# Algebraic and Topological Properties of Unitary Groups of $\mathrm{II}_1$ Factors

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# Abstract

We are concerned with several group theoretical questions in the context of unitary groups of functional analytic type. Our main focus lies on unitary groups of $\mathrm{II}_1$ factors. $\mathrm{II}_1$ factors are special von Neumann algebras, which are by reduction theory of von Neumann algebras often and by many concerns the most interesting case of those mathematical objects.

For any (noncommutative) group $G$ one can ask under which condition an element $g \in G$ is a product of conjugates of another element $h \in G$. We are able to provide a necessary and sufficient criterion for

(i) the projective unitary group $\mathrm{PU}(n)$ of the $n \times n$ matrix algebra over $\mathbb{C}$;

(ii) the connected component of the identity of the projective unitary group of the Calkin algebra;

(iii) the projective unitary group $\mathrm{PU}(\mathcal{M})$ of a $\mathrm{II}_1$ factor $\mathcal{M}$.

Our criteria are formulated in terms of so called projective generalized $s$-numbers. It is known that one can generalize the classical $s$-numbers for compact operators to the case of semifinite factors $\mathcal{M}$ with faithful normal semifinite trace $\tau$ by setting

$$\mu_t(x) := \inf_{p \in \mathrm{Proj}(\mathcal{M}),\ \tau(1-p) \le t} \|xp\| \qquad \text{for } x \in \mathcal{M},\ t \ge 0.$$

Using this we define the $t$-th projective generalized $s$-number of $x \in \mathcal{M}$ by

$$\ell_t(x) := \inf_{\lambda \in \mathrm{U}(1)} \mu_t(1 - \lambda x) \qquad \text{for } t \ge 0.$$

A typical criterion on products of conjugates in this thesis then reads:
Let $u, v \in G$. If $\ell_0(u) \le m\ell_t(v)$ for all $t \in [0, s]$ and for some $m \in \mathbb{N}$, then

$$u \in (v^G \cup v^{-G})^{cm\lceil 1/s \rceil},$$

where $c \in \mathbb{N}$ is a constant independent of $m, s, u$ and $v$. On the other hand, if $u$ is a product of $k$ conjugates of $v$, then $\ell_{kt}(u) \le k\ell_t(v)$ for all $t \ge 0$.

Having such a criterion, one may further ask if a given element $u \in G$ is a uniform normal

generator for $G$, i.e. if there exists $k \in \mathbb{N}$ such that

$$G = (u^G \cup u^{-G})^k.$$

Analogously we say that an element $u$ in a *topological* group $G$ is a topological uniform normal generator if there exists $k \in \mathbb{N}$ such that

$$G = \overline{(u^G \cup u^{-G})^k}.$$

It turns out that the unitary group $\mathrm{U}(\mathcal{H})_{\mathcal{K}(\mathcal{H})}$ of compact perturbations from the identity, despite being topologically simple, does not have any topological uniform normal generator. In contrast, the projective unitary group $\mathrm{PU}(\mathcal{H})$ on a separable Hilbert space is not simple but does have uniform normal generators (e.g. symmetries with two infinite-dimensional eigenspaces by a modification of a theorem of Halmos and Kakutani).

If every nontrivial element in $G$ is a uniform normal generator, then we say that $G$ has the bounded normal generation property, or property (BNG). Using our result on products of conjugates in $\mathrm{PU}(\mathcal{H})$ we deduce that the connected component of the identity of the projective unitary group of the Calkin algebra has property (BNG). A modification of a theorem of Broise shows that every symmetry of trace 0 is a uniform normal generator for the projective unitary group $\mathrm{PU}(\mathcal{M})$ of a (separable) $\mathrm{II}_1$ factor. As a vast generalization of this result we show that $\mathrm{PU}(\mathcal{M})$ has property (BNG) for any separable $\mathrm{II}_1$ factor.

A group property, which recently has drawn a lot of attention of several experts in descriptive set theory, is the so called automatic continuity property, or property (AC). Automatic continuity comes out of a question of Cauchy, asking whether every endomorphism of the additive group of the reals is continuous. We say that a topological group $G$ has property (AC) if every homomorphism from $G$ to any separable topological group is continuous. Motivated by a recent result of Tsankov showing that $\mathrm{U}(\mathcal{H})$, endowed with the strong operator topology, has property (AC), we attack this question for projective unitary groups $\mathrm{PU}(\mathcal{M})$ of separable $\mathrm{II}_1$ factors. We are able to show that any homomorphism from the groups

- $\mathrm{PU}(n)$, $n \in \mathbb{N}$, endowed with the uniform topology,

- $\mathrm{PU}(\mathcal{M})$, endowed with the strong operator topology,

into any separable topological group with bi-invariant metric is continuous. Our proof uses our results on products of conjugates for $\mathrm{PU}(\mathcal{M})$. As an easy application we obtain the uniqueness of the bi-invariant Polish group topology on these groups. Our techniques allow us to further show that $\mathrm{PU}(\mathcal{M})$ has a unique Polish group topology - this has previously been unknown even in the hyperfinite case. It is worthwile mentioning that $\mathrm{PU}(n)$, $n \in \mathbb{N}$, does not have property

(AC).

A group is called extremely amenable if any continuous action on a compact space has a fixed point. This is a very rigid property which can never be observed in the universe of locally compact groups by a theorem of Veech. Gromov and Milman discovered that the unitary group $\mathrm{U}(\mathcal{H})$ of a separable infinite-dimensional Hilbert space is extremely amenable, when endowed with the strong operator topology. In modern formulation the proof of this can be boiled down to showing first that $\mathrm{U}(\mathcal{H})$ is a Lévy group and second that every Lévy group is extremely amenable. The first step in Gromov and Milman's work is based on growth of the (infimum over unit tangent vectors of the) Ricci curvature of $\mathrm{SU}(n)$ with $n$. We present an alternative and elementary proof of the first step by estimating concentration inequalities on the groups $\mathrm{U}(n)$ of unitary $n \times n$ matrices. Our proof yields extreme amenability of the unitary group of the hyperfinite $\mathrm{II}_1$ factor with the strong operator topology, which was first observed, building on results of Gromov and Milman, by Giordano and Pestov.

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# Contents

# 1 Introduction

The topic of this PhD thesis is about algebraic and topological properties of unitary groups of $\mathrm{II}_1$ factors. Before we come to the specific algebraic and topological properties that we prove in this thesis, we explain why one should generally care about these groups. Let us first encourage the readers interest in type $\mathrm{II}_1$ factors.

Motivated by the study of group representation theory, ergodic theory and quantum mechanics, John von Neumann introduced in [Ne 30] the so called rings of operators on a Hilbert space. These objects are nowadays called von Neumann algebras. Large parts of the theory of von Neumann algebras were developed in a series of papers of Murray and von Neumann, see [MN 36], [MN 37], [Ne 40], [MN 43], [Ne 43], [Ne 49]. By reduction theory the study of von Neumann algebras can be reduced to the study of so called factors, which are von Neumann algebras with trivial center. There are three types of factors - type I factors (these are algebras of bounded operators on a finite- or infinite-dimensional Hilbert space), type II factors and type III factors. The study of type III factors can be reduced to the study of type II factors by the Tomita-Takesaki theory. The class of type II factors can be split into $\mathrm{II}_1$ factors and $\mathrm{II}_\infty$ factors, the latter being tensor products of a type I and a type $\mathrm{II}_1$ factor. Thus the study of von Neumann algebras can basically be reduced to the study of $\mathrm{II}_1$ factors.

One of the major open problems in the theory of operator algebras is the so called Connes' embedding problem, raised in his ingenious article [Co 76]. It asks whether every $\mathrm{II}_1$ factor can be embedded into an ultrapower of the hyperfinite $\mathrm{II}_1$ factor. It is worthwile mentioning that this problem has equivalent reformulations in many different areas of mathematics, see e.g. the recent accounts [Oz 13] and [CL 13]. An open question connected to Connes' embedding problem is if every countable discrete group is hyperlinear (i.e. its group von Neumann algebra embeds into an ultrapower of the hyperfinite $\mathrm{II}_1$ factor).

Another famous open problem is the isomorphism problem for free group factors by Murray and von Neumann. It asks if the group von Neumann algebras of nonabelian free groups (which are factors of type $\mathrm{II}_1$) of diverse rank are isomorphic. This led Voiculescu to the discovery of free probability theory around 1985.

As every von Neumann algebra is linearly spanned by its unitary group, the above explains why the study unitary groups of $\mathrm{II}_1$ factors is crucial to understand von Neumann algebras.

However, the unitary group $\mathrm{U}(\mathcal{M})$ of $\mathrm{II}_1$ factor $\mathcal{M}$ is an interesting object to study not only from an operator algebraic point of view. For example, $\mathrm{U}(\mathcal{M})$ is a non-locally compact Polish group in the strong operator topology, which makes it an interesting object in descriptive set theory. Even more, the strong operator topology on $\mathrm{U}(\mathcal{M})$ is induced from a bi-invariant metric. A problem of Popa (coming from the study of cocycle superrigidity theory) asks for a necessary and sufficient criterion for a Polish group to be isomorphic as a topological group to a closed subgroup of the unitary group of some $\mathrm{II}_1$ factor. Further motivation for (and account on) Popa's problem, due to Ando and Matsuzawa [AM 12], can be found in the theory of infinite-dimensional Lie algebras associated with unitary groups of $\mathrm{II}_1$ factors.

Another reason for studying unitary groups of $\mathrm{II}_1$ factors can be found in representation theory. In some cases the representations of a group on the whole unitary group on a Hilbert space cannot be classified while representations on the unitary group of a $\mathrm{II}_1$ factor can - cf. [PT 13] and references therein.

Let us explain the main questions that we explore in this thesis. In order to keep this introduction at reasonable length we refer the reader for detailed introductions with historical background and precise statements of our results to Chapters 3, 4 and 5.

In Chapter 3 we are concerned with the property of extreme amenability. A topological group is extremely amenable if every continuous action on a compact space has a fixed point. This a very rigid property only occuring in the universe of non-locally compact topological groups. A milestone was set by Gromov and Milman [GM 83], who proved the extreme amenability of the unitary group on a separable infinite-dimensional Hilbert space. Building on the seminal work of Gromov and Milman, Giordano and Pestov [GP 06] have shown (amongst other results in this area of research) that the unitary group of the hyperfinite $\mathrm{II}_1$ factor, endowed with the strong operator topology, is extremely amenable. We present a more elementary and natural proof of this result in Chapter 3. Using estimates on the Ricci curvature of $\mathrm{SU}(n)$, Gromov and Milman have shown that $(\mathrm{SU}(n), \mu_n, d_n)_{n\in\mathbb{N}}$ forms a Lévy family, where $\mu_n$ denotes the Haar measure and $d_n$ the unnormalized Hilbert-Schmidt metric. We instead focus on estimates of concentration functions for $\mathrm{U}(n)$ with regard to the normalized trace metric $d_{1,n}$ to obtain that $(\mathrm{U}(n), \mu_n, d_{1,n})_{n\in\mathbb{N}}$ forms a Lévy family.

It is a fundamental question in group theory to ask under which conditions one element of a given group $G$ is the product of conjugates of another element of $G$. We are interested in this question in the context of unitary groups of functional analytic type, see Chapter 4. Particular examples are finite-dimensional projective unitary groups, the projective unitary group of the connected component of the identity of the Calkin algebra and the projective unitary group of a $\mathrm{II}_1$ factor. In all of these cases we provide necessary and sufficient criteria for an element to be a product of conjugates of another element. Let us call such a result a (PC)-criterion for short. However, in some cases our criterion allows only to decide whether an element in a topological group $G$ is in a certain closure of some power of the conjugacy class of some $g \in G$ and of $g^{-1}$.

Finding such a result allows us to ask further if the conjugacy class of an element and of its inverse generate the whole group in finitely many steps. We call such an element a uniform normal generator. If every nontrivial element of a group $G$ is a uniform normal generator we say that the group has the bounded normal generation property, or property (BNG) for short.

Using a Baire category argument it is not hard to show property (BNG) for compact simple groups. But to quantify the number of steps one needs to generate the whole group with a conjugacy class, i.e. to find a normal generation function in our terminology (see Definition 4.7), is hard work even in the case of finite simple groups - it was done by Liebeck and Shalev in the main theorem in their seminal article [LS 01]. The normal generation they found is basically given by $\log\left|g^G\right| / \log|G|$ for $g \in G \setminus \{1\}$. In the context of compact connected simple Lie groups, Nikolov and Segal [NS 12] managed to obtain quantitative estimates, their normal generation function is given by averaging over angles in maximal tori. Using some ideas of Stolz and Thom [ST 14] we prove a rank-independent criterion for an element in the projective unitary group $\mathrm{PU}(n)$ of finite rank to be a product of conjugates of another. In general, rank-independent versions of finite-dimensional results give hope for infinite-dimensional analogues.

In contrast to compact simple groups it is not at all clear for a non-locally compact simple groups to have property (BNG) (even qualitatively). There are examples of simple groups which do not have property (BNG), e.g., the group of finitely supported permutations on $\mathbb{N}$. In Section 4.5 we prove a (PC)-criterion for $\mathrm{PU}(n)$ in terms of what we call projective generalized $s$-numbers for $\mathrm{PU}(n)$, the reason being that this setting is suitable for a generalization to semifinite von Neumann algebras. With the help of this result we can prove topological (PC)-criteria. This is done in Section 4.7 for $\mathrm{II}_1$ factors. The topological (PC)-criterion for unitary groups of $\mathrm{II}_1$ factors allows us to conclude that the projective unitary group of any $\mathrm{II}_1$ has the topological bounded normal generation property (in the strong operator topology).

We are able to provide algebraic (PC)-criteria for

- the projective unitary group $\mathrm{PU}(n)$, where $n \in \mathbb{N}$;
- the connected component $\mathrm{PU}_1(\mathcal{C})$ of the identity of the projective unitary group of the Calkin algebra in Section 4.6;
- the projective unitary group of a separable $\mathrm{II}_1$ factor in Section 4.8.

We also have a topological (PC)-criterion for the unitary group $\mathrm{U}(\mathcal{H})$ on a separable infinite-dimensional Hilbert space $\mathcal{H}$ for elements that are not compact perturbations of the identity. The corresponding statements are roughly of the following form:

> An element $u \in G$ is a product of conjugates of $v \in G$ and $v^{-1}$ if the graph of the projective generalized $s$-numbers of $v$ covers the box of height determined by the projective operator norm distance from 1 and length determined by the rank of the group.
> Conversely, if $u$ is a product of conjugates of $v$, then the graph of the projective generalized $s$-numbers of $u$ can be covered by that of $v$ by finite expansion along the axes.

It is worthwhile noting that the unitary group $\mathrm{U}(\mathcal{H})_{\mathcal{K}(\mathcal{H})}$ of compact perturbations of the identity, despite being topologically simple, does not possess any topological uniform normal generator in the uniform topology. In contrast, the group $\mathrm{U}(\mathcal{H})$ has uniform normal generators, but is not simple since it contains the normal subgroup $\mathrm{U}(\mathcal{H})_{\mathcal{K}(\mathcal{H})}$. A well-known theorem of Halmos and Kakutani [HK 58] states that every unitary operator on $\mathcal{H}$ is the product of four symmetries (having infinite-dimensional eigenspaces corresponding to 1 and $-1$). Since those symmetries are conjugate, their result can be reformulated into: every symmetry $s \in \mathrm{U}(\mathcal{H})$ with infinite-dimensional eigenspaces corresponding to 1 and $-1$ is a 4-uniform normal generator for $\mathrm{U}(\mathcal{H})$. From the algebraic (PC)-criterion for $\mathrm{PU}_1(\mathcal{C})$ we deduce that this group has property (BNG).

In Section 4.3 we obtain the following modification of Broise's result [Br 67, Theorem 1]: Every symmetry $s$ in the projective unitary group $\mathrm{PU}(\mathcal{M})$ of a $\mathrm{II}_1$ factor $\mathcal{M}$ is a 32-uniform normal generator. Using our algebraic (PC)-criterion for unitary groups of $\mathrm{II}_1$ factors together with the just mentioned result we can show that the projective unitary group $\mathrm{PU}(\mathcal{M})$ has property (BNG). Our work on the bounded normal generation property for projective unitary groups of $\mathrm{II}_1$ factors is applied in Chapter 5, whose content we will describe now.

Chapter 5 is concerned with the phenomenon of automatic continuity, which goes back to the work of A. L. Cauchy. Cauchy analyzed the question whether every endomorphism of the additive group of the reals is continuous. However, using the axiom of choice one can show that there are discontinuous homomorphisms $\mathbb{R} \to \mathbb{R}$. Cauchy's problem drew a lot of attention around the beginning of the 20th century - for example, M. Fréchet, S. Banach, W. Sierpiński and H. Steinhaus published articles around 1920. In the 1930's A. Weil extended some results of Steinhaus to all locally compact groups.

A very general form of Cauchy's question reads:

> When is a homomorphism $\pi : G \to H$ between separable topological groups continuous?

We say that a topological group $G$ has the automatic continuity property, or property (AC), if every homomorphism from $G$ to any separable topological group is continuous. This is not at all a trivial property, for example, some matrix groups such as $\mathrm{SO}(3, \mathbb{R})$ embed discontinuously into the group $\mathrm{S}_\infty$ of all bijections on $\mathbb{N}$ by the work of R. R. Kallman [Ka 00] and S. Thomas [Th 99], cf. [Ro 09b, Example 1.5] in Rosendal's survey article on automatic continuity.

One of the first general automatic continuity theorems was found around 1950 by Pettis [Pe 50]. He proved that any Baire measurable homormophism between Polish groups is continuous. Since then important contributions were made by J. P. R. Christensen [Ch 71], R. M. Dudley [Du 61], A. S. Kechris and C. Rosendal [KR 07], C. Rosendal and S. Solecki [RS 07], I. Ben Yaacov, A. Berenstein and J. Melleray [BYBM

13], T. Tsankov [Ts 13], and M. Sabok [Sa 13], to name a few. Kechris and Rosendal [KR 07] proved that groups with ample generics (that is, for each $n \in \mathbb{N}$ there is a comeager orbit for the diagonal conjugacy action of $G$ on $G^n$) have the automatic continuity property. A countably syndetic set for a group $G$ is a subset of $G$ spanning $G$ with countably many left-translates. A group is called Steinhaus if some fixed power of every countably syndetic set contains an open neighborhood of the identity. Rosendal and Solecki [RS 07] found that Steinhaus groups have the automatic continuity property. Steinhaus groups are the largest known class of groups to have this property. Of particular interest to us is the recent [Ts 13, Theorem 1], stating that the unitary group on a separable infinite-dimensional Hilbert space, endowed with the strong operator topology, has the automatic continuity property (by showing that it is Steinhaus). Many of the results from Kechris, Rosendal, Solecki, Tsankov and Sabok crucially depend on the existence of comeager conjugacy classes. In some cases, e.g. $\mathrm{U}(\mathcal{H})$ with the strong operator topology, the group itself does not have comeager conjugacy classes, but can be compared to a group (e.g. by a homeomorphic embedding) having comeager conjugacy classes (or a weaker form, so called ample topometric generics, see [BYBM 13]).

