related approach has been presented in [TS14], but following the ideas exposed in [Sun13]. As we will show in Chapter 7, such kind of relaxed instance queries can be expressed as instance queries with respect to threshold concepts of the form $C_{>t}$. However, the new family of DLs introduced in this thesis is considerably more expressive than just such threshold concepts since we also allow the use of comparison operators other than $>$ in threshold concepts, and the threshold concepts can be embedded in complex $\mathcal{EL}$ concepts.

1.2 Structure of the Thesis

In the following, we briefly describe the contents of each chapter of the thesis.

- Chapter 2 formally introduces the lightweight Description Logic $\mathcal{EL}$. We start by presenting the syntax and semantics of $\mathcal{EL}$, as well as defining some technical notions that will be important for the rest of the thesis. To conclude, we then recall the well-known characterization of element-hood in $\mathcal{EL}$ concepts via existence of homomorphisms between $\mathcal{EL}$ description graphs (which can express both $\mathcal{EL}$ concepts and interpretations in a graphical way).

- In Chapter 3, we introduce our new family of DLs $\tau\mathcal{EL}(m)$. We extend $\mathcal{EL}$ by new threshold concept constructors which are based on an arbitrary, but fixed graded membership function m (hence the name $\tau\mathcal{EL}(m)$). We will impose some minimal requirements on such membership functions, and show the consequences that these conditions have for our threshold logic. Afterwards, we define description graphs and the notion of τ-homomorphisms for $\tau\mathcal{EL}(m)$. Based on them we show that membership in $\tau\mathcal{EL}(m)$ concept descriptions can be characterized by the existence of τ-homomorphisms. Such a characterization is independent of the used graded membership function, and will be crucial for the study of the computational complexity of inference problems carried out in subsequent chapters. Finally, we provide algorithms that for finite interpretations, can be used to decide membership in $\tau\mathcal{EL}(m)$ concepts according to the given characterization.

- Chapter 4 introduces the graded membership function *deg*. We show that *deg* is well-defined and satisfies the properties required for membership functions in Chapter 3. In the last part of the chapter, we look at the relationship between its induced threshold logic $\tau\mathcal{EL}(deg)$ and the DL $\mathcal{ALC}$ [SS91]. On the one hand, we show that full negation is not expressible in $\tau\mathcal{EL}(deg)$, and thus there are $\mathcal{ALC}$ concept descriptions that cannot be expressed in $\tau\mathcal{EL}(deg)$. On the other hand, we prove that $\tau\mathcal{EL}(deg)$ is a fragment of $\mathcal{ALC}$.

- Chapter 5 investigates the computational properties of $\tau\mathcal{EL}(deg)$. We start by considering *satisfiability* and *subsumption* as the standard reasoning tasks concerning terminological reasoning. In contrast to $\mathcal{EL}$, the satisfiability problem is not trivial and it turns out to be NP-hard. A matching upper bound is obtained due to the existence of polynomial size models for all satisfiable concepts. Then, we demonstrate, that the ideas used to construct such small models can be extended to concepts of the form $\widehat{C} \sqcap \neg\widehat{D}$ where $\widehat{C}$ and $\widehat{D}$ are $\tau\mathcal{EL}(deg)$ concepts. Since $\tau\mathcal{EL}(deg)$ cannot express negation of $\tau\mathcal{EL}(deg)$ concepts, this comes in handy to prove that subsumption is a complete problem for the class coNP. Finally, we are able to extend these ideas further to deal with assertional knowledge, and show that ABox consistency is NP-complete whereas the instance problem is coNP-complete (w.r.t. *data complexity*).

- Chapter 6 is concerned with extending our logic $\tau\mathcal{EL}(deg)$ to consider concept descriptions defined in a background TBox. We first extend well-defined graded membership functions to compute membership degrees with respect to acyclic $\mathcal{EL}$ TBoxes. Subsequently, $\tau\mathcal{EL}(m)$ and $\tau\mathcal{EL}(deg)$ TBoxes are defined taking into account some necessary restrictions. We will see that the presence of TBoxes apparently increases the computational complexity of the *satisfiability* and *subsumption* problems, namely, they become Π_2^P- and Σ_2^P-hard, respectively. These hardness results hold already with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes. Regarding upper bounds, we design a non-deterministic polynomial space algorithm that solves both problems, thus providing membership in PSPACE for both of them. Moreover, these PSPACE upper bounds carry over to reasoning with respect to acyclic $\tau\mathcal{EL}(deg)$ knowledge bases.

- In Chapter 7, we study the relationship between our threshold DLs $\tau\mathcal{EL}(m)$ and concept similarity measures. The chapter is organized into three main parts. To start, we show that a variant of the relaxed instance query approach of [EPT14] can be used to turn a similarity measure $\bowtie$ into a well-defined graded membership function $m_{\bowtie}$, and consequently $\bowtie$ induces a threshold logic $\tau\mathcal{EL}(m_{\bowtie})$. In addition, we show that the relaxed instance queries of [EPT14] can be expressed as instance queries w.r.t. threshold concepts of the form $C_{>t}$. The second part of the chapter explores the computational complexity landscape of reasoning in such a big family of threshold logics. We obtain undecidability and decidability results, as well as more precise complexity results for logics induced by a particular class of measures satisfying certain properties. Last, we present the framework *simi* introduced in [LT12] for defining similarity measures, and identify a concrete subclass of its instances exhibiting those properties. Moreover, it turns out that, applied to a simple instance $\bowtie^1$ of *simi*, our construction actually yields our membership function *deg*.

- In Chapter 8, we summarize our results and point out several directions for future work.

- Appendix A contains missing proofs of some results needed along this document.

The results of this thesis consisting of the definition of the family of DLs $\tau\mathcal{EL}(m)$, the graded membership function *deg* and the computational properties of $\tau\mathcal{EL}(deg)$ studied in Chapter 5, have previously been published in [BBG15a] and [BBG15b].

Chapter 2

The Description Logic $\mathcal{EL}$

We start by introducing the Description Logic $\mathcal{EL}$. Starting with finite sets of concept names $\mathsf{N_C}$ and role names $\mathsf{N_R}$, the set $\mathcal{C}_{\mathcal{EL}}$ of $\mathcal{EL}$ concept descriptions is obtained by using the concept constructors *conjunction* $(C \sqcap D)$, *existential restriction* $(\exists r.C)$ and *top* $(\top)$, in the following way:

$$C ::= \top \mid A \mid C \sqcap C \mid \exists r.C$$

where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$ and $C \in \mathcal{C}_{\mathcal{EL}}$.

An *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consists of a non-empty domain $\Delta^{\mathcal{I}}$ and an interpretation function $\cdot^{\mathcal{I}}$ that assigns subsets of $\Delta^{\mathcal{I}}$ to each concept name and binary relations over $\Delta^{\mathcal{I}}$ to each role name. The interpretation function $\cdot^{\mathcal{I}}$ is inductively extended to concept descriptions in the usual way:

$$\begin{aligned} \top^{\mathcal{I}} &:= \Delta^{\mathcal{I}} \\ (C \sqcap D)^{\mathcal{I}} &:= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\ (\exists r.C)^{\mathcal{I}} &:= \{x \in \Delta^{\mathcal{I}} \mid \exists y.\big[(x,y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\big]\} \end{aligned}$$

Given $C, D \in \mathcal{C}_{\mathcal{EL}}$, we say that C is *subsumed by* D (denoted as $C \sqsubseteq D$) iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for every interpretation $\mathcal{I}$. These two concept descriptions are *equivalent* (denoted as $C \equiv D$) iff $C \sqsubseteq D$ and $D \sqsubseteq C$. Finally, C is *satisfiable* iff $C^{\mathcal{I}} \neq \emptyset$ for some interpretation $\mathcal{I}$.

Information about specific individuals can be expressed in an ABox. An ABox $\mathcal{A}$ is a finite set of *assertions* of the form $C(a)$ or $r(a,b)$, where C is an $\mathcal{EL}$ concept description, $r \in \mathsf{N_R}$, and a, b are individual names. For example, if HUGUITO, JULIA and SANTIAGO are individual names, one can state that HUGUITO is a human male, JULIA is his daughter and SANTIAGO his son, through the following ABox $\mathcal{A}$:

$$\begin{aligned} \mathcal{A} := \{&\mathsf{Human}(\mathsf{HUGUITO}), \mathsf{Male}(\mathsf{HUGUITO}), \mathsf{Male}(\mathsf{SANTIAGO}), \mathsf{Female}(\mathsf{JULIA}), \\ &\mathsf{child}(\mathsf{HUGUITO}, \mathsf{JULIA}), \mathsf{child}(\mathsf{HUGUITO}, \mathsf{SANTIAGO})\} \end{aligned} \tag{2.1}$$

Concerning the semantics, in addition to concept and role names, an interpretation $\mathcal{I}$ now assigns domain elements $a^{\mathcal{I}}$ to individual names a. An assertion $C(a)$ is satisfied by $\mathcal{I}$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$, and $r(a,b)$ is satisfied by $\mathcal{I}$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$. The interpretation $\mathcal{I}$ is a *model* of $\mathcal{A}$ iff $\mathcal{I}$ satisfies all assertion in $\mathcal{A}$. The ABox $\mathcal{A}$ is *consistent* iff it has a model, and the individual a is an *instance* of the concept C in $\mathcal{A}$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$ holds in all models of $\mathcal{A}$. We denote the set of individual names occurring in $\mathcal{A}$ as $\mathsf{Ind}(\mathcal{A})$.

We now define some notions related to $\mathcal{EL}$ concept descriptions that will be useful for subsequent chapters.

Definition 2.1 (sub-description). Let C be an $\mathcal{EL}$ concept description. The set $\mathsf{sub}(C)$ of sub-descriptions of C is defined in the following way:

$$\mathsf{sub}(C) := \begin{cases} \{C\} & \text{if } C = \top \text{ or } C \in \mathsf{N_C}, \\ \{C\} \cup \mathsf{sub}(C_1) \cup \mathsf{sub}(C_2) & \text{if } C \text{ is of the form } C_1 \sqcap C_2, \\ \{C\} \cup \mathsf{sub}(D) & \text{if } C \text{ is of the form } \exists r.D. \end{cases}$$

Note that the number of sub-descriptions $|\mathsf{sub}(C)|$ of a concept C is linear in the size of C. Next, we define the *role depth* of a concept description C.

Definition 2.2 (role depth). The *role depth* $\mathsf{rd}(C)$ of an $\mathcal{EL}$ concept description C is inductively defined as follows:

$$\begin{aligned} \mathsf{rd}(\top) = \mathsf{rd}(A) &:= 0, \\ \mathsf{rd}(C_1 \sqcap C_2) &:= \max(\mathsf{rd}(C_1), \mathsf{rd}(C_2)), \\ \mathsf{rd}(\exists r.C) &:= \mathsf{rd}(C) + 1. \end{aligned}$$ ♢

A concept description is called an *atom* iff it is a concept name or an existential restriction. The set of all $\mathcal{EL}$ atoms is denoted by N_A. Additionally, every $\mathcal{EL}$ concept description is a conjunction $C_1 \sqcap \ldots \sqcap C_n$ of atoms. These conjuncts are called the *top-level* atoms of C and the set $\{C_1, \ldots, C_n\}$ is denoted as $tl(C)$.

Finally, given two interpretations $\mathcal{I}$ and $\mathcal{J}$, we say that $\mathcal{I}$ *is contained* in $\mathcal{J}$ (denoted $\mathcal{I} \subseteq \mathcal{J}$) iff $\Delta^{\mathcal{I}} \subseteq \Delta^{\mathcal{J}}$ and $X^{\mathcal{I}} \subseteq X^{\mathcal{J}}$ for all $X \in (\mathsf{N_C} \cup \mathsf{N_R})$.

2.1 Characterization of membership in $\mathcal{EL}$

Our definition of graded membership will be based on graphical representations of concepts and interpretations, and on homomorphisms between such representations. For this reason, we recall these notions together with the pertinent results. They are all taken from [BKM99, Küs01, Baa03].

Definition 2.3 ($\mathcal{EL}$ description graph). An $\mathcal{EL}$ description graph is a graph of the form $G = (V_G, E_G, \ell_G)$ where:

- V_G is a set of nodes.
- $E_G \subseteq V_G \times \mathsf{N_R} \times V_G$ is a set of edges labeled by role names,
- $\ell_G : V_G \to 2^{\mathsf{N_C}}$ is a function that labels nodes with sets of concept names. ♢

The empty label corresponds to the top concept $\top$. In particular, an $\mathcal{EL}$ description tree T is a description graph that is a tree with a distinguished element v_0 representing its root. In [BKM99], it was shown the correspondence that exists between $\mathcal{EL}$ concept descriptions and $\mathcal{EL}$ description trees, i.e., every $\mathcal{EL}$ concept description C can be translated into a corresponding description tree T_C and vice versa. Furthermore, every interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ can be translated into an $\mathcal{EL}$ description graph $G_{\mathcal{I}} = (V_{\mathcal{I}}, E_{\mathcal{I}}, \ell_{\mathcal{I}})$ in the following way [Baa03]:

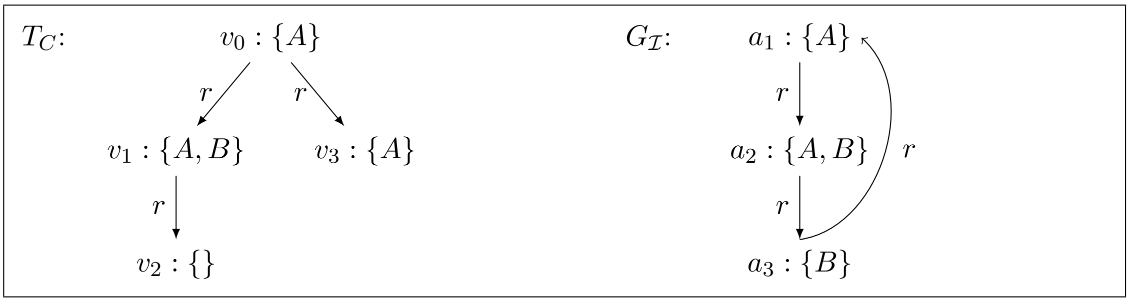


Figure 2.1: $\mathcal{EL}$ description graphs.

- $V_\mathcal{I} = \Delta^\mathcal{I}$,
- $E_\mathcal{I} = \{(vrw) \mid (v, w) \in r^\mathcal{I}\}$,
- $\ell_\mathcal{I}(v) = \{A \mid v \in A^\mathcal{I}\}$ for all $v \in V_\mathcal{I}$.

The following example illustrates the relation between concept descriptions and description trees, and interpretations and description graphs.

Example 2.4. The $\mathcal{EL}$ concept description

$$C := A \sqcap \exists r.(A \sqcap B \sqcap \exists r.\top) \sqcap \exists r.A$$

yields the $\mathcal{EL}$ description tree T_C depicted on the left-hand side of Figure 2.1. The description graph on the right-hand side corresponds to the following interpretation:

- $\Delta^\mathcal{I} := \{a_1, a_2, a_3\}$,
- $A^\mathcal{I} := \{a_1, a_2\}$ and $B^\mathcal{I} := \{a_2, a_3\}$,
- $r^\mathcal{I} := \{(a_1, a_2), (a_2, a_3), (a_3, a_1)\}$. ♢

Now, we generalize homomorphisms between $\mathcal{EL}$ description trees [BKM99] to arbitrary graphs.

Definition 2.5 (Homomorphisms on $\mathcal{EL}$ description graphs). Let $G = (V_G, E_G, \ell_G)$ and $H = (V_H, E_H, \ell_H)$ be two $\mathcal{EL}$ description graphs. A mapping $\varphi : V_G \to V_H$ is a *homomorphism* from G to H iff the following conditions are satisfied:

1. $\ell_G(v) \subseteq \ell_H(\varphi(v))$ for all $v \in V_G$, and
2. $vrw \in E_G$ implies $\varphi(v)r\varphi(w) \in E_H$.

This homomorphism is an *isomorphism* iff it is bijective, equality instead of just inclusion holds in 1), and biimplication instead of just implication holds in 2). ♢

In Example 2.4, the mapping φ with $\varphi(v_i) = a_{i+1}$ for $i = 0, 1, 2$ and $\varphi(v_3) = a_2$ is a homomorphism. Homomorphisms between $\mathcal{EL}$ description trees can be used to characterize subsumption in $\mathcal{EL}$.

Theorem 2.6 ([BKM99]). *Let C, D be $\mathcal{EL}$ concept descriptions and T_C, T_D the corresponding $\mathcal{EL}$ description trees. Then $C \sqsubseteq D$ iff there exists a homomorphism from T_D to T_C that maps the root of T_D to the root of T_C.*

The proof of this result can be easily adapted to obtain a similar characterization of *element-hood* in $\mathcal{EL}$, i.e., whether $d \in C^{\mathcal{I}}$ for some $d \in \Delta^{\mathcal{I}}$.

Theorem 2.7. *Let $\mathcal{I}$ be an interpretation, $d \in \Delta^{\mathcal{I}}$, and C an $\mathcal{EL}$ concept description. Then, $d \in C^{\mathcal{I}}$ iff there exists a homomorphism φ from T_C to $G_{\mathcal{I}}$ such that $\varphi(v_0) = d$.*

In Example 2.4, the existence of the homomorphism φ defined above thus shows that $a_1 \in C^{\mathcal{I}}$. Equivalence of $\mathcal{EL}$ concept descriptions can be characterized via the existence of isomorphisms, but for this the concept descriptions first need to be normalized by removing redundant existential restrictions. To be more precise, the *reduced form* of an $\mathcal{EL}$ concept description is obtained by applying the rewrite rule $\exists r.C \sqcap \exists r.D \longrightarrow \exists r.C$ if $C \sqsubseteq D$ as long as possible. This rule is applied modulo associativity and commutativity of $\sqcap$, and not only on the top-level conjunction of the description, but also under the scope of existential restrictions. Since every application of the rule decreases the size of the description, it is easy to see that the reduced form can be computed in polynomial time. We say that an $\mathcal{EL}$ concept description is *reduced* iff this rule does not apply to it. In our Example 2.4, the reduced form of C is the reduced description $A \sqcap \exists r.(A \sqcap B \sqcap \exists r.\top)$.

Theorem 2.8 ([Küs01]). *Let C, D be $\mathcal{EL}$ concept descriptions, C^r, D^r their reduced forms, and T_{C^r}, T_{D^r} the corresponding $\mathcal{EL}$ description trees. Then $C \equiv D$ iff there exists an isomorphism between T_{C^r} and T_{D^r}.*

Chapter 3

The Logic $\tau\mathcal{EL}(m)$

Our new logic will allow us to take an arbitrary $\mathcal{EL}$ concept C and turn it into a threshold concept. To this end we introduce a family of constructors that are based on the membership degree of individuals in C. For instance, the threshold concept $C_{>.8}$ represents the individuals that belong to C with degree $> .8$. The semantics of the new threshold concepts depends on a (graded) membership function m. Given an interpretation $\mathcal{I}$, this function takes a domain element $d \in \Delta^{\mathcal{I}}$ and an $\mathcal{EL}$ concept C as input, and returns a value between 0 and 1, representing the extent to which d belongs to C in $\mathcal{I}$.

The choice of the membership function obviously has a great influence on the semantics of the threshold concepts. In Chapter 4 we will propose one specific such function *deg*, but we do not claim this is the only reasonable way to define such a function. Rather, the membership function is a parameter in defining the logic. To highlight this dependency, we call the logic $\tau\mathcal{EL}(m)$.

Nevertheless, membership functions are not arbitrary. There are two properties we require such functions to satisfy:

Definition 3.1. A graded membership function m is a family of functions that contains for every interpretation $\mathcal{I}$ a function $m^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \mathcal{C}_{\mathcal{EL}} \to [0, 1]$ satisfying the following conditions (for $C, D \in \mathcal{C}_{\mathcal{EL}}$):

$$\begin{aligned} &M1 \ : d \in C^{\mathcal{I}} \Leftrightarrow m^{\mathcal{I}}(d, C) = 1 \ \text{ for all } d \in \Delta^{\mathcal{I}}, \\ &M2 \ : C \equiv D \Leftrightarrow \forall \mathcal{I}\, \forall d \in \Delta^{\mathcal{I}} : m^{\mathcal{I}}(d, C) = m^{\mathcal{I}}(d, D). \end{aligned} \qquad \Diamond$$

Property *M1* requires that the value 1 is a distinguished value reserved for proper containment in a concept. Property *M2* requires equivalence invariance. It expresses the intuition that the membership value should not depend on the syntactic form of a concept, but only on its semantics. Note that the right to left implication in *M2* is already a consequence of *M1*: suppose for a contradiction that $C \not\equiv D$. This would imply that for some interpretation $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$, $d \in C^{\mathcal{I}}$ and $d \notin D^{\mathcal{I}}$ (or the opposite). Then, by *M1* and the *right-hand* side of *M2* we would obtain $m^{\mathcal{I}}(d, C) = 1 = m^{\mathcal{I}}(d, D)$, which clearly yields a contradiction against $d \notin D^{\mathcal{I}}$ and property *M1*.

We now turn to the syntax of $\tau\mathcal{EL}(m)$. Given finite sets of concept names $\mathsf{N_C}$ and role names $\mathsf{N_R}$, $\tau\mathcal{EL}(m)$ *concept descriptions* are defined as follows:

$$\widehat{C} ::= \top \mid A \mid \widehat{C} \sqcap \widehat{C} \mid \exists r.\widehat{C} \mid E_{\sim t}$$

where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, $\sim \in \{<, \leq, >, \geq\}$, $t \in [0, 1] \cap \mathbb{Q}$, E is an $\mathcal{EL}$ concept description and $\widehat{C}$ is a $\tau\mathcal{EL}(m)$ concept description. Concepts of the form $E_{\sim t}$ are called *threshold concepts*. We denote by $\widehat{\mathsf{N}}_\mathsf{E}$ the set of all threshold concepts.

Using this newly introduced constructors, we can define ABoxes in $\tau\mathcal{EL}(m)$ as a natural extension of $\mathcal{EL}$ ABoxes. A $\tau\mathcal{EL}(m)$ ABox is an $\mathcal{EL}$ ABox that, in addition, is allowed to contain assertions of the form $\widehat{C}(a)$, where $\widehat{C}$ is a $\tau\mathcal{EL}(deg)$ concept description. Hence, if we know that another individual JACINTA is *healthy* and *handsome* with degree at least .8, we can now enrich the information provided in the ABox (2.1) by adding the assertion $\big((\mathsf{Healthy} \sqcap \mathsf{Handsome})_{\geq .8}\big)(\mathsf{JACINTA})$.

The semantics of the new threshold concepts is defined in the following way:

$$(E_{\sim t})^\mathcal{I} := \{d \in \Delta^\mathcal{I} \mid m^\mathcal{I}(d, E) \sim t\}$$

The extension of $\cdot^\mathcal{I}$ to more complex concepts is defined as in $\mathcal{EL}$ by additionally considering the underlying semantics of the newly introduced threshold concepts.

Requiring property *M1* has the following consequences for the semantics of threshold concepts.

Proposition 3.2. *For every $\mathcal{EL}$ concept description E we have*

$$E_{\geq 1} \equiv E \quad \textit{and} \quad E_{<1} \equiv \neg E,$$

where the semantics of negation is defined as usual, i.e., $[\neg E]^\mathcal{I} := \Delta^\mathcal{I} \setminus E^\mathcal{I}$.

The second equivalence basically says that $\tau\mathcal{EL}(m)$ can express negation of $\mathcal{EL}$ concept descriptions. This does not imply that $\tau\mathcal{EL}(m)$ is closed under negation since the threshold constructors can only be applied to $\mathcal{EL}$ concept descriptions. Thus, negation cannot be nested using these constructors. A formal proof that $\tau\mathcal{EL}(deg)$ for the membership function *deg* introduced in the next section cannot express full negation can be found in Section 4.3.1. However, atomic negation (i.e., negation applied to concept names) can obviously be expressed. Consequently, unlike $\mathcal{EL}$ concept descriptions, not all $\tau\mathcal{EL}(m)$ concept descriptions are satisfiable (i.e., can be interpreted by a non-empty set). A simple example is the concept description $A_{\geq 1} \sqcap A_{<1}$, which is equivalent to $A \sqcap \neg A$.

Last, some other notions defined for $\mathcal{EL}$ in Chapter 2 extend naturally to $\tau\mathcal{EL}(m)$:

- *role depth*: extends to $\tau\mathcal{EL}(m)$ concept descriptions by defining $\mathsf{rd}(E_{\sim t}) := 0$ for all threshold concept $E_{\sim t} \in \widehat{\mathsf{N}}_\mathsf{E}$,
- *sub-description:* for all $E_{\sim t} \in \widehat{\mathsf{N}}_\mathsf{E}$, $\mathsf{sub}(E_{\sim t}) := \{E_{\sim t}\}$.

3.1 Description graphs and homomorphisms in $\tau\mathcal{EL}(m)$

Our next goal is to extend the characterization of membership in $\mathcal{EL}$ (see Theorem 3.8) to $\tau\mathcal{EL}(m)$. In addition, we will show that given a $\tau\mathcal{EL}(m)$ ABox $\mathcal{A}$ and an interpretation $\mathcal{I}$, the satisfaction relation $\mathcal{I} \models \mathcal{A}$ can also be characterized by the existence of homomorphisms. Such characterizations will be useful later on to provide decision procedures for specific instances of $\tau\mathcal{EL}(m)$.

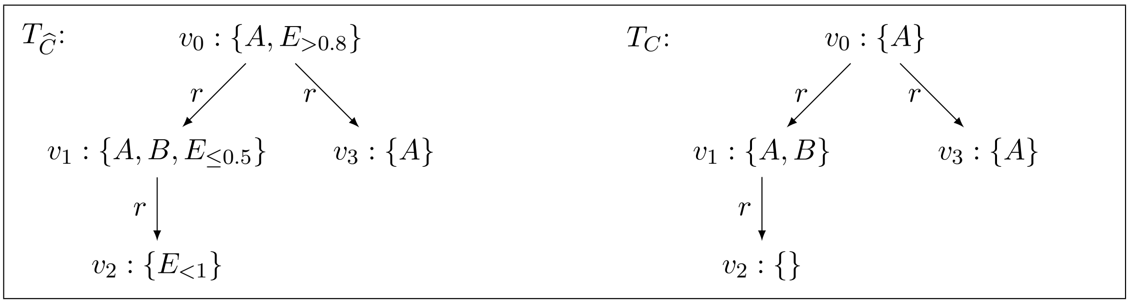


Figure 3.1: $\tau\mathcal{EL}(m)$ description trees.

We start by extending the notion of *description graphs* from $\mathcal{EL}$ to $\tau\mathcal{EL}(m)$. This is done by allowing the use of threshold concepts as labels.

Definition 3.3 ($\tau\mathcal{EL}(m)$ description graph). A $\tau\mathcal{EL}(m)$ description graph is a graph of the form $\widehat{G} = (V_G, E_G, \widehat{\ell}_G)$ where:

- V_G is a set of nodes,
- $E_G \subseteq V_G \times \mathsf{N_R} \times V_G$ is a set of edges labeled by role names, and
- $\widehat{\ell}_G : V_G \to 2^{\mathsf{N_C} \cup \widehat{\mathsf{N}}_\mathsf{E}}$ is a function that labels nodes with subsets of $\mathsf{N_C} \cup \widehat{\mathsf{N}}_\mathsf{E}$. ♢

Likewise for $\mathcal{EL}$ (see Definition 2.3), a $\tau\mathcal{EL}(m)$ description tree $\widehat{T}$ is a $\tau\mathcal{EL}(m)$ description graph that is a tree with a distinguished element v_0 representing its root. Therefore, we can establish a similar relationship between concept descriptions and description trees in $\tau\mathcal{EL}(m)$, i.e., every $\tau\mathcal{EL}(m)$ concept description $\widehat{C}$ can be translated into a $\tau\mathcal{EL}(m)$ description tree $T_{\widehat{C}}$ and vice versa. The following example illustrates such a relationship.

Example 3.4. Let E be an $\mathcal{EL}$ concept description. The $\tau\mathcal{EL}(m)$ concept description

$$\widehat{C} := A \sqcap E_{>0.8} \sqcap \exists r.(A \sqcap B \sqcap E_{\leq 0.5} \sqcap \exists r.E_{<1}) \sqcap \exists r.A$$

yields the $\tau\mathcal{EL}(m)$ description tree $T_{\widehat{C}}$ depicted on the left-hand side of Figure 3.1. Note that the translation of $\widehat{C}$ into $T_{\widehat{C}}$ is an extension of the one used for $\mathcal{EL}$ concept descriptions, where threshold concepts $E_{\sim t}$ are treated like concept names. The $\mathcal{EL}$ description tree T_C depicted in the right-hand side of Figure 3.1 corresponds to the $\mathcal{EL}$ description tree that results by ignoring the threshold concepts in the labels of $T_{\widehat{C}}$. ♢

Now, for ABoxes, the use of individual names and role assertions excludes the possibility of representing them as a description trees. Individuals in the ABox may have no relation at all or it could also happen that role assertions enforce the existence of a cycle involving some of them. In fact, the translation of concept descriptions into description trees in $\mathcal{EL}$ is adapted in [KM02] for an ABox $\mathcal{A}$ into a description graph $G(\mathcal{A})$.

We lift the very same translation (see Section 3 in [KM02]) to ABoxes and description graphs in $\tau\mathcal{EL}(m)$. Some of the notation used in [KM02] is slightly changed for the sake of readability within this document.

Definition 3.5 (ABoxes and $\tau\mathcal{EL}(m)$ description graphs). Let $\mathcal{A}$ be a $\tau\mathcal{EL}(m)$ ABox. $\mathcal{A}$ is translated into a $\tau\mathcal{EL}(m)$ description graph $\widehat{G}(\mathcal{A})$ in the following way:

$\widehat{C}_{a_1} := A \sqcap B$ $\quad \widehat{C}_{a_3} := \top$

$\widehat{C}_{a_2} := E_{<1}$ $\quad \widehat{C}_{a_4} := \exists r.A$

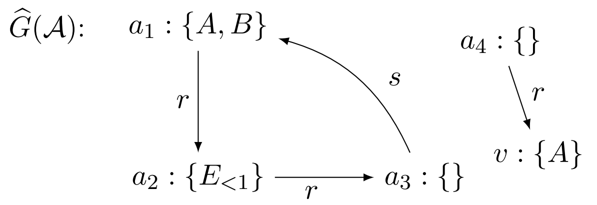


Figure 3.2: $\tau\mathcal{EL}(m)$ description graph associated to an ABox.

- For all $a \in \mathsf{Ind}(\mathcal{A})$, the $\tau\mathcal{EL}(m)$ concept description $\widehat{C}_a$ is defined as:

$$\widehat{C}_a := \bigsqcap_{\widehat{D}(a) \in \mathcal{A}} \widehat{D}$$

  If there exists no assertion of the form $\widehat{D}(a)$ in $\mathcal{A}$, then $\widehat{C}_a := \top$.

- For all $a \in \mathsf{Ind}(\mathcal{A})$, let $\widehat{T}(a) = (V_a, E_a, a, \widehat{\ell}_a)$ be the $\tau\mathcal{EL}(m)$ description tree corresponding to the concept $\widehat{C}_a$ where a itself represents its root. Without loss of generality let the sets V_a with $a \in \mathsf{Ind}(\mathcal{A})$ be pairwise disjoint. Then, $\widehat{G}(\mathcal{A}) = (V_{\mathcal{A}}, E_{\mathcal{A}}, \widehat{\ell}_{\mathcal{A}})$ is defined as:
  - $V_{\mathcal{A}} := \bigcup_{a \in \mathsf{Ind}(\mathcal{A})} V_a$,
  - $E_{\mathcal{A}} := \bigcup_{a \in \mathsf{Ind}(\mathcal{A})} E_a \cup \{arb \mid r(a, b) \in \mathcal{A}\}$, and
  - $\widehat{\ell}_{\mathcal{A}}(v) := \widehat{\ell}_a(v)$ for $v \in V_a$. ♢

The following example shows the idea of the previous construction.

Example 3.6. Let E be an $\mathcal{EL}$ concept description and $\mathcal{A}$ the following ABox:

$$\mathcal{A} := \{A(a_1), B(a_1), E_{<1}(a_2), (\exists r.A)(a_4), r(a_1, a_2), r(a_2, a_3), s(a_3, a_1)\}$$

The corresponding $\tau\mathcal{EL}(m)$ description graph $\widehat{G}(\mathcal{A})$ is depicted in Figure 3.2. ♢

Based on the notion of $\tau\mathcal{EL}(m)$ description graphs, we define homomorphisms from $\tau\mathcal{EL}(m)$ description graphs to the associated $\mathcal{EL}$ description graph of an interpretation $\mathcal{I}$. To differentiate these kinds of homomorphisms from the classical ones, we name them τ-homomorphisms and use the Greek letter ϕ (possibly with subscripts) to denote them.

Definition 3.7. Let $\widehat{H} = (V_H, E_H, \widehat{\ell}_H)$ be a $\tau\mathcal{EL}(m)$ description graph and $\mathcal{I}$ an interpretation. The mapping $\phi : V_H \to V_{\mathcal{I}}$ is a τ-homomorphism from $\widehat{H}$ to $G_{\mathcal{I}}$ iff:

1. ϕ is a homomorphism from $\widehat{H}$ to $G_{\mathcal{I}}$ in the sense of Definition 2.5 (ignoring threshold concepts in the labeling of V_H), and
2. for all $v \in V_H$: if $E_{\sim t} \in \widehat{\ell}_H(v)$, then $\phi(v) \in (E_{\sim t})^{\mathcal{I}}$. ♢

We denote by $\mathsf{dom}(\phi)$ and $\mathsf{img}(\phi)$ the domain and image of ϕ, respectively. This is, $\mathsf{dom}(\phi) := V_H$ and $\mathsf{img}(\phi) := \{\phi(v) \mid v \in V_H\}$.

We now use τ-homomorphisms to characterize membership in $\tau\mathcal{EL}(m)$. Such a characterization is based on the existence of a τ-homomorphism and generalizes Lemma 2.7 from $\mathcal{EL}$ to $\tau\mathcal{EL}(m)$.

Theorem 3.8. *Let $\widehat{C}$ be a $\tau\mathcal{EL}(m)$ concept description and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ an interpretation. The following statements are equivalent for all $d \in \Delta^{\mathcal{I}}$:*

1. *$d \in \widehat{C}^{\mathcal{I}}$.*

2. *there exists a τ-homomorphism ϕ from $T_{\widehat{C}}$ to $G_{\mathcal{I}}$ with $\phi(v_0) = d$.*

Proof. The 1) → 2) direction is shown by induction on the role depth of $\widehat{C}$, while the other direction is proved by induction on the number of nodes in $T_{\widehat{C}}$. The details of the proof are deferred to the Appendix A. □

Using the previous lemma we give a similar characterization for the satisfaction relation between interpretations and ABoxes in $\tau\mathcal{EL}(m)$.

Theorem 3.9. *Let $\mathcal{A}$ be a $\tau\mathcal{EL}(m)$ ABox and $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation. The following statements are equivalent:*

1. *$\mathcal{I}$ is a model of $\mathcal{A}$.*

2. *there exists a τ-homomorphism ϕ from $\widehat{G}(\mathcal{A})$ to $G_{\mathcal{I}}$ such that $\phi(a) = a^{\mathcal{I}}$ for all $a \in \mathsf{Ind}(\mathcal{A})$.*

Proof. 1) → 2). Assume that $\mathcal{I}$ is a model of $\mathcal{A}$. Then, $a^{\mathcal{I}} \in \widehat{D}^{\mathcal{I}}$ and $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ hold for all assertions $\widehat{D}(a) \in \mathcal{A}$ and $r(a, b) \in \mathcal{A}$, respectively. Consequently, by definition of $\widehat{C}_a$ we have that $a^{\mathcal{I}} \in (\widehat{C}_a)^{\mathcal{I}}$ for all $a \in \mathsf{Ind}(\mathcal{A})$. Hence, we can apply Theorem 3.8 to obtain a τ-homomorphism ϕ_a from $\widehat{T}(a)$ to $G_{\mathcal{I}}$ with $\phi_a(a) = a^{\mathcal{I}}$ (recall that a is the root of $\widehat{T}(a)$).

Finally, since $a^{\mathcal{I}} r b^{\mathcal{I}} \in E_{\mathcal{I}}$ for all $r(a, b) \in \mathcal{A}$, and the sets V_a used in the construction of $\widehat{G}(\mathcal{A})$ are pairwise disjoint, it is easy to verify that $\phi := \bigcup_{a \in \mathsf{Ind}(\mathcal{A})} \phi_a$ is a τ-homomorphism from $\widehat{G}(\mathcal{A})$ to $G_{\mathcal{I}}$ such that $\phi(a) = a^{\mathcal{I}}$ for all $a \in \mathsf{Ind}(\mathcal{A})$.

2) → 1). Assume that the statement 2) holds. We show that $\mathcal{I}$ satisfies all assertions in $\mathcal{A}$:

- $r(a, b) \in \mathcal{A}$. By construction of $\widehat{G}(\mathcal{A})$ we know that $arb \in E_{\mathcal{A}}$. Since ϕ is a homomorphism from $\widehat{G}(\mathcal{A})$ to $G_{\mathcal{I}}$, this means that $\phi(a) r \phi(b) \in E_{\mathcal{I}}$ as well. Consequently, it follows from $\phi(a) = a^{\mathcal{I}}$ and $\phi(b) = b^{\mathcal{I}}$ that $a^{\mathcal{I}} r b^{\mathcal{I}} \in E_{\mathcal{I}}$. Thus, $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$.

- $\widehat{D}(a) \in \mathcal{A}$. By construction of $\widehat{G}(\mathcal{A})$ one can see that the description tree $\widehat{T}(a)$ is a sub-graph of $\widehat{G}(\mathcal{A})$. Therefore, ϕ is also a τ-homomorphism from $\widehat{T}(a)$ to $G_{\mathcal{I}}$ with $\phi(a) = a^{\mathcal{I}}$. An application of Theorem 3.8 then yields: $a^{\mathcal{I}} \in (\widehat{C}_a)^{\mathcal{I}}$. Thus, since $\widehat{D}$ is one of the conjuncts in the definition of $\widehat{C}_a$, it follows that $a^{\mathcal{I}} \in \widehat{D}^{\mathcal{I}}$. □

3.2 Deciding the existence of a τ-homomorphism

If the interpretation $\mathcal{I}$ is finite and m is computable, then the existence of a τ-homomorphism can be decided. We present two algorithms that (under the previous conditions) can be used to decide the relations characterized by Theorems 3.8 and 3.9. Our starting point is the polynomial time algorithm (Algorithm 1 below) introduced in [BKM99] to decide the existence of a homomorphism between two $\mathcal{EL}$ description trees.

Algorithm 1 Homomorphisms between $\mathcal{EL}$ description trees.

Input: Two $\mathcal{EL}$ description trees T_1 and T_2.
Output: "yes", if there exists a homomorphism from T_1 to T_2; "no", otherwise.

```
1:  Let T_1 = (V_1, E_1, v_0, ℓ_1) and T_2 = (V_2, E_2, w_0, ℓ_2). Further, let {v_1, ..., v_n} be a
    post-order sequence of V_1, i.e., v_1 is a leaf and v_n = v_0.
2:  Define a labeling δ : V_2 → 2^{V_1} as follows.
3:  Initialize δ by δ(w) := ∅ for all w ∈ V_2.
4:  for all 1 ≤ i ≤ n do
5:      for all w ∈ V_2 do
6:          if (ℓ_1(v_i) ⊆ ℓ_2(w) and for all v_i r v ∈ E_1 there is w' ∈ V_2 such that
7:              v ∈ δ(w') and w r w' ∈ E_2) then
8:              δ(w) := δ(w) ∪ {v_i}
9:          end if
10:     end for
11: end for
12: If v_0 ∈ δ(w_0) then return "yes", else return "no".
```

Theorem 3.8 characterizes *elementhood* in $\tau\mathcal{EL}(m)$ concept descriptions via the existence of a τ-homomorphism from a $\tau\mathcal{EL}(m)$ description tree $T_{\widehat{C}}$ to an $\mathcal{EL}$ description graph $G_{\mathcal{I}}$ associated to an interpretation $\mathcal{I}$. If $\mathcal{I}$ is finite, then Algorithm 1 can be used to decide whether there exists a mapping satisfying Condition 1 in Definition 3.7. One needs only to replace the last line by $v_0 \in \delta(d)$ for some $d \in \Delta^{\mathcal{I}}$, since now T_2 becomes $G_{\mathcal{I}}$. In order to verify the second condition in Definition 3.7, we modify the test in line 6 to also consider whether $m^{\mathcal{I}}(d, E) \sim t$ for all $E_{\sim t} \in \widehat{\ell}_{T_{\widehat{C}}}(v_i)$. Algorithm 2 implements this modification.

Then, if one wants to know whether a precise element $e \in \Delta^{\mathcal{I}}$ belongs to $(\widehat{C})^{\mathcal{I}}$, Algorithm 2 shall be invoked on $T_{\widehat{C}}$ and $\mathcal{I}$. Note that a simple modification in line 12, namely testing whether $v_0 \in \delta(e)$, adapts the algorithm to answer the question for e.

Now, the main difference between Algorithms 1 and 2 is that the latter might need to compute $m^{\mathcal{I}}$ to verify whether $m^{\mathcal{I}}(d, E) \sim q$. Therefore, its computational complexity may depend on how difficult is to compute $m^{\mathcal{I}}$ for a chosen m. In particular, if $m^{\mathcal{I}}$ can be computed in polynomial time as for the graded membership function *deg* introduced in the next section, Algorithm 2 will run in polynomial time.

Regarding the characterization given in Theorem 3.9 for the satisfaction relation between interpretations and ABoxes, note that the description graph $\widehat{G}(\mathcal{A})$ associated to an ABox $\mathcal{A}$ is not necessarily a tree. Therefore, finding a τ-homomorphism ϕ from $\widehat{G}(\mathcal{A})$ to $G_{\mathcal{I}}$ includes finding a homomorphism between two graphs, which in general

Algorithm 2 τ-homomorphism from a $\tau\mathcal{EL}(m)$ description tree to $G_{\mathcal{I}}$.

Input: A $\tau\mathcal{EL}(m)$ description tree $\widehat{T}$ and a finite interpretation $\mathcal{I}$.
Output: "yes", if there exists a τ-homomorphism from $\widehat{T}$ to $G_{\mathcal{I}}$; "no", otherwise.

Let $\widehat{T} = (V_T, E_T, v_0, \widehat{\ell}_T)$ and $G_{\mathcal{I}} = (V_{\mathcal{I}}, E_{\mathcal{I}}, \ell_{\mathcal{I}})$. Further, let $\{v_1, \ldots, v_n\}$ be a post-order sequence of V_T, i.e., v_1 is a leaf and $v_n = v_0$.
Define a labeling $\delta : V_{\mathcal{I}} \to 2^{V_T}$ as follows.
Initialize δ by $\delta(w) := \emptyset$ for all $w \in V_{\mathcal{I}}$.
for all $1 \leq i \leq n$ **do**
    for all $d \in \Delta^{\mathcal{I}}$ **do**
        if $(\ell_T(v_i) \subseteq \ell_{\mathcal{I}}(d)$ **and** $[E_{\sim t} \in \widehat{\ell}_T(v_i) \Rightarrow m^{\mathcal{I}}(d, E) \sim t]$ **and**
            $[v_i r v \in E_T \Rightarrow \exists d' \in \Delta^{\mathcal{I}} : v \in \delta(d')]$ **and** $d r d' \in E_{\mathcal{I}})$ **then**
            $\delta(d) := \delta(d) \cup \{v_i\}$
        end if
    end for
end for
If there exists $d \in \Delta^{\mathcal{I}}$ such that $v_0 \in \delta(d)$ then return "yes", else return "no".

is an NP-complete problem [GJ79]. However, by Definition 3.5 it can be seen that $\widehat{G}(\mathcal{A})$ has a particular form where cycles only involve nodes and edges corresponding to the individual elements and role assertions, respectively, occurring in $\mathcal{A}$. Moreover, since Theorem 3.9 requires $\phi(a) = a^{\mathcal{I}}$ for all $a \in \mathsf{Ind}(\mathcal{A})$, this means that the wanted τ-homomorphism is partially fixed with respect to those elements. Hence, it suffices to check whether the interpretation of the individual names satisfies the role assertions in $\mathcal{A}$ and $a^{\mathcal{I}} \in (\widehat{C}_a)^{\mathcal{I}}$ (see Definition 3.5), for all $a \in \mathsf{Ind}(\mathcal{A})$. The following algorithm uses Algorithm 2 to decide whether a finite interpretation $\mathcal{I}$ satisfies an ABox $\mathcal{A}$.

Algorithm 3 τ-homomorphisms for ABoxes and interpretations.

Input: An ABox $\mathcal{A}$ and a finite interpretation $\mathcal{I}$.
Output: "yes", if there exists a τ-homomorphism ϕ from $\widehat{G}(\mathcal{A})$ to $G_{\mathcal{I}}$ with $\phi(a) = a^{\mathcal{I}}$ for all $a \in \mathsf{Ind}(\mathcal{A})$; "no", otherwise.

Let $\widehat{G}(\mathcal{A})$ be as in Definition 3.5.
for all $r(a, b) \in \mathcal{A}$ **do**
    if $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin r^{\mathcal{I}}$ **then**
        return "no"
    end if
end for
for all $a \in \mathsf{Ind}(\mathcal{A})$ **do**
    if $a^{\mathcal{I}} \notin (\widehat{C}_a)^{\mathcal{I}}$ **then**   // this can be checked using Algorithm 2
        return "no"
    end if
end for
return "yes"

Chapter 4

The membership function *deg*

To make things more concrete, we now introduce a specific membership function, denoted *deg*. Given an interpretation $\mathcal{I}$, an element $d \in \Delta^{\mathcal{I}}$, and an $\mathcal{EL}$ concept description C, this function is supposed to measure to which degree d satisfies the conditions for membership expressed by C. To come up with such a measure, we use the homomorphism characterization of membership in $\mathcal{EL}$ concepts as starting point (see Theorem 2.7). Basically, we consider all partial mappings from T_C to $G_{\mathcal{I}}$ that map the root of T_C to d and respect the edge structure of T_C. For each of these mappings we then calculate to which degree it satisfies the homomorphism conditions, and take the degree of the best such mapping as the membership degree $deg^{\mathcal{I}}(d, C)$.

Example 4.1. Figure 4.1 shows the $\mathcal{EL}$ description tree corresponding to the $\mathcal{EL}$ concept description $C := A \sqcap B \sqcap \exists s.(B_1 \sqcap \exists r.B_3 \sqcap \exists r.B_2)$ and a fragment of an interpretation graph $G_{\mathcal{I}}$. In addition, it depicts two mappings from V_{T_C} to $V_{\mathcal{I}}$. The one represented by the *dashed* lines and a variation represented with the *dotted* lines. One can see that none of them is a homomorphism from T_C to $G_{\mathcal{I}}$ in the sense of Definition 2.5. In fact, since obviously $d \notin C^{\mathcal{I}}$, by Theorem 2.7 there exists no such homomorphism.

To compute the membership value induced by an specific mapping, we count the number of properties of v_0 (say m), see how many of those does d in $\mathcal{I}$ actually have (say n) and give $\frac{n}{m}$ as the membership degree value. In our example v_0 has three properties, e.g., A, B and the existence of an s-successor (represented by v_1) with certain properties. Interesting to see is that for both mappings, the selected s-successor of d does not satisfy all the properties of v_1. Should we just assume that d does not have this last property and give $\frac{1}{3}$ as the membership degree value? Instead of that, we would like to compute a value that expresses to which degree the s-successor of d (to which v_1 is mapped to), satisfies the conditions for membership expressed by the subtree of T_C rooted at v_1. This will be done using the very same idea recursively. ♢

As mentioned before, we consider partial mappings rather than total ones since one of the violations of properties demanded by C could be that a required role successor does not exist at all.

Example 4.2. Consider the description tree T_C and the interpretation $\mathcal{I}$ depicted in Figure 4.2. Obviously, there exists no total mapping from T_C to $G_{\mathcal{I}}$ since neither d_1 nor d_2 have a successor. Thus, restricting to consider only total mappings would give *zero* as the membership degree value of d in C. This is not desired, since just like concept names may be missing and the membership value does not become zero, also role successors (required by C) may be missing and the membership degree need not be zero. ♢

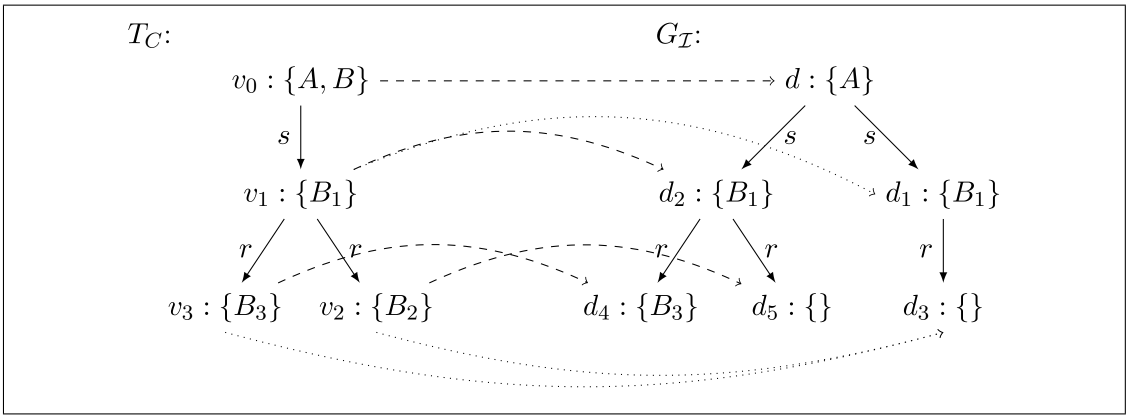


Figure 4.1: Mappings from T_C to $G_{\mathcal{I}}$.

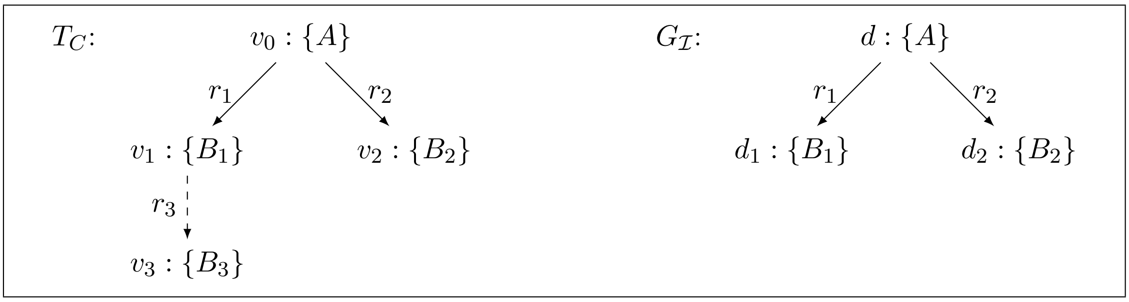


Figure 4.2: An example where no total mapping exists from T_C to $G_{\mathcal{I}}$.

4.1 The membership function *deg*

To formalize the previously exposed ideas, we first define the notion of *partial tree-to-graph homomorphisms* from description trees to description graphs. In this definition, the node labels are ignored (they will be considered in the next step).

Definition 4.3 (Partial tree-to-graph homomorphisms). Let $T = (V_t, E_t, \ell_t, v_0)$ and $G = (V_g, E_g, \ell_g)$ be a description tree (with root v_0) and a description graph, respectively. A partial mapping $h : V_t \to V_g$ is a *partial tree-to-graph homomorphism* (*ptgh*) from T to G iff the following conditions are satisfied:

1. $\mathsf{dom}(h)$ is a subtree of T with root v_0, i.e., $v_0 \in \mathsf{dom}(h)$ and if $(v, r, w) \in E_t$ and $w \in \mathsf{dom}(h)$, then $v \in \mathsf{dom}(h)$;

2. for all edges $(v, r, w) \in E_t$, $w \in \mathsf{dom}(h)$ implies $(h(v), r, h(w)) \in E_g$.

To abbreviate, from now on we will write *ptgh*(*ptghs* for the plural) instead of *partial tree-to-graph homomorphism.* ♢

In order to measure how far away from a homomorphism (in the sense of Definition 2.5) such a *ptgh* is, we define the notion of a *weighted* homomorphism between a finite $\mathcal{EL}$ description tree and an $\mathcal{EL}$ description graph.

Definition 4.4. Let T be a finite $\mathcal{EL}$ description tree, G an $\mathcal{EL}$ description graph and $h : V_T \to V_G$ a *ptgh* from T to G. We define the *weighted* homomorphism induced by h from T to G as a recursive function $h_w : \mathsf{dom}(h) \to [0..1]$ in the following way:

$$h_w(v) := \begin{cases} 1 & \text{if } |\ell_T(v)| + k^*(v) = 0 \\ \dfrac{|\ell_T(v) \cap \ell_G(h(v))| + \sum\limits_{1 \le i \le k} h_w(v_i)}{|\ell_T(v)| + k^*(v)} & \text{otherwise.} \end{cases}$$

The elements used to define h_w have the following meaning. For a given $v \in \mathsf{dom}(h)$, $k^*(v)$ denotes the number of successors of v in T, and $v_1, \ldots, v_k$ with $0 \le k \le k^*(v)$ are the children of v in T such that $v_i \in \mathsf{dom}(h)$. ♢

It is easy to see that h_w is well-defined. In fact, T is a finite tree, which ensures that the recursive definition of h_w is well-founded. In addition, the base case of the definition guarantees that division by zero is avoided. Using value 1 in this case is justified since then no property is required. In the second case, missing concept names and missing successors decrease the weight of a node since then the required name or successor contributes to the denominator, but not to the numerator. Required successors that are there are only counted if they are successors for the correct role, and then they do not contribute with value 1 to the numerator, but only with their weight (i.e., the degree to which they match the requirements for this successor).

When defining the value of the membership function $deg^{\mathcal{I}}(d, C)$, we do not use the concept C directly, but rather its reduced from C^r. This will ensure that *deg* satisfies property *M2*.

Definition 4.5. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation, d an element of $\Delta^{\mathcal{I}}$ and C an $\mathcal{EL}$ concept description with reduced form C^r. In addition, let $\mathcal{H}(T_{C^r}, G_{\mathcal{I}}, d)$ be the set of all *ptghs* from T_{C^r} to $G_{\mathcal{I}}$ with $h(v_0) = d$. The set $\mathcal{V}^{\mathcal{I}}(d, C^r)$ of all relevant values is defined as:

$$\mathcal{V}^{\mathcal{I}}(d, C^r) := \{q \mid h_w(v_0) = q \text{ and } h \in \mathcal{H}(T_{C^r}, G_{\mathcal{I}}, d\}$$

Then we define $deg^{\mathcal{I}}(d, C) := \max \mathcal{V}^{\mathcal{I}}(d, C^r)$. ♢

In case the interpretation $\mathcal{I}$ is infinite, there may exist infinitely many *ptghs* from T_{C^r} to $G_{\mathcal{I}}$ with $h(v_0) = d$. Therefore, it is not immediately clear whether the maximum in the above definition actually exists, and thus whether $deg^{\mathcal{I}}(d, C)$ is well-defined. To prove that the maximum exists also for infinite interpretations, we show that the set $\mathcal{V}^{\mathcal{I}}(d, C^r)$ is actually a finite set. To this end, we introduce canonical interpretations induced by *ptghs*.

Definition 4.6 (Canonical interpretation). Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation, C an $\mathcal{EL}$ concept description and h be a *ptgh* from T_{C^r} to $G_{\mathcal{I}}$. The *canonical interpretation* $\mathcal{I}_h$ induced by h is the one having the description tree $T_{\mathcal{I}_h} = (V_{\mathcal{I}_h}, E_{\mathcal{I}_h}, v_0, \ell_{\mathcal{I}_h})$ with

$$\begin{aligned} V_{\mathcal{I}_h} &:= \mathsf{dom}(h), \\ E_{\mathcal{I}_h} &:= \{vrw \in E_{T_{C^r}} \mid v, w \in \mathsf{dom}(h)\} \\ \ell_{\mathcal{I}_h}(v) &:= \ell_{T_{C^r}}(v) \cap \ell_{\mathcal{I}}(h(v)) \text{ for all } v \in \mathsf{dom}(h). \end{aligned}$$

Remark 4.7. One can see that $T_{\mathcal{I}_h}$ satisfies $V_{\mathcal{I}_h} \subseteq V_{T_{C^r}}$, $E_{\mathcal{I}_h} \subseteq E_{T_{C^r}}$, $\ell_{\mathcal{I}_h}(v) \subseteq \ell_{T_{C^r}}(v)$ and $\ell_{\mathcal{I}_h}(v) \subseteq \ell_{\mathcal{I}}(h(v))$ for all $v \in \mathsf{dom}(h)$. Moreover, the construction of $\mathcal{I}_h$ verifies that the mapping h is a homomorphism from $T_{\mathcal{I}_h}$ to $G_{\mathcal{I}}$. ♢

Lemma 4.8. *Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation, $d \in \Delta^{\mathcal{I}}$ and C an $\mathcal{EL}$ concept description. The set $\mathcal{V}^{\mathcal{I}}(d, C^r)$ contains finitely many elements.*

Proof. Let $\mathcal{I}_{\mathcal{H}}$ be the set of all canonical interpretations induced by all $h \in \mathcal{H}(T_{C^r}, G_{\mathcal{I}}, d)$, i.e.,

$$\mathcal{I}_{\mathcal{H}} := \{\mathcal{I}_h \mid h \in \mathcal{H}(T_{C^r}, G_{\mathcal{I}}, d)\}$$

From Remark 4.7, we have that $V_{\mathcal{I}_h} \subseteq V_{T_{C^r}}$, $E_{\mathcal{I}_h} \subseteq E_{T_{C^r}}$ and $\ell_{\mathcal{I}_h}(v) \subseteq \ell_{T_{C^r}}(v)$ for all $v \in \mathsf{dom}(h)$. This implies that the description graph $T_{\mathcal{I}_h}$ induced by $\mathcal{I}_h$ is a subtree of T_{C^r}. Hence, the set $\mathcal{I}_{\mathcal{H}}$ must be finite, i.e., there are only finitely many different canonical interpretations induced by *ptghs* $h \in \mathcal{H}(T_{C^r}, G_{\mathcal{I}}, d)$.

Now, consider any $h \in \mathcal{H}(T_{C^r}, G_{\mathcal{I}}, d)$ and let $i^{\mathcal{I}_h} : \mathsf{dom}(h) \to V_{\mathcal{I}_h}$ be a mapping such that $i^{\mathcal{I}_h}(v) = v$ for all $v \in \mathsf{dom}(h)$. Note that $i^{\mathcal{I}_h}$ is well-defined by definition of $\mathcal{I}_h$, and it is easy to see that it is a *ptgh* from T_{C^r} to $T_{\mathcal{I}_h}$. Furthermore, let $\mathcal{V}^{\mathcal{I}_{\mathcal{H}}}$ be the set:

$$\mathcal{V}^{\mathcal{I}_{\mathcal{H}}} := \{q \mid i_w^{\mathcal{I}_h}(v_0) = q \text{ for all } h \in \mathcal{H}(T_{C^r}, G_{\mathcal{I}}, d)\}$$

Since $\mathsf{dom}(h) \subseteq V_{T_{C^r}}$, there are finitely many sets that could act as the source for a mapping $i^{\mathcal{I}_h}$. Moreover, $\mathcal{I}_{\mathcal{H}}$ is a finite set of finite interpretations. Hence, there can only be finitely many different mappings $i^{\mathcal{I}_h}$. Consequently, the set $\mathcal{V}^{\mathcal{I}_{\mathcal{H}}}$ must be finite. In addition, one can see that the following three properties hold:

- $\mathsf{dom}(i^{\mathcal{I}_h}) = \mathsf{dom}(h)$,
- $\ell_{\mathcal{I}_h}(i^{\mathcal{I}_h}(v)) = \ell_{T_{C^r}}(v) \cap \ell_{\mathcal{I}}(h(v))$ for all $v \in \mathsf{dom}(h)$, and
- for all $v, w \in \mathsf{dom}(h)$: if $vrw \in E_{T_{C^r}}$, then $h(v)rh(w) \in E_{\mathcal{I}}$ and $i^{\mathcal{I}_h}(v)ri^{\mathcal{I}_h}(w) \in E_{\mathcal{I}_h}$.

Therefore, from Definition 4.4 it follows that $h_w(v_0) = i_w^{\mathcal{I}_h}(v_0)$. This means that for all $h \in \mathcal{H}(T_{C^r}, G_{\mathcal{I}}, d)$ it is the case that $h_w(v_0) \in \mathcal{V}^{\mathcal{I}_{\mathcal{H}}}$. Hence, $\mathcal{V}^{\mathcal{I}}(d, C^r) \subseteq \mathcal{V}^{\mathcal{I}_{\mathcal{H}}}$ and $\mathcal{V}^{\mathcal{I}}(d, C^r)$ is a finite set.

Thus, $\max \mathcal{V}^{\mathcal{I}}(d, C^r)$ exists and $deg^{\mathcal{I}}(d, C)$ is well-defined. □

If the interpretation $\mathcal{I}$ is finite, $deg^{\mathcal{I}}(d, C)$ can be computed in polynomial time for all $d \in \Delta^{\mathcal{I}}$ and all $\mathcal{EL}$ concept descriptions C. The polynomial time algorithm described below (Algorithm 4) is inspired by the polynomial time algorithm for checking the existence of a homomorphism between $\mathcal{EL}$ description trees [BKM98, BKM99], and similar to the algorithm for computing the similarity degree between $\mathcal{EL}$ concept descriptions introduced in [Sun13].

Algorithm 4 considers each pair (v, e) with $v \in V_{T_{C^r}}$ and $e \in \Delta^{\mathcal{I}}$ only once. Therefore, it is easy to see that it runs in polynomial time in the size of C and $\mathcal{I}$. The following lemma shows that Algorithm 4 computes the value of $deg^{\mathcal{I}}$, i.e., $S(v_0, d) = deg^{\mathcal{I}}(d, C^r)$ (see Appendix A).

Algorithm 4 Computation of $deg^{\mathcal{I}}$.

Input: An $\mathcal{EL}$ concept description C, a finite interpretation $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$.
Output: $deg^{\mathcal{I}}(d, C)$.

Let C^r be the reduced form of C, $G_{\mathcal{I}} = (V_{\mathcal{I}}, E_{\mathcal{I}}, \ell_{\mathcal{I}})$ and $\{v_1, \ldots, v_n\}$ be a post-order sequence of $V_{T_{C^r}}$ where $v_n = v_0$.
The assignment $S : V_{T_{C^r}} \times V_{\mathcal{I}} \to [0..1]$ is computed as follows:
for all $1 \leq i \leq n$ **do**
  if $|\ell_{T_{C^r}}(v_i)| + k^*(v_i) = 0$ **then**
    $S(v_i, e) := 1$ for all $e \in \Delta^{\mathcal{I}}$
  else
    for all $e \in \Delta^{\mathcal{I}}$ **do**
      $c := |\ell_{T_{C^r}}(v_i) \cap \ell_{\mathcal{I}}(e)|$
      for all $v_i r v \in E_{T_{C^r}}$ **do**
        $c := c + \max\{S(v, e') \mid (e, e') \in r^{\mathcal{I}}\}$
      end for
      $S(v_i, e) := \frac{c}{|\ell_{T_{C^r}}(v_i)| + k^*(v_i)}$
    end for
  end if
end for
return $S(v_0, d)$

Lemma 4.9. *Let C be an $\mathcal{EL}$ concept description, $\mathcal{I}$ a finite interpretation and $d \in \Delta^{\mathcal{I}}$. Then, Algorithm 4 terminates on input $(C, \mathcal{I}, d)$ and outputs $deg^{\mathcal{I}}(d, C)$, i.e., $S(v_0, d) = deg^{\mathcal{I}}(d, C^r)$.*

Finally, it remains to show that *deg* satisfies the properties required for graded membership functions.

Proposition 4.10. *The function deg satisfies the properties M1 and M2.*

Proof. We first show that *M1* is satisfied by *deg*. Assume that $d \in C^{\mathcal{I}}$. Since C is equivalent to its reduced form, we also have $d \in (C^r)^{\mathcal{I}}$. The application of Theorem 2.7 yields that there exists a homomorphism φ from T_{C^r} to $G_{\mathcal{I}}$ with $\varphi(v_0) = d$. Then it is easy to verify from Definition 4.4 that $\varphi_w(v_0) = 1$ and consequently, $\max \mathcal{V}^{\mathcal{I}}(d, C^r) = 1$. Thus, $deg^{\mathcal{I}}(d, C) = 1$. Conversely, assume that $deg^{\mathcal{I}}(C, d) = 1$. This means that there exists a *ptgh* h from T_{C^r} to $G_{\mathcal{I}}$ with $h(v_0) = d$ and $h_w(v_0) = 1$. Similar as before, it is easy to see that h is a homomorphism according to Definition 2.5. The application of Theorem 2.7 yields $d \in (C^r)^{\mathcal{I}}$ and consequently, $d \in C^{\mathcal{I}}$.

Concerning *M2*, as mentioned in Chapter 3 the right to left implication is already a consequence of *M1*, which we just proved to be satisfied by *deg*. Assume that $C \equiv D$, then by Theorem 2.8 there exists an isomorphism ψ between T_{C^r} and T_{D^r}. Consider an arbitrary interpretation $\mathcal{I}$ and any element $d \in \Delta^{\mathcal{I}}$. We show that $deg^{\mathcal{I}}(d, C^r) = deg^{\mathcal{I}}(d, D^r)$, which obviously implies $deg^{\mathcal{I}}(d, C) = deg^{\mathcal{I}}(d, D)$ (see Definition 4.5).

Let h be a *ptgh* from T_{C^r} to $G_{\mathcal{I}}$ with $h(v_0) = d$ and $h_w(v_0) = \max \mathcal{V}^{\mathcal{I}}(d, C^r)$. Since ψ is an isomorphism, the composition $h \circ \psi$ is a *ptgh* from T_{D^r} to $G_{\mathcal{I}}$, with $(h \circ \psi)(v_0) = d$

and $(h \circ \psi)_w(v_0) = h_w(v_0)$. This means that $deg^{\mathcal{I}}(d, C^r) \leq deg^{\mathcal{I}}(d, D^r)$. The same reasoning can be applied starting with T_{D^r} to obtain $deg^{\mathcal{I}}(d, D^r) \leq deg^{\mathcal{I}}(d, C^r)$. Thus, we have shown that $deg^{\mathcal{I}}(d, C^r) = deg^{\mathcal{I}}(d, D^r)$. □

Note that *M2* follows from the fact that we use the reduced form of a concept description rather than the description itself. Otherwise, *M2* would not hold. For example, consider the concept description $C := \exists r.A \sqcap \exists r.(A \sqcap B)$, which is equivalent to its reduced form $C^r = \exists r.(A \sqcap B)$. Let d be an individual that has a single r-successor belonging to A, but not to B. Then using C instead of C^r would yield membership degree $\frac{3}{4}$, whereas the use of C^r yields the degree $\frac{1}{2}$.

4.2 Two useful properties of *deg*

The following lemma shows that *deg* satisfies a monotonicity property with respect to two interpretations $\mathcal{I}$ and $\mathcal{J}$ which are related by a homomorphism.

Lemma 4.11. *Let $\mathcal{I}$ and $\mathcal{J}$ be two interpretations such that there exists a homomorphism φ from $G_{\mathcal{I}}$ to $G_{\mathcal{J}}$. Then, for any individual $d \in \Delta^{\mathcal{I}}$ and any $\mathcal{EL}$ concept description C it holds: $deg^{\mathcal{I}}(d, C) \leq deg^{\mathcal{J}}(\varphi(d), C)$.*

Proof. Let C^r be the reduced form of C and h be any *ptgh* from T_{C^r} to $G_{\mathcal{I}}$ with $h(v_0) = d$. Since φ is a homomorphism from $G_{\mathcal{I}}$ to $G_{\mathcal{J}}$, the mapping $\varphi \circ h$ is a *ptgh* from T_{C^r} to $G_{\mathcal{J}}$ with $(\varphi \circ h)(v_0) = \varphi(d)$.

Then, we have that for each $v \in \mathsf{dom}(h)$ the homomorphism φ makes $\ell_{\mathcal{I}}(h(v)) \subseteq \ell_{\mathcal{J}}((\varphi \circ h)(v))$. In addition, for each r-successor $w \in \mathsf{dom}(h)$ of v in T_{C^r}, we have that $h(w)$ is an r-successor of $h(v)$ in $G_{\mathcal{I}}$. Therefore, $(\varphi \circ h)(w)$ is also an r-successor of $(\varphi \circ h)(v)$ in $G_{\mathcal{J}}$. Hence, it follows from Definition 4.4 that $h_w(v_0) \leq (\varphi \circ h)_w(v_0)$ for all *ptghs* h from T_{C^r} to $G_{\mathcal{I}}$ with $h(v_0) = d$.

Thus, we can conclude that $deg^{\mathcal{I}}(d, C) \leq deg^{\mathcal{J}}(\varphi(d), C)$. □

Now, using this monotonicity property and elements from the proof of Lemma 4.8, we can show that the value $deg^{\mathcal{I}}(d, C)$ is preserved by the canonical interpretation corresponding to a *ptgh* h such that $h_w(v_0) = deg^{\mathcal{I}}(d, C)$.

Lemma 4.12. *Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation, d be an individual of $\Delta^{\mathcal{I}}$ and C an $\mathcal{EL}$ concept description. Let h be a ptgh from T_{C^r} to $G_{\mathcal{I}}$ such that $h(v_0) = d$ and $h_w(v_0) = deg^{\mathcal{I}}(d, C)$. In addition, let $\mathcal{I}_h$ be the canonical interpretation induced by h. Then, $deg^{\mathcal{I}_h}(v_0, C) = deg^{\mathcal{I}}(d, C)$.*

Proof. Assume that $deg^{\mathcal{I}}(d, C) = q$. From Definition 4.5 we have:

$$deg^{\mathcal{I}}(d, C) = \max \mathcal{V}^{\mathcal{I}}(d, C^r) = h_w(v_0) = q$$

In the proof of Lemma 4.8 we saw that $i^{\mathcal{I}_h}$ is a *ptgh* from T_{C^r} to $T_{\mathcal{I}_h}$ with $i^{\mathcal{I}_h}(v_0) = v_0$ and $h_w(v_0) = i^{\mathcal{I}_h}_w(v_0)$. Hence, $deg^{\mathcal{I}_h}(v_0, C) \geq q$. Remark 4.7 tells us that h is a homomorphism from $T_{\mathcal{I}_h}$ to $G_{\mathcal{I}}$ with $h(v_0) = d$. Then, the application of Lemma 4.11 yields:

$$deg^{\mathcal{I}_h}(v_0, C) \leq deg^{\mathcal{I}}(d, C)$$

Thus, $deg^{\mathcal{I}_h}(v_0, C) = q = deg^{\mathcal{I}}(d, C)$. □

4.3 Relation to the Description Logic $\mathcal{ALC}$

We now investigate the relation between our threshold logic $\tau\mathcal{EL}(deg)$ and the DL $\mathcal{ALC}$ [SS91]. On the one side, we show that full negation of $\mathcal{EL}$ concept descriptions cannot be expressed in $\tau\mathcal{EL}(deg)$, and consequently there are $\mathcal{ALC}$ concept descriptions that cannot be expressed in $\tau\mathcal{EL}(deg)$. On the other side, we will see that every $\tau\mathcal{EL}(deg)$ concept description has its corresponding equivalent concept in $\mathcal{ALC}$, but the provided translation involves an exponential blow up.

Let us start by briefly introducing the DL $\mathcal{ALC}$. The set of $\mathcal{ALC}$ concept descriptions is the smallest set such that:

- $\top$ is an $\mathcal{ALC}$ concept description,
- if $A \in \mathsf{N_C}$, then A is an $\mathcal{ALC}$ concept description,
- if C, D are $\mathcal{ALC}$ concept descriptions and $r \in \mathsf{N_R}$, then $\neg C$, $C \sqcap D$ and $\exists r.C$ are $\mathcal{ALC}$ concept descriptions.

The semantics of the negation constructor under an interpretation $\mathcal{I}$ is given as:

$$(\neg C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid d \notin C^{\mathcal{I}}\}$$

As usual, $\forall r.C$ is an abbreviation for $\neg\exists r.\neg C$ and $C \sqcup D$ for $\neg(\neg C \sqcap \neg D)$.