Motivated by the automatic continuity property of $\mathrm{U}(\mathcal{H})$ [Ts 13, Theorem 1], we analyze unitary groups $\mathrm{U}(\mathcal{M})$ of separable $\mathrm{II}_1$ factors $\mathcal{M}$, endowed with the strong operator topology, with respect to property (AC). A difficulty to handle this case stems from the fact that every conjugacy class is meager. Our approach uses instead our results on products of conjugates for $\mathrm{PU}(\mathcal{M})$. However, we are forced by our approach in showing property (AC) to restrict our attention to conjugacy-invariant countably syndetic sets and separable SIN target groups (i.e. separable topological groups such that the topology is induced from a bi-invariant metric). We call this phenomenon invariant automatic continuity. In particular, we show that these groups are invariant Steinhaus in the sense that we need the additional condition of conjugacy-invariance on countably syndetic sets. We are able to show that the groups

- $\mathrm{PU}(n)$ and $\mathrm{SU}(n)$, $2 \leq n \in \mathbb{N}$, with the uniform topology,
- $\mathrm{U}(\mathcal{M})$ and $\mathrm{PU}(\mathcal{M})$, $\mathcal{M}$ a separable $\mathrm{II}_1$ factor, endowed with the strong operator topology,

have the invariant automatic continuity property. This allows us to conclude the uniqueness of the Polish SIN topology on these groups. In particular this shows that

$\mathrm{PU}(n)$ is a group with invariant automatic continuity but not automatic continuity. In the $\mathrm{II}_1$ factor case it is open if it has property (AC).

Combining techniques from our proof of invariant automatic continuity with a result of Gartside and Pejić [GP 08] we are able to prove the uniqueness of the Polish group topology for the projective unitary group of a separable $\mathrm{II}_1$ factor. This was previously unknown even in the hyperfinite case.

## Structure of the thesis

The introduction in Chapter 1 is followed by a survey on the necessary preliminaries in Chapter 2. In Chapter 3 we treat the topic of extreme amenability and provide an alternative proof to the original one by Giordano and Pestov. The main part of this thesis is formed by Chapter 4, where we prove for several unitary groups $G$ of functional analytic type necessary and sufficient criteria for an element of $G$ to be a product of conjugates of another element in $G$. Moreover, we are concerned with (topological) uniform normal generators and the (topological) bounded normal generation property. Finally, in Chapter 5, applying some results of Chapter 4, we prove invariant automatic continuity for $\mathrm{PU}(n)$ and $\mathrm{SU}(n)$, $2 \leq n \in \mathbb{N}$, endowed with the uniform topology, and the projective unitary group of a separable $\mathrm{II}_1$ factor, endowed with the strong operator topology. This again is applied to prove the uniqueness of the Polish SIN topology on $\mathrm{PU}(\mathcal{M})$. We conclude our thesis with an outlook on open problems either arising from Chapters 4 and 5 or being closely linked, see Chapter 6. At the end of the thesis the reader will find a list of symbols, a detailed bibliography and an index of notation.

## Notation

Some remarks on notation. We try to use standard notation whenever possible. We denote the natural numbers $\{1, 2, \ldots\}$, integers, rationals, reals and complex numbers by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively. The symbol $\mathcal{H}$ always denotes a (complex) Hilbert space, which is usually assumed to be separable. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$. The operator norm on $\mathcal{B}(\mathcal{H})$ is denoted by $\|\cdot\|$. $\mathcal{M}$ always denotes a von Neumann algebra, often assumed to be semifinite or more specifically a $\mathrm{II}_1$ factor. A semifinite von Neumann algebra $\mathcal{M}$ is always equipped with a faithful normal semifinite trace $\tau$. If $\mathcal{M}$ is a $\mathrm{II}_1$ factor, it is always assumed to be the unique normalized faithful normal finite trace. The unitary group of a von Neumann algebra $\mathcal{M}$ is written as $\mathrm{U}(\mathcal{M})$, the projective unitary group of $\mathcal{M}$ as $\mathrm{PU}(\mathcal{M})$. The conjugacy class of an element $g$ in a group $G$ is denoted by $g^G$ and the conjugacy class of $g^{-1}$ by $g^{-G}$. For a metric space $(X, d)$ we define the $\varepsilon$-neighborhood of a subset $A \subseteq X$ by

$$(A)_\varepsilon \coloneqq \{x \in X \mid d(x,a) \leq \varepsilon \text{ for some } a \in A\} .$$

If we want to specify the metric $d$ (or norm $\|\cdot\|$) used we write $(A)_{\varepsilon,d}$ (or $(A)_{\varepsilon,\|\cdot\|}$). We will often use the ceiling function for $x \in \mathbb{R}$:

$$\lceil x \rceil \coloneqq \min \{n \in \mathbb{Z} \mid n \geq x\} .$$

# 2 Preliminaries

This chapter represents a short survey on some definitions and facts from topological group theory and operator algebra theory that will be freely used in the remaining chapters.

## 2.1 Topological groups

We assume some familiarity with general topology. Nonetheless we will repeat a few important definitions and results from [Bo 89, Chapters II and III], [Ga 09, Chapters 1 and 2], [Ke 95, Chapter I] and [Pe 06, Chapter 1].

Let us start right out with a crucial definition.

**Definition 2.1.** A **topological group** is a group $G$ with a topology such that the map $(g, h) \mapsto gh^{-1}$ of $G \times G$ into $G$ is continuous.

For every $u \in G$, the **left translation** $g \mapsto ug$ (respectively the **right translation**) is a homeomorphism. Moreover, the mappings $g \mapsto ugv$ with $u, v$ running through $G$ form a group of homeomorphisms. The mappings $g \mapsto ugu^{-1}$ with $u$ running through $G$ form a subgroup.

Clearly every group is a topological group when endowed with the discrete topology. This is not an interesting topology for the purpose of our questions - we will be concerned with more sophisticated topologies (e.g. the strong operator topology and the uniform topology on the unitary group of a Hilbert space) introduced in Section 2.2.

A topological space $X$ is called **Hausdorff** if for any given distinct points $x, y \in X$, there are open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. $X$ is called **homogeneous** if for every $x, y \in X$, there exists a homeomorphism $f$ of $X$ to itself such that $f(x) = y$. Every topological group $G$ is homogeneous, since, given $g, h \in G$, the map $x \mapsto hg^{-1}x$ is a homeomorhism from $G$ to $G$ mapping $g$ to $h$. For a topological group, being Hausdorff is equivalent to $\{e\}$ being a closed set in $G$, by homogeneity.

Recall that in a topological space $X$, a **fundamental system of neighborhoods** of a point $x \in X$ (respectively a subset $U \subseteq X$) is a set of neighborhoods $\mathcal{N}$ of $x$ (respectively $U$) such that for each neighborhood $V$ of $x$ (respectively $U$) there is a neighborhood $W \in \mathcal{N}$ such that $W \subseteq V$. A **basis $\mathcal{B}$ of the topology** can be characterized as a set of open subsets of $X$ such that for every $x \in X$ the set $\{V \in \mathcal{B} \mid x \in V\}$ is a fundamental system of neighborhoods of $x$.

A topological group $G$ is first countable if and only if the identity element 1 of $G$ has a countable neighborhood base. Every topological group $G$ has an open base at the identity consisting of symmetric neighborhoods. A connected topological group is generated by any neighborhood of the identity element.

If $H$ is a normal subgroup of a topological group $G$, then the quotient by $H$ of the topology of $G$ is compatible with the group structure of $G/H$. The quotient group $G/H$ is Hausdorff if and only if $H$ is closed in $G$. It is discrete if and only if $H$ is open in $G$. In our situation, $H$ will always be normal and closed - usually $H$ is the center of a unitary group $G$ of functional analytic type.

**Definition 2.2.** A group $G$ is **simple** if it has no nontrivial normal subgroup, i.e., for every normal subgroup $H$ of $G$ one has either $H = \{1\}$ or $H = G$. A topological group is **topologically simple** if it has no nontrivial closed normal subgroup.

### 2.1.1 Uniform spaces

In the context of Lévy groups (cf. Chapter 3) we will make use of the concept of uniform spaces. The conceptual advantage over topological spaces is that one has a notion of closeness between points. We repeat the definition of a uniform space. More information can be found in [Bo 89, Chapter II] and [Pe 06, Chapter 1].

**Definition 2.3.** A **uniform space** is a pair $(X,\mathcal{U})$ consisting of a set $X$ and a **uniform structure** (or **uniformity**) $\mathcal{U}$ on $X$. A uniform structure is a family of subsets of $X \times X$, called **entourages**, satisfying the following properties:

(i) $\mathcal{U}$ is closed under finite intersections and supersets (if $V \in \mathcal{U}$ and $V \subseteq U \subseteq X \times X$, then $U \in \mathcal{U}$).

(ii) Every $V \in \mathcal{U}$ contains the diagonal $\triangle := \{(x, x) \mid x \in X\}$.

(iii) If $V \in \mathcal{U}$, then $V^{-1} := \{(x, y) \mid (y, x) \in V\} \in \mathcal{U}$.

(iv) For every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that

$$U \circ U := \{(x, z) \mid \exists y \in X : \ (x, y) \in U, \ (y, z) \in U\} \subseteq V.$$

For an entourage $V \in \mathcal{U}$ we express the relation $(x, y) \in V$ by saying that $x$ and $y$ are $V$-close.
A subfamily $\mathcal{B} \subseteq \mathcal{U}$ is said to be a **basis** of the uniformity $\mathcal{U}$ if for every $U, V \in \mathcal{B}$ there exists $W \in \mathcal{B}$ with $W \subseteq U \cap V$, and every entourage $V \in \mathcal{U}$ contains an element $U \in \mathcal{B}$ as a subset.

Given an element $V$ of a uniform structure $\mathcal{U}$ on a set $X$ we define the **$V$-neighborhood $V[x]$ of a point** $x \in X$ by

$$V[x] := \{y \in X \mid (x, y) \in V\}.$$

The sets $V[x]$ with $V \in \mathcal{U}$ form a neighborhood basis for $x$ with regard to a certain topology on $X$ that we call the topology **determined** by $\mathcal{U}$. We say that an uniformity $\mathcal{U}$ determining the topology of a given topological space $X$ is **compatible**. Every compact space admits a unique compatible uniformity consisting of all neighborhoods of the diagonal.

Of course, the notion of an uniformity can be carried over to the context of topological groups, see [Bo 89, Section III.3] and [Pe 06, Chapter 1]. The **left uniform structure** of a topological group $G$, denoted by $\mathcal{U}_L(G)$, is an uniformity on $G$ which has as a basis of entourages of the diagonal the sets

$$V_L := \left\{(x, y) \in G \times G \mid x^{-1}y \in V\right\},$$

where $V$ is a neighborhood of the identity. Analogously one can define the **right uniform structure** of a topological group.

### 2.1.2 Metrizable groups

Let $G$ be a metrizable group, that is, $G$ is a topological group admitting a compatible metric $d$. By compatible we mean that it induces the group topology. Then $d$ is called **left-invariant** if

$$d(gu, gv) = d(u, v) \quad \text{for all } g, u, v \in G.$$

Analogously $d$ is called **right-invariant** if

$$d(uh, vh) = d(u, v) \quad \text{for all } h, u, v \in G.$$

Finally, $d$ is called **bi-invariant** if it is both left- and right-invariant, that is,

$$d(guh, gvh) = d(u, v) \quad \text{for all } g, h, u, v \in G.$$

If $G$ be a topological group and $d$ a left-invariant metric generating the topology of $G$, then the corresponding uniform structure is the left uniform structure.

The Birkhoff-Kakutani theorem states that a topological group $G$ is metrizable if and only if it is Hausdorff and first countable. Moreover, if $G$ is metrizable, then $G$ admits a compatible left-invariant metric.

If $G$ is a group and $d$ a bi-invariant metric on $G$, then $G$ is a topological group in the topology induced by $d$. Any compact metrizable group admits a compatible bi-invariant metric.

The groups we are concerned with all admit a bi-invariant metric - however this metric does not in all cases introduce the topology of our interest. For example, unitary groups of von Neumann algebras (see Section 2.2) can be endowed with the topology induced from the operator norm, which is bi-invariant but not separable if the von Neumann algebra is not of type $I_n$, $n \in \mathbb{N}$.

**Definition 2.4.** A neighborhood $V$ at the identity of a topological group $G$ is called **invariant** if it is invariant under all inner automorphisms, that is, if $gVg^{-1} = V$ for all $g \in G$. A topological group $G$ is called **SIN** if it has a neighborhood basis of the identity consisting of invariant neighborhoods of the identity.

In the above definition SIN stands for small invariant neighborhoods. Note that a topological group $G$ is SIN if and only if the left and right uniformities on $G$ coincide. In particular, for every SIN group we have $\mathrm{LUCB}(G) = \mathrm{RUCB}(G)$, where $\mathrm{LUCB}(G)$ (respectively $\mathrm{RUCB}(G)$) stands for the space of left (respectively right) uniformly continuous bounded functions on $G$.

A first countable Hausdorff topological group is SIN if and only if it admits a compatible bi-invariant metric.

### 2.1.3 Polish groups

**Definition 2.5.** A topological space is **Polish** if it is separable and completely metrizable. A topological group is called **Polish** if it is Polish as a topological space.

Any Polish group $G$ admits a compatible complete metric. The Birkhoff-Kakutani theorem implies that it admits a compatible left-invariant metric, which is not necessarily complete. If a Polish group $G$ admits a compatible bi-invariant metric $d$, then $d$ is complete. If $G$ is a Polish group with bi-invariant metric $d$ and $H$ is a closed normal subgroup of $G$, then $G/H$ as a topological group admits a compatible bi-invariant metric.

*Examples.* (i) $(\mathbb{R}^n, +)$ and $(\mathbb{C}^n, +)$ are Polish groups. But $(\mathbb{Q}, +)$ with the topology induced by the absolute value is no Polish group.

(ii) All countable groups with discrete topology are Polish groups.

(iii) All Lie groups (see Subsection 2.1.4) are Polish groups.

(iv) $S_\infty := \{f : \mathbb{N} \to \mathbb{N} \mid f \text{ is a bijection}\}$ with compatible complete metric $\rho(x, y) = d(x, y) + d(x^{-1}, y^{-1})$, where $d(x, y) = 2^{-n-1}$ for $x \neq y$ and $n$ the least number such that $x_n \neq y_n$, is a Polish group.

(v) The unitary group $\mathrm{U}(\mathcal{H})$ of a separable infinite-dimensional Hilbert space, endowed with the strong operator topology, is a Polish group.

(vi) The unitary group $\mathrm{U}(\mathcal{M})$ of a separable $\mathrm{II}_1$ factor $\mathcal{M}$ (see Section 2.2), endowed with the strong operator topology, is a Polish SIN group. The same holds for the projective unitary group $\mathrm{PU}(\mathcal{M}) = \mathrm{U}(\mathcal{M})/\mathrm{U}(1) \cdot 1$.

(vii) The universal Urysohn space is a universal Polish space, i.e., it contains every Polish space as a closed subspace. The isometry group of the Urysohn space is a universal Polish group, that is, every Polish group is isomorphic to a closed subgroup of it. For the construction and interesting facts see [Ga 09, Chapter 1 and 2] and [Pe 06, Chapter 5].

[BK 96, Theorem 1.2.6] implies that every bijective continuous homomorphism between Polish groups is a homeomorphism.

Not every group can be be endowed with a Polish group topology. For example, the free group on a continuum of generators cannot be equipped with a Polish group topology, see Corollary 3.3 in [Ro 09b].

### 2.1.4 Lie groups

Let us briefly collect some information about Lie groups, taken from [Kn, Chapter IV].

**Definition 2.6.** A **Lie group** is a separable topological group with the additional structure of a smooth manifold such that the multiplication and inversion are smooth.

We will deal with some special Lie groups in Section 4.5. For $n \in \mathbb{N}$ denote by $M_{n\times n}(\mathbb{C})$ the algebra of $n \times n$ matrices over the complex field $\mathbb{C}$. Given a group $G$ we write $\mathcal{Z}(G)$ for the center of $G$. We are interested in the Lie groups

$$\begin{aligned} \mathrm{U}(n) &:= \{u \in M_{n\times n}(\mathbb{C}) \mid u^*u = uu^* = 1\}, \ n \in \mathbb{N}, \\ \mathrm{SU}(n) &:= \{u \in \mathrm{U}(n) \mid \det(u) = 1\}, \\ \mathrm{PU}(n) &:= \mathrm{U}(n)/\mathcal{Z}(\mathrm{U}(n)) = \mathrm{U}(n)/\mathrm{U}(1)\cdot 1, \end{aligned}$$

all of which are compact and connected. One can show that $\mathrm{PU}(n) = \mathrm{SU}(n)/\mathcal{Z}(\mathrm{SU}(n))$. A **torus** is a product of circle groups $\mathrm{U}(1)$. Every compact connected abelian Lie group is a torus. In a compact Lie group $G$ one can look for tori as subgroups. Tori are partially ordered by inclusion and thus any torus is contained in a maximal torus of $G$. Any two maximal tori in a compact connected Lie group are conjugate. For example, the diagonal matrices in $\mathrm{U}(n)$ form a maximal torus in $\mathrm{U}(n)$.

**Theorem 2.7.** *Let $G$ be a compact connected Lie group and $T$ a maximal torus in $G$. Then each element of $G$ is conjugate to an element of $T$.*

## 2.2 Von Neumann algebras

The material presented in this section can be found in [Bl 06], [Di 81], [KR 83], [KR 86] and [Ta 03].

We assume the reader to be familiar with basic functional analysis as found e.g. in [Ru 73]. Some of the results that are assumed to be known by the reader are the Baire category theorem, Borel functional calculus and the spectral theorem for normal operators on a Hilbert space. Let $\mathcal{H}$ be an finite- or infinite-dimensional separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|_{\mathcal{H}}$. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators. The identity operator is denoted by $1 \in \mathcal{B}(\mathcal{H})$ and it will be always clear from the context whether 1 refers to the identity operator or the complex number. If not stated explicitly, we will always denote by $\|\cdot\|$ the **operator norm** on $\mathcal{B}(\mathcal{H})$ defined by

$$\|x\| := \sup_{\xi \in \mathcal{H}, \|\xi\|_{\mathcal{H}}=1} \|x\xi\|_{\mathcal{H}} .$$

There are several interesting topologies on $\mathcal{B}(\mathcal{H})$. The topology induced from the operator norm (i.e. the topology of uniform convergence) is called the **uniform topology**, or **norm topology**. The **strong operator topology** on $\mathcal{B}(\mathcal{H})$ is the topology of pointwise operator norm convergence, i.e. the topology induced from the separating family of semi-norms $\|\cdot\xi\|_{\mathcal{H}}$ with $\xi \in \mathcal{H}$. A net $\{x_i\}_{i \in I}$ is strong operator convergent to $x_0$ if and only if $\|(x_j - x_0)\xi\|_{\mathcal{H}}$ converges to 0 for every $\xi \in \mathcal{H}$. The **weak operator topology** on $\mathcal{B}(\mathcal{H})$ is the topology induced by the family $\{\langle \cdot\xi, \eta \rangle\}_{\xi,\eta \in \mathcal{H}}$. It is readily checked that uniform convergence implies strong operator convergence, which in turn implies weak operator convergence. For other interesting topologies as for example the $\sigma$-strong-, $\sigma$-weak-, and $\sigma$-strong* operator topology we refer the reader to the literature.