Before going on to the main details of this section, we need to define the notion of a *concept part* of an $\mathcal{EL}$ concept description.

Definition 4.13 (concept part). Let C be an $\mathcal{EL}$ concept description. The set of *concept parts* $\mathrm{Part}(C)$ of C is the smallest set such that:

- $\{\top, C\} \subseteq \mathrm{Part}(C)$.
- if $\exists r.D \in \mathrm{Part}(C)$, then $\exists r.D' \in \mathrm{Part}(C)$ where $D' \in \mathrm{Part}(D)$.
- if $C_1 \sqcap C_2 \in \mathrm{Part}(C)$, then $C_1' \sqcap C_2' \in \mathrm{Part}(C)$ where $C_1' \in \mathrm{Part}(C_1)$ and $C_2' \in \mathrm{Part}(C_2)$. ♢

4.3.1 Full negation is not expressible in $\tau\mathcal{EL}(deg)$

In Chapter 3 we mentioned that although $\tau\mathcal{EL}(m)$ can express negation of $\mathcal{EL}$ concept descriptions, negation cannot be nested using the constructors of $\tau\mathcal{EL}(m)$. We prove that full negation cannot be expressed in $\tau\mathcal{EL}(deg)$, by showing that $\tau\mathcal{EL}(deg)$ cannot express the simple $\mathcal{ALC}$ concept description $\forall r.A$.

The semantics of $\forall r.C$ can be expressed as follows:

$$(\forall r.C)^{\mathcal{I}} := \{d \in \Delta^{\mathcal{I}} \mid \forall e \in \Delta^{\mathcal{I}}.\big[(d,e) \in r^{\mathcal{I}} \Rightarrow e \in C^{\mathcal{I}}\big]\}$$

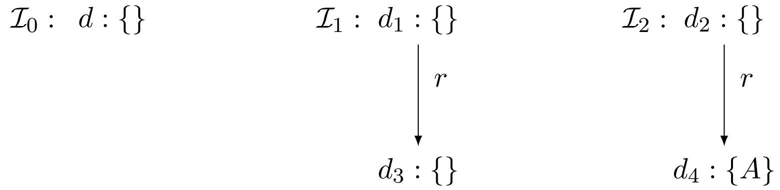
Lemma 4.14. *In $\tau\mathcal{EL}(deg)$, there is no concept description $\widehat{C}$ such that $\forall r.A \equiv \widehat{C}$, where $A \in \mathsf{N_C}$.*

Proof. Suppose that there exists a $\tau\mathcal{EL}(deg)$ concept description $\widehat{C}$ such that $\forall r.A \equiv \widehat{C}$. Then, for all interpretations $\mathcal{I}$ we have $(\forall r.A)^{\mathcal{I}} = \widehat{C}^{\mathcal{I}}$. Consider the interpretation $\mathcal{I}_0 = (\{d\}, \cdot^{\mathcal{I}_0})$ such that $X^{\mathcal{I}_0} = \emptyset$ for all $X \in \mathsf{N_C} \cup \mathsf{N_R}$. Obviously, $d \in (\forall r.A)^{\mathcal{I}_0}$ and by our initial assumption it also holds $d \in \widehat{C}^{\mathcal{I}_0}$.

By Theorem 3.8 there exists a τ-homomorphism ϕ from $T_{\widehat{C}}$ to $G_{\mathcal{I}_0}$ with $\phi(v_0) = d$. Since d has no r-successors in $\Delta^{\mathcal{I}_0}$ nor it is an instance of any concept name, this means that $\widehat{C}$ must be of the following form:

$$(E^1)_{\sim t_1} \sqcap \ldots \sqcap (E^q)_{\sim t_q}$$

where each E^i is an $\mathcal{EL}$ concept description. Let us now consider the interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ which have the description graphs shown below.



In addition to $d \in (\forall r.A)^{\mathcal{I}_0}$, it is also the case that $d_2 \in (\forall r.A)^{\mathcal{I}_2}$. Hence, since $\widehat{C} \equiv \forall r.A$, we also have $d_2 \in \widehat{C}^{\mathcal{I}_2}$. This means that $d \in [(E^i)_{\sim t_i}]^{\mathcal{I}_0}$ and $d_2 \in [(E^i)_{\sim t_i}]^{\mathcal{I}_2}$ for all $1 \leq i \leq q$. Further, it is easy to see that Lemma 4.11 can be applied to obtain for all $1 \leq i \leq q$:

$$deg^{\mathcal{I}_0}(d, E^i) \leq deg^{\mathcal{I}_1}(d_1, E^i) \leq deg^{\mathcal{I}_2}(d_2, E^i)$$

Therefore, it is immediate to see that $d_1 \in [(E^i)_{\sim t_i}]^{\mathcal{I}_1}$ for all conjuncts $(E^i)_{\sim t_i}$ of $\widehat{C}$, and consequently $d_1 \in \widehat{C}^{\mathcal{I}_1}$. Our initial assumption $\forall r.A \equiv \widehat{C}$ implies that $d_1 \in (\forall r.A)^{\mathcal{I}_1}$, but this is a contradiction since d_1 has an r-successor d_3 and $d_3 \notin A^{\mathcal{I}_1}$. Thus, there is no $\tau\mathcal{EL}(deg)$ concept description $\widehat{C}$ such that $\widehat{C} \equiv \forall r.A$. □

Lemma 4.14 implies that full negation of $\mathcal{EL}$ concept descriptions cannot be expressed in $\tau\mathcal{EL}(deg)$. Otherwise, since $\forall r.A \equiv \neg\exists r.\neg A$, there would be a $\tau\mathcal{EL}(deg)$ concept description $\widehat{D}$ such that $\widehat{D} \equiv \neg\exists r.\neg A$ contradicting the lemma. Moreover, since $\exists r.\neg A \equiv \exists r.A_{<1}$, this implies that neither negation of $\tau\mathcal{EL}(deg)$ concept descriptions can be expressed.

4.3.2 Expressing $\tau\mathcal{EL}(deg)$ concept descriptions in $\mathcal{ALC}$

Since $\mathcal{EL}$ is a fragment of $\mathcal{ALC}$, the concept constructors we need to look at are the ones corresponding to threshold concepts $E_{\sim t}$. In particular, a direct consequence of the semantics corresponding to such constructors are the equivalences:

$$E_{<t} \equiv \neg E_{\geq t} \quad \text{and} \quad E_{\leq t} \equiv \neg E_{>t}$$

The four possibilities are gathered in the following proposition.

Proposition 4.15. *Let $E_{\sim t}$ be a threshold concept. The negated concept $\neg E_{\sim t}$ is equivalent to the threshold concept $E_{\chi(\sim)t}$, where χ is the following mapping:*

$$\chi(<) := \geq \quad \chi(\leq) := > \quad \chi(>) := \leq \quad \chi(\geq) := <$$

Thus, since $\mathcal{ALC}$ allows full negation of concept descriptions, we can restrict our attention to threshold concepts of the form $E_{\sim t}$ with $\sim \in \{>, \geq\}$. For all $\tau\mathcal{EL}(deg)$ concept descriptions $\widehat{C}$, its corresponding $\mathcal{ALC}$ concept description $[\widehat{C}]^*$ is recursively defined as follows:

$$\begin{aligned}
[\top]^* &:= \top \\
[A]^* &:= A, \text{ if } A \in \mathsf{N_C} \\
[E_{\sim t}]^* &:= \neg[E_{\chi(\sim)t}]^*, \text{ if } \sim \in \{<, \leq\} \\
[\widehat{C} \sqcap \widehat{D}]^* &:= [\widehat{C}]^* \sqcap [\widehat{D}]^* \\
[\exists r.\widehat{C}]^* &:= \exists r.[\widehat{C}]^*
\end{aligned}$$

It remains to define the transformation $[E_{\sim t}]^*$ for $E_{\sim t}$, when $\sim \in \{>, \geq\}$. We show that such threshold concepts $E_{\sim t}$ can be equivalently expressed in $\mathcal{ALC}$ as a disjunction of $\mathcal{EL}$ concept descriptions $E_1 \sqcup \ldots \sqcup E_q$, such that E_i $(1 \leq i \leq q)$ is a *concept part* of E^r.

Let $\mathcal{I}$ be an interpretation and $d \in \Delta^{\mathcal{I}}$ such that $d \in (E_{\sim t})^{\mathcal{I}}$. We make two observations about *ptghs* h and their induced interpretations $\mathcal{I}_h$.

1. Let h be *ptgh* in $\mathcal{H}(T_{E^r}, G_{\mathcal{I}}, d)$ and $\mathcal{I}_h$ its induced canonical interpretation. Since $T_{\mathcal{I}_h}$ is a tree, we can speak of its associated $\mathcal{EL}$ concept description $C_{\mathcal{I}_h}$. Furthermore, from Remark 4.7 we know that h is a homomorphism from $T_{\mathcal{I}_h}$ to $G_{\mathcal{I}}$. Thus, since $h(v_0) = d$, we can apply Theorem 2.7 to obtain $d \in (C_{\mathcal{I}_h})^{\mathcal{I}}$.

2. It is clear from the construction of $\mathcal{I}_h$ in Definition 4.6 that $C_{\mathcal{I}_h}$ is a *concept part* of E^r.

In view of Lemma 4.12 and the fact that $\sim \in \{>, \geq\}$, the first observation tells us that: $d \in (E_{\sim} t)^{\mathcal{I}}$ iff $d \in C^{\mathcal{I}}$ for some $\mathcal{EL}$ concept description C, whose associated $\mathcal{EL}$ description tree T_C corresponds to an interpretation $\mathcal{I}_C$ such that $deg^{\mathcal{I}_C}(v_0, E) \sim t$. In addition, the second observation implies that it is sufficient to consider *concept parts* of E^r. We now formally define the set of such relevant concepts.

Definition 4.16. Let $E_{\sim t}$ be a threshold concept with $\sim \in \{>, \geq\}$. For all $X \in \text{Part}(E^r)$, we assign to X the value $v(X) \in [0..1]$ computed as:

$$v(X) := deg^{\mathcal{I}_X}(v_0, E)$$

Then, the subset $\mathcal{R}(E_{\sim t})$ of relevant concepts in $\text{Part}(E^r)$ is defined as follows:

$$\mathcal{R}(E_{\sim t}) := \{X \mid X \in \text{Part}(E^r) \text{ and } v(X) \sim t\} \qquad \diamondsuit$$

The following lemma shows that membership in $E_{\sim t}$ is equivalent to membership in at least one concept description from $\mathcal{R}(E_{\sim t})$.

Lemma 4.17. *Let $\mathcal{I}$ be an interpretation, $d \in \Delta^{\mathcal{I}}$ and $E_{\sim t}$ a threshold concept with $\sim \in \{>, \geq\}$. The following statements are equivalent.*

1. $d \in (E_{\sim t})^{\mathcal{I}}$.

2. *There exists $X \in \mathcal{R}(E_{\sim t})$ such that $d \in X^{\mathcal{I}}$.*

Proof. 1) $\rightarrow$ 2). Assume that $d \in (E_{\sim t})^{\mathcal{I}}$. Then, there exists a *ptgh* h in $\mathcal{H}(T_{E^r}, G_{\mathcal{I}}, d)$ such that:

$$h(v_0) = d \quad \text{and} \quad h_w(v_0) = deg^{\mathcal{I}}(d, E) \sim t$$

By Lemma 4.12, the canonical interpretation $\mathcal{I}_h$ satisfies

$$deg^{\mathcal{I}_h}(v_0, E) = deg^{\mathcal{I}}(d, E) \sim t$$

As observed above, we have that $d \in (C_{\mathcal{I}_h})^{\mathcal{I}}$ and $C_{\mathcal{I}_h}$ is a *concept part* of E^r. Hence, since $v(C_{\mathcal{I}_h}) = deg^{\mathcal{I}_h}(v_0, E) \sim t$, this means that $C_{\mathcal{I}_h} \in \mathcal{R}(E_{\sim t})$.

2) $\rightarrow$ 1). Assume that there exists $X \in \mathcal{R}(E_{\sim t})$ such that $d \in X^{\mathcal{I}}$. By definition of $\mathcal{R}(E_{\sim t})$ we know that $deg^{\mathcal{I}_X}(v_0, E) \sim t$. Moreover, since $d \in X^{\mathcal{I}}$, there exists a homomorphism φ from $T_{\mathcal{I}_X}$ to $G_{\mathcal{I}}$ with $\varphi(v_0) = d$ (Theorem 2.7). Hence, the application of Lemma 4.11 to $\mathcal{I}_X$ and $\mathcal{I}$ yields $deg^{\mathcal{I}_X}(v_0, E) \leq deg^{\mathcal{I}}(d, E)$.

Thus, $deg^{\mathcal{I}}(d, E) \sim t$ and $d \in (E_{\sim t})^{\mathcal{I}}$. □

The previous lemma tells us how to build an equivalent $\mathcal{ALC}$ concept description $[E_{\sim t}]^*$ for $E_{\sim t}$. The existential quantification in the second statement is expressed using *disjunction*, and since $\mathcal{R}(E_{\sim t})$ is a finite set, we translate $E_{\sim t}$ into the following $\mathcal{ALC}$ concept description:

$$[E_{\sim t}]^* := \bigsqcup_{X \in \mathcal{R}(E_{\sim t})} X$$

One can still reduce the size of $[E_{\sim t}]^*$. Let $(\mathcal{R}(E_{\sim t}), \sqsubseteq)$ be the partially ordered set defined by $\sqsubseteq$ on $\mathcal{R}(E_{\sim t})$. Using Lemma 4.11 and the characterization of subsumption in $\mathcal{EL}$ (Theorem 2.6), it is easy to prove that for all pairs of concepts $X, Y \in \mathcal{R}(E_{\sim t})$

$$X \sqsubseteq Y \;\Rightarrow\; v(X) \geq v(Y)$$

This means that it is enough to consider the concept descriptions in $\mathcal{R}(E_{\sim t})$ that are *maximal* (or the most general ones) with respect to $\sqsubseteq$. Let $\mathcal{R}_{\max}(E_{\sim t})$ be the set of maximal concepts in $\mathcal{R}(E_{\sim t})$ with respect to $\sqsubseteq$. We redefine $[E_{\sim t}]^*$ as:

$$[E_{\sim t}]^* := \bigsqcup_{X \in \mathcal{R}_{\max}(E_{\sim t})} X$$

Lemma 4.18. *Let $E_{\sim t}$ be a $\tau\mathcal{EL}(deg)$ threshold concept with $\sim \in \{>, \geq\}$. Then,*

$$E_{\sim t} \equiv [E_{\sim t}]^*$$

Proof. Let $\mathcal{I}$ be an interpretation and $d \in \Delta^{\mathcal{I}}$. Assume that $d \in (E_{\sim t})^{\mathcal{I}}$. The application of Lemma 4.17 yields that there exists $X \in \mathcal{R}(E_{\sim t})$ such that $d \in X^{\mathcal{I}}$. Obviously, there exists $Y \in \mathcal{R}_{\max}(E_{\sim t})$ such that $X \sqsubseteq Y$. Consequently, $d \in Y^{\mathcal{I}}$ and $d \in ([E_{\sim t}]^*)^{\mathcal{I}}$. Therefore, $(E_{\sim t})^{\mathcal{I}} \subseteq ([E_{\sim t}]^*)^{\mathcal{I}}$.

Conversely, suppose that $d \in ([E_{\sim t}]^*)^{\mathcal{I}}$. This means that there is $X \in \mathcal{R}_{\max}(E_{\sim t})$ such that $d \in X^{\mathcal{I}}$. Hence, since $\mathcal{R}_{\max}(E_{\sim t}) \subseteq \mathcal{R}(E_{\sim t})$, the application of Lemma 4.17 yields $d \in (E_{\sim t})^{\mathcal{I}}$.

Thus, we have shown that $E_{\sim t} \equiv [E_{\sim t}]^*$. □

Lemma 4.18 completes the translation $[.]^*$ presented above. Then, one can easily show by induction on the structure of $\widehat{C}$ that $\widehat{C} \equiv [\widehat{C}]^*$ for all $\tau\mathcal{EL}(deg)$ concept descriptions $\widehat{C}$. Thus, we have shown that $\tau\mathcal{EL}(deg)$ is a fragment of the DL $\mathcal{ALC}$. However, as we will see in the following, this translation may produce a concept $[\widehat{C}]^*$ of size exponential in the size of $\widehat{C}$.

Let C_n $(n \geq 1)$ be the $\mathcal{EL}$ concept description $C_n^r \sqcap C_n^s$, where C_n^x $(x \in \{r, s\})$ is inductively defined as follows:

$$C_n^x := \begin{cases} \exists x.A & \text{if } n = 1 \\ \exists x.(A \sqcap C_{n-1}^x) & \text{if } n > 1 \end{cases}$$

The size of C_n is linear in n, i.e., $\mathsf{s}(C_n) = \mathcal{O}(n)$. Our translation into $\mathcal{ALC}$ of the threshold concept $(C_n)_{\geq \frac{1}{2}}$ yields an $\mathcal{ALC}$ concept description $[(C_n)_{\geq \frac{1}{2}}]^*$ of size exponential in n. Let us explain this in the following example for $n = 3$.

Example 4.19. The $\mathcal{EL}$ description tree depicted on the right-hand side of Figure 4.3 corresponds to the concept description C_3. Now, the left-hand side of the same figure contains the representation of four $\mathcal{EL}$ description trees. In particular, we can say the following about $T_{\frac{4}{4}}$:

- its associated concept description $D_{\frac{4}{4}}$ is a *concept part* of C_3, and
- $deg^{\mathcal{I}_{\frac{4}{4}}}(v_0, C_3) = \frac{1}{2}$, where $\mathcal{I}_{\frac{4}{4}}$ denotes the interpretation with description graph $T_{\frac{4}{4}}$ and v_0 its root. Consequently, there exists $h \in \mathcal{H}(T_{C_3}, T_{\frac{4}{4}}, v_0)$ such that $h_w(w_0) = 1/2$.

Therefore, $D_{\frac{4}{4}} \in \mathcal{R}((C_3)_{\geq \frac{1}{2}})$. Furthermore, the r-branch in T_{C_3} is fully present for v_0 in $T_{\frac{4}{4}}$, whereas the s-branch is completely missing. This means that they contribute to the top-level of the computation of $h_w(w_0)$ with the values $v_r = 1$ and $v_s = 0$, respectively. Hence, $D_{\frac{4}{4}}$ must be maximal in $\mathcal{R}((C_3)_{\geq \frac{1}{2}})$ with respect to $\sqsubseteq$, for otherwise any concept $X \in \mathcal{R}((C_3)_{\geq \frac{1}{2}})$ satisfying $D_{\frac{4}{4}} \sqsubseteq X$ and $X \not\sqsubseteq D_{\frac{4}{4}}$ is such that $v_r < 1$ and $v_s = 0$. This would imply that $deg^{\mathcal{I}_X}(v_0, (C_3)_{\geq \frac{1}{2}}) < \frac{1}{2}$ which contradicts $X \in \mathcal{R}((C_3)_{\geq \frac{1}{2}})$. Thus, $D_{\frac{4}{4}} \in \mathcal{R}_{\max}((C_3)_{\geq \frac{1}{2}})$ and it is one of the disjuncts in $[(C_3)_{\geq \frac{1}{2}}]^*$.

The same conclusion can be drawn for the other three description trees. Basically, the values of the pair (v_r, v_s) for $T_{\frac{3}{4}}, T_{\frac{2}{4}}$ and $T_{\frac{1}{4}}$ will be $(3/4, 1/4), (2/4, 2/4)$ and $(1/4, 3/4)$, respectively. Then, finding a more general concept X for $D_{\frac{i}{4}}$ $(1 \leq i \leq 3)$ would mean

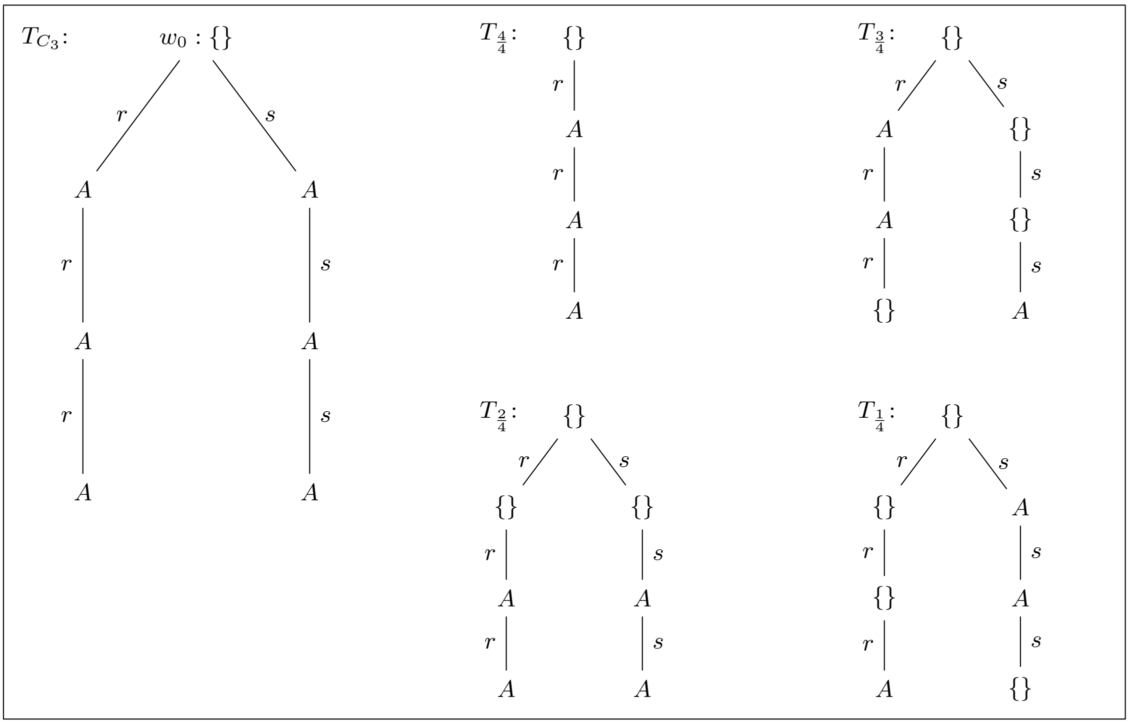


Figure 4.3: Exponentially many concepts in $\mathcal{R}_{\max}((C_3)_{\geq\frac{1}{2}})$.

that one of v_r, v_s decreases while the other one remains the same. Overall, this means that $D_{\frac{1}{4}} \sqcup D_{\frac{2}{4}} \sqcup D_{\frac{3}{4}} \sqcup D_{\frac{4}{4}}$ is a fragment of $[(C_3)_{\geq\frac{1}{2}}]^*$.

Thus, generalizing this idea for all n we obtain that $\mathsf{s}([(C_n)_{\geq\frac{1}{2}}]^*) \geq 2^{n-1}$. ♢

In conclusion, the DL $\tau\mathcal{EL}(deg)$ is a fragment of $\mathcal{ALC}$, but so far we do not know whether it is more succinct than $\mathcal{ALC}$.

Chapter 5

Reasoning in $\tau\mathcal{EL}(deg)$

We now study the complexity of reasoning in the DL $\tau\mathcal{EL}(deg)$. We start with investigating the complexity of terminological reasoning (*satisfiability*, *subsumption*), and then turn to assertional reasoning (*consistency*, *instance checking*).

Using a very simple reduction from a variant of the propositional satisfiability problem, we show that *satisfiability* and *non-subsumption* in $\tau\mathcal{EL}(deg)$ are NP-hard. To provide an NP upper bound for satisfiability, we establish a polynomial bounded model property for satisfiable $\tau\mathcal{EL}(deg)$ concept descriptions. A key ingredient to obtain this property is the characterization of membership in $\tau\mathcal{EL}(deg)$ concept descriptions settled in Theorem 3.8. Afterwards, starting with a polynomial size model of a concept $\widehat{C}$, we show how to extend it into a model of a concept $\widehat{C} \sqcap \neg\widehat{D}$ that is still polynomial in the size of $\widehat{C}$ and $\widehat{D}$. This will gives us membership in NP for the non-subsumption problem, and thus a matching coNP upper bound for subsumption in $\tau\mathcal{EL}(deg)$.

Regarding assertional reasoning, the consistency problem can be tackled in a similar way as the satisfiability problem, by using Theorem 3.9 as a characterization of the satisfaction relation for $\tau\mathcal{EL}(deg)$ ABoxes. Then, similar to our treatment of subsumption, the bounded model of an ABox can be used to obtain a bounded model property for the *non-instance* problem. Therefore, we obtain that ABox consistency is NP-complete and the instance problem is coNP-complete (w.r.t. *data complexity*).

5.1 Terminological reasoning

We start by recalling the first two decision problems we will look at:

- *concept satisfiability*: Let $\widehat{C}$ be a $\tau\mathcal{EL}(deg)$ concept description $\widehat{C}$. The concept $\widehat{C}$ is satisfiable iff there exists an interpretation $\mathcal{I}$ such that $\widehat{C}^{\mathcal{I}} \neq \emptyset$.

- *subsumption*: Let $\widehat{C}$ and $\widehat{D}$ be two $\tau\mathcal{EL}(deg)$ concept descriptions. $\widehat{C}$ is subsumed by $\widehat{D}$ iff $\widehat{C}^{\mathcal{I}} \subseteq \widehat{D}^{\mathcal{I}}$ for every interpretation $\mathcal{I}$.

The size $\mathsf{s}(\widehat{C})$ of a $\tau\mathcal{EL}(deg)$ concept description $\widehat{C}$ is the number of occurrences of symbols needed to write $\widehat{C}$.

In contrast to $\mathcal{EL}$, where every concept description is satisfiable, we have seen in Chapter 3 that there are unsatisfiable $\tau\mathcal{EL}(deg)$ concept descriptions such as $A_{\geq 1} \sqcap A_{<1}$. Thus, the satisfiability problem is non-trivial in $\tau\mathcal{EL}(deg)$. In fact, by a simple reduction from the well-known NP-complete problem ALL-POS ONE-IN-THREE 3SAT (see [GJ79], page 259) we can show that testing $\tau\mathcal{EL}(deg)$ concept descriptions for satisfiability is actually NP-hard.

Definition 5.1 (ALL-POS ONE-IN-THREE 3SAT). Let U be a set of propositional variables and $\mathcal{C}$ be a finite set of propositional clauses over U such that:

- each clause in $\mathcal{C}$ is a set of three literals over U, and
- no $c \in \mathcal{C}$ contains a negated literal.

ALL-POS ONE-IN-THREE 3SAT is the problem of deciding whether there exists a truth assignment to the variables in U, such that each clause in $\mathcal{C}$ has exactly one true literal. $\diamondsuit$

Let $\mathcal{C} = \{c_1, \ldots, c_n\}$ be a set of clauses over U. We now show how to build a $\tau\mathcal{EL}(deg)$ concept description $\widehat{C}_{\mathcal{C}}$ such that U has a truth assignment where exactly one literal per clause in $\mathcal{C}$ is true iff $\widehat{C}_{\mathcal{C}}$ is satisfiable. Each propositional variable $u \in U$ is identified with the concept name A_u. In addition, to each clause $c_i = \{u_{i1}, u_{i2}, u_{i3}\}$ in $\mathcal{C}$ we associate an $\mathcal{EL}$ concept description D_i of the form $A_{u_{i1}} \sqcap A_{u_{i2}} \sqcap A_{u_{i3}}$. Then the concept $\widehat{C}_{\mathcal{C}}$ is defined as follows:

$$\widehat{C}_{\mathcal{C}} := \prod_{i=1}^{n} [(D_i)_{\leq \frac{1}{3}} \sqcap (D_i)_{\geq \frac{1}{3}}]$$

The main idea underlying this reduction is that for any three distinct concept names A_i, A_j, A_k, an individual belongs to $(A_i \sqcap A_j \sqcap A_k)_{\leq \frac{1}{3}} \sqcap (A_i \sqcap A_j \sqcap A_k)_{\geq \frac{1}{3}}$ iff it belongs to exactly one of these three concepts.

Lemma 5.2. *$\widehat{C}_{\mathcal{C}}$ is satisfiable iff there exists a truth assignment to the variables in U such that each clause in $\mathcal{C}$ has exactly one true literal.*

Proof. ($\Rightarrow$) Assume that $\widehat{C}_{\mathcal{C}}$ is satisfiable. Then, there exists an interpretation $\mathcal{I}$ such that $(\widehat{C}_{\mathcal{C}})^{\mathcal{I}} \neq \emptyset$, i.e., $d \in (\widehat{C}_{\mathcal{C}})^{\mathcal{I}}$ for some $d \in \Delta^{\mathcal{I}}$. We construct an assignment $\mathfrak{t}$ for U in the following way:

$$\mathfrak{t}(u) = \mathit{true} \text{ iff } d \in (A_u)^{\mathcal{I}}, \text{ for all } u \in U \tag{5.1}$$

Now, let $c_i = \{u_{i1}, u_{i2}, u_{i3}\}$ be any clause in $\mathcal{C}$. Since $d \in (\widehat{C}_{\mathcal{C}})^{\mathcal{I}}$, this means that $d \in [(D_i)_{\leq \frac{1}{3}}]^{\mathcal{I}}$ and $d \in [(D_i)_{\geq \frac{1}{3}}]^{\mathcal{I}}$. Therefore, $deg^{\mathcal{I}}(d, D_i) = \frac{1}{3}$ and by definition of $deg^{\mathcal{I}}$, d is an instance of exactly one of the concept names $A_{u_{i1}}, A_{u_{i2}}, A_{u_{i3}}$. Thus, by construction of $\mathfrak{t}$ in (5.1), exactly one literal in c_i is assigned to true.

($\Leftarrow$) We assume that there exists a truth assignment $\mathfrak{t}$ to the variables in U such that exactly one literal is true for each clause in $\mathcal{C}$. Then, we build a *single-pointed* interpretation $\mathcal{I} = (\{d\}, \cdot^{\mathcal{I}})$ in the following way:

$$d \in (A_u)^{\mathcal{I}} \text{ iff } \mathfrak{t}(u) = \mathit{true}, \text{ for all } u \in U$$

The properties satisfied by $\mathfrak{t}$ (with respect to $\mathcal{C}$) imply that d is an instance of exactly one concept name in the definition of D_i. Hence, for all $1 \leq i \leq n$ we have $deg^{\mathcal{I}}(d, D_i) = \frac{1}{3}$. Thus, $d \in (\widehat{C}_{\mathcal{C}})^{\mathcal{I}}$ and $\mathcal{I}$ satisfies $\widehat{C}_{\mathcal{C}}$. $\square$

This also yields coNP-hardness for subsumption in $\tau\mathcal{EL}(deg)$ since unsatisfiability can be reduced to subsumption: $\widehat{C}$ is *unsatisfiable* iff $\widehat{C} \sqsubseteq A_{\geq 1} \sqcap A_{<1}$.

Lemma 5.3. *In $\tau\mathcal{EL}(deg)$, satisfiability is NP-hard and subsumption is coNP-hard.*

To show an NP upper bound for satisfiability, we use the τ-homomorphism characterization of membership for $\tau\mathcal{EL}(m)$ concept descriptions introduced in Chapter 3. Using Theorem 3.8 we prove a bounded model property for $\tau\mathcal{EL}(deg)$ concept descriptions.

Lemma 5.4. *Let $\widehat{C}$ be a $\tau\mathcal{EL}(deg)$ concept description of size $\mathsf{s}(\widehat{C})$. If $\widehat{C}$ is satisfiable, then there exists an interpretation $\mathcal{J}$ such that $\widehat{C}^{\mathcal{J}} \neq \emptyset$ and $|\Delta^{\mathcal{J}}| \leq \mathsf{s}(\widehat{C})$.*

Proof. Since $\widehat{C}$ is satisfiable, there exists an interpretation $\mathcal{I}$ such that $d \in \widehat{C}^{\mathcal{I}}$ for some $d \in \Delta^{\mathcal{I}}$. Therefore, there exists a τ-homomorphism ϕ from $T_{\widehat{C}}$ to $G_{\mathcal{I}}$ with $\phi(v_0) = d$ (Theorem 3.8). The idea is to use ϕ and *small* fragments of $\mathcal{I}$ to build $\mathcal{J}$ and a τ-homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{J}}$, and then apply Theorem 3.8 to $\widehat{C}$ and $\mathcal{J}$.

The interpretation $\mathcal{J}$ is built in two steps. We first use as base interpretation $\mathcal{I}_0$ the one associated to the description tree $T_{\widehat{C}}$, where we ignore the labels of the form $E_{\sim t}$ (i.e. the description tree T_C, see Figure 3.1). It is easy to see that the identity mapping ϕ_{id} is a homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{I}_0}$. However, this interpretation and homomorphism need not satisfy Condition 2 of Definition 3.7. There may exist $v \in \Delta^{\mathcal{I}_0}$ such that $E_{\sim t} \in \widehat{\ell}_{T_{\widehat{C}}}(v)$, but $v \notin (E_{\sim t})^{\mathcal{I}_0}$. To repair this we extend $\mathcal{I}_0$ to $\mathcal{J}$ by adding appropriate fragments of $\mathcal{I}$.

More precisely, for such a node v in $\mathcal{I}_0$ we know that $\phi(v) \in (E_{\sim t})^{\mathcal{I}}$, and consequently $deg^{\mathcal{I}}(\phi(v), E) \sim t$. By Lemma 4.12 we do not need all of $\mathcal{I}$ to obtain $deg^{\mathcal{I}}(\phi(v), E)$ for v in $\mathcal{J}$. It is sufficient to use the canonical interpretation $\mathcal{I}_h$ induced by a *ptgh* h from T_{E^r} to $G_{\mathcal{I}}$ such that:

- $h(w_0) = \phi(v)$, and
- $deg^{\mathcal{I}}(\phi(v), E) = h_w(w_0)$.

Here, w_0 is the root of T_{E^r}. We rename it as v (the corresponding problematic node in $\mathcal{I}_0$) for the rest of the proof. We denote $\mathcal{I}_h$ as $\mathcal{I}_v^E$ and the *ptgh* h which induces $\mathcal{I}_h$ as h_v^E. Now, let $\mathfrak{I}$ be the family of all interpretations $\mathcal{I}_v^E$ needed to repair the inconsistencies in $\mathcal{I}_0$, i.e.,

$$\mathfrak{I} := \{\mathcal{I}_v^E \mid v \in \Delta^{\mathcal{I}_0}, E_{\sim t} \in \widehat{\ell}_{T_{\widehat{C}}}(v) \text{ and } v \notin (E_{\sim t})^{\mathcal{I}_0}\}$$

For all pairs $\mathcal{I}_v^{E_1}, \mathcal{I}_w^{E_2} \in \mathfrak{I}$ we assume $\Delta^{\mathcal{I}_v^{E_1}}$ and $\Delta^{\mathcal{I}_w^{E_2}}$ to be pairwise disjoint in the following sense: if $v \neq w$ they have no element in common, otherwise only v is shared. In addition, for all $\mathcal{I}_v^E \in \mathfrak{I}$ the sets $\Delta^{\mathcal{I}_v^E}$ share only the distinguished element v with $\Delta^{\mathcal{I}_0}$. Once these disjointness assumptions have been established, $\mathcal{J}$ is constructed as follows:

- $\Delta^{\mathcal{J}} := \Delta^{\mathcal{I}_0} \cup \bigcup\limits_{\mathcal{K} \in \mathfrak{I}} \Delta^{\mathcal{K}}$,
- $X^{\mathcal{J}} := X^{\mathcal{I}_0} \cup \bigcup\limits_{\mathcal{K} \in \mathfrak{I}} X^{\mathcal{K}}$ for all $X \in (\mathsf{N_C} \cup \mathsf{N_R})$.

We now prove that Condition 2 of Definition 3.7 is satisfied by ϕ_{id} and $\mathcal{J}$. For all $v \in V_{T_{\widehat{C}}}$ and $E_{\sim t} \in \widehat{\ell}_{T_{\widehat{C}}}(v)$, we distinguish two cases:

- $\sim \in \{>, \geq\}$. Suppose that $v \in (E_{\sim t})^{\mathcal{I}_0}$. Since $\mathcal{I}_0 \subseteq \mathcal{J}$, this makes Lemma 4.11 to be applicable to $\mathcal{I}_0, \mathcal{J}$ and v[1]. Hence, we have $deg^{\mathcal{I}_0}(v, E) \leq deg^{\mathcal{J}}(v, E)$ and

[1]The identity mapping from $\Delta^{\mathcal{I}_0}$ to $\Delta^{\mathcal{J}}$ is a homomorphism from $G_{\mathcal{I}_0}$ to $G_{\mathcal{J}}$ (recall the definition of $\mathcal{I} \subseteq \mathcal{J}$ in Chapter 2).

obviously $v \in (E_{\sim t})^{\mathcal{J}}$. Conversely, assume that $v \notin (E_{\sim t})^{\mathcal{I}_0}$. The selection of $\mathcal{I}_v^E$ to build $\mathcal{J}$ and the application of Lemma 4.12 yields:

$$deg^{\mathcal{I}_v^E}(v, E) = deg^{\mathcal{I}}(\phi(v), E)$$

This means that $v \in (E_{\sim t})^{\mathcal{I}_v^E}$, since $\phi(v) \in (E_{\sim t})^{\mathcal{I}}$. Moreover, from the construction of $\mathcal{J}$ it follows that $\mathcal{I}_v^E \subseteq \mathcal{J}$. Then, a second application of Lemma 4.11 to $\mathcal{I}_v^E$, $\mathcal{J}$ and v yields $v \in (E_{\sim t})^{\mathcal{J}}$.

- $\sim \in \{<, \leq\}$. Since $\phi(v) \in (E_{\sim t})^{\mathcal{I}}$, we intend to use again Lemma 4.11 with respect to $\mathcal{J}$ and $\mathcal{I}$. For this, we build a mapping φ from $V_{\mathcal{J}}$ to $V_{\mathcal{I}}$ such that $\varphi(w) = \phi(w)$ for all $w \in \Delta^{\mathcal{I}_0}$, and show that it is a homomorphism from $G_{\mathcal{J}}$ to $G_{\mathcal{I}}$.

$$\varphi := \phi \cup \bigcup_{\mathcal{I}_w^E \in \mathfrak{I}} h_w^E$$

Recall from Remark 4.7 that h_w^E is a homomorphism from $T_{\mathcal{I}_w^E}$ to $G_{\mathcal{I}}$. Consequently, φ is defined for all $d \in \Delta^{\mathcal{J}}$ and since ϕ and each h_w^E have $V_{\mathcal{I}}$ as their images, φ is certainly a mapping from $V_{\mathcal{J}}$ to $V_{\mathcal{I}}$. In addition, by the disjointness assumptions made to build $\Delta^{\mathcal{J}}$ and the fact that h_w^E is chosen such that $h_v^E(w) = \phi(w)$, we further have that φ is unambiguous and $\varphi(w) = \phi(w)$ for all $w \in \Delta^{\mathcal{I}_0}$.

Let us now see why φ is really a homomorphism in the sense of Definition 2.5:

  - For all $d \in \Delta^{\mathcal{J}}$, we either have $d \in \Delta^{\mathcal{I}_0}$ and

$$\ell_{\mathcal{J}}(d) = \ell_{\mathcal{I}_0}(d) \cup \bigcup_{\mathcal{I}_w^E \in \mathfrak{I}} \ell_{\mathcal{I}_w^E}(d) \quad (w = d)$$

    or $d \in \Delta^{\mathcal{I}_w^E}$ for some $\mathcal{I}_w^E \in \mathfrak{I}$ and $\ell_{\mathcal{J}}(d) = \ell_{\mathcal{I}_w^E}(d)$, where $w \neq d$. Since ϕ is a homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{I}}$, this means that $\ell_{\mathcal{I}_0}(d) \subseteq \ell_{\mathcal{I}}(\phi(d))$. Since each h_w^E is also a homomorphism from $T_{\mathcal{I}_w^E}$ to $G_{\mathcal{I}}$, this means that $\ell_{\mathcal{I}_w^E}(d) \subseteq \ell_{\mathcal{I}}(h_w^E(d))$. Hence, by the way φ has been defined we can conclude that $\ell_{\mathcal{J}}(d) \subseteq \ell_{\mathcal{I}}(\varphi(d))$ for all $d \in \Delta^{\mathcal{J}}$.
  - $d_1 r d_2 \in E_{\mathcal{J}}$. If $d_1, d_2 \in \Delta^{\mathcal{I}_0}$, then ϕ implies that $\varphi(d_1) r \varphi(d_2) \in E_{\mathcal{I}}$. Otherwise, $d_1, d_2 \in \Delta^{\mathcal{I}_w^E}$ for some $\mathcal{I}_w^E \in \mathfrak{I}$. Then, the corresponding homomorphism h_w^E guarantees that $\varphi(d_1) r \varphi(d_2) \in E_{\mathcal{I}}$.

  Consequently, φ is a homomorphism from $G_{\mathcal{J}}$ to $G_{\mathcal{I}}$. Since $\varphi(w) = \phi(w)$ for all $w \in \Delta^{\mathcal{I}_0}$ and $v \in \Delta^{\mathcal{I}_0}$, the rest relies in applying Lemma 4.11 with respect to $\mathcal{J}$, $\mathcal{I}$ and v to obtain $v \in (E_{\sim t})^{\mathcal{J}}$.

Thus, we have shown that ϕ_{id} is τ-homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{J}}$. Since $\phi_{id}(v_0) = v_0$, the application of Theorem 3.8 yields $v_0 \in \widehat{C}^{\mathcal{J}}$.

To conclude, we look at the size of $\mathcal{J}$. By construction of $\mathcal{J}$ we have:

$$|\Delta^{\mathcal{J}}| = |\Delta^{\mathcal{I}_0}| + \sum_{\mathcal{K} \in \mathfrak{I}} |\Delta^{\mathcal{K}}|$$

It is not hard to see that the size of $\mathcal{I}_0$ is bounded by the size of $\widehat{C}$ (without counting the threshold concepts). In addition, any occurrence of a threshold concept $E_{\sim t}$ in $\widehat{C}$ is considered at most once to build $\mathcal{J}$. Moreover, each canonical interpretation $\mathcal{I}_v^E \in \mathfrak{I}$ is selected with respect to E^r and its size is bounded by the size of E^r (see Definition 4.6). Since E^r is obviously not bigger than E, this implies $|\Delta^{\mathcal{I}_v^E}| \leq \mathsf{s}(E_{\sim t})$. Thus, it is clear that $|\Delta^{\mathcal{J}}| \leq \mathsf{s}(\widehat{C})$. □

This lemma yields a standard *guess-and-check* NP-algorithm to decide satisfiability of a concept $\widehat{C}$. The algorithm first guesses an interpretation $\mathcal{J}$ of size at most $\mathsf{s}(\widehat{C})$, and then checks whether there exists a τ-homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{J}}$. To verify the existence of a τ-homomorphism it uses Algorithm 2 in Section 3.2. Since *deg* can be computed in polynomial time (Chapter 4), Algorithm 2 runs in polynomial time with respect to *deg*.

Remark 5.5. We would like to point out that the construction presented in the previous lemma yields a *tree-shaped* interpretation $\mathcal{J}$, i.e., $G_{\mathcal{J}}$ is a tree. The base interpretation $\mathcal{I}_0$ is tree-shaped since its description graph has the structure of $T_{\widehat{C}}$, and so are the canonical interpretations used to extend $\mathcal{I}_0$ into $\mathcal{J}$. This combined with the applied disjointness assumptions guarantee that the resulting graph $G_{\mathcal{J}}$ is a tree. Additionally, the element $v_0 \in \Delta^{\mathcal{J}}$ corresponding to the root of $G_{\mathcal{J}}$ satisfies $v_0 \in \widehat{C}^{\mathcal{J}}$. ♢

A coNP upper bound for subsumption cannot directly be obtained from the fact that satisfiability is in NP. In fact, though we have $\widehat{C} \sqsubseteq \widehat{D}$ iff $\widehat{C} \sqcap \neg\widehat{D}$ is unsatisfiable, this equivalence cannot be used directly since $\neg\widehat{D}$ need not be a $\tau\mathcal{EL}(deg)$ concept description as shown in Section 4.3.1. Nevertheless, we can extend the ideas used in the proof of Lemma 5.4 to obtain a bounded model property for satisfiability of concepts of the form $\widehat{C} \sqcap \neg\widehat{D}$.

Lemma 5.6. *Let $\widehat{C}$ and $\widehat{D}$ be $\tau\mathcal{EL}(deg)$ concept descriptions of respective sizes $\mathsf{s}(\widehat{C})$ and $\mathsf{s}(\widehat{D})$. If $\widehat{C} \sqcap \neg\widehat{D}$ is satisfiable, then there exists an interpretation $\mathcal{J}$ such that $\widehat{C}^{\mathcal{J}} \setminus \widehat{D}^{\mathcal{J}} \neq \emptyset$ and $|\Delta^{\mathcal{J}}| \leq \mathsf{s}(\widehat{C}) \times \mathsf{s}(\widehat{D})$.*

Proof. Assume that $\widehat{C} \sqcap \neg\widehat{D}$ is satisfiable. Then, there exists an interpretation $\mathcal{I}$ such that $d \in \widehat{C}^{\mathcal{I}}$ and $d \notin \widehat{D}^{\mathcal{I}}$ for some $d \in \Delta^{\mathcal{I}}$. We first apply the construction used in Lemma 5.4 to build (with respect to $\mathcal{I}$) an interpretation $\mathcal{J}_0$ such that $\widehat{C}^{\mathcal{J}_0} \neq \emptyset$ and $|\Delta^{\mathcal{J}_0}| \leq \mathsf{s}(\widehat{C})$. From Lemma 5.4 we know:

- $G_{\mathcal{J}_0}$ is a tree and $v_0 \in \widehat{C}^{\mathcal{J}_0}$.
- ϕ is a τ-homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{I}}$ with $\phi(v_0) = d$.
- ϕ_{id} is a τ-homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{J}_0}$.
- φ is a homomorphism from $G_{\mathcal{J}_0}$ to $G_{\mathcal{I}}$ with $\varphi(w) = \phi(w)$ for all $w \in \Delta^{\mathcal{I}_0}$.

Since $\varphi(v_0) \notin \widehat{D}^{\mathcal{I}}$, the idea is to use φ to extract from $\mathcal{I}$ the necessary information to extend $\mathcal{J}_0$ into an interpretation $\mathcal{J}$ that falsifies $\widehat{D}$ in v_0, while keeping $v_0 \in \widehat{C}^{\mathcal{J}}$. In order to do this, we consider all the nodes in $\Delta^{\mathcal{J}_0}$ in a top-down manner starting with the root v_0.

We construct a series of pairs $(\mathcal{J}_0, S_0)(\mathcal{J}_1, S_1)\ldots$, where each $\mathcal{J}_i$ is an interpretation and S_i is a set of pairs of the form $(v, \widehat{F})$ being $v \in \Delta^{\mathcal{J}_0}$ and $\widehat{F}$ a $\tau\mathcal{EL}(deg)$ concept description. The initial pair $(\mathcal{J}_0, S_0)$ is set as $(\mathcal{J}_0, \{(v_0, \widehat{D})\})$. The sequence is built such that $\varphi(v) \notin \widehat{F}^{\mathcal{I}}$ represents an invariant for all pairs $(v, \widehat{F}) \in S_i$. This will then be used to show that $v \notin \widehat{F}^{\mathcal{J}}$, and hence $v_0 \notin \widehat{D}^{\mathcal{J}}$.

Each pair $(\mathcal{J}_i, S_i)$ $(i > 0)$ is computed from the pair $(\mathcal{J}_{i-1}, S_{i-1})$ as follows:

- First, an auxiliary set S_i^* is computed with the purpose to decompose concepts $\widehat{F}$ of the form $\widehat{F}_1 \sqcap \ldots \sqcap \widehat{F}_n$. More precisely, for all $(v, \widehat{F}) \in S_{i-1}$ exactly one conjunct $\widehat{F}' = \widehat{F}_j$ $(1 \leq j \leq n)$ is selected such that $\varphi(v) \notin (\widehat{F}_j)^{\mathcal{I}}$. The set S_i^* is defined as follows:
$$S_i^* := S_{i-1} \cup \bigcup_{(v,\widehat{F}) \in S_{i-1}} \{(v, \widehat{F}')\}$$
- Then, S_i is obtained from S_i^* as:
$$S_i := \{(u, \widehat{F}) \mid (v, \exists r.\widehat{F}) \in S_i^*, (v, u) \in r^{\mathcal{J}_0} \text{ and } u \in \widehat{F}^{\mathcal{J}_0}\}$$
- Regarding $\mathcal{J}_i$, for all $(v, E_{\sim t}) \in S_i^*$ such that $v \in (E_{\sim t})^{\mathcal{J}_0}$ we select a canonical interpretation $\mathcal{I}_v^E$ (as for the proof of Lemma 5.4) with $h_v^E(w_0) = \varphi(v)$. Now, let $\mathfrak{I}_i$ be the following set:
$$\mathfrak{I}_i := \{\mathcal{I}_v^E \mid (v, E_{\sim t}) \in S_i^* \text{ and } v \in (E_{\sim t})^{\mathcal{J}_0}\}$$
Using the same disjointness assumptions as in Lemma 5.4, $\mathcal{J}_i$ is built as follows:
  - $\Delta^{\mathcal{J}_i} := \Delta^{\mathcal{J}_{i-1}} \cup \bigcup_{\mathcal{K} \in \mathfrak{I}_i} \Delta^{\mathcal{K}}$,
  - $X^{\mathcal{J}_i} := X^{\mathcal{J}_{i-1}} \cup \bigcup_{\mathcal{K} \in \mathfrak{I}_i} X^{\mathcal{K}}$ for all $X \in (\mathsf{N_C} \cup \mathsf{N_R})$.

As we will see later, whenever $(v, \widehat{F}_1 \sqcap \ldots \sqcap \widehat{F}_n) \in S_{i-1}$ for some $i > 0$, there always exists $1 \leq j \leq n$ such that $\varphi(v) \notin (\widehat{F}_j)^{\mathcal{I}}$. Moreover, the tree shape of $\mathcal{J}_0$ makes this construction to consider every node in $\Delta^{\mathcal{J}_0}$ at most once in the following sense. On the one hand, a node v does not occur in more than one set S_i $(i \geq 0)$. Therefore, at some point the iteration terminates for some p where $S_p = \emptyset$. On the other hand, if $(v, \widehat{F}) \in S_i$, there is no other pair $(v, \_)$ occurring in S_i. This further implies that at most one canonical interpretation is added for each $v \in \Delta^{\mathcal{J}_0}$. Moreover, observe that for all $(v, \widehat{F}) \in S_i^*$ the concept description $\widehat{F}$ is a sub-description of $\widehat{D}$. In particular, for $(v, E_{\sim t}) \in S_i^*$ it follows that $|\Delta^{\mathcal{I}_v^E}| \leq \mathsf{s}(\widehat{D})$. Then, since $|\Delta^{\mathcal{J}_0}| \leq \mathsf{s}(\widehat{C})$, once the iteration finishes we will have $|\Delta^{\mathcal{J}_p}| \leq \mathsf{s}(\widehat{C}) \times \mathsf{s}(\widehat{D})$.

The next step is to show that $v_0 \in \widehat{C}^{\mathcal{J}_p}$ and $v_0 \notin \widehat{D}^{\mathcal{J}_p}$. Consider the mapping φ^* from $V_{\mathcal{J}_p}$ to $V_{\mathcal{I}}$:
$$\varphi^* := \varphi \cup \bigcup_{i=1}^{p} \bigcup_{\mathcal{I}_v^E \in \mathfrak{I}_i} h_v^E$$
One can show that φ^* is a homomorphism from $G_{\mathcal{J}_p}$ to $G_{\mathcal{I}}$ with $\varphi^*(w) = \phi(w)$ for all $w \in \Delta^{\mathcal{I}_0}$. The proof uses the same arguments showing that φ is a homomorphism

from $G_{\mathcal{J}}$ to $G_{\mathcal{I}}$ in Lemma 5.4. Then, ϕ_{id} remains a τ-homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{J}_p}$. Similar as in Lemma 5.4, one can use Lemma 4.11 to prove that $v \in (E_{\sim t})^{\mathcal{J}_p}$ for all $E_{\sim t} \in \widehat{\ell}_{T_{\widehat{C}}}(v)$. If $\sim \in \{>, \geq\}$, it follows from the fact that $\mathcal{J}_0 \subseteq \mathcal{J}_p$ and $v \in (E_{\sim t})^{\mathcal{J}_0}$. Otherwise, the argument relies on the homomorphism φ^* from $G_{\mathcal{J}_p}$ to $\mathcal{I}$, $\varphi^*(v) = \phi(v)$ and $\phi(v) \in (E_{\sim t})^{\mathcal{I}}$. We thus have $v_0 \in \widehat{C}^{\mathcal{J}_p}$.

Before going into the main details of why $v_0 \notin \widehat{D}^{\mathcal{J}_p}$, we clarify why the invariant mentioned before is satisfied along the construction of $\mathcal{J}_p$:

$$(v, \widehat{F}) \in S_i \;\Rightarrow\; \varphi(v) \notin \widehat{F}^{\mathcal{I}} \tag{5.2}$$

The initial pair $(v_0, \widehat{D})$ satisfies it, since $\varphi(v_0) = d$ and $d \notin \widehat{D}^{\mathcal{I}}$. By definition, all the pairs in S_1^* clearly satisfy the property. Now, let $(v, \exists r.\widehat{F}) \in S_1^*$. Starting with $\varphi(v) \notin (\exists r.\widehat{F})^{\mathcal{I}}$, for any r-successor u of v the homomorphism makes $(\varphi(v), \varphi(u)) \in r^{\mathcal{I}}$. Therefore, $\varphi(u) \notin \widehat{F}^{\mathcal{I}}$ and $(u, \widehat{F})$ satisfies the property as well. Consequently, S_1 satisfies (5.2). Applying the same reasoning inductively shows that (5.2) remains invariant for all S_i. Note that this additionally implies that $\widehat{F}'$ can always be selected when constructing S_i^*.

To finally prove that $v_0 \notin \widehat{D}^{\mathcal{J}_p}$, we show the following more general claim.

Claim: for all $0 < i \leq p$, if $(v, \widehat{F}) \in S_i^*$ then $v \notin \widehat{F}^{\mathcal{J}_p}$

The proof goes by induction on the structure of $\widehat{F}$. Let $(v, \widehat{F}) \in S_i^*$ for some $0 < i \leq p$:

- $\widehat{F}$ is of the form $\top$ or $A \in \mathsf{N_C}$. The case $\widehat{F} = \top$ never occurs, since $\varphi(v) \notin \widehat{F}^{\mathcal{I}}$. Otherwise, if $\widehat{F} = A$ this means that $\varphi(v) \notin A^{\mathcal{I}}$. Since φ^* is a homomorphism from $G_{\mathcal{J}_p}$ to $G_{\mathcal{I}}$ with $\varphi^*(v) = \varphi(v)$ for all $v \in \Delta^{\mathcal{J}_0}$, it must be that $v \notin A^{\mathcal{J}_p}$.

- $\widehat{F}$ is of the form $E_{\sim t}$. By (5.2) we have $\varphi(v) \notin (E_{\sim t})^{\mathcal{I}}$. Moreover, we know that $\mathcal{J}_0 \subseteq \mathcal{J}_p$ and φ^* is a homomorphism from $G_{\mathcal{J}_p}$ to $G_{\mathcal{I}}$ with $\varphi^*(v) = \varphi(v)$ for all $v \in \Delta^{\mathcal{J}_0}$. Hence, when $v \notin (E_{\sim t})^{\mathcal{J}_0}$, Lemma 4.11 ensures that $v \notin (E_{\sim t})^{\mathcal{J}_p}$. Otherwise, $v \in (E_{\sim t})^{\mathcal{J}_0}$ and the construction of $\mathcal{J}_p$ adds an interpretation $\mathcal{I}_v^E$ such that $deg^{\mathcal{I}_v^E}(v, E) = deg^{\mathcal{I}}(\varphi(v), E)$. Since $\mathcal{I}_v^E \subseteq \mathcal{J}_p$, again we obtain $v \notin (E_{\sim t})^{\mathcal{J}_p}$.

- $\widehat{F} = \widehat{F}_1 \sqcap \ldots \sqcap \widehat{F}_n$. By construction of S_i^* there is $\widehat{F}_j$ $(1 \leq j \leq n)$, such that $\varphi(v) \notin (\widehat{F}_j)^{\mathcal{I}}$ and $(v, \widehat{F}_j) \in S_i^*$. The application of induction to $\widehat{F}_j$ yields $v \notin (\widehat{F}_j)^{\mathcal{J}_p}$. Hence, $v \notin \widehat{F}^{\mathcal{J}_p}$.

- $\widehat{F}$ is of the form $\exists r.\widehat{F}'$. Since each node is considered only once while building $\mathcal{J}_p$, one can see that each direct r-successor of v in $G_{\mathcal{J}_p}$ is a node in $\Delta^{\mathcal{J}_0}$. Let $u \in \Delta^{\mathcal{J}_0}$ such that $(v, u) \in r^{\mathcal{J}_0}$. We distinguish two cases:
  - $u \in (\widehat{F}')^{\mathcal{J}_0}$. This means that $(u, \widehat{F}') \in S_i$ and consequently $(u, \widehat{F}') \in S_{i+1}^*$. Then, the application of induction hypothesis yields $u \notin (\widehat{F}')^{\mathcal{J}_p}$.
  - $u \notin (\widehat{F}')^{\mathcal{J}_0}$. This means that u is not relevant to obtain S_i from S_i^*, and since $G_{\mathcal{J}_0}$ is a tree neither of its successors is considered in the construction of $\mathcal{J}_p$. Therefore, the elements reachable from u in $\mathcal{J}_p$ are exactly the same as in $\mathcal{J}_0$. Suppose now that $u \in (\widehat{F}')^{\mathcal{J}_p}$, then by Theorem 3.8 there exists a

τ-homomorphism ϕ' from $T_{\widehat{F}}$ to $G_{\mathcal{J}_p}$ with $\phi'(w_0) = u$ (w_0 is the root of $T_{\widehat{F}}$). But then, it would also be a τ-homomorphism from $T_{\widehat{F}}$ to $G_{\mathcal{J}_0}$ contradicting $u \notin (\widehat{F}')^{\mathcal{J}_0}$. Consequently $u \notin (\widehat{F}')^{\mathcal{J}_p}$.

In conclusion, we have that for any $u \in \Delta^{\mathcal{J}_p}$ such that $(v, u) \in r^{\mathcal{J}_p}$ it is the case that $u \notin (\widehat{F}')^{\mathcal{J}_p}$. Hence, $v \notin (\exists r.\widehat{F}')^{\mathcal{J}_p}$.

Then, $(v_0, \widehat{D}) \in S_1^*$ implies $v_0 \notin \widehat{D}^{\mathcal{J}_p}$. All in all, we have $(\widehat{C} \sqcap \neg\widehat{D})^{\mathcal{J}_p} \neq \emptyset$ and $|\Delta|^{\mathcal{J}_p} \leq \mathsf{s}(\widehat{C}) \times \mathsf{s}(\widehat{D})$. Thus, $\mathcal{J}_p$ is the interpretation $\mathcal{J}$ satisfying our main claim. □

The lemma yields an obvious guess-and-check NP-algorithm for non-subsumption, which shows that subsumption is in coNP. Like for the satisfiability problem, the algorithm guesses an interpretation $\mathcal{J}$ of size $\mathsf{s}(\widehat{C}) \times \mathsf{s}(\widehat{D})$, and then checks if $d \in \widehat{C}^{\mathcal{J}}$ and $d \notin \widehat{D}^{\mathcal{J}}$ for some element $d \in \Delta^{\mathcal{J}}$. This can obviously be done, in polynomial time, by using Algorithm 2.

Overall, we thus have shown:

Theorem 5.7. *In $\tau\mathcal{EL}(deg)$, satisfiability is NP-complete and subsumption is coNP-complete.*

5.2 Assertional reasoning

Let us now look at reasoning in the presence of $\tau\mathcal{EL}(deg)$ ABoxes. We study the following two decision problems.

- *ABox consistency:* Let $\mathcal{A}$ be a $\tau\mathcal{EL}(deg)$ ABox. The ABox $\mathcal{A}$ is consistent iff there exists an interpretation $\mathcal{I}$ which is a model of $\mathcal{A}$ (denoted $\mathcal{I} \models \mathcal{A}$).
- *instance checking:* Let $\mathcal{A}$ be $\tau\mathcal{EL}(deg)$ ABox, $\widehat{C}$ a $\tau\mathcal{EL}(deg)$ concept description and a an individual. The individual a is an instance of $\widehat{C}$ in $\mathcal{A}$ (denoted $\mathcal{A} \models \widehat{C}(a)$) iff $a^{\mathcal{I}} \in \widehat{C}^{\mathcal{I}}$ holds in all models of $\mathcal{A}$.

We define the size $\mathsf{s}(A)$ of an ABox $\mathcal{A}$ as:

$$\mathsf{s}(\mathcal{A}) := \sum_{\substack{\widehat{C}(a)\in\mathcal{A} \\ a\in\mathsf{Ind}(\mathcal{A})}} \mathsf{s}(\widehat{C}) + \sum_{\substack{r(a,b)\in\mathcal{A} \\ a,b\in\mathsf{Ind}(\mathcal{A})}} 1$$

Since satisfiability can obviously be reduced to consistency ($\widehat{C}$ is satisfiable iff $\{\widehat{C}(a)\}$ is consistent), and subsumption to the instance problem ($\widehat{C} \sqsubseteq \widehat{D}$ iff $\{\widehat{C}(a)\} \models \widehat{D}(a)$), the lower bounds from Lemma 5.3 also hold for assertional reasoning.

Lemma 5.8. *In $\tau\mathcal{EL}(deg)$, ABox consistency is NP-hard and instance checking is coNP-hard.*

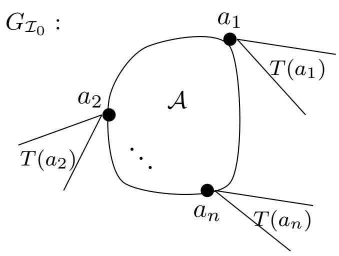
Regarding upper bounds, we proceed in the same way as for concept satisfiability and subsumption. We first show a bounded model property for consistent ABoxes, which yields an NP upper bound for ABox consistency. Then, similar to our treatment of subsumption, this bounded model can be used to obtain a bounded model property for the complement of the instance problem (a is not an instance of $\widehat{C}$ in $\mathcal{A}$). However, as we will show, the bound of the model has the size of $\widehat{C}$ in the exponent. For this reason, we obtain a coNP upper bound for the instance problem only if we consider *data complexity* [DLNS94], where the size of the query concept $\widehat{C}$ is assumed to be constant.

The consistency problem can be tackled in a similar way as the satisfiability problem. As we have shown in Section 3.1, based on the translation given in [KM02], $\tau\mathcal{EL}(m)$ ABoxes can be translated into $\tau\mathcal{EL}(m)$ description graphs and consistency can be characterized using τ-homomorphisms (see Theorem 3.9). We use this characterization to prove the following bounded model property.

Lemma 5.9. *Let $\mathcal{A}$ be an ABox in $\tau\mathcal{EL}(deg)$ of size $\mathsf{s}(\mathcal{A})$. If $\mathcal{A}$ is consistent, then there exists an interpretation $\mathcal{J}$ such that $\mathcal{J} \models \mathcal{A}$ and $|\Delta^{\mathcal{J}}| \leq \mathsf{s}(\mathcal{A})$.*

Proof. Assume that $\mathcal{A}$ is consistent, then there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{A}$. Therefore, there exists a τ-homomorphism ϕ from $\widehat{G}(\mathcal{A})$ to $G_{\mathcal{I}}$ such that $\phi(a) = a^{\mathcal{I}}$ for all $a \in \mathsf{Ind}(\mathcal{A})$ (Theorem 3.9).

We proceed in the same way as in Lemma 5.4. The base interpretation $\mathcal{I}_0$ is the one having the description graph $\widehat{G}(\mathcal{A})$, where we ignore the labels of the form $E_{\sim t}$. Again, the identity mapping ϕ_{id} is a homomorphism from $\widehat{G}(\mathcal{A})$ to $G_{\mathcal{I}_0}$, but need not satisfy Condition 2 of Definition 3.7. The interpretation $\mathcal{I}_0$ has the following shape:



Here, $\{a_1, a_2, \ldots, a_n\} = \mathsf{Ind}(\mathcal{A})$ and $T(a_1), T(a_2), \ldots, T(a_n)$ are the $\tau\mathcal{EL}(m)$ description trees corresponding to $\widehat{C}_{a_1}, \widehat{C}_{a_2}, \ldots, \widehat{C}_{a_n}$, respectively (see Definition 3.5). The inner area of the diagram consists of the role assertions in $\mathcal{A}$, i.e.,

$$(a, b) \in E_{\mathcal{A}} \text{ iff } r(a, b) \in \mathcal{A}$$

We extend $\mathcal{I}_0$ into $\mathcal{J}$ using the same construction of Lemma 5.4, i.e., a canonical interpretation $\mathcal{I}_v^E$ is attached to $\mathcal{I}_0$ for all $v \in V_{\mathcal{A}}$ such that $E_{\sim t} \in \widehat{\ell}_{\mathcal{A}^+}(v)$ and $v \notin (E_{\sim t})^{\mathcal{I}_0}$. Note that besides the structure required by the role assertions in $\mathcal{A}$, the rest of $G_{\mathcal{I}_0}$ consists of disjoint description trees whose roots are individual elements of $\mathcal{A}$. Therefore, for two different individuals $a, b \in \mathsf{Ind}(\mathcal{A})$, reparations needed in $T(a)$ and $T(b)$ can be done independently of each other. Then, one can show that there is also a homomorphism φ from $G_{\mathcal{J}}$ to $G_{\mathcal{I}}$ with $\varphi(w) = \phi(w)$ for all $w \in \Delta^{\mathcal{I}_0}$. Once we have this homomorphism, the same arguments used in Lemma 5.4 will show that ϕ_{id} is a τ-homomorphism from $\widehat{G}(\mathcal{A})$ to $G_{\mathcal{J}}$. Finally, by setting $a^{\mathcal{J}} = a$ we obtain $\phi_{id}(a) = a^{\mathcal{J}}$ for all $a \in \mathsf{Ind}(\mathcal{A})$. Thus, the application of Theorem 3.9 yields $\mathcal{J} \models \mathcal{A}$.

Now, similarly as for $\widehat{C}$ in Lemma 5.4, the size of $\mathcal{I}_0$ is bounded by the size of $\mathcal{A}$ without counting the threshold concepts. Moreover, threshold concepts occurring in concept assertions of $\mathcal{A}$ are also used at most once to build $\mathcal{J}$. Thus, it easily follows that $|\Delta^{\mathcal{J}}| \leq \mathsf{s}(\mathcal{A})$. □

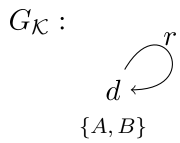
Using this lemma we can design an NP-algorithm to decide the consistency problem. The algorithm guesses an interpretation $\mathcal{J}$ of size at most $\mathsf{s}(\mathcal{A})$. Afterwards, it checks using Algorithm 3 in polynomial time, whether there exists a τ-homomorphism ϕ from $\widehat{G}(\mathcal{A})$ to $G_{\mathcal{J}}$ with $\phi(a) = a^{\mathcal{J}}$ for all $a \in \mathsf{Ind}(\mathcal{A})$.

We now turn into the instance checking problem. The model $\mathcal{J}$ of $\mathcal{A}$ obtained in the previous lemma can be used as starting point to obtain a bounded model property for *non-instance*, i.e., a is not an instance of $\widehat{C}$ with respect to $\mathcal{A}$ iff $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$ is consistent. However, different from the interpretation $\mathcal{J}_0$ used in the construction of Lemma 5.6, the bounded model for an ABox obtained in Lemma 5.9 does not necessarily have a tree shape. As a consequence, using the procedure described in Lemma 5.6 to construct a model $\mathcal{J}$ of $\widehat{C} \sqcap \neg\widehat{D}$ would require to consider nodes from $\Delta^{\mathcal{J}_0}$ more than one time.

Example 5.10. Let E be the $\mathcal{EL}$ concept description $\exists r.A \sqcap \exists r.B$. Consider the following ABox $\mathcal{A}$ and $\tau\mathcal{EL}(deg)$ concept description $\widehat{C}$:

$$\mathcal{A} := \{r(a,a)\} \text{ and } \widehat{C} := \underbrace{\exists r \ldots \exists r}_{p}.E_{<1}$$

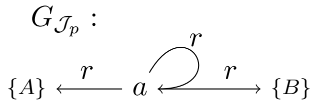
It is easy to see that a is not an instance of $\widehat{C}$ with respect to $\mathcal{A}$. The following single-pointed interpretation $\mathcal{K}$ (with $a^{\mathcal{K}} = d$) is a model of $\mathcal{A}$ not satisfying $\widehat{C}(a)$.



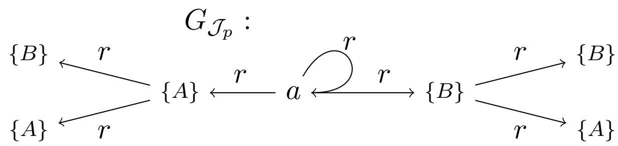
This means that $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$ is consistent. Let us now try to adapt the construction in Lemma 5.6 to $\mathcal{A}$ and $\neg\widehat{C}(a)$. It starts by choosing $\mathcal{J}_0$ as the bounded model of $\mathcal{A}$ given by Lemma 5.9. Such a model has a similar shape as $\mathcal{K}$, but with $A^{\mathcal{J}_0} = B^{\mathcal{J}_0} = \emptyset$. The iteration is then guided by an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{A}$ and $a^{\mathcal{I}} \notin \widehat{C}$, and generates the following sequence of sets:

$$\begin{aligned} S_0 &= \{(a, \widehat{C})\} \\ &\ldots \\ S_i &= \{(a, \underbrace{\exists r \ldots \exists r}_{p-i}.E_{<1})\} \quad (1 \leq i < p) \\ &\ldots \\ S_p &= \{(a, E_{<1})\} \\ S_{p+1} &= \emptyset \end{aligned}$$

One can see that the iteration still terminates. The difference now is that not being $G_{\mathcal{J}_0}$ a tree, the element a is considered several times. In particular, since $S_p = \{(a, E_{<1})\}$ and $a \in (E_{<1})^{\mathcal{J}_0}$, this means that $\mathcal{J}_0$ will be extended by adding a canonical interpretation which has the same description tree as E:



Since $\varphi(a) \notin (E_{<1})^{\mathcal{I}}$, this will ensure that $a \notin (E_{<1})^{\mathcal{J}_p}$. Unfortunately this is not sufficient to achieve $a \notin \widehat{C}^{\mathcal{J}_p}$. The set S_{p-1} contains the pair $(a, \exists r.E_{<1})$, which intuitively asks for a to satisfy $a \notin (\exists r.E_{<1})^{\mathcal{J}_p}$. Clearly, the addition of the two r-successors of a implies that this is not the case. To repair this new problem, the natural extension of the procedure is to reconsider S_{p-1} with respect to the newly added elements. Such a repetition would then yield the following interpretation:



Note that after fixing the problem for S_{p-1}, the same issue will arise with respect to $(a, \exists r \exists r.E_{<1}) \in S_{p-2}$ and so on. Therefore, whenever a node v requires the addition of a canonical interpretation and has additional constraints (as just explained), the same idea needs to be recursively applied with respect to its new successors and those constraints.

Finally, one can see that this recursive application of the procedure leads to a model of size exponential in the size of $\widehat{C}$. This, however, does not necessarily imply that this is the best bound we can hope for. In fact, as illustrated above, $\mathcal{K}$ is a very small model satisfying $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$. It is just that the procedure does not realize that a can be an instance of A and B in $\mathcal{J}_0$ without contradicting $\mathcal{J}_0 \models \mathcal{A}$. We do not yet know whether there is a better bound which applies to all possible combinations of $\mathcal{A}$ and $\widehat{C}$. ♢

Based on the intuition given in Example 5.10, we extend the construction from Lemma 5.6 to ABoxes of the form $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$. We introduce a set of rules to transform $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$ into an ABox $\mathcal{A}'$, which contains additional assertions that (when consistent with $\mathcal{A}$) are sufficient to falsify $\widehat{C}(a)$ in a model of $\mathcal{A}$. These rules are similar to some of the *pre-processing* rules defined in [BH91, Hol96], with the addition of specific rules to deal with the negation of threshold concepts. For the rest of this section we will use ABoxes that may also contain assertions of the form $\neg\widehat{C}(a)$. In case we want to refer to an ABox strictly in $\tau\mathcal{EL}(deg)$ we will mention it explicitly.

Definition 5.11 (*pre-processing rules*). Let $\mathcal{A}$ be an ABox. We define the following pre-processing rules:

- $\mathcal{A} \rightarrow_{\neg\sqcap} \mathcal{A} \cup \{\neg\widehat{D}(a)\}$
  if $\neg\widehat{C}(a) \in \mathcal{A}$ where $\widehat{C}$ is of the form $\widehat{C}_1 \sqcap \ldots \sqcap \widehat{C}_n$, $\neg\widehat{C}_i(a) \notin \mathcal{A}$ for all $i \in \{1 \ldots n\}$ and $\widehat{D} = \widehat{C}_i$ for some $i \in \{1 \ldots n\}$.
- $\mathcal{A} \rightarrow_{\neg\exists} \mathcal{A} \cup \{\neg\widehat{D}(b)\}$
  if $(\neg\exists r.\widehat{D})(a) \in \mathcal{A}$, $r(a,b) \in \mathcal{A}$ and $\neg\widehat{D}(b) \notin \mathcal{A}$.
- $\mathcal{A} \rightarrow_{\neg\sim} \mathcal{A} \cup \{E_{\chi(\sim)t}(a)\}$
  if $\neg E_{\sim t}(a) \in \mathcal{A}$ and $E_{\chi(\sim)t}(a) \notin \mathcal{A}$.

- $\mathcal{A} \rightarrow_{\neg A} \mathcal{A} \cup \{A_{<1}(a)\}$
  if $A \in \mathsf{N_C}$, $\neg A(a) \in \mathcal{A}$ and $A_{<1}(a) \notin \mathcal{A}$. ♢

A *pre-processing* of $\mathcal{A}$ is an ABox $\mathcal{A}'$ obtained by a sequence of rule applications such that no further rule application is possible over $\mathcal{A}'$. Note that if $\mathcal{A}$ is a $\tau\mathcal{EL}(deg)$ ABox, the unique *pre-processing* of $\mathcal{A}$ is $\mathcal{A}$ itself. The rules $\rightarrow_{\neg\sim}$ and $\rightarrow_{\neg A}$ are supported by the equivalences $\neg E_{\sim t} \equiv E_{\chi(\sim)t}$ and $\neg A \equiv A_{<1}$ (see Proposition 4.15 and Chapter 3). The rule $\rightarrow_{\neg\sqcap}$ has a non-deterministic flavor. It can be seen as the counterpart of the guided choice made in Lemma 5.6 to obtain a set S_i^*. Regarding $\rightarrow_{\neg\exists}$, it has a similar aim as the construction of S_i from S_i^* in Lemma 5.6.

One can see that a rule application introduces neither a new individual nor a new role assertion. Therefore, $\mathcal{A}$ and $\mathcal{A}'$ have the same set of individuals and role assertions. Furthermore, only new assertions of the form $\neg\widehat{C}(a)$, $E_{\chi(\sim)t}(a)$ or $A_{<1}(a)$ results from a rule application. In the first case $\widehat{C}$ is a *sub-description* of some concept $\widehat{D}$ such that $\neg\widehat{D}(b)$ is an assertion initially in $\mathcal{A}$, whereas no rule is applicable to the other two cases. Hence, since $\mathcal{A}$ is finite, there can never be an infinite sequence of rule applications.

Now, we can prove the following proposition (see Appendix A).

Proposition 5.12. *Let $\mathcal{A}$ be an ABox. Then, $\mathcal{A}$ is consistent iff there exists a consistent pre-processing $\mathcal{A}'$ of $\mathcal{A}$.*

The following remark is a direct consequence from the proof of the previous proposition.

Remark 5.13. Let $\mathcal{A}$ be an ABox and $\mathcal{I}$ an interpretation. If $\mathcal{I} \models \mathcal{A}$, then there exists a pre-processing $\mathcal{A}'$ of $\mathcal{A}$ such that $\mathcal{I} \models \mathcal{A}'$. ♢

Before moving on to the last results of the section, it would be useful to introduce some notation. An ABox containing only one individual name and no role assertions is called a *single-element* ABox. Additionally, given an ABox $\mathcal{A}$, the ABox $\mathcal{A}(a)$ consists of all the concept assertions $\widehat{D}(a)$ or $\neg\widehat{D}(a)$ occurring in $\mathcal{A}$. Furthermore, $\mathcal{A}^+$ is defined as:

$$\mathcal{A}^+ := \bigcup_{\substack{\widehat{D}(a)\in\mathcal{A}\\ a\in\mathsf{Ind}(\mathcal{A})}} \{\widehat{D}(a)\} \cup \bigcup_{\substack{r(a,b)\in\mathcal{A}\\ a,b\in\mathsf{Ind}(\mathcal{A})}} \{r(a,b)\}$$

and $\mathcal{A}^-$ is defined as:

$$\mathcal{A}^- := \bigcup_{\substack{\neg\widehat{D}(a)\in\mathcal{A}\\ a\in\mathsf{Ind}(\mathcal{A})}} \{\neg\widehat{D}(a)\}$$

We are now ready to show a bounded model property for consistency of ABoxes of the form $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$. The proof consists of three lemmas. Given a consistent ABox $\mathcal{A}$ and an arbitrary model $\mathcal{I}$ of it, one can select a *pre-processing* $\mathcal{A}'$ of $\mathcal{A}$ such that $\mathcal{I} \models \mathcal{A}'$. In particular, we focus on the ABoxes $\mathcal{A}'(a)$ for all $a \in \mathsf{Ind}(\mathcal{A})$. Even when such ABoxes may contain assertions over negated concepts, they are nevertheless simpler than $\mathcal{A}$, in the sense that only contain one individual name and no role assertions. Our first step is to show how to provide a model of bounded size for this particular type of ABox. The following lemma offers such a construction (its proof is deferred to the Appendix A).

Lemma 5.14. *Let $\mathcal{A}$ be a consistent single-element ABox and $\mathcal{I}$ an interpretation such that $\mathcal{I} \models \mathcal{A}$. In addition, let $\mathcal{J}$ be the bounded model of $\mathcal{A}^+$ obtained in Lemma 5.9 with respect to $\mathcal{I}$. Then, there exists a tree-shaped interpretation $\mathcal{K}$ such that:*

1. $\mathcal{K} \models \mathcal{A}$,
2. *there exists a homomorphism φ from $G_\mathcal{K}$ to $G_\mathcal{I}$ with $\varphi(a^\mathcal{K}) = a^\mathcal{I}$, and*
3. $|\Delta^\mathcal{K}| \leq |\Delta^\mathcal{J}| \times p$, *where:*

$$p := \begin{cases} 1, & \textit{if } \mathcal{A}^- = \emptyset \\ \prod\limits_{\neg\widehat{D}(a)\in\mathcal{A}^-} \mathsf{s}(\widehat{D}), & \textit{otherwise.} \end{cases}$$

Once the previous lemma is applied to all the ABoxes $\mathcal{A}'(a)$, the second step is to combine all those models into a model of $\mathcal{A}$ of bounded size. More precisely, we show that the disjoint union of all those models together with the role assertions in $\mathcal{A}$ yield the wanted model. This is formalized in the following lemma (see Appendix A for its proof).

Lemma 5.15. *Let $\mathcal{A}$ be an ABox, $\mathcal{I}$ an interpretation satisfying $\mathcal{A}$ and $\mathcal{A}'$ a pre-processing of $\mathcal{A}$ such that $\mathcal{I} \models \mathcal{A}'$. Moreover, for all $a \in \mathsf{Ind}(\mathcal{A})$, let $\mathcal{I}_a$ be a tree-shaped interpretation satisfying the following:*

- $\mathcal{I}_a \models \mathcal{A}'(a)$,
- *there exists a homomorphism φ_a from $G_{\mathcal{I}_a}$ to $G_\mathcal{I}$ with $\varphi_a(a^{\mathcal{I}_a}) = a^\mathcal{I}$.*

Last, let $\mathcal{J}$ be the following interpretation:

- $\Delta^\mathcal{J} := \bigcup\limits_{a\in\mathsf{Ind}(\mathcal{A})} \Delta^{\mathcal{I}_a}$,
- $A^\mathcal{J} := \bigcup\limits_{a\in\mathsf{Ind}(\mathcal{A})} A^{\mathcal{I}_a}$ *for all* $A \in \mathsf{N_C}$,
- $r^\mathcal{J} := \{(a^{\mathcal{I}_a}, b^{\mathcal{I}_b}) \mid r(a,b) \in \mathcal{A}\} \cup \bigcup\limits_{a\in\mathsf{Ind}(\mathcal{A})} r^{\mathcal{I}_a}$ *for all* $r \in \mathsf{N_R}$, *and*
- $a^\mathcal{J} := a^{\mathcal{I}_a}$, *for all* $a \in \mathsf{Ind}(\mathcal{A})$.

where the sets $\Delta^{\mathcal{I}_a}$ are pairwise disjoint. Then, $\mathcal{J} \models \mathcal{A}$.

Using these two lemmas we can now established the final result. Recall that $\mathsf{sub}(\widehat{C})$ denotes the set of sub-descriptions of a concept description $\widehat{C}$.

Lemma 5.16. *Let $\mathcal{A}$ be an ABox in $\tau\mathcal{EL}(deg)$ of size $\mathsf{s}(\mathcal{A})$, $\widehat{C}$ a $\tau\mathcal{EL}(deg)$ concept description of size $\mathsf{s}(\widehat{C})$ and $a \in \mathsf{N_I}$. If $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$ is consistent, then there exists an interpretation $\mathcal{J}$ such that:*

1. $\mathcal{J} \models \mathcal{A} \cup \{\neg\widehat{C}(a)\}$,

2. *$\mathcal{J}$ is the result of the construction in Lemma 5.15,*

3. *for all $a \in \mathsf{Ind}(\mathcal{A})$:*

$$|\Delta^{\mathcal{I}_a}| \leq \mathsf{s}(\mathcal{A}(a)) \times [\mathsf{s}(\widehat{C})]^u, \text{ where } u = |\mathsf{sub}(\widehat{C})|$$

Proof. Let $\mathcal{I}$ be an interpretation satisfying $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$. By Remark 5.13 there exists a *pre-processing* $\mathcal{A}'$ of $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$ such that $\mathcal{I} \models \mathcal{A}'$. We apply Lemma 5.14 to $\mathcal{A}'(a)$ for all $a \in \mathsf{Ind}(\mathcal{A})$, and obtain a *tree-shaped* interpretation $\mathcal{I}_a$ such that:

- $\mathcal{I}_a \models \mathcal{A}'(a)$,
- there exists a homomorphism φ_a from $G_{\mathcal{I}_a}$ to $G_{\mathcal{I}}$ with $\varphi_a(a^{\mathcal{I}_a}) = a^{\mathcal{I}}$.

Then, we can apply Lemma 5.15 to obtain an interpretation $\mathcal{J}$ such that:

$$\mathcal{J} \models \mathcal{A} \text{ and } \Delta^{\mathcal{J}} = \bigcup_{a \in \mathsf{Ind}(\mathcal{A})} \Delta^{\mathcal{I}_a}$$

We now look at the size of $\mathcal{I}_a$. For all $a \in \mathsf{Ind}(\mathcal{A})$, let $\mathcal{J}_a$ denote the bounded model of ${\mathcal{A}'}^+(a)$ obtained in Lemma 5.9 with respect to $\mathcal{I}$. The construction of $\mathcal{I}_a$ in Lemma 5.14 yields:

$$|\Delta^{\mathcal{I}_a}| \leq |\Delta^{\mathcal{J}_a}| \times \prod_{\neg\widehat{D}(a) \in {\mathcal{A}'}^-(a)} \mathsf{s}(\widehat{D}) \tag{5.3}$$

One can see that each assertion in ${\mathcal{A}'}^+(a)$ is either of the form $\widehat{D}(a) \in \mathcal{A}(a)$ or $E_{\chi(\sim)t}$. The latter case results from applications of the rules $\rightarrow_{\neg\sim}$ and $\rightarrow_{\neg A}$. For the rule $\rightarrow_{\neg A}$, $A_{<1}$ corresponds to $A_{\chi(\geq)1}$. In Lemma 5.9, the interpretation $\mathcal{J}_a$ is built starting with the interpretation $\mathcal{I}_0$ which have the description graph $\widehat{G}({\mathcal{A}'}^+(a)) = (V_{{\mathcal{A}'}^+(a)}, \ldots)$ (without threshold concepts), and it is then extended by considering the threshold concepts occurring in $\widehat{G}({\mathcal{A}'}^+(a))$. We know the following about them:

- for all threshold concepts $E_{\sim t}$ occurring in ${\mathcal{A}'}^+(a)$, either $E_{\sim t}$ occurs in an assertion of $\mathcal{A}(a)$ or it has been introduced by an application of $\rightarrow_{\neg\sim}$ or $\rightarrow_{\neg A}$ (i.e., it is of the form $E_{\chi(\sim)t}(a)$),
- except for assertions of the form $E_{\chi(\sim)t}(a)$,

$$\widehat{D}(a) \in {\mathcal{A}'}^+(a) \text{ iff } \widehat{D}(a) \in \mathcal{A}(a)$$

Thus, $|V_{\mathcal{A}(a)}| = |V_{{\mathcal{A}'}^+(a)}|$ and by construction of $\mathcal{J}_a$ we obtain:

$$|\Delta^{\mathcal{J}_a}| \leq |V_{\mathcal{A}(a)}| + \sum_{E_{\sim t} \in \widehat{G}(\mathcal{A}(a))} \mathsf{s}(E_{\sim t}) + \sum_{E_{\chi(\sim)t}(a) \notin \widehat{G}(\mathcal{A}(a))} \mathsf{s}(E_{\chi(\sim)t})$$

Note that the partial sum of the first two elements in the right-hand side of the inequality is actually bounded by the size of $\mathcal{A}(a)$. In addition, since $\mathsf{s}(E_{\chi(\sim)t}) > 1$ we further have:

$$|\Delta^{\mathcal{J}_a}| \leq \mathsf{s}(\mathcal{A}(a)) \times \prod_{E_{\chi(\sim)t}(a) \notin \widehat{G}(\mathcal{A}(a))} \mathsf{s}(E_{\chi(\sim)t}) \tag{5.4}$$

It is not hard to see that for all $\neg\widehat{D}(a) \in \mathcal{A}'^{-}(a)$ and $E_{\chi(\sim)t}(a) \notin \widehat{G}(\mathcal{A}(a))$, the concepts $\widehat{D}$ and $E_{\sim t}$ (or A for $\rightarrow_{\neg A}$) are sub-descriptions of $\widehat{C}$. Hence, the combinations of inequalities (5.3) and (5.4) yields:

$$|\Delta^{\mathcal{I}_a}| \leq \mathsf{s}(\mathcal{A}(a)) \times [\mathsf{s}(\widehat{C})]^u$$ □

Based on the previous results, we devise the following non-deterministic procedure to decide consistency of an ABox of the form $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$.

1. For all $a \in \mathsf{Ind}(\mathcal{A})$, guess an interpretation $\mathcal{I}_a$ of size at most:

$$\mathsf{s}(\mathcal{A}(a)) \times [\mathsf{s}(\widehat{C})]^u$$

2. Construct $\mathcal{J}$ using all the interpretations $\mathcal{I}_a$ and $\mathcal{A}$, as described in Lemma 5.15.

3. Check whether $\mathcal{J} \models \mathcal{A}$. This can be done in polynomial time (in the size of $\mathcal{J}$ and $\mathcal{A}$) by using Algorithm 3. If it is not the case, then the algorithm answers "no". Otherwise, it remains to verify whether $a^{\mathcal{J}} \notin \widehat{C}^{\mathcal{J}}$.

4. To verify $a^{\mathcal{J}} \notin \widehat{C}^{\mathcal{J}}$, by Theorem 3.8 it is enough to check that there is no τ-homomorphism ϕ from $T_{\widehat{C}}$ to $G_{\mathcal{J}}$ with $\phi(v_0) = a^{\mathcal{J}}$. This can also be checked in polynomial time by Algorithm 2. If there is no such τ-homomorphism the algorithm answers "yes", and "no"otherwise.

If the size of $\widehat{C}$ is considered as a constant, this algorithm becomes an NP-procedure for consistency of $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$, and consequently a coNP-procedure to decide instance checking with respect to data complexity. Altogether, we thus have shown:

Theorem 5.17. *In $\tau\mathcal{EL}(deg)$, consistency is NP-complete, and instance checking is coNP-complete w.r.t. data complexity.*

The instance problem becomes simpler if we consider only $\mathcal{EL}$ ABoxes and *positive* $\tau\mathcal{EL}(deg)$ concept descriptions, i.e., concept descriptions $\widehat{C}$ that only contain threshold concepts of the form $E_{\geq t}$ or $E_{>t}$. Basically, given an $\mathcal{EL}$ ABox $\mathcal{A}$, a positive $\tau\mathcal{EL}(deg)$ concept description $\widehat{C}$, and an individual a, one considers the interpretation $\mathcal{I}$ corresponding to the description graph $G(\mathcal{A})$ of $\mathcal{A}$, and then checks whether there is a τ-homomorphism ϕ from $T_{\widehat{C}}$ to $G_{\mathcal{I}}$ with $\phi(v_0) = a$. The following lemma supports the previous idea.

Lemma 5.18. *Let $\mathcal{A}$ be an $\mathcal{EL}$ ABox, $a \in \mathsf{Ind}(\mathcal{A})$ and $\widehat{C}$ a positive $\tau\mathcal{EL}(deg)$ concept description. Additionally, let $\mathcal{I}_{\mathcal{A}}$ be the interpretation corresponding to the description graph $G(\mathcal{A})$ with $a^{\mathcal{I}_{\mathcal{A}}} = a$ for all $a \in \mathsf{Ind}(\mathcal{A})$. Then, the following statements are equivalent:*

1. $\mathcal{A} \models \widehat{C}(a)$*, and*

2. $a \in \widehat{C}^{\mathcal{I}_\mathcal{A}}$.

Proof. 1) $\rightarrow$ 2). Assume that $\mathcal{A} \models \widehat{C}(a)$. Then, for every model $\mathcal{I}$ of $\mathcal{A}$ we have $a^\mathcal{I} \in \widehat{C}^\mathcal{I}$. Since $\mathcal{I}_\mathcal{A}$ is obviously a model of $\mathcal{A}$ and $a^{\mathcal{I}_\mathcal{A}} = a$, this means that $a \in \widehat{C}^{\mathcal{I}_\mathcal{A}}$.

2) $\rightarrow$ 1). Assume that $a \in \widehat{C}^{\mathcal{I}_\mathcal{A}}$. The characterization for membership in $\tau\mathcal{EL}(deg)$ given in Theorem 3.8, yields a τ-homomorphism ϕ from $T_{\widehat{C}}$ to $G(\mathcal{A})$ with $\phi(v_0) = a$. Now, consider any model $\mathcal{I}$ of $\mathcal{A}$. The application of Theorem 3.9 yields the existence of a τ-homomorphism φ from $G(\mathcal{A})$ to $G_\mathcal{I}$ such that $\varphi(a) = a^\mathcal{I}$ for all $a \in \mathsf{Ind}(\mathcal{A})$. We then show that the mapping $\varphi \circ \phi$ is a τ-homomorphism from $T_{\widehat{C}}$ to $G_\mathcal{I}$:

- From ϕ we know that $\ell_{T_{\widehat{C}}}(v) \subseteq \ell_\mathcal{A}(\phi(v))$ for all $v \in V_{T_{\widehat{C}}}$. Similarly, φ implies that $\ell_\mathcal{A}(a) \subseteq \ell_\mathcal{I}(\varphi(a))$ for all $a \in V_\mathcal{A}$. Hence, $\ell_{T_{\widehat{C}}}(v) \subseteq \ell_\mathcal{I}((\varphi \circ \phi)(v))$ for all $v \in V_{T_{\widehat{C}}}$. The edge preserving relation can be verified in a similar way.

- Let $v \in V_{T_{\widehat{C}}}$ and $E_{\sim t} \in \widehat{\ell}_{T_{\widehat{C}}}(v)$. Since ϕ is a τ-homomorphism, this means that $\phi(v) \in (E_{\sim t})^{\mathcal{I}_\mathcal{A}}$. Furthermore, the application of Lemma 4.11 to $\mathcal{I}_\mathcal{A}$, $\mathcal{J}$ and φ yields:
$$deg^{\mathcal{I}_\mathcal{A}}(\phi(v), E) \leq deg^\mathcal{I}(\varphi(\phi(v)), E)$$
Since $\widehat{C}$ is positive, this means that $\sim$ is either $>$ or $\geq$. Consequently, $(\varphi \circ \phi)(v) \in (E_{\sim t})^\mathcal{I}$.

Hence, $\varphi \circ \phi$ is a τ-homomorphism from $T_{\widehat{C}}$ to $G_\mathcal{I}$ with $(\varphi \circ \phi)(v_0) = a^\mathcal{I}$. Altogether, this means that $a^\mathcal{I} \in \widehat{C}^\mathcal{I}$ for all models $\mathcal{I}$ of $\mathcal{A}$. Thus, $\mathcal{A} \models \widehat{C}(a)$. □

Finally, since $\mathcal{I}_\mathcal{A}$ is linear on the size of $\mathcal{A}$, checking whether $a \in \widehat{C}^{\mathcal{I}_\mathcal{A}}$ can be done in polynomial time in the size of $\mathcal{A}$ and $\widehat{C}$ by using Algorithm 2. Therefore, we obtain the following proposition.

Proposition 5.19. *For* positive $\tau\mathcal{EL}(deg)$ *concept descriptions and* $\mathcal{EL}$ *ABoxes, the instance checking problem can be decided in polynomial time.*

Chapter 6

Adding Terminologies to $\tau\mathcal{EL}(m)$

Until now, we have only considered the basic concept language and background knowledge represented in the form of assertions about specific individuals. Nevertheless, most DLs also allows to store terminological knowledge about the application domain in a TBox. The aim of this chapter is to take initial steps in extending our logic $\tau\mathcal{EL}(deg)$ towards background knowledge represented in the form of axioms in a TBox.

We start by introducing $\mathcal{EL}$ TBoxes, and some related properties and technical notions concerning them. We will then turn to the definition of $\tau\mathcal{EL}(m)$ and $\tau\mathcal{EL}(deg)$ TBoxes. To accomplish this, there are two important aspects that we take into account. On the one hand, graded membership functions will now compute membership degrees to $\mathcal{EL}$ concepts defined with respect to an $\mathcal{EL}$ TBox. To handle this, we propose a general way to extend all functions m through unfolding, and hence restrict threshold concepts $E_{\sim t}$ to have E defined with respect to an acyclic $\mathcal{EL}$ TBox. On the other hand, since we also intend to use TBoxes to define $\tau\mathcal{EL}(m)$ concept descriptions, further constraints are needed to exclude definitions of not well-formed $\tau\mathcal{EL}(m)$ concepts.

Once $\tau\mathcal{EL}(deg)$ TBoxes are defined, we direct our attention to study the computational complexity of reasoning in the presence of acyclic $\tau\mathcal{EL}(deg)$ TBoxes. It turns out that the possibility of succinctly representing exponentially large concept descriptions in a TBox, combined with the semantics of threshold concepts in $\tau\mathcal{EL}(deg)$, makes *satisfiability* and *subsumption* to be Π_2^P- and Σ_2^P-hard, respectively. Additionally, we provide a *sound* and *complete* non-deterministic PSPACE procedure to solve both problems, and later extend it to also consider assertional knowledge in an ABox. Such an extension keeps the use of space polynomial, an thus yields a PSPACE upper bound for all the standard reasoning tasks (including instance checking w.r.t. *combined complexity*).

6.1 $\mathcal{EL}$ TBoxes

A *concept definition* is of the form $A \doteq C_A$, where A is a concept name and C_A an $\mathcal{EL}$ concept description. An $\mathcal{EL}$ TBox $\mathcal{T}$ is a finite set of concept definitions such that no concept name occurs more than once on the left-hand side of a definition in $\mathcal{T}$. Concept names occurring on the left hand side of a definition are called *defined concepts* while all other concept names are called *primitive concepts*. The sets of defined and primitive concepts are denoted as N_{def} and N_{prim}, respectively. A knowledge base (KB) $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ consists of a TBox and an ABox.

We denote by $\mathfrak{T}$ the set of all $\mathcal{EL}$ TBoxes. Given $\mathcal{T} \in \mathfrak{T}$, $\mathsf{def}(\mathcal{T})$ stands for the set of defined concepts in $\mathcal{T}$. Moreover, the size $\mathsf{s}(\mathcal{T})$ of $\mathcal{T}$ corresponds to the following expression:

$$\mathsf{s}(\mathcal{T}) := |\mathsf{def}(\mathcal{T})| + \sum_{A \doteq C_A \in \mathcal{T}} \mathsf{s}(C_A)$$

Finally, the size $\mathsf{s}(\mathcal{K})$ of a KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is simply $\mathsf{s}(\mathcal{T}) + \mathsf{s}(\mathcal{A})$.

Concerning the semantics, an interpretation $\mathcal{I}$ is a model of a TBox $\mathcal{T}$ (in symbols $\mathcal{I} \models \mathcal{T}$) iff

$$A^{\mathcal{I}} = (C_A)^{\mathcal{I}} \text{ for all } A \doteq C_A \in \mathcal{T}$$

We denote as $\mathfrak{T}(\mathcal{I}) \subseteq \mathfrak{T}$ the set of all $\mathcal{EL}$ TBoxes $\mathcal{T}$ such that $\mathcal{I} \models \mathcal{T}$. The satisfaction relation for KBs is defined in the usual way: $\mathcal{I}$ is a model of a KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ iff $\mathcal{I}$ satisfies both $\mathcal{T}$ and $\mathcal{A}$. Now, given two $\mathcal{EL}$ concept descriptions C and D, C is satisfiable with respect to $\mathcal{K}$ iff $C^{\mathcal{I}} \neq \emptyset$ for some model $\mathcal{I}$ of $\mathcal{K}$. In addition, we say that C is subsumed by D with respect to $\mathcal{K}$ (denoted $C \sqsubseteq_{\mathcal{K}} D$) iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for all models $\mathcal{I}$ of $\mathcal{K}$. They are equivalent with respect to $\mathcal{K}$ (denoted $C \equiv_{\mathcal{K}} D$) iff $C \sqsubseteq_{\mathcal{K}} D$ and $D \sqsubseteq_{\mathcal{K}} C$.

TBoxes can be classified regarding the dependencies between its concept definitions. More precisely,

Definition 6.1 ($\mathcal{EL}$ cyclic/acyclic TBoxes). Let $\mathcal{T}$ be an $\mathcal{EL}$ TBox. We define $\rightarrow$ as a binary relation over the set $\mathsf{def}(\mathcal{T})$ to represent *direct dependency* between defined concepts in the following way.

A defined concept A directly depends on a defined concept B (denoted as $A \rightarrow B$) iff $A \doteq C_A \in \mathcal{T}$ and B occurs in C_A. Let $\rightarrow^+$ be the transitive closure of $\rightarrow$. The TBox $\mathcal{T}$ contains a terminological cycle iff there exists a defined concept A in $\mathcal{T}$ that depends on itself, i.e., $A \rightarrow^+ A$. Then, $\mathcal{T}$ is called *cyclic* if it contains a terminological cycle. Otherwise, it is called *acyclic*. ♢

For acyclic TBoxes, the relation $\rightarrow^+$ induces a *well-founded* partial order $\preceq$ on the set $\mathsf{def}(\mathcal{T})$, i.e., $A \preceq B$ iff $B \rightarrow^+ A$. Furthermore, the unfolding $u_{\mathcal{T}}(C)$ of an $\mathcal{EL}$ concept description C with respect to $\mathcal{T}$ can be defined as follows:

$$\begin{aligned}
u_{\mathcal{T}}(C \sqcap D) &:= u_{\mathcal{T}}(C) \sqcap u_{\mathcal{T}}(D) \\
u_{\mathcal{T}}(\exists r.C) &:= \exists r.u_{\mathcal{T}}(C) \\
u_{\mathcal{T}}(A) &:= \begin{cases} A & \text{if } A \in \mathsf{N}_{prim}, \\ u_{\mathcal{T}}(C_A) & \text{if } A \doteq C_A \in \mathcal{T}. \end{cases}
\end{aligned}$$

It is well known that, regarding acyclic TBoxes, the meaning of concept descriptions follows directly from the meaning of their corresponding unfolded descriptions. The following is the equivalent, for $\mathcal{EL}$, of Proposition 1 in [Neb90].

Proposition 6.2. *For every acyclic $\mathcal{EL}$ TBox $\mathcal{T}$, every $\mathcal{EL}$ concept description C and every model $\mathcal{I}$ of $\mathcal{T}$:*

$$C^{\mathcal{I}} = [u_{\mathcal{T}}(C)]^{\mathcal{I}}$$

As an immediate consequence of this equality, we obtain $C \equiv_{\mathcal{T}} u_{\mathcal{T}}(C)$. This also has its counterpart from the model-theoretical point of view. Similar to Proposition 2 in [Neb91], we have the following for $\mathcal{EL}$.

Proposition 6.3. *Let $\mathcal{T}$ be an acyclic $\mathcal{EL}$ TBox. Any interpretation $\mathcal{J}$ of N_{prim} and $\mathsf{N_R}$ can be uniquely extended to a model of $\mathcal{T}$.*

These type of partial interpretations are called *primitive*. They do not assign any meaning to the defined concepts in $\mathcal{T}$. We say that an interpretation $\mathcal{I}$ is based on a primitive interpretation $\mathcal{J}$ iff it has the same domain as $\mathcal{J}$ and coincides with $\mathcal{J}$ on $\mathsf{N_R}$ and N_{prim}.

6.2 TBoxes for $\tau\mathcal{EL}(m)$ and $\tau\mathcal{EL}(deg)$

We would now like to use sets of concept definitions to define $\tau\mathcal{EL}(m)$ concept descriptions. These concept definitions come in two forms. For example, the $\mathcal{EL}$ concept definition $E \doteq \exists r.A \sqcap \exists r.B$, can be used to build the threshold concept $E_{\leq .8}$. Furthermore, on top of that, one could also have $\tau\mathcal{EL}(m)$ concept definitions of the form:

$$\alpha \doteq \widehat{C}_\alpha \tag{6.1}$$

where $\alpha \in \mathsf{N}_{def}$ and $\widehat{C}_\alpha$ is a $\tau\mathcal{EL}(m)$ concept description. For instance, the definition of E together with $\alpha \doteq A \sqcap E_{\leq .8}$, can be used to define the $\tau\mathcal{EL}(m)$ concept description $\exists s.A \sqcap \exists r.\alpha$.

In what follows, we first revisit the notion of membership degree functions from Definition 3.1 and define $\tau\mathcal{EL}(m)$ TBoxes. Afterwards, we provide a general way to extend such functions to consider concept descriptions defined in acyclic $\mathcal{EL}$ TBoxes. In particular, these two aspects are combined to extend our logic $\tau\mathcal{EL}(deg)$ towards $\tau\mathcal{EL}(deg)$ TBoxes. Last, preliminary aspects related to reasoning in $\tau\mathcal{EL}(deg)$ with respect to acyclic TBoxes are discussed as a starting point for subsequent sections.

The use of defined $\mathcal{EL}$ concepts to build threshold concepts compels us to revisit the definition of membership degree functions. In this new setting, the equivalence relation between concept descriptions is defined modulo the TBox definitions, i.e., $\equiv_\mathcal{T}$. Therefore, to maintain the equivalence invariance property, the condition *M2* from Definition 3.1 should be redefined with respect to $\equiv_\mathcal{T}$. This means that the definition of $m^\mathcal{I}$ with respect to the set of definitions in a TBox $\mathcal{T}$ only makes sense for models $\mathcal{I}$ of $\mathcal{T}$.

Definition 6.4. A graded membership function m is a family of functions that contains for every interpretation $\mathcal{I}$ a function $m^\mathcal{I} : \Delta^\mathcal{I} \times \mathcal{C}_\mathcal{EL} \times \mathfrak{T}(\mathcal{I}) \to [0,1]$ satisfying the following conditions (for $C, D \in \mathcal{C}_\mathcal{EL}$ and $\mathcal{T} \in \mathfrak{T}(\mathcal{I})$):

$$M1^\mathcal{T} : d \in C^\mathcal{I} \Leftrightarrow m^\mathcal{I}(d, C, \mathcal{T}) = 1 \text{ for all } d \in \Delta^\mathcal{I},$$
$$M2^\mathcal{T} : C \equiv_\mathcal{T} D \Leftrightarrow \forall \mathcal{I} \models \mathcal{T}\ \forall d \in \Delta^\mathcal{I} : m^\mathcal{I}(d, C, \mathcal{T}) = m^\mathcal{I}(d, D, \mathcal{T}).$$

Note that this is a generalization of Definition 3.1, where $\mathcal{T}$ is the empty TBox. ♢

Now, our idea of a $\tau\mathcal{EL}(m)$ TBox is to combine definitions from $\mathcal{EL}$ TBoxes with concept definitions having the form of (6.1). The following example shows that such a combination should not be arbitrary, for otherwise not every defined concept will be a well-formed $\tau\mathcal{EL}(m)$ concept description.

Example 6.5. Let $\mathcal{T}_{\mathcal{EL}}$ be the following $\mathcal{EL}$ TBox:

$$\mathcal{T}_{\mathcal{EL}} := \{E \doteq \exists r.A \sqcap \exists r.B\}$$

and $\mathcal{T}_\tau$ the following set of concept definitions:

$$\mathcal{T}_\tau := \left\{ \begin{array}{l} \alpha \doteq \exists s.A \sqcap \exists r.\beta \\ \beta \doteq A \sqcap E_{\leq .8} \end{array} \right\}$$

Here, the definition of E corresponds to the $\mathcal{EL}$ concept description $\exists r.A \sqcap \exists r.B$. Moreover, α and β define well-formed $\tau\mathcal{EL}(m)$ concept descriptions. For example, the unfolding of α with respect to $\mathcal{T}_\tau \cup \mathcal{T}_{\mathcal{EL}}$ yields the $\tau\mathcal{EL}(m)$ concept description $\exists s.A \sqcap \exists r.(A \sqcap (\exists r.A \sqcap \exists r.B)_{\leq .8})$. Suppose now, that α has the following definition in $\mathcal{T}_\tau$:

$$\alpha \doteq \exists s.A \sqcap \exists r.(\beta_{<1})$$

Even though $\widehat{C}_\alpha$ looks like a syntactically well-formed $\tau\mathcal{EL}(m)$ concept description, the unfolding of α yields $\exists s.A \sqcap \exists r.[A \sqcap (\exists r.A \sqcap \exists r.B)_{\leq .8}]_{<1}$ and the problem arise immediately: $[A \sqcap (\exists r.A \sqcap \exists r.B)_{\leq .8}]_{<1}$ is not a valid threshold concept. Consequently, α does not define a well-formed $\tau\mathcal{EL}(m)$ concept description. Thus, we should require for all $E_{\sim t}$ occurring in definitions of $\mathcal{T}_\tau$ that E is not defined in terms of any other threshold concept. $\diamondsuit$

Definition 6.6. Let $\{\mathsf{N}^\tau_{def}, \mathsf{N}^0_{def}\}$ be a partition of N_{def}. A $\tau\mathcal{EL}(m)$ TBox $\widehat{\mathcal{T}}$ is a pair $(\mathcal{T}_\tau, \mathcal{T}_{\mathcal{EL}})$ satisfying the following conditions:

- $\mathcal{T}_\tau$ is a set of concept definitions of the form $\alpha \doteq \widehat{C}_\alpha$ such that:
  - $\alpha \in \mathsf{N}^\tau_{def}$ and $\widehat{C}_\alpha$ is a $\tau\mathcal{EL}(m)$ concept description.
  - for all threshold concepts $E_{\sim t}$ occurring in a definition of $\mathcal{T}_\tau$, E is defined over $\mathsf{N}^0_{def} \cup \mathsf{N}_{prim}$.
- $\mathcal{T}_{\mathcal{EL}}$ is an $\mathcal{EL}$ TBox such that:
  - $E \in \mathsf{N}^0_{def}$, for all defined concepts in $\mathcal{T}_{\mathcal{EL}}$.
  - for all $\alpha \in \mathsf{N}^\tau_{def}$, α does not occur in any definition of $\mathcal{T}_{\mathcal{EL}}$. $\diamondsuit$

Restricting threshold concepts to be defined over $\mathsf{N}^0_{def} \cup \mathsf{N}_{prim}$ and $\mathcal{T}_{\mathcal{EL}}$ not to contain occurrences of defined concepts in $\mathcal{T}_\tau$ guarantees the α always defines a well-formed $\tau\mathcal{EL}(m)$ concept description for all $\alpha \doteq \widehat{C}_\alpha \in \mathcal{T}_\tau$.

Remark 6.7. We have not been very precise about well-formed $\tau\mathcal{EL}(m)$ concept descriptions externally defined over the signature of a $\tau\mathcal{EL}(m)$ TBox $\widehat{\mathcal{T}}$. Hereafter, we understand by that any string of symbols $\widehat{C}$ generated by the grammar in Chapter 3, such that $\widehat{C}$ complies with the same restrictions imposed on the defined concepts in $\widehat{\mathcal{T}}$. More precisely, the set of definitions $\widehat{\mathcal{T}} \cup \{\alpha \doteq \widehat{C}\}$ is still a $\tau\mathcal{EL}(m)$ TBox, where α is a fresh concept name from N^τ_{def}. $\diamondsuit$

Consequently, we define a $\tau\mathcal{EL}(m)$ knowledge base $\mathcal{K}$ as a pair $\mathcal{K} = (\widehat{\mathcal{T}}, \mathcal{A})$ where $\widehat{\mathcal{T}}$ is a $\tau\mathcal{EL}(m)$ TBox, and for all $\widehat{C}(a) \in \mathcal{A}$ the concept description $\widehat{C}$ is defined over

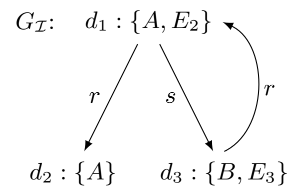
$\widehat{\mathcal{T}}$. As we will introduce next, the satisfaction relation between interpretations and KBs allows to replace the assertions $\widehat{C}(a)$ in $\mathcal{A}$ with $\alpha_{\widehat{C}}(a)$, by adding the concept definition $\alpha_{\widehat{C}} \doteq \widehat{C}$ to $\mathcal{T}_\tau$ where $\alpha_{\widehat{C}}$ is a fresh concept name from N^τ_{def}. Therefore, from now on we assume that all the concept assertions in $\mathcal{A}$ are of the form $\alpha(a)$ where $\alpha \in \mathsf{def}(\widehat{\mathcal{T}})$.

The satisfaction relation for $\tau\mathcal{EL}(m)$ TBoxes depends on the chosen function m. An interpretation $\mathcal{I}$ satisfies a $\tau\mathcal{EL}(m)$ TBox $\widehat{\mathcal{T}} = (\mathcal{T}_\tau, \mathcal{T}_{\mathcal{EL}})$ iff $\mathcal{I} \models \mathcal{T}_{\mathcal{EL}}$ and $\alpha^\mathcal{I} = (\widehat{C}_\alpha)^\mathcal{I}$ in $\tau\mathcal{EL}(m)$ for all $\alpha \doteq \widehat{C}_\alpha \in \mathcal{T}_\tau$. Finally, $\mathcal{I}$ satisfies a knowledge base $\mathcal{K} = (\widehat{\mathcal{T}}, \mathcal{A})$ iff $\mathcal{I} \models \widehat{\mathcal{T}}$ and $\mathcal{I} \models \mathcal{A}$.

We now turn to extending graded membership functions to deal with concept descriptions defined with respect to a background $\mathcal{EL}$ TBox $\mathcal{T}_{\mathcal{EL}}$. Since m is a parameter for the logic, besides the properties required in Definition 6.4, there is not much information about how defined concepts should be taken into account to compute membership degrees. Initially, one idea could be to treat defined concepts simply as concept names in the computation of m. This, however, would mean that the definition of a concept E in $\mathcal{T}_{\mathcal{EL}}$ is not really used to compute $m^\mathcal{I}(d, E, \mathcal{T}_{\mathcal{EL}})$ whenever $d \notin E^\mathcal{I}$, i.e., $m^\mathcal{I}(d, E, \mathcal{T}_{\mathcal{EL}}) = 0$, rather than giving a more approximate value based on the definition of E. The following example explains this situation for the membership function deg.

Example 6.8. Consider the following $\mathcal{EL}$ TBox $\mathcal{T}_{\mathcal{EL}}$ and interpretation $\mathcal{I}$:

$$\mathcal{T}_{\mathcal{EL}} := \left\{ \begin{array}{l} E_1 \doteq \exists r.A \sqcap \exists r.B \\ E_2 \doteq A \sqcap \exists s.E_3 \\ E_3 \doteq B \sqcap \exists r.E_2 \end{array} \right\}$$



Here, $\mathcal{I} \models \mathcal{T}_{\mathcal{EL}}$ and $d_1 \notin (E_1)^\mathcal{I}$. Treating E_1 as a concept name in the computation of deg yields $deg^\mathcal{I}(d_1, E_1, \mathcal{T}_{\mathcal{EL}}) = 0$. In other words, deg would ignore the fact that d_1 has an r-successor which is an instance of A. In contrast, using directly the definition of E_1, we have $deg^\mathcal{I}(d_1, \exists r.A \sqcap \exists r.B, \mathcal{T}_{\mathcal{EL}}) = 1/2$. This shows the limitations of treating defined concepts as concept names when computing deg, but more importantly it tells us that deg would then violate property $M2^\mathcal{T}$ since $E_1 \equiv_{\mathcal{T}_{\mathcal{EL}}} \exists r.A \sqcap \exists r.B$.

One way to repair this is to consider the unfolding $u_{\mathcal{T}_{\mathcal{EL}}}(E_1)$ of E_1 to compute $deg^\mathcal{I}(d_1, E_1, \mathcal{T}_{\mathcal{EL}})$, i.e., $deg^\mathcal{I}(d_1, E_1, \mathcal{T}_{\mathcal{EL}}) := deg^\mathcal{I}(d_1, u_{\mathcal{T}_{\mathcal{EL}}}(E_1))$. Obviously, this will not work for E_2 and E_3 since they are defined in a cyclic way, and this means that they cannot be unfolded. ♢

Based on the previous arguments, we extend the computation of graded membership functions towards $\mathcal{EL}$ concept descriptions defined over acyclic $\mathcal{EL}$ TBoxes.

Definition 6.9. Let $\mathcal{T}$ be an *acyclic* $\mathcal{EL}$ TBox, C an $\mathcal{EL}$ concept description and m a graded membership function. m is extended to compute membership degree values with respect to $\mathcal{T}$ as follows:

$$m^\mathcal{I}(d, C, \mathcal{T}) := m^\mathcal{I}(d, u_\mathcal{T}(C)) \qquad \diamondsuit$$

Being m a graded membership function in the sense of Definition 3.1, it satisfies *M1* and *M2*. Hence, since $C \equiv_\mathcal{T} u_\mathcal{T}(C)$, the definition of m with respect to $\mathcal{T}$ satisfies $M1^\mathcal{T}$.

Moreover, $C \equiv_{\mathcal{T}} D$ implies that $u_{\mathcal{T}}(C) \equiv u_{\mathcal{T}}(D)$. From this it is easy to verify that m also satisfies $M2^{\mathcal{T}}$. Additionally, since the unfolding of an $\mathcal{EL}$ concept description with respect to an acyclic TBox yields another $\mathcal{EL}$ concept description, well-definedness of m implies its well-definedness with respect to $\mathcal{T}$.

So far, we have first restricted sets of definitions in order to avoid nesting of threshold concepts. Afterwards, we have extended the computation of membership degree functions to concept descriptions defined with respect to acyclic $\mathcal{EL}$ TBoxes. Now, since *deg* is a well-defined graded membership function in the sense of Definition 3.1, its extension according to Definition 6.9 is also well-defined. Thus, we can now consider threshold concepts defined with respect to a background TBox in $\tau\mathcal{EL}(deg)$. To emphasize that *deg* is defined for acyclic $\mathcal{EL}$ TBoxes, we define a $\tau\mathcal{EL}(deg)$ TBox as a $\tau\mathcal{EL}(m)$ TBox $\widehat{\mathcal{T}} = (\mathcal{T}_\tau, \mathcal{T}_{\mathcal{EL}})$ where $\mathcal{T}_{\mathcal{EL}}$ is an acyclic $\mathcal{EL}$ TBox.

Despite the acyclicity restriction on $\mathcal{T}_{\mathcal{EL}}$, $\mathcal{T}_\tau$ is allowed to have terminological cycles in the sense of Definition 6.1. This is considering the relation $\rightarrow$ and its transitive closure $\rightarrow^+$ over defined concepts in $\mathcal{T}_\tau$. Consequently, we can talk about cyclic and acyclic $\tau\mathcal{EL}(deg)$ TBoxes. In particular, the notion of unfolding transfers naturally to acyclic ones. The constructors $\sqcap$ and $\exists r.C$ are treated in the same way and two new rules are added:

$$\begin{aligned} u_{\widehat{\mathcal{T}}}(\alpha) &:= u_{\widehat{\mathcal{T}}}(\widehat{C}_\alpha), \text{ for all } \alpha \doteq \widehat{C}_\alpha \in \mathcal{T}_\tau \\ u_{\widehat{\mathcal{T}}}(E_{\sim t}) &:= [u_{\mathcal{T}_{\mathcal{EL}}}(E)]_{\sim t} \end{aligned}$$

Since $\mathcal{T}_{\mathcal{EL}}$ is an acyclic TBox, this means that $u_{\mathcal{T}_{\mathcal{EL}}}(E)$ is an $\mathcal{EL}$ concept description. Consequently, $(u_{\mathcal{T}_{\mathcal{EL}}}(E))_{\sim t}$ is a well-formed threshold concept, and thus the unfolding $u_{\widehat{\mathcal{T}}}(E_{\sim t})$ is well-defined. Then, the counterparts of Propositions 6.2 and 6.3 also hold for acyclic $\tau\mathcal{EL}(deg)$ TBoxes.

Proposition 6.10. *For every acyclic $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}}$, every $\tau\mathcal{EL}(deg)$ concept description $\widehat{C}$ and every model $\mathcal{I}$ of $\widehat{\mathcal{T}}$:*

$$\widehat{C}^{\mathcal{I}} = [u_{\widehat{\mathcal{T}}}(\widehat{C})]^{\mathcal{I}}$$

Proof. The proof is the same as for $\mathcal{EL}$, except that in addition, one has to consider the unfolding of threshold concepts $E_{\sim t}$ where E is defined with respect to $\mathcal{T}_{\mathcal{EL}}$. This is not a problem, since $E \equiv_{\mathcal{T}_{\mathcal{EL}}} u_{\mathcal{T}_{\mathcal{EL}}}(E)$ and therefore, property $M2^{\mathcal{T}}$ implies that $E_{\sim t} \equiv_{\widehat{\mathcal{T}}} [u_{\mathcal{T}_{\mathcal{EL}}}(E)]_{\sim t}$. □

Proposition 6.11. *Let $\widehat{\mathcal{T}}$ be an acyclic $\tau\mathcal{EL}(deg)$ TBox. Any primitive interpretation $\mathcal{I}$ can be uniquely extended to a model of $\widehat{\mathcal{T}}$.*

Proposition 6.10 tells us that the unfolding of concepts preserves equivalence, i.e., $\widehat{C} \equiv_{\widehat{\mathcal{T}}} u_{\widehat{\mathcal{T}}}(\widehat{C})$. This together with Proposition 6.11 allow us to reduce reasoning with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes to reasoning in the empty terminology, by using unfolding. However, there are two reasons why this could not yield *worst-case* optimal decision procedures for the different reasoning tasks. On the one hand, as shown in [Neb90], the unfolding of a concept description may result in a concept description of exponential size. This was actually shown for the description logic $\mathcal{FL}_0$. The following example shows the corresponding version for $\mathcal{EL}$.

Example 6.12. For all $n \geq 0$, the $\mathcal{EL}$ TBox $\mathcal{T}_n$ is inductively defined as follows:

$$\begin{aligned}
\mathcal{T}_0 &:= \{\alpha_0 \doteq \top\} \\
\mathcal{T}_1 &:= \mathcal{T}_0 \cup \{\alpha_1 \doteq \exists r.\alpha_0 \sqcap \exists s.\alpha_0\} \\
&\vdots \\
\mathcal{T}_n &:= \mathcal{T}_{n-1} \cup \{\alpha_n \doteq \exists r.\alpha_{n-1} \sqcap \exists s.\alpha_{n-1}\}
\end{aligned}$$

Regarding the size of $\mathcal{T}_n$ and $u_{\mathcal{T}_n}(\alpha_n)$, we have $\mathsf{s}(\mathcal{T}_n) = \Theta(n)$ and $\mathsf{s}(u_{\mathcal{T}_n}(\alpha_n)) \geq 2^n$. ◇

On the other hand, concept satisfiability is NP-complete in $\tau\mathcal{EL}(deg)$ with respect to the empty TBox. Therefore, given a $\tau\mathcal{EL}(deg)$ acyclic TBox $\widehat{\mathcal{T}}$ and a $\tau\mathcal{EL}(deg)$ concept description $\widehat{C}$, unfolding $\widehat{C}$ with respect to $\widehat{\mathcal{T}}$ and then using the NP decision procedure from Chapter 5, yields in general a *non-deterministic exponential time* algorithm for concept satisfiability with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes.

Two natural questions arise from the previous discussion: Can we do it better than in NEXP?, and more hopefully, could it still be decided in *non-deterministic polynomial time*? We give a positive and a negative answer, respectively, to these questions. In fact, we will see that the possibility of using acyclic TBoxes to express exponentially large concept descriptions in a succinct way combined with the use of threshold concepts, makes the concept satisfiability problem harder than all the problems in NP (unless NP=Π_2^P).

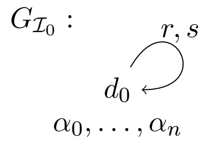
6.3 Models of non-polynomial size

We start by showing that, different from the empty TBox case, $\tau\mathcal{EL}(deg)$ concept descriptions do not enjoy the polynomial model property when defined with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes. More precisely, for all $n \geq 0$ there is a TBox $\widehat{\mathcal{T}_n}$ and a defined concept α_n such that α_n is satisfiable with respect to $\widehat{\mathcal{T}_n}$, but not in models of size polynomial in $\mathsf{s}(\widehat{\mathcal{T}_n})$. There are two purposes in doing this. It automatically rules out the possibility of designing an algorithm that searches for a model of polynomial size, as for the case where $\widehat{\mathcal{T}} = \emptyset$. Further, the structure that an interpretation $\mathcal{I}$ needs to have in order to satisfy α_n in $\widehat{\mathcal{T}_n}$ will be suitable to show that concept satisfiability is at least as hard as the problems contained in the class Π_2^P.

Consider the $\mathcal{EL}$ TBox $\mathcal{T}_n$ from Example 6.12 and let T_{α_n} be the description tree corresponding to $u_{\mathcal{T}_n}(\alpha_n)$. Additionally, let $\mathcal{I}_{\alpha_n}$ be the primitive interpretation with description graph T_{α_n}, and $d_0 \in \Delta^{\mathcal{I}_{\alpha_n}}$ the element representing its root. $\mathcal{I}_{\alpha_n}$ can be uniquely extended to a model of $\mathcal{T}_n$ (Proposition 6.3), and this extension is such that $d_0 \in (\alpha_n)^{\mathcal{I}_{\alpha_n}}$. Moreover, the following is an easy consequence from the definition of $\mathcal{T}_n$.

Proposition 6.13. *Let $\mathcal{I}$ be a model of $\mathcal{T}_n$ and $d \in \Delta^{\mathcal{I}}$. For all $0 \leq j \leq n$: if $d \in (\alpha_j)^{\mathcal{I}}$, then for each word $x \in \{r, s\}^j$ there exists a path $dx_1d_1 \ldots x_jd_j$ in $G_{\mathcal{I}}$ such that $d_i \in (\alpha_{j-i})^{\mathcal{I}}$ for all $1 \leq i \leq j$.*

The reason why $|\Delta^{\mathcal{I}_{\alpha_n}}| \geq 2^n$ is that all pair of paths (π_1, π_2) in T_{α_n} corresponding to two different words $x_{\pi_1}, x_{\pi_2} \in \{r, s\}^n$, are disjoint in their last nodes. This can obviously be avoided, the interpretation $\mathcal{I}_0$ (in the picture below) is a model of $\mathcal{T}_n$ satisfying α_n and has size $\mathcal{O}(1)$ with respect to the size of $\mathcal{T}_n$:



In fact, in $\mathcal{EL}$ regardless which type of TBox is considered, every satisfiable concept description is satisfiable in an interpretation of polynomial size.

Our aim is to transform $\mathcal{T}_n$ into a $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}_n}$ such that each model $\mathcal{I}$ of $\widehat{\mathcal{T}_n}$ satisfying α_n is of size at least 2^n. To this end, we use $2n$ auxiliary primitive concept names $A_1, \ldots, A_n, \bar{A}_1, \ldots, \bar{A}_n$. The intention is to enforce for each word $x \in \{r, s\}^n$ the existence of at least one path $d_0x_1d_1 \ldots x_nd_n$ in $G_\mathcal{I}$ such that:

$$d_n \in (A_i)^\mathcal{I} \text{ iff } x_i = r \quad (1 \leq i \leq n) \tag{6.2}$$

Note that for two different words $x, y \in \{r, s\}^n$ and the corresponding paths $\pi_x = d_0x_1 \ldots x_nd_{nx}$, $\pi_y = d_0y_1 \ldots y_nd_{ny}$ in $G_\mathcal{I}$ satisfying the equivalence in (6.2), there must exist i such that $x_i \neq y_i$, and this would imply:

$$d_{nx} \in (A_i)^\mathcal{I} \text{ iff } d_{ny} \notin (A_i)^\mathcal{I}$$

Hence, d_{nx} and d_{ny} must be two different domain elements in $\Delta^\mathcal{I}$. This argument extends to all pair of words in $\{r, s\}^n$. Thus, since there are 2^n words in $\{r, s\}^n$, in this way $\Delta^\mathcal{I}$ would need to have at least 2^n elements.

So far, the structure of $\mathcal{T}_n$ already guarantees the existence of a path from d_0 for all $x \in \{r, s\}^n$, but not the satisfaction of (6.2). One needs to be able to express within the logic the correct propagation of the concept names along each path. For example, for $n = 3$ and x_1, one possible way to do it is redefining α_3 as:

$$\alpha_3 \doteq \exists r.\Big(\alpha_2 \sqcap \bigsqcap_{x_2,x_3 \in \{r,s\}} \forall x_2x_3.A_1\Big) \sqcap \exists s.\Big(\alpha_2 \sqcap \bigsqcap_{x_2,x_3 \in \{r,s\}} \forall x_2x_3.\neg A_1\Big) \tag{6.3}$$

Unfortunately, as shown in Section 4.3.1, the simple concept $\forall r.A$ cannot be expressed in $\tau\mathcal{EL}(deg)$. Moreover, in general this idea would require the use of exponentially many $\forall$-restrictions. Nevertheless, $\forall r.\neg A$ can actually be expressed, and this is where the concept names $\bar{A}_i$ come into play. Their role is to be complementary with A_i at d_n.

Our first step is to assert a weaker version of the equivalence in (6.2) using A_i and $\bar{A}_i$. For each $1 \leq j \leq n$, we define two TBoxes $\mathcal{T}^j$ and $\mathcal{T}^{\bar{j}}$ as follows. We select $\mathcal{T}^j$ and $\mathcal{T}^{\bar{j}}$ as two copies of $\mathcal{T}_{j-1}$ (from Example 6.12), where each defined concept α_i $(0 \leq i \leq j-1)$ is renamed as E^j_i and $E^{\bar{j}}_i$, in $\mathcal{T}^j$ and $\mathcal{T}^{\bar{j}}$, respectively. Then, E^j_0 and $E^{\bar{j}}_0$ are redefined as $E^j_0 \doteq \bar{A}_{n-j+1}$ and $E^{\bar{j}}_0 \doteq A_{n-j+1}$. The union of all these TBoxes is denoted as $\mathcal{T}_{n,paths}$:

$$\mathcal{T}_{n,paths} := \bigcup_{j=1}^{n} (\mathcal{T}^j \cup \mathcal{T}^{\bar{j}})$$

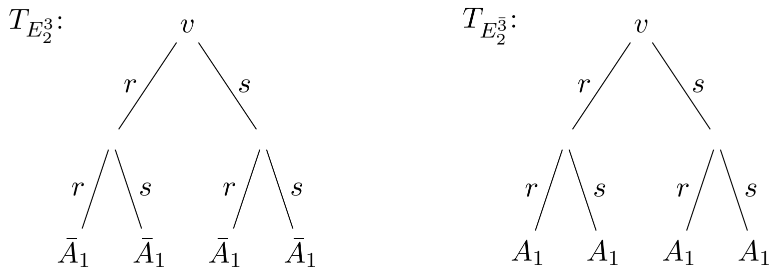
Let us illustrate this construction and explain why it will be useful.

Example 6.14. Let $n = 3$. Starting from $\mathcal{T}_2, \mathcal{T}_1$ and $\mathcal{T}_0$, the $\mathcal{EL}$ TBoxes $\mathcal{T}^3, \mathcal{T}^2$ and $\mathcal{T}^1$ consist of the following set of definitions, respectively:

$$
\begin{array}{lll}
E_2^3 \doteq \exists r.E_1^3 \sqcap \exists s.E_1^3 & E_1^2 \doteq \exists r.E_0^2 \sqcap \exists s.E_0^2 & E_0^1 \doteq \bar{A}_3 \\
E_1^3 \doteq \exists r.E_0^3 \sqcap \exists s.E_0^3 & E_0^2 \doteq \bar{A}_2 & \\
E_0^3 \doteq \bar{A}_1 & &
\end{array}
$$

The TBox $\mathcal{T}^3$ corresponds to the case where $j = 3$. Note that $n - j + 1 = 1$ matches the index of $\bar{A}_i$ in the definition of E_0^3. The same applies for E_0^2 and E_0^1, since as j decreases the index i increases accordingly. TBoxes $\mathcal{T}^{\bar{3}}, \mathcal{T}^{\bar{2}}$ and $\mathcal{T}^{\bar{1}}$ have the same structure except that A_1, A_2 and A_3 are used instead.

Let now $T_{E_2^3}$ and $T_{E_2^{\bar{3}}}$ be the $\mathcal{EL}$ description trees corresponding to the unfolding of E_2^3 and $E_2^{\bar{3}}$ in $\mathcal{T}^3$ and $\mathcal{T}^{\bar{3}}$, respectively:



We exploit the structure of these trees in two directions. First, both of them provide a succinct representation of all the possible paths corresponding to words in $\{r, s\}^2$. Second, for an interpretation $\mathcal{I}$ the threshold concept $(E_2^3)_{\leq 0}$ tells the following about any $d \in \Delta^{\mathcal{I}}$:

- if $dx_2d_2x_3d_3$ is a path in $G_{\mathcal{I}}$ where $\{x_2, x_3\} \subseteq \{r, s\}$ and $d_3 \in (\bar{A}_1)^{\mathcal{I}}$, then there is an equal path $vx_2v_2x_3v_3$ in $T_{E_2^3}$. Now, the partial mapping h from $T_{E_2^3}$ to $G_{\mathcal{I}}$ with $h(v) = d$ and $h(v_i) = d_i$ $(i = 2, 3)$ satisfies $h_w(v) > 0$. Therefore, $d \notin [(E_2^3)_{\leq 0}]^{\mathcal{I}}$.
- Conversely, if no such path exists for d, then for all paths $vx_2v_2x_3v_3$ in $T_{E_2^3}$ and all partial mappings h from $T_{E_2^3}$ to $G_{\mathcal{I}}$ such that $v_3 \in \mathsf{dom}(h)$, it is the case that $h(v_3) \notin (\bar{A}_1)^{\mathcal{I}}$. Therefore, by definition of h_w, it must be the case that $h_w(v) = 0$. Consequently, $deg^{\mathcal{I}}(d, E_2^3, \mathcal{T}^3) = 0$ and $d \in [(E_2^3)_{\leq 0}]^{\mathcal{I}}$.

The same reasoning applies for $E_2^{\bar{3}}$ and A_1. The good that comes from this is that we obtain the following equivalences:

$$(E_2^3)_{\leq 0} \equiv_{\mathcal{T}^3} \bigsqcap_{x_2, x_3 \in \{r,s\}} \forall x_2 x_3.\neg \bar{A}_1 \quad \text{and} \quad (E_2^{\bar{3}})_{\leq 0} \equiv_{\mathcal{T}^{\bar{3}}} \bigsqcap_{x_2, x_3 \in \{r,s\}} \forall x_2 x_3.\neg A_1$$

Since $\neg \bar{A}_1$ is meant to represent A_1, as in (6.3) it would be possible to propagate correctly A_1 according to the value of x_1 in all paths. ♢

Based on this, one can in general use the threshold concepts $(E^{j}_{j-1})_{\leq 0}$ and $(E^{\bar{j}}_{j-1})_{\leq 0}$ to represent the generalization of the value restrictions used in (6.3) to arbitrary lengths.

Proposition 6.15. *For all models $\mathcal{I}$ of $\mathcal{T}_{n,paths}$, $d \in \Delta^{\mathcal{I}}$ and $1 \leq j \leq n$:*

1. $d \in [(E^{j}_{j-1})_{\leq 0}]^{\mathcal{I}}$ *iff* $d \in \Big(\bigsqcap_{x\in\{r,s\}^{j-1}} \forall x.\neg \bar{A}_{n-j+1}\Big)^{\mathcal{I}}$.

2. $d \in [(E^{\bar{j}}_{j-1})_{\leq 0}]^{\mathcal{I}}$ *iff* $d \in \Big(\bigsqcap_{x\in\{r,s\}^{j-1}} \forall x.\neg A_{n-j+1}\Big)^{\mathcal{I}}$.

Proof. We only give the proof for the first statement (the second one can be shown using the same argument). We denote as $T_{E^{j}_{j-1}}$ the description tree corresponding to the unfolding $u_{\mathcal{T}^j}(E^{j}_{j-1})$ of E^{j}_{j-1} in $\mathcal{T}^j$. For simplicity, we use just ℓ (without subscript) to refer to the labeling of $T_{E^{j}_{j-1}}$.

($\Rightarrow$) Assume that $d \in [(E^{j}_{j-1})_{\leq 0}]^{\mathcal{I}}$. Since E^{j}_{j-1} is a defined concept in $\mathcal{T}^j$, this implies:

$$deg^{\mathcal{I}}(d, E^{j}_{j-1}, \mathcal{T}^j) = deg^{\mathcal{I}}(d, u_{\mathcal{T}^j}(E^{j}_{j-1})) = 0$$

For a contradiction, suppose that:

$$d \notin \Big(\bigsqcap_{x\in\{r,s\}^{j-1}} \forall x.\neg \bar{A}_{n-j+1}\Big)^{\mathcal{I}}$$

Then, there is a word $x_1 \dots x_{j-1} \in \{r,s\}^{j-1}$ such that $d \notin (\forall x_1 \dots x_{j-1}.\neg \bar{A}_{n-j+1})^{\mathcal{I}}$. The semantics of the value restriction constructor yields the existence of a path of the form $dx_1d_1 \dots x_{j-1}d_{j-1}$ in $G_{\mathcal{I}}$ such that $d_{j-1} \in (\bar{A}_{n-j+1})^{\mathcal{I}}$.

By definition of E^{j}_{j-1} in $\mathcal{T}^j$, there is a path $v_0x_1v_1 \dots x_{j-1}v_{j-1}$ in $T_{E^{j}_{j-1}}$ with $\ell(v_{j-1}) = \{\bar{A}_{n-j+1}\}$, where v_0 is the root of $T_{E^{j}_{j-1}}$. Therefore, the *ptgh* h from $T_{E^{j}_{j-1}}$ to $G_{\mathcal{I}}$ with $h(v_0) = d$ and $h(v_i) = d_i$ $(1 \leq i \leq j-1)$ induces a weighted homomorphism h_w such that: $h_w(v_0) > 0$. This contradicts our initial assumption since it implies:

$$deg^{\mathcal{I}}(d, u_{\mathcal{T}^j}(E^{j}_{j-1})) > 0$$

Thus, the left to right implication holds.

($\Leftarrow$) Assume that

$$d \in \Big(\bigsqcap_{x\in\{r,s\}^{j-1}} \forall x.\neg \bar{A}_{n-j+1}\Big)^{\mathcal{I}}$$

This implies that $d \in (\forall x_1 \dots x_{j-1}.\neg \bar{A}_{n-j+1})^{\mathcal{I}}$ for all words $x_1, \dots, x_{j-1} \in \{r,s\}^{j-1}$. Hence, any path of the form $dx_1d_1 \dots x_{j-1}d_{j-1}$ in $G_{\mathcal{I}}$ is restricted to have:

$$d_{j-1} \notin (\bar{A}_{n-j+1})^{\mathcal{I}}$$

Let now $v_0x_1v_1 \dots x_{j-1}v_{j-1}$ be any path in $T_{E^{j}_{j-1}}$. By definition of E^{j}_{j-1} in $\mathcal{T}^j$ we know that $x_1 \dots x_{j-1} \in \{r,s\}^{j-1}$ and $\ell(v_{j-1}) = \{\bar{A}_{n-j+1}\}$. Therefore, for all *ptgh* h from $T_{E^{j}_{j-1}}$

to $G_{\mathcal{I}}$ having $h(v_0) = d$ and $v_{j-1} \in \mathsf{dom}(h)$, it is the case that $\bar{A}_{n-j+1} \notin \ell_{\mathcal{I}}(h(v_{j-1}))$. Hence, since v_{j-1} is a leaf in $T_{E^j_{j-1}}$, this means that $h_w(v_{j-1}) = 0$.

Overall, we have shown that for all leaves v in $T_{E^j_{j-1}}$ and all *ptgh* h with $v \in \mathsf{dom}(h)$, it holds that $h_w(v) = 0$. Then, since $\ell(v) = \emptyset$ iff v is a non-leaf node, there is no possible way in which $h_w(v_0) > 0$. Consequently, it follows:

$$deg^{\mathcal{I}}(d, u_{\mathcal{T}^j}(E^j_{j-1})) = 0$$

Thus, $d \in [(E^j_{j-1})_{\leq 0}]^{\mathcal{I}}$. □

Having these equivalences, the next step is to generalize the intuition expressed by the combination of (6.3) and Example 6.14. More precisely, we integrate the threshold concepts of the form $(E^j_{j-1})_{\leq 0}$ and $(E^{\bar{j}}_{j-1})_{\leq 0}$ into $\mathcal{T}_n$ as follows. For all $1 \leq j \leq n$:

$$\alpha_j \doteq \exists r.(\alpha_{j-1} \sqcap (E^j_{j-1})_{\leq 0}) \sqcap \exists s.(\alpha_{j-1} \sqcap (E^{\bar{j}}_{j-1})_{\leq 0})$$

We name the resulting TBox as $\mathcal{T}_{n,\tau}$. Note that $\mathcal{T}_{n,paths}$ is acyclic and $(\mathcal{T}_{n,\tau}, \mathcal{T}_{n,paths})$ satisfies the conditions required in Definition 6.6. Therefore, $(\mathcal{T}_{n,\tau}, \mathcal{T}_{n,paths})$ is a $\tau\mathcal{EL}(deg)$ TBox. We can now state for $(\mathcal{T}_{n,\tau}, \mathcal{T}_{n,paths})$ the equivalent of Proposition 6.13 .

Proposition 6.16. *Let $\mathcal{I}$ be a model of $(\mathcal{T}_{n,\tau}, \mathcal{T}_{n,paths})$ and $d \in \Delta^{\mathcal{I}}$. For all $0 \leq j \leq n$: if $d \in (\alpha_j)^{\mathcal{I}}$, then for each word $x \in \{r, s\}^j$ there exists a path $dx_1d_1 \ldots x_jd_j$ in $G_{\mathcal{I}}$ such that for all $1 \leq i \leq j$,*

- *$d_i \in (\alpha_{j-i})^{\mathcal{I}}$,*
- *$d_i \in [(E^{j-i+1}_{j-i})_{\leq 0}]^{\mathcal{I}}$ if $x_i = r$, otherwise $d_i \in [(E^{\overline{j-i+1}}_{j-i})_{\leq 0}]^{\mathcal{I}}$.*

We continue with the previous example to see how $\mathcal{T}_{3,\tau}$ looks like, and explain what is still missing to achieve our goal.

Continuation of Example 6.14. After integrating the new threshold concepts into $\mathcal{T}_3$, the $\tau\mathcal{EL}(deg)$ TBox $\mathcal{T}_{3,\tau}$ consists of the following set of definitions:

$$\begin{aligned}
\alpha_3 &\doteq \exists r.(\alpha_2 \sqcap (E^3_2)_{\leq 0}) \sqcap \exists s.(\alpha_2 \sqcap (E^{\bar{3}}_2)_{\leq 0}) \\
\alpha_2 &\doteq \exists r.(\alpha_1 \sqcap (E^2_1)_{\leq 0}) \sqcap \exists s.(\alpha_1 \sqcap (E^{\bar{2}}_1)_{\leq 0}) \\
\alpha_1 &\doteq \exists r.(\alpha_0 \sqcap (E^1_0)_{\leq 0}) \sqcap \exists s.(\alpha_0 \sqcap (E^{\bar{1}}_0)_{\leq 0}) \\
\alpha_0 &\doteq \top
\end{aligned}$$

Let $\mathcal{I}$ be an interpretation such that $\mathcal{I} \models (\mathcal{T}_{3,\tau}, \mathcal{T}_{3,paths})$, $d_0 \in \Delta^{\mathcal{I}}$ and $d_0 \in (\alpha_3)^{\mathcal{I}}$. The addition of the threshold concepts gives us the following. For all words $x \in \{r, s\}^3$ there is at least one path $d_0x_1d_1x_2d_2x_3d_3$ such that:

$$\begin{aligned}
(x_i = r) &\Rightarrow d_3 \notin (\bar{A}_i)^{\mathcal{I}} \\
(x_i = s) &\Rightarrow d_3 \notin (A_i)^{\mathcal{I}}
\end{aligned} \tag{6.4}$$

For instance, the words rrr and srr yield two paths $d_0rd_1rd_2rd_3$ and $d_0se_1re_2re_3$ in $G_\mathcal{I}$, where $d_3 \in (\neg\bar{A}_1)^\mathcal{I}$ and $e_3 \in (\neg A_1)^\mathcal{I}$. However, since no relationship has yet been established between $\bar{A}_1$ and A_1, there is no inconsistency between $\neg\bar{A}_1$ and $\neg A_1$. Hence, d_3 and e_3 can be merged into one while keeping $d_0 \in (\alpha_3)^\mathcal{I}$. Overall only non-membership is required at the end of each path. Consequently, $\mathcal{I}_0$ is still good enough to be a model of $(\mathcal{T}_{3,\tau}, \mathcal{T}_{3,paths})$ satisfying α_3. ♢

In general, if $d_0 \in (\alpha_n)^\mathcal{I}$, this construction tells us that for each word $x \in \{r, s\}^n$ there exists at least one path $d_0x_1d_1 \dots x_nd_n$ in $G_\mathcal{I}$ such that:

$$d_n \in (A_i)^\mathcal{I} \Rightarrow x_i = r \text{ (the contraposition of 6.4)}$$

To have this implication also valid in the opposite direction, we use again threshold concepts to make A_i and $\bar{A}_i$ complementary at d_n, i.e., $d_n \in (A_i)^\mathcal{I}$ iff $d_n \notin (\bar{A}_i)^\mathcal{I}$. To each pair $(A_i, \bar{A}_i)$ we associate the concept definition $F_i \doteq A_i \sqcap \bar{A}_i$. The TBox $\mathcal{T}_{n,comp}$ is defined as:

$$\mathcal{T}_{n,comp} := \bigcup_{i=1}^{n} \{F_i \doteq A_i \sqcap \bar{A}_i\}$$

Using this, α_0 is redefined in $\mathcal{T}_{n,\tau}$ as:

$$\alpha_0 \doteq \prod_{i=1}^{n} [(F_i)_{\leq \frac{1}{2}} \sqcap (F_i)_{\geq \frac{1}{2}}]$$

Remark 6.17. Defining $\bar{A}_i \doteq (A_i)_{<1}$ $(1 \leq i \leq n)$ in $\mathcal{T}_{n,\tau}$, makes $d \in (A_i)^\mathcal{I}$ iff $d \notin (\bar{A}_i)^\mathcal{I}$ not only for $d = d_n$, but for the whole interpretation domain. This is simpler than the definition of α_0. However, it would make $(E^j_{j-1})_{\leq 0}$ (with $j = n-i+1$) not a well-formed threshold concept since $\bar{A}_i$ is used to define E^j_{j-1}. ♢

Finally, putting all these parts together, we end up with the $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}}_n := (\mathcal{T}_{n,\tau}, \mathcal{T}_{n,\mathcal{EL}})$ where:

$$\mathcal{T}_{n,\mathcal{EL}} := \mathcal{T}_{n,comp} \cup \mathcal{T}_{n,paths}$$

We now proceed to show that satisfying α_n in $\widehat{\mathcal{T}}_n$ requires interpretations of size exponential in n.