Let us repeat the definitions of important classes of operators. An operator $x \in \mathcal{B}(\mathcal{H})$ is said to be

- **self-adjoint** if $x = x^*$;
- **normal** if $x^*x = xx^*$;
- **positive** if $\langle x\xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{H}$;
- a **projection** if $x = x^* = x^2$;
- an **isometry** if $x^*x = 1$;
- a **partial isometry** if $x^*x$ is a projection;
- **unitary** if $x^*x = xx^* = 1$.
- a **symmetry** if $x = x^* = x^{-1}$.

A **Banach algebra** is a (complex) algebra $\mathcal{A}$ which is a Banach space under a submultiplicative norm $\|\cdot\|$ (i.e. $\|xy\| \leq \|x\| \|y\|$ for all $x, y \in \mathcal{A}$). An **involution** on a Banach algebra $\mathcal{A}$ is a conjugate-linear isometric anti-automorphism $^* : \mathcal{A} \to \mathcal{A},\ x \mapsto x^*$, of order two, that is, for all $x, y \in \mathcal{A},\ \lambda \in \mathbb{C}$ one has $(x+y)^* = x^* + y^*,\ (xy)^* = y^* x^*,\ (\lambda x)^* = \overline{\lambda} x^*,\ (x^*)^* = x,\ \|x^*\| = \|x\|$. A $C^*$**-algebra** is a Banach algebra $\mathcal{A}$ with an involution $^*$ such that

$$\|x^* x\| = \|x\|^2 \qquad \text{for all } x \in \mathcal{A}.$$

Every $C^*$-algebra can be realized as a (not necessarily unital) operator norm closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ by the Gelfand-Naimark-Segal construction - see Theorem I.9.18 in [Ta 03]. For our purposes, the most important examples of (concrete) $C^*$-algebras are

- the two-sided ideal $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ of compact operators;
- the **Calkin algebra** $\mathcal{C}(\mathcal{H})$ defined by $\mathcal{C}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

The compact operators form the largest two-sided ideal in $\mathcal{B}(\mathcal{H})$.

**Definition 2.8.** A **norm ideal** $K$ is a two-sided ideal of $\mathcal{B}(\mathcal{H})$ equipped with a norm $\|\cdot\|_K$ such that $\|x\| \leq \|x\|_K = \|x^*\|_K$ for $x \in K$ and $\|axb\|_K \leq \|a\| \|x\|_K \|b\|$ for $a, b \in \mathcal{B}(\mathcal{H})$.

All nontrivial norm ideals in $\mathcal{B}(\mathcal{H})$ are contained in $\mathcal{K}(\mathcal{H})$. Some particularly important examples of norm ideals are the $C^*$-algebra of compact operators and the Hilbert-Schmidt operators (or Schatten 2-class operators).

We state a result showing the importance of unitary operators in unital $C^*$-algebras.

**Proposition 2.9.** Every element in a unital $C^*$-algebra $\mathcal{A}$ is a linear combination of four unitary elements of $\mathcal{A}$.

The **commutant** of a set $M \subseteq \mathcal{B}(\mathcal{H})$ is

$$M' := \{x \in \mathcal{M} \mid xy = yx \text{ for all } y \in \mathcal{B}(\mathcal{H})\}.$$

The **bicommutant** of $M$ is then defined as $M'' := (M')'$. It is obvious that $M \subseteq M''$ and $M_2' \subseteq M_1'$ if $M_1 \subseteq M_2$. Thus $M' = (M'')'$ for any set $M \subseteq \mathcal{B}(\mathcal{H})$.

**Definition 2.10.** A $*$-subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ such that $\mathcal{M} = \mathcal{M}''$ is called **von Neumann algebra**. A **factor** is a von Neumann algebra such that $\mathcal{M} \cap \mathcal{M}' = \mathbb{C} \cdot 1$.

The definition implies that every von Neumann algebra is unital. We say that a von Neumann algebra is **separable** if it acts on a separable Hilbert space. We will usually be interested in separable von Neumann algebras. Von Neumann algebras were introduced by John von Neumann in [Ne 30]. He was motivated by his study of group representation theory, ergodic theory and quantum mechanics. Much progress on the theory von Neumann algebras was achieved in a series of papers of Murray and von Neumann, see [MN 36], [MN 37], [Ne 40], [MN 43], [Ne 43], [Ne 49].

One of the most crucial results is the double commutant theorem.

**Theorem 2.11.** *If $\mathcal{M}$ is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ then $\mathcal{M}''$ coincides with the weak-, strong-, $\sigma$-weak- and $\sigma$-strong closure of $\mathcal{M}$.*

In particular every von Neumann algebra is closed in all above topologies. We note here that the original definition of a von Neumann algebra involves the weak closure. In particular, every von Neumann algebra is a $C^*$-algebra (i.e. it is closed with respect to the operator norm). But, for example, the $C^*$-algebras $\mathcal{K}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$ are not $*$-isomorphic to any von Neumann algebra.

The double commutant theorem indicates that there is a rich interplay between algebraic and topological techniques in analyzing von Neumann algebras. Any von Neumann algebra $\mathcal{M}$ contains an abundance of projections, in particular $\mathcal{M}$ contains all spectral projections of any element in $\mathcal{M}$. Denote the set of unitary operators in $\mathcal{M}$ by $\mathrm{U}(\mathcal{M})$ and the set of projections in $\mathcal{M}$ by $\mathrm{P}(\mathcal{M})$. Then we have

$$\mathcal{M} = \mathrm{U}(\mathcal{M})'' = \mathrm{P}(\mathcal{M})''.$$

*Remark.* The last mentioned fact marks a major difference between general $C^*$-algebras and von Neumann algebras. A $C^*$-algebra may contain no other projections than 0 and 1 - e.g. the $C^*$-algebra of continuous functions on the unit interval $[0, 1]$.

Simple examples of von Neumann algebras are finite-dimensional matrix algebras $M_{n\times n}(\mathbb{C})$ over $\mathbb{C}$ and the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on an infinite-dimensional Hilbert space. If $\mathcal{M}$ is an abelian von Neumann algebra on a separable Hilbert space $\mathcal{H}$, then there exists a second-countable compact Hausdorff space $X$ and a positive measure $\mu$ on $X$ such that $\mathcal{M}$ is $*$-isomorphic to the algebra $L^\infty(X,\mu)$ of (equivalence classes of) essentially bounded complex-valued measurable functions on $X$ (with the usual pointwise-defined operations and the essential supremum norm). More sophisticated examples are presented in Subsection 2.2.1 below.

### 2.2.1 Type classification

Von Neumann algebras can be characterized by their projections. Before being able to give a complete classification of factors, we need to distinguish various classes of projections. To this end, we introduce an equivalence relation on $\mathrm{P}(\mathcal{M})$. The set of projections in a von Neumann algebra forms a complete lattice.

**Definition 2.12.** Two projections $p$ and $q$ in a von Neumann algebra $\mathcal{M}$ are called **equivalent** if there exists a partial isometry $x \in \mathcal{M}$ such that $x^*x = p$ and $xx^* = q$. In this case we write

$$p \sim q.$$

$p$ and $q$ are called, respectively, the **initial projection** and the **final projection**. If $p \sim q_1 \in \mathcal{M}$ with $q_1 \leq q$ then we write $p \precsim q$ or $q \succsim p$. If $p \precsim q$ and $p \nsim q$ ($p$ is not equivalent to $q$) then we write $p \prec q$ or $q \succ p$.

The relation $\sim$ defines an equivalence relation on the set of projections in $\mathcal{M}$. Furthermore, one can show that $p \precsim q$ and $p \succsim q$ imply $p \sim q$. Let $x \in \mathcal{M}$. The smallest projection $p \in \mathcal{M}$ such that $px = x$ is called **left support** of $x$ and is denoted by $s_l(x)$. The **right support** $s_r(x)$ of $x$ is the smallest projection $q \in \mathcal{M}$ such that $xq = x$. By polar decomposition, $s_l(x) \sim s_r(x)$.

Let us state the following powerful theorem.

**Theorem 2.13.** *(Comparability Theorem) Let $\mathcal{M}$ be a von Neumann algebra and $p, q$ projections in $\mathcal{M}$. There exists a central projection $z \in \mathcal{M}$ such that*

$$zp \precsim zq \quad \text{and} \quad (1-z)p \succsim (1-z)q.$$

*If $\mathcal{M}$ is a factor, then exactly one of the following relations holds:*

$$p \prec q; \quad p \sim q; \quad p \succ q.$$

**Definition 2.14.** A projection $p$ in a von Neumann algebra $\mathcal{M}$ is said to be

- **finite** if $p \sim q \leq p$ implies $p = q$;
- **infinite** if $p$ is not finite;
- **purely infinite** if there is no nonzero finite subprojection $q \leq p$ in $\mathcal{M}$;
- **properly infinite** if $zp$ is infinite for every central projection $z \in \mathcal{M}$ with $zp \neq 0$;
- **abelian** if $p\mathcal{M}p$ is abelian.

With these definitions in hand, we can completely classify von Neumann algebras.

**Definition 2.15.** A von Neumann algebra $\mathcal{M}$ is called **finite**, **infinite**, **properly infinite** or **purely infinite** if its identity 1 has the corresponding property.

**Definition 2.16.** A von Neumann algebra $\mathcal{M}$ is of

- **type** I if every nonzero central projection in $\mathcal{M}$ majorizes a nonzero abelian projection in $\mathcal{M}$. $\mathcal{M}$ is of **type** $\mathrm{I_n}$ for some cardinal $n$ (finite or infinite) if 1 is the sum of $n$ equivalent abelian projections;
- **type** II if $\mathcal{M}$ has no nonzero abelian projection and every nonzero central projection in $\mathcal{M}$ majorizes a nonzero finite projection of $\mathcal{M}$;
  - $\mathcal{M}$ is of **type** $\mathrm{II}_1$ if $\mathcal{M}$ is of type II and finite;
  - $\mathcal{M}$ is of **type** $II_\infty$ if $\mathcal{M}$ is of type II and has no nonzero central finite projection;
- **type** III if there is no nonzero finite projection in $\mathcal{M}$ (i.e. $\mathcal{M}$ is purely infinite).

Any von Neumann algebra is uniquely decomposable into the direct sum of von Neumann algebras of type I, type $\mathrm{II}_1$, type $\mathrm{II}_\infty$ and type III. A von Neumann algebra without type III summand is called **semifinite**. There is a crucial structural difference between semifinite and type III von Neumann algebras - it is given by the existence of a so called semifinite trace.

**Definition 2.17.** Let $\mathcal{M}$ be a von Neumann algebra. We say that a positive linear functional $\tau : \mathcal{M} \to \mathbb{C}$ is a **trace** if $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{M}$. A trace $\tau$ is

- **faithful** if $\tau(x^*x) = 0$ implies $x = 0$;
- **normal** if $\tau(\sup_i x_i) = \sup_i \tau(x_i)$ for every bounded increasing net $\{x_i\}_{i\in I}$ of positive operators in $\mathcal{M}$;
- **finite** if $\tau(1) < \infty$.

A **semifinite trace** is a positive linear functional $\tau : \mathcal{M} \to \mathbb{C} \cup \{\infty\}$ such that $\tau(xy) = \tau(yx)$ for all $x, y \in \mathcal{M}$ and such that every nonzero positive $x \in \mathcal{M}$ majorizes some nonzero positive $y \in \mathcal{M}$ with $\tau(y) < \infty$;

For a von Neumann algebra it is equivalent to be semifinite and to admit a faithful normal semifinite trace.

A semifinite von Neumann algebra $\mathcal{M}$ with faithful normal semifinite trace $\tau$ naturally acts on the Hilbert space $L^2(\mathcal{M}, \tau)$ obtained from the GNS construction (named after Gelfand, Naimark and Segal). (Since $\tau$ is a positive linear functional, one can define a pre-inner product on $\mathfrak{n}_\tau := \{x \in \mathcal{M} \mid \tau(x^*x) < \infty\}$ by $\langle x, y\rangle_\tau = \tau(y^*x)$ for $x, y \in \mathcal{M}$. Faithfulness of $\tau$ implies that $\langle \cdot, \cdot \rangle_\tau$ is an inner product. The completion of $\mathfrak{n}_\tau$ with respect to $\langle \cdot, \cdot \rangle_\tau$ is then a Hilbert space denoted by $L^2(\mathcal{M}, \tau)$.)

It is worth mentioning that a priori, the existence of type II and type III von Neumann algebras is not at all clear. Since we will be interested in $\mathrm{II}_1$ factors, we will focus on examples of those in Subsection 2.2.2.

### 2.2.2 Type $\mathrm{II}_1$ factors

A $\mathrm{II}_1$ factor $\mathcal{M}$ admits a unique faithful normal normalized trace $\tau$. Any isometry in $\mathcal{M}$ is unitary. The notions of (Murray von Neumann) equivalence of projections and unitary equivalence of projections coincide. Denote the set of projections of $\mathcal{M}$ by $\mathrm{Proj}(\mathcal{M})$. Elements in $\mathrm{Proj}(\mathcal{M})$ can take any value in $[0, 1]$. For $p \in \mathrm{Proj}(\mathcal{M})$ one has $p = 0$ if $\tau(p) = 0$ by faithfulness and $p = 1$ if $\tau(p) = 1$. A projection $p \in \mathrm{Proj}(\mathcal{M})$ can always be halved, i.e. one can find subprojections $p_1, p_2 \precsim p$ such that $p_1 + p_2 = p$ and $\tau(p_1) = \tau(p_2)$. The fact that $\tau$ takes continuous values on $\mathrm{Proj}(\mathcal{M})$ is often referred to as continuous dimension. For any nonzero projection $p \in \mathrm{Proj}(\mathcal{M})$ the algebra $p\mathcal{M}p$ is a $\mathrm{II}_1$ factor (acting on $p\mathcal{H}$). For a spectral projection $p \in \mathrm{Proj}(\mathcal{M})$ of an operator we call $\tau(p) \in [0, 1]$ the **spectral weight** or **weight** for short.

For $x \in \mathcal{M}$ we let $|x| := (x^*x)^{1/2}$. We define the **1-norm** $\|\cdot\|_1$ on $\mathcal{M}$ by

$$\|x\|_1 := \tau(|x|) \text{ for } x \in \mathcal{M}.$$

The **2-norm** $\|\cdot\|_2$ is defined by

$$\|x\|_2 := \tau(x^*x)^{1/2} \text{ for } x \in \mathcal{M}.$$

We will use without further notice that the metric $d_2$ induced from the 2-norm defines a bi-invariant metric on $\mathcal{M}$, which follows from the trace property $\tau(xy) = \tau(yx)$ for $x, y \in \mathcal{M}$. Note that the metrics induced from $\|\cdot\|$ and $\|\cdot\|_1$ are also bi-invariant.

We will need the following inequalities between the 1-norm and the 2-norm. For convenience, we provide its easy proof.

**Proposition 2.18.** Let $\mathcal{M}$ be a $\mathrm{II}_1$ factor. Assume that $u, v \in \mathrm{U}(\mathcal{M})$ and $x \in \mathcal{M}$.

(i) $|\tau(x)| \leq \tau(|x|)$.

(ii) $\|u - v\|_1^2 \leq \|u - v\|_2^2 \leq 2 \cdot \|u - v\|_1$.

(iii) $\|\cdot\|_1$ and $\|\cdot\|_2$ induce the same topology.

***Proof.*** (i) Let $x = w\,|x| = w(x^*x)^{1/2}$ be the polar decomposition of $x$. Using the Cauchy-Schwarz inequality [Ta 03, Proposition I.9.5], we obtain

$$\begin{aligned}
|\tau(x)| &= |\tau(w\,|x|)| \\
&= \left|\tau(w\,|x|^{1/2}\,|x|^{1/2})\right| \\
&\leq \tau(|x|^{1/2}\,w^*w\,|x|^{1/2})^{1/2}\tau(|x|^{1/2}\,|x|^{1/2})^{1/2} \\
&= \tau(w^*w\,|x|)^{1/2}\tau(|x|)^{1/2} \\
&= \tau(|x|).
\end{aligned}$$

In the last step, we used that $w^*w$ is unitary on $\ker(|x|)^\perp = \ker(x)^\perp$.

(ii) Using the Cauchy-Schwarz inequality, we obtain

$$\|u - v\|_1 \leq \tau(|u - v|^*\,|u - v|)^{1/2}\tau(1^*1)^{1/2} = \|u - v\|_2\,.$$

The second inequality follows from (i) and the bi-invariance of $\|\cdot\|_1$:

$$\begin{aligned}
\|u - v\|_2^2 &= \tau(1 - u^*v + 1 - v^*u) \\
&= \tau(1 - u^*v) + \tau(1 - v^*u) \\
&\leq |\tau(1 - u^*v)| + |\tau(1 - v^*u)|
\end{aligned}$$

$$\leq \tau(|1-u^*v|) + \tau(|1-v^*u|)$$
$$= 2 \cdot \|u-v\|_1 .$$

(iii) This is clear from (ii). □

Now we describe a way to construct $\mathrm{II}_1$ factors. Let $G$ be a countable discrete group. Then the left regular representation $\lambda : G \to \mathrm{U}(\ell^2(G))$ is defined by

$$\lambda(g)\xi_h = \xi_{gh} \quad \text{for } g, h \in G,\ \xi_h \in \ell^2(G).$$

The group von Neumann algebra of $G$ is defined as

$$L(G) \coloneqq \{\lambda(g) \mid g \in G\}'' .$$

The canonical trace on $L(G)$ is given by $\tau(\cdot) = \langle \cdot \xi_1, \xi_1 \rangle$. We say that a group is ICC if the conjugacy class of every nontrivial element is infinite. It can be checked that $L(G)$ is a $\mathrm{II}_1$ factor if and only if $G$ is ICC. Elementary examples of ICC groups are

- the group $S_{fin}$ of finitely supported permutations on $\mathbb{N}$;
- the free group $\mathbb{F}_n$ on $n \geq 2$ generators;
- the "$ax+b$" group $\left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mid a, b \in \mathbb{Q}, a > 0 \right\}$.

An interesting fact is that all *amenable* ICC groups (e.g. $S_{fin}$) give isomorphic group von Neumann algebras, called the **hyperfinite** $\mathrm{II}_1$ **factor**, usually denoted by $\mathcal{R}$. The hyperfinite $\mathrm{II}_1$ is the unique smallest $\mathrm{II}_1$ factor in the sense that every $\mathrm{II}_1$ factor contains a copy of $\mathcal{R}$. We note that Ozawa [Oz 03] has proved that there are uncountably many non-isomorphic separable $\mathrm{II}_1$ factors (and that there is no universal separable $\mathrm{II}_1$ factor). There are several equivalent ways of describing $\mathcal{R}$, see [Ta 03, Theorem XIV.2.4]. For example, $\mathcal{R}$ can be constructed as an infinite tensor product (see [Ta 03, Section XIV.1]) of the matrix algebras $M_{2\times 2}(\mathbb{C})$ with unique normalized trace. Or equivalently, $\mathcal{R}$ is generated by an increasing sequence of finite-dimensional $*$-subalgebras of $\mathcal{R}$.