Lemma 6.18. *For all $n \geq 0$ and all interpretations $\mathcal{I}$ such that $\mathcal{I} \models \widehat{\mathcal{T}}_n$ and $(\alpha_n)^\mathcal{I} \neq \emptyset$, we have $\Delta^\mathcal{I} \geq 2^n$.*

Proof. Let $\mathcal{I}$ be an interpretation such that $\mathcal{I} \models \widehat{\mathcal{T}}_n$ and $(\alpha_n)^\mathcal{I} \neq \emptyset$.

The case $n = 0$ is trivial since $\Delta^\mathcal{I}$ is a non-empty domain. To see that the statement of the lemma is also true for an arbitrary $n > 0$, we show that for all subsets X of $\{A_1, \dots, A_n\}$ there exists an element $d_X \in \Delta^\mathcal{I}$ such that:

$$d_X \in (A_i)^\mathcal{I} \text{ iff } A_i \in X \quad (1 \leq i \leq n) \tag{6.5}$$

Let $d \in \Delta^\mathcal{I}$ be such that $d \in (\alpha_n)^\mathcal{I}$. In addition, let us fix a set $Y \subseteq \{A_1, \dots, A_n\}$ and define its corresponding word $y \in \{r, s\}^n$ as:

$$y_i = r \text{ iff } A_i \in Y \quad (1 \leq i \leq n) \tag{6.6}$$

By applying Proposition 6.16 to d, we know that there is a path $dy_1d_1 \dots y_nd_n$ in $G_{\mathcal{I}}$ such that for all $1 \leq i \leq n$:

- $d_i \in (\alpha_{n-i})^{\mathcal{I}}$,
- $d_i \in [(E^{n-i+1}_{n-i})_{\leq 0}]^{\mathcal{I}}$ if $y_i = r$, otherwise $d_i \in [(E^{\overline{n-i+1}}_{n-i})_{\leq 0}]^{\mathcal{I}}$

In particular, the suffix $d_iy_{i+1} \dots y_nd_n$ is of length $n - i$. Therefore, we further have:

$$\begin{aligned} y_i = r &\Rightarrow d_i \in [(E^{n-i+1}_{n-i})_{\leq 0}]^{\mathcal{I}} \\ &\Rightarrow d_i \in (\forall y_{i+1} \dots y_n . \neg \bar{A}_i)^{\mathcal{I}} \qquad \text{(Proposition 6.15 applied to } d_i \text{ and } E^{n-i+1}_{n-i}) \\ &\Rightarrow d_n \notin (\bar{A}_i)^{\mathcal{I}} \end{aligned}$$

Symmetrically, $y_i = s$ implies $d_n \notin (A_i)^{\mathcal{I}}$. Now, we know that $F_i \doteq A_i \sqcap \bar{A}_i \in \mathcal{T}_{n,comp}$ and α_0 is of the form:

$$\alpha_0 \doteq \prod_{i=1}^{n} [(F_i)_{\leq \frac{1}{2}} \sqcap (F_i)_{\geq \frac{1}{2}}]$$

Since $d_n \in (\alpha_0)^{\mathcal{I}}$ it follows:

$$d_n \in (A_i)^{\mathcal{I}} \text{ iff } y_i = r \quad (1 \leq i \leq n)$$

From the way the word y is defined in (6.6), we can conclude that d_n is an element of $\Delta^{\mathcal{I}}$ satisfying (6.5) with respect to Y.

Thus, one can easily see why $\Delta^{\mathcal{I}} \geq 2^n$. □

To finally fulfill the initial aim of this section, it remains to show that α_n is indeed satisfiable with respect to $\widehat{\mathcal{T}}_n$.

Lemma 6.19. *α_n is satisfiable with respect to $\widehat{\mathcal{T}}_n$.*

Proof. We take the interpretation $\mathcal{I}_{\alpha_n}$ and extend it into a model $\widehat{\mathcal{I}}_{\alpha_n}$ of $\widehat{\mathcal{T}}_n$ satisfying α_n. By construction, $\mathcal{I}_{\alpha_n}$ is tree-shaped and has 2^n leaves. Moreover, there is a one-to-one correspondence between words in $\{r, s\}^n$ and the leaves in T_{α_n}. The leaf d_x corresponding to the word x is the one reached from d_0 through the path $d_0x_1d_1 \dots x_nd_x$.

Let $\mathcal{L}_{\alpha_n}$ denote the set of leaves of T_{α_n}. The interpretation of $A_i, \bar{A}_i$ under $\widehat{\mathcal{I}}_{\alpha_n}$ is defined as follows. For all $1 \leq i \leq n$:

$$\begin{aligned} (A_i)^{\widehat{\mathcal{I}}_{\alpha_n}} &:= \{d_x \mid d_x \in \mathcal{L}_{\alpha_n} \text{ and } x_i = r\} \\ (\bar{A}_i)^{\widehat{\mathcal{I}}_{\alpha_n}} &:= \{d_x \mid d_x \in \mathcal{L}_{\alpha_n} \text{ and } x_i = s\} \end{aligned} \tag{6.7}$$

Hence, for all leaves d of T_{α_n} and all $i \in \{1 \dots n\}$ we have:

$$d \in (A_i)^{\widehat{\mathcal{I}}_{\alpha_n}} \text{ iff } d \notin (\bar{A}_i)^{\widehat{\mathcal{I}}_{\alpha_n}} \tag{6.8}$$

Since $\mathcal{T}_{n,\tau}$ and $\mathcal{T}_{n,\mathcal{EL}}$ are both acyclic, there is a unique way to extend $\widehat{\mathcal{I}}_{\alpha_n}$ into a model of $\widehat{\mathcal{T}}_n$. Having done so, let $\eta(d)$ denote the *height* of a domain element d in T_{α_n}. We show by induction on $\eta(d)$ the following claim:

$$\textit{for all } d \in \Delta^{\widehat{\mathcal{I}}_{\alpha_n}}\textit{: } d \in (\alpha_{\eta(d)})^{\widehat{\mathcal{I}}_{\alpha_n}}$$

Induction Base. $d \in \Delta^{\widehat{\mathcal{I}}_{\alpha_n}}$ and $\eta(d) = 0$. Then, d is a leaf in T_{α_n}. Recall that α_0 is defined in $\mathcal{T}_{n,\tau}$ as:

$$\alpha_0 \doteq \prod_{i=1}^{n} [(F_i)_{\leq\frac{1}{2}} \sqcap (F_i)_{\geq\frac{1}{2}}]$$

Consequently, since F_i is defined in $\mathcal{T}_{n,comp}$ as $F_i \doteq A_i \sqcap \bar{A}_i$, using (6.8) we obtain $d \in [(F_i)_{\leq\frac{1}{2}} \sqcap (F_i)_{\geq\frac{1}{2}}]^{\widehat{\mathcal{I}}_{\alpha_n}}$. Thus, $d \in (\alpha_0)^{\widehat{\mathcal{I}}_{\alpha_n}}$ holds.

Induction Step. Let $d \in \Delta^{\widehat{\mathcal{I}}_{\alpha_n}}$ with $0 < \eta(d) \leq n$. We assume our claim holds for all $e \in \Delta^{\widehat{\mathcal{I}}_{\alpha_n}}$ with $\eta(e) < \eta(d)$.

To start, $\alpha_{\eta(d)}$ is defined in $\mathcal{T}_{n,\tau}$ as:

$$\alpha_{\eta(d)} \doteq \exists r.(\alpha_{\eta(d)-1} \sqcap (E^{\eta(d)}_{\eta(d)-1})_{\leq 0}) \sqcap \exists s.(\alpha_{\eta(d)-1} \sqcap (E^{\overline{\eta(d)}}_{\eta(d)-1})_{\leq 0})$$

By construction of T_{α_n}, there exists $e \in \Delta^{\widehat{\mathcal{I}}_{\alpha_n}}$ such that $(d,e) \in r^{\widehat{\mathcal{I}}_{\alpha_n}}$ and $\eta(e) = \eta(d) - 1$. The application of induction hypothesis to e yields $e \in (\alpha_{\eta(d)-1})^{\widehat{\mathcal{I}}_{\alpha_n}}$.

Consider now any word $y \in \{r,s\}^{\eta(d)-1}$. Since $\eta(e) = \eta(d) - 1$, by definition of T_{α_n} there is a unique path of the form $ey_1e_1 \ldots y_{\eta(d)-1}e_{\eta(d)-1}$ in T_{α_n}, where $e_{\eta(d)-1}$ is a leaf. Moreover, such a path is suffix of a path $d_0x_1d_1 \ldots d_jx_jex_{j+1} \ldots x_nd_n$, where $d_j = d$, $x_j = r$ and $e_{\eta(d)-1} = d_n$. Then, we obtain the following equalities:

$$\begin{aligned} n - (j+1) &= (\eta(d) - 1) - 1 \\ n - \eta(d) + 1 &= j \end{aligned}$$

Since $x_j = r$, by (6.7) we obtain that $d_n \in (A_{n-\eta(d)+1})^{\widehat{\mathcal{I}}_{\alpha_n}}$, and by (6.8) $d_n \notin (\bar{A}_{n-\eta(d)+1})^{\widehat{\mathcal{I}}_{\alpha_n}}$. Hence, as y was chosen arbitrarily from $\{r,s\}^{\eta(d)-1}$, we have just shown that:

$$e \in \left(\prod_{y \in \{r,s\}^{\eta(d)-1}} \forall y.\neg\bar{A}_{n-\eta(d)+1} \right)^{\mathcal{I}_{\alpha_n}}$$

The application of Proposition 6.15 then yields $e \in [(E^{\eta(d)}_{\eta(d)-1})_{\leq 0}]^{\widehat{\mathcal{I}}_{\alpha_n}}$. For the $\exists s$ restriction in the definition of $\alpha_{\eta(d)}$, the corresponding result can be shown in the same way. Therefore, $d \in (\alpha_{\eta(d)})^{\widehat{\mathcal{I}}_{\alpha_n}}$.

Using this result and the fact that d_0 is of height n in T_{α_n}, we can conclude that $d_0 \in (\alpha_n)^{\widehat{\mathcal{I}}_{\alpha_n}}$. Thus, α_n is satisfiable with respect to $\widehat{\mathcal{T}}_n$. □

Finally, let us look at the size of $\widehat{\mathcal{T}}_n$. $\mathcal{T}_{n,\tau}$ and $\mathcal{T}_{n,comp}$ are both of size $\mathcal{O}(n)$. Let us recall the definition of $\mathcal{T}_{n,paths}$:

$$\mathcal{T}_{n,paths} = \bigcup_{j=1}^{n} (\mathcal{T}^j \cup \mathcal{T}^{\bar{j}})$$

Then, it can be equivalently expressed as follows:

$$\mathcal{T}_{n,paths} = \mathcal{T}_{n-1,paths} \cup \mathcal{T}^n \cup \mathcal{T}^{\bar{n}}$$

Since $\mathcal{T}^n$ and $\mathcal{T}^{\bar{n}}$ have the same size as the $\mathcal{EL}$ TBox $\mathcal{T}_{n-1}$, and $\mathsf{s}(\mathcal{T}_n) = \mathcal{O}(n)$ for all $n \geq 1$, we obtain:

$$\mathsf{s}(\mathcal{T}_{n,paths}) = \mathsf{s}(\mathcal{T}_{n-1,paths}) + 2 * \mathcal{O}(n-1)$$

where $\mathsf{s}(\mathcal{T}_{1,paths}) = c \geq 1$ and $c \in \mathbb{N}$ is a constant.

Hence, $\mathsf{s}(\widehat{\mathcal{T}_n}) = \mathcal{O}(n^2)$ for all $n \geq 1$. Now let $\mathcal{I}$ be a model of $\widehat{\mathcal{T}_n}$ satisfying α_n. By Lemma 6.18 we know that $|\Delta^{\mathcal{I}}| \geq 2^n$, and consequently $|\Delta^{\mathcal{I}}| \geq 2^{\mathcal{O}(\sqrt{\mathsf{s}(\widehat{\mathcal{T}_n})})}$. Therefore, the size of $\mathcal{I}$ is not polynomial in the size of $\widehat{\mathcal{T}_n}$.

Theorem 6.20. *For all $n \geq 0$ there exists a $\tau\mathcal{EL}(deg)$ acyclic TBox $\widehat{\mathcal{T}_n}$ with a defined concept α_n, such that $\mathsf{s}(\widehat{\mathcal{T}})$ is polynomial in n, but all models satisfying α_n are of size at least exponential in n.*

6.4 Reasoning with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes

The result obtained in Theorem 6.20 does not imply that there is no NP decision procedure for concept satisfiability. Even when, in general, models of polynomial size satisfying a concept description do not exist, it may well be the case that such very large models have abstract representations of polynomial size which can be used to design an NP procedure, or simply there is a different way to do it. Unfortunately, this seems to be very unlikely. We show that concept satisfiability and subsumption are Π_2^P-hard and Σ_2^P-hard, respectively, with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes. Additionally, we provide a PSPACE algorithm that is sound and complete for both problems. Finally, the algorithm will be extended to reasoning with respect to acyclic $\tau\mathcal{EL}(deg)$ KBs without giving up its polynomial space property.

6.4.1 Lower bounds

We reduce the problem $\forall\exists$**3SAT** to concept satisfiability with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes. This problem is well-known to be complete for the class Π_2^P (see [Sto76], Section 4).

Definition 6.21 ($\forall\exists$3SAT). Let $u = \{u_1, \ldots, u_n\}$ and $v = \{v_1, \ldots, v_m\}$ be two disjoint sets of propositional variables. Additionally, let $\varphi(u, v)$ be a formula in 3CNF defined over $u \cup v$, i.e., $\varphi(u, v)$ is a finite set of propositional clauses $\mathcal{C} = \{c_1, \ldots, c_k\}$ such that:

- Each clause c_i is a set of three literals $\{\ell_{i1}, \ell_{i2}, \ell_{i3}\}$ over $u \cup v$.

A formula $(\forall u)(\exists v)\varphi(u,v)$ is satisfiable iff for all truth assignments $\mathfrak{t}$ for the variables in u there is an extension of $\mathfrak{t}$ for the variables in v such that it satisfies $\varphi(u,v)$. $\forall\exists$**3SAT** is then the problem of deciding whether given a 3CNF formula $\varphi(u,v)$, the formula $(\forall u)(\exists v)\varphi(u,v)$ is satisfiable or not. $\diamondsuit$

The idea for the reduction goes as follows. Each 3CNF formula $\varphi(u,v)$ is translated into a $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}}_n^{\varphi}$ containing a defined concept α_n such that: $(\forall u)(\exists v)\varphi(u,v)$ is satisfiable iff α_n is satisfiable with respect to $\widehat{\mathcal{T}}_n^{\varphi}$ (here, the value n corresponds to the number of universally quantified variables). We have seen in the proof of Lemma 6.18 that satisfiability of α_n with respect to $\widehat{\mathcal{T}}_n$ requires interpretations $\mathcal{I}$ containing for all subsets X of $\{A_1,\ldots,A_n\}$ an element $d_X \in \Delta^{\mathcal{I}}$ such that: $d_X \in (A_i)^{\mathcal{I}}$ iff $A_i \in X$. We take advantage of this to encode the universal quantification $(\forall u)$. The existential quantification can be simulated by the very nature of the concept satisfiability problem. We will obtain $\widehat{\mathcal{T}}_n^{\varphi}$ from $\widehat{\mathcal{T}}_n$, by modifying the definition of α_0 in $\mathcal{T}_{n,\tau}$, and adding new definitions to $\mathcal{T}_{n,comp}$. In the following we provide the details of the translation and prove its correctness.

For each variable $u_i \in u$, the literals u_i and $\neg u_i$ are identified with the concept names A_i and $\bar{A}_i$, respectively. Similarly, for literals over v we introduce new primitive concept names $B_1,\bar{B}_1,\ldots,B_m,\bar{B}_m$. More formally, to each literal ℓ over $u\cup v$, the mapping γ assigns a primitive concept name as follows:

$$\gamma(\ell) = \begin{cases} A_i & \text{if } \ell = u_i, \\ \bar{A}_i & \text{if } \ell = \neg u_i, \\ B_j & \text{if } \ell = v_j, \\ \bar{B}_j & \text{if } \ell = \neg v_j. \end{cases}$$

To encode $\varphi(u,v)$, each clause $c_i = \{\ell_{i1},\ell_{i2},\ell_{i3}\}$ in $\mathcal{C}$ is represented by the $\mathcal{EL}$ concept description $D_i := \gamma(\ell_{i1}) \sqcap \gamma(\ell_{i2}) \sqcap \gamma(\ell_{i3})$. Then, we define the $\tau\mathcal{EL}(deg)$ concept description $\widehat{C}_{\varphi}$ corresponding to $\varphi(u,v)$ as:

$$\widehat{C}_{\varphi} := \prod_{i=1}^{k} (D_i)_{\geq \frac{1}{3}}$$

The idea is that an individual d_X belongs to $(D_i)_{\geq\frac{1}{3}}$ iff it belongs to at least one concept name $\gamma(\ell_{il})$ $(1 \leq l \leq 3)$. To constrain B_j and $\bar{B}_j$ to be complementary at d_X, for all $1 \leq j \leq m$ we add to $\mathcal{T}_{n,comp}$ the concept definition $G_j \doteq B_j \sqcap \bar{B}_j$. The last step is to adjust the definition of α_0 in $\mathcal{T}_{n,\tau}$ to take into account the formula $\varphi(u,v)$ and the newly introduced concept names corresponding to the variables in v:

$$\alpha_0 \doteq \widehat{C}_{\varphi} \sqcap \prod_{i=1}^{n} [(F_i)_{\leq\frac{1}{2}} \sqcap (F_i)_{\geq\frac{1}{2}}] \sqcap \prod_{j=1}^{m} [(G_j)_{\leq\frac{1}{2}} \sqcap (G_j)_{\geq\frac{1}{2}}] \tag{6.9}$$

To avoid confusions we denote by $\mathcal{T}_{n,\tau}^{\varphi}$ and $\mathcal{T}_{n,comp}^{\varphi}$ the modified TBoxes, and by α_0^{φ} the altered α_0. Then, $\widehat{\mathcal{T}}_n^{\varphi} := (\mathcal{T}_{n,\tau}^{\varphi}, \mathcal{T}_{n,comp}^{\varphi} \cup \mathcal{T}_{n,paths})$.

Lemma 6.22. *Let $u = \{u_1, \ldots, u_n\}$, $v = \{v_1, \ldots, v_m\}$ be sets of propositional variables, and $\varphi(u, v)$ a formula in 3CNF defined over $u \cup v$. Then, $(\forall u)(\exists v)\varphi(u, v)$ is satisfiable iff α_n is satisfiable with respect to $\widehat{\mathcal{T}}_n^{\varphi}$.*

Proof. ($\Rightarrow$) Assume that $(\forall u)(\exists v)\varphi(u, v)$ is satisfiable. In the previous section (see Lemma 6.19) we have constructed an interpretation $\widehat{\mathcal{I}}_{\alpha_n}$ such that $d_0 \in (\alpha_n)^{\widehat{\mathcal{I}}_{\alpha_n}}$, and $d \in (\alpha_0)^{\widehat{\mathcal{I}}_{\alpha_n}}$ for all leaves d in T_{α_n}. We extend $\widehat{\mathcal{I}}_{\alpha_n}$ into a model $\widehat{\mathcal{I}}^{\varphi}_{\alpha_n}$ of $\widehat{\mathcal{T}}_n^{\varphi}$ satisfying α_n.

The new interpretation $\widehat{\mathcal{I}}^{\varphi}_{\alpha_n}$ extends $\widehat{\mathcal{I}}_{\alpha_n}$ with the interpretation of the concept names $B_1, \ldots, B_m, \bar{B}_1, \ldots, \bar{B}_m$. The positive side of using $\widehat{\mathcal{I}}_{\alpha_n}$ as a starting point is that since $\mathcal{T}_{n,paths}$ does not change and $\mathcal{T}_{n,\tau}$ only changes in the definition of α_0, it is enough to extend $\widehat{\mathcal{I}}_{\alpha_n}$ in such a way that $d \in (\alpha_0^{\varphi})^{\widehat{\mathcal{I}}^{\varphi}_{\alpha_n}}$ holds for all leaves d in T_{α_n}.

Let d be a leaf of T_{α_n}. We define the assignment $\mathfrak{t}_d$ for $u \cup v$ as follows. First,

$$\mathfrak{t}_d(u_i) = \mathit{true} \text{ iff } d \in (A_i)^{\widehat{\mathcal{I}}_{\alpha_n}} \quad (1 \leq i \leq n)$$

Second, $\mathfrak{t}_d$ assigns truth values to the variables in v such that it satisfies $\varphi(u, v)$. This is always possible because $(\forall u)(\exists v)\varphi(u, v)$ is satisfiable. If there is more than one possible way any of them can be used. Then, $\widehat{\mathcal{I}}^{\varphi}_{\alpha_n}$ extends $\widehat{\mathcal{I}}_{\alpha_n}$ as follows. For all $1 \leq j \leq m$:

$$\begin{aligned} (B_j)^{\widehat{\mathcal{I}}^{\varphi}_{\alpha_n}} &:= \{d \mid d \in \mathcal{L}_{\alpha_n} \text{ and } \mathfrak{t}_d(v_j) = \mathit{true}\} \\ (\bar{B}_j)^{\widehat{\mathcal{I}}^{\varphi}_{\alpha_n}} &:= \{d \mid d \in \mathcal{L}_{\alpha_n} \text{ and } \mathfrak{t}_d(v_j) = \mathit{false}\} \end{aligned} \tag{6.10}$$

Now, let us see why $d \in (\alpha_0^{\varphi})^{\widehat{\mathcal{I}}^{\varphi}_{\alpha_n}}$. From (6.10) it follows directly that for all $1 \leq j \leq m$:

$$d \in (B_j)^{\widehat{\mathcal{I}}^{\varphi}_{\alpha_n}} \text{ iff } d \notin (\bar{B}_j)^{\widehat{\mathcal{I}}^{\varphi}_{\alpha_n}}$$

A similar relationship exists between d and $A_i, \bar{A}_i$ $(1 \leq i \leq n)$, since $d \in (\alpha_0)^{\widehat{\mathcal{I}}_{\alpha_n}}$. Thus, we have:

$$d \in \left(\prod_{i=1}^{n} [(F_i)_{\leq \frac{1}{2}} \sqcap (F_i)_{\geq \frac{1}{2}}] \sqcap \prod_{j=1}^{m} [(G_j)_{\leq \frac{1}{2}} \sqcap (G_j)_{\geq \frac{1}{2}}] \right)^{\widehat{\mathcal{I}}^{\varphi}_{\alpha_n}}$$

Regarding $\widehat{C}_{\varphi}$, let $(D_i)_{\geq \frac{1}{3}}$ be any of its conjuncts and $c_i = \{\ell_{i1}, \ell_{i2}, \ell_{i3}\}$ its associated clause in $\varphi(u, v)$. Since $\mathfrak{t}_d$ satisfies $\varphi(u, v)$, this means that there is ℓ_{il} $(1 \leq l \leq 3)$ such that $\mathfrak{t}_d(\ell_{il}) = \mathit{true}$. Once we know that, the constructions of γ and $\mathfrak{t}_d$ in combination with the properties of d mentioned above imply that $d \in [\gamma(\ell_{il})]^{\widehat{\mathcal{I}}^{\varphi}_{\alpha_n}}$. Consequently, $d \in [(D_i)_{\geq \frac{1}{3}}]^{\widehat{\mathcal{I}}^{\varphi}_{\alpha_n}}$ for all $1 \leq i \leq k$, and thus $d \in (\widehat{C}_{\varphi})^{\widehat{\mathcal{I}}^{\varphi}_{\alpha_n}}$. Hence, $d \in (\alpha_0^{\varphi})^{\widehat{\mathcal{I}}^{\varphi}_{\alpha_n}}$.

Since d is an arbitrary leaf, the same result is valid for all the leaves in T_{α_n}. As already mentioned, this guarantees that α_n is satisfiable with respect to $\widehat{\mathcal{T}}_n^{\varphi}$.

($\Leftarrow$) Conversely, assume that α_n is satisfiable with respect to $\widehat{\mathcal{T}}_n^{\varphi}$. This means that there exists a model $\mathcal{I}$ of $\widehat{\mathcal{T}}_n^{\varphi}$ and $d \in \Delta^{\mathcal{I}}$ such that $d \in (\alpha_n)^{\mathcal{I}}$.

Let us fix a partial truth assignment $\mathfrak{t}$ covering all the variables in u. We show that $\mathfrak{t}$ can be extended to v in such a way that it satisfies $\varphi(u, v)$. The subset $X_{\mathfrak{t}}$ of $\{A_1, \ldots, A_n\}$ is induced by $\mathfrak{t}$ as follows:

$$X_{\mathfrak{t}} := \{A_i \mid \mathfrak{t}(u_i) = \mathit{true}\} \quad (1 \leq i \leq n)$$

Now, since $\widehat{\mathcal{T}}_n^\varphi$ only differs from $\widehat{\mathcal{T}}_n$ in the definition of α_0 and the inclusion of the G_j's in $\mathcal{T}_{n,comp}$, Propositions 6.15 and 6.16 still apply to $\mathcal{I}$ as a model of $\widehat{\mathcal{T}}_n^\varphi$. Following the proof of Lemma 6.18 with respect to $\mathcal{I}$ and $\widehat{\mathcal{T}}_n^\varphi$, we obtain that there exists $d_\mathfrak{t} \in \Delta^\mathcal{I}$ such that:

- $d_\mathfrak{t} \in (A_i)^\mathcal{I}$ iff $A_i \in X_\mathfrak{t}$ (iff $\mathfrak{t}(u_i) = \top$),
- $d_\mathfrak{t} \in (\alpha_0^\varphi)^\mathcal{I}$.

We use $d_\mathfrak{t}$ to extend $\mathfrak{t}$ to v as follows. For all $1 \leq j \leq m$:

$$\mathfrak{t}(v_j) = \top \text{ iff } d_\mathfrak{t} \in (B_j)^\mathcal{I}$$

Therefore, since $d_\mathfrak{t}$ satisfies the complementary restrictions required in the definition of α_0^φ for $A_1, \ldots, A_n, \bar{A}_1, \ldots, \bar{A}_n$ and $B_1, \ldots, B_m, \bar{B}_1, \ldots, \bar{B}_m$, we further obtain for all literals ℓ over $u \cup v$:

$$\mathfrak{t}(\ell) = \mathit{true} \text{ iff } d_\mathfrak{t} \in (\gamma(\ell))^\mathcal{I} \tag{6.11}$$

Moreover, since $d_\mathfrak{t} \in (\widehat{C}_\varphi)^\mathcal{I}$ we have that $d_\mathfrak{t} \in [(D_i)_{\geq \frac{1}{3}}]^\mathcal{I}$ for all $1 \leq i \leq k$. By definition of *deg* and D_i there must exist ℓ_{il} in c_i such that $d_\mathfrak{t} \in (\gamma(\ell_{il}))^\mathcal{I}$. It then follows from (6.11) that $\mathfrak{t}$ satisfies every clause $c_i \in \mathcal{C}$, and consequently it satisfies $\varphi(u, v)$.

Since the partial truth assignment $\mathfrak{t}$ for u was chosen arbitrarily, we thus have shown that $(\forall u)(\exists v)\varphi(u, v)$ is satisfiable. □

The construction of $\widehat{\mathcal{T}}_n^\varphi$ modifies $\widehat{\mathcal{T}}_n$ in two ways. First, $\mathcal{T}_{n,comp}$ is extended by adding the definitions of the concepts G_j for all $1 \leq j \leq m$. This yields a TBox $\mathcal{T}_{n,comp}^\varphi$ such that:

$$\mathsf{s}(\mathcal{T}_{n,comp}^\varphi) = \mathsf{s}(\mathcal{T}_{n,comp}) + \mathcal{O}(m)$$

Second, $\mathcal{T}_{n,\tau}^\varphi$ results from $\mathcal{T}_{n,\tau}$ by redefining α_0 (renamed as α_0^φ) as described in (6.9). It is not hard to see that the definition of α_0^φ is of size polynomial in $\varphi(u, v)$. Recall that $\mathsf{s}(\widehat{\mathcal{T}}_n)$ is a polynomial in n. Hence, since n and m are the number of variables in u and v, respectively, this means that $\mathsf{s}(\widehat{\mathcal{T}}_n^\varphi)$ is a polynomial in the size of $(\forall u)(\exists v)\varphi(u, v)$. Thus, $\forall\exists$**3SAT** is polynomial-time reducible to satisfiability in $\tau\mathcal{EL}(deg)$ with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes. The reduction of satisfiability to subsumption still holds, and therefore we obtain the following lower bounds.

Lemma 6.23. *In $\tau\mathcal{EL}(deg)$, satisfiability is Π_2^P-hard and subsumption is Σ_2^P-hard, with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes.*

6.4.2 Normalization

To simplify the technical development of the decision procedures presented in the next section, it is convenient to use TBoxes in a special form. We now introduce *normalized* $\tau\mathcal{EL}(deg)$ TBoxes in *reduced* form, and show that one can (without loss of generality) restrict the attention to this kind of TBoxes.

Let us start by recalling the normal form for $\mathcal{EL}$ TBoxes introduced in [Baa02]. An $\mathcal{EL}$ TBox $\mathcal{T}$ is said to be normalized iff $\alpha \doteq C_\alpha \in \mathcal{T}$ implies that C_α is of the form:

$$P_1 \sqcap \ldots \sqcap P_m \sqcap \exists r_1.\beta_1 \sqcap \ldots \sqcap \exists r_n.\beta_n$$

where $m, n \geq 0$, $P_1, \ldots, P_m \in \mathsf{N}_{prim}$, and $\beta_1, \ldots, \beta_n \in \mathsf{N}_{def}$. We extend this form to $\tau\mathcal{EL}(deg)$, and say that a $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}} = (\mathcal{T}_\tau, \mathcal{T}_{\mathcal{EL}})$ is normalized iff $\mathcal{T}_{\mathcal{EL}}$ is normalized and $\alpha \doteq \widehat{C}_\alpha \in \mathcal{T}_\tau$ implies that $\widehat{C}_\alpha$ is of the form:

$$\widehat{P}_1 \sqcap \ldots \sqcap \widehat{P}_m \sqcap \exists r_1.\beta_1 \sqcap \ldots \sqcap \exists r_n.\beta_n$$

where $m, n \geq 0$, for all $1 \leq i \leq m$ either $\widehat{P}_i \in \mathsf{N}_{prim}$ or it is of the form $E_{\sim t}$ with $E \in \mathsf{N}^0_{def}$, and $\beta_1, \ldots, \beta_n \in \mathsf{N}^\tau_{def} \cup \mathsf{N}^0_{def}$.

To illustrate this normalization process we start with a simpler version of Example 12 in [Baa02].

Example 6.24. Let $\mathcal{T}$ be the $\mathcal{EL}$ TBox consisting of the following definitions:

$$\begin{aligned}
\alpha_1 &\doteq P_1 \sqcap \alpha_2 \sqcap \exists r_1.\exists r_2.\alpha_3 \\
\alpha_2 &\doteq P_2 \sqcap \alpha_3 \sqcap \exists s.(\alpha_3 \sqcap P_3) \\
\alpha_3 &\doteq P_4
\end{aligned}$$

Using auxiliary definitions we obtain a new TBox $\mathcal{T}'$:

$$\begin{aligned}
\alpha_1 &\doteq P_1 \sqcap \alpha_2 \sqcap \exists r_1.\beta_1 \\
\beta_1 &\doteq \exists r_2.\alpha_3 \\
\alpha_2 &\doteq P_2 \sqcap \alpha_3 \sqcap \exists s.\beta_2 \\
\beta_2 &\doteq \alpha_3 \sqcap P_3 \\
\alpha_3 &\doteq P_4
\end{aligned}$$

This step is formalized as the exhaustive application of the rule $R_\exists$.

Condition: applies to concept definitions of the form $\alpha \doteq C_1 \sqcap \ldots \sqcap C_n$ if there is an index $i \in \{1, \ldots, n\}$ with $C_i = \exists r.D$ and $D \notin \mathsf{N}_{def}$.

Action: its application replaces the conjunct C_i by $\exists r.\beta$, and introduces a new definition $\beta \doteq D$, where $\beta \in \mathsf{N}_{def}$ is a fresh concept name.

Since α_1, α_2 and β_2 contain top-level atoms which are defined concepts, $\mathcal{T}'$ is not yet normalized. The original normalization process is devised to handle cyclic $\mathcal{EL}$ TBoxes that can be interpreted by different types of semantics. Consequently, the approach used to overcome this problem varies according to each semantics. In our case, however, this becomes simpler since the $\mathcal{EL}$ TBox $\mathcal{T}_{\mathcal{EL}}$ we are dealing with is acyclic. The solution for this follows from the discussion presented in [Baa02] for the general case, and consists of substituting these occurrences of defined concepts by their definitions. Following the example we obtain the following TBox:

$$\begin{aligned}
\alpha_1 &\doteq P_1 \sqcap P_2 \sqcap P_4 \sqcap \exists s.\beta_2 \sqcap \exists r_1.\beta_1 \\
\beta_1 &\doteq \exists r_2.\alpha_3 \\
\alpha_2 &\doteq P_2 \sqcap P_4 \sqcap \exists s.\beta_2 \\
\beta_2 &\doteq P_4 \sqcap P_3
\end{aligned}$$

$$\alpha_3 \doteq P_4$$

We name the corresponding rule R_α and formally define it as follows.

Condition: applies to concept definitions of the form $\alpha \doteq C_1 \sqcap \ldots \sqcap C_n$ if there is an index $i \in \{1, \ldots, n\}$ with $C_i = \beta$ and $\beta \doteq C_\beta \in \mathcal{T}$.

Action: its application replaces C_i by C_β.

Then, once $R_\exists$ can no longer be applied, an exhaustive application of the rule R_α will produce a normalized acyclic $\mathcal{EL}$ TBox. However, to have a polynomial time procedure generating a new TBox of polynomial size, the sequence of applications of R_α should not be arbitrary. This is achieved by following the order $\preceq$ induced by $\rightarrow^+$, i.e., R_α can be applied to a concept definition $\alpha \doteq C_\alpha$ only if it has already been applied to all $\beta \in \mathsf{def}(\mathcal{T})$ such that $\beta \preceq \alpha$. ♢

Each application of $R_\exists$ replaces a top-level atom of the form $\exists r.D$ with a new atom $\exists r.\beta$, and introduces a simpler definition $\beta \doteq D$. Concerning R_α, such an ordered sequence of rule applications will always terminate since we are dealing with acyclic TBoxes. Moreover, the *idempotency* of $\sqcap$ can be exploited to avoid duplications. Hence, R_α is only applied one time for each top-level atom of the form $\beta \in \mathsf{N}_{def}$ occurring in the TBox that results from the application of $R_\exists$, and it does not cause an exponential *blow-up* of the size of the TBox. Thus, the described normalization procedure runs in polynomial time and produces a TBox $\mathcal{T}'$ of size polynomial in the size of $\mathcal{T}$.

This procedure can be easily adapted to normalize acyclic $\tau\mathcal{EL}(deg)$ TBoxes. The rules $R_\exists$ and R_α can be applied to $\mathcal{T}_\tau$ in the same way. The only difference is that to apply R_α in $\mathcal{T}_\tau$, the definition $\beta \doteq C_\beta$ may also occur in $\mathcal{T}_{\mathcal{EL}}$. Additionally, it is required that all occurrences of threshold concepts $E_{\sim t}$ in $\widehat{\mathcal{T}}$ are such that E is a defined concept in $\mathcal{T}_{\mathcal{EL}}$. For example, α_1 could have been defined as:

$$\alpha_1 \doteq P_1 \sqcap \exists r_1.[(P_2 \sqcap \exists r_2.P_3)_{\leq .8}] \sqcap \exists r_1.\exists r_2.\alpha_3$$

To handle this we use a new rule $R_\sim$.

Condition: applies to concept definitions of the form $\alpha \doteq \widehat{C}_1 \sqcap \ldots \sqcap \widehat{C}_n \in \mathcal{T}_\tau$ if there is an index $i \in \{1, \ldots, n\}$ with $\widehat{C}_i = D_{\sim t}$ and $D \notin \mathsf{N}^0_{def}$.

Action: its application replaces the conjunct $\widehat{C}_i$ by $(E_D)_{\sim t}$, and adds a new definition $E_D \doteq D$ to $\mathcal{T}_{\mathcal{EL}}$, being E_D a fresh concept name in N^0_{def}.

Thus, the normalization will yield the $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}} = (\mathcal{T}_\tau, \mathcal{T}_{\mathcal{EL}})$ consisting of the following two sets of definitions:

$$\begin{aligned}
\alpha_1 &\doteq P_1 \sqcap \exists r_1.\beta_4 \sqcap \exists r_1.\beta_1 \\
\beta_4 &\doteq E_{\leq .8} \\
\beta_1 &\doteq \exists r_2.\alpha_3 \\
\alpha_2 &\doteq P_2 \sqcap P_4 \sqcap \exists s.\beta_2 \\
\beta_2 &\doteq P_4 \sqcap P_3 \\
\alpha_3 &\doteq P_4
\end{aligned}$$

and $\mathcal{T}_{\mathcal{EL}}$ the following set:

$$E \doteq P_2 \sqcap \exists r_2.P_3$$

Notice that in order to trigger the application of $R_\sim$, the concerned existential restriction in the definition of α_1 had to be first decomposed by applying $R_\exists$. With this in mind, we define the normalization procedure for acyclic $\tau\mathcal{EL}(deg)$ TBoxes as the execution of the following steps.

1. Apply the rule $R_\exists$ exhaustively to $\mathcal{T}_\tau$.

2. Apply the rule $R_\sim$ to $\mathcal{T}_\tau$ as long as possible.

3. Normalize the augmented $\mathcal{EL}$ TBox $\mathcal{T}_{\mathcal{EL}}$.

4. Apply the rule R_α exhaustively to $\mathcal{T}_\tau$.

The applications of $R_\exists$ and R_α to $\mathcal{T}_\tau$ modify only $\mathcal{T}_\tau$, while no new threshold expressions are introduced. Regarding the second step, as $D_{\sim t}$ is such that D is defined over $\mathsf{N}^0_{def} \cup \mathsf{N}_{prim}$, the threshold concept $(E_D)_{\sim t}$ introduced by the application of the rule $R_{\sim t}$ is still defined over $\mathsf{N}^0_{def} \cup \mathsf{N}_{prim}$. Furthermore, adding $E_D \doteq D$ to $\mathcal{T}_{\mathcal{EL}}$ does not introduce any concept name $\alpha \in \mathsf{N}^\tau_{def}$ in definitions of $\mathcal{T}_{\mathcal{EL}}$. Finally, the normalization of $\mathcal{T}_{\mathcal{EL}}$ only transforms the structure of $\mathcal{T}_{\mathcal{EL}}$. Therefore, $\widehat{\mathcal{T}}'$ satisfies the restrictions required for $\tau\mathcal{EL}(m)$ TBoxes in Definition 6.6, and it is easy to see that no cycles are introduced in it.

Now, after the first step has been executed, all occurrences of threshold concepts in $\mathcal{T}_\tau$ appear as top-level atoms on its concept definitions. Consequently, the application of $R_\sim$ in the second step will cover all of them. Moreover, the normalization of $\mathcal{T}_{\mathcal{EL}}$ before the final step guarantees that $R_\exists$ need not be applied in case R_α applies to a defined concept in $\mathcal{T}_{\mathcal{EL}}$. Overall, this implies that the resulting TBox $\widehat{\mathcal{T}}'$ is normalized.

Last, one can see that the rule $R_\sim$ is applied at most one time for each threshold concept $D_{\sim t}$ occurring in a definition of the initial TBox $\mathcal{T}_\tau$. Consequently, at most polynomially many new definitions of the form $E_D \doteq D$ are added to $\mathcal{T}_{\mathcal{EL}}$. Thus, using the same arguments given for the application of $R_\exists$ and R_α in the $\mathcal{EL}$ setting, the devised normalization procedure runs in polynomial time and yields a normalized acyclic $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}}'$ of size polynomial in the size of $\widehat{\mathcal{T}}$.

We now show that normalization preserves the unfolding of defined concepts.

Lemma 6.25. *Let $\widehat{\mathcal{T}}$ be an acyclic $\tau\mathcal{EL}(deg)$ TBox and $\widehat{\mathcal{T}}'$ the $\tau\mathcal{EL}(deg)$ TBox that results from a single application of a normalization rule. Then, for all defined concepts α in $\widehat{\mathcal{T}}$, $u_{\widehat{\mathcal{T}}}(\alpha) = u_{\widehat{\mathcal{T}}'}(\alpha)$*

Proof. Let R be a normalization rule and $\beta \doteq \widehat{C}_\beta \in \widehat{\mathcal{T}}$ the concept definition that R has been applied to. We use *well-founded* induction on the partial order induced by $\rightarrow^+$ on $\mathsf{def}(\widehat{\mathcal{T}})$. For all defined concepts α in $\widehat{\mathcal{T}}$ we distinguish two cases:

- $\alpha \neq \beta$. This means that R was not applied to $\alpha \doteq \widehat{C}_\alpha$, and consequently $\alpha \doteq \widehat{C}_\alpha \in \widehat{\mathcal{T}}'$. The top-down recursive application of unfolding through the structure of $\widehat{C}_\alpha$

with respect to $\widehat{\mathcal{T}}$ and $\widehat{\mathcal{T}}'$ may only result in different concept descriptions if:

$$u_{\widehat{\mathcal{T}}}(\alpha') \neq u_{\widehat{\mathcal{T}}'}(\alpha')$$

for some symbol α' occurring in $\widehat{C}_\alpha$ that corresponds to a defined concept name in $\widehat{\mathcal{T}}$. However, $\alpha \to^+ \alpha'$ and the application of the induction hypothesis to α' imply that this is never the case. Hence, $u_{\widehat{\mathcal{T}}}(\alpha) = u_{\widehat{\mathcal{T}}'}(\alpha)$.

- $\alpha = \beta$. Let $\widehat{C}_\beta$ be of the form $\widehat{C}_1 \sqcap \ldots \sqcap \widehat{C}_n$. We analyze the outcome of applying each of the three possible rules to $\beta \doteq \widehat{C}_\beta$:
  - $R_\sim$: the rule was applied to a conjunct $\widehat{C}_i$ such that $\widehat{C}_i = D_{\sim t}$ and $D \notin \mathsf{N}^0_{def}$. Its application replaces $\widehat{C}_i$ by $(E_D)_{\sim t}$ in $\widehat{C}_\beta$, and adds $E_D \doteq D$ to $\mathcal{T}'_{\mathcal{EL}}$ where E_D is a fresh concept name. By definition of unfolding we have:

$$u_{\widehat{\mathcal{T}}}(\beta) = \prod_{j=1}^{i-1} u_{\widehat{\mathcal{T}}}(\widehat{C}_j) \sqcap [u_{\widehat{\mathcal{T}}}(D)]_{\sim t} \sqcap \prod_{j=i+1}^{n} u_{\widehat{\mathcal{T}}}(\widehat{C}_j)$$

    and,

$$u_{\widehat{\mathcal{T}}'}(\beta) = \prod_{j=1}^{i-1} u_{\widehat{\mathcal{T}}'}(\widehat{C}_j) \sqcap [u_{\widehat{\mathcal{T}}'}(E_D)]_{\sim t} \sqcap \prod_{j=i+1}^{n} u_{\widehat{\mathcal{T}}'}(\widehat{C}_j)$$

    Applying the same inductive argument used above we obtain $u_{\widehat{\mathcal{T}}}(\widehat{C}_j) = u_{\widehat{\mathcal{T}}'}(\widehat{C}_j)$ for all $j \neq i$ (likewise for D). Thus, since $u_{\widehat{\mathcal{T}}'}(E_D) = u_{\widehat{\mathcal{T}}'}(D)$ it follows that $u_{\widehat{\mathcal{T}}}(\beta) = u_{\widehat{\mathcal{T}}'}(\beta)$.
  - $R_\exists$: the rule has been applied to an atom $\widehat{C}_i$ of the form $\exists r.\widehat{D}$ such that $\widehat{D} \notin \mathsf{N}_{def}$. Hence, $\widehat{C}_i$ is substituted in $\widehat{C}_\beta$ by $\exists r.\beta_1$ with β_1 being a fresh concept name and $\beta_1 \doteq \widehat{D} \in \widehat{\mathcal{T}}'$. By definition of unfolding we have:

$$u_{\widehat{\mathcal{T}}}(\beta) = \prod_{j=1}^{i-1} u_{\widehat{\mathcal{T}}}(\widehat{C}_j) \sqcap \exists r.u_{\widehat{\mathcal{T}}}(\widehat{D}) \sqcap \prod_{j=i+1}^{n} u_{\widehat{\mathcal{T}}}(\widehat{C}_j)$$

    and,

$$u_{\widehat{\mathcal{T}}'}(\beta) = \prod_{j=1}^{i-1} u_{\widehat{\mathcal{T}}'}(\widehat{C}_j) \sqcap \exists r.u_{\widehat{\mathcal{T}}'}(\beta_1) \sqcap \prod_{j=i+1}^{n} u_{\widehat{\mathcal{T}}'}(\widehat{C}_j)$$

    We know that $u_{\widehat{\mathcal{T}}'}(\beta_1) = u_{\widehat{\mathcal{T}}'}(\widehat{D})$. Hence, the same reasoning used for $R_\sim$ yields $u_{\widehat{\mathcal{T}}}(\alpha) = u_{\widehat{\mathcal{T}}'}(\alpha)$.
  - R_α: there is an index $1 \leq i \leq n$ such that $\widehat{C}_i$ is of the form β_1, and $\beta_1 \doteq \widehat{C}_{\beta_1} \in \widehat{\mathcal{T}}$. The application of R_α to β_1 replaces it in $\widehat{C}_\beta$ with $\widehat{C}_{\beta_1}$. Since $\beta \to^+ \beta_1$, the application of induction hypothesis yields $u_{\widehat{\mathcal{T}}}(\beta_1) = u_{\widehat{\mathcal{T}}'}(\beta_1) = u_{\widehat{\mathcal{T}}'}(\widehat{C}_{\beta_1})$. Again, $u_{\widehat{\mathcal{T}}}(\widehat{C}_j) = u_{\widehat{\mathcal{T}}'}(\widehat{C}_j)$ for all $j \neq i$, and the rest follows from the definition of unfolding on β. □

This property is then invariant under any number of rule applications. Therefore, the following proposition is a direct consequence of Lemma 6.25.

Proposition 6.26. *Let $\widehat{\mathcal{T}}$ be an acyclic $\tau\mathcal{EL}(deg)$ TBox and $\widehat{\mathcal{T}}'$ the normal form of $\widehat{\mathcal{T}}$. For all defined concepts α in $\widehat{\mathcal{T}}$, $u_{\widehat{\mathcal{T}}}(\alpha) = u_{\widehat{\mathcal{T}}'}(\alpha)$.*

Proposition 6.26 implies that reasoning with respect to an acyclic $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}}$ can be reduced to reasoning with respect to its normal form $\widehat{\mathcal{T}}'$. Therefore, from now on we only consider normalized TBoxes.

We still require one more transformation. Recall that for acyclic $\mathcal{EL}$ TBoxes, the value of $deg^{\mathcal{I}}(d, C, \mathcal{T})$ is defined in terms of applying the basic definition of *deg* (Chapter 4) to the unfolding of C in $\mathcal{T}$. Moreover, *deg* needs to further translate $u_{\mathcal{T}}(C)$ into its reduced form $[u_{\mathcal{T}}(C)]^r$. Since $u_{\mathcal{T}}(C)$ may result in a concept of exponential size, it is certainly not a good idea to unfold and then compute the reduced form. To have this issue handled in a more transparent way by the decision procedures presented in the next section, we introduce the reduced form for acyclic $\mathcal{EL}$ TBoxes. The ideas that follow are based on the results shown by Küsters in [Küs01].

Definition 6.27. Let $\mathcal{T}$ be an acyclic $\mathcal{EL}$ TBox and C an $\mathcal{EL}$ concept description. Then, C is *reduced* with respect to $\mathcal{T}$ iff:

- C is *reduced* according to Küsters' definition modulo $\sqsubseteq_{\mathcal{T}}$ (i.e., $\sqsubseteq_{\mathcal{T}}$ is used to identify redundancies instead of $\sqsubseteq$).

We say that $\mathcal{T}$ is in reduced form iff for all $\alpha \doteq C_\alpha \in \mathcal{T}$ the concept C_α is reduced with respect to $\mathcal{T}$. ♢

The benefit of using these type of TBoxes is that the unfolding of a defined concept will always result in a reduced concept description.

Lemma 6.28. *Let $\mathcal{T}$ be a normalized acyclic $\mathcal{EL}$ TBox in reduced form. Then, for all $\alpha \doteq C_\alpha$ the $\mathcal{EL}$ concept description $u_{\mathcal{T}}(\alpha)$ is reduced.*

Proof. We use *well-founded* induction on $\rightarrow^+$ over $\mathsf{def}(\mathcal{T})$. Since $\mathcal{T}$ is normalized, C_α has the following structure:

$$P_1 \sqcap \ldots \sqcap P_m \sqcap \exists r_1.\alpha_1 \sqcap \ldots \sqcap \exists r_n.\alpha_n$$

Clearly, $\alpha \rightarrow^+ \alpha_i$ for all $1 \leq i \leq n$. Therefore, the application of induction hypothesis yields that $u_{\mathcal{T}}(\alpha_i)$ is reduced. Now, since C_α is reduced with respect to $\mathcal{T}$, for all pairs $(\exists r_i.\alpha_i, \exists r_j.\alpha_j)$ we have:

- $r_i \neq r_j$, or
- $\alpha_i \not\sqsubseteq_{\mathcal{T}} \alpha_j$ and $\alpha_j \not\sqsubseteq_{\mathcal{T}} \alpha_i$.

In addition, we know that $\alpha_i \equiv_{\mathcal{T}} u_{\mathcal{T}}(\alpha_i)$ and $\alpha_j \equiv_{\mathcal{T}} u_{\mathcal{T}}(\alpha_j)$. This means that having $r_i = r_j$, it will be the case that $u_{\mathcal{T}}(\alpha_i) \not\sqsubseteq u_{\mathcal{T}}(\alpha_j)$ and $u_{\mathcal{T}}(\alpha_j) \not\sqsubseteq u_{\mathcal{T}}(\alpha_i)$. Finally, since $u_{\mathcal{T}}(\alpha)$ is the following concept description

$$P_1 \sqcap \ldots P_m \sqcap \exists r_1.u_{\mathcal{T}}(\alpha_1) \sqcap \ldots \sqcap \exists r_n.u_{\mathcal{T}}(\alpha_n)$$

we can conclude that $u_{\mathcal{T}}(\alpha)$ is reduced. □

To translate acyclic $\mathcal{EL}$ TBoxes into its reduced form, the algorithm sketched in [Küs01] (derived from Proposition 6.3.1.) to compute the reduced form of $\mathcal{EL}$ concept descriptions comes in handy. By using $\sqsubseteq_{\mathcal{T}}$ instead of $\sqsubseteq$, it will be able to compute the reduced form $C^{r(\mathcal{T})}$ of a concept C with respect to $\mathcal{T}$. Since $\sqsubseteq_{\mathcal{T}}$ is decidable in polynomial time in $\mathcal{EL}$ [Baa03], the modified procedure also runs in polynomial time. Moreover, the concept $C^{r(\mathcal{T})}$ satisfies $C \equiv_{\mathcal{T}} C^{r(\mathcal{T})}$.

Based on this we can devise a very simple polynomial time transformation that given an acyclic $\mathcal{EL}$ TBox $\mathcal{T}$ outputs and equivalent TBox $\mathcal{T}'$ in reduced form. The translation and its correctness are given in the following lemma.

Lemma 6.29. *Let $\mathcal{T}$ be a normalized acyclic $\mathcal{EL}$ TBox. The TBox $\mathcal{T}'$ obtained from $\mathcal{T}$ by the substitution of $\alpha \doteq C_\alpha$ for $\alpha \doteq (C_\alpha)^{r(\mathcal{T})}$ (for all $\alpha \doteq C_\alpha \in \mathcal{T}$) satisfies the following:*

1. *$\mathcal{T}$ and $\mathcal{T}'$ are equivalent.*
2. *$\mathcal{T}'$ is in reduced form.*

Proof. 1) We show that every model of $\mathcal{T}$ is a model of $\mathcal{T}'$ and vice versa. Let $\mathcal{I}$ be a model of $\mathcal{T}$, then $\alpha^{\mathcal{I}} = (C_\alpha)^{\mathcal{I}}$ for all $\alpha \doteq C_\alpha \in \mathcal{T}$. Since $C_\alpha \equiv_{\mathcal{T}} (C_\alpha)^{r(\mathcal{T})}$, this means that $\alpha^{\mathcal{I}} = \left[(C_\alpha)^{r(\mathcal{T})}\right]^{\mathcal{I}}$ for all $\alpha \doteq (C_\alpha)^{r(\mathcal{T})} \in \mathcal{T}'$. Hence, $\mathcal{I} \models \mathcal{T}'$.

Conversely, let $\mathcal{I}'$ be a model of $\mathcal{T}'$. We take a model $\mathcal{I}$ of $\mathcal{T}$ such that $\Delta^{\mathcal{I}} = \Delta^{\mathcal{I}'}$ and $X^{\mathcal{I}} = X^{\mathcal{I}'}$, for all $X \in \mathsf{N}_{prim} \cup \mathsf{N}_{\mathsf{R}}$. Such a model exists because of Proposition 6.3. We prove that $\alpha^{\mathcal{I}} = \alpha^{\mathcal{I}'}$ for all $\alpha \doteq C_\alpha \in \mathcal{T}$. The proof goes by induction on the partial order induced by $\rightarrow^+$. Since $\mathcal{T}$ is normalized, each top-level atom of C_α is of the form $A \in \mathsf{N}_{prim}$ or $\exists r.\beta$, where $\beta \doteq C_\beta \in \mathcal{T}$. Moreover, the set of atoms occurring in $(C_\alpha)^{r(\mathcal{T})}$ is a subset of the corresponding set for C_α. Therefore, we distinguish two cases for all top-level atoms At of C_α:

- At occurs in $(C_\alpha)^{r(\mathcal{T})}$. If $\mathsf{At} = A$, by selection of $\mathcal{I}$ we have $A^{\mathcal{I}} = A^{\mathcal{I}'}$. Otherwise, $\mathsf{At} = \exists r.\beta$ and $\alpha \rightarrow^+ \beta$. The application of induction yields $\beta^{\mathcal{I}} = \beta^{\mathcal{I}'}$ and thus $(\exists r.\beta)^{\mathcal{I}} = (\exists r.\beta)^{\mathcal{I}'}$. Hence, it is not hard to see that for all $d \in \Delta^{\mathcal{I}}$, $d \in \alpha^{\mathcal{I}}$ implies $d \in \alpha^{\mathcal{I}'}$.
- At only occurs in C_α. There must be a top-level atom At' in C_α such that $\mathsf{At}' \sqsubseteq_{\mathcal{T}} \mathsf{At}$ and At' does occur in $(C_\alpha)^{r(\mathcal{T})}$. From the previous point we know that $(\mathsf{At}')^{\mathcal{I}} = (\mathsf{At}')^{\mathcal{I}'}$. Therefore, if $d \in (\mathsf{At}')^{\mathcal{I}'}$ we also have $d \in (\mathsf{At}')^{\mathcal{I}}$ and $d \in \mathsf{At}^{\mathcal{I}}$. Hence, $d \in \alpha^{\mathcal{I}'}$ implies $d \in \alpha^{\mathcal{I}}$.

Thus, we have shown that $\alpha^{\mathcal{I}} = \alpha^{\mathcal{I}'}$. This implies the following equalities:

$$\alpha^{\mathcal{I}'} = \alpha^{\mathcal{I}} = (C_\alpha)^{\mathcal{I}} = (u_{\mathcal{T}}(C_\alpha))^{\mathcal{I}}$$

Then, since $\mathcal{I}$ and $\mathcal{I}'$ have the same interpretation for $\mathsf{N}_{prim} \cup \mathsf{N}_{\mathsf{R}}$, this means that $[u_{\mathcal{T}}(C_\alpha)]^{\mathcal{I}} = [u_{\mathcal{T}}(C_\alpha)]^{\mathcal{I}'}$. Hence, for all $\alpha \doteq C_\alpha \in \mathcal{T}$ we have:

$$\alpha^{\mathcal{I}'} = [u_{\mathcal{T}}(C_\alpha)]^{\mathcal{I}'}$$

Consequently, $\alpha^{\mathcal{I}'} = (C_\alpha)^{\mathcal{I}'}$ and $\mathcal{I}'$ is a model of $\mathcal{T}$.

2) Assume that $\mathcal{T}'$ is not in reduced form. Then, there exists $\alpha \doteq (C_\alpha)^{r(\mathcal{T})} \in \mathcal{T}'$ such that $(C_\alpha)^{r(\mathcal{T})}$ is reducible with respect to $\mathcal{T}'$. This means that there are two top-level atoms At_1 and At_2 in $(C_\alpha)^{r(\mathcal{T})}$ such that $\mathsf{At}_1 \sqsubseteq_{\mathcal{T}'} \mathsf{At}_2$. Since we just have shown that $\mathcal{T}$ and $\mathcal{T}'$ are equivalent from a model-theoretic point of view, we also have $\mathsf{At}_1 \sqsubseteq_{\mathcal{T}} \mathsf{At}_2$. Hence, we obtain a contradiction against the fact that $(C_\alpha)^{r(\mathcal{T})}$ is reduced with respect to $\mathcal{T}$. Thus, $\mathcal{T}'$ is in reduced form. □

To sum up, given an acyclic $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}} = (\mathcal{T}_\tau, \mathcal{T}_{\mathcal{EL}})$, we have demonstrated the following along this section:

- $\widehat{\mathcal{T}}$ can be normalized in polynomial time into an acyclic TBox $\widehat{\mathcal{T}}' = (\mathcal{T}'_\tau, \mathcal{T}'_{\mathcal{EL}})$, such that reasoning w.r.t. $\widehat{\mathcal{T}}$ can be reduced to reasoning w.r.t. $\widehat{\mathcal{T}}'$.
- The new TBox $\mathcal{T}'_{\mathcal{EL}}$ can be translated in polynomial time into an equivalent $\mathcal{EL}$ TBox $\mathcal{T}''_{\mathcal{EL}}$ in reduced form.
- The computation of the reduced form only removes atoms from concept definitions. Therefore, $\mathcal{T}''_{\mathcal{EL}}$ remains normalized.

Hence, reasoning in $\tau\mathcal{EL}(deg)$ with respect to acyclic TBoxes can be restricted to normalized acyclic TBoxes in reduced form.

Proposition 6.30. *Satisfiability and subsumption on concepts defined in an acyclic $\tau\mathcal{EL}(deg)$ TBox can be reduced in polynomial time to satisfiability and subsumption on concepts defined in a normalized acyclic $\tau\mathcal{EL}(deg)$ TBox in reduced form.*

6.4.3 Upper bounds

We now provide a PSPACE algorithm to decide satisfiability and subsumption with respect to an acyclic $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}}$. Note that one can focus on satisfiability of concepts α and subsumption questions of the form $\beta_1 \sqsubseteq_{\widehat{\mathcal{T}}} \beta_2$, where $\alpha, \beta_1, \beta_2 \in \mathsf{def}(\widehat{\mathcal{T}})$. Any concept description $\widehat{C}$ can be equivalently replaced with a *fresh* defined concept $\alpha_{\widehat{C}}$, by adding the definition $\alpha_{\widehat{C}} \doteq \widehat{C}$ to $\mathcal{T}_\tau$.

As explained in Section 6.2, by using unfolding, *satisfiability* and *subsumption* with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes can be reduced to reasoning with the empty TBox. In addition, in Chapter 5 we showed that a concept description of the form $\widehat{C} \sqcap \neg\widehat{D}$ is satisfiable in $\tau\mathcal{EL}(deg)$ iff there exists an interpretation $\mathcal{I}$ such that $\widehat{C}^\mathcal{I} \setminus \widehat{D}^\mathcal{I} \neq \emptyset$ and $|\Delta^\mathcal{I}| \leq \mathsf{s}(\widehat{C}) \times \mathsf{s}(\widehat{D})$ (see Lemma 5.6). Hence, given an acyclic $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}}$ and two of its defined concepts α_1 and α_2: $\alpha_1 \sqcap \neg\alpha_2$ is satisfiable with respect to $\widehat{\mathcal{T}}$ iff there exists an interpretation $\mathcal{I}$ over $\mathsf{N}_{prim} \cup \mathsf{N_R}$ such that:

- $[u_{\widehat{\mathcal{T}}}(\alpha_1)]^\mathcal{I} \setminus [u_{\widehat{\mathcal{T}}}(\alpha_2)]^\mathcal{I} \neq \emptyset$, and
- $|\Delta^\mathcal{I}| \leq \mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_1)) \times \mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_2))$.

Now, there is unique way to extend $\mathcal{I}$ into a model of $\widehat{\mathcal{T}}$ (Proposition 6.11). Therefore, we obtain the following bounded model property for satisfiability of concepts of the form $\alpha_1 \sqcap \neg\alpha_2$ in the presence of acyclic $\tau\mathcal{EL}(deg)$ TBoxes.

Proposition 6.31. *Let $\widehat{\mathcal{T}}$ be an acyclic $\tau\mathcal{EL}(deg)$ TBox and α_1, α_2 two defined concepts in $\widehat{\mathcal{T}}$. If $\alpha_1 \sqcap \neg\alpha_2$ is satisfiable in $\widehat{\mathcal{T}}$, then there exists an interpretation $\mathcal{I} \models \widehat{\mathcal{T}}$ such that $(\alpha_1)^{\mathcal{I}} \setminus (\alpha_2)^{\mathcal{I}} \neq \emptyset$ and $|\Delta^{\mathcal{I}}| \leq \mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_1)) \times \mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_2))$.*

For the empty terminology $|\Delta^{\mathcal{I}}|$ is polynomial in the size of $\widehat{C}$ and $\widehat{D}$. This was used to provide an NP-algorithm for satisfiability of concepts of the form $\widehat{C} \sqcap \neg\widehat{D}$, which uses non-determinism to guess the whole interpretation $\mathcal{I}$. Therefore, since $u_{\widehat{\mathcal{T}}}(\alpha_1)$ may result in a concept description of size exponential in $\mathsf{s}(\widehat{\mathcal{T}})$, the same procedure applied to $u_{\widehat{\mathcal{T}}}(\alpha_1)$ or $u_{\widehat{\mathcal{T}}}(\alpha_1) \sqcap \neg u_{\widehat{\mathcal{T}}}(\alpha_2)$ would give a NEXP-algorithm for concept satisfiability and non-subsumption with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes.

Our aim is to design a PSPACE algorithm that solves these problems. Obviously, such a procedure cannot store the whole interpretation $\mathcal{I}$. However, the proof of Lemma 5.6 tells us the following:

- $\mathcal{I}$ is *tree-shaped*,
- the depth of the associated description tree $T_{\mathcal{I}}$ is bounded by:

$$\mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha_1)) + \mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha_2))$$

- the domain element d_0 of $\mathcal{I}$ corresponding to the root of $T_{\mathcal{I}}$ satisfies:

$$d_0 \in [u_{\widehat{\mathcal{T}}}(\alpha_1) \sqcap \neg u_{\widehat{\mathcal{T}}}(\alpha_2)]^{\mathcal{I}}$$

Fortunately, the depth $\mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha_1)) + \mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha_2))$ is always polynomial in $\mathsf{s}(\widehat{\mathcal{T}})$. Thus, despite its size, it is possible to non-deterministically generate $\mathcal{I}$ in a top-down fashion, while keeping the used space polynomial in $\mathsf{s}(\widehat{\mathcal{T}})$. Let $\mathfrak{d}$ and $\mathfrak{b} > 0$ be natural numbers. The following procedure is meant to generate all the *tree-shaped* interpretations $\mathcal{I}$ over $\mathsf{N}_{prim} \cup \mathsf{N_R}$, such that $|\Delta^{\mathcal{I}}| \leq \mathfrak{b}$ and the depth of $T_{\mathcal{I}}$ is not greater than $\mathfrak{d}$:

procedure A($\mathfrak{d} : \mathbb{N}$, $\mathfrak{b}$: *binary*)
    $\mathfrak{b} := \mathfrak{b} - 1$ // counts the individual represented by the current call
    non-deterministically choose a subset $\mathcal{P}$ of N_{prim}
    if $(\mathfrak{d} \neq 0)$ **and** $(\mathfrak{b} \neq 0)$ **then**
        for all $r \in \mathsf{N_R}$ **do**
            non-deterministically choose $0 \leq \mathfrak{b}_r \leq \mathfrak{b}$ // $\mathfrak{b}_r$: *binary*
            $\mathfrak{b} := \mathfrak{b} - \mathfrak{b}_r$
            for all $1 \leq i \leq \mathfrak{b}_r$ **do**
                non-deterministically choose $0 \leq \mathfrak{b}_r^i \leq \mathfrak{b}$ // $\mathfrak{b}_r^i$: *binary*
                $\mathfrak{b} := \mathfrak{b} - \mathfrak{b}_r^i$
                A$(\mathfrak{d} - 1, \mathfrak{b}_r^i + 1)$
            end for
        end for
    end if
end procedure

Note that each recursive call decreases the value of $\mathfrak{d}$, which implies that this is a terminating procedure executing at most $\mathfrak{d}$ nested recursive calls. Moreover, the parameter

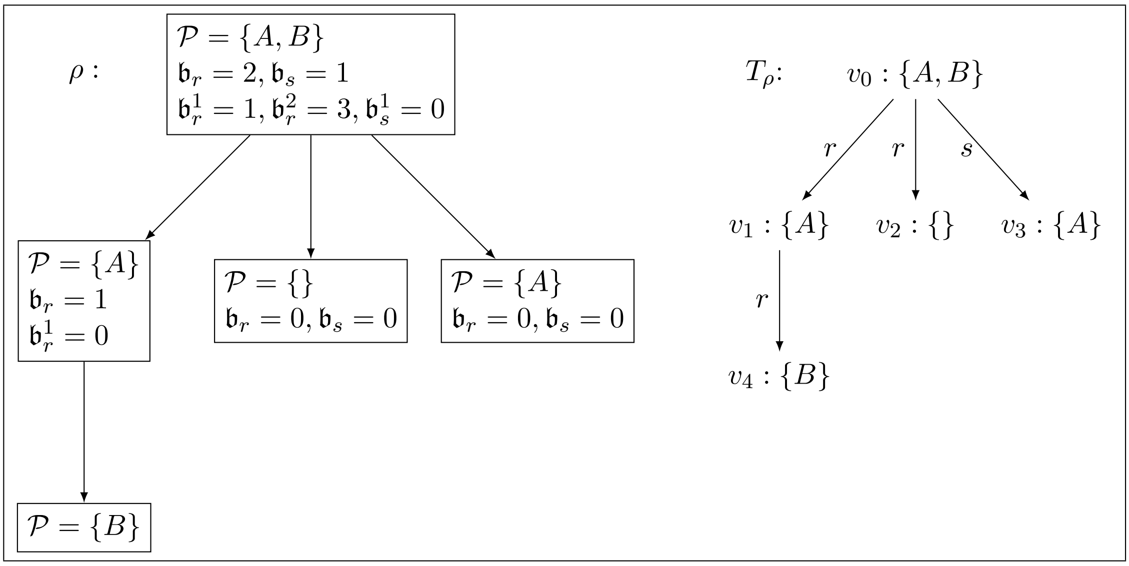


Figure 6.1: A run ρ of A and its induced $\mathcal{EL}$ description tree T_ρ.

declaration $\mathfrak{b}$: *binary* states that A works with the binary representation of the value $\mathfrak{b}$. As mentioned above, we are dealing with interpretations that may have size exponential in $\mathsf{s}(\widehat{\mathcal{T}})$, and that is why the use of a binary counter to represent the value of $\mathfrak{b}$. Finally, the set of variables $\mathfrak{b}_r$ and $\mathfrak{b}_r^i$ can be reduced to two variables since they are only used within the scope of the **for** loops. Therefore, each run of A uses space polynomial on $\mathfrak{d}$ and the number of bits needed to represent $\mathfrak{b}$.

The general idea of the procedure is as follows: each recursive call represents an individual of the domain and the recursion tree lays out the tree-shaped form of an interpretation. The set $\mathcal{P}$ contains the primitive concept names that a domain element is an instance of, the number $\mathfrak{b}_r$ stands for the number of r-successors, and $\mathfrak{b}_r^i$ means that the interpretation rooted at the i-th r-successor has at most $\mathfrak{b}_r^i + 1$ elements. To formalize this intuition we define the notion of a run of A.

Definition 6.32. A run ρ of A on $(\mathfrak{d}, \mathfrak{b})$ is a tree of recursive calls $T_{(\mathfrak{d},\mathfrak{b})}$ such that:

- its root v_0 is labeled by the non-deterministic choices $\mathcal{P}$, $\mathfrak{b}_r$ for all $r \in \mathsf{N_R}$, and $\mathfrak{b}_r^i$ for all $1 \leq i \leq \mathfrak{b}_r$.
- for all $r \in \mathsf{N_R}$, there are exactly $\mathfrak{b}_r$ successors $v_{r1}, \ldots, v_{r\mathfrak{b}_r}$ of v_0 such that, the tree rooted at v_{ri} is a run of A on $(\mathfrak{d} - 1, \mathfrak{b}_r^i + 1)$. ♢

Figure 6.1 depicts a run ρ of A (left-hand side). Such a run induces the $\mathcal{EL}$ description tree T_ρ (right-hand side) with the same structure, where its nodes are labeled with the corresponding sets $\mathcal{P}$ chosen by ρ and the edges with the role names generating the corresponding recursive call (line 5 in A). Therefore, we say that ρ induces the interpretation $\mathcal{I}_\rho$ that has the description tree T_ρ.

Conversely, for all tree-shaped interpretations $\mathcal{I}$ of size at most $\mathfrak{b}$ and depth not greater than $\mathfrak{d}$, there is always a run of A describing $\mathcal{I}$.

Lemma 6.33. *Let $\mathfrak{d} \geq 0$ and $\mathfrak{b} > 0$ be two natural numbers. For all tree-shaped interpretations $\mathcal{I}$ over $\mathsf{N}_{prim} \cup \mathsf{N_R}$ with at most $\mathfrak{b}$ elements and depth not greater than $\mathfrak{d}$, there exists a run ρ of A on $(\mathfrak{d}, \mathfrak{b})$ such that $\mathcal{I} = \mathcal{I}_\rho$.*

Proof. Let $\mathcal{I}$ be a tree-shaped interpretation of depth $\mathsf{d}(\mathcal{I})$ such that $|\Delta^{\mathcal{I}}| \leq \mathfrak{b}$ and $\mathsf{d}(\mathcal{I}) \leq \mathfrak{d}$. We show how to guide a run ρ of A such that $\mathcal{I}_\rho = \mathcal{I}$. The proof goes by induction on the number $\mathsf{d}(\mathcal{I})$.

Let $d_0 \in \Delta^{\mathcal{I}}$ be the root of $T_{\mathcal{I}}$. For all $r \in \mathsf{N_R}$ we denote as $r(d_0) = \{e_1, \dots, e_n\}$ $(n \geq 0)$ the set of r-successors of d_0 in $\mathcal{I}$. In addition, for an r-successor e_i of d_0, $T_{\mathcal{I}}[e_i]$ denotes the subtree of $T_{\mathcal{I}}$ rooted at e_i, and $\mathcal{I}_{e_i}$ the associated interpretation. Then, when A is invoked on $(\mathfrak{d}, \mathfrak{b})$ it makes the following non-deterministic choices:

- $\mathcal{P} = \ell_{\mathcal{I}}(d_0)$,
- for all $r \in \mathsf{N_R}$: $b_r = |r(d_0)|$,
- for all $r \in \mathsf{N_R}$ and $e_i \in r(d_0)$: $\mathfrak{b}^i_r = |\Delta^{\mathcal{I}_{e_i}}| - 1$,
- for all $r \in \mathsf{N_R}$ and $1 \leq i \leq \mathfrak{b}_r$, the recursive call $\mathrm{A}(\mathfrak{d} - 1, \mathfrak{b}^i_r + 1)$ follows a run ρ^i_r such that $\mathcal{I}_{\rho^i_r} = \mathcal{I}_{e_i}$.

Since $|\Delta^{\mathcal{I}}| \leq \mathfrak{b}$ and $\mathsf{d}(\mathcal{I}) \leq \mathfrak{d}$, the first three choices are consistent with the execution of A. Regarding the last choice, since $T_{\mathcal{I}}$ is a tree we know that $\mathsf{d}(\mathcal{I}_{e_i}) < \mathsf{d}(\mathcal{I})$. Consequently, $\mathsf{d}(\mathcal{I}_{e_i}) \leq \mathfrak{d} - 1$ and the induction hypothesis can be applied to obtain the proper run ρ^i_r. Therefore, ρ induces an $\mathcal{EL}$ description tree T_ρ such that:

- its root v_0 is labelled with $\ell_{\mathcal{I}}(d_0)$,
- for all $r \in \mathsf{N_R}$: v_0 has exactly $|r(d_0)|$ children $v_1, \dots, v_{|r(d_0)|}$, each edge (v_0, v_i) $(1 \leq i \leq |r(d_0)|)$ is labelled with r, and the subtree $T_\rho[v_i]$ rooted at v_i in T_ρ is equal to $T_{\mathcal{I}}[e_i]$.

Thus, we can conclude that $\mathcal{I}_\rho = \mathcal{I}$. □

The previous lemma ensures that, by choosing $\mathfrak{d}$ as $\mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha_1)) + \mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha_2))$ and $\mathfrak{b}$ as $\mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_1)) \times \mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_2))$, the set of runs of A on $(\mathfrak{d}, \mathfrak{b})$ covers all the relevant interpretations to find out if $u_{\widehat{\mathcal{T}}}(\alpha_1) \sqcap \neg u_{\widehat{\mathcal{T}}}(\alpha_2)$ is satisfiable. Therefore, it remains to see how to verify for each run ρ of A, whether its induced interpretation $\mathcal{I}_\rho$ fulfills $d_0 \in [u_{\widehat{\mathcal{T}}}(\alpha_1) \sqcap \neg u_{\widehat{\mathcal{T}}}(\alpha_2)]^{\mathcal{I}_\rho}$. We have already provided an algorithm to do that (Algorithm 2 in Chapter 3), which is based on Theorem 3.8. Nevertheless, on the one hand it is not immediate to link Algorithm 2 to the way A generates $\mathcal{I}_\rho$; and on the other hand due to the possible exponential size of $u_{\widehat{\mathcal{T}}}(\alpha_1)$ and $u_{\widehat{\mathcal{T}}}(\alpha_2)$, special care would be required in doing so.

To address these concerns, we go back to the initial formulation of our problem: *search for a model $\mathcal{J}$ of $\widehat{\mathcal{T}}$ such that $(\alpha_1 \sqcap \neg\alpha_2)^{\mathcal{J}} \neq \emptyset$.* Contrary to models of $\widehat{\mathcal{T}}$, interpretations induced by runs of A do not interpret defined concepts in $\widehat{\mathcal{T}}$. However, there is a unique way to extend each of them into a model of $\widehat{\mathcal{T}}$ (see Proposition 6.11). Hence, since $\mathcal{I}_\rho$ is tree-shaped, it is possible to compute such an extension in a bottom-up manner. From now on we will use indistinctively $\mathcal{I}_\rho$ to identify both, a primitive interpretation and its unique extension into a model of $\widehat{\mathcal{T}}$.

The idea is then to compute for all $\alpha \doteq \widehat{C}_\alpha \in \widehat{\mathcal{T}}$, whether $d_0 \in \alpha^{\mathcal{I}_\rho}$. To this end, we modify *procedure* A such that each run ρ additionally computes a set $\mathsf{Ex} \subseteq \mathsf{N}_{def}$ with the following meaning:

$$\mathsf{Ex} := \{\alpha \mid \alpha \doteq \widehat{C}_\alpha \in \widehat{\mathcal{T}} \text{ and } d_0 \in \alpha^{\mathcal{I}_\rho}\}$$

The special forms introduced in Section 6.4.2 for acyclic TBoxes are of great help in computing Ex. In particular, the normal form of $\widehat{\mathcal{T}}$ provides the following shape for $\widehat{C}_\alpha$:

$$\widehat{P}_1 \sqcap \ldots \sqcap \widehat{P}_n \sqcap \exists r_1.\alpha_1 \sqcap \ldots \sqcap \exists r_m.\alpha_m$$

Consequently for all $d \in \Delta^{\mathcal{I}_\rho}$, $d \in \alpha^{\mathcal{I}_\rho}$ iff:

1. $d \in (\widehat{P}_i)^{\mathcal{I}_\rho}$ for all $1 \leq i \leq n$, and
2. for all $1 \leq i \leq m$, there exists $d_i \in \Delta^{\mathcal{I}_\rho}$ such that $(d, d_i) \in (r_i)^{\mathcal{I}_\rho}$ and $d_i \in (\alpha_i)^{\mathcal{I}_\rho}$.

The computation of Ex will be based on checking these two conditions for d_0. If $\widehat{P}_i$ is of the form $A \in \mathsf{N}_{prim}$, verifying whether $d_0 \in A^{\mathcal{I}_\rho}$ is simple since $\mathcal{I}_\rho$ already contains that information (the non-deterministic choice in line 3). To check whether $d_0 \in (E_{\sim t})^{\mathcal{I}_\rho}$, we further extend A to compute for all runs ρ an assignment $D : \mathsf{def}(\mathcal{T}_{\mathcal{EL}}) \to [0,1]$ such that:

$$D(E) := deg^{\mathcal{I}_\rho}(d_0, u_{\mathcal{T}_{\mathcal{EL}}}(E))$$

Once D is computed for d_0, it is immediate to verify whether $d_0 \in (E_{\sim t})^{\mathcal{I}_\rho}$. Regarding Condition 2, as explained before the successors e of d_0 in $\mathcal{I}_\rho$ are the roots of the interpretations induced by runs corresponding to the recursive calls triggered by ρ. Hence, the sets Ex_e computed by such calls provide the necessary information to determine whether $d_0 \in (\exists r_i.\alpha_i)^{\mathcal{I}_\rho}$ for all $1 \leq i \leq m$. However, since d_0 may have exponentially many *direct* successors in $\mathcal{I}_\rho$, a PSPACE procedure cannot store all the corresponding sets Ex_e. To deal with this, A will compute a relation of the form $z \subseteq (\mathsf{N}_\mathsf{R} \times \mathsf{def}(\widehat{\mathcal{T}})) \cup (\epsilon \times \mathsf{N}_{prim})$ such that: $(r, \alpha) \in z$ iff there is $e \in \Delta^{\mathcal{I}_\rho}$ satisfying $(d_0, e) \in r^{\mathcal{I}_\rho}$ and $\alpha \in \mathsf{Ex}_e$. In this way we can keep the relevant information needed to verify whether $d_0 \in (\exists r_i.\alpha_i)^{\mathcal{I}_\rho}$, while using polynomial space.

Putting all these ideas together, we transform A into the following function:

1: **function** A($\mathfrak{d}$: *integer*, $\mathfrak{b}$: *binary*)
2: $\quad$ $\mathfrak{b} := \mathfrak{b} - 1$
3: $\quad$ *non-deterministically* choose a subset $\mathcal{P}$ of N_{prim}
4: $\quad$ initialize v and z
5: $\quad$ **if** $(\mathfrak{d} \neq 0)$ **and**$(\mathfrak{b} \neq 0)$ **then**
6: $\quad\quad$ **for all** $r \in \mathsf{N}_\mathsf{R}$ **do**
7: $\quad\quad\quad$ *non-deterministically* choose $0 \leq \mathfrak{b}_r :\leq \mathfrak{b}$
8: $\quad\quad\quad$ $\mathfrak{b} := \mathfrak{b} - \mathfrak{b}_r$
9: $\quad\quad\quad$ **for all** $1 \leq i \leq \mathfrak{b}_r$ **do**
10: $\quad\quad\quad\quad$ *non-deterministically* choose $0 \leq \mathfrak{b}_r^i \leq \mathfrak{b}$
11: $\quad\quad\quad\quad$ $\mathfrak{b} := \mathfrak{b} - \mathfrak{b}_r^i$
12: $\quad\quad\quad\quad$ $(\mathsf{Ex}_r^i, D_r^i) :=A(\mathfrak{d} - 1, \mathfrak{b}_r^i + 1)$
13: $\quad\quad\quad\quad$ update v

14: update z
15: **end for**
16: **end for**
17: **end if**
18: $D :=$ SUBDEG(v)
19: $\mathsf{Ex} :=$ SUBEX(D, z)
20: **return** (Ex, D)
21: **end function**

The subroutines SUBDEG and SUBEX invoked in lines 18 and 19 correspond to the computation of the assignment D and the set Ex, respectively. The execution of line 14 updates the relation z using the content of Ex_r^i after each recursive call has been executed. Regarding the symbol v in line 13, as we explain below it represents a table used to help the computation of D.

Let us now move on to the details of the computation of Ex and D. We start with the computation of D, and afterwards explain how to compute Ex.

Due to the normal form of $\widehat{\mathcal{T}}$, the $\mathcal{EL}$ concept E in $E_{\sim t}$ is a defined concept in $\mathcal{T}_{\mathcal{EL}}$. Therefore, by Definition 6.9 for all $d \in \Delta^{\mathcal{I}_\rho}$:

$$d \in (E_{\sim t})^{\mathcal{I}_\rho} \text{ iff } deg^{\mathcal{I}_\rho}(d, u_{\mathcal{T}_{\mathcal{EL}}}(E)) \sim t$$

Coming back to Chapter 4 we know that $deg^{\mathcal{I}_\rho}(d, u_{\mathcal{T}_{\mathcal{EL}}}(E))$ is the maximal value of $h_w(v_0)$ among all *ptghs* $h \in \mathcal{H}(T_{u_{\mathcal{T}_{\mathcal{EL}}}(E)}, G_{\mathcal{I}_\rho}, d)$, where v_0 is the root of the description tree $T_{u_{\mathcal{T}_{\mathcal{EL}}}(E)}$. Note that we use directly $T_{u_{\mathcal{T}_{\mathcal{EL}}}(E)}$, since being $\mathcal{T}_{\mathcal{EL}}$ in reduced form implies that $u_{\mathcal{T}_{\mathcal{EL}}}(E)$ is reduced (see Lemma 6.28). Now, E is defined in $\mathcal{T}_{\mathcal{EL}}$ as follows:

$$E \doteq P_1 \sqcap \ldots \sqcap P_m \sqcap \exists r_1.E_1 \sqcap \ldots \sqcap \exists r_n.E_n$$

This gives us the following information regarding $T_{u_{\mathcal{T}_{\mathcal{EL}}}(E)}$:

- the label of v_0 in $T_{u_{\mathcal{T}_{\mathcal{EL}}}(E)}$ is the set $\{P_1, \ldots, P_m\}$,
- v_0 has exactly n ($n \geq 0$) successors $v_1, \ldots, v_n$ in $T_{u_{\mathcal{T}_{\mathcal{EL}}}(E)}$,
- for all $1 \leq i \leq n$, the subtree $T_{u_{\mathcal{T}_{\mathcal{EL}}}(E)}[v_i]$ of $T_{u_{\mathcal{T}_{\mathcal{EL}}}(E)}$ rooted at v_i is exactly the description tree associated to $u_{\mathcal{T}_{\mathcal{EL}}}(E_i)$.

Additionally, the computation of $h_w(v_0)$ is based on the following expression:

$$h_w(v_0) = \begin{cases} 1 & \text{if } m + n = 0 \\ \frac{|\{P_1,\ldots,P_m\} \cap \ell_{\mathcal{I}_\rho}(d)| + \sum_{1 \leq i \leq k} h_w(w_i)}{m+n} & \text{otherwise.} \end{cases}$$

where $w_1, \ldots, w_k$ are the children of v_0 in $T_{u_{\mathcal{T}_{\mathcal{EL}}}(E)}$ mapped by h. Now, regarding a *ptgh* h yielding a maximal value for $h_w(v_0)$ we observe the following:

- if $(d, e) \in (r_i)^{\mathcal{I}_\rho}$ for some $e \in \Delta^{\mathcal{I}_\rho}$, then we can assume that $v_i \in \mathsf{dom}(h)$.

- Let $h(v_i) = e_i$ where $e_i \in \Delta^{\mathcal{I}_\rho}$. Then, $h_w(v_i) = deg^{\mathcal{I}_\rho}(e_i, u_{\mathcal{T}_{\mathcal{EL}}}(E_i))$. This is a consequence of v_i being the root of the description tree corresponding to $u_{\mathcal{T}_{\mathcal{EL}}}(E_i)$, and the fact that $h_w(v_0)$ is maximal.

Therefore, $deg^{\mathcal{I}_\rho}(d, u_{\mathcal{T}_{\mathcal{EL}}}(E))$ can be expressed as:

$$\frac{|\{P_1, \ldots, P_m\} \cap \ell_{\mathcal{I}_\rho}(d)| + \sum_{i=1}^{n} \max\{deg^{\mathcal{I}_\rho}(e, u_{\mathcal{T}_{\mathcal{EL}}}(E_i)) \mid (d,e) \in (r_i)^{\mathcal{I}_\rho}\}}{m+n} \tag{6.12}$$

Thus, knowing the values $deg^{\mathcal{I}_\rho}(e, u_{\mathcal{T}_{\mathcal{EL}}}(F))$ for all successors e of d in $\mathcal{I}_\rho$ and all $F \in \{E_1, \ldots, E_n\}$, it is straightforward to compute $deg^{\mathcal{I}_\rho}(d, u_{\mathcal{T}_{\mathcal{EL}}}(E))$. Therefore, similar to the computation of Ex the assignment D for $\mathfrak{d}_0$ can be computed by using all the assignments D recursively computed for all successors of d_0 in $\mathcal{I}_\rho$. Once more, the problem related to the possible exponentially many successors of d_0 needs to be addressed. Here is where the aforementioned table v comes into play. It is defined as $v : (\mathsf{N_R} \times \mathsf{def}(\mathcal{T}_{\mathcal{EL}})) \cup (\epsilon \times \mathsf{N}_{prim}) \to [0,1]$ and each entry $v[r,E]$ stores the value $\max\{D_e(E) \mid (d_0, e) \in r^{\mathcal{I}_\rho}\}$, where D_e is the assignment D for e, and $v[\epsilon, P] = 1$ iff $P \in \mathcal{P}$ (0 otherwise). The following fragment of *pseudo*-code updates v within a run of A:

$v[r,E] = 0$ for all $(r,E) \in (\mathsf{N_R} \times \mathsf{def}(\mathcal{T}_{\mathcal{EL}})) \cup (\epsilon \times \mathsf{N}_{prim})$ // Initialization
$v[\epsilon, P] = 1$ iff $P \in \mathcal{P}$
$\vdots$
$D^i_r := \mathrm{A}(\mathfrak{d} - 1, \mathfrak{b}^i_r + 1)$
for all $(E \doteq C_E \in \mathcal{T}_{\mathcal{EL}})$ **do**
  if $D^i_r(E) > v[r,E]$ **then**
    $v[r,E] := D^i_r(E)$
  end if
end for

Here, D^i_r stands for the assignment D corresponding to the root element of the interpretation induced by the recursive call. In other words, the i-th r-successor of d_0 in $\mathcal{I}_\rho$. After all the recursive calls have been executed, v is used to compute D as described in the following subroutine:

procedure SUBDEG($v : (\mathsf{N_R} \times \mathsf{def}(\mathcal{T}_{\mathcal{EL}})) \cup (\epsilon \times \mathsf{N}_{prim}) \to [0,1]$)
  for all $(E \doteq C_E \in \mathcal{T}_{\mathcal{EL}})$ **do**
    $c := |\{P \mid P \in tl(C_E)$ and $v[\epsilon, P] = 1\}|$
    for all $\exists r.E' \in tl(C_E)$ **do**
      $c := c + v[r, E']$
    end for
    $D(E) := \frac{c}{|tl(C_E)|}$
  end for
  return D
end procedure

It remains to see the details of the computation of Ex. The updating of the relation z in A is carried out as follows:

$z := \{(\epsilon, P) \mid P \in \mathcal{P}\}$ // Initialization
$\vdots$
$\mathsf{Ex}_r^i := \mathrm{A}(\mathfrak{d} - 1, \mathfrak{b}_r^i + 1)$
for all $(\alpha \doteq \widehat{C}_\alpha \in \widehat{\mathcal{T}})$ **do**
  if $\alpha \in \mathsf{Ex}_r^i$ **then**
    $z := z \cup \{(r, \alpha)\}$
  end if
end for

Then, using D and z Conditions 1 and 2 can be verified, and Ex can be computed in the following way:

procedure SUBEX($D : \mathsf{def}(\mathcal{T}_{\mathcal{EL}}) \rightarrow [0, 1]$, $z \subseteq (\mathsf{N_R} \times \mathsf{def}(\widehat{\mathcal{T}})) \cup (\epsilon \times \mathsf{N}_{prim})$)
  $s := \emptyset$
  for all $(\alpha \doteq \widehat{C}_\alpha \in \widehat{\mathcal{T}})$ **do**
    if $([P \in tl(\widehat{C}_\alpha)] \Rightarrow (\epsilon, P) \in z)$ **and** $([E_{\sim t} \in tl(\widehat{C}_\alpha)] \Rightarrow D(E) \sim t)$ **and** $([\exists r.\beta \in tl(\widehat{C}_\alpha)] \Rightarrow (r, \beta) \in z)$ **then**
      $s := s \cup \{\alpha\}$
    end if
  end for
  return s
end procedure

Thus, using the function A we define our non-deterministic algorithm to decide satisfiability of concepts of the form $\alpha_1 \sqcap \neg\alpha_2$ with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes.

Algorithm 5 Satisfiability of $\alpha_1 \sqcap \neg\alpha_2$ w.r.t. acyclic $\tau\mathcal{EL}(deg)$ TBoxes.

Input: An acyclic $\tau\mathcal{EL}(deg)$ TBox $\widehat{\mathcal{T}}$ and two defined concepts α_1, α_2 in $\widehat{\mathcal{T}}$.
Output: "yes", if $\alpha_1 \sqcap \neg\alpha_2$ is satisfiable in $\widehat{\mathcal{T}}$, "no" otherwise.

$\mathfrak{b} := \mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_1)) \times \mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_2))$ // $\mathfrak{b}$ is stored in binary
$\mathfrak{d} := \mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha_1)) + \mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha_2))$
$(\mathsf{Ex}, D) := \mathrm{A}(\mathfrak{d}, \mathfrak{b})$
if $\alpha_1 \in \mathsf{Ex}$ **and** $\alpha_2 \notin \mathsf{Ex}$ **then**
  return "yes"
end if
return "no"

Since A terminates, this implies that Algorithm 5 terminates as well. In the following, we show that Algorithm 5 is sound and complete. Let us start by showing that A computes the right values for D and Ex.

Lemma 6.34. *Let $\mathfrak{d} \geq 0$ and $\mathfrak{b} > 0$ be two natural numbers, and ρ be a run of A on $(\mathfrak{d}, \mathfrak{b})$. Then,*

1. $D(E) = deg^{\mathcal{I}_\rho}(d_0, u_{\mathcal{T}_{\mathcal{EL}}}(E))$*, for all* $E \doteq C_E \in \mathcal{T}_{\mathcal{EL}}$.
2. $\mathsf{Ex} = \{\alpha \mid \alpha \doteq \widehat{C}_\alpha \in \widehat{\mathcal{T}}$ *and* $d_0 \in \alpha^{\mathcal{I}_\rho}\}$

Proof. Let $\mathsf{d}(\mathcal{I}_\rho)$ denote the depth of $T_{\mathcal{I}_\rho}$. We prove our claims by induction on $\mathsf{d}(\mathcal{I}_\rho)$. To start, we fix a role name $r \in \mathsf{N_R}$ and define $r(d_0) = \{e_1, \ldots, e_n\}$ to be the set of r-successors of d_0 in $\mathcal{I}_\rho$ (with $n \geq 0$). By construction of $T_{\mathcal{I}_\rho}$, ρ does exactly n recursive calls $\mathrm{A}(\mathfrak{d} - 1, \mathfrak{b}^i_r)$ $(1 \leq i \leq n)$. Let ρ^i_r denote the run corresponding to the i-th call. Then, the interpretation $\mathcal{I}_{\rho^i_r}$ induced by ρ^i_r is the one having the description tree $T_{\mathcal{I}_\rho}[e_i]$, i.e., the subtree of $T_{\mathcal{I}_\rho}$ rooted at e_i.

The tree shape of $\mathcal{I}_\rho$ implies that $\mathsf{d}(\mathcal{I}_{\rho^i_r}) < \mathsf{d}(\mathcal{I}_\rho)$. Therefore, induction hypothesis can be applied to obtain:

$$D^i_r(E) = deg^{\mathcal{I}_{\rho^i_r}}(e_i, u_{\mathcal{T}_{\mathcal{EL}}}(E))$$
$$\mathsf{Ex}^i_r = \{\alpha \mid \alpha \doteq \widehat{C}_\alpha \in \widehat{\mathcal{T}} \text{ and } e_i \in \alpha^{\mathcal{I}_{\rho^i_r}}\}$$

The same reasoning applies for all the other role names $s \in \mathsf{N_R}$. Note that since $\mathcal{I}_{\rho^i_r}$ is a subtree of $\mathcal{I}_\rho$, those two equalities are also valid for $\mathcal{I}_\rho$, i.e.:

$$D^i_r(E) = deg^{\mathcal{I}_\rho}(e_i, u_{\mathcal{T}_{\mathcal{EL}}}(E))$$
$$\mathsf{Ex}^i_r = \{\alpha \mid \alpha \doteq \widehat{C}_\alpha \in \widehat{\mathcal{T}} \text{ and } e_i \in \alpha^{\mathcal{I}_\rho}\}$$

Therefore, after all the recursive calls have been executed and the values in table v and relation z have been fully updated, we have for all $(r, E) \in \mathsf{N_R} \times \mathsf{def}(\mathcal{T}_{\mathcal{EL}})$:

$$v[r, E] = \max\{deg^{\mathcal{I}_\rho}(e, u_{\mathcal{T}_{\mathcal{EL}}}(E)) \mid (d_0, e) \in r^{\mathcal{I}_\rho}\} \tag{6.13}$$

and,

$$z = \{(r, \alpha) \mid e \in \Delta^{\mathcal{I}_\rho}, (d_0, e) \in r^{\mathcal{I}_\rho} \text{ and } e \in \alpha^{\mathcal{I}_\rho}\} \tag{6.14}$$

Looking at the subroutine SUBDEG, for all $E \doteq C_E \in \mathcal{T}_{\mathcal{EL}}$ the value $D(E)$ is computed by the following expression:

$$D(E) = \frac{|tl(C_E) \cap \mathcal{P}| + \sum_{\exists r.E' \in tl(C_E)} v[r, E']}{tl(C_E)}$$

Now, by construction of $\mathcal{I}_\rho$ we have that $\ell_{\mathcal{I}_\rho}(d_0) = \mathcal{P}$. Hence, replacing $v[r, E']$ by the right-hand side of the equality in (6.13) we obtain the expression in (6.12). Consequently, we have shown that:

$$D(E) = deg^{\mathcal{I}_\rho}(d_0, u_{\mathcal{T}_{\mathcal{EL}}}(E))$$

Last, let $\alpha \doteq \widehat{C}_\alpha \in \widehat{\mathcal{T}}$ with $\widehat{C}_\alpha$ of the form:

$$\widehat{P}_1 \sqcap \ldots \sqcap \widehat{P}_n \sqcap \exists r_1.\alpha_1 \sqcap \ldots \sqcap \exists r_m.\alpha_m$$

According to SUBEX, $\alpha \in \mathsf{Ex}$ iff:

- for all $1 \leq i \leq n$: if $\widehat{P}_i$ is of the form $E_{\sim t}$ then $D(E) \sim t$, otherwise $\widehat{P}_i \in \mathcal{P}$, and
- $(r_j, \alpha_j) \in z$, for all $1 \leq j \leq m$.

Since $\ell_{\mathcal{I}_\rho}(d_0) = \mathcal{P}$ and $D(E) = deg^{\mathcal{I}_\rho}(d_0, u_{\mathcal{T}_{\mathcal{EL}}}(E))$, the first statement is equivalent to have $d_0 \in (\widehat{P}_i)^{\mathcal{I}_\rho}$ $(1 \leq i \leq n)$. Furthermore, (6.14) makes the second statement equivalent to having $d_0 \in (\exists r_j.\alpha_j)^{\mathcal{I}_\rho}$ $(1 \leq j \leq m)$. Thus, $\alpha \in \mathsf{Ex}$ iff $d_0 \in \alpha^{\mathcal{I}_\rho}$.

Note that the base case for the induction is already contained in the proof. □

Using Lemma 6.34 we now prove that Algorithm 5 is sound and complete.

Lemma 6.35. *Let $\widehat{\mathcal{T}}$ be an acyclic $\tau\mathcal{EL}(deg)$ TBox and α_1, α_2 two defined concepts in $\widehat{\mathcal{T}}$. Then,*

Algorithm 5 answers "yes" iff $\alpha_1 \sqcap \neg\alpha_2$ is satisfiable in $\widehat{\mathcal{T}}$.

Proof. (⇒) Suppose that the algorithm gives a positive answer and let ρ be the run of function A that leads to it. Then, we can talk about the interpretation $\mathcal{I}_\rho$ induced by ρ. The "yes" answer means that for ρ, $\alpha_1 \in \mathsf{Ex}$ and $\alpha_2 \notin \mathsf{Ex}$. Then, the application of Lemma 6.34 yields:

$$d_0 \in (\alpha_1 \sqcap \neg\alpha_2)^{\mathcal{I}_\rho}$$

with $d_0 \in \Delta^{\mathcal{I}_\rho}$. Hence, $\alpha_1 \sqcap \neg\alpha_2$ is satisfiable with respect to $\widehat{\mathcal{T}}$.

(⇐) Assume that $\alpha_1 \sqcap \neg\alpha_2$ is satisfiable with respect to $\widehat{\mathcal{T}}$. This means that there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \widehat{\mathcal{T}}$ and $(\alpha_1 \sqcap \neg\alpha_2)^{\mathcal{I}} \neq \emptyset$. By Proposition 6.31 and its subsequent remarks one can assume that $\mathcal{I}$ is tree-shaped and satisfies the following properties:

1. $\Delta^{\mathcal{I}}$ has at most $\mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_1)) \times \mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_2))$ elements,

2. the depth of $T_{\mathcal{I}}$ is not greater than $\mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha_1)) + \mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha_2))$, and

3. its root element d_0 satisfies: $d_0 \in (\alpha_1 \sqcap \neg\alpha_2)^{\mathcal{I}}$.

The selection of $\mathfrak{d}$ and $\mathfrak{b}$ in Algorithm 5 and the application of Lemma 6.33 guarantee the existence of a run ρ of A on $(\mathfrak{d}, \mathfrak{b})$ generating the restriction of $\mathcal{I}$ to $\mathsf{N}_{prim} \cup \mathsf{N}_{\mathsf{R}}$. Hence, the application of Lemma 6.34 implies that the conditional in line 4 must evaluate to *true* for such a run ρ. Thus, Algorithm 5 answers "yes". □

Algorithm 5 uses space polynomial in the size of $\widehat{\mathcal{T}}$ to store the binary representation of $\mathfrak{b}$. Furthermore, z and v are also stored within polynomial space, and the two subroutines run in polynomial time. Therefore, since each run ρ of A on $(\mathfrak{d}, \mathfrak{b})$ does at most $\mathfrak{d}$ many nested recursive calls, ρ uses space polynomial in $\mathsf{s}(\widehat{\mathcal{T}})$. In addition, it is easy to see that both $\mathfrak{b}$ and $\mathfrak{d}$ can be computed in time polynomial in $\mathsf{s}(\widehat{\mathcal{T}})$. Thus, Algorithm 5 is a non-deterministic polynomial space decision procedure for satisfiability of concepts of the form $\alpha_1 \sqcap \neg\alpha_2$ with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes. This means that satisfiability and non-subsumption are in NPSPACE. Then, by Savitch's theorem [Sav70] and since PSPACE is closed under complement, we obtain the following results.

Lemma 6.36. *In $\tau\mathcal{EL}(deg)$, satisfiability and subsumption are in PSPACE, with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes.*

6.4.4 Reasoning with acyclic knowledge bases

We show in this section that satisfiability and subsumption are still decidable in PSPACE with respect to acyclic knowledge bases. Furthermore, we also consider the *consistency* and the *instance* problem. Let $\mathcal{K} = (\widehat{\mathcal{T}}, \mathcal{A})$ be an acyclic $\tau\mathcal{EL}(deg)$ knowledge base:

- $\mathcal{K}$ is *consistent* iff there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{K}$.

Additionally, let $a \in \mathsf{N_I}$ be an individual name and α a defined concept in $\widehat{\mathcal{T}}$:

- a is an instance of α with respect to $\mathcal{K}$ iff for all models $\mathcal{I}$ of $\mathcal{K}$ it holds that $a^{\mathcal{I}} \in \alpha^{\mathcal{I}}$.

Without loss of generality, we can restrict our attention to the consistency problem for KBs of the form $(\widehat{\mathcal{T}}, \mathcal{A} \cup \{\neg\alpha(a)\})$, since all the other problems can be reduced to it.

Proposition 6.37. *Let $\mathcal{K} = (\widehat{\mathcal{T}}, \mathcal{A})$ be an acyclic $\tau\mathcal{EL}(deg)$ KB, α, α_1 and α_2 defined concepts in $\widehat{\mathcal{T}}$ and $a \in \mathsf{N_I}$. Then,*

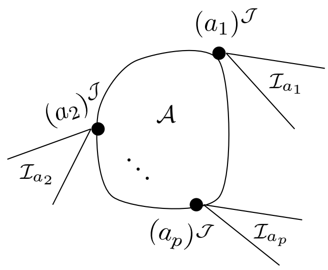
1. *α is satisfiable with respect to $\mathcal{K}$ iff $(\widehat{\mathcal{T}}, \mathcal{A} \cup \{\alpha(b)\})$ is consistent, where b is an individual name not occurring in $\mathcal{A}$.*
2. *α_1 is subsumed by α_2 with respect to $\mathcal{K}$ (in symbols $\alpha_1 \sqsubseteq_{\mathcal{K}} \alpha_2$) iff the knowledge base $(\widehat{\mathcal{T}}, \mathcal{A} \cup \{\alpha_1(b), \neg\alpha_2(b)\})$ is inconsistent, where b is an individual name not occurring in $\mathcal{A}$.*
3. *a is an instance of α in $\mathcal{K}$ (in symbols $\mathcal{K} \models \alpha(a)$) iff $(\widehat{\mathcal{T}}, \mathcal{A} \cup \{\neg\alpha(a)\})$ is not consistent.*

Further, since $\widehat{\mathcal{T}}$ is acyclic, by using unfolding we can again get rid of the TBox and reduce reasoning to consistency with respect to the empty terminology. The unfolding of a $\tau\mathcal{EL}(deg)$ ABox $\mathcal{A}$ with respect to $\widehat{\mathcal{T}}$ is defined as follows:

$$u_{\widehat{\mathcal{T}}}(\mathcal{A}) := \bigcup_{\substack{\alpha(a)\in\mathcal{A}\\ a\in\mathsf{Ind}(\mathcal{A})}} \{[u_{\widehat{\mathcal{T}}}(\alpha)](a)\} \cup \bigcup_{\substack{r(a,b)\in\mathcal{A}\\ a,b\in\mathsf{Ind}(\mathcal{A})}} \{r(a,b)\}$$

Proposition 6.38. *Let $\mathcal{K} = (\widehat{\mathcal{T}}, \mathcal{A})$ be an acyclic $\tau\mathcal{EL}(deg)$ KB, α a defined concept in $\widehat{\mathcal{T}}$ and $a \in \mathsf{N_I}$. $(\widehat{\mathcal{T}}, \mathcal{A} \cup \{\neg\alpha(a)\})$ is consistent iff $u_{\widehat{\mathcal{T}}}(\mathcal{A}) \cup \{[\neg u_{\widehat{\mathcal{T}}}(\alpha)](a)\}$ is consistent.*

In what follows, we show how to reuse the idea of Algorithm 5 to decide consistency of $u_{\widehat{\mathcal{T}}}(\mathcal{A}) \cup \{[\neg u_{\widehat{\mathcal{T}}}(\alpha)](a)\}$. Lemma 5.16 tells us that if $u_{\widehat{\mathcal{T}}}(\mathcal{A}) \cup \{[\neg u_{\widehat{\mathcal{T}}}(\alpha)](a)\}$ is consistent, then it has a model $\mathcal{J}$ of the following form,



where $\mathsf{Ind}(\mathcal{A}) = \{a_1, a_2, \ldots, a_p\}$ and $\mathcal{I}_{a_1}, \mathcal{I}_{a_2}, \ldots, \mathcal{I}_{a_p}$ are tree-shaped interpretations. The inner area of the diagram consists of the satisfaction of the role assertions in $\mathcal{A}$, i.e., $(a^{\mathcal{J}}, b^{\mathcal{J}}) \in r^{\mathcal{J}}$ iff $r(a,b) \in \mathcal{A}$. Additionally, Lemma 5.14 provides an upper bound for the size of these tree-shaped interpretations. We will later talk about how big this bound could be, but for the moment let us focus in how to reuse Algorithm 5.

To start, it is clear that by choosing the appropriate values for $\mathfrak{d}$ and $\mathfrak{b}$, the interpretations $\mathcal{I}_a$ can be independently generated using the function A. It is important to keep in mind that $a^{\mathcal{I}_a}$ is the root of $\mathcal{I}_a$. Consequently, a run ρ_a of A inducing $\mathcal{I}_a$ will compute two sets Ex_a and D_a with the following meaning:

$$D_a(E) = deg^{\mathcal{I}_a}(a^{\mathcal{I}_a}, u_{\mathcal{T}_{\mathcal{EL}}}(E)), \text{ for all } E \doteq C_E \in \mathcal{T}_{\mathcal{EL}}$$

$$\mathsf{Ex}_a = \{\beta \mid \beta \doteq \widehat{C}_\beta \in \widehat{\mathcal{T}} \text{ and } a^{\mathcal{I}_a} \in \beta^{\mathcal{I}_a}\}$$

Recall that technically $\mathcal{I}_a$ (as generated by A) only interprets symbols from $\mathsf{N}_{prim} \cup \mathsf{N}_\mathsf{R}$, but when writing $\beta^{\mathcal{I}_a}$ we meant its unique extension to a model of $\widehat{\mathcal{T}}$. The veracity of the previous two equalities has been shown in Lemma 6.34. Now, the construction of the model $\mathcal{J}$ depicted above combines all those interpretations in the following way (see Lemma 5.15):

- $\Delta^{\mathcal{J}} = \bigcup_{a \in \mathsf{Ind}(\mathcal{A})} \Delta^{\mathcal{I}_a}$,
- $A^{\mathcal{J}} = \bigcup_{a \in \mathsf{Ind}(\mathcal{A})} A^{\mathcal{I}_a}$ for all $A \in \mathsf{N}_{prim}$,
- $r^{\mathcal{J}} = \bigcup_{a \in \mathsf{Ind}(\mathcal{A})} r^{\mathcal{I}_a} \cup \{(a^{\mathcal{I}_a}, b^{\mathcal{I}_b}) \mid r(a,b) \in \mathcal{A}\}$ for all $r \in \mathsf{N}_\mathsf{R}$, and
- $a^{\mathcal{J}} = a^{\mathcal{I}_a}$, for all $a \in \mathsf{Ind}(\mathcal{A})$.

This means that given an individual $a \in \mathsf{Ind}(\mathcal{A})$, a defined concept β and an element $d \in \Delta^{\mathcal{I}_a}$, it is not necessarily the case that $d^{\mathcal{J}} \in \beta^{\mathcal{J}}$ iff $\beta \in \mathsf{Ex}_d$ (similarly for the membership degrees and the assignment D_d). The reason is that the role assertions between individual names are used to build $\mathcal{J}$, but they are not taken into account by ρ_a to compute Ex_d and D_d. Fortunately, this could only be the case for the domain elements $a^{\mathcal{J}} = a^{\mathcal{I}_a}$ corresponding to the individual names of $\mathcal{A}$. This is a consequence of something that we have already observed in Chapter 5: for all $a \in \mathsf{Ind}(\mathcal{A})$ and $d \in \Delta^{\mathcal{I}_a}$ such that $d \neq a^{\mathcal{I}_a}$, no path in $G_{\mathcal{J}}$ starting at d reaches a domain element $b^{\mathcal{J}}$ ($b \in \mathsf{Ind}(\mathcal{A})$). As a result we obtain the following:

$$deg^{\mathcal{J}}(d, u_{\widehat{\mathcal{T}}}(E)) = deg^{\mathcal{I}_a}(d, u_{\widehat{\mathcal{T}}}(E)), \text{ for all } E \doteq C_E \in \mathcal{T}_{\mathcal{EL}}$$

$$d \in \beta^{\mathcal{J}} \text{ iff } d \in \beta^{\mathcal{I}_a}, \text{ for all } \beta \doteq \widehat{C}_\beta \in \widehat{\mathcal{T}}$$

Therefore, if we can compute the correct content/values of Ex_a and D_a for the unique extension of $\mathcal{J}$ satisfying $\widehat{\mathcal{T}}$, it will be possible to verify whether $\mathcal{J}$ satisfies $u_{\widehat{\mathcal{T}}}(\mathcal{A}) \cup \{[\neg u_{\widehat{\mathcal{T}}}(\alpha)](a)\}$ (as it is done for subsumption in the previous section). There are two obstacles that we need to overcome. The first one is that Ex_a and D_a, as computed by ρ_a, do not contain enough information to obtain the ones corresponding to $\mathcal{J}$.

Example 6.39. Let $a_1, a_2 \in \mathsf{Ind}(\mathcal{A})$ and $r(a_1, a_2) \in \mathcal{A}$. Suppose that a run ρ_{a_1} of A representing $\mathcal{I}_{a_1}$ yields $D_{a_1}(E) = t_1$ for some $E \doteq C_E \in \mathcal{T}_{\mathcal{EL}}$. Likewise, $D_{a_2}(E') = t_2$ for some run ρ_{a_2} representing $\mathcal{I}_{a_2}$ and $E' \doteq C_{E'} \in \mathcal{T}_{\mathcal{EL}}$. In addition, there is a top-level atom in C_E of the form $\exists r.E'$.

As explained above, the value of $D_{a_2}(E')$ has not been considered in the computation of $D_{a_1}(E)$, and it may well be the case that it actually affects $D_{a_1}(E)$ in the big model $\mathcal{J}$, i.e., $deg^{\mathcal{J}}((a_1)^{\mathcal{J}}, u_{\widehat{\mathcal{T}}}(E)) > t_1$. This could happen if for all r-successors d of a_1 in $\mathcal{I}_{a_1}$, we have that $deg^{\mathcal{I}_{a_1}}(d, u_{\widehat{\mathcal{T}}}(E')) < t_2$. Clearly, this is not something that can be inferred from D_{a_1}, but from the table v computed for a_1 by ρ_{a_1}.

Similarly, assume that $\beta \notin \mathsf{Ex}_{a_1}$ for some $\beta \doteq \widehat{C}_\beta \in \mathcal{T}_\tau$. This means that $a^{\mathcal{I}_a} \notin \beta^{\mathcal{I}_a}$. It could happen that a_2 satisfies properties in $\mathcal{J}$ that would make $(a_1)^{\mathcal{J}} \in \beta^{\mathcal{J}}$. Then, we would need to look into the relation z computed for a_1 by ρ_{a_1}, to discern such a change. ♢

To deal with that, we rearrange the structure of function A such that it returns the pair (z, v) instead of (Ex, D). The following sketches how to modify A accordingly.

function A($\mathfrak{d}$: *integer*, $\mathfrak{b}$: *binary*)
    ⋮
    initialize v and z
    ⋮
    $(z^i_r, v^i_r) :=$A$(\mathfrak{d} - 1, \mathfrak{b}^i_r + 1)$
    $D^i_r :=$ SUBDEG(v^i_r)
    $\mathsf{Ex}^i_r :=$ SUBEX(D^i_r, z^i_r)
    update v
    update z
    ⋮
    return (z, v)
end function

Note that in the previous version of A, the computation of D^i_r and Ex^i_r are the last operations executed inside the recursive call A$(\mathfrak{d} - 1, \mathfrak{b}^i_r + 1)$, and v, z are updated right away after that. This order of actions is kept in the new definition given above. Since the computation of D^i_r and Ex^i_r only requires of v^i_r and z^i_r, and these are returned by A, the new modifications preserve the properties of A.

The next step is to recompute z_a and v_a for all $a \in \mathsf{Ind}(\mathcal{A})$ using the information provided by the role assertions in $\mathcal{A}$. Following Example 6.39, since $b^{\mathcal{J}}$ is related to $a^{\mathcal{J}}$ by the role name r, this means that v_a and z_a must be updated with respect to r, Ex_b and D_b. Obviously, changes in v_a and z_a should be propagated to the individuals that a is related to, and so on. The function A can cope with such propagation in a bottom-up form, because it is *computing* a tree-shaped structure. However, this is no longer the case for the individuals in $\mathcal{A}$, since role assertions can define cycles involving them.

To solve this we appeal to the acyclic nature of $\widehat{\mathcal{T}_\tau}$ and $\mathcal{T}_{\mathcal{EL}}$. It allows to traverse the structure of any defined concept (bottom-up) based on the partial order $\preceq$ induced by $\rightarrow^+$ on $\mathsf{def}(\widehat{\mathcal{T}})$. Note that now we limit our attention to the fragment of $\mathcal{J}$ corresponding to the role assertions in $\mathcal{A}$, which is part of the input. Therefore, provided that (z_a, v_a)

has been computed for all $a \in \mathsf{Ind}(\mathcal{A})$, the following subroutine updates all those pairs with respect to the combined interpretation $\mathcal{J}$.

```
procedure Update( )
    compute D_a := SUBdeg(v_a)        // for all a ∈ Ind(A)
    let {E_1, ..., E_n} be a post-order of ⪯ (induced by →⁺ on def(T_EL))
    for all 1 ≤ i ≤ n do
        for all r(a, b) ∈ A do
            if D_b(E_i) > v_a[r, E_i] then
                v_a[r, E_i] := D_b(E_i)
            end if
        end for
        re-compute D_a        // for all a ∈ Ind(A)
    end for
    compute Ex_a := SUBex(D_a, z_a)        // for all a ∈ Ind(A)
    let {β_1, ..., β_n} be a post-order of ⪯ on def(T̂)
    for all 1 ≤ i ≤ n do
        for all r(a, b) ∈ A do
            if β_i ∈ Ex_b then
                z_a := z_a ∪ {(r, β_i)}
            end if
        end for
        re-compute Ex_a        // for all a ∈ Ind(A)
    end for
end procedure
```

Let us prove that Update does what we have claimed.

Lemma 6.40. *For all $a \in \mathsf{Ind}(\mathcal{A})$, let ρ_a be a run of A and $\mathcal{I}_a$ its induced interpretation. Moreover, based on these interpretations let $\mathcal{J}$ be the interpretation that results from the combination presented in Lemma 5.15. Then,*

1. $D_a(E) = deg^{\mathcal{J}}(a^{\mathcal{J}}, u_{\mathcal{T}_{\mathcal{EL}}}(E))$, *for all* $E \doteq C_E \in \mathcal{T}_{\mathcal{EL}}$.
2. $\mathsf{Ex}_a = \{\beta \mid \beta \doteq \widehat{C}_\beta \in \widehat{\mathcal{T}}$ *and* $a^{\mathcal{J}} \in \beta^{\mathcal{J}}\}$

Proof. We give the proof for the assignments D_a. The case for Ex_a can be done using the same idea and Lemma 6.34. To differentiate the final assignment D_a from the initial one computed by ρ_a, we denote the latter as D_a^0 (likewise for v_a and v_a^0). We show the claim by well-founded induction on the partial order $\preceq$.

Let $a \in \mathsf{Ind}(\mathcal{A})$ and $E \doteq C_E \in \mathcal{T}_{\mathcal{EL}}$. Since $\mathcal{T}_{\mathcal{EL}}$ is normalized, the concept description C_E has the following structure:

$$P_1 \sqcap \ldots \sqcap P_n \sqcap \exists r_1.E_1 \sqcap \ldots \sqcap \exists r_m.E_m$$

Clearly, when $m = 0$ the value $deg^{\mathcal{J}}(a^{\mathcal{J}}, u_{\mathcal{T}_{\mathcal{EL}}}(E))$ does not depend on any successor of $a^{\mathcal{J}}$. Moreover, by construction of $\mathcal{J}$ we know that $a^{\mathcal{J}} \in (P_i)^{\mathcal{J}}$ iff $a^{\mathcal{I}_a} \in (P_i)^{\mathcal{I}_a}$ for all $1 \leq i \leq n$. This implies that:

$$deg^{\mathcal{J}}(a^{\mathcal{J}}, u_{\mathcal{T}_{\mathcal{EL}}}(E)) = deg^{\mathcal{I}_a}(a^{\mathcal{I}_a}, u_{\mathcal{T}_{\mathcal{EL}}}(E))$$

Then, by Lemma 6.34 we obtain that:

$$D_a^0(E) = deg^{\mathcal{J}}(a^{\mathcal{J}}, u_{\mathcal{T}_{\mathcal{EL}}}(E))$$

Looking at SUBDEG one can see that the computation of $D_a^0(E)$ depends only on the values $v_a^0[\epsilon, P]$. Furthermore, it is easy to see that those values are never changed by a run of UPDATE. Hence, $v_a[\epsilon, P] = v_a^0[\epsilon, P]$ and $D_a(E) = D_a^0(E)$. Thus, $D_a(E)$ is the right number.

Now, to show the claim for $m > 0$ we start by making some observations for all $b \in \mathsf{Ind}(\mathcal{A})$. Let F be a defined concept in $\mathcal{T}_{\mathcal{EL}}$:

- By Lemma 6.34, the initial table v_b^0 satisfies the following:

$$v_b^0[r, F] = \max\{deg^{\mathcal{I}_b}(d, u_{\mathcal{T}_{\mathcal{EL}}}(F)) \mid d \in \Delta^{\mathcal{I}_b} \text{ and } (b^{\mathcal{I}_b}, d) \in r^{\mathcal{I}_b}\}$$

  As explained above, since $d \neq b^{\mathcal{I}_b}$ it further satisfies:

$$v_b^0[r, F] = \max\{deg^{\mathcal{J}}(d, u_{\mathcal{T}_{\mathcal{EL}}}(F)) \mid d \in \Delta^{\mathcal{I}_b} \text{ and } (b^{\mathcal{I}_b}, d) \in r^{\mathcal{I}_b}\} \qquad (6.15)$$

Additionally, let j be the index of F in the post-order created in line 3. Then,

- the value of $v_b[r, F]$ only changes at the j^{th} iteration of the outer-loop in line 4.
- let k be the largest index of F' among all the top-level atoms of the form $\exists r.F'$ in the definition of F. Then, taking into account the previous statement, the value of $D_b(F)$ never changes after the k^{th} iteration of the outer-loop.
- since $F' \preceq F$, this means that $j > k$. Consequently, the final value of $D_b(F)$ is computed before the iteration corresponding to F.

Coming back to the defined concept E, we know that $E \preceq E_j$ for all $1 \leq j \leq m$. Then, the application of induction hypothesis yields:

$$D_a(E_j) = deg^{\mathcal{J}}(a^{\mathcal{J}}, u_{\mathcal{T}_{\mathcal{EL}}}(E_j)) \qquad (6.16)$$

Moreover, since at the moment of updating $v_a[r, E_j]$ the value of $D_b(E_j)$ is the one in (6.16) for all $b \in \mathsf{Ind}(\mathcal{A})$, using (6.15) we obtain:

$$v_a[r, E_j] = \max\{deg^{\mathcal{J}}(d, u_{\mathcal{T}_{\mathcal{EL}}}(E_j)) \mid d \in \Delta^{\mathcal{J}} \text{ and } (a^{\mathcal{J}}, d) \in r^{\mathcal{J}}\}$$

Thus, by the same arguments given in Lemma 6.34 it follows:

$$D_a(E) = deg^{\mathcal{J}}(a^{\mathcal{J}}, u_{\mathcal{T}_{\mathcal{EL}}}(E)) \qquad \square$$

By the previous lemma, once (Ex_a, D_a) has been computed by UPDATE for all $a \in \mathsf{Ind}(\mathcal{A})$, it is easy to verify whether $\mathcal{J}$ satisfies $\mathcal{A} \cup \{\neg\alpha(a)\}$. Therefore, it remains to make sure that enough candidates $\mathcal{J}$ are considered to decide the satisfiability status of $u_{\widehat{\mathcal{T}}}(\mathcal{A}) \cup \{[\neg u_{\widehat{\mathcal{T}}}(\alpha)](a)\}$. This relies on estimating the appropriate values for $\mathfrak{d}$ and $\mathfrak{b}$.

- Let $m_{\mathsf{rd}}(\mathcal{A})$ be the maximal role depth of a concept $\widehat{D}$ occurring in an ABox $\mathcal{A}$, i.e.,

$$m_{\mathsf{rd}}(\mathcal{A}) := \max\{\mathsf{rd}(\widehat{D}) \mid \widehat{D}(a) \in \mathcal{A}\}$$

  Coming back to Chapter 5, the construction described in Lemma 5.16 to obtain a bounded model $\mathcal{J}$ of $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$, uses Lemma 5.14 to obtain the interpretations $\mathcal{I}_a$ for all $a \in \mathsf{Ind}(\mathcal{A})$. Basically, $\mathcal{I}_a$ is built by extending the description tree of $\mathcal{A}(a)$ with canonical interpretations representing threshold concepts that either occur in $\mathcal{A}$ or are sub-descriptions of $\widehat{C}$. Hence, it is not hard to see that the depth $\mathsf{d}(\mathcal{I}_a)$ of $\mathcal{I}_a$ can be bounded by:

$$\mathsf{d}(\mathcal{I}_a) \leq m_{\mathsf{rd}}(\mathcal{A}(a)) + \mathsf{rd}(\widehat{C})$$

  In the present context this means that $\mathfrak{d}_a$ can be chosen as:

$$m_{\mathsf{rd}}(u_{\widehat{\mathcal{T}}}(\mathcal{A}(a))) + \mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha))$$

- By Lemma 5.16 we have an upper-bound for $|\Delta^{\mathcal{I}_a}|$, namely,

$$|\Delta^{\mathcal{I}_a}| \leq \mathsf{s}(\mathcal{A}(a)) \times [\mathsf{s}(\widehat{C})]^u$$

  where $u = |\mathsf{sub}(\widehat{C})|$. Translating this bound to our current setting, we obtain:

$$|\Delta^{\mathcal{I}_a}| \leq \mathsf{s}(u_{\widehat{\mathcal{T}}}(\mathcal{A}(a))) \times [\mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha))]^{u^*}$$

  with u^* now being $|\mathsf{sub}(u_{\widehat{\mathcal{T}}}(\alpha))|$.

Putting all the given arguments together, we devise Algorithm 6 as a non-deterministic procedure to decide consistency of $(\widehat{\mathcal{T}}, \mathcal{A} \cup \{\neg\alpha(a)\})$. The following lemma shows that it is correct.

Lemma 6.41. *Let $\mathcal{K} = (\widehat{\mathcal{T}}, \mathcal{A})$ be an acyclic $\tau\mathcal{EL}(deg)$ KB, α a defined concept in $\widehat{\mathcal{T}}$ and $a \in \mathsf{Ind}(\mathcal{A})$. Then,*

Algorithm 6 answers "yes" iff $(\widehat{\mathcal{T}}, \mathcal{A} \cup \{\neg\alpha(a)\})$ is consistent.

Proof. ($\Rightarrow$) Suppose that the algorithm gives a positive answer, and for all $a \in \mathsf{Ind}(\mathcal{A})$ let ρ_a be the run of A that leads to it. Then, we can talk about the interpretation $\mathcal{I}_a$ induced by ρ_a. Now, let $\mathcal{J}$ be the interpretation that results from the combination of all the fragments $\mathcal{I}_a$ and the role assertions occurring in $\mathcal{A}$ (as done in Lemma 5.15). A "yes" answer implies that the **for** loop described between lines 7 and 13 never falsifies $\beta \in \mathsf{Ex}_b$ for all concept assertions $\beta(b) \in \mathcal{A}$. By Lemma 6.40, this means that the extension of $\mathcal{J}$ satisfying $\widehat{\mathcal{T}}$ is also a model of $\mathcal{A}$.

In addition, the conditional in line 14 must evaluate to *true*. Consequently, for the same reasons explained above, we obtain that $a^{\mathcal{J}} \notin \alpha^{\mathcal{J}}$. Thus, $(\widehat{\mathcal{T}}, \mathcal{A} \cup \{\neg\alpha(a)\})$ is consistent.

($\Leftarrow$) Conversely, assume that $(\widehat{\mathcal{T}}, \mathcal{A} \cup \{\neg\alpha(a)\})$ is consistent. This means that there is an interpretation $\mathcal{J} \models \mathcal{K}$ such that $a^{\mathcal{J}} \notin \alpha^{\mathcal{J}}$. By Proposition 6.38 and Lemma 5.16, one can assume that $\mathcal{J}$ is of the form described in Lemma 5.15. Therefore, for all $a \in \mathsf{Ind}(\mathcal{A})$ the corresponding interpretation $\mathcal{I}_a$ is tree-shaped and satisfies:

- $\mathsf{d}(\mathcal{I}_a) \leq m_{\mathsf{rd}}(u_{\widehat{\mathcal{T}}}(\mathcal{A})) + \mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha))$, and
- $|\Delta^{\mathcal{I}_a}| \leq \mathsf{s}(u_{\widehat{\mathcal{T}}}(\mathcal{A})) \times [\mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha))]^{u^*}$ (note that $\mathsf{s}(u_{\widehat{\mathcal{T}}}(\mathcal{A}(a))) \leq \mathsf{s}(u_{\widehat{\mathcal{T}}}(\mathcal{A}))$).

By the selection of $\mathfrak{d}$ and $\mathfrak{b}$ in Algorithm 6 and an application of Lemma 6.33, there is always a run ρ_a of A generating $\mathcal{I}_a$ for all $a \in \mathsf{Ind}(\mathcal{A})$. Then, by Lemma 6.40, after executing UPDATE none of the subsequent conditionals could evaluate to *false*. Thus, the algorithm answers "yes". □

Algorithm 6 Consistency of $(\widehat{\mathcal{T}}, \mathcal{A} \cup \{\neg\alpha(a)\})$.

Input: An acyclic KB $(\widehat{\mathcal{T}}, \mathcal{A})$, a defined concept α in $\widehat{\mathcal{T}}$ and $a \in \mathsf{N_I}$.
Output: "yes", if $(\widehat{\mathcal{T}}, \mathcal{A} \cup \{\neg\alpha(a)\})$ is consistent, "no" otherwise.

$\mathfrak{b} := \mathsf{s}(u_{\widehat{\mathcal{T}}}(\mathcal{A})) \times [\mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha))]^{u^*}$ // $\mathfrak{b}$ is represented in binary
$\mathfrak{d} := m_{\mathsf{rd}}(u_{\widehat{\mathcal{T}}}(\mathcal{A})) + \mathsf{rd}(u_{\widehat{\mathcal{T}}}(\alpha))$
for all $b \in \mathsf{Ind}(\mathcal{A})$ **do**
  $(z_b, v_b) :=$ A$(\mathfrak{d}, \mathfrak{b})$
end for
UPDATE( )
for all $b \in \mathsf{Ind}(\mathcal{A})$ **do**
  for all $\beta(b) \in \mathcal{A}$ **do**
    if $\beta \notin \mathsf{Ex}_b$ **then**
      return "no"
    end if
  end for
end for
if $\alpha \notin \mathsf{Ex}_a$ **then**
  return "yes"
end if
return "no"

Regarding the computational complexity of Algorithm 6, one can see that the value of $\mathfrak{d}$ is a polynomial in the size of $\mathcal{K}$. Furthermore, since there are polynomially many individual names, this means that any run of the algorithm uses polynomial space (including the execution of UPDATE), except maybe for the number of bits needed to represent $\mathfrak{b}$. Indeed, the expression that calculates $\mathfrak{b}$ is exponential in u^*. To give a preliminary approximation of how big $\mathfrak{b}$ could be, we observe that due to unfolding we may end up with the following worst-case lower bounds:

$$2^{\mathsf{s}(\widehat{\mathcal{T}})} \leq \mathsf{s}(u_{\widehat{\mathcal{T}}}(\mathcal{A})) \text{ and } 2^{\mathsf{s}(\widehat{\mathcal{T}})} \leq \mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha))$$

In particular, u^* corresponds to the number of sub-descriptions of $u_{\widehat{\mathcal{T}}}(\alpha)$. Hence, in view of the lower bound for the size of $u_{\widehat{\mathcal{T}}}(\alpha)$ one might think that the following lower bound also holds:

$$2^{2^{\mathsf{s}(\widehat{\mathcal{T}})}} \leq [\mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha))]^{u^*} \tag{6.17}$$

Therefore, in the worst-case we would end up with an EXPSPACE non-deterministic procedure. However, on the one side, a closer look at the reductions in Proposition 6.37 reveals that there are better choices for $\mathfrak{b}$ depending on the reasoning problem. On the other side, the statement in (6.17) is actually false.

- *Knowledge base consistency and satisfiability*: in these cases the problem reduces to consistency of a $\tau\mathcal{EL}(deg)$ ABox. Consequently, such double exponential explosion does not exist. Thus, $\mathfrak{b}$ simply becomes $\mathsf{s}(u_{\widehat{\mathcal{T}}}(\mathcal{A}))$ or $\mathsf{s}(u_{\widehat{\mathcal{T}}}(\mathcal{A} \cup \{\alpha(b)\}))$.
- *Subsumption*: the reduction produces an ABox of the form:

  $$\mathcal{A} \cup \{\alpha_1(b), \neg\alpha_2(b)\}$$

  The key aspect is that b does not occur in $\mathcal{A}$. This means that the pre-processing propagation of the negative assertions does not go through the cycles that may occur in $\mathcal{A}$. This obviously avoids the exponential explosion and $\mathfrak{b}$ can be selected as:

  $$\mathsf{s}(u_{\widehat{\mathcal{T}}}(\mathcal{A})) + [\mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_1)) \times \mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha_2))]$$

- *Instance checking*: According to (6.17), in this case the algorithm would need to store a value $\mathfrak{b} \geq 2^{2^{\mathsf{s}(\widehat{\mathcal{T}})}}$. However, one can show that the number of sub-descriptions in $u_{\widehat{\mathcal{T}}}(\alpha)$ is actually bounded by $\mathsf{s}(\widehat{\mathcal{T}})$ (see Corollary A.3 in Appendix A). Hence, the statement made in (6.17) is false and $\mathfrak{b}$ can be chosen as:

  $$\mathsf{s}(u_{\widehat{\mathcal{T}}}(\mathcal{A})) \; \times \; [\mathsf{s}(u_{\widehat{\mathcal{T}}}(\alpha))]^{\mathsf{s}(\widehat{\mathcal{T}})}$$

  Consequently, the binary representation of $\mathfrak{b}$ needs only polynomially many bits in the size of $\widehat{\mathcal{T}}$.

Thus, reasoning in $\tau\mathcal{EL}(deg)$ with respect to acyclic KBs is in PSPACE.

Theorem 6.42. *In $\tau\mathcal{EL}(deg)$, satisfiability, subsumption, consistency and instance checking are in PSPACE with respect to acyclic $\tau\mathcal{EL}(deg)$ knowledge bases.*

Chapter 7

Concept similarity measures, relaxed instance queries and $\tau\mathcal{EL}(m)$

This chapter consists of three sections. First, we show how to use the relaxed instance query approach from [EPT14] to turn a concept similarity measure (CSM) $\bowtie$ into a membership degree function $m_{\bowtie}$. Such a membership degree function, however, need not be well-defined. We present two properties that when satisfied by $\bowtie$, are sufficient to obtain well-definedness for $m_{\bowtie}$. Consequently, such CSMs induce a family of DLs $\tau\mathcal{EL}(m_{\bowtie})$. Additionally, we show that the relaxed instance queries from [EPT14] can be expressed as instance queries with respect to threshold concepts of the form $C_{>t}$.

Afterwards, in Section 7.2 we investigate the computational properties of such induced family of threshold DLs. We will see that there are undecidable threshold logics, but also show that computability of a CSM $\bowtie$ is sufficient to have a decidable DL $\tau\mathcal{EL}(m_{\bowtie})$. Moreover, we will present more specific results for logics belonging to a particular subclass of the considered family.

Last, we present the framework *simi* introduced in [LT12], which can be used to define a variety of CSMs. It turns out that all instances of *simi* satisfy the properties required to obtain well-defined graded membership functions. Then, we consider their induced threshold DLs and see how the previously investigated computational properties apply to them. We further show that a particular instance $\bowtie^1$ of this framework turns out to be equivalent to our membership degree function *deg*.

7.1 Defining membership degree functions

In its most general form, a concept similarity measure $\bowtie$ is a function that maps each pair of concepts C, D (of a given DL) to a value $C \bowtie D \in [0, 1]$ such that $C \bowtie C = 1$. Intuitively, the higher the value of $C \bowtie D$ is, the more similar the two concepts are supposed to be. Such measures can in principle be defined for arbitrary DLs, but here we restrict the attention to CSMs between $\mathcal{EL}$ concepts, i.e., a CSM is a mapping $\bowtie : \mathcal{C}_{\mathcal{EL}} \times \mathcal{C}_{\mathcal{EL}} \to [0, 1]$.

Ecke et al. [EPT14, EPT15] use CSMs to relax instance queries, i.e., instead of requiring that an individual is an instance of the query concept, they only require that it is an instance of a concept that is "similar enough" to the query concept.

Definition 7.1 ([EPT14, EPT15]). Let $\bowtie$ be a CSM, $\mathcal{A}$ an $\mathcal{EL}$ ABox, and $t \in [0, 1)$. The individual $a \in \mathsf{N_I}$ is a *relaxed instance* of the $\mathcal{EL}$ query concept Q w.r.t. $\mathcal{A}$, $\bowtie$, and the threshold t iff there exists an $\mathcal{EL}$ concept description X such that $Q \bowtie X > t$ and

$\mathcal{A} \models X(a)$. The set of all individuals occurring in $\mathcal{A}$ that are relaxed instances of Q w.r.t. $\mathcal{A}$, $\bowtie$, and t is denoted by $\mathsf{Relax}_t^{\bowtie}(Q, \mathcal{A})$. ♢

We apply the same idea on the semantic level of an interpretation rather than the ABox level to obtain graded membership functions from similarity measures.

Definition 7.2. Let $\bowtie$ be a CSM. Then, for each interpretation $\mathcal{I}$, we define the function $m_{\bowtie}^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \mathcal{C}_{\mathcal{EL}} \to [0, 1]$ as

$$m_{\bowtie}^{\mathcal{I}}(d, C) := \max\{C \bowtie D \mid D \in \mathcal{C}_{\mathcal{EL}} \text{ and } d \in D^{\mathcal{I}}\}.$$ ♢

For an arbitrary CSM $\bowtie$, the maximum in this definition need not exist since D ranges over infinitely many concept descriptions. However, two properties that are satisfied by many similarity measures considered in the literature are sufficient to obtain well-definedness for $m_{\bowtie}$. The first is equivalence invariance:

- The CSM $\bowtie$ is *equivalence invariant* iff $C \equiv C'$ and $D \equiv D'$ implies $C \bowtie D = C' \bowtie D'$ for all $C, C', D, D' \in \mathcal{C}_{\mathcal{EL}}$.

To formulate the second property, we need to recall that the *role depth* of an $\mathcal{EL}$ concept description C is the maximum nesting of existential restrictions in C (see Chapter 2 for the formal definition); equivalently, it is the height of the description tree T_C. The *restriction C_k of C to role depth k* is the concept description whose description tree is obtained from T_C by removing all the nodes (and edges leading to them) whose distance from the root is larger than k. More formally,

$$\begin{aligned} C_k &:= C && \text{if } C \in \mathsf{N_C} \text{ or } C = \top,\\ C_k &:= [C_1]_k \sqcap \ldots \sqcap [C_n]_k && \text{if } C = C_1 \sqcap \ldots \sqcap C_n,\\ [\exists r.C]_k &:= \begin{cases} \top & \text{if } k = 0,\\ \exists r.[C]_{k-1} & \text{otherwise.} \end{cases} \end{aligned}$$

- The CSM $\bowtie$ is *role-depth bounded* iff $C \bowtie D = C_k \bowtie D_k$ for all $C, D \in \mathcal{C}_{\mathcal{EL}}$ and any k that is larger than the minimal role depth of C, D.

Role-depth boundedness implies that, in Definition 7.2, we can restrict the maximum computation to concepts D whose role depth is at most $\mathsf{rd}(C) + 1$. Since it is well-known that, up to equivalence, $\mathcal{C}_{\mathcal{EL}}$ contains only finitely many concept descriptions of any fixed role depth (see Proposition 13 in [BST07]), these two properties yield well-definedness for $m_{\bowtie}$. For $m_{\bowtie}$ to be a graded membership function, it also needs to satisfy the properties *M1* and *M2*. To obtain these two properties for $m_{\bowtie}$, we must require that $\bowtie$ satisfies the following additional property:

- The CSM $\bowtie$ is *equivalence closed* iff the following equivalence holds: $C \equiv D$ iff $C \bowtie D = 1$.

Proposition 7.3. *Let $\bowtie$ be an equivalence invariant, role-depth bounded, and equivalence closed CSM. Then $m_{\bowtie}$ is a well-defined graded membership function.*

Proof. Let $\mathcal{I}$ be an interpretation, $d \in \Delta^{\mathcal{I}}$ and C an $\mathcal{EL}$ concept description of role-depth k. Since $\bowtie$ is role-depth bounded, this means that $m_{\bowtie}^{\mathcal{I}}(d, C)$ can be equivalently expressed as:

$$\max\{C \bowtie D \mid D \in \mathcal{C}_{\mathcal{EL}}, d \in D^{\mathcal{I}} \text{ and } \mathsf{rd}(D) \leq k+1\}$$

Now, let D_1 be an $\mathcal{EL}$ concept description such that $d \in [D_1]^{\mathcal{I}}$. Since $\bowtie$ is equivalence invariant, this means that for any other $\mathcal{EL}$ concept D_2 such that $D_1 \equiv D_2$, the values $C \bowtie D_1$ and $C \bowtie D_2$ are the same. Therefore, since there are finitely many concepts in $\mathcal{C}_{\mathcal{EL}}$ of depth at most $k+1$ (up to equivalence), it follows that the maximum always exists.

Since $\bowtie$ is equivalence closed, it easily follows that $m_{\bowtie}$ satisfies property *M1*. As mentioned in Chapter 3, the right to left implication in *M2* already follows from *M1*. The left to right direction is a consequence of the definition of $m_{\bowtie}$ and the fact that $\bowtie$ is equivalence invariant. Hence, $m_{\bowtie}$ satisfies property *M2*.

Thus, $m_{\bowtie}$ is a well-defined graded membership function. □

Consequently, an equivalence invariant, role-depth bounded, and equivalence closed CSM $\bowtie$ induces a DL $\tau\mathcal{EL}(m_{\bowtie})$. Moreover, as we show in the following, computing instances of threshold concepts of the form $Q_{>t}$ in this logic corresponds to answering relaxed instance queries with respect to $\bowtie$.

Proposition 7.4. *Let $\bowtie$ be an equivalence invariant, role-depth bounded, and equivalence closed CSM, $\mathcal{A}$ an $\mathcal{EL}$ ABox, and $t \in [0, 1)$. Then*

$$\mathsf{Relax}_t^{\bowtie}(Q, \mathcal{A}) = \{a \mid \mathcal{A} \models Q_{>t}(a) \textit{ and } a \textit{ occurs in } \mathcal{A}\},$$

where the semantics of the threshold concept $Q_{>t}$ is defined as in $\tau\mathcal{EL}(m_{\bowtie})$.

Proof. ($\Rightarrow$) Let $a \in \mathsf{Ind}(\mathcal{A})$ such that $a \in \mathsf{Relax}_t^{\bowtie}(Q, \mathcal{A})$. Then, there exists an $\mathcal{EL}$ concept description X such that $\mathcal{A} \models X(a)$ and $Q \bowtie X > t$. Since $\mathcal{A} \models X(a)$, this means that for each interpretation $\mathcal{J}$ such that $\mathcal{J} \models \mathcal{A}$, it happens that $a^{\mathcal{J}} \in X^{\mathcal{J}}$. Hence, by definition of $m_{\bowtie}$ we have $m_{\bowtie}^{\mathcal{J}}(d, Q) > t$ for all models of $\mathcal{A}$. Thus, $\mathcal{A} \models Q_{>t}(a)$.

($\Leftarrow$) Conversely, assume that $\mathcal{A} \models Q_{>t}(a)$. By definition of $m_{\bowtie}$, we know that for each model $\mathcal{J}$ of $\mathcal{A}$ there exists $X_{\mathcal{J}}$ such that $a^{\mathcal{J}} \in (X_{\mathcal{J}})^{\mathcal{J}}$ and $Q \bowtie X_{\mathcal{J}} > t$. However, to guarantee that $a \in \mathsf{Relax}_t^{\bowtie}(Q, \mathcal{A})$, we need to show that there exists one such concept which is common for all models of $\mathcal{A}$.

To this end, consider the description graph $G(\mathcal{A})$ induced by $\mathcal{A}$. Additionally, let $\mathcal{I}_{\mathcal{A}}$ denote the interpretation corresponding to $G(\mathcal{A})$ such that $a^{\mathcal{I}_{\mathcal{A}}} = a$ for all $a \in \mathsf{Ind}(\mathcal{A})$. The following facts are easy consequences of Theorem 3.9:

- $\mathcal{I}_{\mathcal{A}} \models \mathcal{A}$, and
- for each $\mathcal{J}$ such that $\mathcal{J} \models \mathcal{A}$, there exists a homomorphism $\varphi_{\mathcal{J}}$ from $G(\mathcal{A})$ to $G_{\mathcal{J}}$ with $\varphi(a) = a^{\mathcal{J}}$ for all $a \in \mathsf{Ind}(\mathcal{A})$.

Since $\mathcal{I}_{\mathcal{A}} \models \mathcal{A}$, this means that there exists an $\mathcal{EL}$ concept description X such that $Q \bowtie X > t$ and $a^{\mathcal{I}_{\mathcal{A}}} \in X^{\mathcal{I}_{\mathcal{A}}}$. The membership characterization via homomorphism in Theorem 2.7, yields the existence of a homomorphism φ_1 from T_X to $G(\mathcal{A})$ with

$\varphi_1(v_0) = a$. Then, the composition $\varphi_{\mathcal{J}} \circ \varphi_1$ yields a similar homomorphism to each model $\mathcal{J}$ of $\mathcal{A}$, which implies $a^{\mathcal{J}} \in X^{\mathcal{J}}$. Therefore, $\mathcal{A} \models X(a)$ and thus, $a \in \mathsf{Relax}_t^{\bowtie}(Q, \mathcal{A})$. $\square$

7.2 Reasoning in $\tau\mathcal{EL}(m_{\bowtie})$

Definition 7.2 allows to create a wide range of well-defined graded membership functions $m_{\bowtie}$ and their corresponding DLs $\tau\mathcal{EL}(m_{\bowtie})$. In this section, we carry out a preliminary study of the computational properties of such a big family of threshold DLs. We will present undecidability and decidability results, as well as more fine-grained complexity results for specific classes within this family.

7.2.1 Undecidability

We present some uncomputability results concerning the type of CSMs being considered and their induced threshold DLs. To start, based on a specific kind of binary relations between $\mathcal{EL}$ concept descriptions, we introduce a very simple form of CSMs satisfying the three properties required in the previous section. We will see that it is not difficult to put a subset of such measures into a one-to-one correspondence with the *power set* of the natural numbers.