The von Neumann algebraic infinite tensor product $\mathcal{R}_\lambda$ of $M_{2\times 2}(\mathbb{C})$ with the state $\phi_\lambda, \lambda \in (0,1)$, defined by

$$\phi_\lambda((a_{ij})) \coloneqq \frac{\lambda}{1+\lambda} a_{11} + \frac{1}{1+\lambda} a_{22}.$$

gives a type III factor. The factor $\mathcal{R}_\lambda$ is called Powers factor after its inventor R. Powers. There is a finer classification of type factors into type $\mathrm{III}_\lambda, \lambda \in [0,1]$, with the help of Connes' invariant $S(\mathcal{M})$. It turns out that $\mathcal{R}_\lambda, \lambda \in (0,1)$, is of type $\mathrm{III}_\lambda$ and that $\mathcal{R}_\lambda$ is not isomorphic to $\mathcal{R}_\mu$ for $\lambda \neq \mu \in (0,1)$.

Historically the group measure space construction by Murray and von Neumann was the first construction which proved the existence of non-type I factors. The construction is a special case of a $W^*$-crossed product. The interested reader is referred to the literature.

A longstanding open problem (the so called free factor problem) asks whether $L(\mathbb{F}_n) = L(\mathbb{F}_m)$ for $n \neq m$. Note here that $\mathbb{F}_n,\ n \geq 2$, is not amenable and thus $L(\mathbb{F}_n) \neq \mathcal{R}$. It led Voiculescu to the discovery of free probability theory, a noncommutative analogue of classical probability theory.

### 2.2.3 Reduction theory

The reduction theory of von Neumann algebras is concerned with the decomposition of von Neumann algebras into smallest parts. We mentioned already that every von Neumann algebra can be uniquely decomposed into the direct sum of those of type I, $\mathrm{II}_1$, $\mathrm{II}_\infty$ and III. Every factor is of one of these types. On a separable Hilbert space, any von Neumann algebra is a direct integral (a "measurable direct sum") of factors. The theory of direct integrals requires considerable technical effort and can be found in [Di 81, Part II], [KR 86, Chapter 14] and [Ta 03, Chapter IV]. Our short account is taken from [Bl 06, Section III.1.6].

Let $(X,\mu)$ be a standard measure space and $(\mathcal{H}_x, \langle\cdot,\cdot\rangle_x)_{x\in X}$ a separable Hilbert space for $\mu$-almost all $x \in X$. A **measurable field** is a vector subspace $V \subseteq \prod_x \mathcal{H}_x$ closed under multiplication by $L^\infty(X,\mu)$, such that $x \mapsto \langle \xi(x), \eta(x)\rangle_x$ is measurable for all $\xi, \eta \in V$ and $\int_X \langle \xi(x), \xi(x)\rangle_x d\mu(x) < \infty$ and such that $V$ is generated as an $L^\infty(X,\mu)$-module by a countable subset $\{\xi_n\}_{n\in\mathbb{N}} \subseteq V$ such that the completion of span $\{\xi_n(x)\}_{n\in\mathbb{N}}$ is $\mathcal{H}_x$ for $\mu$-almost all $x \in X$. The completion of $V$ is a separable Hilbert space $\mathcal{H}$, which can be identified with the space of equivalence classes of measurable section of the field $(\mathcal{H}_x)$) written

$$\mathcal{H} = \int_X^\oplus \mathcal{H}_x d\mu(x).$$

For $T_x \in \mathcal{B}(\mathcal{H}_x)$, $(T_x)$ is a **measurable field of bounded operators** if $(T_x\xi(x))$ is a measurable section for each measurable section $\xi$. Assume that $\|T_x\|$ is uniformly bounded. Then $(T_x)$ defines an operator $T \in \mathcal{B}(\mathcal{H})$, which is called **decomposable** and written

$$T = \int_X^{\oplus} T_x d\mu(x).$$

The **algebra of diagonalizable operators** is the image of $L^\infty(X, \mu)$ via

$$f \mapsto \int_X^{\oplus} f(x) 1_x d\mu(x).$$

Let $\mathcal{M}$ be a von Neumann algebra on a separable Hilbert space $\mathcal{H}$ and $\mathcal{A}$ an commutative von Neumann subalgebra of $\mathcal{M}'$. As $\mathcal{A}$ is an abelian von Neumann algebra on a separable Hilbert space $\mathcal{H}$, it is generated (as a von Neumann algebra) by a single bounded operator $T$ on $\mathcal{H}$. There exists a measure $\mu$ on $X = \sigma(T)$, a measurable field of Hilbert spaces $(\mathcal{H}_x)$ over $(X, \mu)$, and a unitary

$$u : \mathcal{H} \to \int_X^{\oplus} \mathcal{H}_x d\mu(x)$$

such that $\mathcal{A}$ is mapped onto the set of diagonalizable operator and for all $T \in \mathcal{A}'$ the element

$$uTu^* = \int_X^{\oplus} T_x d\mu(x)$$

is measurable.

### 2.2.4 Generalized $s$-numbers

We summarize some facts on generalized $s$-numbers for semifinite von Neumann algebras, collected mainly from [FK 86]. The definition and several properties of generalized $s$-numbers are crucial for understanding Chapter 4.
Throughout this section, let $\mathcal{M}$ denote a semifinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with faithful semifinite normal trace $\tau$. Fack and Kosaki provide a more general framework in [FK 86], using $\tau$-measurable operators (which are special possibly unbounded operators affiliated with $\mathcal{M}$). For our purposes, it suffices to consider operators in $\mathcal{M}$ itself.
The classical $s$-numbers of compact operators can be generalized in the following way.

**Definition 2.19.** Let $T \in \mathcal{M}$ and $t > 0$. We define the **$t$-th generalized $s$-number**

$\mu_t(T)$ of $T$ as

$$\mu_t(T) := \inf \{\|Tp\| \mid p \in \mathrm{Proj}(\mathcal{M}) \text{ such that } \tau(1-p) \leq t\}.$$

We aim at presenting various expressions for $\mu_t$ in this section. For $T \in \mathcal{M}$ we define the **distribution function** of $T$ by

$$\lambda_t(T) = \tau\left(E_{(t,\infty)}(|T|)\right), \ t \geq 0,$$

where $E_{(t,\infty)}(|T|)$ is the spectral projection of $|T|$ corresponding to the interval $(t, \infty)$.

If $T \in \mathcal{M}$, then $\lambda_t(T) < \infty$ for $t$ large enough and $\lim_{t\to} \lambda_t(T) = 0$. Since $\tau$ is normal and $E_{(t_n,\infty)}(|T|) \nearrow E_{(t,\infty)}(|T|)$ strongly as $t_n \searrow t$ (by strong right continuity of $E_{(t,\infty)}(|T|)$), the map $[0, \infty) \ni t \mapsto \lambda_t(T)$ is non-increasing and continuous from the right.

For $t > 0$, let $\mathfrak{R}_t := \left\{S \in \overline{\mathcal{M}} \mid \tau\left(\mathrm{supp}(|S|)\right) \leq t\right\}$, where $\mathrm{supp}(|S|)$ denotes the support projection of $|S|$. The **approximation number** $d(T, \mathfrak{R}_t)$ of $T \in \mathcal{M}$ is defined as

$$d(T, \mathfrak{R}_t) := \inf \{\|T - S\| \mid S \in \mathfrak{R}_t\}.$$

Let us collect important characterizations of the above defined generalized $s$-numbers.

**Proposition 2.20.** Let $T$ be an element of the semifinite von Neumann algebra $\mathcal{M}$.

(i) If $\mathcal{N}$ is any von Neumann subalgebra of $\mathcal{M}$ containing the spectral projections of $|T|$, then

$$\mu_t(T) = \inf_{p \in \mathrm{Proj}(\mathcal{N}),\ \tau(1-p) \leq t} \left( \sup_{\xi \in p\mathcal{H},\ \|\xi\|=1} \|T\xi\| \right).$$

(ii) For any $t > 0$, we have

$$\mu_t(T) = \inf \{s \geq 0 \mid \lambda_s(T) \leq t\},$$

and this infimum is attained.

(iii) $\mu_t(T) = d(T, \mathfrak{R}_t)$.

We list some more important properties of generalized $s$-numbers.

**Lemma 2.21.** Assume that $S, T \in \mathcal{M}$.

(i) The map $(0,\infty) \ni t \mapsto \mu_t(T)$ is non-increasing and right continuous. Moreover,

$$\lim_{t \searrow 0} \mu_t(T) = \|T\| \in [0,\infty].$$

(ii) $\mu_t(T) = \mu_t(|T|) = \mu_t(T^*)$ and $\mu_t(\alpha T) = |\alpha| \, \mu_t(T)$ for $t > 0$ and $\alpha \in \mathbb{C}$.

(iii) $\mu_t(T) \leq \mu_t(S)$ for $t > 0$, if $0 \leq T \leq S$.

(iv) $\mu_t\left(f(|T|)\right) = f\left(\mu_t(|T|)\right), \ t > 0$, for any continuous increasing function $f$ on $[0,\infty)$ with $f(0) \geq 0$.

(v) $\mu_{t+s}(T+S) \leq \mu_t(T) + \mu_s(S)$ for $s, t > 0$.

(vi) $\mu_t(STR) \leq \|S\| \, \|R\| \, \mu_t(T), \ t > 0$.

(vii) $\mu_{t+s}(TS) \leq \mu_t(T)\mu_s(S), \ s, t > 0$.

Lemma 2.21(i) shows that we can actually define $\mu_t$ for all $t \geq 0$. The following lemma contains the following statement, which will be implicitly used in Chapter 4: If $\mathcal{M}$ is a $\mathrm{II}_1$ factor and $T \in \mathcal{M}$ then we have $\mu_t(T) = 0$ for $t \geq 1$.

**Lemma 2.22.** If $T \in \mathcal{M}$ and $p \in \mathrm{Proj}(\mathcal{M})$, then we have

$$\mu_t(Tp) = 0 \text{ for } t \geq \tau(p).$$

In particular, if $\tau(1) = \alpha < \infty$, then

$$\mu_t(T) = 0 \text{ for } t \geq \alpha.$$

We actually get a new expression for the trace $\tau$.

**Proposition 2.23.** (i) Assume that $T \in \mathcal{M}$ is positive. Then

$$\tau(T) = \int_0^\infty \mu_t(T) dt.$$

(ii) Assume that $T \in \mathcal{M}$ and let $f$ be a continuous increasing function on $[0,\infty)$ satisfying $f(0) = 0$. Then

$$\tau\left(f(|T|)\right) = \int_0^\infty f(\mu_t(T)) dt,$$

and in particular,

$$\|T\|_p = \left( \int_0^\infty \mu_t(T)^p dt \right)^{1/p} \quad \text{for } p \in (0, \infty).$$

**Corollary 2.24.** For positive operators $S, T \in \mathcal{M}$, the following conditions are equivalent:

(i) $\mu_t(T) \leq \mu_t(S),\ t > 0$;

(ii) $\lambda_s(T) \leq \lambda_s(S),\ s \geq 0$;

(iii) $\tau(f(T)) \leq \tau(f(S))$ for any continuous increasing function $f$ on $[0, \infty)$ with $f(0) = 0$.

If $\mathcal{M}$ is a factor, then the above are equivalent to

(iv) $E_{(s,\infty)}(T) \precsim E_{(s,\infty)}(S),\ s \geq 0$.

## 2.3 Basic properties of unitary groups of $\mathrm{II}_1$ factors

In this section we collect some fundamental and important known properties of unitary groups of $\mathrm{II}_1$ factors. Throughout this section, $\mathcal{M}$ denotes a $\mathrm{II}_1$ factor if not explicitly stated.

As mentioned already in Section 2.2, every element in $\mathcal{M}$ is a linear combination of four unitary elements in $\mathcal{M}$. Any unitary operator in a von Neumann algebra is the exponential of a self-adjoint operator in the algebra. (This is wrong for general $C^*$-algebras due to the lack of enough spectral projections.) It is not hard to see that the unitary group of a von Neumann algebra is pathwise connected in its norm topology and thus in particular in the strong operator topology, see e.g. [KR 86, Exercise 5.7.24].

One can show that on $\mathrm{U}(\mathcal{M})$ the strong-, weak-, $\sigma$-weak, $\sigma$-strong and $\sigma$-strong-$*$ topologies coincide. We will implicitly use the following result.

**Proposition 2.25.** The topology induced by the 2-norm coincides with the strong operator topology on $\mathrm{U}(\mathcal{M})$. Moreover, $\mathrm{U}(\mathcal{M})$ is complete in this topology.

Let $\mathcal{M}$ for a moment be a separable $\mathrm{II}_1$ factor, that is, $\mathcal{H} := L^2(\mathcal{M}, \tau)$ is separable. To obtain that $\mathrm{U}(\mathcal{M})$ is a Polish group with the strong operator topology, it only remains to show that $\mathrm{U}(\mathcal{M})$ is separable. By Corollary 3 to Theorem IV.2.1 in [Bo 66], the unit ball $\mathcal{B}(H)_1$ of the bounded operators on a separable Hilbert space is compact (separable, since metrizable) for the weak topology. As $\mathrm{U}(\mathcal{M})$ is a subset of the separable set $\mathcal{B}(\mathcal{H})_1$, it is again separable for the weak topology, and hence for the strong topology. It is well-known that $\mathrm{U}(\mathcal{M})$ is not locally compact in the strong operator topology.

**Theorem 2.26.** *The unitary group* $\mathrm{U}(\mathcal{M})$ *(respectively projective unitary group* $\mathrm{PU}(\mathcal{M})$*) of a separable* $\mathrm{II}_1$ *factor, endowed with the strong operator topology, is a non-locally compact Polish group.*

We remark here that if $\mathrm{U}(\mathcal{M})$ is equipped with the uniform topology, then it is not separable and hence not a Polish group.

The following [Br 67, Theorem 1] of Broise plays an important role in Chapter 4 and in particular in Section 4.3.

**Theorem 2.27** (Broise)**.** *Every unitary operator of* $\mathcal{M}$ *is the product of* 32 *symmetries.*

Broise concluded from the above theorem that $\mathrm{U}(\mathcal{M})$ admits no nontrivial character. In Section 4.3 we present the proof of Theorem 2.27 with slight modifications in order to obtain that every unitary in $\mathrm{U}(\mathcal{M})$ is the product of 32 conjugates of any symmetry of trace 0.

Kadison [Ka 52] has shown that $\mathrm{PU}(\mathcal{M})$ is topologically simple in the uniform topology. De la Harpe [Ha 79] was able to show more:

**Theorem 2.28** (de la Harpe)**.** $\mathrm{PU}(\mathcal{M})$ *is simple.*

The proof of Theorem 2.28 crucially depends on Theorem 2.27. Let us explain de la Harpe's strategy. He first shows that any normal subgroup $H \neq \{1\}$ of $\mathrm{PU}(\mathcal{M})$ contains at least one nontrivial symmetry, then concluding that $H$ contains all symmetries of $\mathrm{PU}(\mathcal{M})$ which in turn completes the proof by Theorem 2.28.

It was also noticed by de la Harpe that Theorem 2.28 implies that $\mathrm{U}(\mathcal{M})$ admits no nontrivial finite-dimensional unitary representation.

We will reprove the following result of Giordano and Pestov [GP 06] in Chapter 3.

**Theorem 2.29.** *Let* $\mathcal{R}$ *be the hyperfinite* $\mathrm{II}_1$*-factor. Then* $\mathrm{U}(\mathcal{R})$*, endowed with the strong topology, is extremely amenable.*

We close this section by mentioning that Popa and Takesaki [PT 93] managed to prove contractibility of $\mathrm{U}(\mathcal{M})$ for some classes of $\mathrm{II}_1$ factors. It is very surprising that this result could not be proved by $\mathrm{II}_1$ factor techniques but instead by type III factor techniques. In the following theorem, we call a $\mathrm{II}_1$ factor strongly stable if it is isomorphic to the tensor product of itself with the hyperfinite $\mathrm{II}_1$ factor.

**Theorem 2.30** (Popa-Takesaki). *The unitary group* $\mathrm{U}(\mathcal{M})$ *of a* $\mathrm{II}_1$*-factor* $\mathcal{M}$ *is contractible in the strong operator topology if* $\mathcal{M}$ *is either hyperfinite, strongly stable, isomorphic to the factor* $L(F_\infty)$ *associated with the free group* $F_\infty$ *of infinite generators, or isomorphic to the tensor product of* $L(F_\infty)$ *with any other factor.*

It is still open whether $\mathrm{U}(\mathcal{M})$ is contractible in the strong operator topology for any $\mathrm{II}_1$ factor.

# 3 Extreme Amenability

This chapter is concerned with an alternative proof of extreme amenability of the unitary group $\mathrm{U}(\mathcal{R})$ of the hyperfinite $\mathrm{II}_1$ factor, endowed with the strong operator topology. Let us start right out with the main definition of this chapter.

**Definition 3.1.** A topological group $G$ is **extremely amenable** (or has the **fixed point on compacta property**) if every continuous action of $G$ on a compact space $X$ admits a fixed point.

Recall that a topological group $G$ is **amenable** if every affine continuous action $G$ on a compact convex set $X$ admits a fixed point. Thus, extreme amenability is a considerably stronger property.

Let us present a short account on the history of the subject following Pestov [Pe 06]. Extreme amenability for semigroups was first considered by Granirer [Gr 65], [Gr 66] and Mitchell [Mi 66]. Granirer and Mitchell found examples of semigroups (but not groups) which are extremely amenable. Mitchell [Mi 70, Footnote 2] asked whether extremely amenable groups exist at all.

Ellis has proved in [El 60] that every discrete group acts freely on a compact space, i.e. no discrete group can be extremely amenable. Granirer and Lau [Gr 71] showed that no locally compact group can be extremely amenable. A celebrated theorem of Veech, see [Ve 77] (or [Pe 06, Section 3.3] for a simplified proof) states that any locally compact group acts freely on a compact space.

Herer and Christensen [HC 75] provided the first example of a nontrivial extremely amenable group in 1975. Their example was more in the spirit of a counterexample than a naturally occuring topological group: Let $G$ be a topological group and $\mu$ a non-atomic probability measure on $\mathbb{R}$. We denote by $L_0(\mathbb{R}, \mu; G)$ the group of all $\mu$-equivalence classes of measurable maps from $\mathbb{R}$ to $G$. Herer and Christensen showed that $L_0(\mathbb{R}, \mu; \mathbb{R})$ is extremely amenable whenever $\mu$ is a pathological submeasure (i.e.

$\mu$ is nontrivial but dominates no nontrivial measure).

A milestone was set by Gromov and Milman [GM 83] - they proved (answering a question of Furstenberg) that the unitary group $\mathrm{U}(\ell^2)$ with the strong operator topology has the fixed point on compacta property. $\mathrm{U}(\ell^2)$ is the first known example of a Lévy group, see Definition 3.14. It is worthwile noting that de la Harpe has shown that $\mathrm{U}(\ell^2)$ is amenable in the strong operator topology, cf. [Ha 73], and non-amenable in the uniform operator topology, cf. [Ha 78].