Definition 7.5. Let R be a binary relation over $\mathcal{C}_{\mathcal{EL}}$ and $0 < a < 1$ a fixed rational number. Then, R induces the following CSM $\bowtie_R$:

$$C \bowtie_R D := \begin{cases} 1 & \text{if } C \equiv D \\ \mu(C, D) & \text{otherwise.} \end{cases}$$

where μ is defined as follows:

$$\mu(C, D) := \begin{cases} a & \text{if } \mathsf{rd}(C) = \mathsf{rd}(D) \text{ and } (C, D) \in R \\ 0 & \text{otherwise.} \end{cases}$$

In addition, we say that R is equivalence invariant (w.r.t. $\equiv$) iff $C \equiv C'$ and $D \equiv D'$ implies:

$$(C, D) \in R \Leftrightarrow (C', D') \in R \qquad \diamondsuit$$

For equivalence invariant relations R, the induced CSM $\bowtie_R$ satisfies the three properties required in Proposition 7.3.

Lemma 7.6. *Let $R \subseteq \mathcal{C}_{\mathcal{EL}} \times \mathcal{C}_{\mathcal{EL}}$ be equivalence invariant. Then, $\bowtie_R$ is an equivalence invariant, role-depth bounded and equivalence closed CSM.*

Proof. That $\bowtie_R$ is equivalence closed follows directly from its definition. Let us look at the other two properties.

1. *equivalence invariance*: let $C, C', D, D' \in \mathcal{C}_{\mathcal{EL}}$ such that $C \equiv C'$ and $D \equiv D'$. According to the definition of $\bowtie_R$ there are three possible cases for the value $C \bowtie_R D$:

- $C \bowtie_R D = 1$. This means that $C \equiv C' \equiv D \equiv D'$, and by definition $C \bowtie_R D = C' \bowtie_R D' = 1$.
- $C \bowtie_R D = 0$. There are two possibilities:
  - $\mathsf{rd}(C) \neq \mathsf{rd}(D)$. Since $C \equiv C'$ and $D \equiv D'$, this means that $\mathsf{rd}(C') \neq \mathsf{rd}(D')$. Hence, $C' \bowtie_R D' = 0$.
  - $(C, D) \notin R$. Since R is equivalence invariant, $C \equiv C'$ and $D \equiv D'$ imply that $(C', D') \notin R$. Therefore, $C' \bowtie_R D' = 0$.
- $C \bowtie_R D = a$. Then, $C \not\equiv D$, $\mathsf{rd}(C) = \mathsf{rd}(D)$ and $(C, D) \in R$. Similarly as in the previous case, we obtain $C' \not\equiv D'$, $\mathsf{rd}(C') = \mathsf{rd}(D')$ and $(C', D') \in R$. Thus, $C' \bowtie_R D' = a$.

2. *role-depth boundedness*: let $C, D \in \mathcal{C}_{\mathcal{EL}}$. Whenever $\mathsf{rd}(C) = \mathsf{rd}(D)$ the role-depth boundedness condition trivially holds for C and D, since for any $k > \mathsf{rd}(C)$ it is the case that $C = C_k$ and $D = D_k$. It remains to look at the case where $\mathsf{rd}(C) \neq \mathsf{rd}(D)$. It follows from the definition of $\bowtie_R$ that $C \bowtie_R D = 0$. Now, without loss of generality, let $\mathsf{rd}(C) < \mathsf{rd}(D)$. For any value $k > \mathsf{rd}(C)$ we have $\mathsf{rd}(C_k) < \mathsf{rd}(D_k)$. Then, $\mathsf{rd}(C_k) \neq \mathsf{rd}(D_k)$, and consequently $C_k \bowtie_R D_k = 0 = C \bowtie_R D$. □

Now, let us fix the sets $\mathsf{N_C} = \{A\}$ and $\mathsf{N_R} = \{r\}$. For all $N \subseteq \mathbb{N}$, its corresponding binary relation R_N on $\mathcal{EL}$ concept descriptions defined over $\mathsf{N_C} \cup \mathsf{N_R}$, is built as follows:

$$(C, D) \in R_N \Leftrightarrow \mathsf{rd}(C) \in N \tag{7.1}$$

Obviously, since $C \equiv C'$ implies that $\mathsf{rd}(C) = \mathsf{rd}(C')$ and membership in R_N only depends on the role depth of C, it follows that R_N is equivalence invariant. Hence, each subset N of the natural numbers induces an equivalence closed, equivalence invariant and role-depth bounded CSM $\bowtie_{R_N}$. More importantly, for all pairs of distinct subsets $N_1, N_2 \in \mathbb{N}$, the induced CSMs $\bowtie_{R_{N_1}}$ and $\bowtie_{R_{N_2}}$ are different. Just take a number n such that $n \in N_1$ and $n \notin N_2$ (or vice versa). Then, take two concepts C and D such that $\mathsf{rd}(C) = \mathsf{rd}(D) = n$ and $C \not\equiv D$ (the fixed signature $\mathsf{N_C} \cup \mathsf{N_R}$ ensures that this is always possible). By definition we will obtain $C \bowtie_{R_{N_1}} D = a$ and $C \bowtie_{R_{N_2}} D = 0$.

Hence, there are as many CSMs of this type as subsets of the natural numbers, namely, *uncountably* many. Since there are only countable many Turing Machines, there must be non-computable CSMs which are equivalence invariant, role-depth bounded and equivalence closed.

Proposition 7.7. *The set of equivalence invariant, role-depth bounded and equivalence closed CSMs on $\mathcal{EL}$ concept descriptions, contains non-computable functions.*

On the side of the induced threshold DLs, Proposition 7.3 implies that $m_{\bowtie_{R_N}}$ is a well-defined graded membership function for all $N \subseteq \mathbb{N}$, and it induces the DL $\tau\mathcal{EL}(m_{\bowtie_{R_N}})$. Moreover, the very simple definition of $\bowtie_{R_N}$ makes possible to use *satisfiability* in $\tau\mathcal{EL}(m_{\bowtie_{R_N}})$ as a component of an algorithm computing $\bowtie_{R_N}$. More precisely, given two $\mathcal{EL}$ concept descriptions C and D:

1. $C \equiv D \Rightarrow C \bowtie_{R_N} D = 1$, and $\mathsf{rd}(C) \neq \mathsf{rd}(D) \Rightarrow C \bowtie_{R_N} D = 0$.

2. Otherwise, the computation of $C \bowtie_{R_N} D$ solely depends on whether $\mathsf{rd}(C) \in N$. This can be alternatively solved by asking for satisfiability of the concept $C_{\leq a} \sqcap C_{\geq a}$ in $\tau\mathcal{EL}(m_{\bowtie_{R_N}})$ whenever $C \not\equiv \top$. A positive answer corresponds to $C \bowtie_{R_N} D = a$, while the opposite one yields $C \bowtie_{R_N} D = 0$. Why is this true?
   - Satisfiability of $C_{\leq a} \sqcap C_{\geq a}$ implies that for some interpretation $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$:

   $$m^{\mathcal{I}}_{\bowtie_{R_N}}(d, C) = a$$

     This means that for some concept F, $C \bowtie_{R_N} F = a$ which by definition of $\bowtie_{R_N}$ implies $\mathsf{rd}(C) \in N$.
   - Conversely, let $C_{\leq a} \sqcap C_{\geq a}$ be unsatisfiable. Except for $C \equiv \top$, for all $\mathcal{EL}$ concept descriptions C there is always an interpretation $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$ such that $d \notin C^{\mathcal{I}}$. This means that $m^{\mathcal{I}}_{\bowtie_{R_N}}(d, C) < 1$ for such a particular case. Since we are in the unsatisfiability case, it must be that $m^{\mathcal{I}}_{\bowtie_{R_N}}(d, C) = 0$. Moreover, since $d \in \top^{\mathcal{I}}$, the computation of $C \bowtie_{R_N} \top$ must have value 0. Thus, again by definition of $\bowtie_{R_N}$ it follows that $\mathsf{rd}(C) \notin N$.
3. If $C \equiv \top$, the dichotomy used in the previous step cannot be directly applied since $\top_{\leq a} \sqcap \top_{\geq a}$ is actually unsatisfiable. However, once the algorithm reaches the second step, the goal is to decide whether $\mathsf{rd}(C) \in N$. Hence, since $\mathsf{rd}(\top) = \mathsf{rd}(A)$, A can be used instead of $\top$ to solve the issue.

The first step of the previously describe procedure consists of solving "fairly" easy tasks. Consequently, it becomes clear that decidability of the satisfiability problem in a DL $\tau\mathcal{EL}(m_{\bowtie_{R_N}})$ implies computability of the CSM $\bowtie_{R_N}$. Hence, the following undecidability result follows.

Proposition 7.8. *Let $N \subseteq \mathbb{N}$ and R_N its corresponding relation defined as in (7.1). If $\bowtie_{R_N}$ is a non-computable CSM, then it induces an undecidable threshold DL $\tau\mathcal{EL}(m_{\bowtie_{R_N}})$.*

Summing up, on the one hand, we have seen that there are non-computable CSMs that are equivalence invariant, role-depth bounded and equivalence closed. This has been established by setting a one-to-one correspondence with the power set of the natural numbers. On the other hand, a subset of all non-computable CSMs induces a set of undecidable DLs that are constructed as described in Definition 7.2. Nevertheless, it is not yet clear to us whether non-computability of a CSM $\bowtie$ always implies undecidability of the induced DL $\tau\mathcal{EL}(m_{\bowtie})$.

7.2.2 Decidability

We will now show that whenever $\bowtie$ is computable, the standard reasoning problems in the corresponding logic $\tau\mathcal{EL}(m_{\bowtie})$ are decidable. To this end, we establish the following three properties for $m_{\bowtie}$ and $\tau\mathcal{EL}(m_{\bowtie})$. First, we prove that computability of $\bowtie$ implies that $m_{\bowtie}$ is computable with respect to finite interpretations. Second, $\tau\mathcal{EL}(m_{\bowtie})$ enjoys the *finite* model property. Last, we show that there is a computable function that given a concept $\widehat{C}$ finds a number representing a sufficiently large upper bound for the size of models satisfying $\widehat{C}$.

Lemma 7.9. *Let $\bowtie$ be an equivalence invariant, role-depth bounded and equivalence closed CSM. Further, let $\mathcal{I}$ be a finite interpretation, C an $\mathcal{EL}$ concept description and $d \in \Delta^{\mathcal{I}}$. If $\bowtie$ is computable, then $m^{\mathcal{I}}_{\bowtie}(d, C)$ is computable.*

Proof. By definition of $m_{\bowtie}$ we know that:

$$m^{\mathcal{I}}_{\bowtie}(d, C) = \max\{C \bowtie D \mid D \in \mathcal{C}_{\mathcal{EL}} \text{ and } d \in D^{\mathcal{I}}\}$$

Since $\bowtie$ is equivalence invariant and role-depth bounded, we can restrict our attention to concepts D in reduced form whose role depth is at most $\mathsf{rd}(C) + 1$. As explained before, up to equivalence, $\mathcal{C}_{\mathcal{EL}}$ contains finitely many concept descriptions of role depth at most $\mathsf{rd}(C) + 1$. Therefore, it is enough to consider the concepts D in reduced form identifying the corresponding equivalence classes.

Now, the set of such concept descriptions can be enumerated in finite time. Let $[\mathcal{C}^k_{\mathcal{EL}}]$ denote the set of all the representatives of role depth at most $k \geq 0$. For role depth 0, there are exactly $2^{|\mathsf{N_C}|}$ equivalence classes. These are represented by all the concept descriptions of the form $A_1 \sqcap \ldots \sqcap A_n$, where $n \geq 0$, $\{A_1, \ldots, A_n\} \subseteq \mathsf{N_C}$ and $A_i \neq A_j$ (for all $i \neq j$). The particular case of $n = 0$ corresponds to the $\top$ concept. Consequently, $[\mathcal{C}^0_{\mathcal{EL}}]$ is the following set:

$$[\mathcal{C}^0_{\mathcal{EL}}] := \{\top\} \cup \bigcup_{\substack{S \subseteq \mathsf{N_C} \\ S \neq \emptyset}} \left\{ \bigsqcap_{A \in S} A \right\}$$

To continue the enumeration for larger values of k, we inductively describe how to generate $[\mathcal{C}^k_{\mathcal{EL}}]$ from $[\mathcal{C}^{k-1}_{\mathcal{EL}}]$. First, every concept description C of role depth $k > 0$ is of the following form:

$$A_1 \sqcap \ldots \sqcap A_n \sqcap \exists s_1.C_1 \sqcap \ldots \sqcap \exists s_q C_q$$

where $n \geq 0$, $q \geq 1$ and for all $i \in \{1, \ldots, q\}$, $\mathsf{rd}(C_i) < k$. In addition, at least one C_i must have role depth equal to $k - 1$. Moreover, since we are interested only on concepts in reduced form, C satisfies the following conditions:

- for all $1 \leq i \leq q$, C_i is a concept in reduced form.
- for all $s \in \mathsf{N_R}$, let $s(C)$ denote the following set:

  $$s(C) := \{D \mid \exists s.D \in tl(C)\}$$

  Then, $s(C)$ must be an *antichain* with respect to the subsumption relation, i.e., if $C_1, C_2 \in s(C)$ neither $C_1 \sqsubseteq C_2$ nor $C_2 \sqsubseteq C_1$ holds. The same must be true for the set $\{A_1, \ldots, A_n\}$.

Then, once $[\mathcal{C}^{k-1}_{\mathcal{EL}}]$ has been generated, it can be extended to $[\mathcal{C}^k_{\mathcal{EL}}]$ as follows:

1: $Aux := \emptyset$
2: Let $\{r_1, \ldots, r_{|\mathsf{N_R}|}\}$ be the enumeration of the role names in $\mathsf{N_R}$.
3: **for all** $(S_\epsilon, S_1, \ldots, S_{|\mathsf{N_R}|}) \in 2^{\mathsf{N_C}} \times \underbrace{2^{[\mathcal{C}^{k-1}_{\mathcal{EL}}]} \times \ldots \times 2^{[\mathcal{C}^{k-1}_{\mathcal{EL}}]}}_{|\mathsf{N_R}|}$ **do**

if (S_i is an *antichain* for all $1 \leq i \leq |\mathsf{N_R}|$) **and**
($\exists i \, \exists D$ s.t. $D \in S_i$ and $\mathsf{rd}(D) = k-1$) **then**
construct the $\mathcal{EL}$ concept description X as follows:

$$X := \bigsqcap_{A \in S_\epsilon} A \sqcap \bigsqcap_{i=1}^{|\mathsf{N_R}|} \bigsqcap_{Y \in S_i} \exists r_i.Y$$

$Aux := Aux \cup \{X\}$
end if
end for
$[\mathcal{C}^k_{\mathcal{EL}}] := [\mathcal{C}^{k-1}_{\mathcal{EL}}] \cup Aux$

Starting from $[\mathcal{C}^0_{\mathcal{EL}}]$, the iteration of this procedure can be used to enumerate all the concepts D identifying the equivalence classes in $\mathcal{C}^k_{\mathcal{EL}}$. Hence, computing $m^{\mathcal{I}}_{\bowtie}(d, C)$ reduces to use this enumeration up to $\mathsf{rd}(C) + 1$, and keep the maximum value $C \bowtie D$ among those satisfying $d \in D^{\mathcal{I}}$. Checking for $d \in D^{\mathcal{I}}$ in $\mathcal{EL}$ can be done in polynomial time in the size of D and $\mathcal{I}$, whenever $\mathcal{I}$ is finite. Thus, since $\bowtie$ is computable, $m^{\mathcal{I}}_{\bowtie}(d, C)$ can always be computed. □

Let us now turn into the finite model property. We will see that the method used to provide a small model property for $\tau\mathcal{EL}(deg)$ can be used to establish the finite model property for $\tau\mathcal{EL}(m_{\bowtie})$. The base argument for this comes again from the definition of $m_{\bowtie}$ and the basic properties required of $\bowtie$. There is always an $\mathcal{EL}$ concept description D of role depth at most $\mathsf{rd}(C) + 1$ such that:

$$d \in D^{\mathcal{I}} \text{ and } m^{\mathcal{I}}_{\bowtie}(d, C) = C \bowtie D$$

Membership of d into $D^{\mathcal{I}}$ implies that the structure of T_D can be extracted from $G_{\mathcal{I}}$. The idea is that T_D can play the same role as the canonical interpretations $\mathcal{I}_h$ do for *deg*, in the construction introduced in Lemma 5.4. In what follows, after formally defining the analogous of canonical interpretations for the current scenario, we show that such interpretations and $m_{\bowtie}$ exhibit the necessary properties to achieve the correctness of the construction in Lemma 5.4.

Definition 7.10. Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation, $d \in \Delta^{\mathcal{I}}$ and D an $\mathcal{EL}$ concept description such that $d \in D^{\mathcal{I}}$. The canonical interpretation $\mathcal{I}_D$ induced by D is the one having the description tree T_D. ♢

Like Lemma 4.11 for *deg*, the monotonicity property generalizes easily to all graded membership functions $m_{\bowtie}$.

Lemma 7.11. *Let $\bowtie$ be an equivalence invariant, role-depth bounded, and equivalence closed CSM. Additionally, let $\mathcal{I}$ and $\mathcal{J}$ be two interpretations such that there exists a homomorphism φ from $G_{\mathcal{I}}$ to $G_{\mathcal{J}}$. Then, for all $d \in \Delta^{\mathcal{I}}$ and all $\mathcal{EL}$ concept descriptions C it holds:*

$$m^{\mathcal{I}}_{\bowtie}(d, C) \leq m^{\mathcal{J}}_{\bowtie}(\varphi(d), C)$$

Proof. By definition of $m_{\bowtie}$ we know that:

$$m_{\bowtie}^{\mathcal{I}}(d, C) = \max\{C \bowtie D \mid D \in \mathcal{C}_{\mathcal{EL}} \text{ and } d \in D^{\mathcal{I}}\}$$

Let D be one such maximal concept description. Then, $d \in D^{\mathcal{I}}$ implies the existence of a homomorphism φ_D from T_D to $G_{\mathcal{I}}$ such that $\varphi_D(v_0) = d$, where v_0 is the root of T_D. The composition $\varphi \circ \varphi_D$ yields a homomorphism from T_D to $G_{\mathcal{J}}$ with $(\varphi \circ \varphi_D)(v_0) = \varphi(d)$. Hence, $\varphi(d) \in D^{\mathcal{J}}$ and we have:

$$C \bowtie D \leq m_{\bowtie}^{\mathcal{J}}(\varphi(d), C)$$

Thus, $m_{\bowtie}^{\mathcal{I}}(d, C) \leq m_{\bowtie}^{\mathcal{J}}(\varphi(d), C)$ follows. □

The next step is to show that the value of $m_{\bowtie}^{\mathcal{I}}(d, C)$ is preserved by canonical interpretations $\mathcal{I}_D$ corresponding to a concept D, such that the value $C \bowtie D$ is the maximum with respect to the definition of $m_{\bowtie}$.

Lemma 7.12. *Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation, $d \in \Delta^{\mathcal{I}}$ and C an $\mathcal{EL}$ concept description. For all equivalence invariant, role-depth bounded and equivalence closed CSM $\bowtie$:*

$$m_{\bowtie}^{\mathcal{I}_D}(v_0, C) = m_{\bowtie}^{\mathcal{I}}(d, C)$$

for all $D \in \mathcal{C}_{\mathcal{EL}}$ such that $d \in D^{\mathcal{I}}$ and $m_{\bowtie}^{\mathcal{I}}(d, C) = C \bowtie D$.

Proof. Since $\mathcal{I}_D$ corresponds to T_D and $d \in D^{\mathcal{I}}$, this means that there is a homomorphism φ from $G_{\mathcal{I}_D}$ to $G_{\mathcal{I}}$ having $\varphi(v_0) = d$. Then, applying Lemma 7.11 we obtain:

$$m_{\bowtie}^{\mathcal{I}_D}(v_0, C) \leq m_{\bowtie}^{\mathcal{I}}(d, C)$$

On the other side, we know that $m_{\bowtie}^{\mathcal{I}}(d, C) = C \bowtie D$. Since $v_0 \in D^{\mathcal{I}_D}$, the maximum in the definition of $m_{\bowtie}$ implies:

$$m_{\bowtie}^{\mathcal{I}}(d, C) \leq m_{\bowtie}^{\mathcal{I}_D}(v_0, C)$$ □

At this point we observe the following commonalities between *deg* and $m_{\bowtie}$.

- The characterization of membership for $\tau\mathcal{EL}(m)$ given in Theorem 3.8 applies to all graded membership functions m. Therefore, it holds for $\tau\mathcal{EL}(m_{\bowtie})$ as well.
- Lemmas 7.11 and 7.12 are for $m_{\bowtie}$, the same as Lemmas 4.11 and 4.12 are for *deg* in Section 4.2.
- If $d \in D^{\mathcal{I}}$, there is always a homomorphism φ from $\mathcal{I}_D$ to $\mathcal{I}$ with $\varphi_D(v_0) = d$.
- Canonical interpretations $\mathcal{I}_D$ are tree-shaped as $\mathcal{I}_h$ in Definition 4.6.

Hence, the same construction used in Lemma 5.4 for $\tau\mathcal{EL}(deg)$ applies to $\tau\mathcal{EL}(m_{\bowtie})$ by using interpretations $\mathcal{I}_D$ instead of the canonical interpretations $\mathcal{I}_h$. More precisely, suppose that $E_{\sim t} \in \widehat{\ell}_{T_{\widehat{C}}}(v)$ and $v \notin (E_{\sim t})^{\mathcal{I}_0}$ for some $v \in \Delta^{\mathcal{I}_0}$. Then, the interpretation

$\mathcal{I}_D$ used to repair this problem would be such that $\phi(v) \in D^{\mathcal{I}}$ and $m_{\bowtie}^{\mathcal{I}}(\phi(v), E) = E \bowtie D$. Since all those interpretations $\mathcal{I}_D$ are finite and tree-shaped, we obtain a finite tree model property for $\tau\mathcal{EL}(m_{\bowtie})$.

Proposition 7.13. *Let $\bowtie$ be an equivalence invariant, role-depth bounded, and equivalence closed CSM. For all $\tau\mathcal{EL}(m_{\bowtie})$ concept descriptions $\widehat{C}$, if $\widehat{C}$ is satisfiable then there is a finite tree-shaped interpretation $\mathcal{J}$ such that $\widehat{C}^{\mathcal{J}} \neq \emptyset$.*

This form of finite model property is not sufficient to obtain decidability of the satisfiability problem in $\tau\mathcal{EL}(m_{\bowtie})$. To achieve that, we show that a bound for the size of such models can always be computed. Let $\mathcal{I}$ be a model of $\widehat{C}$. Following the construction in Lemma 5.4, the size of the finite model $\mathcal{J}$ resulting from Proposition 7.13 corresponds to the following expression:

$$|\Delta^{\mathcal{J}}| = |\Delta^{\mathcal{I}_0}| + \sum_{\mathcal{I}_D \in \mathfrak{I}} |\Delta^{\mathcal{I}_D}|$$

Recall that $\mathcal{I}_0$ corresponds to the description tree T_C (this is $T_{\widehat{C}}$ without labels of the form $E_{\sim t}$) and $\mathfrak{I}$ the set of canonical interpretations used to extend $\mathcal{I}_0$ into $\mathcal{J}$. The unclear part is to know how big $\Delta^{\mathcal{I}_D}$ can be. Those interpretations $\mathcal{I}_D$ are introduced for threshold concepts of the form $E_{>t}$ or $E_{\geq t}$ occurring in $\widehat{C}$. Moreover, they correspond to the description tree of a concept D satisfying $d \in D^{\mathcal{I}}$ and $m_{\bowtie}^{\mathcal{I}}(d, E) = E \bowtie D$, for some $d \in \Delta^{\mathcal{I}}$. Hence, a trivial choice to provide such a bound is the size of the largest concept D in reduced form, whose role depth is at most $\mathsf{rd}(E) + 1$.

We have already seen in Lemma 7.9 that the set of all those concepts can be enumerated in finite time. Then, the algorithm computing the bound $b(\widehat{C})$ for the size of models satisfying $\widehat{C}$ does the following:

1. list the occurrences in $\widehat{C}$ of threshold concepts $\widehat{F}_1, \ldots, \widehat{F}_q$ of the form $E_{>t}$ or $E_{\geq t}$.
2. let $k_i = \mathsf{rd}(\widehat{F}_i)$ for all $1 \leq i \leq q$, and k the largest value among them.
3. enumerate the set of $\mathcal{EL}$ concept descriptions in $[\mathcal{C}_{\mathcal{EL}}^{k+1}]$. For all $1 \leq i \leq q$, let D_i be one of largest size among those with role depth at most $k_i + 1$.
4. the bound $b(\widehat{C})$ for the size of $\mathcal{J}$ is given as:

$$|\Delta^{\mathcal{I}_0}| + \sum_{i=1}^{q} \mathsf{s}(D_i)$$

Thus, satisfiability of a concept $\widehat{C}$ in $\tau\mathcal{EL}(m_{\bowtie})$ can be decided by first computing $b(\widehat{C})$, and then looking for an interpretation $\mathcal{J}$ of size at most $b(\widehat{C})$ satisfying $\widehat{C}$. Checking whether $\mathcal{J}$ satisfies $\widehat{C}$ can be done using Algorithm 2, since $m_{\bowtie}$ has been proven to be computable in Lemma 7.9.

With respect to the other reasoning problems, observe that the interpretation $\mathcal{J}$ obtained in Proposition 7.13 is tree-shaped. Therefore, it can be used as a base to extend decidability to the other reasoning problems by following the same constructions provided for $\tau\mathcal{EL}(deg)$ in Chapter 5.

- *non-subsumption:* Lemma 5.6 describes how to build a bounded model for satisfiable concepts of the form $\widehat{C} \sqcap \neg\widehat{D}$. The construction starts with a model $\mathcal{J}$ of $\widehat{C}$ that is extended into a model $\mathcal{J}_p$ of $\widehat{C} \sqcap \neg\widehat{D}$, by attaching canonical interpretations $\mathcal{I}_h$. The arguments used to show $(\widehat{C} \sqcap \neg\widehat{D})^{\mathcal{J}_p} \neq \emptyset$ that depends on the nature of *deg* can be separated into two groups.
  1. The results from Lemmas 4.11, 4.12 and 5.4. They all have a corresponding version for $\tau\mathcal{EL}(m_{\bowtie})$.
  2. The value $deg^{\mathcal{I}}(d, C)$ only depends on the fragment of $G_{\mathcal{I}}$ that is reachable from d. This property is exploited at the end of Lemma 5.6. Now, the computation of $m^{\mathcal{I}}_{\bowtie}(d, C)$ depends on the $\mathcal{EL}$ concept descriptions D satisfying $d \in D^{\mathcal{I}}$. Hence, we can also say that $m^{\mathcal{I}}_{\bowtie}(d, C)$ only depends on the fragment of $G_{\mathcal{I}}$ that is reachable from d.

  Thus, the construction of Lemma 5.6 can be applied to $\tau\mathcal{EL}(m_{\bowtie})$ to obtain finite models for satisfiable concepts of the form $\widehat{C} \sqcap \neg\widehat{D}$.

- *consistency:* uses Theorem 3.9 as a characterization of ABox satisfaction for all DLs $\tau\mathcal{EL}(m)$. The construction of the corresponding bounded model in Lemma 5.9 uses basically the same arguments as the ones provided in Lemma 5.4 for *satisfiability.*

- *non-instance:* the bounded model obtained for $\tau\mathcal{EL}(deg)$ is a combination of Lemmas 5.14 and 5.15 (see Lemma 5.16). The properties of *deg* needed in the proofs are the same as the ones used for *non-subsumption.* Hence, the same construction is also valid for $\tau\mathcal{EL}(m_{\bowtie})$.

A common aspect of all these constructions is that they extend $\mathcal{J}$ by plugging canonical interpretations $\mathcal{I}_h$. Moreover, the proofs in Chapter 5 are constructive and describe how those canonical interpretations are obtained from the threshold concepts occurring in an instance of a problem. Hence, the procedure computing $b(\widehat{C})$ can be adapted to estimate a sufficient upper bound for the size of models satisfying concepts of the form $\widehat{C} \sqcap \neg\widehat{D}$ and/or ABoxes of the form $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$ in $\tau\mathcal{EL}(m_{\bowtie})$.

Theorem 7.14. *Let $\bowtie$ be an equivalence invariant, role-depth bounded and equivalence closed CSM. If $\bowtie$ is a computable function, then in $\tau\mathcal{EL}(m_{\bowtie})$ satisfiability, subsumption, consistency and instance checking are decidable problems.*

Overall, we have provided decidability for $\tau\mathcal{EL}(m_{\bowtie})$ based on a *strong* form of the finite model property. It comes as a result of adapting the methods used to obtain "small" models for $\tau\mathcal{EL}(deg)$. However, other than decidability, the previous construction does not give us much insight on how difficult is to reason in a particular logic $\tau\mathcal{EL}(m_{\bowtie})$. In fact, the computation of the upper bound $b(\widehat{C})$ is merely based on the structure of $\widehat{C}$ and the described enumeration, not to mention how big it could be in its general form. In conclusion, CSMs are treated just as "black boxes" satisfying the properties required in Proposition 7.3 to induce the corresponding threshold DL, and none of its internal technicalities are taken into account.

For a particular CSM $\bowtie$ there are two main aspects to be considered in this regard:

- the complexity of computing $\bowtie$,

- the machinery that results from the interaction between the internal characteristics of $\bowtie$ and the maximization mechanism defining $\tau\mathcal{EL}(m_{\bowtie})$.

Obviously, the set of CSMs of this kind is very wide, and as shown in the previous section it even contains functions that are uncomputable. From now on we focus our attention to CSMs that can be computed in polynomial time. In the next two sections we intend to take some initial steps towards understanding the computational properties of a logic $\tau\mathcal{EL}(m_{\bowtie})$ where $\bowtie$ is polynomial time computable. First, we will show that this low-complexity family of CSMs has members whose induced threshold DL is at least PSPACE-hard. Notice that this will be the case, despite our initial requirement of CSMs to be defined over finite alphabets of concept and role names. Afterwards, we will provide a sufficient condition on CSMs that determines a better behaved (in terms of worst case complexity) family of threshold DLs.

7.2.3 A polynomial time CSM and its PSPACE-hard threshold DL

In the following, we define a polynomial time computable CSM satisfying the properties required in Proposition 7.3, such that the satisfiability problem in the induced threshold logic is at least PSPACE-hard. To define such a CSM we start by defining a particular relation R_s, and then follow the construction provided in Definition 7.5 to obtain $\bowtie_{R_s}$.

Note that the abstract definition of $\bowtie_R$ sets up a special connection between the value a and membership in R. This permits to fix any subset of $\mathcal{EL}$ concepts (provided that the resulting relation R is equivalence invariant) as the relevant ones to obtain the similarity value a when compared to a concept description C. In particular, there are concepts in reduced form that grow exponentially (with respect to the size of C) in its width, having in this way description trees that represent exponentially large structures. Then, asking for satisfiability of the $\tau\mathcal{EL}(m)$ concept $C_{\leq a} \sqcap C_{\geq a}$ in $\tau\mathcal{EL}(m_{\bowtie_R})$ could be used to test whether C satisfies a specific property on such type of structures. We will exploit this to obtain our PSPACE-hard threshold logic. The hardness result will be established by a reduction from the problem of deciding the validity of quantified Boolean formulas (QBF), which is introduced in the next definition.

Definition 7.15. A quantified Boolean formula consists of a pair $P.\varphi$ where:

- φ is a propositional formula, and
- the *prefix* P is a sequence of the form $Q_1x_1, \ldots, Q_nx_n$, where $x_1, \ldots, x_n$ are the propositional variables occurring in φ and $Q_i \in \{\exists, \forall\}$ $(1 \leq i \leq n)$. We say that P is of length n.

A quantified Boolean formula $P.\varphi$ can be seen as a first-order logic closed formula, where its variables $x_1, \ldots, x_n$ are interpreted over a two-element domain $\{true, false\}$. For simplicity, we use the semantics defined in [DLNS94] (Section 5.2.2), where QBF is used to establish a PSPACE-hardness result for the DL $\mathcal{ALE}$.

A P-assignment is a mapping $\mathfrak{t} : \{x_1, \ldots, x_n\} \to \{true, false\}$. An assignment $\mathfrak{t}$ satisfies a literal x_i if $\mathfrak{t}(x_i) = true$, and its negation $\neg x_i$ if $\mathfrak{t}(x_i) = false$. An assignment $\mathfrak{t}$ satisfies a clause c if $\mathfrak{t}(\ell) = true$ for at least one literal ℓ of c. A set S of P-assignments is *canonical* for P if it satisfies the following conditions:

1. S is non-empty,

2. $P = \exists x_1.P'$:

   - for all $\mathfrak{t}_1, \mathfrak{t}_2 \in S$, it holds $\mathfrak{t}_1(x_1) = \mathfrak{t}_2(x_1)$.
   - if P' is non-empty, then the set $\{\mathfrak{t}_{|\{x_2,\ldots,x_n\}} \mid \mathfrak{t} \in S\}$ is canonical for P'.

3. $P = \forall x_1.P'$:

   - S contains an assignment $\mathfrak{t}$ such that $\mathfrak{t}(x_1) = \mathit{true}$, and if P' is not empty the set $\{\mathfrak{t}_{|\{x_2,\ldots,x_n\}} \mid \mathfrak{t} \in S$ and $\mathfrak{t}(x_1) = \mathit{true}\}$ is *canonical* for P'.
   - S contains an assignment $\mathfrak{t}$ such that $\mathfrak{t}(x_1) = \mathit{false}$, and if P' is not empty the set $\{\mathfrak{t}_{|\{x_2,\ldots,x_n\}} \mid \mathfrak{t} \in S$ and $\mathfrak{t}(x_1) = \mathit{false}\}$ is *canonical* for P'.

Then, $P.\varphi$ is *valid* if there exists a set S of P-assignments that is canonical for P such that every assignment in S satisfies every clause in φ. ♢

QBF is a PSPACE-complete problem [GJ79], and this is still the case even if φ is in *conjunctive normal form* (CNF) and the quantifiers in P alternate. Moreover, by using *dummy* variables when needed, we can assume without loss of generality that P begins with $\exists$. Consequently, we denote as P_n "the" prefix of length n.

We now move into the details of the PSPACE-hardness result. First, we need to fix our particular threshold logic. To this end, a relation R_s is defined to obtain the CSM $\bowtie_{R_s}$ (according to Definition 7.5), and the logic $\tau\mathcal{EL}(m_{\bowtie_{R_s}})$. Afterwards, we provide the translation reducing QBF to concept satisfiability in $\tau\mathcal{EL}(m_{\bowtie_{R_s}})$. The reduction is based on the following ideas.

- Each set of P-assignments S that is canonical for P can be represented as a concept description D_S. As one may expect, such a concept D_S is of size exponential on the size of $P.\varphi$. Nevertheless, we want to stress that they are not involved in the translation, but are used to define the relation R_s and subsequently the target DL $\tau\mathcal{EL}(m_{\bowtie_{R_s}})$.

- Likewise, a propositional formula φ in CNF can be translated into an $\mathcal{EL}$ concept description C_φ, but this time C_φ is polynomial on the size of φ.

- R_s can be defined such that $(C_\varphi, D_S) \in R_s$ iff every assignment in S satisfies φ. Then, the singularity of the value a in the definition of $\bowtie_{R_s}$ can be used to link validity of $P.\varphi$ to satisfiability of $(C_\varphi)_{\leq a} \sqcap (C_\varphi)_{\geq a}$ in $\tau\mathcal{EL}(m_{\bowtie_{R_s}})$.

Let us start with the encoding of a set S of P-assignments. Let A be a distinguished concept name, for all $n > 0$ we inductively define the string $D_n := \exists r.D_1^0$ as follows:

$$\begin{aligned}
D_1^0 &:= X_1^0 \sqcap \exists r.(A \sqcap D_2^0) \sqcap \exists s.D_2^1 \\
D_2^0 &:= \exists r.D_3^0 \\
D_2^1 &:= \exists r.D_3^1 \\
&\vdots \\
D_{2i+1}^j &:= X_{2i+1}^j \sqcap \exists r.(A \sqcap D_{2(i+1)}^{2j}) \sqcap \exists s.D_{2(i+1)}^{2j+1} \quad (0 \leq j < 2^i) \\
D_{2i}^j &:= \exists r.D_{2i+1}^j \quad (0 \leq j < 2^i) \\
&\vdots \\
D_n^j &:= \begin{cases} \top & \text{if n is even} \\ X_n^j & \text{otherwise.} \end{cases}
\end{aligned} \tag{7.2}$$

The symbols X_{2i+1}^j represent *variables* that are to be instantiated to obtain $\mathcal{EL}$ concept descriptions. Let X_n be the set of all variables occurring in D_n, i.e.:

$$X_n = \{X_{2i+1}^j \mid 1 \leq 2i+1 \leq n \text{ and } 0 \leq j < 2^i\}$$

We denote by $\mathfrak{X}_n$ the set of all total mappings $\theta : X_n \to \{\top, A\}$. The application of one such θ to D_n is denoted as $\theta[D_n]$ and consists of substituting each variable X_{2i+1}^j in D_n by $\theta(X_{2i+1}^j)$. Then, $\mathfrak{X}_n$ generates the following family of $\mathcal{EL}$ concept descriptions:

$$\mathfrak{D}_n := \{\theta[D_n] \mid \theta \in \mathfrak{X}_n\}$$

To be consistent later on, we define $\mathfrak{D}_0$ as the empty set. Additionally, since the "branching" in the definition of the string D_n is defined using two different role names r and s, one can see that all concept descriptions in $\mathfrak{D}_n$ are in reduced form. The purpose of these sets is that each concept in $\mathfrak{D}_n$ identifies a set of P-assignments that is canonical for the prefix P of length n. The following example gives the intuition underlying such a correspondence.

Example 7.16. Let P_4 be the prefix of length 4, i.e., $P_4 = \exists x_1 \forall x_2 \exists x_3 \forall x_4$. Let $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3$ and $\mathfrak{t}_4$ be the following P_4-assignments:

$$\begin{array}{llll}
\mathfrak{t}_1(x_1) = \textit{true} & \mathfrak{t}_1(x_2) = \textit{true} & \mathfrak{t}_1(x_3) = \textit{false} & \mathfrak{t}_1(x_4) = \textit{true} \\
\mathfrak{t}_2(x_1) = \textit{true} & \mathfrak{t}_2(x_2) = \textit{true} & \mathfrak{t}_2(x_3) = \textit{false} & \mathfrak{t}_2(x_4) = \textit{false} \\
\mathfrak{t}_3(x_1) = \textit{true} & \mathfrak{t}_3(x_2) = \textit{false} & \mathfrak{t}_3(x_3) = \textit{true} & \mathfrak{t}_3(x_4) = \textit{true} \\
\mathfrak{t}_4(x_1) = \textit{true} & \mathfrak{t}_4(x_2) = \textit{false} & \mathfrak{t}_4(x_3) = \textit{true} & \mathfrak{t}_4(x_4) = \textit{false}
\end{array}$$

One can easily see that the set $S = \{\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4\}$ is canonical for P_4. Now, the string D_4 contains the set of variables $X_4 = \{X_1^0, X_3^0, X_3^1\}$. Let $\theta \in \mathfrak{X}_4$ be the mapping such that $\theta(X_1^0) = A$, $\theta(X_3^0) = \top$ and $\theta(X_3^1) = A$. This yields the $\mathcal{EL}$ concept description $D_S := \theta[D_4]$ having the description tree depicted on the left-hand side of Figure 7.1.

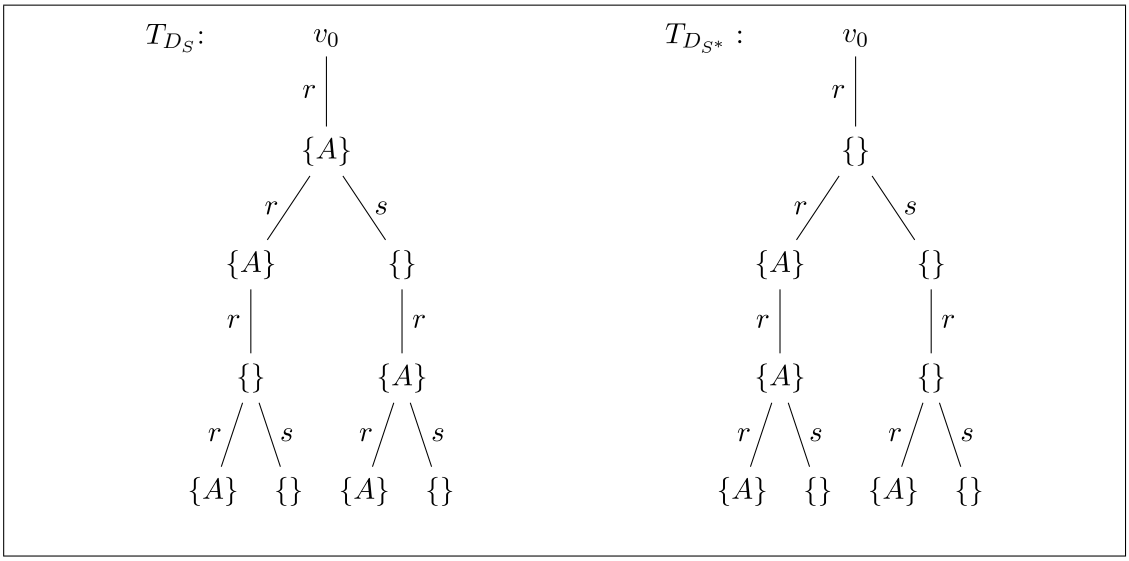


Figure 7.1: $\mathcal{EL}$ description trees corresponding to canonical sets of P-assignments.

Consider the *left-most* path in T_{D_S}:

$$\pi_{lm}: \quad \{\} \xrightarrow{\ r\ } \{A\} \xrightarrow{\ r\ } \{A\} \xrightarrow{\ r\ } \{\} \xrightarrow{\ r\ } \{A\}$$

Denoting the nodes in π_{lm} from left to right as $v_0, \ldots, v_4$, the assignment $\mathfrak{t}_1$ can be obtained as follows. For all $1 \leq i \leq 4$:

$$\mathfrak{t}_1(x_i) := \begin{cases} \mathit{true} & \text{if } \ell_{\pi_{lm}}(v_i) = \{A\} \\ \mathit{false} & \text{if } \ell_{\pi_{lm}}(v_i) = \{\} \end{cases}$$

Conversely, the path π_{lm} can be constructed from $\mathfrak{t}_1$ using the inverse correspondence between $\{\mathit{true}, \mathit{false}\}$ and $\{A, \top\}$. The same relationship can also be established between the other three paths in T_{D_S} and the assignments $\mathfrak{t}_2, \mathfrak{t}_3, \mathfrak{t}_4$, respectively. Hence, the idea is that for all sets S that are canonical for P_4, there is an instance $\theta[D_4]$ such that each assignment $\mathfrak{t} \in S$ corresponds to a path $\pi_{\mathfrak{t}}$ in $T_{\theta[D_4]}$ and vice versa. For example, the variation of S where $\mathfrak{t}_1(x_1) = \mathfrak{t}_2(x_1) = \mathfrak{t}_3(x_1) = \mathfrak{t}_4(x_1) = \mathit{false}$, $\mathfrak{t}_1(x_3) = \mathfrak{t}_2(x_3) = \mathit{true}$ and $\mathfrak{t}_3(x_3) = \mathfrak{t}_4(x_3) = \mathit{false}$, would correspond to the concept $\theta^*[D_4]$ where $\theta^*(X_1^0) = \top$, $\theta^*(X_3^0) = A$ and $\theta^*(X_3^1) = \top$. The description tree in the right-hand side of the same figure is the one associated to $\theta^*[D_4]$. $\diamondsuit$

Let us formally define the correspondence illustrated in the previous example, and then show that it actually exists.

Definition 7.17. Let $n > 0$ be a natural number, $\pi = v_0 r_1 v_1 \ldots r_n v_n$ a path of length n in some $\mathcal{EL}$ description tree T and $\mathfrak{t}$ a truth assignment of the variables $x_1, \ldots, x_n$. We say that π and $\mathfrak{t}$ are *corresponding* iff for all $1 \leq i \leq n$:

$$\mathfrak{t}(x_i) = \mathit{true} \Leftrightarrow \ell_T(v_i) \cap \{A\} = \{A\}$$

Additionally, let S be a set of P_n-assignments that is canonical for P_n and $D \in \mathfrak{D}_n$. We further say that S and D are *corresponding* iff:

- for all $\mathfrak{t} \in S$ exists a path $\pi_\mathfrak{t}$ in T_D of length n, such that $\mathfrak{t}$ and $\pi_\mathfrak{t}$ are corresponding, and
- for all paths π of length n in T_D there is $\mathfrak{t}_\pi \in S$, such that $\mathfrak{t}_\pi$ and π are corresponding. ♢

The proof of the following lemma is deferred to the Appendix A.

Lemma 7.18. *Let $n > 0$ be a natural number. Then,*

1. *for all sets S of P_n-assignments that are canonical for P_n, there exists $D_S \in \mathfrak{D}_n$ such that S and D_S are corresponding, and*
2. *for all $D \in \mathfrak{D}_n$, exists a set S_D of P_n-assignments that is canonical for P_n such that S_D and D are corresponding.*

Next, we describe how to encode the structure of a propositional formula φ in CNF into an $\mathcal{EL}$ concept description C_φ. We need to use an additional concept name I. Let φ be the conjunction of clauses $c_1 \wedge \ldots \wedge c_q$, and $x_1, \ldots, x_n$ the propositional variables occurring in φ. For each c_j $(1 \leq j \leq q)$ we define its corresponding $\mathcal{EL}$ concept description C_j as $\exists r_1^j.E_1^j$, where E_1^j is of the following form:

$$
\begin{aligned}
&E_1^j := \gamma_1^j \sqcap \exists r_2^j.E_2^j \\
&\ldots \\
&E_i^j := \gamma_i^j \sqcap \exists r_{i+1}^j.E_{i+1}^j \quad (1 \leq i < n) \\
&\ldots \\
&E_n^j := \gamma_n^j
\end{aligned}
$$

Here $\gamma_i^j = I$ if x_i does not occur in c_j. Otherwise $\gamma_i^j = A$ whenever x_i is a literal in c_j and $\gamma_i^j = \top$ for $\neg x_i$. One can assume that x_i and $\neg x_i$ do not occur at the same time in any set c_j, since otherwise c_j is always satisfied and it can be removed from φ. Regarding $r_1^j, \ldots, r_n^j$, they correspond to any of two fixed role names r and s as follows: if $\gamma_i^j = A$ or $\gamma_i^j = I$ then $r_i^j = r$. Otherwise, $r_i^j = s$. Then, the $\mathcal{EL}$ concept description C_φ encoding the structure of φ is defined as:

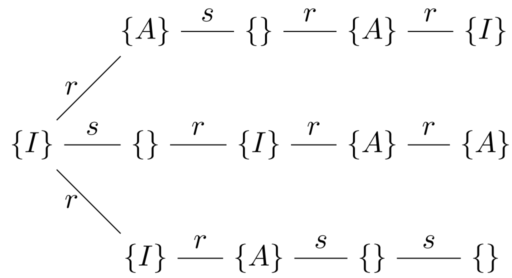
$$C_\varphi := I \sqcap \prod_{j=1}^{q} C_j$$

Example 7.19. Let φ be the following propositional formula in CNF:

$$\{x_1, \neg x_2, x_3\} \wedge \{\neg x_1, x_4, x_3\} \wedge \{\neg x_4, x_2, \neg x_3\}$$

A total of four propositional variables occur in φ. Then, the concept description C_φ is the one having the following $\mathcal{EL}$ description tree:

T_{C_φ}:



Each branch of the tree corresponds to a clause in φ. In particular, the nodes at the i^{th} level (except for the root) tell us in which form the variable x_i occurs in each clause of φ. For x_2, the empty set (or $\top$) is used in the *upper* branch to represent a negative occurrence in c_1, I in the *middle* branch expresses that x_2 is irrelevant for c_2, and A is used in the last branch to state that x_2 occurs in c_3. The same idea applies for the rest of the variables occurring in φ. $\diamondsuit$

So far, we have a way to encode propositional formulas and sets of P-assignments into concept descriptions. Then, the role of R_s is to verify whether a formula φ is satisfied by all the assignments of a set S. The definition of R_s is based on the representation of concepts as description trees. Let T_1 and T_2 be two $\mathcal{EL}$ description trees, and $\pi_1 = v_0 r_1 v_1 \dots r_n v_n, \pi_2 = w_0 s_1 w_1 \dots s_n w_n$ two paths of length n in T_1 and T_2, respectively. We say that π_1 has a *coincidence* in π_2 (denoted $\pi_1 \rhd \pi_2$) iff there is $0 \leq i \leq n$ such that:

$$\ell_{T_1}(v_i) \cap \{A\} = \ell_{T_2}(w_i) \cap \{A\} \text{ and } I \notin \ell_{T_1}(v_i) \tag{7.3}$$

For all $\mathcal{EL}$ description trees T we denote by $\Pi(T)$ the set of all paths in T starting at its root, and by $\Pi_p(T) \subseteq \Pi(T)$ the subset of those having length $p \geq 0$. Then, using the relation $\rhd$ we define a family of binary relations $\rhd_p$ (for all $p \in \mathbb{N}$) over the set of $\mathcal{EL}$ description trees as follows:

$$(T_1, T_2) \in \rhd_p$$
$$\text{iff}$$
$$\forall \pi_2 \in \Pi_p(T_2)\, \forall \pi_1 \in \Pi_p(T_1), \text{ it holds } \pi_1 \rhd \pi_2$$

Using this family of relations together with the family of sets $\mathfrak{D}_n$, the relation R_s is defined as follows:

$$R_s := \{(C, D) \mid \exists D^* \in \mathfrak{D}_{\mathsf{rd}(C)} \text{ s.t. } D \equiv D^* \text{ and } (T_{C^r}, T_{D^*}) \in \rhd_{\mathsf{rd}(C)}\}$$

The following lemma shows that R_s is equivalence invariant (see Definition 7.5).

Lemma 7.20. *The relation R_s is equivalence invariant.*

Proof. Let C, C', D and D' be $\mathcal{EL}$ concepts such that $C \equiv C'$ and $D \equiv D'$. We need to show that $(C, D) \in R_s$ iff $(C', D') \in R_s$. Some simple facts follow:

- Since $C \equiv C'$, we have that $\mathsf{rd}(C) = \mathsf{rd}(C') = k$ for some $k \geq 0$. Moreover, by Theorem 2.8 there is an isomorphism between T_{C^r} and $T_{(C')^r}$.
- Let $D^* \in \mathfrak{D}_k$, $k > 0$. Then, $D \equiv D'$ implies that $D \equiv D^*$ iff $D' \equiv D^*$.

For $k = 0$, the claim trivially holds since $\mathfrak{D}_0$ is empty and there is no D^*. Otherwise, suppose that $(T_{C^r}, T_{D^*}) \in \rhd_k$. To see that also $(T_{(C')^r}, T_{D^*}) \in \rhd_k$, let $\pi_{C'}$ and π_{D^*} be arbitrary paths of length k in $T_{(C')^r}$ and T_{D^*}, respectively. Using the isomorphism mentioned above, one can find a path π_C in T_{C^r} such that $\pi_C = \pi_{C'}$. Since $(T_{C^r}, T_{D^*}) \in \rhd_k$, this means that $\pi_C \rhd \pi_{D^*}$. Consequently $\pi_{C'} \rhd \pi_{D^*}$, and we have thus shown that $(T_{C^r}, T_{D^*}) \in \rhd_k \Rightarrow (T_{(C')^r}, T_{D^*}) \in \rhd_k$. The implication in the opposite direction can be obtained in a similar way.

All these elements combined imply that R_s is equivalence invariant. □

Hence, R_s induces a CSM $\bowtie_{R_s}$ that is equivalence invariant, role-depth bounded and equivalence closed (see Lemma 7.6). To see how difficult it is to compute $\bowtie_{R_s}$, let us look at the abstract formulation of $\bowtie_R$ in Definition 7.5. The computation of $C \bowtie_R D$ first discriminates between whether $C \equiv D$ or not. Checking equivalence of concept descriptions in $\mathcal{EL}$ is a polynomial time problem. In case of a negative answer, $C \bowtie_R D$ corresponds to the value $\mu(C, D)$. Since verifying whether $\mathsf{rd}(C) = \mathsf{rd}(D)$ is also a polynomial time issue, the difficulty of computing $\bowtie_R$ will then depend on how hard it is to check for membership in R.

Let $\mathsf{rd}(C) = k$. In particular, checking for membership in R_s consists of two steps.

1. Test whether there is $D^* \in \mathfrak{D}_k$ such that $D \equiv D^*$. If such D^* exists, then there is an isomorphism between $(D^*)^r$ and D^r (by Theorem 2.8). The computation of reduced forms is in polynomial time, and we know that D^* is already in reduced form. Moreover, concepts in $\mathfrak{D}_k$ result from different instantiations of the variables occurring in D_k. Fortunately, the definition of D_k follows a very simple pattern that simplifies the quest of whether such concept description D^* exists.

   From a graphical point of view, this means that in T_{D^*} every node at an *even* level (the root node is at level 0) has exactly one r-successor, whereas the ones at *odd* levels have exactly one r-successor and one s-successor labeled with $\{A\}$ and $\{\}$, respectively. Testing whether D^r has such a shape (*up to isomorphism*) can be done by traversing its structure. Since $\mathsf{s}(D^r) \leq \mathsf{s}(D)$, this is a polynomial time procedure in the size of D.

2. Check if $(T_{C^r}, T_{D^*}) \in \rhd_k$. Note that D^r and D^* need not be syntactically equal, but they have nevertheless exactly the same paths. Hence, this step is equivalent to verifying whether $(T_{C^r}, T_{D^r}) \in \rhd_k$.

   - deciding $\rhd$ for paths of length k is obviously linear in k.
   - the number of paths of length k in T_{C^r} and T_{D^r} can be bounded by $\mathsf{s}(C^r)$ and $\mathsf{s}(D^r)$, respectively.

   Therefore, checking whether $(T_{C^r}, T_{D^r}) \in \rhd_k$ can be done in time $\mathcal{O}(\mathsf{s}(C^r) \times \mathsf{s}(D^r) \times k)$.

Finally, counting that concepts in reduced form are the smallest elements in their equivalence classes, we have that $\bowtie_{R_s}$ is computable in polynomial time.

Next, we continue Example 7.19 to see how all these pieces fit together.

Continuation of Example 7.19. Consider the set $S = \{\mathsf{t}_1, \mathsf{t}_2, \mathsf{t}_3, \mathsf{t}_4\}$ from Example 7.16 and the corresponding concept description D_S. One can easily see that t_1 satisfies the clause c_3 in φ whereas $\mathfrak{t}_3$ does not. From a graphical perspective, c_3 corresponds to the third path (from *top to bottom*) of the description tree T_{C_φ} and $\mathfrak{t}_1$ ($\mathfrak{t}_3$) to the first (third) one (from *left to right*) in T_{C_S}, i.e.:

$$\pi_{c_3}:\quad \{I\} \xrightarrow{\;r\;} \{I\} \xrightarrow{\;r\;} \{A\} \xrightarrow{\;s\;} \{\} \xrightarrow{\;s\;} \{\}$$

$$\pi_{\mathfrak{t}_1}:\quad \{\} \xrightarrow{\;r\;} \{A\} \xrightarrow{\;r\;} \{A\} \xrightarrow{\;r\;} \{\} \xrightarrow{\;r\;} \{A\}$$

$$\pi_{\mathfrak{t}_3}:\quad \{\} \xrightarrow{\;r\;} \{A\} \xrightarrow{\;s\;} \{\} \xrightarrow{\;r\;} \{A\} \xrightarrow{\;r\;} \{A\}$$

Note that π_{c_3} and $\pi_{\mathfrak{t}_1}$ agree on the third position according to (7.3), but this is not the case for any position regarding π_{c_3} and $\pi_{\mathfrak{t}_3}$. This means that $\pi_{c_3} \not\rhd \pi_{\mathfrak{t}_3}$ and $\pi_{c_3} \rhd \pi_{\mathfrak{t}_1}$. The intuition here is that $\rhd$ can be used to verify whether a clause c is satisfied by an assignment $\mathfrak{t}$. Hence, membership/non-membership of (T_{C_φ}, T_{D_S}) in $\rhd_4$ would determine whether every assignment in S satisfies φ or not. Contrary to T_{D_S}, for the description tree $T_{D_{S^*}}$ corresponding to $\theta^*[D_4]$ (see Figure 7.1) it is the case that $(T_{C_\varphi}, T_{D_{S^*}}) \in \rhd_4$. This would mean that the canonical set corresponding to $\theta^*[D_4]$ certifies the validity of the formula $\exists x_1 \forall x_2 \exists x_3 \forall x_4.\varphi$.

Overall, membership in $\rhd_4$ leads to membership in R_s, and subsequently to the similarity value a when computing $\bowtie_{R_s}$. Since R_s emphasizes that only those concepts in $\mathfrak{D}_4$ are relevant, this will make satisfiability of $(C_\varphi)_{\leq a} \sqcap (C_\varphi)_{\geq a}$ in $\tau\mathcal{EL}(m_{\bowtie_{R_s}})$ to be equivalent to validity of $\exists x_1 \forall x_2 \exists x_3 \forall x_4.\varphi$. $\diamondsuit$

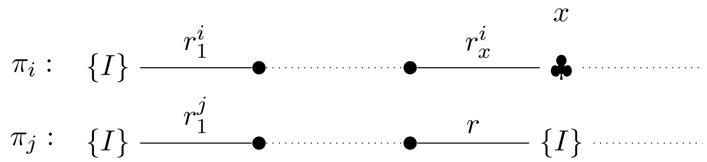
Obviously, the correctness of the previous idea relies on comparing all the paths in T_{C_φ} to all the paths in T_{D_S}, when assessing the value of $C_\varphi \bowtie_{R_s} D_S$. Since the definition of R_s uses $\rhd_p$ with respect to concept descriptions in reduced form, it would be a problem to have a *reducible* concept C_φ. The following lemma shows that the particular use of r and s when building C_φ guarantees that this is never the case.

Lemma 7.21. *Let φ be a propositional formula in CNF. The concept description C_φ is in reduced form.*

Proof. Recall that we restrict our attention to clauses where x and $\neg x$ do not occur at the same time. Additionally, one can also assume that no two clauses of φ are equal. We denote by $V(c)$ the set of variables occurring in a clause c.

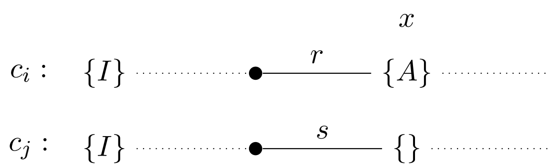
Now, let c_i and c_j be two clauses of φ. Following the construction of C_φ, they correspond to the top-level atoms $\exists r_1^i.E_1^i$ and $\exists r_1^j.E_1^j$ in C_φ. We want to prove that $r_1^i = r_1^j$ implies $E_1^i \not\sqsubseteq E_1^j$ and $E_1^j \not\sqsubseteq E_1^i$. To this end, we distinguish two cases regarding $V(c_i)$ and $V(c_j)$:

- $V(c_i) \neq V(c_j)$. This means that there is at least one variable x such that x occurs in c_i and not in c_j (or vice versa). By construction of C_φ, c_i and c_j contribute to T_{C_φ} with two paths of the form:



where ♣ stands for $\{A\}$ or $\{\}$ and r_x^i for r or s, depending on how x occurs in c_i. By construction of C_φ, the possible combinations for $(♣, r_x^i)$ are $(\{\}, s)$ and $(\{A\}, r)$. Then, it is not hard to verify that no subsumption relation exists between E_1^i and E_1^j.

- $V(c_i) = V(c_j)$. Since $c_i \neq c_j$, this means that there is $x \in V(c_i)$ such that x occurs in c_i and $\neg x$ in c_j (or vice versa). Thus, the corresponding paths have the following structure:



Again, it is immediate to see why $E_1^i \not\sqsubseteq E_1^j$ and $E_1^j \not\sqsubseteq E_1^i$.

Hence, these arguments prove that C_φ is already in reduced form. □

Altogether we have a polynomial time computable CSM $\bowtie_{R_s}$ that is equivalence invariant, role-depth bounded and equivalence closed. Moreover, $\bowtie_{R_s}$ has been defined over the finite sets $\{A, I\}$ and $\{r, s\}$ of concept and role names, respectively. Consequently, it induces the DL $\tau\mathcal{EL}(m_{\bowtie_{R_s}})$ as described in Section 7.1. Then, we reduce QBF to satisfiability in $\tau\mathcal{EL}(m_{\bowtie_{R_s}})$ as follows. Given a quantified Boolean formula $P.\varphi$, it is translated into the $\tau\mathcal{EL}(m)$ concept description $\widehat{C}_\varphi$ defined as follows:

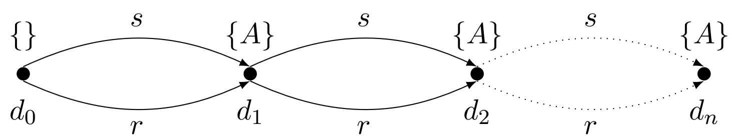
$$\widehat{C}_\varphi := (C_\varphi)_{\leq a} \sqcap (C_\varphi)_{\geq a}$$

The following lemma shows the correctness of the reduction.

Lemma 7.22. *Let $P.\varphi$ be a quantified Boolean formula. Then, $P.\varphi$ is valid iff $\widehat{C}_\varphi$ is satisfiable in $\tau\mathcal{EL}(m_{\bowtie_{R_s}})$.*

Proof. Let $n > 0$ be the length of P, $c_1, \ldots, c_q$ the clauses of φ, and $x_1, \ldots, x_n$ the propositional variables occurring in φ.

($\Rightarrow$) Assume that $P.\varphi$ is valid. To prove that $\widehat{C}_\varphi$ is satisfiable in $\tau\mathcal{EL}(m_{\bowtie_{R_s}})$ we select the interpretation $\mathcal{I}$ having the following $\mathcal{EL}$ description graph:



Our goal is to show that $d_0 \in (\widehat{C}_\varphi)^\mathcal{I}$. By Definition 7.2,

$$m^\mathcal{I}_{\bowtie_{R_s}}(d_0, C_\varphi) = \max\{C_\varphi \bowtie_{R_s} D \mid D \in \mathcal{C}_{\mathcal{EL}} \text{ and } d_0 \in D^\mathcal{I}\}$$

Since $d_0 \notin I^\mathcal{I}$, this means that for all the candidate concepts D we have $C_\varphi \not\equiv D$. Therefore, $m^\mathcal{I}_{\bowtie_{R_s}}(d_0, C_\varphi) < 1$. In particular, if $D \in \mathfrak{D}_n$ it is easy to see that there is a homomorphism φ from T_D to $G_\mathcal{I}$ mapping the root v_0 of T_D to d_0. Hence, by Theorem 2.7 we obtain $d_0 \in D^\mathcal{I}$.

Now, since $P.\varphi$ is valid, there is a set S of P-assignments that is canonical for P such that every truth assignment $\mathfrak{t} \in S$ satisfies φ. Let $D_S \in \mathfrak{D}_n$ be an $\mathcal{EL}$ concept description such that S and D_S are *corresponding* (see Lemma 7.18). Then, we know the following about C_φ and D_S:

- $\mathsf{rd}(C_\varphi) = \mathsf{rd}(D_S) = n$,
- $d_0 \in (D_S)^\mathcal{I}$, and
- $C_\varphi \bowtie_{R_s} D_S < 1$.

Let us now establish that $(T_{C_\varphi}, T_{D_S}) \in \rhd_n$. Consider two arbitrary paths π and π_j of length n in T_{D_S} and T_{C_φ}, respectively. By construction of C_φ and Definition 7.17 we have:

- π_j corresponds to the clause c_j in φ.
- since S and D_S are corresponding, there exists $\mathfrak{t}_\pi \in S$ such that $\mathfrak{t}_\pi$ and π are *corresponding.*

As $\mathfrak{t}_\pi$ satisfies c_j, it must exist at least one literal ℓ in c_j such that $\mathfrak{t}_\pi(\ell) = \mathit{true}$. Let x_i be the variable corresponding to ℓ. Then, for the i^{th} node v_i of π_j:

$$\ell_{T_{C_\varphi}}(v_i) = \{A\} \text{ if } \ell = x_i, \text{ and } \ell_{T_{C_\varphi}}(v_i) = \{\} \text{ if } \ell = \neg x_i \tag{7.4}$$

and (since $\mathfrak{t}_\pi$ and π are corresponding) for the node w_i of π:

$$\mathfrak{t}_\pi(x_i) = \mathit{true} \;\Leftrightarrow\; \ell_{T_{D_S}}(w_i) \cap \{A\} = \{A\} \tag{7.5}$$

Hence, one can see that $\mathfrak{t}_\pi$ satisfies ℓ iff $\ell_{T_{C_\varphi}}(v_i) \cap \{A\} = \ell_{T_{D_S}}(w_i) \cap \{A\}$. Consequently, $\pi_j \rhd \pi$. Since these two paths have been chosen arbitrarily, we just have shown that $(T_{C_\varphi}, T_{D_S}) \in \rhd_n$. Having $D_S \in \mathfrak{D}_n$ further implies that $(C_\varphi, D_S) \in R_s$, and this means that $C_\varphi \bowtie_{R_s} D_S = a$ (see the expression $\mu(C, D)$ in Definition 7.5). Thus, $m^\mathcal{I}_{\bowtie_{R_s}}(d_0, C_\varphi) = a$ and $d_0 \in (\widehat{C}_\varphi)^\mathcal{I}$.

($\Leftarrow$) Assume that $\widehat{C}_\varphi$ is satisfiable in $\tau\mathcal{EL}(m_{\bowtie_{R_s}})$. Then, there exists an interpretation $\mathcal{I}$ and $d \in \Delta^\mathcal{I}$ such that $d \in (\widehat{C}_\varphi)^\mathcal{I}$. This means that $m^\mathcal{I}_{\bowtie_{R_s}}(d, C_\varphi) = a$. Thus, there exists a concept D such that $d \in D^\mathcal{I}$ and $C_\varphi \bowtie_{R_s} D = a$. By definition of $\bowtie_{R_s}$ this is the case if $(C_\varphi, D) \in R_s$, and membership in R_s for (C_φ, D) implies the existence of a concept $D^* \in \mathfrak{D}_n$ such that:

$$D \equiv D^* \;\text{ and }\; (T_{C_\varphi}, T_{D^*}) \in \rhd_n$$

Now, by Lemma 7.18 there is a set of P-assignments S_{D^*} such that:

- S_{D^*} is canonical for P, and
- S_{D^*} and D^* are *corresponding.*

The rest consists of proving that each assignment in S_{D^*} satisfies φ. Let $\mathfrak{t} \in S_{D^*}$ and c_j $(1 \leq j \leq q)$ be a clause of φ. Again, by construction of C_φ and Definition 7.17 we obtain:

- c_j corresponds to a path π_j in T_{C_φ}.
- since S_{D^*} and D^* are corresponding, there exists a path $\pi_\mathfrak{t}$ in T_{D^*} of length n such that $\mathfrak{t}$ and $\pi_\mathfrak{t}$ are also corresponding.

From $(T_{C_\varphi}, T_{D^*}) \in \rhd_n$, it follows that there is $0 \leq i \leq n$ such that (7.3) holds with respect to π_j and $\pi_\mathfrak{t}$, i.e.:

$$\ell_{T_{C_\varphi}}(v_i) \cap \{A\} = \ell_{T_{D^*}}(w_i) \cap \{A\} \text{ and } I \notin \ell_{T_{C_\varphi}}(v_i)$$

Note that $I \in \ell_{T_{C_\varphi}}(v_0)$, and consequently $i > 0$. Let ℓ be the literal corresponding to the occurrence of the variable x_i in c_j. The relationships from (7.4) and (7.5) are also valid in this case. Then, combining them with the previous equality ensures that $\mathfrak{t}$ satisfies c_j. This will always be the case for all $\mathfrak{t} \in S_{D^*}$ and c_j in φ. Thus, $P.\varphi$ is valid.□

Thus, we have shown PSPACE-hardness for *satisfiability* in $\tau\mathcal{EL}(m_{\bowtie R_s})$.

Corollary 7.23. *In $\tau\mathcal{EL}(m_{\bowtie R_s})$, concept satisfiability is PSPACE-hard.*

The standard reductions from *satisfiability* to the other reasoning problems (*subsumption, consistency* and *instance*) yield PSPACE-hardness for them as well.

Now, some additional information emerges from the previous result and its proof. The specific set of all satisfiable $\tau\mathcal{EL}(m_{\bowtie R_s})$ concept descriptions of the form $\widehat{C}_\varphi$ constitutes a PSPACE-hard language. Moreover, the proof of Lemma 7.22 shows that all these concepts are satisfiable in a model of polynomial size. Intuitively, the source of complexity resides on the fact that *model checking* $(C_\varphi)_{\leq a} \sqcap (C_\varphi)_{\geq a}$ requires to consider concepts of size exponential in $\widehat{C}_\varphi$. These are the ones fixed by the definition of R_s, and its structure is succinctly encoded within the shape of the interpretation $\mathcal{I}$ selected in Lemma 7.22.

Based on these observations, we now move into defining a *bounding* condition that is sufficient to safely disregard such a big concept descriptions.

7.2.4 Bounded CSMs

This section is organized as follows. We start right away by defining the bounded condition on CSMs. Afterwards, we shall restrict our attention to polynomially bounded CSMs, and study the computational aspects of the DLs induced by such a family of measures.

Definition 7.24. A *filter* F is a subset of $\mathcal{C}_{\mathcal{EL}}$ such that for all $C, D \in F$:

- $C \sqcap D \in F$, and
- $C \sqsubseteq C'$ implies $C' \in F$.

The set of all filters is denoted as $\mathfrak{F}$. Now, let $g : \mathbb{N} \to \mathbb{N}$ be a function. We say that a CSM $\bowtie$ is *g-bounded* iff for all $\mathcal{EL}$ concept descriptions C and all filters $f \in \mathfrak{F}$, there is $D \in f$ such that:

- $C \bowtie D = \max\{C \bowtie X \mid X \in f\}$, and
- $\mathsf{s}(D) \leq g(\mathsf{s}(C))$. ♢

Establishing g-boundedness for a CSM $\bowtie$ provides a form to estimate a more accurate bound for the size of models resulting from the general construction offered in Section 7.2.2. In particular, when g is a polynomial $p(n) = n^k$, it yields a polynomial model property for the induced logic $\tau\mathcal{EL}(m_{\bowtie})$.

Proposition 7.25. *Let $p(n) = n^k$ be a polynomial and $\widehat{C}$ a $\tau\mathcal{EL}(m)$ concept description. Moreover, let $\bowtie$ be an equivalence invariant, role-depth bounded, equivalence closed and $p(n)$-bounded CSM. If $\widehat{C}$ is satisfiable in $\tau\mathcal{EL}(m_{\bowtie})$, then there exists a tree-shaped interpretation $\mathcal{J}$ such that $\widehat{C}^{\mathcal{J}} \neq \emptyset$ and $|\Delta^{\mathcal{J}}| \leq p(\mathsf{s}(\widehat{C}))$.*

Proof. Let $(E_1)_{\sim t_1}, \ldots, (E_n)_{\sim t_n}$ be the threshold concepts occurring in $\widehat{C}$ with $\sim \in \{>, \geq\}$. Conversely, let $(F_1)_{\sim s_1}, \ldots, (F_q)_{\sim s_q}$ be the ones where $\sim \in \{<, \leq\}$. Then, the size of $\widehat{C}$ can be expressed as:

$$|\Delta^{\mathcal{I}_0}| + \sum_{i=1}^{n} \mathsf{s}((E_i)_{\sim t_i}) + \sum_{j=1}^{q} \mathsf{s}((F_j)_{\sim s_j})$$

Proposition 7.13 in the previous section shows that the construction used in Lemma 5.4 can be applied to $\tau\mathcal{EL}(m_{\bowtie})$ to obtain a finite interpretation $\mathcal{J}$ satisfying $\widehat{C}$ such that:

$$|\Delta^{\mathcal{J}}| = |\Delta^{\mathcal{I}_0}| + \sum_{i=1}^{n} |\Delta^{\mathcal{I}_{D_i}}|$$

Here, $\mathcal{I}_{D_i}$ is a canonical interpretation in the sense of Definition 7.10, that by construction of $\mathcal{J}$ has been "extracted" from $\mathcal{I}$. This means that there is an element $d \in \Delta^{\mathcal{I}}$ such that $d \in (D_i)^{\mathcal{I}}$. Moreover, D_i satisfies $m^{\mathcal{I}}_{\bowtie}(d, E_i) = E_i \bowtie D_i$, and by definition of $m_{\bowtie}$ we then have:

$$E_i \bowtie D_i = \max\{E_i \bowtie D \mid D \in \mathcal{C}_{\mathcal{EL}} \text{ and } d \in D^{\mathcal{I}}\} \tag{7.6}$$

Now, D_i generates the filter f_i containing all the concepts X such that $D_i \sqsubseteq X$. Therefore, there are two things one can say about f_i:

- $d \in X^{\mathcal{I}}$ for all $X \in f_i$, and
- $p(n)$-boundedness of $\bowtie$ yields a concept $D_i^* \in f_i$ such that:
  - $E_i \bowtie D_i^* = \max\{E_i \bowtie X \mid X \in f_i\}$,
  - $\mathsf{s}(D_i^*) \leq p(\mathsf{s}(E_i))$.

Since $D_i \in f_i$ and f_i is a subset of the set considered in (7.6), this means that $E_i \bowtie D_i = E_i \bowtie D_i^*$. Hence, Lemma 7.12 makes possible to choose D_i^* in the place of D_i to build $\mathcal{J}$. Therefore, without loss of generality, we can assume that $|\Delta^{\mathcal{I}_{D_i}}| \leq p(\mathsf{s}(E_i))$ for all $1 \leq i \leq n$. Taking this into account, the size of $\mathcal{J}$ can be bounded by the following expression:

$$|\Delta^{\mathcal{J}}| \leq |\Delta^{\mathcal{I}_0}| + \sum_{i=1}^{n} p(\mathsf{s}(E_i)) \tag{7.7}$$

Thus, since $p(n) = n^k$, this means that $|\Delta^{\mathcal{J}}| \leq p(\mathsf{s}(\widehat{C}))$. □

Hence, $p(n)$-boundedness defines a family of CSMs for which the logic $\tau\mathcal{EL}(m_{\bowtie})$ enjoys the polynomial model property. The proof of Proposition 7.25 describes how to compute such a bound, provided that $p(n)$ is known. Furthermore, contrary to $\bowtie_{R_s}$ and $\tau\mathcal{EL}(m_{\bowtie_{R_s}})$, model checking a concept $\widehat{C}$ on an interpretation $\mathcal{I}$ will not need to consider exponentially large concepts in the size of $\widehat{C}$. We denote by $\mathcal{F}_{\bowtie}[poly]$ the family of equivalence invariant, role-depth bounded and equivalence closed CSMs, that are polynomially bounded. Later in Section 7.3 we will identify a concrete set of CSMs that are part of this family.

Now, since we have focused our interest in CSMs that are computable in polynomial time, one might think of the algorithm presented in Chapter 5 (for *deg*) as a *general purpose* NP-algorithm to decide satisfiability in $\tau\mathcal{EL}(m_{\bowtie})$. However, differently from $\tau\mathcal{EL}(deg)$ a polynomial bound does not ensure that such algorithm will always run in non-deterministic polynomial time. Intuitively, there are two reasons for this:

- $m_{\bowtie}$ is defined as maximization on top of $\bowtie$, and
- the NP-decision procedure from Chapter 5 uses Algorithm 2 to check the existence of a τ-homomorphism. In particular, in line 6 it is required to check whether $m^{\mathcal{I}}_{\bowtie}(d, E) \sim t$ for some $d \in \Delta^{\mathcal{I}}$ and a threshold concept $E_{\sim t}$.

Where could the interaction of these two aspects be harmful? Being the threshold concept of the form $E_{>t}$ or $E_{\geq t}$, the maximization in the definition of $m_{\bowtie}$ allows to handle this by simply guessing an $\mathcal{EL}$ concept description D such that $d \in D^{\mathcal{I}}$ and $E \bowtie D > t$ (or $\geq$). Here, the second *benefit* of having $p(n)$-boundedness comes into play: the size of such D is polynomial in the size of E. The problem arises, however, in the presence of threshold concepts of the form $E_{<t}$ or $E_{\leq t}$. In this case the same strategy will not suffice, since for instance having $E \bowtie D < t$ for a particular D does not ensure $m^{\mathcal{I}}_{\bowtie}(d, E) < t$.

A natural way to repair this problem is to use an NP-algorithm as an *oracle*, which verifies whether there exists D such that $d \in D^{\mathcal{I}}$ and $E \bowtie D \geq t$. Then, a "no" answer from the oracle will definitely certify that $m^{\mathcal{I}}_{\bowtie}(d, E) < t$. The following lemma provides an NP^{NP}-algorithm that decides concept satisfiability in $\tau\mathcal{EL}(m_{\bowtie})$, provided that $\bowtie$ is polynomially bounded and polynomial time computable.

Lemma 7.26. *Let $p(n) = n^k$ be a polynomial and $\bowtie$ an equivalence invariant, role-depth bounded, equivalence closed and $p(n)$-bounded CSM. Additionally, $\bowtie$ can be computed in polynomial time. Then, in $\tau\mathcal{EL}(m_{\bowtie})$ it is in NP^{NP} to decide whether a concept is satisfiable.*

Proof. Assume that we want to decide satisfiability of the concept $\widehat{C}$ in $\tau\mathcal{EL}(m_{\bowtie})$. The NP-algorithm we are going to use as an oracle solves the following problem:

- **Instance:** A tuple $(\mathcal{I}, d, C, t, \sim)$ where $\mathcal{I}$ is a finite interpretation, $d \in \Delta^{\mathcal{I}}$, C a concept description, $t \in \mathbb{Q} \cap [0, 1]$ and $\sim \in \{>, \geq\}$.
- **Question:** Is there an $\mathcal{EL}$ concept description D such that $d \in D^{\mathcal{I}}$ and $C \bowtie D \sim t$?

Based on the definition of $m_{\bowtie}$ and the special properties satisfied by $\bowtie$, we observe the following:

- $m^{\mathcal{I}}_{\bowtie}(d, C) \sim t$ iff there exists such a concept D. This is a consequence of the maximization used to define $m_{\bowtie}$ and $\sim \in \{>, \geq\}$.
- $p(n)$-boundedness of $\bowtie$ means that $m^{\mathcal{I}}_{\bowtie}(d, C) = C \bowtie D'$, where $d \in (D')^{\mathcal{I}}$ and $\mathsf{s}(D') \leq p(\mathsf{s}(C))$.

These observations permit to reduce the search space to concepts of size at most $p(\mathsf{s}(C))$. Moreover, testing for $d \in D^{\mathcal{I}}$ and computing $C \bowtie D$ are both polynomial time tasks in the size of $\mathcal{I}$, C and D. Hence, the algorithm first guesses a concept description D of size at most $p(\mathsf{s}(C))$, and then verifies whether $d \in D^{\mathcal{I}}$ and $C \bowtie D \sim t$. It answers "yes" if both checks succeed, and "no" otherwise.

Now, the NP^{NP}-procedure behaves as follows:

1. Guess an interpretation $\mathcal{J}$ of size at most $p(\mathsf{s}(\widehat{C}))$ (or use expression (7.7) for a tighter bound).
2. Use Algorithm 2 to check whether there exists a τ-homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{J}}$ with the following modifications. Whenever the test $m^{\mathcal{J}}(w, E) \sim t$ in line 6 needs to be executed, it is handled by calling the oracle on $(\mathcal{J}, w, E, t, \diamond)$ where $\diamond$ is selected as $\geq$ if $\sim\in \{<, \geq\}$, or as $>$ otherwise. The resulting pair $(\sim, oracle'sanswer)$ determines the truth of the aforementioned test as follows:
   - the pairs ($<$,"no"), ($\leq$,"no"), ($\geq$,"yes") and ($>$,"yes") result in a positive answer to the question of whether $m^{\mathcal{J}}(w, E) \sim t$. In any other case the statement is false.
3. Answer $\widehat{C}$ is satisfiable iff there exists a τ-homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{J}}$.

By Proposition 7.25 it is sufficient to look only at interpretations of size at most $p(\mathsf{s}(\widehat{C}))$. The characterization of membership for $\tau\mathcal{EL}(m)$ given in Theorem 3.8 does not depend on which graded membership function m is considered. Hence, it is correct to use Algorithm 2 for $\tau\mathcal{EL}(m_{\bowtie})$. Furthermore, it is not hard to see that the introduced modification is consistent with whether $m^{\mathcal{J}}_{\bowtie}(w, E) \sim t$. For instance, to check $m^{\mathcal{J}}_{\bowtie}(w, E) < t$ the oracle is invoked with $\diamond = \geq$. A "no" answer means that there is no D such that $w \in D^{\mathcal{J}}$ and $E \bowtie D \geq t$, which clearly implies $m^{\mathcal{J}}_{\bowtie}(w, E) < t$. Finally, the number of calls to the oracle is at most $\mathsf{s}(\widehat{C}) \times |\Delta^{\mathcal{J}}|$. These arguments prove that concept satisfiability in $\tau\mathcal{EL}(m_{\bowtie})$ is in NP^{NP}. □

The use of an oracle by this procedure is somehow forced by the fact that we stick to a specific approach, and in particular such method uses Algorithm 2 to decide a model checking problem. Moreover, as we will see in Section 7.3.2, *deg* is a function that can be obtained from a CSM $\bowtie_1$ of the kind being considered, but its satisfiability problem is in NP. Therefore, an obvious question is whether this is a really "naive" way to decide satisfiability for the family of logics induced by such a class of CSMs.

We will now define a CSM that, at least in terms of worst-case complexity, suggests that the previous algorithm may not be so bad. More precisely, we slightly modify R_s into R_s^* such that the CSM $\bowtie_{R_s^*}$ obtained as in Definition 7.5, is polynomially bounded and induces a logic $\tau\mathcal{EL}(m_{\bowtie_{R_s^*}})$ where concept satisfiability is both NP-hard and coNP-hard. The problems we are going to use for the reductions are *satisfiability* and *unsatisfiability* of propositional formulas in conjunctive normal form, which are well-known to be complete for the classes NP and coNP, respectively [GJ79].

Since propositional satisfiability corresponds to validity of quantified Boolean formulas of the form $P.\varphi$ where P only contains existential quantifiers, the elements defining R_s can be modified to obtain R_s^*. Basically, starting with the relation $\rhd$ a family of binary relations $\rhd_p^*$ (for all $p \in \mathbb{N}$) is built as follows:

$$(T_1, T_2) \in \rhd_p^* \\ \text{iff} \\ \exists \pi_2 \in \Pi_p(T_2) \text{ s.t. } \forall \pi_1 \in \Pi_p(T_1), \text{ it holds } \pi_1 \rhd \pi_2$$

Then, the relation R_s^* is defined in the following form:

$$R_s^* := \{(C, D) \mid (T_{C^r}, T_{D^r}) \in \rhd^*_{\mathsf{rd}(C)}\}$$

There are two main differences between R_s and R_s^*. First, the definition of $\rhd_p^*$ poses an existential quantification over $\Pi_p(T_2)$. This has to do with the fact that for propositional satisfiability only one "good" assignment (*path in a description tree*) needs to be found. Second, the special concepts used to represent the structure of certificates for QBF are no longer needed. Therefore, the final definition of R_s^* is limited to checking for membership into $\rhd^*_{\mathsf{rd}(C)}$.

Concerning its computational properties, checking for membership into $\rhd_p^*$ requires at most the same number of comparisons as for $\rhd_p$, and as explained in the previous section for $\rhd_p$, it can be done in polynomial time. Therefore, $\bowtie_{R_s^*}$ is a polynomial time computable CSM. In addition, the following lemma shows that $\bowtie_{R_s^*}$ is polynomially bounded.

Lemma 7.27. *The CSM $\bowtie_{R_s^*}$ is* linear *bounded.*

Proof. Let us fix a filter $f \in \mathfrak{F}$ and a concept description C with $\mathsf{rd}(C) = k$. Now, let $D \in f$ such that $C \bowtie_{R_s^*} D = \max\{C \bowtie_{R_s^*} X \mid X \in f\}$. We make a case distinction on the possible values of $C \bowtie_{R_s^*} D$ (see Definition 7.5):

- $C \bowtie_{R_s^*} D = 1$. Then, $C \equiv D$ and obviously $C \in f$.
- $C \bowtie_{R_s^*} D = a$. This means that $C \not\equiv D$, $\mathsf{rd}(C) = \mathsf{rd}(D)$ and $(C, D) \in R_s^*$. By definition of R_s^* we know that $(T_{C^r}, T_{D^r}) \in \rhd_k^*$. Then, there exists a path π_D of

length k in T_{D^r} such that for all paths $\pi_C \in \Pi_k(T_{C^r})$, it is true that $\pi_C \rhd \pi_D$. Let $\pi_{D'}$ be the path that results by replacing the labels in π_D by their intersection with $\{A\}$ (as required in the definition of $\rhd$). We denote as D' the $\mathcal{EL}$ concept description corresponding to $\pi_{D'}$ (when seen as a description tree).

- Clearly, $\pi_C \rhd \pi_{D'}$ still holds for all paths $\pi_C \in \Pi_k(T_{C^r})$. This means that $(T_{C^r}, T_{D'}) \in \rhd_k^*$, and consequently $(C, D') \in R_s^*$ (notice that D' is already in reduced form).
- It is not hard to see that $D \sqsubseteq D'$. This implies that $D' \in f$, and since D has been assumed to be maximal within f for $C \bowtie_{R_s^*} D$, it must be the case that $C \not\equiv D'$. For otherwise, it would contradict $C \bowtie_{R_s^*} D = a$.

Overall, this means that $C \bowtie_{R_s^*} D' = a$.

- $C \bowtie_{R_s^*} D = 0$. Clearly, $\top \in f$ since $D \sqsubseteq \top$. Moreover, the maximality of D implies that $C \bowtie_{R_s^*} \top = 0$.

In any case, there is always a concept D' such that $D' \in f$, $C \bowtie_{R_s^*} D'$ is maximal in f and $\mathsf{s}(D') \leq g(\mathsf{s}(C))$ where $g(n) = n$. Thus, $\bowtie_{R_s^*}$ is linear bounded. □

Finally, likewise R_s the relation R_s^* is equivalence invariant.

Lemma 7.28. *The relation R_s^* is equivalence invariant.*

Proof. Let C, C', D and D' be $\mathcal{EL}$ concepts such that $C \equiv C'$ and $D \equiv D'$. We show that $(C, D) \in R_s^*$ iff $(C', D') \in R_s^*$. As pointed out for R_s:

- Since $C \equiv C'$ and $D \equiv D'$, we have that $\mathsf{rd}(C) = \mathsf{rd}(C') = k$ for some $k \geq 0$. Moreover, by Theorem 2.8 there are isomorphisms between T_{C^r} and $T_{(C')^r}$, and between T_{D^r} and $T_{(D')^r}$.

Suppose that $(T_{C^r}, T_{D^r}) \in \rhd_k^*$. This means that there is a path π_D in T_{D^r} of length k, such that for all paths $\pi_C \in \Pi_k(T_{C^r})$ it holds $\pi_C \rhd \pi_D$. Using the isomorphism mentioned above, one can find a path $\pi_{D'}$ in $T_{(D')^r}$ such that $\pi_D = \pi_{D'}$. To see that also $(T_{(C')^r}, T_{(D')^r}) \in \rhd_k^*$, let $\pi_{C'}$ be an arbitrary path of length k in $T_{(C')^r}$. It will be enough to show that $\pi_{C'} \rhd \pi_{D'}$. Again, the isomorphism yields a path $\pi_C \in \Pi_k(T_{C^r})$ such that $\pi_C = \pi_{C'}$. Hence, since $\pi_C \rhd \pi_D$ and $\pi_D = \pi_{D'}$, this implies that $\pi_{C'} \rhd \pi_{D'}$.

We have thus shown that $(T_{C^r}, T_{D^r}) \in \rhd_k^* \Rightarrow (T_{(C')^r}, T_{(D')^r}) \in \rhd_k^*$. The implication in the opposite direction can be obtained in a similar way. Therefore, $(C, D) \in R_s^* \Leftrightarrow (C', D') \in R_s^*$, and R_s^* is equivalence invariant. □

Thus, $m_{\bowtie_{R_s^*}}$ is a well-defined graded membership function and it induces the DL $\tau\mathcal{EL}(m_{\bowtie_{R_s^*}})$. To show NP-hardness of *satisfiability* in $\tau\mathcal{EL}(m_{\bowtie_{R_s^*}})$, we use exactly the same translation as in the previous section: given a propositional formula φ in conjunctive normal form its corresponding $\tau\mathcal{EL}(m)$ concept description $\widehat{C}_\varphi$ is of the form $(C_\varphi)_{\leq a} \sqcap (C_\varphi)_{\geq a}$.

Lemma 7.29. *Let φ be a propositional formula in CNF of the form $c_1 \wedge \ldots \wedge c_q$, and $x_1, \ldots, x_n$ the variables occurring in φ. Then, φ is satisfiable iff $\widehat{C}_\varphi$ is satisfiable in $\tau\mathcal{EL}(m_{\bowtie_{R_s^*}})$.*

Proof. ($\Rightarrow$) Assume that φ is satisfiable. To show that $\widehat{C}_\varphi$ is satisfiable in $\tau\mathcal{EL}(m_{\bowtie_{R_s^*}})$ we choose the interpretation $\mathcal{I}$ that has the following $\mathcal{EL}$ description graph:

$$d_0 : \{\} \xrightarrow{\;r\;} d_1 : \{A\} \xrightarrow{\;r\;} d_2 : \{A\} \overset{r}{\cdots\cdots} d_n : \{A\}$$

We want to show that $d_0 \in (\widehat{C}_\varphi)^\mathcal{I}$. By Definition 7.2 we have:

$$m^\mathcal{I}_{\bowtie_{R_s^*}}(d_0, C_\varphi) = \max\{C_\varphi \bowtie_{R_s^*} D \mid D \in \mathcal{C}_{\mathcal{EL}} \text{ and } d_0 \in D^\mathcal{I}\}$$

Now, since φ is satisfiable, there is a truth assignment $\mathfrak{t}$ satisfying each clause in φ. Such an assignment induces the $\mathcal{EL}$ concept description $D_\mathfrak{t} = \exists r.F_1^\mathfrak{t}$:

$$\begin{aligned}
&F_1^\mathfrak{t} := \lambda_1 \sqcap \exists r.F_2^\mathfrak{t}\\
&\dots\\
&F_i^\mathfrak{t} := \lambda_i \sqcap \exists r.F_{i+1}^\mathfrak{t} \quad (1 \le i < n)\\
&\dots\\
&F_n^\mathfrak{t} := \lambda_n
\end{aligned}$$

where $\mathfrak{t}(x_i) = \mathit{true}$ implies $\lambda_i = A$, and $\lambda_i = \top$ otherwise. It is straightforward to see that there is a homomorphism from $T_{D_\mathfrak{t}}$ to $G_\mathcal{I}$ mapping the root of $T_{D_\mathfrak{t}}$ to d_0. Therefore, $d_0 \in (D_\mathfrak{t})^\mathcal{I}$. Let us now look at the value $C_\varphi \bowtie_{R_s^*} D_\mathfrak{t}$.

- The description tree $T_{D_\mathfrak{t}}$ has a single path $\pi_\mathfrak{t} = w_0 r w_1 \dots r w_n$, and its labeling is determined by the values $\lambda_1, \dots, \lambda_n$.
- There are q paths $\pi_1, \dots, \pi_q$ in T_{C_φ} such that π_j is induced by the top-level atom C_j in C_φ. At the same time, C_j corresponds to the clause c_j of φ.
- Let ℓ be a literal in c_j such that $\mathfrak{t}(\ell) = \mathit{true}$ and x_i the corresponding variable (it exists because $\mathfrak{t}$ satisfies φ). By construction of C_j, we have two possibilities:
  - $\ell = x_i$ and $\ell_{T_{C_\varphi}}(v_i) = \{A\}$. Since $\mathfrak{t}(\ell) = \mathit{true}$, this means that $\lambda_i = A$ and $\ell_{T_{C_\varphi}}(v_i) = \ell_{T_{D_\mathfrak{t}}}(w_i) = \{A\}$. Thus, according to (7.3) it follows $\pi_j \rhd \pi_\mathfrak{t}$.
  - $\ell = \neg x_i$ and $\ell_{T_{C_\varphi}}(v_i) = \{\}$. The same argument as before yields $\lambda_i = \top$ and $\ell_{T_{C_\varphi}}(v_i) = \ell_{T_{D_\mathfrak{t}}}(w_i) = \{\}$. Consequently, $\pi_j \rhd \pi_\mathfrak{t}$.

Overall, this means that $\pi_j \rhd \pi_\mathfrak{t}$ for all $1 \le j \le q$. Hence, $(T_{C_\varphi}, T_{D_\mathfrak{t}}) \in \rhd_n^*$ and $C_\varphi \bowtie_{R_s^*} D_\mathfrak{t} = a$. Moreover, $d_0 \notin I^\mathcal{I}$ implies $C_\varphi \not\equiv D$ for all D such that $d_0 \in D^\mathcal{I}$. Thus, we can conclude that $m^\mathcal{I}_{\bowtie_{R_s^*}}(d_0, C_\varphi) = a$ and $d_0 \in (\widehat{C}_\varphi)^\mathcal{I}$.

($\Leftarrow$) Assume that $\widehat{C}_\varphi$ is satisfiable in $\tau\mathcal{EL}(m_{\bowtie_{R_s^*}})$. Then, there exists an interpretation $\mathcal{I}$ and $d \in \Delta^\mathcal{I}$ such that $d \in (\widehat{C}_\varphi)^\mathcal{I}$. This means that $m^\mathcal{I}_{\bowtie_{R_s^*}}(d, C_\varphi) = a$. By definition of $m_{\bowtie_{R_s^*}}$ there must exist a concept D such that $d \in D^\mathcal{I}$ and $C \bowtie_{R_s^*} D = a$. Moreover, since $\bowtie_{R_s^*}$ is based on the relation R_s^* as constructed in Definition 7.5, we further have that $\mathsf{rd}(C_\varphi) = \mathsf{rd}(D) = n$ and $(T_{C_\varphi}, T_{D^r}) \in \rhd_n^*$. Hence, there exists a path π in T_{D^r} such that for all $1 \le j \le q$ it holds $\pi_j \rhd \pi$, where π_j is the path in T_{C_φ} corresponding the clause c_j of φ.

Let π be of the form $w_0r_1w_1\dots r_nw_n$, the assignment $\mathfrak{t}_\pi$ is built as follows. For all $1 \le i \le n$:

$$\mathfrak{t}_\pi(x_i) := \begin{cases} \mathit{true} & \text{if } A \in \ell_{T_{D^r}}(w_i) \\ \mathit{false} & \text{otherwise.} \end{cases}$$

Then, we show that $\mathfrak{t}_\pi$ satisfies φ. For any clause c_j of φ, its corresponding top-level atom C_j in C_φ induces a path $\pi_j = v_0r_1v_1\dots r_nv_n$ in T_{C_φ}. We have already seen that $\pi_j \rhd \pi$, and this means that there is $0 \le i \le n$ such that:

$$\ell_{T_{C_\varphi}}(v_i) \cap \{A\} = \ell_{T_{D^r}}(w_i) \cap \{A\} \text{ and } I \notin \ell_{T_{C_\varphi}}(v_i)$$

Since $I \in \ell_{T_{C_\varphi}}(v_0)$, we know that $i > 0$. This means that the variable x_i occurs in c_j. If x_i occurs in a positive form, by construction of C_φ we have that $\ell_{T_{C_\varphi}}(v_i) = \{A\}$ and $A \in \ell_{T_{D^r}}(w_i)$. Hence, it must be the case that $\mathfrak{t}_\pi(x_i) = \mathit{true}$ and $\mathfrak{t}_\pi$ satisfies c_j. The case where $\neg x_i$ occurs in c_j can be treated in a similar way.

Thus, we have shown that $\mathfrak{t}_\pi$ satisfies φ. □

Next, we establish the coNP lower bound by a reduction from the non-satisfiability problem. Based on the previous reduction, notice that $(C_\varphi)_{<a}$ represents that an assignment $\mathfrak{t}$ does not satisfy φ. However, since unsatisfiability means that all possible assignments fail to satisfy φ, we additionally need to ensure that all of them are taken into account. To this end, we introduce the concept $C^n_{all} := \exists r.A^n_1$ where A^n_1 is of the following form:

$$\begin{aligned} &A^n_1 := A \sqcap \exists r.A^n_2 \\ &\dots \\ &A^n_i := A \sqcap \exists r.A^n_{i+1} \quad (1 \le i < n) \\ &\dots \\ &A^n_n := A \end{aligned}$$

Then, given a propositional formula φ in CNF its corresponding $\tau\mathcal{EL}(m)$ concept description $\widehat{C}_{\varphi^*}$ has the following definition:

$$\widehat{C}_{\varphi^*} := (C_\varphi)_{<a} \sqcap C^n_{all}$$

Lemma 7.30. *φ is unsatisfiable iff $\widehat{C}_{\varphi^*}$ is satisfiable in $\tau\mathcal{EL}(m_{\bowtie_{R^*_s}})$.*

Proof. ($\Rightarrow$) Assume that φ is unsatisfiable and let $\mathcal{I}$ be the interpretation having the following description graph:

$$d_0 : \{\} \xrightarrow{\;r\;} d_1 : \{A\} \xrightarrow{\;r\;} d_2 : \{A\} \cdots\overset{r}{\cdots} d_n : \{A\}$$

We want to show that $d_0 \in (\widehat{C}_{\varphi^*})^{\mathcal{I}}$. Notice that this is exactly the description tree associated to the concept C^n_{all}, and consequently $d_0 \in (C^n_{all})^{\mathcal{I}}$. Hence, it remains to show that $d_0 \in [(C_\varphi)_{<a}]^{\mathcal{I}}$. By Lemma 7.29 we obtain that $\widehat{C}_\varphi$ is unsatisfiable in $\tau\mathcal{EL}(m_{\bowtie_{R^*_s}})$. Looking at the definition of $\widehat{C}_\varphi$, this means that for all interpretations $\mathcal{J}$ and $d \in \Delta^{\mathcal{J}}$ it

holds:

$$d \notin [(C_\varphi)_{\leq a} \sqcap (C_\varphi)_{\geq a}]^{\mathcal{J}}$$

Therefore, there are two possible scenarios for d:

$$m^{\mathcal{J}}_{\bowtie_{R_s^*}}(d, C_\varphi) < a \quad \text{or} \quad m^{\mathcal{J}}_{\bowtie_{R_s^*}}(d, C_\varphi) > a$$

Since the similarity values computed by $\bowtie_{R_s^*}$ range over the set $\{0, a, 1\}$, the second case is only valid when $d \in (C_\varphi)^{\mathcal{J}}$ ($\bowtie_{R_s^*}$ is equivalence closed). Now, the concept name I is a top-level atom of C_φ. This means, that whenever $d \notin I^{\mathcal{J}}$ it must be that $m^{\mathcal{J}}_{\bowtie_{R_s^*}}(d, C_\varphi) < a$. This is actually the case for d_0 in $\mathcal{I}$. Thus, $m^{\mathcal{I}}_{\bowtie_{R_s^*}}(d_0, C_\varphi) < a$ and $d_0 \in [(C_\varphi)_{<a}]^{\mathcal{I}}$.

($\Leftarrow$) Conversely, suppose that φ is satisfiable. Based on a truth assignment $\mathfrak{t}$ satisfying φ, in the proof of Lemma 7.29 a concept $D_\mathfrak{t}$ is built such that:

$$C_\varphi \bowtie_{R_s^*} D_\mathfrak{t} = a$$

Moreover, it can also be seen that $D_\mathfrak{t}$ is such that $C^n_{all} \sqsubseteq D_\mathfrak{t}$. Hence, for all interpretations $\mathcal{I}$ and $d \in \Delta^{\mathcal{I}}$, having $d \in C^n_{all}$ implies:

$$m^{\mathcal{I}}_{\bowtie_{R_s^*}}(d, C_\varphi) \geq a$$

Thus, $\widehat{C}_{\varphi^*}$ is unsatisfiable in $\tau\mathcal{EL}(m_{\bowtie_{R_s^*}})$. □

As a consequence of the previous two lemmas, we obtain the following computational lower bounds for *satisfiability* in $\tau\mathcal{EL}(m_{\bowtie_{R_s^*}})$.

Lemma 7.31. *In $\tau\mathcal{EL}(m_{\bowtie_{R_s^*}})$, satisfiability is NP-hard and coNP-hard.*

Overall, $p(n)$-boundedness of a CSM $\bowtie$ yields the following results for $\tau\mathcal{EL}(m_\bowtie)$.

Theorem 7.32.

1. *For all $\bowtie \in \mathcal{F}_\bowtie[poly]$, if $\bowtie$ is polynomial time computable and the polynomial $p(n)$ corresponding to its boundedness is known, then in $\tau\mathcal{EL}(m_\bowtie)$* satisfiability *is in Σ^p_2.*

2. *There is at least one CSM $\bowtie \in \mathcal{F}_\bowtie[poly]$ (for instance $\bowtie_{R_s^*}$), such that in $\tau\mathcal{EL}(m_\bowtie)$* satisfiability *is NP-hard and coNP-hard.*

Similar to the decidability results from Section 7.2.2, the base model built in Proposition 7.25 and the $p(n)$-boundedness property can be used to obtain a polynomial model property for satisfiability of concepts of the form $\widehat{C} \sqcap \neg\widehat{D}$. Hence, the procedure described in Lemma 7.26 can easily be extended to obtain an NP^{NP}-decision procedure for the complement of the subsumption problem. Likewise, such a small model property exists also for consistency of ABoxes of the form $\mathcal{A} \cup \{\neg\widehat{C}(a)\}$ with respect to the size of $\mathcal{A}$. Therefore, by using Algorithm 3 we obtain an NP^{NP}-algorithm to solve the consistency and the *non-instance* problem (data complexity). Thus, the first result in Theorem 7.32 can be extended to include the rest of the reasoning tasks.

Theorem 7.33. *For all* $\bowtie \in \mathcal{F}_{\bowtie}[poly]$*, if* $\bowtie$ *is polynomial time computable and the polynomial* $p(n)$ *corresponding to its boundedness is known, then in* $\tau\mathcal{EL}(m_{\bowtie})$ *consistency is in* Σ_2^p*, and subsumption and instance checking (data complexity) are in* Π_2^p*.*

Summing up, based on the polynomial boundedness property we have obtained a family of DLs $\tau\mathcal{EL}(m_{\bowtie})$ with a satisfiability problem in Σ_2^p. This upper bound has been established by applying the methods introduced in Chapter 5 for $\tau\mathcal{EL}(deg)$ to the class of polynomially bounded and polynomial time computable CSMs. Nevertheless, this only represents a sufficient condition to obtain our results, and it does not prevent CSMs outside $F_{\bowtie}[poly]$ to induce equally behaved threshold logics.

7.3 The *simi* framework

Lehmann and Turhan [LT12] introduced a framework (called *simi framework*) that can be used to define a variety of similarity measures between $\mathcal{EL}$ concepts satisfying the properties required by our Propositions 7.3 and 7.4. They first define a directional measure $simi_d$, and then use a fuzzy connector $\otimes$ to combine the values obtained by comparing the concepts in both directions with $simi_d$. Given two $\mathcal{EL}$ concepts C and D, one could say that *simi* uses $simi_d$ to measure how many properties of C are present in D and vice versa. Then, the bidirectional similarity measure *simi* is defined as:

$$simi(C, D) := simi_d(C^r, D^r) \otimes simi_d(D^r, C^r)$$

The fuzzy connector is an operator $\otimes : [0, 1] \times [0, 1] \to [0, 1]$ satisfying (among others) the following two properties (see [LT12]). For all $x, y \in [0, 1]$:

- $x \otimes y = y \otimes x$ (*commutativity*),
- $x \leq y \Rightarrow 1 \otimes x \leq 1 \otimes y$ (*weak monotonicity*).

In addition, $\otimes$ is *monotonic* if for all $x, y, z \in [0, 1]$:

- $x \leq y \Rightarrow x \otimes z \leq y \otimes z$.

Examples of monotonic fuzzy connectors are the *average* and *minimum* operators, and all bounded *t-norms* (see [LT12] for more information). In the following we recall the general definition of $simi_d$.

Definition 7.34 ([LT12]). Let C, D be two $\mathcal{EL}$ concept descriptions. If one of these two concepts is equivalent to $\top$, then:

$$simi_d(C, D) := \begin{cases} 1 & \text{if } C \equiv \top \\ 0 & \text{if } C \not\equiv \top \text{ and } D \equiv \top \end{cases}$$

Otherwise, let $tl(C)$ and $tl(D)$ be the set of top-level atoms of C and D, respectively.

Then, $simi_d$ is defined as follows:

$$simi_d(C,D) := \begin{cases} \dfrac{\sum\limits_{C' \in tl(C)} \left[g(C') \times \bigoplus\limits_{D' \in tl(D)} simi_d(C',D') \right]}{\sum\limits_{C' \in tl(C)} g(C')} & \text{if } |tl(C)| > 1 \text{ or } |tl(D)| > 1 \\ pm(C,D) & \text{if } C, D \in \mathsf{N_C} \\ pm(r,s)[w + (1-w) simi_d(E,F)] & \text{if } C = \exists r.E \text{ and } D = \exists s.F \\ 0 & \text{otherwise.} \end{cases}$$

Let us now explain the meaning of the parameters used in the definition of $simi_d$.

- The symbol g stands for a function mapping the set of $\mathcal{EL}$ atoms N_A to a value in $\mathbb{R}_{>0}$. The idea is that $g : N_A \to \mathbb{R}_{>0}$ assigns a weight to each atom in N_A. This could be helpful, for instance, if one wants to express that some atom contributes more (is more important) to the similarity than others.

- The purpose of the value $w \in (0,1)$ is the following. Given two concept descriptions $\exists r.C$ and $\exists s.D$, if $simi_d(C,D) = 0$, having $w > 0$ allows to distinguish between the cases $r = s$ and $r \neq s$.

- $pm : (\mathsf{N_C} \times \mathsf{N_C}) \cup (\mathsf{N_R} \times \mathsf{N_R}) \to [0,1]$ is a primitive measure for concept and role names satisfying the following basic properties (different from [LT12] we do not deal with role inclusion axioms):

  - $pm(A,B) = 1$ iff $A = B$ for all $A, B \in \mathsf{N_C}$,
  - $pm(r,s) = 1$ iff $r = s$ for all $r, s \in \mathsf{N_R}$.

  In particular the *default* primitive measure pm_d is defined as:

  $$pm_d(A,B) := \begin{cases} 1 & \text{if } A = B \\ 0 & \textit{otherwise.} \end{cases}$$

  and

  $$pm_d(r,s) := \begin{cases} 1 & \text{if } r = s \\ 0 & \textit{otherwise.} \end{cases}$$

- Finally, the operator $\oplus$ represents a *bounded* triangular-conorm. One can find in [LT12] arguments in favor of using this type of operator. The max operator is a particular case of a bounded t-conorm. ♢

The following two properties of $simi_d$ are presented in [LT12] (see Lemma 1). They will be useful later on to obtain our results. Let C, D and E be $\mathcal{EL}$ concept descriptions, then:

$$simi_d(C,D) = 1 \text{ iff } D \sqsubseteq C \tag{7.8}$$

$$D \sqsubseteq E \Rightarrow simi_d(C,E) \leq simi_d(C,D) \tag{7.9}$$

The proofs can be found in the extended version [Leh12] of [LT12] (Lemma 14 and Lemma 15). They indicate that these properties hold regardless of whether the concepts C, D and E are in reduced form or not.

Finally, one can easily see that *simi* only defines CSMs that are equivalence invariant, role-depth bounded and equivalence closed. The equivalence invariance property follows from the fact that $simi_d$ is computed using the reduced forms of C and D, and the fact that $C \equiv C'$ implies that the structures of C^r and $(C')^r$ are isomorphic (see Theorem 2.8). In addition, its structural definition implies that *simi* is role-depth bounded. Regarding the third property, it has been shown already in [LT12] that this is the case for any instance of *simi*. Hence, for all instances $\bowtie$ of *simi* the induced $m_{\bowtie}$ is a well-defined graded membership function (Proposition 7.3). From now on, for any instance $\bowtie$ of *simi* we denote as $\bowtie_d$ the corresponding instance of $simi_d$, and will use $\bowtie_d$ in *infix* notation.

7.3.1 A *polynomially bounded* family of instances of *simi*

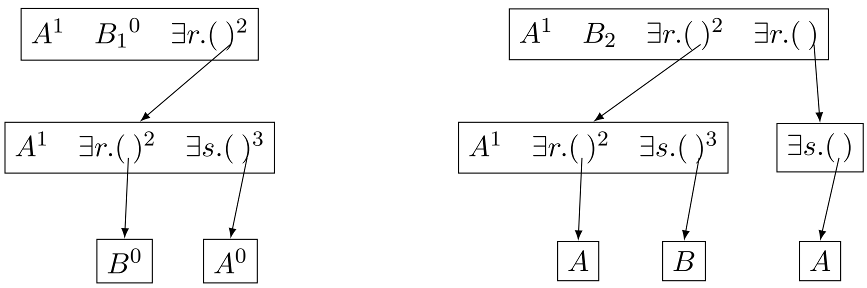
We now identify a family of instances of *simi* that are polynomially bounded. Let $\mathcal{F}_1$ be the family of CSMs that are instances of *simi*, where $\oplus$ is selected as max, $\otimes$ is a monotonic fuzzy connector and *pm* is the default primitive measure pm_d. The following example gives an intuition of why CSMs in $\mathcal{F}_1$ are polynomially-bounded.

Example 7.35. Let $\bowtie^x \in \mathcal{F}_1$ such that g assigns value 1 to every atom and $w = 0.5$. In addition, let C and D be the following concept descriptions:

$$C := A \sqcap B_1 \sqcap \exists r.(A \sqcap \exists r.B \sqcap \exists s.A)$$

$$D := A \sqcap B_2 \sqcap \exists r.(A \sqcap \exists r.A \sqcap \exists s.B) \sqcap \exists rs.A$$

Let us look at the atoms in D chosen by $\oplus = \max$ along the computation of $C \bowtie_d^x D$. To illustrate this, we use the following picture:



The left-hand side of the picture depicts the structure of C and the right-hand side does the same for D. The superscripts are used to denote the pairings done by $\oplus = \max$ in the computation of $C \bowtie_d^x D$. For instance, at the top level of C, A^1 means that A is paired with the top-level atom of D exhibiting the 1 superscript (which is also A). The superscript 0 is used to denote that no such match exists, i.e., every possible match gives value 0. This is the case for B_1 at the top-level of C, since $B_1 \bowtie_d^x A = B_1 \bowtie_d^x B_2 = B_1 \bowtie_d^x \exists r.(\ldots) = 0$.

Our interest is to see what is the effect of removing the unmatched atoms from D. In our example, doing that yields the following concept description:

$$Y := A \sqcap \exists r.(A \sqcap \exists r.\top \sqcap \exists s.\top)$$

From the definition of $simi_d$ and the particular characteristics of $\bowtie^x$, it is easy to see that $C \bowtie_d^x D = C \bowtie_d^x Y = \frac{8}{9}$. This means that the unmatched atoms are actually irrelevant to obtain the value $C \bowtie_d^x D$. However, this need not be the case for the computation of $C \bowtie^x D$. In fact, one must not forget that $\bowtie_d^x$ is used in both directions to compute $C \bowtie^x D$. But still, there is something special in the structure of Y: it is a *concept part* of both C and D (see Definition 4.13). Some consequences follow from it:

- $\mathsf{s}(Y) \leq \mathsf{s}(C)$,
- $D \sqsubseteq Y$. This means that for all filters f, $D \in f$ implies $Y \in f$,
- $C \sqsubseteq Y$. By property (7.8), it is the case that $Y \bowtie_d^x C = 1$.

Therefore, although the relationship between $C \bowtie^x D$ and $C \bowtie^x Y$ (if $\otimes$ were not monotonic) is not clear in general, for a *monotonic* fuzzy connector it holds $C \bowtie^x D \leq C \bowtie^x Y$. Consequently, even though $C \bowtie^x Y$ may not preserve the value $C \bowtie^x D$, the concept Y represents a better choice towards bounding $\bowtie$ for C and a filter f containing D (as required in Definition 7.24). ♢

Let us now generalize the intuition presented in the previous example. First, we show that such a concept Y always exists. Afterwards, we use its properties to establish that all CSMs in $\mathcal{F}_1$ are linear bounded.

Lemma 7.36. *Let $\bowtie$ be a CSM in $\mathcal{F}_1$. For all $\mathcal{EL}$ concept descriptions C and D, there exists a concept description Y such that:*

- *$C \sqsubseteq Y$ and $D \sqsubseteq Y$,*
- *$C \bowtie_d D = C \bowtie_d Y$, and*
- *$\mathsf{s}(Y) \leq \mathsf{s}(C)$.*

Proof. We use induction on the structure of C to prove the claim.

- C is of the form $A \in \mathsf{N_C}$ or $\top$. For $C = A$, the value $C \bowtie_d D$ is the result of the following expression:

  $$\frac{g(A) \times \max\{A \bowtie_d D' \mid D' \in tl(D)\}}{g(A)}$$

  The use of the primitive default measure in $\bowtie$ implies that $A \bowtie_d D = 1$ if $A \in tl(D)$, otherwise $A \bowtie_d D = 0$. Choosing $Y := A$ or $Y := \top$, accordingly, ensures that the claim is true. If $C = \top$, then the definition of $simi_d$ implies $C \bowtie_d X = 1$ for all concept descriptions X. Thus, setting $Y := \top$ satisfies the claim.

- $C = C_1 \sqcap \ldots \sqcap C_n$ with $n > 1$. In this case we have:

$$C \bowtie_d D = \frac{\sum_{i=1}^{n} \Big[g(C_i) \times \max\{C_i \bowtie_d D' \mid D' \in tl(D)\} \Big]}{\sum_{i=1}^{n} g(C_i)}$$

Let D_i ($1 \leq i \leq n$) be the top-level atom of D that maximizes the value $C_i \bowtie_d D'$ among all $D' \in tl(D)$. The application of the induction hypothesis to C_i and D_i yields a concept description Y_i such that:

  - $C_i \sqsubseteq Y_i$ and $D_i \sqsubseteq Y_i$,
  - $C_i \bowtie_d D_i = C_i \bowtie_d Y_i$,
  - $\mathsf{s}(Y_i) \leq \mathsf{s}(C_i)$.

Obviously, $C_1 \sqcap \ldots \sqcap C_n \sqsubseteq Y_1 \sqcap \ldots \sqcap Y_n$ and $D_1 \sqcap \ldots \sqcap D_n \sqsubseteq Y_1 \sqcap \ldots \sqcap Y_n$. Therefore, the concept description $Y := Y_1 \sqcap \ldots \sqcap Y_n$ satisfies $C \sqsubseteq Y$, $D \sqsubseteq Y$ and $\mathsf{s}(Y) \leq \mathsf{s}(C)$. Now, the value of $C \bowtie_d Y$ is computed by the following expression:

$$C \bowtie_d Y = \frac{\sum_{i=1}^{n} \Big[g(C_i) \times \max\{C_i \bowtie_d Y' \mid Y' \in tl(Y)\} \Big]}{\sum_{i=1}^{n} g(C_i)}$$

Suppose that for some C_i ($1 \leq i \leq n$), $C_i \bowtie_d Y_i$ is not the maximum among all the values $C_i \bowtie_d Y'$. Then, there is $Y_j \in tl(Y)$ such that $i \neq j$ and $C_i \bowtie_d Y_i < C_i \bowtie_d Y_j$. From this we obtain:

$$\begin{aligned} C_i \bowtie_d D_i &= C_i \bowtie_d Y_i \\ &< C_i \bowtie_d Y_j \\ &\leq C_i \bowtie_d D_j && (D_j \sqsubseteq Y_j \text{ and } (7.9)) \end{aligned}$$

Hence, it follows that $C_i \bowtie_d D_i < C_i \bowtie_d D_j$ which contradicts the maximality of D_i with respect to C_i. Hence, $C_i \bowtie_d Y_i$ is actually the maximum and once this is true, it is easy to see that $C \bowtie_d D = C \bowtie_d Y$.

- C is of the form $\exists r.C'$. Let D^* be the top-level atom of D maximizing the value $C \bowtie D^*$. If D^* is not of the form $\exists r.D'$, then $C \bowtie_d D = 0$. This is a consequence of the general definition of $simi_d$ and the use of pm_d. Then, choosing $Y := \top$ is enough. Otherwise, $C \bowtie_d D$ can be expressed as:

$$C \bowtie_d D = [w + (1 - w) \times (C' \bowtie_d D')]$$

The application of induction hypothesis to C' and D' yields a concept description Y' such that:

  - $C' \sqsubseteq Y'$ and $D' \sqsubseteq Y'$
  - $C' \bowtie_d D' = C' \bowtie_d Y'$, and

– $\mathsf{s}(Y') \le \mathsf{s}(C')$.

Then, for the concept $Y := \exists r.Y'$ we have that $C \sqsubseteq Y$, $D \sqsubseteq \exists r.D' \sqsubseteq Y$ and $\mathsf{s}(Y) \le \mathsf{s}(C)$. Additionally,

$$C \bowtie_d Y = [w + (1 - w) \times (C' \bowtie_d Y')]$$

Thus, $C \bowtie_d D = C \bowtie_d Y$. □

Next, using Lemma 7.36 we show linear boundedness for the family $\mathcal{F}_1$.

Corollary 7.37. *Let $\bowtie$ be a CSM in $\mathcal{F}_1$. Then, $\bowtie$ is linear bounded.*

Proof. Let f be a filter and C a concept description. Moreover, let $D \in f$ be a concept description such that $C \bowtie D = \max\{C \bowtie X \mid X \in f\}$. From the abstract definition of *simi* we have:

$$C \bowtie D = (C^r \bowtie_d D^r) \otimes (D^r \bowtie_d C^r) \tag{7.10}$$

The application of Lemma 7.36 to C^r and D^r yields a concept description Y such that:

$$C^r \sqsubseteq Y,\ D^r \sqsubseteq Y,\ C^r \bowtie_d D^r = C^r \bowtie_d Y \text{ and } \mathsf{s}(Y) \le \mathsf{s}(C^r)$$

From $C^r \sqsubseteq Y \equiv Y^r$, it follows that $Y^r \bowtie_d C^r = 1$ (see property (7.8)). In addition, property (7.9) and $Y \equiv Y^r$ imply that $C^r \bowtie_d Y = C^r \bowtie_d Y^r$. Hence, $C \bowtie Y$ can be expressed as follows:

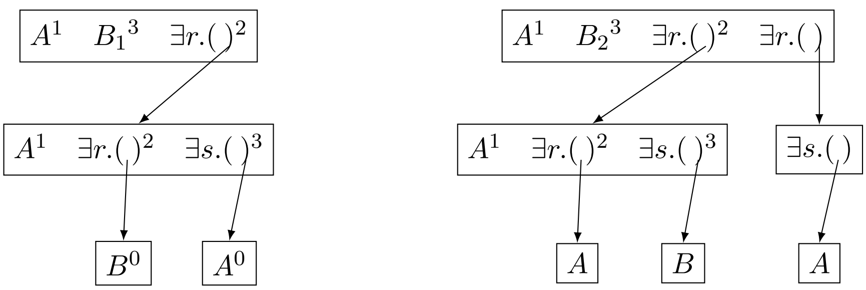
$$C \bowtie Y = (C^r \bowtie_d D^r) \otimes 1 \tag{7.11}$$

Since fuzzy connectors are commutative, the monotonicity of $\otimes$ implies that it is monotone in both arguments. Then, due to (7.10) and (7.11) we obtain $(C \bowtie D) \le (C \bowtie Y)$. Hence, $(C \bowtie D) = (C \bowtie Y)$, for otherwise it would contradict the maximality of $C \bowtie D$ ($D^r \sqsubseteq Y$ implies that $Y \in f$). Finally, since reduced forms are the smallest concepts in their equivalence classes, we have $\mathsf{s}(Y) \le \mathsf{s}(C^r) \le \mathsf{s}(C)$. Thus, the concept Y witnesses that $\bowtie$ is *linear* bounded. □

Corollary 7.37 implies that $\mathcal{F}_1 \subseteq \mathcal{F}_{\bowtie}[poly]$. Then, since all its elements are linear bounded CSMs, the upper bounds shown in Section 7.2.4 with respect to $\mathcal{F}_{\bowtie}[poly]$ also apply for any DL $\tau\mathcal{EL}(m_{\bowtie})$ induced by a CSM $\bowtie \in \mathcal{F}_1$.

Let us now continue Example 7.35, to illustrate that the same arguments failed for arbitrary primitive measures.

Continuation of Example 7.35. Let us slightly modify $\bowtie^x$ such that $pm(B_1, B_2) = pm(B_2, B_1) = 0.8$. Now, B_2 becomes a relevant atom for the computation of $C \bowtie_d^x D$, since $B_1 \bowtie_d^x B_2 \neq 0$. Like in the first part of the example, the picture below shows the matches performed by $\oplus = \max$.



Following the same idea as before, Y becomes the following concept description:

$$A \sqcap B_2 \sqcap \exists r.(A \sqcap \exists r.\top \sqcap \exists s.\top)$$

Obviously, we still have $C \bowtie_d^x D = C \bowtie_d^x Y$, but now $C \not\sqsubseteq Y$. Hence, $Y^r \bowtie_d^x C^r < 1$ and in principle one can no longer assume that $(C \bowtie^x D) \leq (C \bowtie^x Y)$ as before. In fact, a closer inspection of the computation of $\bowtie_d^x$ shows that $Y^r \bowtie_d^x C^r < D^r \bowtie_d^x C^r$, and by monotonicity of $\otimes$ it follows $(C \bowtie^x Y) \leq (C \bowtie^x D)$. We will not enter into the details of the computation (they are very tedious), but let us briefly explain the idea behind this. The construction of Y excludes the *right-most* top-level atom of D. However, one can see that the structure of $\exists r.\exists s.A$ can be entirely "mapped" into the structure of C. This means that as a top-level atom of D, it contributes with value 1 to the computation of $D^r \bowtie_d^x C^r$. Therefore, we end up with the following expressions:

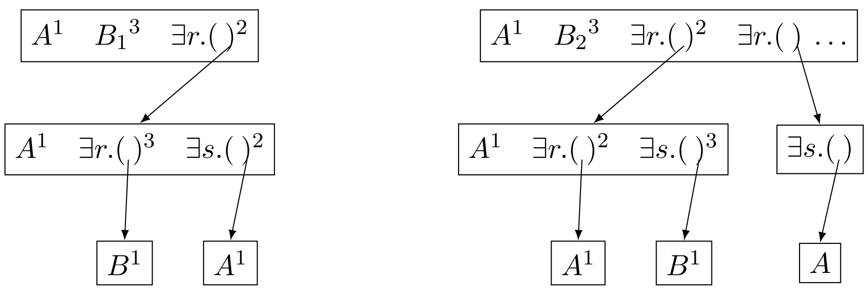
$$Y^r \bowtie_d^x C^r = \frac{a}{3} \;\text{ and }\; D^r \bowtie_d^x C^r = \frac{a+1}{4}$$

where a is a real value smaller than 3, which proves that $Y^r \bowtie_d^x C^r < D^r \bowtie_d^x C^r$. Hence, throwing away the atom $\exists r.\exists s.A$ decreases the value of the right to left comparison when computing $\bowtie^x$. This means that the arguments used to prove linear boundedness in Corollary 7.37 are not valid in this case.

One could still wonder whether it is possible to remove less information from D, while keeping the value $C^r \bowtie_d^x D^r$ and the size of the resulting concept small enough. Notice that the concept description $Y \sqcap \exists r.\exists s.A$ represents such a possibility. Nevertheless, this is a very particular case where the size of D is actually not much bigger than C. Suppose for instance, that $pm(r,s) = pm(s,r) = 0.9$ and D is extended into D' as follows:

$$D' := D \sqcap \exists r.\exists r.B \sqcap \exists s.\exists r.A \sqcap \exists s.\exists s.A$$

A consequence of having such a high similarity between r and s is that now $\exists r.B \bowtie_d^x \exists s.B > \exists r.B \bowtie_d^x \exists r.A$. The picture below shows the change of scenario in the mapping corresponding to the top-level atoms of the second level.



Consequently, the same way of selecting Y would result in the following concept description:

$$A \sqcap B_2 \sqcap \exists r.(A \sqcap \exists r.A \sqcap \exists s.B)$$

Notice that all the newly added existential restrictions are irrelevant for the selection of Y. In addition, there are at least exponentially many top-level atoms in D' (with respect to $\mathsf{rd}(C)$), namely the ones corresponding to the atoms $\exists r.\exists s.A, \exists r.\exists r.B, \exists s.\exists s.A$

and $\exists s.\exists r.A$. Furthermore, due to the primitive similarity between r and s, such atoms contribute with value 1 (or very close to 1) to the computation of $(D')^r \bowtie_d^x C^r$. Therefore, likewise for $\exists r.\exists s.A$ and D, throwing away any of them will decrease the value $(D')^r \bowtie_d^x C^r$. $\diamondsuit$

To conclude, the previous example tells us that if D' is the selected maximal concept with respect to C and a filter f, one cannot use the idea from Corollary 7.37 to extract a "small" fragment Y of D' such that $C^r \bowtie_d^x (D')^r = C^r \bowtie_d^x Y^r$, and then exploit the monotonicity of $\otimes$ to obtain $C \bowtie^x D' \leq C \bowtie^x Y$.

At this moment, it is not clear to us whether for non-default primitive measures the resulting instances of *simi* are polynomially bounded or not. An alternative could be to drop the requirement of having $C^r \bowtie_d (D')^r = C^r \bowtie_d Y^r$, but find a different method to build Y such that at the end $C \bowtie D' \leq C \bowtie Y$ while keeping Y small enough. We do not know if it is possible to do that by only knowing that the fuzzy connector $\otimes$ is monotonic.

7.3.2 Relation to the membership degree function *deg*

To conclude the section, we show that our graded membership function *deg* can be obtained from a CSM $\bowtie^1$, using the construction in Definition 7.2. The function $\bowtie^1$ is defined as the following instance of *simi*:

- the fuzzy connector is defined as $\otimes = \min$ and the bounded t-conorm $\oplus$ as max,
- the function g maps every atom to 1, pm is the default primitive measure pm_d and the value w is selected as 0.

There is a minor detail in the definition of $\bowtie^1$ regarding the *simi* framework, namely, $w = 0$. The *simi* framework defines $w \in (0, 1)$ for two reasons. First, using $w = 1$ would nullify the recursive computation of *simi* on existential restrictions. Secondly, $w > 0$ is desired in order to be able to distinguish between different role names, as explained above. However, any instance of *simi* with $w = 0$ still complies with the basic properties shown in [Leh12] that have been used so far. Therefore, $\bowtie^1$ is equivalence invariant, role-depth bounded, equivalence closed, and induces a well-defined graded membership function $m_{\bowtie^1}$. Moreover, since min is a monotonic fuzzy connector, this means that $\bowtie^1$ satisfies all the same properties as those CSMs in the family $\mathcal{F}_1$. Notice that the value of w is irrelevant for the results shown for CSMs in $\mathcal{F}_1$, and one could say that $\bowtie^1 \in \mathcal{F}_1$.

Our main goal now is to show that $deg = m_{\bowtie^1}$. We start by proving that selecting $\otimes = \min$ makes the value $D^r \bowtie_d^1 C^r$ irrelevant for the computation of $C \bowtie^1 D$. The proof is supported by the application of Lemma 7.36 in the context of $\bowtie^1$.

Lemma 7.38. *For all interpretations $\mathcal{I}$, $d \in \Delta^{\mathcal{I}}$, and $\mathcal{EL}$ concept descriptions C we have:*

$$m^{\mathcal{I}}_{\bowtie^1}(d, C) = \max\{C^r \bowtie_d^1 D^r \mid D \in \mathcal{C}_{\mathcal{EL}} \text{ and } d \in D^{\mathcal{I}}\}$$

Proof. By Definition 7.2

$$m^{\mathcal{I}}_{\bowtie^1}(d, C) = \max\{C \bowtie^1 D \mid D \in \mathcal{C}_{\mathcal{EL}} \text{ and } d \in D^{\mathcal{I}}\}$$

For all concept descriptions D, by definition of $\bowtie^1$ we know that

$$C \bowtie^1 D = \min\{C^r \bowtie^1_d D^r, D^r \bowtie^1_d C^r\}$$

Hence, it follows that $C \bowtie^1 D \leq C^r \bowtie^1_d D^r$ and we obtain:

$$m^{\mathcal{I}}_{\bowtie^1}(d, C) \leq \max\{C^r \bowtie^1_d D^r \mid D \in \mathcal{C}_{\mathcal{EL}} \text{ and } d \in D^{\mathcal{I}}\} \tag{7.12}$$

Now, let X be a concept description such that $d \in X^{\mathcal{I}}$ and $C^r \bowtie^1_d X^r$ gives the maximum in (7.12). The application of Lemma 7.36 to C^r and X^r, yields a concept Y such that:

- $C^r \sqsubseteq Y$ and $X^r \sqsubseteq Y$,
- $C^r \bowtie^1_d X^r = C^r \bowtie^1_d Y$.

Since $Y \equiv Y^r$, having $C^r \sqsubseteq Y$ implies that $Y^r \bowtie^1_d C^r = 1$ (see (7.8)). Additionally, (7.9) further implies $C^r \bowtie^1_d Y = C^r \bowtie^1_d Y^r$. Hence, we obtain the following sequence of equalities:

$$\begin{aligned} C \bowtie^1 Y &= \min\{C^r \bowtie^1_d Y^r, Y^r \bowtie^1_d C^r\} \\ &= C^r \bowtie^1_d Y^r \\ &= C^r \bowtie^1_d Y \\ &= C^r \bowtie^1_d X^r \end{aligned}$$

Moreover, $d \in X$ and $X^r \sqsubseteq Y$ imply that $d \in Y^{\mathcal{I}}$. This means that Y is one of the candidate concepts in the computation of $m^{\mathcal{I}}_{\bowtie^1}(d, C)$. Therefore,

$$C^r \bowtie^1_d X^r \leq m^{\mathcal{I}}_{\bowtie^1}(d, C) \tag{7.13}$$

Thus, by the way X was chosen and the combination of (7.12) and (7.13), our claim follows. □

Once we know that $D^r \bowtie^1_d C^r$ can be forgotten when computing $C \bowtie^1 D$, a basic relationship between $\bowtie^1_d$ and deg is established in the following lemma.

Lemma 7.39. *Let X be an $\mathcal{EL}$ concept description and $\mathcal{I}_X$ be the interpretation corresponding to the $\mathcal{EL}$ description tree T_X. Then, for each $\mathcal{EL}$ concept description C, it holds:*

$$C^r \bowtie^1_d X = deg^{\mathcal{I}_X}(d_0, C)$$

where d_0 is the domain element corresponding to the root of T_X.

Proof. We prove the claim by induction on the structure of C.

Induction Base. $C \in \mathsf{N_C}$ or $C = \top$. Then, $C = C^r$. If C^r is of the form A, then $A \bowtie^1_d X = 1$ when $A \in tl(X)$ and 0 otherwise. A similar relationship holds for $deg^{\mathcal{I}_X}(d_0, A)$, but with respect to whether $d_0 \in A^{\mathcal{I}_X}$. Since $A \in tl(X)$ iff $d_0 \in A^{\mathcal{I}_X}$, this means that $A \bowtie^1_d X = deg^{\mathcal{I}_X}(d_0, A)$. The case for $\top$ is trivial, since $\top \bowtie^1_d X = deg^{\mathcal{I}_X}(d_0, \top) = 1$.

Induction Step. We distinguish two cases:

- C is of the form $\exists r.D$. Then, C^r is of the form $\exists r.D^r$. By definition of $\bowtie_d^1$ and *deg*, it is easy to see that whenever X does not have a top-level atom of the form $\exists r.X'$, it is the case that:

$$\exists r.D^r \bowtie_d^1 X \;=\; deg^{\mathcal{I}_X}(d_0, \exists r.D) = 0$$

Hence, without loss of generality, we focus on the cases where there exists at least one top-level atom in X of the form $\exists r.X'$. Consequently, since $|tl(\exists r.D^r)| = 1$, we have:

$$\exists r.D^r \bowtie_d^1 X = \max\{D^r \bowtie_d^1 X' \mid \exists r.X' \in tl(X)\} \tag{7.14}$$

Since $\mathcal{I}_X$ is induced by T_X, then for each atom $\exists r.X' \in tl(X)$ there exists a corresponding domain element $e \in \Delta^{\mathcal{I}_X}$ such that $(d_0, e) \in r^{\mathcal{I}_X}$. This correspondence also holds in the opposite direction. Moreover, it is easy to see that the tree rooted at e in T_X corresponds to the $\mathcal{EL}$ description tree $T_{X'}$. Hence, the application of induction hypothesis to D yields:

$$D^r \bowtie_d^1 X' = deg^{\mathcal{I}_X}(e, D), \text{ for all } \exists r.X' \in tl(X)$$

Therefore, it follows from the equality in (7.14):

$$\exists r.D^r \bowtie_d^1 X = \max\{deg^{\mathcal{I}_X}(e, D) \mid (d_0, e) \in r^{\mathcal{I}_X}\} \tag{7.15}$$

Now, let $T_{\exists r.D^r}$ be the corresponding $\mathcal{EL}$ description tree of $\exists r.D^r$ and v_0 its root. Obviously, there exists exactly one r-successor v_1 of v_0 in $T_{\exists r.D^r}$ and moreover, the subtree of $T_{\exists r.D^r}$ rooted at v_1 is exactly the $\mathcal{EL}$ description tree T_{D^r} associated to D^r. Consider, then, the set $\mathcal{H}(T_{\exists r.D^r}, G_{\mathcal{I}_X}, d_0)$. By Definition 4.5 we have:

$$deg^{\mathcal{I}_X}(d_0, \exists r.D) = \max\{h_w(v_0) \mid h \in \mathcal{H}(T_{\exists r.D^r}, G_{\mathcal{I}_X}, d_0)\} \tag{7.16}$$

Now, let h be any *ptgh* in $\mathcal{H}(T_{\exists r.D^r}, G_{\mathcal{I}_X}, d_0)$ with $h(v_1) = e$, for some $e \in \Delta^{\mathcal{I}_X}$ such that $(d_0, e) \in r^{\mathcal{I}_X}$. We know that there exists at least one and any *ptgh* h' of a different form will not be interesting, since $h'_w(v_0) = 0$. By definition of h_w (Definition 4.4), it follows that $h_w(v_0) = h_w(v_1)$. Additionally, for any *ptgh* $h \in \mathcal{H}(T_{\exists r.D^r}, G_{\mathcal{I}_X}, d_0)$ with $h(v_1) = e$, its restriction to $(V_{T_{\exists r.D^r}} \setminus \{v_0\})$ is a *ptgh* in $\mathcal{H}(T_{D^r}, G_{\mathcal{I}_X}, e)$. Conversely, any *ptgh* g in $\mathcal{H}(T_{D^r}, G_{\mathcal{I}_X}, e)$ can be extended to a *ptgh* in $\mathcal{H}(T_{\exists r.D^r}, G_{\mathcal{I}_X}, d_0)$, by defining $g(v_0) = d_0$. Hence, (7.16) can be transformed into:

$$deg^{\mathcal{I}_X}(d_0, \exists r.D) = \max_{(d_0,e)\in r^{\mathcal{I}_X}} \{g_w(v_1) \mid g \in \mathcal{H}(T_{D^r}, G_{\mathcal{I}_X}, e)\}$$

Finally, since for each $e \in \Delta^{\mathcal{I}_X}$ there exists a *ptgh* $g \in \mathcal{H}(T_{D^r}, G_{\mathcal{I}_X}, e)$ such that $deg^{\mathcal{I}_X}(e, D) = g_w(v_1)$ and $g_w(v_1)$ gives the maximum value, we further obtain the following equation:

$$deg^{\mathcal{I}_X}(d_0, \exists r.D) = \max\{deg^{\mathcal{I}_X}(e, D) \mid (d_0, e) \in r^{\mathcal{I}_X}\} \tag{7.17}$$

Thus, the combination of (7.15) and (7.17) yields

$$\exists r.D^r \bowtie_d^1 X = deg^{\mathcal{I}_X}(d_0, \exists r.D)$$

- C is of the form $C_1 \sqcap \ldots \sqcap C_k$. Then, its reduced form C^r is of the form $D_1 \sqcap \ldots \sqcap D_n$, where $1 \leq n \leq k$ and each D_j is the reduced form $[C_i]^r$ of some conjunct C_i. Now, it is easy to see from the definition of $\bowtie_d^1$, that $C^r \bowtie_d^1 X$ can be equivalently expressed as:

$$C^r \bowtie_d^1 X = \frac{\sum_{j=1}^{n} (D_j \bowtie_d^1 X)}{n} \tag{7.18}$$

Furthermore, though more involved, it is not hard to see from the definitions of *deg* and h_w, that a similar situation occurs with respect to *deg*:

$$deg^{\mathcal{I}_X}(d_0, C^r) = \frac{\sum_{j=1}^{n} deg^{\mathcal{I}_X}(d_0, D_j)}{n} \tag{7.19}$$

Then, for each D_j one can apply the induction hypothesis to the atom C_i that has $[C_i]^r = D_j$ to obtain $D_j \bowtie_d^1 X = deg^{\mathcal{I}_X}(d_0, C_i)$. Since *deg* is equivalence invariant (in the sense of property *M2*), we have $D_j \bowtie_d^1 X = deg^{\mathcal{I}_X}(d_0, C_i) = deg^{\mathcal{I}_X}(d_0, D_j)$. Hence, the combination of (7.18) and (7.19) yields $C^r \bowtie_d^1 X = deg^{\mathcal{I}_X}(d_0, C)$. □

Finally, using the previous two results, one can show the equivalence between $m_{\bowtie^1}$ and *deg*.

Theorem 7.40. *For all interpretations $\mathcal{I}$, $d \in \Delta^{\mathcal{I}}$, and $\mathcal{EL}$ concept descriptions C we have $m_{\bowtie^1}^{\mathcal{I}}(d, C) = deg^{\mathcal{I}}(d, C)$.*

Proof. (⇒) From Lemma 7.38, we know that there exists an $\mathcal{EL}$ concept description X such that $m_{\bowtie^1}^{\mathcal{I}}(d, C) = C^r \bowtie_d^1 X^r$ and $d \in X^{\mathcal{I}}$. The application of Lemma 7.39 to C and X yields:

$$C^r \bowtie_d^1 X \;=\; deg^{\mathcal{I}_X}(d_0, C)$$

Recall that due to property (7.9), $C^r \bowtie_d^1 X = C^r \bowtie_d^1 X^r$. Since $d \in X^{\mathcal{I}}$, the characterization of crisp membership in $\mathcal{EL}$ yields the existence of a homomorphism φ from $G_{\mathcal{I}_X}$ (or T_X) to $G_{\mathcal{I}}$ with $\varphi(d_0) = d$. Hence, the application of Lemma 4.11 to $\mathcal{I}_X$ and $\mathcal{I}$ implies $deg^{\mathcal{I}_X}(d_0, C) \leq deg^{\mathcal{I}}(d, C)$. Therefore, we obtain:

$$m_{\bowtie^1}^{\mathcal{I}}(d, C) \leq deg^{\mathcal{I}}(d, C) \tag{7.20}$$

(⇐) Consider a *ptgh* $h \in \mathcal{H}(T_{C^r}, G_{\mathcal{I}}, d)$ such that $h_w(v_0) = deg^{\mathcal{I}}(d, C)$. Let $\mathcal{I}_h$ be the canonical interpretation induced by h. Since $T_{\mathcal{I}_h}$ is a tree, we can speak of its corresponding $\mathcal{EL}$ concept description $C_{\mathcal{I}_h}$. Then, we obtain the following equalities:

$$\begin{aligned} deg^{\mathcal{I}}(d, C) &= deg^{\mathcal{I}_h}(v_0, C) && \text{(Lemma 4.12)} \\ &= C^r \bowtie_d^1 C_{\mathcal{I}_h} && \text{(Lemma 7.39)} \\ &= C^r \bowtie_d^1 (C_{\mathcal{I}_h})^r && \text{(property 7.9)} \end{aligned}$$

Furthermore, it is easy to see that by definition of $\mathcal{I}_h$, it holds that $d \in [C_{\mathcal{I}_h}]^{\mathcal{I}}$. Hence, Lemma 7.38 implies that $C^r \bowtie_d^1 (C_{\mathcal{I}_h})^r \leq m^{\mathcal{I}}_{\bowtie^1}(d, C)$ and consequently:

$$deg^{\mathcal{I}}(d, C) \leq m^{\mathcal{I}}_{\bowtie^1}(d, C) \tag{7.21}$$

Thus, our claim follows from the combination of inequalities (7.20) and (7.21). □

Once we have established this equivalence, Proposition 7.4 thus implies that answering of relaxed instance queries w.r.t. $\bowtie^1$ is the same as computing instances for threshold concepts of the form $Q_{>t}$ in $\tau\mathcal{EL}(deg)$. Since such concepts are positive, Proposition 5.19 yields the following corollary.

Corollary 7.41. *Let $\mathcal{A}$ be an $\mathcal{EL}$ ABox, Q an $\mathcal{EL}$ query concept, a an individual name, and $t \in [0, 1)$. Then it can be decided in polynomial time whether $a \in \mathsf{Relax}_t^{\bowtie^1}(Q, \mathcal{A})$ or not.*

Note that Ecke et al. [EPT14, EPT15] show only an NP upper bound w.r.t. data complexity for this problem, albeit for a larger class of instances of the simi framework.

Chapter 8

Conclusions and Future Work

We have introduced a family of DLs $\tau\mathcal{EL}(m)$ parameterized with a graded membership function m, which extends the popular lightweight DL $\mathcal{EL}$ by threshold concepts that can be used to approximate classical concepts. Inspired by the homomorphism characterization of membership in $\mathcal{EL}$ concepts, we have defined a particular membership function *deg* and have investigated the complexity of reasoning in $\tau\mathcal{EL}(deg)$. It turns out that the higher expressiveness takes its toll: whereas reasoning in $\mathcal{EL}$ can be done in polynomial time, it is NP- or coNP-complete in $\tau\mathcal{EL}(deg)$, depending on which inference problem is considered.

The membership function *deg* has been further extended to consider $\mathcal{EL}$ concepts defined with respect to acyclic TBoxes. Based on this, we have defined $\tau\mathcal{EL}(deg)$ TBoxes as pairs $(\mathcal{T}_\tau, \mathcal{T}_{\mathcal{EL}})$, where $\mathcal{T}_\tau$ contains concept definitions that use threshold concepts defined over $\mathcal{T}_{\mathcal{EL}}$. Obviously, reasoning with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes can already be handled by the basic approach through unfolding. We hoped that the possible exponential blow-up due to unfolding could be avoided, but unfortunately this is not the case. In fact, we have seen that the satisfiability and subsumption problems with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes are Π_2^P-hard and Σ_2^P-hard, respectively. In Section 6.4.3 a PSPACE decision procedure is provided to solve these problems, and it is later extended to tackle all the standard reasoning problems with respect to acyclic knowledge bases, while keeping the use of space polynomial in the size of the input.

We have also shown that concept similarity measures satisfying certain properties can be used to define graded membership functions. This extensive family of CSMs contains non-computable functions, and some of them induce undecidable threshold logics. On the positive side, however, a computable CSM $\bowtie$ always induces a decidable threshold DL $\tau\mathcal{EL}(m_\bowtie)$. Decidability is achieved by adapting the decision procedures provided for $\tau\mathcal{EL}(deg)$ to this more general class of DLs. To gain a preliminary insight into the computational complexity landscape exhibited by this family of decidable logics, we restricted our attention to polynomial time computable CSMs. It turns out that the maximization mechanism used to define a membership function $m_\bowtie$ may yield a PSPACE-hard logic $\tau\mathcal{EL}(m_\bowtie)$. A sufficient bounding condition on CSMs is then defined to obtain a subfamily of logics whose satisfiability problem is in Σ_2^P.