**Theorem 3.2** (Gromov-Milman)**.** *The unitary group of a separable infinite-dimensional Hilbert space, endowed with the strong operator topology, is extremely amenable.*

Several other groups have been proved to be extremely amenable in the past decades:

- the group $L_0(\mathbb{R}, \mu; \mathbb{T})$ is extremely amenable if $\mu$ is a non-atomic measure, it is also the second known example of a Lévy group and due to Glasner [Gl 98] and independently Furstenberg and Weiss (unpublished); more examples of $L_0$-groups have been found by Pestov [Pe 02], Farah and Solecki [FS 08] and recently by Sabok [Sa 12];

- the group $\mathrm{Homeo}_+([0,1])$ of order-preserving homeomorphisms of the unit interval, due to Pestov [Pe 98];

- the isometry group $\mathrm{Iso}(\mathbb{U})$ of the Urysohn space $\mathbb{U}$, due to Pestov [Pe 02];

- groups of the form $\mathrm{Aut}(M)$ for a large class of countable structures $M$, due to Kechris, Pestov and Todorcevic [KPT 05];

- the unitary group $\mathrm{U}(\mathcal{M})$ of a finite continuous injective von Neumann algebra $\mathcal{M}$ with separable predual and the group $\mathrm{Aut}([0,1], \lambda)$ of measure-preserving automorphisms of the Lebesgue space, due to Giordano and Pestov [GP 06];

The proof of extreme amenability of $\mathrm{U}(\ell^2)$ by Gromov and Milman linked extreme amenability with the concentration of measure phenomenon. The concentration of measure phenomenon roughly states that for a typical high-dimensional structure $X$ for any $\varepsilon > 0$ the $\varepsilon$-thickening $A_\varepsilon$ of any set $A \subseteq X$ containing at least half of the points of $X$ contains already almost all points. This is capured in the concept of Lévy families and Lévy groups and will be explained in Section 3.1. Gromov and Milman proved that $\mathrm{U}(\ell^2)$ with the strong operator topology is a Lévy group by treating $\mathrm{SU}(n)$ as a

Riemannian manifold and showing that $\inf_t \mathrm{Ric}(t,t) \to \infty$ as $n \to \infty$ (showing that $(\mathrm{SU}(n), d_n, \mu_n)$ forms a Lévy family with the unnormalized Hilbert-Schmidt metric $d_n$ and Haar measure $\mu_n$ on $\mathrm{SU}(n)$), where $t$ runs over all unit tangent vectors in the tangent space of $\mathrm{SU}(n)$. This was also used in the proof of extreme amenability of $\mathrm{U}(\mathcal{R})$ by Giordano and Pestov. Our approach uses instead estimates on the so called concentration functions (see Definition 3.7) and Theorem 3.9 - this seems more natural and clear to us.

Let us lose some words on its history. The concentration of measure phenomenon was seemingly already used by Maxwell to obtain his Maxwell-Boltzmann distribution law (around 1860), for an account on this cf. [Gr 93, Section $3\frac{1}{2}$.22]. The topic was explicitly treated for Euclidean spheres in the book [Le 22] by Lévy. Important results have been discovered by Dvoretzky [Dv 59] and Milman [Mi 67], [Mi 71]. Let us mention a version of a theorem of Milman which has been established via the concentration of measure phenomenon, cf. [Pe 06, Theorem 0.0.2].

**Theorem 3.3** (Milman). *Let $f$ be a uniformly continuous function from the unit sphere $\mathbb{S}^\infty$ of $\ell^2(\mathbb{N})$ to $\mathbb{R}$. For every $\varepsilon > 0$ and every $n \in \mathbb{N}$ there exists a $n$-dimensional linear subspace $V$ of $\ell^2(\mathbb{N})$ such that restriction of $f$ to the unit sphere of $V$ is constant within $\varepsilon$.*

The above theorem says that the unit sphere of $\ell^2(\mathbb{N})$ is finitely oscillation stable, a property closely related to extreme amenability. There is a combinatorial version [Pe 06, Theorem 0.0.4] of Theorem 3.3 deduced from the classical infinite Ramsey theorem (see [Pe 06, Section 1.5] for the classical Ramsey theorem and versions of it).

**Theorem 3.4.** *Let $\gamma$ be a finite coloring (i.e. finite partition) of $\mathbb{S}^\infty$. Then for every $\varepsilon > 0$ there exists a sphere $\mathbb{S}^n \subset \mathbb{S}^\infty$ of arbitrarily high dimension $n \in \mathbb{N}$ which is monochromatic within $\varepsilon$. That is, $\mathbb{S}^n$ is contained in the $\varepsilon$-neighborhood of one of the elements of $\gamma$.*

In [MT 14] Melleray and Tsankov have generalized some ideas of [KPT 05] to establish a characterization of extreme amenability in Fraïsse-theoretic terms in the framework of continuous logic.

A geometric account on the concentration of measure phenomenon can be found in Section [Gr 93, $3\frac{1}{2}$]. Ledoux [Le 01] published a book on the topic more focusing on quantitative and probabilistic aspects as well as applications, e.g., in statistical mechanics, discrete and algorithmic mathematics. Pestov [Pe 06] has written a lecture series volume treating the concentration of measure phenomenon from various perspectives around dynamics of infinite-dimensional groups.

## 3.1 $\mathrm{U}(\mathcal{R})$ is a Lévy group

We provide an alternative proof for the following result.

**Theorem 3.5** (Giordano-Pestov)**.** $\mathrm{U}(\mathcal{R})$*, endowed with the strong operator topology, is extremely amenable.*

The proof consists of two steps. The first is to show that $\mathrm{U}(\mathcal{R})$ is a Lévy group (see Definition 3.14) via showing that $(\mathrm{U}(n), d_{1,n}, \mu_n)$ is a Lévy family (see Definition 3.6), where $d_{1,n}$ is the normalized trace metric and $\mu_n$ denotes the normalized Haar measure on $\mathrm{U}(n)$. The second step is to show that every Lévy group is extremely amenable. While we present a different approach to the first step, the second step is taken from [Pe 06].

Following Gromov and Milman [GM 83] we introduce some definitions. Note that Definition 3.6(iii), taken from [Pe 06, Section 1.2], is a generalization of Definition 3.6(ii).

**Definition 3.6.** (i) A **space with metric and measure**, or a $mm$**-space**, is a triple $(X, d, \mu)$ consisting of a set $X$, a metric $d$ on $X$ and a probability Borel measure on the metric space $(X, d)$.

(ii) A family $(X_n, d_n, \mu_n)_{n\in\mathbb{N}}$ of $mm$-spaces is a **Lévy family** if, whenever Borel subsets $A_n \subseteq X_n$ satisfy
$$\liminf_{n\to\infty} \mu_n(A_n) > 0$$
one has
$$\lim_{n\to\infty} \mu_n((A_n)_\varepsilon) = 1$$
for every $\varepsilon > 0$. Here $(A_n)_\varepsilon$ denotes the $\varepsilon$-neighbourhood of $A_n$, that is,
$$(A_n)_\varepsilon = \{x \in X_n \mid \exists y \in A_n \text{ such that } d_n(x, y) < \varepsilon\}.$$

(iii) A net $(\mu_\alpha)$ of probability Borel measures on a uniform space $(X, \mathcal{U})$ has the **Lévy**

**concentration property**, or **concentrates**, if for every family of Borel subsets $A_\alpha \subseteq X$ satisfying

$$\liminf_{\alpha} \mu_\alpha(A_\alpha) > 0$$

and every entourage $V \in \mathcal{U}_X$ one has

$$\mu_\alpha(V[A_\alpha]) \to_\alpha 1.$$

One can show that in Definition 3.6(ii) it is enough to ensure that the values $\mu_n(A_n)$, $A_n \subseteq X_n$ Borel, are bounded away from zero by any apriori chosen constant, e.g. $\mu_n(A_n) \geq 1/2$. This leads us to the concept of so called concentration functions by Milman and Schechtman [Mi 86], [MS 86].

**Definition 3.7.** The **concentration function** $\alpha_X : [0,\infty) \to [0,1/2]$ of an $mm$-space $X$, defined for $\varepsilon \geq 0$ by

$$\alpha_X(\varepsilon) = \begin{cases} \dfrac{1}{2}, & \text{if } \varepsilon = 0, \\ 1 - \inf\{\mu(A_\varepsilon) \mid A \subseteq X \text{ is Borel }, \mu(A) \geq 1/2\}, & \text{if } \varepsilon > 0. \end{cases}$$

The notion of a concentration function is closely related to that of a Lévy family.

**Lemma 3.8.** A family $(X_n, d_n, \mu_n)_{n\in\mathbb{N}}$ is a Lévy family if and only if $\alpha_{X_n}(\varepsilon) \to_{n\to\infty} 0$ pointwise for all $\varepsilon > 0$.

Assume that $H$ is a closed subgroup of a compact group $G$, equipped with a bi-invariant metric $d$. Then the formula $\widetilde{d}(g_1H, g_2H) := \inf_{h_1,h_2\in H} d(g_1h_1, g_2h_2)$ defines a left-invariant metric on the factor space $G/H$, see Lemma 4.5.2 in [Pe 06]. We refer to $\widetilde{d}$ as the **factor metric**. Define the **diameter** $\operatorname{diam}(G/H)$ of the factor space $G/H$ to be

$$\operatorname{diam}(G/H) := \sup_{g_1,g_2\in G} \inf_{h_1,h_2\in H} d(g_1h_1, g_2h_2).$$

We repeat Theorem 2.9 in [GP 06] which is an improved version of [MS 86, Theorem 7.8]. The proof is based on Martingale techniques, the interested reader finds it in [Pe 06, Sections 4.3, 4.5]. This theorem will be crucial for our proof that $(\mathrm{U}(n), d_{1,n}, \mu_n)$ forms a Lévy family.

**Theorem 3.9.** *Let $G$ be a compact group equipped with a bi-invariant metric $d$ and let $\{e\} = H_0 < H_1 < \ldots < H_n = G$ be a sequence of closed subgroups. Equip every factor space $H_k/H_{k-1}$, $k = 1, \ldots, n$, with the factor metric of $d$ and let $a_k$ denote the diameter of $H_k/H_{k-1}$. Then the concentration function of the mm-space $(G, d, \mu)$, where $\mu$ is the normalized Haar measure, satisfies*

$$\alpha_G(\varepsilon) \leq \exp\left(-\frac{\varepsilon^2}{8\sum_{k=0}^{n-1} a_k^2}\right).$$

Let $d_{1,n}$ denote the normalized trace metric on the space $M_{n\times n}(\mathbb{C})$ of $n\times n$-matrices induced from the normalized trace norm $\|\cdot\|_{1,n}$, where $n \in \mathbb{N}$. That is, with tr the unnormalized trace on $M_{n\times n}(\mathbb{C})$,

$$d_{1,n}(u, v) = \|u - v\|_{1,n} = \frac{1}{n}\operatorname{tr}(|u - v|),\ u, v \in M_{n\times n}(\mathbb{C}).$$

We first prove a inequality between the operator norm $\|\cdot\|_{op,n} := \sup_{\xi\in\mathbb{C}^n, \|\xi\|_n=1} \|\cdot\xi\|_n$ on $M_{n\times n}(\mathbb{C})$ and the normalized trace norm. Here, $\|\cdot\|_n = (\langle\cdot,\cdot\rangle_n)^{1/2}$ denotes the norm induced from the standard scalar product $\langle\cdot,\cdot\rangle_n$ on $\mathbb{C}^n$.

**Lemma 3.10.** Denote by $\operatorname{rk}(x)$ the rank of $x$. For every $x \in M_{n\times n}(\mathbb{C})$ one has

$$\|x\|_{1,n} \leq \frac{\operatorname{rk}(x)}{n}\|x\|_{op,n}.$$

***Proof.*** Let $\{\xi_k\}_{k=1,\ldots,n}$ be an orthonormal base for $\mathbb{C}^n$ such that $\{\xi_k\}_{k=1,\ldots,\operatorname{rk}(x)}$ is an orthonormal base for the range of $x$. Recall that $\operatorname{rk}(x^*x) = \operatorname{rk}(x)$ and hence $\operatorname{rk}(|x|) = \operatorname{rk}(|x|^*|x|) = \operatorname{rk}(x^*x) = \operatorname{rk}(x)$. Using the Cauchy-Schwarz inequality, we conclude

$$\begin{aligned}\operatorname{tr}(|x|) &= \sum_{j=1}^{n}\langle|x|\,\xi_j, \xi_j\rangle_n \\ &= \sum_{j=1}^{n}\sum_{k=1}^{\operatorname{rk}(x)}\langle|x|\,\xi_j, \xi_k\rangle_n\langle\xi_k, \xi_j\rangle_n \\ &= \sum_{k=1}^{\operatorname{rk}(x)}\langle|x|\,\xi_k, \xi_k\rangle_n\end{aligned}$$

$$\leq \sum_{k=1}^{\mathrm{rk}(x)} \| |x| \xi_k \|_n \|\xi_k\|_n$$
$$\leq \mathrm{rk}(x) \|x\|_{op,n} .$$

$\square$

**Proposition 3.11.** Assume that $3 \leq n \in \mathbb{N}$ and $u \in \mathrm{U}(n)$. Then there exists $v \in \mathrm{U}(n-1) \subseteq \mathrm{U}(n)$ via $v \hookrightarrow \left(\begin{smallmatrix} v & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{U}(n)$ such that

$$d_{1,n}(1, uv) \leq \frac{4}{n}.$$

In particular, $vu = 1 - x$ for some operator $x$ of rank at most 2.

***Proof.*** Denote by $\{e_k\}_{k=1,\ldots,n}$ the standard orthonormal basis of $\mathbb{C}^n$. If $ue_n = e_n$, then $u \in \mathrm{U}(n-1)$ and we can choose $v = u^* \in \mathrm{U}(n-1)$ such that $\mathrm{tr}(uv) = \mathrm{tr}(uu^*) = n$. Hence assume that $ue_n = \xi \neq e_n$ and consider $X := \mathrm{span}\langle e_n, \xi \rangle \cong \mathbb{C}^2$. There exists an unitary operator $w : X \to X$ such that $w\xi = e_n$. Define $\widetilde{w} := 1_{X^\perp} \oplus w$ with $X^\perp$ denoting the orthogonal complement of $X$ in $\mathbb{C}^n$ (with respect to the standard scalar product). Then $\widetilde{w}u \in \mathrm{U}(n-1)$, since $\widetilde{w}ue_n = (1_{X^\perp} \oplus w)\xi = e_n$. Define $v := u^*\widetilde{w}^*$ and note that $1 = 1_{X^\perp} \oplus 1_X = \widetilde{w} - 0_{X^\perp} \oplus w + 0_{X^\perp} \oplus 1_X$ to obtain

$$\begin{aligned} \mathrm{tr}(uv) &= \mathrm{tr}(vu) \\ &= \mathrm{tr}(u^*\widetilde{w}^*(\widetilde{w} - 0_{X^\perp} \oplus w + 0_{X^\perp} \oplus 1_X)u) \\ &= \mathrm{tr}(u^*\widetilde{w}^*\widetilde{w}u) + \mathrm{tr}(u^*\widetilde{w}^*(0_{X^\perp} \oplus 1_X - 0_{X^\perp} \oplus w)u) \\ &= n - \mathrm{tr}(u^*\widetilde{w}^*(0_{X^\perp} \oplus w - 0_{X^\perp} \oplus 1_X)u). \end{aligned}$$

The rank $\mathrm{rk}(x)$ of the operator $x := u^*\widetilde{w}^*(0_{X^\perp} \oplus w - 0_{X^\perp} \oplus 1_X)u$ is at most 2, since $\mathrm{rk}(0_{X^\perp} \oplus w - 0_{X^\perp} \oplus 1_X) \leq 2$. The bi-invariance of $d_{1,n}$ and Lemma 3.10 imply that

$$d_{1,n}(1, uv) = \|1 - vu\|_{1,n} = \|x\|_{1,n} \leq \frac{4}{n},$$

since $vu = 1 - x$ and $\|x\|_{op,n} \leq 2$. $\square$

Actually the proof shows that Proposition 3.11 is also valid for the orthogonal groups $\mathrm{O}(n)$, $n \geq 3$. The proof of Proposition 3.11 additionally shows the following.

**Corollary 3.12.** For every $u \in \mathrm{U}(n),\ n \geq 3$, there exists $v \in \mathrm{U}(n-1) \subseteq \mathrm{U}(n)$ such that $\mathrm{Re}(\mathrm{tr}(uv)) \geq n-4$ and $|\mathrm{Im}(\mathrm{tr}(uv))| \leq \mathrm{tr}(|\mathrm{Im}(uv)|) \leq 4$.

***Proof.*** Retain the notation of the above proof. We conclude $\mathrm{Re}(\mathrm{tr}(uv)) \geq n-4$ from the calculation . Since $\mathrm{rk}(x) \leq 2$, $x$ has at most two nonzero eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$. The inequality $\mathrm{sprad}(x) \leq \|x\|_{op} \leq 2$, see [Be 09, Proposition 9.3.2], implies that $|\lambda_1|, |\lambda_2| \leq 2$, where $\mathrm{sprad}(x)$ denotes the spectral radius of $x$. We conclude that

$$|\mathrm{Im}(\mathrm{tr}(uv))| = |\mathrm{Im}(\mathrm{tr}(x))| \leq |\mathrm{tr}(x)| \leq \mathrm{tr}(|x|) = |\lambda_1| + |\lambda_2| \leq 4.$$

□

*Remark.* If $u \in \mathrm{U}(2)$ and $v \in \mathrm{U}(1) \subseteq \mathrm{U}(2)$, then $\mathrm{tr}(uv)$ might be 0, independ of $v$. Indeed, this is true for every $u \in \mathrm{U}(2)$ of the form $u = \begin{pmatrix} 0 & -za \\ a & 0 \end{pmatrix}$, where $|a|, |z| = 1,\ a, z \in \mathbb{C}$.

**Theorem 3.13.** $(\mathrm{U}(n), d_{1,n}, \mu_n)_{n\in\mathbb{N}}$ *forms a Lévy family, where* $d_{1,n}$ *denotes the normalized trace metric on* $\mathrm{U}(n)$ *and* $\mu_n$ *is the normalized Haar measure on* $\mathrm{U}(n)$.