Concrete examples of polynomially bounded CSMs have been presented in Section 7.3.1 as a particular subset of instances of the *simi* framework of Lehmann and Turhan [LT12]. Their induced threshold logics inherit the computational complexity results derived for the whole class of polynomially bounded CSMs. In particular, our function *deg* can be constructed from a polynomially bounded CSM $\bowtie^1$. Nevertheless, our direct definition of *deg* based on homomorphisms is important since the partial tree-to-graph homomorphisms used there are the main technical tool for showing our decidability and complexity results. For instance, satisfiability in $\tau\mathcal{EL}(deg)$ is shown to be NP-complete, in contrast to the general NP^{NP} upper bound obtained from the polynomial boundedness property.

While introduced as a formalism for defining concepts by approximation, a possible use-case for $\tau\mathcal{EL}(deg)$ is relaxation of instance queries, as motivated and investigated in [EPT14, EPT15]. Compared to the setting considered in [EPT14, EPT15], $\tau\mathcal{EL}(deg)$ yields a considerably more expressive query language since we can combine threshold concepts using the constructors of $\mathcal{EL}$ and can also forbid that thresholds are reached. Restricted to the setting of relaxed instance queries, our approach actually allows relaxed instance checking in polynomial time. On the other hand, [EPT14, EPT15] can also deal with other instances of the *simi* framework.

8.1 Future Work

Last, we sketch some ideas and point out several directions for future work.

Membership functions for cyclic and general TBoxes. We would like to extend our function *deg* to be able to compute membership degrees for concepts defined with respect to cyclic and general TBoxes. To do this, homomorphisms probably need to be replaced by simulations [Baa03]. On the side of concept similarity measures for DLs, a specific measure has been proposed in [EPT15] to deal with general TBoxes. In particular, such a CSM is akin to the *simi* framework in the sense that it also combines directional values to compute the similarity between two concepts. We believe that it is possible to exploit the ideas from [EPT15], and use the directional computation to extend *deg* towards concepts defined with respect to general TBoxes. This is joint work in progress with Andreas Ecke.

Nesting of threshold concepts. Extending our introduced family of DLs with nesting of threshold concepts is an interesting topic for future work. To go further in this direction, the initial step is to understand how to come up with a well-defined and meaningful semantics to interpret the resulting concept descriptions. Since a graded membership function m provides the interpretation for simple threshold concepts in a logic $\tau\mathcal{EL}(m)$, one idea that seems natural is to interpret nested threshold concepts by recursively applying the definition of m bottom-up. More precisely, suppose we have a nested threshold concept $X_{>.5}$ where X is of the following form:

$$\mathsf{Healthy} \sqcap (\exists \mathsf{spouse}.(\mathsf{Rich} \sqcap \mathsf{Intelligent} \sqcap \mathsf{Female}))_{\geq .7}$$

To compute $m^{\mathcal{I}}(d, X)$ the function would first calculate $m^{\mathcal{I}}(d, \exists\mathsf{spouse}.(\ldots))$ using the base definition of m to obtain the corresponding value t. Afterwards, m is applied one more time to compute the value $m^{\mathcal{I}}(d, X)$. Here, the inner threshold concept in $X_{>.5}$ would be treated as an atom, where the previously computed value t determines whether d has the property $(\exists\mathsf{spouse}.(\ldots)))_{\geq .7}$ or not. For example, let d be the following element in some interpretation $\mathcal{I}$:

$$d : \{\mathsf{Healthy}\} \xrightarrow{\mathsf{spouse}} \{\mathsf{Intelligent}, \mathsf{Female}\}$$

In our logic $\tau\mathcal{EL}(deg)$ we have $deg^{\mathcal{I}}(d, \exists\mathsf{spouse}.(\ldots)) = 2/3$. This means that $d \notin [(\exists\mathsf{spouse}.(\ldots))_{\geq .7}]^{\mathcal{I}}$. Therefore, applying the idea presented above we would obtain $deg^{\mathcal{I}}(d, X) = 1/2$, since $d \in (\mathsf{Healthy})^{\mathcal{I}}$ and $d \notin [(\exists\mathsf{spouse}.(\ldots))_{\geq .7}]^{\mathcal{I}}$. Thus, $d \notin (X_{>.5})^{\mathcal{I}}$. Obviously, one can object that d is quite close to the crisp set defined by $(\exists\mathsf{spouse}.(\ldots))_{\geq .7}$, and consequently it should not be considered in such a way. Instead, maybe a more suitable idea could be to give a membership degree value for d in $(\exists\mathsf{spouse}.(\ldots))_{\geq .7}$ and use it to compute $deg^{\mathcal{I}}(d, X)$. At this moment it is still unclear to us which one would be a better choice or if both are useful in different scenarios.

Finally, from a computational point of view, one would expect the reasoning problems to become harder. In fact, looking at the equivalences in Proposition 3.2, it is not hard to see that one can express $\mathcal{ALC}$ concept descriptions by just using the threshold values $\{0, 1\}$. For instance, $\neg\exists r.\neg A$ would correspond to the nested threshold concept $(\exists r.A_{<1})_{<1}$.

Cyclic $\tau\mathcal{EL}(deg)$ TBoxes. Since deg is well-defined with respect to acyclic $\mathcal{EL}$ TBoxes $\mathcal{T}_{\mathcal{EL}}$, there is nothing to prevent us to have cyclic definitions in a TBox $\mathcal{T}_\tau$. We would like to consider this in the future. Note, that since $\forall r_1, \ldots, r_n.\neg A$ can be expressed in $\tau\mathcal{EL}(deg)$, it seems to be possible to encode cyclic TBoxes in the DL $\mathcal{FL}_0$ into cyclic $\tau\mathcal{EL}(deg)$ TBoxes. In particular, subsumption in $\mathcal{FL}_0$ for cyclic terminologies is PSPACE-complete w.r.t. descriptive semantics [KdN03, Baa96]. This would give a preliminary PSPACE-hardness result for the subsumption problem in the presence of cyclic $\tau\mathcal{EL}(deg)$ TBoxes.

Bounded CSMs. The polynomially bounded condition is still too strong to be satisfied by many useful CSMs. It would be important to find out how to relax it, without losing the good properties that it gives for a logic $\tau\mathcal{EL}(m_{\bowtie})$. This could, for example, provide more information about the logics $\tau\mathcal{EL}(m_{\bowtie})$ induced by instances of the *simi* framework that use non-primitive measures pm.

Additionally, polynomial boundedness only gives a general $\mathrm{NP}^{\mathrm{NP}}$ upper bound for the satisfiability problem. It would be interesting to characterize which conditions a CSM in $\mathcal{F}_{\bowtie}[poly]$ must satisfy in order to have a satisfiability problem in NP, like it happens for $\bowtie^1$ and $\tau\mathcal{EL}(deg)$.

Open theoretical problems. The exact computational complexity of reasoning with respect to acyclic $\tau\mathcal{EL}(deg)$ TBoxes (between Σ_2^P/Π_2^P and PSPACE) remains open. Regarding the relationship between $\tau\mathcal{EL}(deg)$ and $\mathcal{ALC}$, we do not know whether $\tau\mathcal{EL}(deg)$ is exponentially more succinct than $\mathcal{ALC}$.

Appendix A

Missing proofs

Missing proofs of Chapter 3

Theorem 3.8. *Let $\widehat{C}$ be a $\tau\mathcal{EL}(m)$ concept description and $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$ an interpretation. The following statements are equivalent for all $d \in \Delta^{\mathcal{I}}$:*

1. *$d \in \widehat{C}^{\mathcal{I}}$.*

2. *there exists a τ-homomorphism ϕ from $T_{\widehat{C}}$ to $G_{\mathcal{I}}$ with $\phi(v_0) = d$.*

Proof. Let $T_{\widehat{C}} = (V_T, E_T, v_0, \widehat{\ell}_T)$ be the description tree associated to $\widehat{C}$ and $\widehat{C}$ be of the form $\widehat{C}_1 \sqcap \ldots \sqcap \widehat{C}_q \sqcap \exists r_1.\widehat{D}_1 \sqcap \ldots \sqcap \exists r_n.\widehat{D}_n$, where each $\widehat{C}_i$ is either a concept name $A \in \mathsf{N_C}$ or a threshold concept $E_{\sim t} \in \widehat{\mathsf{N}}_{\mathsf{E}}$.

($\Rightarrow$) Assume that $d \in \widehat{C}^{\mathcal{I}}$. Then, $d \in (\widehat{C}_i)^{\mathcal{I}}$ and $d \in (\exists r_j.\widehat{D}_j)^{\mathcal{I}}$ for all $1 \leq i \leq q$ and $1 \leq j \leq n$. We show by induction on the *role depth* of $\widehat{C}$ that there exists a τ-homomorphism ϕ from $T_{\widehat{C}}$ to $G_{\mathcal{I}}$ with $\phi(v_0) = d$.

Induction Base. $\mathsf{rd}(\widehat{C}) = 0$. Then, $n = 0$ and $T_{\widehat{C}}$ consists only of one node v_0 (the root), it has no edges and $\widehat{\ell}_T(v_0) = \{\widehat{C}_1, \ldots, \widehat{C}_q\}$. The mapping $\phi(v_0) = d$ is a τ-homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{I}}$. For each $\widehat{C}_i$ of the form $A \in \mathsf{N_C}$ we know that $A \in \ell_{\mathcal{I}}(d)$, and consequently ϕ satisfies Condition 1 in Definition 3.7. In case $\widehat{C}_i$ is of the form $E_{\sim t}$, the fact that $d \in (\widehat{C}_i)^{\mathcal{I}}$ implies that ϕ satisfies Condition 2 in Definition 3.7.

Induction Step. Assume that the claim holds for all the concepts with *role depth* smaller than k. We show that it also holds for $\mathsf{rd}(\widehat{C}) = k$. First, consider the concept $\widehat{D}_0 = \widehat{C}_1 \sqcap \ldots \sqcap \widehat{C}_q$. One can see that $T_{\widehat{D}_0} = (V_0, E_0, v_0, \widehat{\ell}_0)$ is exactly the description tree with $V_0 = \{v_0\}$, $E_0 = \emptyset$ and $\widehat{\ell}_0(v_0) = \widehat{\ell}_T(v_0)$. Since $d \in (\widehat{D}_0)^{\mathcal{I}}$ and $\mathsf{rd}(\widehat{D}_0) = 0$, by induction hypothesis there exists a τ-homomorphism ϕ_0 from $T_{\widehat{D}_0}$ to $G_{\mathcal{I}}$ with $\phi_0(v_0) = d$.

Now, consider any edge $v_0 r_j v_j$ in E_T. By the relationship between $T_{\widehat{C}}$ and $\widehat{C}$, there exists a *top-level* concept $\exists r_j.\widehat{D}_j$ of $\widehat{C}$ such that $T_{\widehat{D}_j} = (V_j, E_j, v_j, \widehat{\ell}_j)$ is precisely the subtree of $T_{\widehat{C}}$ with root v_j. In addition, since $d \in (\exists r_j.\widehat{D}_j)^{\mathcal{I}}$ there exists $d_j \in \Delta^{\mathcal{I}}$ such that $d r_j d_j \in E_{\mathcal{I}}$ and $d_j \in (\widehat{D}_j)^{\mathcal{I}}$. Since $\mathsf{rd}(\widehat{D}_j) < k$, the application of the induction hypothesis on d_j and $\widehat{D}_j$ yields a τ-homomorphism ϕ_j from $T_{\widehat{D}_j}$ to $G_{\mathcal{I}}$ with $\phi_j(v_j) = d_j$.

It is not hard to see that for all nodes $v \in V_T$, there exists exactly one of such τ-homomorphism ϕ_j ($0 \leq j \leq n$) such that $v \in \mathsf{dom}(\phi_j)$. Based on this, we build a mapping ϕ from V_T to $V_{\mathcal{I}}$ as $\phi = \bigcup_{j=0}^{n} \phi_j$. Note that $\phi(v_0) = d$ by definition of ϕ_0. Hence, it remains to show that ϕ is τ-homomorphism.

1. ϕ **is a homomorphism from** T_C **to** $G_{\mathcal{I}}$**:** Let v be any node in V_T. We know that v is a node of one description tree $T_{\widehat{D}_j}$ and $\phi(v) = \phi_j(v)$ for the corresponding mapping ϕ_j. Since ϕ_j is a homomorphism, this means that $\ell_j(v) \subseteq \ell_{\mathcal{I}}(\phi_j(v))$. Therefore, $\ell(v) = \ell_j(v)$ implies $\ell(v) \subseteq \ell_{\mathcal{I}}(\phi(v))$. Now, let vrw be any edge from E_T. There are two possibilities:
   - vrw is of the form $v_0 r_j v_j$. As explained before we have $\phi(v_0) = d$, $\phi_j(v_j) = d_j$ and $dr_jd_j \in E_{\mathcal{I}}$. Hence, $\phi(v_0)r_j\phi(v_j) \in E_{\mathcal{I}}$.
   - $v, w \in \mathsf{dom}(\phi_j)$ for some $j \in \{1 \dots n\}$. By construction of ϕ and the fact that ϕ_j is a homomorphism, it follows that $\phi(v)r\phi(w) \in E_{\mathcal{I}}$.
2. Condition 2 in Definition 3.7 follows from the fact that ϕ is constructed using τ-homomorphisms.

Thus, ϕ is τ-homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{I}}$ with $\phi(v_0) = d$.

($\Leftarrow$) Assume that there exists a τ-homomorphism ϕ from $T_{\widehat{C}}$ to $G_{\mathcal{I}}$ with $\phi(v_0) = d$. We show by induction on the size of V_T that $d \in \widehat{C}^{\mathcal{I}}$.

Induction Base. $|V_T| = 1$. Then, $\widehat{C}$ is of the form $\widehat{C}_1 \sqcap \dots \sqcap \widehat{C}_q$ and $\widehat{\ell}_T(v_0) = \{\widehat{C}_1, \dots, \widehat{C}_q\}$. We distinguish two cases for all $\widehat{C}_i \in \widehat{\ell}_T(v_0)$:

- $\widehat{C}_i$ is of the form $A \in \mathsf{N_C}$. Since ϕ is τ-homomorphism, it is also a classical homomorphism in the sense of Definition 2.5 and hence, ignoring the labels of the form $E_{\sim t}$ we have $\ell_T(v_0) \subseteq \ell_{\mathcal{I}}(d)$. Thus, $d \in A^{\mathcal{I}}$.
- $\widehat{C}_i$ is of the form $E_{\sim t}$. By Definition 3.7 we also have $d \in (E_{\sim t})^{\mathcal{I}}$.

Thus, $d \in (\widehat{C}_i)^{\mathcal{I}}$ for all conjuncts $\widehat{C}_i$ of $\widehat{C}$. Consequently, $d \in \widehat{C}^{\mathcal{I}}$.

Induction Step. Assume that the claim holds for $|V_T| < k$. We show that it also holds for $|V_T| = k$. Since $k > 0$, there exist nodes $v_1, \dots, v_n$ in V_T such that $v_0 r_j v_j \in E_T$. This also means that $\widehat{C}$ is of the form $\widehat{C}_1 \sqcap \dots \sqcap \widehat{C}_q \sqcap \exists r_1.\widehat{D}_1 \sqcap \dots \sqcap \exists r_n.\widehat{D}_n$ with $n > 0$, and the description tree $T_{\widehat{D}_j} = (V_j, E_j, v_j, \widehat{\ell}_j)$ associated to $\widehat{D}_j$ is the subtree of $T_{\widehat{C}}$ rooted at v_j. We consider the following two cases:

- $q > 0$. Then, $d \in (\widehat{C}_i)^{\mathcal{I}}$ can be shown in the same way as for the base case.
- Consider any $\exists r_j.\widehat{D}_j$, with $j \in \{1 \dots n\}$. Since ϕ is also a homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{I}}$ and $v_0 r_j v_j \in E_T$, then there exists $e_j \in \Delta^{\mathcal{I}}$ such that $dr_je_j \in E_{\mathcal{I}}$ and $\phi(v_j) = e_j$. Moreover, it is clear that $|V_j| < |V_T|$ and it is not difficult to see that the restriction of the domain of ϕ to V_j is also a τ-homomorphism from $T_{\widehat{D}_j}$ to $G_{\mathcal{I}}$ with $\phi(v_j) = e_j$. Hence, the induction hypothesis can be applied to obtain that $e_j \in (\widehat{D}_j)^{\mathcal{I}}$. Hence, $d \in (\exists r_j.\widehat{D}_j)^{\mathcal{I}}$.

Thus, we have shown that $d \in \widehat{C}^{\mathcal{I}}$. □

Missing proofs of Chapter 4

Definition A.1. Let C be an $\mathcal{EL}$ concept description and T_C its corresponding $\mathcal{EL}$ description tree. For all nodes $v \in V_{T_C}$ we denote by $T_C[v]$ the subtree of T_C rooted at

v. Furthermore, the $\mathcal{EL}$ concept description $C[v]$ is the one having the description tree $T_C[v]$. Finally, the height $\eta(v)$ of a node v in T_C is the length of the longest path from v to a leaf of T_C. ♢

In the proof of Lemma 4.9, we will use concepts and description trees of the form $T_C[v]$ and $C[v]$. We would like to point out that for all concept descriptions C^r in reduced form, the concepts $C^r[v]$ are also in reduced form (for all $v \in V_{T_{C^r}}$). This is a consequence of the fact that to obtain the reduced form of a concept C the rules are not only applied in the top-level conjunction of C, but also under the scope of existential restrictions (see Chapter 2).

Lemma 4.9. *Let C be an $\mathcal{EL}$ concept description, $\mathcal{I}$ a finite interpretation and $d \in \Delta^{\mathcal{I}}$. Then, Algorithm 4 terminates on input $(C, \mathcal{I}, d)$ and outputs $deg^{\mathcal{I}}(d, C)$, i.e., $S(v_0, d) = deg^{\mathcal{I}}(d, C^r)$.*

Proof. To see that the algorithm terminates, it is enough to observe that T_{C^r} and $G_{\mathcal{I}}$ are finite and the algorithm consists of nested iterations over the nodes and edges in T_{C^r} and $G_{\mathcal{I}}$. To show that $S(v_0, d) = deg^{\mathcal{I}}(d, C^r)$, we prove a more general claim:

Claim: $S(v, e) = deg^{\mathcal{I}}(e, C^r[v])$ for all $v \in V_{T_{C^r}}$ and $e \in \Delta^{\mathcal{I}}$.

Note first, that for each pair (v, e) the value of $S(v, e)$ is assigned only once during a run of the algorithm. We prove the claim by induction on the height $\eta(v)$ of each node v in T_{C^r}.

Induction Base. $\eta(v) = 0$. Then v is a leaf in T_{C^r}. This means that v has no successors and for all $e \in \Delta^{\mathcal{I}}$ there exists a unique *ptgh* h from $T_{C^r}[v]$ to $G_{\mathcal{I}}$ with $h(v) = e$. One can see in Algorithm 4, that the special case where $|\ell_{T_{C^r}}(v)| + k^*(v) = 0$ is properly treated. Otherwise, we have $c = |\ell_{T_{C^r}}(v) \cap \ell_{\mathcal{I}}(e)|$ and $S(v, e) = \frac{c}{|\ell_{T_{C^r}}(v)|}$. Note that this is exactly the value of $h_w(v)$ in Definition 4.4. Since h is unique, this means that $deg^{\mathcal{I}}(e, C^r[v]) = S(v, e)$.

Induction Step. $\eta(v) > 0$. Let $v_1, \ldots, v_k$ be the children of v in T_{C^r} such that if v_1 is an r-successor of v in T_{C^r}, then e has at least one r-successor in $G_{\mathcal{I}}$. The application of the max operator in line 10, selects for each r-successor v_i of v an r-successor e_i of e in $\Delta^{\mathcal{I}}$ that has the maximum value for $S(v_i, e_i)$. Such a value is then used in the computation of c. Let (v_i, e_i) be the pairs representing such a selection for all v_i. Two observations are in order:

- Since v_i is a child of v, it occurs first in the post-oder selected in line 1. Therefore, the value of $S(v_i, e_i)$ is computed before the computation of c for (v, e).
- The value of $S(v, e)$ as computed by Algorithm 4 corresponds to the following expression:

$$S(v, e) = \frac{|\ell_{T_{C^r}}(v) \cap \ell_{\mathcal{I}}(e)| \; + \; \sum_{i=1}^{k} S(v_i, e_i)}{|\ell_{T_{C^r}}(v) + k^*(v)|} \tag{A.1}$$

- Since $\eta(v_i) < \eta(v)$, the application of the induction hypothesis yields

$$S(v_i, e_i) = deg^{\mathcal{I}}(e_i, C^r[v_i]) \tag{A.2}$$

Now, let h_i be a *ptgh* from $T_{C^r}[v_i]$ to $G_{\mathcal{I}}$ such that $h_i(v_i) = e_i$ and $h_{i_w}(v_i) = deg^{\mathcal{I}}(e_i, C^r[v_i])$ for all $1 \leq i \leq k$. It is easy to see that the mapping $h = h_1 \cup \ldots \cup h_k \cup \{(v, e)\}$ is a *ptgh* from $T_{C^r}[v]$ to $G_{\mathcal{I}}$ with $h(v) = e$. Moreover, combining (A.1) and (A.2) it is also true that $h_w(v) = S(v, e)$. Hence, by Definition 4.5 we have $S(v, e) \in \mathcal{V}^{\mathcal{I}}(e, C^r[v])$. Suppose, however, that $S(v, e) < \max \mathcal{V}^{\mathcal{I}}(e, C^r[v])$. We show that this is not the case by reaching a contradiction.

Having $S(v, e) < \max \mathcal{V}^{\mathcal{I}}(e, C^r[v])$ implies the existence of *ptgh* h' from $T_{C^r}[v]$ to $G_{\mathcal{I}}$ with $h'(v) = e$ such that $h'_w(v) > h_w(v)$. Looking at h_w in Definition 4.4, the fact that $h(v) = h'(v)$ implies that the difference must be on the values of $h_w(v_i)$ and $h'_w(v_i)$. More precisely, there must exist at least one successor v_i of v such that $h'_w(v_i) > h_{i_w}(v_i)$. Based on this, we distinguish two cases:

- $h'(v_i) \neq h_i(v_i)$, i.e., the *ptgh* h' maps v_i to a different element in $\Delta^{\mathcal{I}}$. But, if that were the case, then the application of the max operator in line 10 would have chosen $h'(v_i)$ as the pairing for v_i, instead of e_i.

- $h'(v_i) = h(v_i) = e_i$. This case would contradict the induction hypothesis, since $h'_w(v_i) > h_{i_w}(v_i)$ would imply $S(v_i, e_i) < deg^{\mathcal{I}}(e_i, C^r[v_i])$.

Hence, we obtain by contradiction that $S(v, e) = \max \mathcal{V}^{\mathcal{I}}(e, C^r[v])$ and consequently, $S(v, e) = deg^{\mathcal{I}}(e, C^r[v])$. Since $S(v_0, d)$ is a particular case, we thus have shown that $S(v_0, d) = deg^{\mathcal{I}}(d, C^r)$. □

Missing proofs of Chapter 5

Proposition 5.12. *Let $\mathcal{A}$ be an ABox. Then, $\mathcal{A}$ is consistent iff there exists a consistent pre-processing $\mathcal{A}'$ of $\mathcal{A}$.*

Proof. ($\Rightarrow$) Let $\mathcal{I}$ be an interpretation such that $\mathcal{I} \models \mathcal{A}$. One can see that for any assertion $\neg\widehat{C}(a)$ that a rule is applicable to, if $\mathcal{I} \models \neg\widehat{C}(a)$ there is a way to apply the rule such that $\mathcal{I}$ also satisfies the newly introduced assertion. The case for $\rightarrow_{\neg\exists}$ is clear. For the rule $\rightarrow_{\neg\sqcap}$, if $\mathcal{I} \models \neg\widehat{C}(a)$ then there exists a conjunct $\widehat{C}_i$ such that $\mathcal{I} \models \neg\widehat{C}_i(a)$. This can be the *non-deterministic* choice made by the application of $\rightarrow_{\neg\sqcap}$. Last, for assertions of the form $\neg E_{\sim t}$ and $\neg A$ the applicable rules are $\rightarrow_{\neg\sim}$ and $\rightarrow_{\neg A}$, respectively. Since $\neg E_{\sim t} \equiv E_{\chi(\sim)t}$ and $\neg A \equiv A_{<1}$, we have that $\mathcal{I}$ satisfies $E_{\chi(\sim)t}$ and $A_{<1}$.

Thus, since $\mathcal{I}$ satisfies every assertion in $\mathcal{A}$ we can conclude that there exists a *pre-processing* $\mathcal{A}'$ of $\mathcal{A}$ such that $\mathcal{I} \models \mathcal{A}'$.

($\Leftarrow$) This direction is trivial since $\mathcal{A} \subseteq \mathcal{A}'$. □

Lemma 5.15. *Let $\mathcal{A}$ be an ABox, $\mathcal{I}$ an interpretation satisfying $\mathcal{A}$ and $\mathcal{A}'$ a pre-processing of $\mathcal{A}$ such that $\mathcal{I} \models \mathcal{A}'$. Moreover, for all $a \in \mathsf{Ind}(\mathcal{A})$, let $\mathcal{I}_a$ be a tree-shaped interpretation satisfying the following:*

- *$\mathcal{I}_a \models \mathcal{A}'(a)$,*

- *there exists a homomorphism φ_a from $G_{\mathcal{I}_a}$ to $G_{\mathcal{I}}$ with $\varphi_a(a^{\mathcal{I}_a}) = a^{\mathcal{I}}$.*

Last, let $\mathcal{J}$ be the following interpretation:

- $\Delta^{\mathcal{J}} := \bigcup_{a \in \mathsf{Ind}(\mathcal{A})} \Delta^{\mathcal{I}_a}$,
- $A^{\mathcal{J}} := \bigcup_{a \in \mathsf{Ind}(\mathcal{A})} A^{\mathcal{I}_a}$ *for all* $A \in \mathsf{N_C}$,
- $r^{\mathcal{J}} := \{(a^{\mathcal{I}_a}, b^{\mathcal{I}_b}) \mid r(a,b) \in \mathcal{A}\} \cup \bigcup_{a \in \mathsf{Ind}(\mathcal{A})} r^{\mathcal{I}_a}$ *for all* $r \in \mathsf{N_R}$, *and*
- $a^{\mathcal{J}} := a^{\mathcal{I}_a}$, *for all* $a \in \mathsf{Ind}(\mathcal{A})$.

where the sets $\Delta^{\mathcal{I}_a}$ are pairwise disjoint. Then, $\mathcal{J} \models \mathcal{A}$.

Proof. We start by considering the following mapping from $V_{\mathcal{J}}$ to $V_{\mathcal{I}}$:

$$\varphi^* := \bigcup_{a \in \mathsf{Ind}(\mathcal{A})} \varphi_a$$

Since the sets $\Delta^{\mathcal{I}_a}$ are pairwise disjoint, the mapping φ^* is unambiguous. Moreover, we know that $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ for all $r(a,b) \in \mathcal{A}$, $\varphi_a(a^{\mathcal{I}_a}) = a^{\mathcal{I}}$ and $\varphi_b(b^{\mathcal{I}_b}) = b^{\mathcal{I}}$. Therefore, $(\varphi^*(a^{\mathcal{I}_a}), \varphi^*(b^{\mathcal{I}_b})) \in r^{\mathcal{I}}$ for all $(a^{\mathcal{I}_a}, b^{\mathcal{I}_b}) \in r^{\mathcal{J}}$. Consequently, it is clear that φ^* is a homomorphism from $G_{\mathcal{J}}$ to $G_{\mathcal{I}}$ with $\varphi^*(a^{\mathcal{J}}) = a^{\mathcal{I}}$ for all $a \in \mathsf{Ind}(\mathcal{A})$.

We now show that $\mathcal{J} \models \mathcal{A}'$. Since $\mathcal{A} \subseteq \mathcal{A}'$, this will imply $\mathcal{J} \models \mathcal{A}$. Recall that $r(a,b) \in \mathcal{A}'$ iff $r(a,b) \in \mathcal{A}$. By construction of $\mathcal{J}$ we have $(a^{\mathcal{I}_a}, b^{\mathcal{I}_b}) \in r^{\mathcal{J}}$ for all $r(a,b) \in \mathcal{A}$, and $a^{\mathcal{J}} = a^{\mathcal{I}_a}$ for all $a \in \mathsf{Ind}(\mathcal{A})$. Hence, $r(a,b) \in \mathcal{A}'$ implies $(a^{\mathcal{J}}, b^{\mathcal{J}}) \in r^{\mathcal{J}}$. Thus, it remains to show that each concept assertion in $\mathcal{A}'$ is satisfied by $\mathcal{J}$.

We first prove that $\mathcal{J} \models \mathcal{A}'^{+}$. Let $a \in \mathsf{Ind}(\mathcal{A})$ and $\widehat{C}(a) \in \mathcal{A}'^{+}$. From $\mathcal{I}_a \models \mathcal{A}'(a)$ we know that $\mathcal{I}_a \models \widehat{C}(a)$ and $a^{\mathcal{I}_a} \in \widehat{C}^{\mathcal{I}_a}$. Then, the application of Theorem 3.8 yields a τ-homomorphism ϕ from $T_{\widehat{C}}$ to $G_{\mathcal{I}_a}$ with $\phi(v_0) = a^{\mathcal{I}_a}$. We want to show that ϕ is also a τ-homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{J}}$. The construction of $\mathcal{J}$ indicates that $\mathcal{I}_a \subseteq \mathcal{J}$. This means that ϕ is a classical homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{J}}$, which means that Condition 1 in Definition 3.7 is satisfied. Hence, it remains to show that the second condition is also satisfied.

Since $\mathcal{I}_a$ is required to be tree-shaped, it is clear that $\phi(v) = a^{\mathcal{I}_a}$ only if $v = v_0$. Let $v \in V_{T_{\widehat{C}}}$ and $E_{\sim t} \in \widehat{\ell}_{T_{\widehat{C}}}(v)$, we distinguish two cases:

- $v = v_0$. By the relationship that exists between $\tau\mathcal{EL}(m)$ concept descriptions and $\tau\mathcal{EL}(m)$ description trees (see Section 3.1), we have that $E_{\sim t}$ is a top-level atom of $\widehat{C}$. Therefore, $a^{\mathcal{I}_a} \in (E_{\sim t})^{\mathcal{I}_a}$ and $a^{\mathcal{I}} \in (E_{\sim t})^{\mathcal{I}}$. Additionally, we have that $\mathcal{I}_a \subseteq \mathcal{J}$ and φ^* is homomorphism from $G_{\mathcal{J}}$ to $G_{\mathcal{I}}$ with $\varphi^*(a^{\mathcal{J}}) = a^{\mathcal{I}}$. Thus, Lemma 4.11 can be applied with respect to $\mathcal{I}_a$ and $\mathcal{J}$ (if $\sim \in \{>, \geq\}$) or to $\mathcal{J}$ and $\mathcal{I}$ (if $\sim \in \{<, \leq\}$), to obtain $a^{\mathcal{J}} \in (E_{\sim t})^{\mathcal{J}}$.
- $v \neq v_0$. As said before, we have $\phi(v) = e$ with $e \neq a^{\mathcal{I}_a}$ and $e \in \Delta^{\mathcal{I}_a}$. Since $G_{\mathcal{I}_a}$ is a tree, the reachable elements from e in $\Delta^{\mathcal{J}}$ through role relations are exactly the same as in $\Delta^{\mathcal{I}_a}$. Then, it is easy to see that $deg^{\mathcal{I}_a}(e, E) = deg^{\mathcal{J}}(e, E)$, and $e \in (E_{\sim t})^{\mathcal{I}_a}$ implies $e \in (E_{\sim t})^{\mathcal{J}}$.

Thus, ϕ is τ-homomorphism from $T_{\widehat{C}}$ to $G_{\mathcal{J}}$ with $\phi(v_0) = a^{\mathcal{J}}$. The application of Theorem 3.8 yields $a^{\mathcal{J}} \in \widehat{C}^{\mathcal{J}}$. Since we have chosen a and $\widehat{C}(a)$ arbitrarily, we can conclude that $\mathcal{J} \models \mathcal{A}'^{+}$.

We now turn into $\mathcal{A}'^{-}$, i.e., we prove $\mathcal{J} \models \neg\widehat{C}(a)$ for all assertions $\neg\widehat{C}(a) \in \mathcal{A}'$. The proof is very similar to the analogous case in Lemma 5.6 for the non-subsumption problem. We use induction on the structure of $\widehat{C}$.

- $\widehat{C}$ is of the form $E_{\sim t}$ or A. Then, rules $\rightarrow_{\neg\sim}$ and $\rightarrow_{\neg A}$ are applicable, and its application yields $E_{\chi(\sim)t}(a) \in \mathcal{A}'^{+}$ and $A_{<1}(a) \in \mathcal{A}'^{+}$. Since $\neg E_{\sim t} \equiv E_{\chi(\sim)t}$ and $\neg A \equiv A_{<1}$ (see Propositions 4.15 and 3.2, respectively) and $\mathcal{J} \models \mathcal{A}'^{+}$, this means that $a^{\mathcal{J}} \notin \widehat{C}^{\mathcal{J}}$.
- $\widehat{C}$ is of the form $\widehat{C}_1 \sqcap \ldots \sqcap \widehat{C}_n$. By the definition of pre-processing, the rule $\rightarrow_{\neg\sqcap}$ must have been applied adding an assertion of the form $\neg\widehat{C}_i(a)$ to $\mathcal{A}'$ for some $i \in \{1, \ldots, n\}$. The application of the induction hypothesis to $\widehat{C}_i$ yields that $\mathcal{J} \models \neg\widehat{C_i}(a)$ and $a^{\mathcal{J}} \notin (\widehat{C_i})^{\mathcal{J}}$. Thus, $a^{\mathcal{J}} \notin \widehat{C}^{\mathcal{J}}$ and $\mathcal{J} \models \neg\widehat{C}(a)$.
- $\widehat{C}$ is of the form $\exists r.\widehat{D}$ and $(\neg\exists r.\widehat{D})(a) \in \mathcal{A}'$. Assume that $(a^{\mathcal{J}}, d) \in r^{\mathcal{J}}$ for some $d \in \Delta^{\mathcal{J}}$. We have two cases:
  - $d = b^{\mathcal{J}}$ for some $b \in \mathsf{Ind}(\mathcal{A})$. By construction of $\mathcal{J}$ we have $r(a, b) \in \mathcal{A}$. Hence, the rule $\rightarrow_{\neg\exists}$ is applicable and its application adds $\neg\widehat{D}(b)$ to $\mathcal{A}'$. The application of induction to $\widehat{D}$ yields $\mathcal{J} \models \neg\widehat{D}(b)$, and therefore $b^{\mathcal{J}} \notin \widehat{D}^{\mathcal{J}}$.
  - $d \neq b^{\mathcal{J}}$ for all $b \in \mathsf{Ind}(\mathcal{A})$. Then, by construction of $\mathcal{J}$ we have $d \in \Delta^{\mathcal{I}_a}$. Since, $(\neg\exists r.\widehat{D})(a) \in \mathcal{A}'(a)$ and $\mathcal{I}_a \models \mathcal{A}'(a)$, it holds that $d \notin \widehat{D}^{\mathcal{I}_a}$. Now, suppose that $d \in \widehat{D}^{\mathcal{J}}$. By Theorem 3.8 there exists a τ-homomorphism ϕ from $T_{\widehat{D}}$ to $G_{\mathcal{J}}$ with $\phi(v_0) = d$. But, if that is the case, by the disjointness assumptions made to build $\mathcal{J}$ and the fact that $G_{\mathcal{I}_a}$ is a tree, we would have that ϕ is also a τ-homomorphism from $T_{\widehat{D}}$ to $G_{\mathcal{I}_a}$, contradicting the fact that $d \notin \widehat{D}^{\mathcal{I}_a}$. Thus, $d \notin \widehat{D}^{\mathcal{J}}$.

  Overall, we just have shown that for each r-successor d of $a^{\mathcal{J}}$ it is the case that $d \notin \widehat{D}^{\mathcal{J}}$. Hence, $a^{\mathcal{J}} \notin (\exists r.\widehat{D})^{\mathcal{J}}$ and $\mathcal{J} \models \neg\exists r.\widehat{D}(a)$.

Thus, $\mathcal{J} \models \mathcal{A}'^{-}$ and consequently $\mathcal{J} \models \mathcal{A}'$. □

Lemma 5.14. *Let $\mathcal{A}$ be a consistent single-element ABox and $\mathcal{I}$ an interpretation such that $\mathcal{I} \models \mathcal{A}$. In addition, let $\mathcal{J}$ be the bounded model of $\mathcal{A}^{+}$ obtained in Lemma 5.9 with respect to $\mathcal{I}$. Then, there exists a tree-shaped interpretation $\mathcal{K}$ such that:*

1. *$\mathcal{K} \models \mathcal{A}$,*
2. *there exists a homomorphism φ from $G_{\mathcal{K}}$ to $G_{\mathcal{I}}$ with $\varphi(a^{\mathcal{K}}) = a^{\mathcal{I}}$, and*
3. *$|\Delta^{\mathcal{K}}| \leq |\Delta^{\mathcal{J}}| \times p$, where:*

$$p := \begin{cases} 1, & \textit{if } \mathcal{A}^{-} = \emptyset \\ \prod\limits_{\neg\widehat{D}(a) \,\in\, \mathcal{A}^{-}} \mathsf{s}(\widehat{D}), & \textit{otherwise.} \end{cases}$$

Proof. We start by recalling some elements from the proof of Lemma 5.9 that are useful to prove our claims.

- ϕ is a τ-homomorphism from $\widehat{G}(\mathcal{A}^+)$ to $G_\mathcal{I}$ with $\phi(a) = a^\mathcal{I}$.
- ϕ_{id} is a τ-homomorphism from $\widehat{G}(\mathcal{A}^+)$ to $G_\mathcal{J}$ with $\phi_{id}(a) = a^\mathcal{J}$.
- φ is a homomorphism from $G_\mathcal{J}$ to $G_\mathcal{I}$ with $\varphi(v) = \phi(v)$ for all $v \in V_{\mathcal{A}^+}$.
- Since $\mathcal{A}$ contains only one individual name and no role assertions, this means that $\widehat{G}(\mathcal{A}^+)$ is a tree and by construction of $\mathcal{J}$ in Lemma 5.9 $G_\mathcal{J}$ is also a tree.

Let $\#(\mathcal{A})$ denote the number of concept assertions occurring in $\mathcal{A}$. We prove our claim by induction on the number $\#(\mathcal{A}^-)$.

Induction Base. $\#(\mathcal{A}^-) = 0$. Then, we have that $\mathcal{A}^- = \emptyset$. Therefore, $\mathcal{A} = \mathcal{A}^+$ is a $\tau\mathcal{EL}(deg)$ ABox. We choose $\mathcal{K}$ to be the interpretation $\mathcal{J}$. Hence, we have $\mathcal{J} \models \mathcal{A}$, $|\Delta^\mathcal{J}| \leq |\Delta^\mathcal{J}|$, and as explained above $G_\mathcal{J}$ is a tree. Finally, $\mathcal{J}$ interprets the individual name a as $a^\mathcal{J} = a$, which means that $\varphi(a^\mathcal{J}) = a^\mathcal{I}$ (see the mappings ϕ and φ above).

Thus, we have shown our claims for the chosen interpretation $\mathcal{K}$.

Induction Step. Assume that the claim holds for all consistent single-element ABoxes $\mathcal{B}$ with $0 \leq \#(\mathcal{B}^-) < k$. Then, we show that it also holds for consistent single-element ABoxes $\mathcal{A}$ with $\#(\mathcal{A}^-) = k$.

As in the base case, we know that $\mathcal{J} \models \mathcal{A}^+$. However, $\mathcal{J}$ need not satisfy $\mathcal{A}^-$ since the assertions from $\mathcal{A}^-$ were not taken into account to obtain it. The idea for the rest of the proof is to start with an ABox $\mathcal{A}_\mathcal{J}$ reflecting the structure of $\mathcal{J}$. Then, we will consider a *pre-processing* $\mathcal{A}'$ of $\mathcal{A}_\mathcal{J} \cup \mathcal{A}^-$ guided by $\mathcal{I}$, and show how to use it to extend $\mathcal{J}$ into an interpretation $\mathcal{K}$ satisfying our claims.

Let $G_\mathcal{J}$ be the description graph associated to $\mathcal{J}$ (recall that it is a tree). The ABox $\mathcal{A}_\mathcal{J}$ is built as follows:

$$\mathcal{A}_\mathcal{J} := \bigcup_{\substack{b \in V_\mathcal{J} \\ A \in \ell_\mathcal{J}(b)}} \{A(b)\} \cup \bigcup_{brc \in E_\mathcal{J}} \{r(b,c)\}$$

where $\mathsf{Ind}(\mathcal{A}_\mathcal{J}) = \Delta^\mathcal{J} = V_\mathcal{J}$.

We name the element $a^\mathcal{J}$ in $\mathcal{J}$ as a in the new ABox $\mathcal{A}_\mathcal{J}$. In addition, for all $b \in \mathsf{Ind}(\mathcal{A}_\mathcal{J})$ such that $b \neq a$, we make $b^\mathcal{J} = b$. Then, since all the concept assertions in $\mathcal{A}_\mathcal{J}$ are of the form $A(a)$ with $A \in \mathsf{N_C}$, it is easy to see that $\mathcal{J} \models \mathcal{A}_\mathcal{J}$. We now extend the interpretation $\mathcal{I}$ to the individual names in $\mathcal{A}_\mathcal{J}$ to make $\mathcal{I}$ a model of $\mathcal{A}_\mathcal{J}$, namely, $b^\mathcal{I} = \varphi(b^\mathcal{J})$ for all $b \in \mathsf{Ind}(\mathcal{A}_\mathcal{J})$. Since $\varphi(a^\mathcal{J}) = a^\mathcal{I}$, this means that the element $a^\mathcal{I}$ does not change. Hence, φ is a homomorphism from $G_\mathcal{J}$ to $G_\mathcal{I}$ with $\varphi(b^\mathcal{J}) = b^\mathcal{I}$ for all $b \in \mathsf{Ind}(\mathcal{A}_\mathcal{J})$. Using φ, from $b^\mathcal{J} \in A^\mathcal{J}$ we get $b^\mathcal{I} \in A^\mathcal{I}$ for all $A(b) \in \mathcal{A}_\mathcal{J}$. Similarly, we obtain $(a^\mathcal{I}, b^\mathcal{I}) \in r^\mathcal{I}$ for all $r(a,b) \in \mathcal{A}_\mathcal{J}$. Thus, $\mathcal{I} \models \mathcal{A}_\mathcal{J}$ and consequently $\mathcal{I} \models \mathcal{A}_\mathcal{J} \cup \mathcal{A}^-$.

By Remark 5.13 there exists a *pre-processing* $\mathcal{A}'$ of $\mathcal{A}_\mathcal{J} \cup \mathcal{A}^-$ such that $\mathcal{I} \models \mathcal{A}'$. Additionally, we have that $\mathsf{Ind}(\mathcal{A}_\mathcal{J}) = \mathsf{Ind}(\mathcal{A}')$ (recall that $\mathsf{Ind}(\mathcal{A}) = \{a\}$). Based on $\mathcal{A}'$, our first goal is to find interpretations $\mathcal{I}_b$ for all $b \in \mathsf{Ind}(\mathcal{A}_\mathcal{J})$, such that $\mathcal{I}_b \models \mathcal{A}'(b)$ and they can be combined using Lemma 5.15 into a model of $\mathcal{A}_\mathcal{J} \cup \mathcal{A}^-$.

For all individuals $b \in \mathsf{Ind}(\mathcal{A}_\mathcal{J})$, let $\mathcal{A}'_b$ be the following ABox:

$$\mathcal{A}'_b := \bigcup_{E_{\chi(\sim)t}(b)\in\mathcal{A}'} \{E_{\chi(\sim)t}(b)\} \cup \bigcup_{\neg\exists r.\widehat{D}(b)\in\mathcal{A}'} \{\neg\exists r.\widehat{D}(b)\}$$

Here, $E_{\chi(\sim)t}(b)$ is an assertion that results from the application of rule $\rightarrow_{\neg\sim}$ or rule $\rightarrow_{\neg A}$. For the rule $\rightarrow_{\neg A}$, we also represent $A_{<1}$ as $E_{\chi(\sim)t}$, since it is obtained from $\neg A$ and $A \equiv A_{\geq 1}$. Then, the ABox $\mathcal{A}_b$ is defined as:

$$\mathcal{A}_b := \mathcal{A}'_b \cup \bigcup_{A(b)\in\mathcal{A}'} \{A(b)\}$$

The difference between $\mathcal{A}_b$ and $\mathcal{A}'(b)$ is that $\mathcal{A}_b$ does not contain assertions of the form $\neg\widehat{C}(b)$ where $\widehat{C}$ is a conjunction or a threshold concept $E_{\sim t}$. Let us now show that $\#(\mathcal{A}'_b) \leq \#(\mathcal{A}^-)$.

- $\mathcal{J}$ is *tree-shaped* and $\Delta^\mathcal{J} = \mathsf{Ind}(\mathcal{A}_\mathcal{J})$.
- Let a be the individual in $\mathcal{A}_\mathcal{J}$ corresponding to the root of $T_\mathcal{J}$. If $E_{\chi(\sim)t}(a) \in \mathcal{A}'$, it must have been obtained by an application of $\rightarrow_{\neg\sim}$ ($\rightarrow_{\neg A}$) to an assertion of the form $\neg E_{\sim t}(a)$ ($\neg A(a)$). Since a is the root element in the tree structure of $\mathcal{A}_\mathcal{J} \cup \mathcal{A}^-$, such a negative assertion is either initially in $\mathcal{A}^-$ or results from the application of $\rightarrow_{\neg\sqcap}$ to $\neg\widehat{C}(a) \in \mathcal{A}^-$. This last argument also applies to the assertions $\neg\exists r.\widehat{D}(a) \in \mathcal{A}'_a$. Since $\rightarrow_{\neg\sqcap}$ can be applied only once to $\widehat{C}(a)$, this implies that $\#(\mathcal{A}'_a) \leq \#(\mathcal{A}^-)$.
- Taking a as the base case, the same can be shown for the rest of the individuals using induction on the depth[1] of each node in $V_\mathcal{J}$.

Once it is known that $\#(\mathcal{A}'_b) \leq \#(\mathcal{A}^-)$, we can then find the interpretations $\mathcal{I}_b$. Let $B \subseteq \mathsf{Ind}(\mathcal{A}_\mathcal{J})$ be a set such that $b \in B$ *if, and only if,* $\mathcal{A}'_b$ contains at least one assertion of the form $E_{\chi(\sim)t}(b)$. We distinguish two cases:

1. $b \in B$. This means that $\#(\mathcal{A}^-_b) < \#(\mathcal{A}^-)$. Hence, we can apply induction to $\mathcal{A}_b$ to obtain a tree-shaped interpretation $\mathcal{I}_b$ and a homomorphism φ_b from $G_{\mathcal{I}_b}$ to $G_\mathcal{I}$ such that: $\mathcal{I}_b \models \mathcal{A}_b$ and $\varphi_b(b^{\mathcal{I}_b}) = b^\mathcal{I}$.
2. $b \notin B$. Consider the single-pointed interpretation $\mathcal{I}_b = (\{b\}, \cdot^{\mathcal{I}_b})$ that is the restriction of $\mathcal{J}$ to $\{b\}$. The ABox $\mathcal{A}_b$ contains only assertions of the form $\neg\exists r.\widehat{D}(b)$ or assertions from $\mathcal{A}_\mathcal{J}$. Since $\mathcal{J} \models \mathcal{A}_\mathcal{J}$, it is clear that $\mathcal{I}_b \models \mathcal{A}_b$ and φ_b with $\varphi_b(b^{\mathcal{I}_b}) = b^\mathcal{I}$ is a homomorphism from $G_{\mathcal{I}_b}$ to $G_\mathcal{I}$.

To fulfill our intermediate goal it remains to show $\mathcal{I}_b$ also satisfies the rest of the assertions in $\mathcal{A}'(b)$. For assertions of the form $\neg E_{\sim t}(b)$ and $\neg A(b)$, the application of the rules $\rightarrow_{\neg\sim}$ and $\rightarrow_{\neg A}$ ensures that $E_{\chi(\sim)t}(b)$ and $A_{<1}(b)$ are in $\mathcal{A}_b$. Since $\mathcal{I}_b \models \mathcal{A}_b$, $\neg E_{\sim t} \equiv E_{\chi(\sim)t}$ and $\neg A \equiv A_{<1}$, it follows that $\mathcal{I}_b \models \neg E_{\sim t}(b)$ and $\mathcal{I}_b \models \neg A(b)$. The other

[1] The depth of a node in a tree is the length of the path from the root of the tree to the node. The root of the tree has depth 0.

case corresponds to $\neg\widehat{C}(b) \in \mathcal{A}'$ where $\widehat{C}$ is of the form $\widehat{C}_1 \sqcap \ldots \sqcap \widehat{C}_n$. By the application of the rule $\rightarrow_{\neg\sqcap}$ we know that there is some $\widehat{C}_i$ such that $\neg\widehat{C}_i(b) \in \mathcal{A}'$. Since $\neg\widehat{C}_i$ is of one of the previously considered forms, it then holds that $\mathcal{I}_b \models \neg\widehat{C}(b)$.

Altogether, we have shown that $\mathcal{I}_b \models \mathcal{A}'(b)$ for all $b \in \mathsf{Ind}(\mathcal{A}_\mathcal{J})$. Therefore, considering the sets $\Delta^{\mathcal{I}_b}$ pairwise disjoint, we can apply Lemma 5.15 to $\mathcal{A}_\mathcal{J} \cup \mathcal{A}^-$ to obtain an interpretation $\mathcal{K}$ such that $\mathcal{K} \models \mathcal{A}_\mathcal{J} \cup \mathcal{A}^-$. Thus, it remains to show that $\mathcal{K}$ is a model of $\mathcal{A}^+$ as well. Note that $\mathcal{K}$ is the result of extending the base interpretation $\mathcal{J}$ satisfying $\mathcal{A}^+$, by attaching to it the interpretations $\mathcal{I}_b$. This means that ϕ_{id} is also a classical homomorphism from $\widehat{G}(\mathcal{A}^+)$ to $G_\mathcal{K}$ with $\phi_{id}(a) = a^\mathcal{J}$. To see that it is also a τ-homomorphism we observe the following.

- $\mathcal{J} \subseteq \mathcal{K}$.
- The homomorphism φ^* from $G_\mathcal{K}$ to $G_\mathcal{I}$ constructed in Lemma 5.15 is such that, $\varphi^*(b) = \varphi_b(b) = b^\mathcal{I}$ for all $b \in \Delta^\mathcal{J}$. Moreover, $b^\mathcal{I}$ was defined as $\varphi(b^\mathcal{J})$ and $b^\mathcal{J} = b$. Hence, $\varphi^*(b) = \varphi(b)$ for all $b \in \Delta^\mathcal{J}$.
- By construction of $\mathcal{J}$ in Lemma 5.9, we know that $G(\mathcal{A}^+)$ is a subgraph of $G_\mathcal{J}$ and $\varphi(v) = \phi(v)$ for all $v \in V_{\mathcal{A}^+}$. Hence, φ^* is a homomorphism from $G_\mathcal{K}$ to $G_\mathcal{I}$ such that $\varphi^*(v) = \phi(v)$ for all $v \in V_{\mathcal{A}^+}$.

Hence, similar to the way it is done for $\mathcal{I}_0$ and its extension $\mathcal{J}$ in Lemma 5.9, we can use the monotonicity property of *deg* introduced in Lemma 4.11 to show that ϕ_{id} is a τ-homomorphism from $\widehat{G}(\mathcal{A}^+)$ to $G_\mathcal{K}$ with $\phi_{id}(a) = a^\mathcal{J}$. Thus, since $a^\mathcal{J} = a^\mathcal{K}$ we can apply Theorem 3.9 to obtain $\mathcal{K} \models \mathcal{A}^+$.

Next, to see that $\mathcal{K}$ is tree-shaped, note that $\mathcal{J}$ and all the interpretations $\mathcal{I}_b$ are tree-shaped. Consequently, since $G_\mathcal{J}$ corresponds to the structure of $\mathcal{A}_\mathcal{J}$, the construction in Lemma 5.15 yields a tree-shaped interpretation $\mathcal{K}$.

Last, let us look at the size of $\mathcal{K}$. If $b \notin B$ we have $|\Delta^{\mathcal{I}_b}| = 1$, otherwise $\mathcal{I}_b$ is obtained by the application of the induction hypothesis to $\mathcal{A}_b$. Let $\mathcal{J}_b$ be the bounded model for $\mathcal{A}_b^+$ constructed in Lemma 5.9. Then,

$$|\Delta^{\mathcal{I}_b}| \leq |\Delta^{\mathcal{J}_b}| \times \prod_{\neg\widehat{D}(b) \in \mathcal{A}_b^-} \mathsf{s}(\widehat{D}) \tag{A.1}$$

A closer look at $\mathcal{A}_b^+$ shows that it only contains assertions of the form $E_{\chi(\sim)t}(b)$, or $A(b)$ with $A(b) \in \mathcal{A}_\mathcal{J}$ and $A \in \mathsf{N_C}$. Furthermore, it contains exactly one individual name and no role assertions. Hence, the construction of $\mathcal{J}_b$ in Lemma 5.9 yields:

$$|\Delta^{\mathcal{J}_b}| \leq \sum_{E_{\chi(\sim)t}(b) \in \mathcal{A}_b^+} \mathsf{s}(E_{\chi(\sim)t})$$

Now, $\mathsf{s}(E_{\chi(\sim)t}) > 1$ allows to transform this inequality into the following one:

$$|\Delta^{\mathcal{J}_b}| \leq \prod_{E_{\chi(\sim)t}(b) \in \mathcal{A}_b^+} \mathsf{s}(E_{\chi(\sim)t}) \tag{A.2}$$

The ABox $\mathcal{A}_b$ is included in the pre-processing $\mathcal{A}'$ of $\mathcal{A}_{\mathcal{J}} \cup \mathcal{A}^-$. Consequently, for all assertions $\neg\exists r.\widehat{D}(b) \in \mathcal{A}_b^-$, the concept $\widehat{D}$ is a sub-description of a concept $\widehat{C}$ such that $\neg\widehat{C}(a) \in \mathcal{A}^-$. In addition, each threshold concept $E_{\chi(\sim)t} \in A_b^+$ is the result of applying the rule $\rightarrow_{\neg\sim}$ to a concept $\neg E_{\sim t}$. Again, $E_{\sim t}$ has to be a sub-description of a concept $\widehat{C}$ such that $\neg\widehat{C}(a) \in \mathcal{A}^-$. Since $G_{\mathcal{J}}$ is a tree, each concept assertion $\neg\widehat{C}(a) \in \mathcal{A}^-$ contributes with at most one of these concepts to $\mathcal{A}_b'$. We have shown above that $\#(\mathcal{A}_b') \leq \#(\mathcal{A}^-)$. Therefore, the combination of (A.1) and (A.2) yields:

$$|\Delta^{\mathcal{I}_b}| \leq \prod_{\neg\widehat{C}(a) \in \mathcal{A}^-} \mathsf{s}(\widehat{C})$$

Finally, since $|\Delta^{\mathcal{J}}| = |\mathsf{Ind}(\mathcal{A}_{\mathcal{J}})|$, the construction of $\Delta^{\mathcal{K}}$ yields:

$$|\Delta^{\mathcal{K}}| \leq \sum_{b \in \mathsf{Ind}(\mathcal{A}_{\mathcal{J}})} |\Delta^{\mathcal{I}_b}| \leq |\Delta^{\mathcal{J}}| \times p$$

□

Missing proofs of Chapter 6

Lemma A.2. *Let $\mathcal{T}$ be an acyclic $\mathcal{EL}$ TBox in normal form. Then, for all $\alpha \in \mathsf{def}(\mathcal{T})$ the number of sub-descriptions of $u_{\mathcal{T}}(\alpha)$ is at most $\mathsf{s}(\mathcal{T})$.*

Proof. Recall the definition of $\mathsf{sub}(C)$ in Definition 2.1. Let $\mathsf{sub}^*(C) \subseteq \mathsf{sub}(C)$ be the following set:

$$\mathsf{sub}^*(C) := \begin{cases} \{C\} & \text{if } C = \top \text{ or } C \in \mathsf{N_C}, \\ \{C\} \cup \mathsf{sub}^*(C_1) \cup \mathsf{sub}^*(C_2) & \text{if } C \text{ is of the form } C_1 \sqcap C_2, \\ \{\exists r.D\} & \text{if } C \text{ is of the form } \exists r.D. \end{cases}$$

Furthermore, for all $\alpha \doteq C_\alpha \in \mathcal{T}$, let $\rightarrow^+(\alpha)$ denotes the set of defined concepts in $\mathcal{T}$ that α depends on, i.e.:

$$\rightarrow^+(\alpha) := \{\beta \mid \beta \in \mathsf{def}(\mathcal{T}) \text{ and } \alpha \rightarrow^+ \beta\}$$

We prove the following claim about the set $\mathsf{sub}(u_{\mathcal{T}}(\alpha))$:

$$\mathsf{sub}(u_{\mathcal{T}}(\alpha)) \;=\; \mathsf{sub}^*(u_{\mathcal{T}}(C_\alpha)) \cup \bigcup_{\substack{\beta \doteq C_\beta \in \mathcal{T} \\ \beta \in \rightarrow^+(\alpha)}} \mathsf{sub}^*(u_{\mathcal{T}}(C_\beta)) \tag{A.3}$$

The proof is by well-founded induction on the partial order $\preceq$ induced by $\rightarrow^+$ on $\mathsf{def}(\mathcal{T})$. Let $\alpha \doteq C_\alpha \in \mathcal{T}$, due to the normal form of $\mathcal{T}$ the concept C_α has the following structure:

$$P_1 \sqcap \ldots P_q \sqcap \exists r_1.\beta_1 \sqcap \ldots \sqcap \exists r_n.\beta_n$$

The unfolding of α with respect to $\mathcal{T}$ is the following concept description:

$$u_{\mathcal{T}}(\alpha) \;=\; P_1 \sqcap \ldots P_q \sqcap \exists r_1.u_{\mathcal{T}}(\beta_1) \sqcap \ldots \sqcap \exists r_n.u_{\mathcal{T}}(\beta_n)$$

By the definitions of sub and sub^*, we can express the set $\mathsf{sub}(u_{\mathcal{T}}(\alpha))$ as follows:

$$\mathsf{sub}(u_{\mathcal{T}}(\alpha)) \;=\; \mathsf{sub}^*(u_{\mathcal{T}}(C_\alpha)) \;\cup\; \bigcup_{i=1}^{n} \mathsf{sub}(u_{\mathcal{T}}(\beta_i)) \tag{A.4}$$

Now, the application of the induction hypothesis to each β_i $(1 \leq i \leq n)$ yields:

$$\mathsf{sub}(u_{\mathcal{T}}(\beta_i)) \;=\; \mathsf{sub}^*(u_{\mathcal{T}}(C_{\beta_i})) \;\cup \bigcup_{\substack{\beta \doteq C_\beta \in \mathcal{T} \\ \beta \in \to^+(\beta_i)}} \mathsf{sub}^*(u_{\mathcal{T}}(C_\beta))$$

Hence, substituting the previous equality in (A.4) we obtain the following one:

$$\mathsf{sub}(u_{\mathcal{T}}(\alpha)) \;=\; \mathsf{sub}^*(u_{\mathcal{T}}(C_\alpha)) \;\cup\; \bigcup_{i=1}^{n} \left[\mathsf{sub}^*(u_{\mathcal{T}}(C_{\beta_i})) \;\cup \bigcup_{\substack{\beta \doteq C_\beta \in \mathcal{T} \\ \beta \in \to^+(\beta_i)}} \mathsf{sub}^*(u_{\mathcal{T}}(C_\beta))\right]$$

Finally, since $\to^+(\alpha) \;=\; \bigcup_{i=1}^{n} \Big(\{\beta_i\} \cup \to^+(\beta_i)\Big)$, it is clear that the set defined by the big union in the previous equality is equal to the one represented by the big union in (A.3). Thus, our claim in (A.3) is true.

According to the definition of sub^*, for a top-level atom $\exists r_i.\beta_i$ of C_α the set of concepts $\mathsf{sub}^*(\exists r_i.u_{\mathcal{T}}(\beta_i))$ corresponds to $\{\exists r_i.u_{\mathcal{T}}(\beta_i)\}$. Hence, it is not hard to see that for all $\alpha \doteq C_\alpha \in \mathcal{T}$ it holds:

$$|\mathsf{sub}^*(u_{\mathcal{T}}(C_\alpha))| \;\leq\; \mathsf{s}(C_\alpha)$$

Thus, using (A.3) we can conclude that $|\mathsf{sub}(u_{\mathcal{T}}(\alpha))| \;\leq\; \mathsf{s}(\mathcal{T})$ for all $\alpha \in \mathsf{def}(\mathcal{T})$. ☐

Now, since $\mathsf{sub}(E_{\sim t})$ is equal to $\{E_{\sim t}\}$, the previous result also applies to acyclic $\tau\mathcal{EL}(deg)$ TBoxes.

Corollary A.3. *Let $\widehat{\mathcal{T}}$ be an acyclic $\tau\mathcal{EL}(deg)$ TBox in normal form. Then, for all $\alpha \in \mathsf{def}(\widehat{\mathcal{T}})$ it holds:*

$$|\mathsf{sub}(u_{\widehat{\mathcal{T}}}(\alpha))| \;\leq\; \mathsf{s}(\widehat{\mathcal{T}})$$

Missing proofs of Chapter 7

Lemma 7.18. *Let $n > 0$ be a natural number. Then,*

1. *for all sets S of P_n-assignments that are canonical for P_n, there exists $D_S \in \mathfrak{D}_n$ such that S and D_S are corresponding, and*

2. *for all $D \in \mathfrak{D}_n$, exists a set S_D of P_n-assignments that is canonical for P_n such that S_D and D are corresponding.*

Proof. We prove the claim by induction on the number n. We start by considering two base cases:

- $n = 1$. The prefix P_1 corresponds to $\exists x_1$. Therefore, there are only two sets of P_1-assignments that are canonical for P_1. Namely, $S_{true} = \{\{\mathfrak{t}(x_1) = true\}\}$ and $S_{false} = \{\{\mathfrak{t}(x_1) = false\}\}$. Now, the string D_1 is of the form $\exists r.X_1^0$ (recall its definition in (7.2)). Hence, the instances $\theta_{true}[D_1]$ and $\theta_{false}[D_1]$ where $\theta_{true}(X_1^0) = A$ and $\theta_{false}(X_1^0) = \top$, are *corresponding* concepts in $\mathfrak{D}_1$ for S_{true} and S_{false}, respectively.

- $n = 2$. The prefix P_2 is of the form $\exists x_1.\forall x_2$. In this case there are also two sets of P_2-assignments that are canonical for P_2, but they are of the following form:

$$S_{true} = \{\{\mathfrak{t}_1(x_1) = true, \mathfrak{t}_1(x_2) = false\}, \{\mathfrak{t}_2(x_1) = true, \mathfrak{t}_2(x_2) = true\}\}$$

$$S_{false} = \{\{\mathfrak{t}_1(x_1) = false, \mathfrak{t}_1(x_2) = false\}, \{\mathfrak{t}_2(x_1) = false, \mathfrak{t}_2(x_2) = true\}\}$$

  The string D_2 is of the form $\exists r.(X_1^0 \sqcap \exists r.A \sqcap \exists s.\top)$. Thus, $\theta_{true}[D_2]$ and $\theta_{false}[D_2]$ are also *corresponding* concepts in $\mathfrak{D}_2$ for S_{true} and S_{false}, respectively.

Notice, that in both cases the selected concepts from $\mathcal{D}_1$ and $\mathcal{D}_2$ are actually the only concepts contained in those sets. Therefore, the statement *2.)* also holds for both base cases.

Induction Step. Let us assume that the claim holds for all natural numbers smaller than n. We show that it also holds for all $n > 2$.

1.) Let S be a set of P_n-assignments that is canonical for P_n. Since P_n is of the form $\exists x_1.P'$, by definition of canonical we have that the set $S' = \{\mathfrak{t}_{|\{x_2,\ldots,x_n\}} \mid \mathfrak{t} \in S\}$ is canonical for P'. Moreover, P' is of the form $\forall x_2.P''$, and P'' is not empty because $n > 2$. Hence, there exist two sets S_{true} and S_{false} that are canonicals for P'' of the following form:

$$S_{true} := \{\mathfrak{t}_{|\{x_3,\ldots,x_n\}} \mid \mathfrak{t} \in S' \text{ and } \mathfrak{t}(x_2) = true\}$$

$$S_{false} := \{\mathfrak{t}_{|\{x_3,\ldots,x_n\}} \mid \mathfrak{t} \in S' \text{ and } \mathfrak{t}(x_2) = false\}$$

Note that P'' is actually the prefix P_{n-2} when shifting the indexes of the variables $\{x_3, \ldots, x_n\}$ to $\{x_1, \ldots, x_{n-2}\}$. Therefore, we can apply the induction hypothesis to obtain two concept descriptions D_{S_1} and D_{S_2} in $\mathfrak{D}_{n-2}$ such that they are *corresponding* concepts for S_{true} and S_{false}, respectively. We now use these two concepts to construct a *corresponding* concept for S. Let us start by observing the following facts about D_{S_1} and D_{S_2}.

- There are mappings $\theta_1, \theta_2 \in \mathfrak{X}_{n-2}$ such that $\theta_1[D_{n-2}] = D_{S_1}$ and $\theta_2[D_{n-2}] = D_{S_2}$.

- For all i such that $1 \leq 2i+1 \leq n-2$, D_{n-2} contains 2^i variables $X_{2i+1}^0, \ldots, X_{2i+1}^{2^i-1}$.

- D_{n-2} can be transformed into the strings $\exists r.D_3^0$ and $\exists r.D_3^1$ that are used to construct the string D_n, by renaming its variables. We define two renamings r_1 and r_2 as follows. For all $i \geq 0$ and all j such that $1 \leq 2i+1 \leq n-2$ and $0 \leq j < 2^i$:

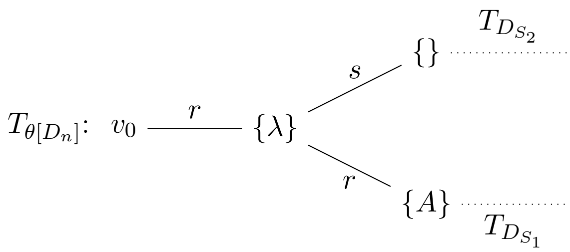
$$r_1(X_{2i+1}^j) := X_{2i+3}^j \quad \text{and} \quad r_2(X_{2i+1}^j) := X_{2i+3}^{2^i+j}$$

  It is not hard to see that applying r_1 (r_2) to D_{n-2} yields the string $\exists r.D_3^0$ ($\exists r.D_3^1$).

Based on r_1 and r_2, we define the mapping $\theta : X_n \to \{\top, A\}$ as:

$$\theta := r_1(\theta_1) \cup r_2(\theta_2) \cup \{(X_1^0, \lambda)\}$$

where $r_1(\theta_1)$ and $r_2(\theta_2)$ stands for the renaming of the variables in the domain sets of θ_1 and θ_2, and $\lambda = A$ if x_1 is mapped to *true* or $\top$ otherwise (recall that x_1 is mapped to the same truth value by all the assignments in S). Hence, $\theta \in \mathfrak{X}_n$ and $\theta[D_n]$ has the following description tree:



We now show that $\theta[D_n]$ and S are *corresponding*.

- Let $\mathfrak{t}$ by an assignment in S. If $\mathfrak{t}(x_2) = true$, then the restriction $\mathfrak{t}_{true} := \mathfrak{t}_{|\{x_3,\ldots,x_n\}}$ of $\mathfrak{t}$ is obviously an assignment in S_{true}. By induction hypothesis, there is a corresponding path of the form $\{\}r\pi$ in $T_{D_{S_1}}$ for $\mathfrak{t}_{true}$. Since $D_{S_1} = \theta_1[D_{n-2}]$, by construction of θ the following is a path in $T_{\theta[D_n]}$:

  $$v_0 : \{\} \xrightarrow{\;r\;} \{\lambda\} \xrightarrow{\;r\;} \{A\} \xrightarrow{\;r\;} \pi$$

  Hence, taking into account the way λ has been selected and the fact that $\mathfrak{t}(x_2) = true$, this is clearly a *corresponding* path for $\mathfrak{t}$. The case where $\mathfrak{t}(x_2) = false$ can be handled symmetrically.

- Conversely, let π be a path in $T_{\theta[D_n]}$. Again, we can consider one of two symmetric cases. For example,

  $$v_0 : \{\} \xrightarrow{\;r\;} \{\lambda\} \xrightarrow{\;s\;} \{\} \xrightarrow{\;r\;} \pi'$$

  By construction of $\theta[D_n]$, $\{\}r\pi'$ is a path in $T_{D_{S_2}}$. Again, by induction hypothesis there is an assignment $\mathfrak{t}' \in S_{false}$ such that $\{\}r\pi'$ and $\mathfrak{t}'$ are *corresponding*. Let t be the truth value of x_1 in S, we build a P_n-assignment $\mathfrak{t}$ as $\mathfrak{t}' \cup \{(x_2, false), (x_1, t)\}$. Obviously, $\mathfrak{t} \in S$, and moreover $\mathfrak{t}$ and π are *corresponding*.

Thus, we have shown that S and $\theta[D_n]$ are *corresponding*, and consequently our first claim is true. Regarding our second claim, a similar line of reasoning as the one just used can be applied. Basically, we start with a concept $\theta[D_n] \in \mathfrak{D}_n$, the mapping θ yields two mappings $\theta_1, \theta_2 \in \mathfrak{X}_{n-2}$, and then the induction hypothesis can be applied to obtain two P_{n-2}-assignments S_{true} and S_{false} with similar properties as the ones discussed above. From them, one can obtain a P_n-assignment S such that it is canonical for P_n, and S and $\theta[D_n]$ are *corresponding*. $\square$

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Scientific Career

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List of Publications

- Franz Baader, Gerhard Brewka, and Oliver Fernández Gil. Adding Threshold Concepts to the Description Logic $\mathcal{EL}$. *In Frontiers of Combining Systems - 10th International Symposium, FroCoS 2015, Wroclaw, Poland, September 21-24, 2015. Proceedings,* pages 33–48, 2015.

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- Oliver Fernández Gil. On the Non-Monotonic Description Logic $\mathcal{ALC}+\mathrm{T}_{\mathsf{min}}$. *In Proceedings of the 15th International Workshop on Non-Monotonic Reasoning, Viena, Austria, July 17-19,* 2014.

- Claudia Carapelle, Shiguang Feng, Oliver Fernández Gil, and Karin Quaas. Ehrenfeucht-Fraisse games for TPTL and MTL over non-monotonic data words. *In Proceedings 14th International Conference on Automata and Formal Languages, AFL'14, Szeged, Hungary, May 27-29, pages 174–187,* 2014.

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- Franz Baader, Oliver Fernández Gil, and Barbara Morawska. Hybrid unification in the Description Logic $\mathcal{EL}$. *In Proceedings of the 9th International Symposium, FroCoS, Nancy, France, September 18-20, pages 295–310,* 2013.

- Franz Baader, Oliver Fernández Gil, and Barbara Morawska. Hybrid $\mathcal{EL}$-unification is NP-complete. *In Proceedings of the 26th International Workshop on Description Logics, Ulm, Germany, July 23-26, pages 29–40,* 2013.

- Franz Baader, Oliver Fernández Gil, and Barbara Morawska. Hybrid unification in the Description Logic $\mathcal{EL}$.*In UNIF@RTA/TLCA, pages 8–12,* 2013.

List of Talks

24.09.2015 FroCoS 2015 - Wroclaw, Poland.
Talk: *Adding Threshold Concepts to the Description Logic* $\mathcal{EL}$.

09.06.2015 DL 2015 - Athens, Greece.
Talk: *Adding Threshold Concepts to the Description Logic* $\mathcal{EL}$.

27.02.2015 Frontiers of Formal Methods 2015 - Aachen, Germany.
Talk: *Threshold Concepts in a Lightweight Description Logic.*

18.07.2014 NMR 2014 - Vienna, Austria
Talk: *On the Non-Monotonic Description Logic* $\mathcal{ALC}+T_{\mathsf{min}}$

20.09.2013 FroCoS 2013 - Nancy, France.
Talk: *Hybrid unification in the Description Logic* $\mathcal{EL}$.

23.07.2013 DL 2013 - Ulm, Germany.
Talk: *Hybrid* $\mathcal{EL}$*-unification is NP-complete.*

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Leipzig, 29. Januar 2016

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