***Proof.*** We want to use Theorem 3.9. Consider the compact Lie group $\mathrm{U}(n),\ 3 \leq n \in \mathbb{N}$, equipped with the bi-invariant trace metric $d_{1,n}$ induced from $\|\cdot\|_{1,n}$. Embed $\mathrm{U}(k)$ in $\mathrm{U}(n)$ via $\mathrm{U}(k) \ni u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1_{n-k} \end{pmatrix} \in \mathrm{U}(n)$, where $k \leq n,\ k \in \mathbb{N}$. We calculate the diameter $a_k := \mathrm{diam}(\mathrm{U}(k)/\mathrm{U}(k-1))$ of the factor space $\mathrm{U}(k)/\mathrm{U}(k-1)$ with regard to the factor metric, where $k = 1, \ldots, n,\ \mathrm{U}(0) := \{1\}$. We use Proposition 3.11 to obtain

$$a_k \leq 2 \sup_{u\in\mathrm{U}(k)} \inf_{v\in\mathrm{U}(k-1)} d_{1,n}(1, uv) \leq \frac{8}{n}.$$

If $k = 1$, we obtain $a_1 \leq 2\sup_{u\in\mathrm{U}(1)} \inf_{v\in\mathrm{U}(0)} d_{1,n}(1, uv) = 2\sup_{u\in\mathrm{U}(1)} d_{1,n}(1, u) = \frac{4}{n}$. If $k = 2$, we obtain $a_2 \leq \frac{8}{n}$, i.e. $a_k \leq \frac{8}{n}$ for all $k = 1, \ldots, n$.

Theorem 3.9 and the above calculations imply that the concentration function of the $mm$-space $(\mathrm{U}(n), d_{1,n}, \mu_n)$ satisfies

$$\begin{aligned}\alpha_{\mathrm{U}(n)}(\varepsilon) &\leq 2\exp\left(-\frac{\varepsilon^2}{8\sum_{k=0}^{n-1} a_k^2}\right) \\ &\leq 2\exp\left(-\frac{n^2\varepsilon^2}{8\sum_{k=0}^{n-1} 64}\right)\end{aligned}$$

$$= 2\exp\left(-\frac{n\varepsilon^2}{512}\right).$$

Hence, $\alpha_{\mathrm{U}(n)} \to_{n\to\infty} 0$ pointwise on $(0,\infty)$ and thus Lemma 3.8 implies that $(\mathrm{U}(n), d_{1,n}, \mu_n)$ is a Lévy family. $\square$

Observe that the proof of Theorem 3.13 holds analogously for the orthogonal groups $\mathrm{O}(n)$, thus showing that $(\mathrm{O}(n), d_{1,n}, \mu_n)_{n\in\mathbb{N}}$ forms a Lévy family.

Let us now define the notion of a Lévy group [Pe 06, Definition 4.1.1], which is basically a group admitting a Lévy family as an approximative structure.

**Definition 3.14.** A topological group $G$ is called **Lévy group** if there is a family of compact subgroups $(G_\alpha)$ of $G$, directed by inclusion, with everywhere dense union and such that the normalized Haar measures on $G_\alpha$ concentrate with respect to the right (or left) uniform structure on $G$.

**Theorem 3.15.** $\mathrm{U}(\mathcal{R})$*, endowed with the strong operator topology, is a Lévy group.*

***Proof.*** The directed family $\{\mathrm{U}(n)\}_{n\in\mathbb{N}}$ of compact subgroups of $\mathrm{U}(\mathcal{R})$ is strongly dense in $\mathrm{U}(\mathcal{R})$ and the strong topology in $\mathrm{U}(\mathcal{R})$ is induced from the 2-metric. Moreover, the trace metric induces the same topology by Lemma 2.18. By Theorem 3.13, the family $(\mathrm{U}(n), d_{1,n}, \mu_n)$ concentrates with regard to the uniform structure of $\mathrm{U}(\mathcal{R})$. $\square$

## 3.2 Every Lévy group is extremely amenable

This section is taken from [Pe 06, Chapter 2 and 4]. It is included for convenience. We need [Pe 06, Lemma 2.1.5]. We omit its proof.

**Lemma 3.16.** Assume that $G$ is a topological group acting continuously on a compact space $X$. Then for every $\xi \in X$, the orbit mapping

$$G \ni g \mapsto g\xi \in X$$

is right uniformly continuous, while the mapping

$$G \ni g \mapsto g^{-1}\xi \in X$$

is left uniformly continuous.

Lemma 3.16 is already enough to give a direct proof that every Lévy group is extremely amenable, cf. [Pe 06, Theorem 4.1.3].

**Theorem 3.17.** *Every Lévy group is extremely amenable.*

***Proof.*** Let $G$ be a Lévy group, $\{G_\alpha\}$ compact subgroups directed by inclusion, equipped with normalized Haar measure $\mu_\alpha$. Choose an arbitrary point $x_0$ in a compact $G$-space $X$. Denote by $\nu_\alpha$ the push-forward measure of the measure $\mu_\alpha$ along the continuous orbit map $G \ni g \mapsto gx_0 \in X$ (Lemma 3.16).

Let $P(X)$ denote the space of all probability measures on $X$. Observe that $P(X)$ is compact since $X$ is compact. Hence the net $\{\nu_\alpha\}$ of elements of $P(X)$ has a cluster point $\nu$, which is left-invariant and non-zero, since all $\mu_\alpha$ are left-invariant and non-zero. Since $G$ is a Lévy group, $\nu$ has the following property: for all $A \subseteq X$ such that $\nu(A) > 0$ and all open neighbourhoods $U$ containing $A$ we have $\nu(U) = 1$ (take $A_n := A$ for all $n \in \mathbb{N}$ in the definition of a Lévy group). To see this, suppose that the support of $\nu$ consists of at least two points $x_1, x_2$. Choose nonempty disjoint neighbourhoods $U_1$ of $x_1$, $U_2$ of $x_2$. Then $1 \geq \nu(U_1 \cup U_2) = \nu(U_1) + \nu(U_2) = 2$. Thus the support of $\nu$ is a singleton, whose only element is a $G$-invariant point, i.e. a fixed point of the continuous action of $G$ on the compact space $X$. □

# 4 Bounded Normal Generation

It is a fundamental question in group theory to ask under which conditions one element of a (noncommutative) group $G$ is the product of conjugates of another element in $G$. If for every $g \in G,\ g \neq 1$, its conjugacy class and that of its inverse generate $G$ in finitely many steps we say that $G$ has the bounded normal generation property, or property (BNG), see Definition 4.7. See Section 4.1 for precise definitions and more details. In this chapter we address and answer both of these questions for many classes of unitary groups of functional analytic type. In particular, we find a normal generation function (see Definition 4.7), i.e. a function which for every $g \in G$ gives the number of steps one needs to generate the whole group with the conjugacy class of $g$ and $g^{-1}$.

For compact metrizable simple groups it is not hard to obtain property (BNG) qualitatively (i.e. without an explicit normal generation function) via a Baire category argument, cf. Proposition 4.9. A finer qualitative result is given by the basic covering lemma in group theory. It states that every finite simple group is generated in finitely many steps by each nontrivial conjugacy class, see e.g. [AHS 85]. However, it is hard to find a normal generation function even in the case of finite simple groups. Liebeck and Shalev provided a minimal normal generation function (see Definition 4.7) for finite simple groups $G$ in the main theorem of their seminal article [LS 01] and used this result to obtain many interesting applications. Their normal generation function is of the form $f(g) = c\log(|G|)/\log(|g^G|)$, where $c$ is a constant and $g^G$ the conjugacy class of $g \in G$. In 2012, Nikolov and Segal proved the bounded normal generation property for compact connected simple Lie groups - see Proposition 5.11 in [NS 12]. They can also provide a normal generation function, it is given by averaging over angles in maximal tori. We use [NS 12, Proposition 5.11] to get Theorem 4.45, i.e., a necessary and sufficient criterion for an element in $\mathrm{PU}(n)$ to be an $k$-uniform normal generator. Our in a sense rank independent version is suitable for generalization to obtain property (topBNG) in the $\mathrm{II}_1$ factor case. While the normal generation functions in [LS 01] and [NS 12] are linear, our normal generation function is quadratic.

In general, having a rank independent version of some finite-dimensional result suggests the existence of an infinite-dimensional analogue. Indeed, using our result for $\mathrm{PU}(n)$ we can prove the topological bounded generation property for the projective unitary group $\mathrm{PU}(\mathcal{M})$ of a $\mathrm{II}_1$ factor $\mathcal{M}$, endowed with the strong operator topology (see Section 4.7). These results use finite-dimensional approximation to basically reduce to the case of the projective unitary group $\mathrm{PU}(n)$, $n \in \mathbb{N}$. We deduce the topological bounded normal generation property in both cases. This result on $\mathrm{II}_1$ factors covers the topological simplicity of their projective unitary groups by Kadison in [Ka 52].

Results on topological uniform normal generators for $\mathrm{PU}(\mathcal{H})$ (e.g. in the uniform topology or finer) in Section 4.6 cannot be settled with finite-dimensional approximation and thus we have to carry the techniques to the infinite-dimensional setting. This allows us to prove the bounded normal generation property for the projective unitary group of the Calkin algebra in Section 4.6. Our result on $\mathrm{PU}(\mathcal{H})$ can be seen as a generalization of the following modified version of Theorem 1 of Halmos and Kakutani in [HK 58]. Every unitary operator on a separable Hilbert space $\mathcal{H}$ is a product of 4 unitary conjugates of a symmetry having infinite-dimensional eigenspaces.

The most interesting and hardest case to handle is that of uniform normal generators (see Definition 4.7) for the projective unitary group of a $\mathrm{II}_1$ factor, see Section 4.8. This is a generalization of several results. For example it implies the algebraic simplicity of projective unitary groups of $\mathrm{II}_1$ factors which was discovered by de la Harpe in [Ha 76] and a modified version of Broise's result in [Br 67] stating that every unitary in a $\mathrm{II}_1$ factor is the product of 32 conjugates of any symmetry with trace 0.

We need new ideas for each of these cases. The most important preliminaries are covered in Section 2.2.4, where we recall the definition and some important properties of the generalized $s$-numbers for semifinite von Neumann algebras based on the article of [FK 86] of Fack and Kosaki. We define generalized projective $s$-numbers in Section 4.4 in this context and prove some properties that are required in the preceding sections.

Let us state the main theorems of this chapter. For $t \geq 0$ and $u$ an element of the unitary group $\mathrm{U}(\mathcal{M})$ of a semifinite von Neumann algebra we define

$$\ell_t(u) \coloneqq \inf_{\lambda \in \mathrm{U}(1)} \mu_t(1 - \lambda u).$$

The definition of $\ell_t$ is also presented in Definition 4.24. For properties of $\ell_t$ we refer the reader to Section 4.4. We remark here that the easy direction in the following theorems can be found as Proposition 4.28 in Section 4.4:

If $u$ is a product of $k$ conjugates of $v$ then $\ell_{kt}(u) \leq k\ell_t(v)$ for all $t \geq 0$.

It is valid for the projective unitary group of any semifinite von Neumann algebra.

In Section 4.5 we present a result that corrects [ST 14, Lemma 4.15], see Theorem 4.45. For property (BNG) for $\mathrm{PU}(n)$ we refer to Corollary 4.44.

**Theorem 4.1.** *Let $G$ denote the projective unitary group of a factor of type $\mathrm{I_n}$, where $n \in \mathbb{N}$, $n \geq 2$. Let $u, v \in G$ and $m \in \mathbb{N}$. If $\ell_0(u) \leq m\ell_t(v)$ for all $t = 0, 1, \ldots, s-1$ then*

$$u \in (v^G \cup v^{-G})^{16m\lceil (n-1)/s \rceil}.$$

*Conversely, if $u$ is a product of $k$ conjugates of $v$ then $\ell_{kt}(u) \leq k\ell_t(v)$ for all $t \geq 0$. Moreover, $G$ has property (BNG).*

The projective unitary group $\mathrm{PU}(\mathcal{H})$ on an infinite-dimensional Hilbert space $\mathcal{H}$, endowed with the uniform topology, is not topologically simple, but there there are uniform normal generators (e.g. symmetries having two infinite-dimensional eigenspaces by [HK 58]). We provide a criterion for an element to be a topological uniform normal generator in the strong operator topology. For the proof and definition of $\|\cdot\|_{HS}$-closure see Section 4.6.

**Theorem 4.2.** *Let $G$ denote the projective unitary group on a separable infinite-dimensional Hilbert space, endowed with the strong operator topology. Assume that $u, v \in G \setminus \mathrm{U}(\mathcal{H})_{\mathcal{K}(\mathcal{H})}$ are elements satisfying $\ell_0(u) \leq m\ell_t(v)$ for all $t \geq 0$ and some $m \in \mathbb{N}$. Then*

$$u \in \overline{((v^G \cup v^{-G})^{20m})}^{\|\cdot\|_{HS}},$$

*where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. Moreover, if the elements $u$ and $v$ are diagonal, then we have*

$$u \in (v^G \cup v^{-G})^{20m}.$$

*If $u$ is a product of $k$ conjugates of $v$, then $\ell_{kt}(u) \leq k\ell_t(v)$ for all $t \geq 0$. Moreover, if $2 \leq m\ell_t(v)$ for all $t \geq 0$, then for any $u \in G$ we have*

$$u \in \overline{((v^G \cup v^{-G})^{20m})}^{\|\cdot\|_{HS}}.$$

Let $\ell_{\mathrm{ess}}(u) := \inf_{\lambda \in \mathrm{U}(1)} \|1 - \lambda u\|_{\mathrm{ess}}$. In Section 4.6 we derive the following collorary of Theorem 4.2.

**Theorem 4.3.** *Let $G$ denote the connected component of the identity of the projective unitary group of the Calkin algebra on a separable infinite-dimensional Hilbert space. Let $u, v \in G$ and assume that $\ell_{ess}(u) \leq m\ell_{ess}(v)$. Then*

$$u \in (v^G \cup v^{-G})^{20m}.$$

*Moreover, $G$ has property (BNG).*

The hardest and most interesting case from our viewpoint is the $\mathrm{II}_1$ factor case. The proof is spread over Sections 4.7 and 4.8. A topological version that also holds for non-separable $\mathrm{II}_1$ factors is given by Theorem 4.57.

**Theorem 4.4.** *Let $G$ denote the projective unitary group of a separable $\mathrm{II}_1$ factor. Let $u, v \in G$ and $m \in \mathbb{N}$. Assume that $u$ has finite spectrum and rational weights. If $\ell_0(u) \leq m\ell_t(v)$ for all $t \in [0, s]$, then*

$$u \in (v^G \cup v^{-G})^{Cm\lceil 1/s \rceil}$$

*for some constant $C \in \mathbb{N}$ independent of $u, v, m$ and $s$. If $u$ is a product of $k$ conjugates $v$ then if $\ell_{kt}(u) \leq k\ell_t(v)$ for all $t \geq 0$.*
*Moreover, $G$ has property (BNG).*

Some of these results allow a formulation in terms of a length function, a notion introduced by Stolz and Thom in [ST 14], see also Definition 4.10. Let us state this in the case of the Calkin algebra. For $u \in \mathrm{PU}(\mathcal{C})$ we let

$$\ell_{\mathrm{ess}}(u) := \inf_{\lambda \in \mathrm{U}(1)} \|1 - \lambda u\|_{\mathrm{ess}}.$$

**Theorem 4.5.** *Let $G$ denote the connected component of the identity of the projective unitary group of the Calkin algebra. If $u \in G$ is nontrivial, then one has*

$$G = (u^G \cup u^{-G})^k.$$

*for every $k \geq 40/\ell_{\mathrm{ess}}(u)$.*

We now present a formulation of Theorem 4.4 with a suitable normal generation function. For $x \in \mathcal{M}$, $\mathcal{M}$ a $\mathrm{II}_1$ factor, we define

$$L(x) := \int_{t\in[0,1]} \ell_t(x)dt.$$

**Corollary 4.6.** Let $G$ denote the projective unitary group of a separable $\mathrm{II}_1$ factor. For some constant $C \in \mathbb{N}$ the function $f : G \setminus \{1\} \to \mathbb{N}$ given by

$$f(v) := \begin{cases} C \cdot \lceil -\ln(L(v)/2)/L(v) \rceil, & \text{if } L(v) \leq 1/3, \\ C, & \text{if } L(v) > 1/3, \end{cases}$$

defines a normal generation function for $G$. That is,

$$G = \left(v^G \cup v^{-G}\right)^k$$

for every $k \geq f(v),\ v \in G \setminus \{1\}$.

Let us state here that as an main application of some of the above main theorems we obtain results on invariant automatic continuity for $\mathrm{PU}(n)$, $\mathrm{SU}(n)$ and $\mathrm{PU}(\mathcal{M})$, where $\mathcal{M}$ is a separable $\mathrm{II}_1$ factor, see Chapter 5.

## 4.1 Bounded normal generation

In this section we define the main notion of this chapter, the so called bounded normal generation property for groups.

**Definition 4.7.** (i) Let $g$ be an element of a group $G$. If there exists $k \in \mathbb{N}$ such that

$$G = (g^G \cup g^{-G})^k$$

then we call $g$ a **uniform normal generator for** $G$. If we want to emphasize the number $k$ we will write that $g$ is a $k$-uniform normal generator .

(ii) A group $G$ has the **bounded normal generation property** or **property (BNG)** if every nontrivial element is a uniform normal generator. That is, there exists a function $f : G \setminus \{1\} \to \mathbb{N}$ such that

$$G = (g^G \cup g^{-G})^{f(g)}$$

for every $g \neq 1$. We call $f$ a **normal generation function**. If there exists a normal generation function $f$ such that $f(g) \leq k$ for all $g \in G \backslash \{1\}$ and some fixed $k \in \mathbb{N}$, then we say that $G$ has the **$k$-bounded normal generation property** or **property $k$-(BNG)**.

(iii) Let $G$ be a topological group and $g \in G$. If there exists $k \in \mathbb{N}$ such that

$$G = \overline{(g^G \cup g^{-G})^k}$$

then we call $g$ a **topological uniform normal generator for** $G$.

(iv) A topological group has the **topological bounded normal generation property** or **property (topBNG)** if every nontrivial element is a topological uniform normal generator of $G$. That is,

$$G = \overline{(g^G \cup g^{-G})^{f(g)}}$$

for every $g \neq 1$, where $f : G \setminus \{1\} \to \mathbb{N}$ is again called normal generation function. If there exists a normal generation function $f$ such that $f(g) \leq k$ for all $g \in G \setminus \{1\}$ and some fixed $k \in \mathbb{N}$, then we say that $G$ has the **topological $k$-bounded normal generation property** or **property $k$-(topBNG)**.

For the sake of completeness we also define a stronger version of property (BNG). Let us call an element $g$ in a group $G$ a **strong uniform normal generator for** $G$ or $k$-uniform normal generator if there exists $k \in \mathbb{N}$ such that

$$G = (g^G)^k.$$

If every $g \in G$ is a strong uniform normal generator, then we say that $G$ has the **strong bounded normal generation property**. Analogously we say that a topological group $G$ has the **strong topological bounded normal generation property** if every $g \in G$ is a **strong topological uniform normal generator**, i.e.

$$G = \overline{(g^G)^k}$$

for some $k \in \mathbb{N}$. We again call a function witnessing the strong (topological) bounded normal generation property a normal generation function.

In the case of compact simple groups one can easily show property (BNG) via a Baire category argument, see Proposition 4.9. However, getting a quantitative result (a normal generation function) is much harder even in the case of finite simple groups.

Given a group with property (BNG) or (topBNG) one can ask for the minimal (up to a universal multiplicative constant) normal generation function. That is, a normal generation function $f_0$ for $G$ is minimal if for every other normal generation function $f$ one has $f(g) \geq cf_0(g)$ for all $g \in G$ and some constant $c \in \mathbb{N}$ independent of $g$. Analogously for a group with property $k$-(BNG) or $k$-(topBNG) one can ask for the minimal $k \in \mathbb{N}$.

Let us list some known examples for the above definitions.

*Examples.* (i) Every finite simple group $G$ has the strong bounded normal generation property by the basic covering theorem, see e.g. [AHS 85]. In fact these groups have strong $k$-bounded normal generation property, where $k$ depends only on the group in question. A minimal (up to a universal multiplicative constant) normal generation function for every finite simple group is given in [LS 01, Theorem 1] - it is given by

$$f(g) := \lceil \log |G| \,/\, \log \left|g^G\right| \rceil \quad \text{for } g \in G \setminus \{1\}\,.$$

(ii) Compact connected simple Lie groups have property (BNG), see Proposition 5.11 of [NS 12]. Nikolov and Segal also provide a normal generation function, it is given by averaging over angles in maximal tori. We provide a different normal generation function for $\mathrm{PU}(n)$ via a study of projective singular values and reprove property (BNG) for $\mathrm{PU}(n)$, see Section 4.5. In Corollary 4.44 we provide the following normal generation function for $\mathrm{PU}(n)$:

$$f(u) := 16n \lceil 1/\inf_{\lambda \in \mathrm{U}(1)} \|1 - \lambda u\| \rceil \quad \text{for } u \in \mathrm{PU}(n) \setminus \{1\}\,.$$

We also list some examples which will be obtained in the following sections of this chapter.

*Examples.* (i) The connected component $\mathrm{PU}_1(\mathcal{C})$ of the identity of the projective unitary group of the Calkin algebra $\mathcal{C}$ has property (BNG), see Section 4.6. A normal generation function is given by

$$f(u) := 40 \lceil 1/\inf_{\lambda \in \mathrm{U}(1)} \|1 - \lambda u\|_{\mathrm{ess}} \rceil \quad \text{for } u \in \mathrm{PU}_1(\mathcal{C}) \setminus \{1\}\,,$$

see Theorem 4.52.

(ii) The projective unitary group of a $\mathrm{II}_1$ factor has property (topBNG), see Section 4.7. If the $\mathrm{II}_1$ factor $\mathcal{M}$ is separable, then $\mathrm{PU}(\mathcal{M})$ has property (BNG), cf. 4.8. To describe our normal generation function in this case, we need the definition of generalized projective $s$-numbers $\ell_t(\cdot)$ from Section 4.4. For $u \in \mathrm{PU}(\mathcal{M})$ we define

$$L(u) := \int_{[0,1]} \ell_t(u)dt.$$

A normal generation function is given by (see Corollary 4.67)

$$f(u) := \begin{cases} C \cdot \lceil - \ln(L(u)/2)/L(u) \rceil, & \text{if } L(u) \leq 1/3, \\ C, & \text{if } L(u) > 1/3, \end{cases}$$

where $u \in \mathrm{PU}(\mathcal{M}) \setminus \{1\}$ and $C \in \mathbb{N}$ is a constant.

(iii) There are groups which have uniform normal generators but do not have property (BNG). The unitary group $\mathrm{U}(\mathcal{H})$, where $\mathcal{H}$ is a separable infinite-dimensional Hilbert space, provides such an example - see Section 4.6. Some special uniform normal generators are symmetries with two infinite eigenspaces, which follows from Theorem 1 in [HK 58] - Halmos and Kakutani could additionally prove that the minimal number such that these symmetries are uniform normal generators is 4.

Let us state some properties of groups which have property (BNG). We omit the proof - (i) and (ii) are easy to see and (iii) is a consequence of Theorem 5.5 in [Dr 87].

**Proposition 4.8.** (i) If a group $G$ has property (BNG) then it is simple. If $G$ is a topological group with property (topBNG) then it is topologically simple.

(i) Assume that $G$ and $H$ are topological groups and $G$ has property (BNG). If $\pi : G \to H$ is a continuous homomorphism with dense image, then $H$ has property (topBNG).

(iii) Every group can be embedded into a group with 2-bounded normal generation property and the same cardinality.

The infinite alternating group of all finitely supported even permutations on $\mathbb{N}$ provides an example of a simple group which does not have property (BNG). In [DTW 99] the authors have shown that there are simple automorphism groups of cycle-free partial orders which do not have property (BNG). The interested reader can find the definitions and details in [DTW 99].

Using a Baire category argument one can show the following qualitative result for compact simple groups. To give an quantitative result with a normal generation function is far more complicated, cf. the proofs of the above mentioned examples.

**Proposition 4.9.** Every compact simple group $G$ has property (BNG).

***Proof.*** The compact simple groups are classified into discrete finite nonabelian simple groups, discrete cyclic groups of prime order, and (centerfree) compact connected simple Lie groups, see [HM 06]. In the first two cases it is obvious that $G$ has property (BNG). So assume that $G$ is a compact connected simple Lie group. Observe that for any $g \in G \setminus 1$ the set $\bigcup_{n\in\mathbb{Z}}(g^G)^n$ forms a nontrivial normal subgroup of $G$ and note that $g^G$ is compact as the continuous image of the compact set $G$ under conjugation. Due to the fact that $G$ is simple we have

$$G = \bigcup_{n\in\mathbb{Z}} (g^G)^n.$$

For $k \in \mathbb{N}$ we define $C^k := \bigcup_{|n|\leq k}(g^G)^n$. Since $G$ is Polish we can apply the Baire category theorem to obtain the existence of $m \in \mathbb{N}$ such that

$$\mathrm{int}(C^m) \neq \emptyset.$$

Assume that $U \subseteq C^m$ is open and let $V := UU^{-1} \subseteq C^{2m}$. Since $1 \in V$ we have $C^m \subseteq VC^m$. Thus $\bigcup_{n\in\mathbb{N}} VC^n$ is an open covering of $G$. Now compactness of $G$ implies that there exists a $m' \in \mathbb{N}$ such that

$$G = \bigcup_{|n|\leq m'} VC^n \subseteq \bigcup_{|n|\leq m'} C^{n+2m}.$$

Thus $G$ has property (BNG). $\square$

Actually every topologically simple compact topological group has property (BNG), since topological simplicity implies algebraic simplicity for compact groups by [HM 06, Theorem 9.90].

## 4.2 Length functions

Natural candidates for a normal generation function are closely related to so called length functions. They were introduced by Stolz and Thom in [ST 14].

**Definition 4.10.** Let $G$ be a group. We say that a function

$$\ell : G \to [0, \infty)$$

is a **pseudo length function** on $G$ if for all $g, h \in G$ the following properties hold:

(i) $\ell(1) = 0$;

(ii) $\ell(g) = \ell(g^{-1})$;

(iii) $\ell(gh) \leq \ell(g) + \ell(h)$.

If $\ell$ is a pseudo length function which additionally satisfies that $\ell(g) = 0$ implies $g = 1$, then $\ell$ is called **length function**. A pseudo length function $\ell$ is called **invariant** if one has

$$\ell(hgh^{-1}) = \ell(g) \text{ for all } g, h \in G.$$

We collect some basic properties of length functions (see [ST 14]), the proofs are easy and will be omitted.

**Proposition 4.11.** Let $\ell$ be a (pseudo) length function on a group $G$.

(i) $d(g, h) := \ell(gh^{-1}),\ g, h \in G$, defines a (pseudo) metric on $G$. Conversely a (pseudo) metric $d$ on $G$ induces a (pseudo) length function on $G$ by $\ell(g) := d(1, g),\ g \in G$.

$\ell$ is invariant if and only if $d$ is bi-invariant.

(ii) Let $H$ be a normal subgroup of $G$ and assume that $\ell$ is invariant. Then

$$\ell_{G/H}(gH) := \inf_{h \in H} \ell(gh)$$

defines an invariant (pseudo) length function on the group $G/H$.

Conversely if $\ell_{G/H}$ is a pseudo length function on $G/H$, then

$$\ell_G(g) := \ell_{G/H}(gH),\ g \in G,$$

defines a pseudo length function on $G$. If $\ell_{G/H}$ is invariant, then $\ell_G$ is invariant, too.

(iii) Assume that $\ell$ is invariant. Then the set $\{g \in G \mid \ell(g) = 0\}$ is a normal subgroup of $G$.

We present some examples of (pseudo) length functions.

*Examples.* (i) Let $G$ be a finite simple group. Then the conjugacy length $\ell_{\mathrm{conj}}(g) := \frac{\log|g^G|}{\log|G|}$ defines an invariant length function. In fact, Liebeck and Shalev [LS 01] showed that $\lceil 1/\ell_{\mathrm{conj}}(\cdot)\rceil$ is (up to a multiplicative constant) the best possible normal generation function for finite simple groups. More precisely, for every nontrivial $g$ in a finite simple group $G$ one has

$$G = (g^G)^m$$

for every $m \geq c/\ell_{\mathrm{conj}}(g)$ and some constant $c \in \mathbb{N}$ independent of $g$.

(ii) Let $\mathcal{C}$ denote the Calkin algebra on the separable infinite-dimensional Hilbert space $\mathcal{H}$. Write $\mathrm{PU}(\mathcal{C})$ for the projective unitary group of $\mathcal{C}$. The essential norm on $\mathcal{C}$ given by $\|x\|_{\mathrm{ess}} = \inf_{y \in \mathcal{K}(\mathcal{H})} \|x - y\|$, $x \in \mathcal{C}$, induces a length function on $\mathrm{PU}(\mathcal{C})$ via

$$\ell_{\mathrm{ess}}(u) := \inf_{\lambda \in \mathrm{U}(1)} \|1 - \lambda u\|_{\mathrm{ess}}, u \in \mathrm{PU}(\mathcal{C}).$$

In Theorem 4.52 we show that $40\lceil 1/\ell_{\mathrm{ess}}(\cdot)\rceil$ defines a normal generation function.

(iii) Let $\mathcal{M}$ be a $\mathrm{II}_1$ factor. By Proposition 4.11(i) the norms $\|\cdot\|$, $\|\cdot\|_1$ and $\|\cdot\|_2$ induce invariant length functions on $\mathrm{U}(\mathcal{M})$. Namely, $\|1 - u\|_1$, $u \in \mathrm{U}(\mathcal{M})$, defines an invariant length function on $\mathrm{U}(\mathcal{M})$. It follows from Proposition 4.11(ii) that

$$\ell(u) := \inf_{\lambda \in \mathrm{U}(1)} \|1 - \lambda u\|_1, \; u \in \mathrm{U}(\mathcal{M}),$$

defines an invariant length function on $\mathrm{PU}(\mathcal{M})$ (and an invariant pseudo length function on $\mathrm{U}(\mathcal{M})$). However, our normal generation function for $\mathrm{PU}(\mathcal{M})$ is of a different form, cf. Corollary 4.67.

Let us present a lower bound on normal generation functions.

**Proposition 4.12.** Let $G$ be a group with property (BNG) and normal generation function $f$. Assume that $\ell$ is an invariant length function on $G$. Then there exists a constant $c \in \mathbb{R}$ such that

$$f(g) \geq \frac{c}{\ell(g)} \quad \text{for every } g \in G \setminus \{1\}.$$

***Proof.*** Let $g \in G$ be a uniform normal generator and assume that $h \in G = (g^G \cup g^{-G})^{f(g)}$. Since $\ell$ is an invariant length function, we have

$$\ell(h) \leq f(g)\ell(g).$$

Thus

$$f(g) \geq \frac{\ell(h)}{\ell(g)}.$$

Since $g \in G$ was an arbitrary uniform normal generator, we are done by taking $c := \ell(h)$ for some fixed nontrivial $h \in G$. $\square$

Proposition 4.12 allows us to conclude that our normal generation function in Theorem 4.52 is the best possible normal generation function (up to a multiplicative constant). In Corollary 4.67 we provide an upper bound on the best possible normal generation function in the case of $\mathrm{II}_1$ factors.

## 4.3 Products of symmetries

This section can be seen as a warm-up for the remainder of this chapter (in particular for our results on $\mathrm{II}_1$ factors). Our aim is to modify [Br 67, Theorem 1] by Broise. The original version states that every unitary element in a $\mathrm{II}_1$ factor can be written as $u = v_1 \cdot \ldots \cdot v_n$, where $v_i = s_i r_i s_i r_i$ and $r_i, s_i$ are symmetries. From his formulation it is not clear whether $n$ depends on $u$. However, going through his proof carefully one finds that $n = 8$, independent of the unitary $u$ in question.

We present his proof with few extra ingredients to obtain that every unitary in a $\mathrm{II}_1$ factor $\mathcal{M}$ is the product of 32 conjugates of any symmetry of trace 0, see Theorem 4.19. In the terminology we have developed in Section 4.1 this means that every symmetry of trace 0 is a 32-uniform normal generator for the projective unitary group of $\mathcal{M}$. We conclude Corollary 4.20 from the proof of Theorem 4.19, which will become useful in the proof of property (BNG) in the $\mathrm{II}_1$ factor case, see Theorem 4.65.

In Subsection 4.3.1 we prove the fact that symmetries in a $\mathrm{II}_1$ factor are conjugate if and only if they have the same trace. This fact is certainly well-known, but we include it for the readers convenience. In Subsection 4.3.2 we present the proof of Broise with few changes in order to combine with the results from Subsection 4.3.1.

### 4.3.1 Conjugate symmetries

In this subsection, we analyze when symmetries in a $\mathrm{II}_1$-factor are conjugate. It turns out, see Proposition 4.15, that this is the case if and only if they have the same trace. This result will help us to refine [Br 67, Theorem 1].

**Lemma 4.13.** Every symmetry $s$ in a von Neumann algebra $\mathcal{M}$ is of the form $s = 1-2p$ for some projection $p \in \mathrm{Proj}(\mathcal{M})$.

***Proof.*** Let $s$ be a symmetry (i.e., a self-adjoint unitary). Define $p = 1/2(1-s)$. We have to show that $p$ is a projection. We have

$$\begin{aligned} p^* &= \frac{1}{2}(1-s^*) = p, \\ p^2 &= \frac{1}{4}(1-2s+s^2) = \frac{1}{4}(2-2s) = p, \end{aligned}$$

which proves the claim. $\square$

We recall that the notions of equivalence of projections and unitary equivalence of projections coincide in finite von Neumann algebras. This statement can be found as Exercise 6.9.11 in [KR 86].

**Proposition 4.14.** Asume that the von Neumann algebra $\mathcal{M}$ is finite. Then for each pair of equivalent projections $p$ and $q$ in $\mathcal{M}$, there is a unitary operator $u \in \mathrm{U}(\mathcal{M})$ such that $upu^* = q$. That is, the notions of equivalence and unitary equivalence of projections coincide in finite von Neumann algebras.

Now we have gathered the necessary results to prove the main result of this subsection.

**Proposition 4.15.** Let $\mathcal{M}$ be a finite von Neumann algebra with faithful finite normal trace $\tau$. Two symmetries $s, t \in \mathcal{M}$ are conjugate if and only if they have the same trace.

***Proof.*** If $s = utu^*$ for some $u \in \mathrm{U}(\mathcal{M})$, then clearly $\tau(s) = \tau(utu^*) = \tau(t)$. Assume that $\tau(s) = \tau(t)$. It follows from Lemma 4.13 that there exist projections $p, q \in \mathrm{Proj}(\mathcal{M})$ such that $s = 1-2p,\ t = 1-2q$. Hence $\tau(p) = \tau(q)$. By Proposition 4.14 there exists a unitary $u \in \mathrm{U}(\mathcal{M})$ such that $q = upu^*$. This implies

$$t = 1-2q = 1-2upu^* = u(1-2p)u^*,$$

as claimed. $\square$

### 4.3.2 Products of symmetries with trace 0

The aim of this subsection is to prove that every unitary element in a $\mathrm{II}_1$-factor is a product of at most 32 symmetries, all of which have trace 0 and are thus conjugate by Proposition 4.15. The results and proofs are taken and refined from [Br 67].

First, let us state [Di 81, Proposition I.12].

**Proposition 4.16.** Every abelian von Neumann subalgebra of a von Neumann algebra $\mathcal{M}$ is contained in a maximal abelian von Neumann subalgebra of $\mathcal{M}$.

We state [Br 67, Lemma 4]. As we do not modify this result we omit its proof.

**Lemma 4.17.** Assume that $\mathcal{M}$ is a $\mathrm{II}_1$-factor with normalized trace $\tau$ and $\mathcal{N}$ a maximal abelian von Neumann subalgebra of $\mathcal{M}$. Let $p \in \mathrm{Proj}(\mathcal{N})$ be a nonzero projection in $\mathcal{N}$. Then there exist orthogonal $p_1, p_2 \in \mathrm{Proj}(\mathcal{N})$ such that $p_1 \sim p_2$ and $p = p_1 + p_2$.

The next lemma is a strengthened version of [Br 67, Lemma 5].

**Lemma 4.18.** Assume that $\mathcal{M}$ is a $\mathrm{II}_1$-factor and $p \in \mathrm{Proj}(\mathcal{M})$. Let $n \in \{2,3\}$. Suppose that $\{w_{i,j}\}_{1\leq i,j\leq n}$ and $\{w_i\}_{1\leq i\leq n}$ are families of elements in $\mathcal{M}$ satisfying the following three conditions:

(1) $w_{i,l}w_{l,j} = w_{i,j}$ and $(w_{i,j})^* = w_{i,j}^* = w_{j,i}$ for all $1 \leq i,j \leq n$.
(2) $p$ and $\{w_{i,i}\}_{1\leq i\leq n}$ are pairwise orthogonal projections.
(3) $p + \sum_{i=1}^n w_i \in \mathrm{U}(\mathcal{M})$ and $w_i w_{i,i} = w_{i,i} w_i = w_i$ for all $1 \leq i \leq n$.

(i) If $n = 2$ and $w_2 = w_{2,1}w_1^* w_{1,2}$, then $p + w_1 + w_2 = stst$ for some symmetries $s, t \in \mathrm{U}(\mathcal{M})$ satisfying $\tau(s) = \tau(t) = 0$.

(ii) If $n = 3$ and $w_3 = w_{3,2}w_2^* w_{2,1} w_1^* w_{1,3}$, then $p + w_1 + w_2 + w_3 = s_1 t_1 s_1 t_1 \cdot s_2 t_2 s_2 t_2$ for some symmetries $s_1, s_2, t_1, t_2 \in \mathrm{U}(\mathcal{M})$ satisfying $\tau(s_i) = \tau(t_i) = 0,\ i = 1,2$.

***Proof.*** Consider first the case $p = 0$. Put $\mathcal{M}_{1,1} := w_{1,1}\mathcal{M}w_{1,1}$ and $\mathcal{N} := \mathcal{M}_{1,1} \otimes M_{n\times n}(\mathbb{C})$. Then $\psi : \mathcal{M} \to \mathcal{N},\ x \mapsto (x_{i,j}) = (w_{1,i}xw_{j,1})$ is a homomorphism from $\mathcal{M}$ to the matrix algebra $\mathcal{N}$. Conditions (1),(2) and (3) imply that $\psi$ is an isomorphism. By corollary to Proposition I.2 in [Di 81], $\mathcal{M}_{1,1}$ is again a $\mathrm{II}_1$-factor, hence $\mathcal{N}$ is a $\mathrm{II}_1$-factor by [KR 86, Proposition 11.2.20].

(i) The assumptions imply that

$$\begin{aligned}\psi(w_1+w_2) &= \begin{pmatrix} w_{1,1}w_1w_{1,1} & w_{1,1}w_1w_{2,1} \\ w_{1,2}w_1w_{1,1} & w_{1,2}w_1w_{2,1}\end{pmatrix} + \begin{pmatrix} w_{1,1}w_2w_{1,1} & w_{1,1}w_2w_{2,1} \\ w_{1,2}w_2w_{1,1} & w_{1,2}w_2w_{2,1}\end{pmatrix} \\ &= \begin{pmatrix} w_{1,1}w_1w_{1,1} & 0 \\ 0 & w_{1,2}w_{2,1}w_1^*w_{1,2}w_{2,1}\end{pmatrix} \\ &= \begin{pmatrix} w_1 & 0 \\ 0 & w_{1,1}w_1^*w_{1,1}\end{pmatrix} \\ &= \begin{pmatrix} w_1 & 0 \\ 0 & w_1^*\end{pmatrix}.\end{aligned}$$

By condition (3), $w_1+w_2$ is unitary in $\mathcal{M}$, hence $w_1 = w_{1,1}(w_1+w_2)w_{1,1}$ is unitary in $\mathcal{M}_{1,1}$. Thus by functional calculus there exists an element $u \in \mathrm{U}(\mathcal{M}_{1,1})$ such that $u^2 = w_1$. It follows that

$$\begin{pmatrix} w_1 & 0 \\ 0 & w_1^*\end{pmatrix} = \begin{pmatrix} u^2 & 0 \\ 0 & u^{*2}\end{pmatrix} = \begin{pmatrix} 0 & u \\ u^* & 0\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} 0 & u \\ u^* & 0\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} = stst,$$

where $s := \begin{pmatrix} 0 & u \\ u^* & 0\end{pmatrix}$ and $t := \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$. Since $\psi$ is an isomorphism,

$$w_1+w_2 = \psi^{-1}(s)\psi^{-1}(t)\psi^{-1}(s)\psi^{-1}(t),$$

and $\psi^{-1}(s)$ and $\psi^{-1}(t)$ are again symmetries. Here we used that although $\psi$ is no $*$-isomorphism, we have that if $\psi(x) = \psi(x)^*$ for $x \in \mathcal{M}$, then $x = x^*$.
Note that the trace of $s$ and $t$ in $\mathcal{M}_{1,1} \otimes M_{n\times n}(\mathbb{C})$ vanishes. The fact that $\psi$ is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$ and that isomorphisms between $\mathrm{II}_1$ factors are trace-preserving imply that the trace of $\psi^{-1}(s)$ and $\psi^{-1}(t)$ also vanish.

(ii) Put $\widetilde{w}_2 := w_{1,2}w_2w_{2,1}$. The condition $w_3 = w_{3,2}w_2^*w_{2,1}w_1^*w_{1,3}$ implies that

$$\psi(w_1+w_2+w_3) = \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_{1,2}w_2w_{2,1} & 0 \\ 0 & 0 & w_{1,3}w_{3,2}w_2^*w_{2,1}w_1^*w_{1,3}w_{3,1}\end{pmatrix}$$

$$= \begin{pmatrix} w_1 & 0 & 0 \\ 0 & \widetilde{w}_2 & 0 \\ 0 & 0 & \widetilde{w}_2^* w_1^* \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \widetilde{w}_2 & 0 \\ 0 & 0 & \widetilde{w}_2^* \end{pmatrix} \begin{pmatrix} w_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & w_1^* \end{pmatrix}.$$

Note that $w_1$ and $\widetilde{w}_2$ are unitary in $\mathcal{M}_{1,1}$. From the above calculation and the case (i) we conclude that $w_1 + w_2 + w_3 = s_1 t_1 s_1 t_1 \cdot s_2 t_2 s_2 t_2$, where $s_i, t_i$ are symmetries in $\mathcal{M}$ having trace 0, $i = 1, 2$.

Now consider the case $p \neq 0$, i.e., $\tau(p) > 0$. Decompose $p$ into $\sum_{i=1}^n p_i$, where $p_i$ are equivalent orthogonal projections. Then define $\widetilde{w}_i := w_i + p_i$, $i = 1, \dots, n$. Hence $\sum_{i=1}^n \widetilde{w}_i \in \mathrm{U}(\mathcal{M})$ by condition (3). Adjust the system $\{w_{i,j}\}$ by setting $\widetilde{w}_{i,j} := w_{i,j} + x_i^* x_j$, where $x_l$ are the partial isometries such that $x_l^* x_l = p_l$ and $x_l x_l^* = p_1$. The families $\{\widetilde{w}_{i,j}\}$ and $\{\widetilde{w}_i\}$ clearly satisfy conditions (2) and (3). We check condition (1). We have $(\widetilde{w}_{i,j})^* = w_{j,i} + x_j^* x_i = \widetilde{w}_{j,i}$ and

$$\begin{aligned} \widetilde{w}_{i,l}\widetilde{w}_{l,j} &= w_{i,j} + x_i^* x_l x_l^* x_j = w_{i,j} + x_i^* x_i x_i^* x_j \\ &= \widetilde{w}_{i,j}. \end{aligned}$$

That is, we may use the first part of the proof on the adjusted families. Finally, let us show that if the assumptions of (i) are satisfied for $w_2$, then we have

$$\begin{aligned} \widetilde{w}_{2,1}\widetilde{w}_1^*\widetilde{w}_{1,2} &= w_{2,1} w_1^* w_{1,2} + x_2^* x_1 p_1 x_1^* x_2 = w_2 + x_2^* x_1 x_1^* x_2 = w_2 + x_2^* x_2 \\ &= \widetilde{w}_2. \end{aligned}$$

Analogously one can check that if $w_3$ satisfies the assumptions of (ii), then

$$\widetilde{w}_3 = \widetilde{w}_{3,2}\widetilde{w}_2^*\widetilde{w}_{2,1}\widetilde{w}_1^*\widetilde{w}_{1,3}.$$

□

We have gathered all necessary results to prove the modified version of [Br 67, Theorem 1]. The main idea of its proof is to construct families of elements in the $\mathrm{II}_1$-factor satisfying the conditions in Lemma 4.18.

**Theorem 4.19** (Broise)**.** *Let $\mathcal{M}$ be a factor of type $\mathrm{II}_1$. Then every element in $\mathrm{U}(\mathcal{M})$ is the product of 32 conjugates of any symmetry $s \in \mathrm{U}(\mathcal{M})$ satisfying $\tau(s) = 0$.*

***Proof.*** Assume that $u \in \mathrm{U}(\mathcal{M})$. Using Lemma 4.17 we conclude that there exists a projection $p_0$ in maximal commutative von Neumann subalgebra containing $u$ such that $p_0 \sim 1 - p_0$. Since $p_0$ commutes with $u$, we have

$$u = (up_0 + 1 - p_0)(p_0 + u(1 - p_0)).$$

Put $u_0 := up_0$. It suffices to show that $u_0 + 1 - p_0$ is a product of 16 conjugates of a symmetry of trace 0 (just replace $p_0$ by $1 - p_0$ in the following construction).

Let now $\{p_0(n)\}_{n\in\mathbb{N}_0}$ be a sequence of pairwise orthogonal projections satisfying

$$p_0(0) = p_0, \quad \tau(p_0(n)) = 2^{-(n+1)}, \quad \sum_{n\in\mathbb{N}_0} p_0(n) = 1.$$

Let $\mathcal{N}_1$ denote the von Neumann algebra generated by $u_0$. Using Lemma 4.17, we conclude that there exist two orthogonal projections $p_1(1), p_2(1) \in \mathcal{N}_1' \cap \mathcal{M}$ such that

$$p_1(1) + p_2(1) = p_0(0), \quad \tau(p_1(1)) = \tau(p_2(1)) = \tau(p_0(1)) = 2^{-2}.$$

Since the projections $p_0(1), p_1(1)$ and $p_2(1)$ are equivalent and pairwise orthogonal, there exists a family $\{v_{i,j}(1)\}_{0\leq i,j\leq 2}$ of elements in $\mathcal{M}$ such that

$$v_{i,i}(1) = p_i(1), \quad v_{i,l}(1)v_{l,j}(1) = v_{i,j}(1), \quad (v_{i,j}(1))^* = v_{j,i} \text{ for all } 0 \leq i, j, l \leq 2.$$

Putting $u_1 \coloneqq v_{0,1}(1)u_0v_{1,2}(1)u_0v_{2,0}(1)$, we obtain

$$\begin{aligned}
u_1u_1^* &= (v_{0,1}(1)u_0v_{1,2}(1)u_0v_{2,0}(1))(v_{0,2}(1)u_0^*v_{2,1}(1)u_0^*v_{1,0}(1)) \\
&= v_{0,1}(1)u_0v_{1,2}(1)p_2(1)u_0u_0^*v_{2,1}(1)u_0^*v_{1,0}(1) \\
&= v_{0,1}(1)u_0v_{1,2}(1)v_{2,2}(1)v_{2,1}(1)u_0^*v_{1,0}(1) \\
&= v_{0,1}(1)v_{1,1}(1)v_{1,0}(1) \\
&= p_0(1) \\
&= u_1^*u_1.
\end{aligned}$$

Inductively on can construct the following objects:

$$\mathcal{N}_n,\ p_1(n),\ p_2(n),\ \{v_{i,j}\}_{0\le i,j\le n},\ u_n.$$

Here $\mathcal{N}_n$ is the von Neumann algebra generated by $u_{n-1}$, $p_1(n)$ and $p_2(n)$ are orthogonal projections in $\mathcal{M}$ satisfying

$$p_1(n)+p_2(n)=p_0(n-1),\quad p_1(n)\sim p_2(n)\sim p_0(n),$$

$\{v_{i,j}\}_{0\le i,j\le n}$ is a family of elements in $\mathcal{M}$ satisfying

$$v_{i,i}(n)=p_i(n),\ v_{i,l}(n)v_{l,j}(n)=v_{i,j}(n),\ (v_{i,j}(n))^*=v_{j,i}(n) \text{ for all } 0\le i,j,l\le 2,$$

and

$$u_n := v_{0,1}(n)u_{n-1}v_{1,2}(n)u_{n-1}v_{2,0}(n).$$

We show that for all $n\in\mathbb{N}$ we can assume the following properties of these objects:

(i) $\mathcal{N}_n$ is commutative;

(ii) $p_1(n)$ and $p_2(n)$ belong to $\mathcal{N}_n'$ (and hence commute with $u_{n-1}$);

(iii) $u_nu_n^*=u_n^*u_n=p_0(n)$.

These properties have been verified for $n=1$. We proceed by induction on $n\in\mathbb{N}$. Suppose the assertion holds for $n\in\mathbb{N}$. We show that it holds for $n+1$.

Property (iii) for $n$ implies property (i) for $n+1$. Lemma 4.17 and property (i) for $n+1$ show that we can suppose that (ii) holds for $n+1$. Since (ii) holds for $n+1$ and (iii) holds for $n\in\mathbb{N}$, we have

$$\begin{aligned}
&u_{n+1}u_{n+1}^*\\
&=(v_{0,1}(n+1)u_nv_{1,2}(n+1)u_nv_{2,0}(n+1))(v_{0,2}(n+1)u_n^*v_{2,1}(n+1)u_n^*v_{1,0}(n+1))\\
&=v_{0,1}(n+1)u_nv_{1,2}(n+1)u_np_2(n+1)u_n^*v_{2,1}(n+1)u_n^*v_{1,0}\\
&=v_{0,1}(n+1)u_nv_{1,2}(n+1)p_0(n)v_{2,1}(n+1)u_n^*v_{1,0}(n+1)\\
&=v_{0,1}(n+1)u_np_1(n+1)u_n^*v_{1,0}(n+1)\\
&=v_{0,1}(n+1)p_0(n)v_{1,0}(n+1)\\
&=v_{0,1}(n+1)v_{1,1}(n+1)v_{1,0}(n+1)\\
&=p_0(n+1),
\end{aligned}$$

that is, (iii) holds for $n+1$, as claimed.

Put

$$w_{i,j} \coloneqq \sum_{m\geq 0} v_{i,j}(2m+1), \quad w'_{i,j} \coloneqq \sum_{m\geq 1} v_{i,j}(2m).$$

Then $w_{0,0},\ w_{1,1},\ w_{2,2}$, respectively $p_0,\ w'_{0,0},\ w'_{1,1},\ w'_{2,2}$ are mutually orthogonal projections and

$$w_{i,l}w_{l,j} = w_{i,j},\ w^*_{i,j} = w_{j,i},\ w'_{i,l}w'_{l,j} = w'_{i,j},\ \text{and } (w'_{i,j})^* = w'_{j,i}.$$

Define the following elements:

$$\begin{aligned} w_0 &\coloneqq \sum_{0\leq m} u^*_{2m+1}, \quad & w_1 &\coloneqq \sum_{0\leq m} u_{2m}p_1(2m+1), \quad & w_2 &\coloneqq \sum_{0\leq m} u_{2m}p_2(2m+1),\\ w'_0 &\coloneqq \sum_{1\leq m} u^*_{2m}, & w'_1 &\coloneqq \sum_{0\leq m} u_{2m+1}p_1(2m+2), & w'_2 &\coloneqq \sum_{0\leq m} u_{2m+1}p_2(2m+2). \end{aligned}$$

The equation $p_1(n+1) + p_2(n+1) = p_0(n)$ implies that

$$w_1 + w_2 = \sum_{m\geq 0} u_{2m}(p_1(2m+1) + p_2(2m+1)) = u_0 + \sum_{m\geq 1} u_{2m}p_0(2m) = u_0 + w_0^{'*},$$

and

$$w'_1 + w'_2 = \sum_{m\geq 0} u_{2m+1}p_0(2m+1) = w^*_0.$$

Using these two formulas as well as $u_n u_m = 0$ for all $n \neq m$ (since $u_n = p_0(n)u_n p_0(n)$), we obtain

$$\begin{aligned} (w_1 + w_2 + w_0)(p_0 + w'_1 + w'_2 + w'_0) &= (u_0 + w_0 + w'^*_0)(p_0 + w^*_0 + w'_0)\\ &= u_0 + w_0 w^*_0 + w'^*_0 w'_0\\ &= u_0 + \sum_{m\geq 1} p_0(m)\\ &= u_0 + 1 - p_0. \end{aligned}$$

Using Properties (ii),(iii) and the fact that $p_1(n)\perp p_2(m)$ for all $n, m \in \mathbb{N}$, we conclude that

$$(w_1 + w_2 + w_0)^*(w_1 + w_2 + w_0)$$

$$\begin{aligned}
&= w_1^* w_1 + w_2^* w_2 + w_0^* w_0 \\
&= \sum_{m\geq 0} p_0(2m)(p_1(2m+1) + p_2(2m+1)) + \sum_{m\geq 0} p_0(2m+1) \\
&= \sum_{m\geq 0} p_0(m) \\
&= 1.
\end{aligned}$$

Analogously one has $(w_1 + w_2 + w_0)(w_1 + w_2 + w_0)^* = 1$. That is, $w_1 + w_2 + w_0$ is unitary. Similarly, $p_0 + w_1' + w_2' + w_0'$ is unitary. Observe that

$$\begin{aligned}
w_{0,2} w_2^* w_{2,1} w_1^* w_{1,0} &= \sum_{m\geq 0} v_{0,2}(2m+1) u_{2m}^* v_{2,1}(2m+1) u_{2m}^* v_{1,0}(2m+1) \\
&= w_0,
\end{aligned}$$

and similarly $w_0' = w_{0,2}' w_2'^* w_{2,1}' w_1'^* w_{1,0}'$. Hence we can apply Lemma 4.18(ii) to obtain that $u_0 + 1 - p_0$ is a product of four elements of the form $stst$, where $s, t \in \mathcal{M}$ are symmetries satisfying $\tau(s) = \tau(t) = 0$. Thus, $u$ is a product of eight such elements, i.e., of 32 conjugates of a symmetry of trace 0 by Proposition 4.15. $\square$

The proofs of Lemma 4.18 and Theorem 4.19 show the following, which will become useful in the proof of Theorem 4.64.

**Corollary 4.20.** Let $\mathcal{M}$ be a $\mathrm{II}_1$ factor. Every $u \in \mathrm{U}(\mathcal{M})$ can be decomposed into factors $u = u_1 \cdot \ldots \cdot u_8$ with $u_i \in \mathrm{U}(\mathcal{M})$, $i = 1, \ldots, 8$, such that for each $u_i$ there is a projection $p_i \in \mathrm{Proj}(\mathcal{M})$, $\tau(p_i) = 1/3$, such that under an isomorphism of $\mathcal{M}$ to $p_i \mathcal{M} p_i \otimes M_{3\times 3}(\mathbb{C})$ such that $u_i$ has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & w_i & 0 \\ 0 & 0 & w_i^* \end{pmatrix}$$

for some $w_i \in \mathrm{U}(p_i \mathcal{M} p_i)$. If $\|1 - u\| < \varepsilon$ for some $\varepsilon > 0$, then

$$\|1 - u_i\|_2 < \sum_{n\geq 0} \min\left\{2, 2^{2n}\varepsilon\right\} \cdot 2^{-(2n+2)}.$$

In particular, for arbitrarily small $\delta > 0$ there exists $\varepsilon > 0$ such that $\|1 - u\| < \varepsilon$

implies

$$\|1-u_i\|_2 < \delta.$$

***Proof*.** The decomposition follows from the proofs of Lemma 4.18 and Theorem 4.19. Now assume that $\|1-u\| < \varepsilon$. We retain the notation of the proof of Theorem 4.19, in particular, $u = (u_0 + p_0^\perp)(p_0 + up^\perp)$ and $p_0(0) = p_0$. It is clear that $\left\|1 - u_0 - p_0^\perp\right\| < \varepsilon$. For $u_1 = v_{0,1}(1)u_0v_{1,2}(1)u_0v_{2,0}$ we then get

$$\begin{aligned}
&\|p_0(1) - u_1\| \\
=& \|v_{0,1}(1)(p_0(0) - u_0)v_{1,2}(1)v_{2,0}(1) + v_{0,1}(1)u_0v_{1,2}(1)(p_0(0) - u_0)v_{2,0}(1)\| \\
\leq& \|v_{1,2}(1)v_{2,0}(1)\| \cdot \|v_{0,1}(1)(p_0 - u_0)\| + \|v_{0,1}(1)u_0v_{1,2}(1)\| \cdot \|(p_0 - u_0)v_{2,0}(1)\| \\
\leq& \|v_{0,1}(1)(p_0 - u_0)\| + \|(p_0 - u_0)v_{2,0}(1)\| \\
\leq& \|p_0 - u_0\| \cdot \|p_0\| + \|p_0 - u_0\| \cdot \|p_0\| \\
<& 2\varepsilon.
\end{aligned}$$

It follows by induction on $n \in \mathbb{N}$ (with the analogous calculation) that for $u_n = v_{0,1}(n)u_{n-1}v_{1,2}(n)u_{n-1}v_{2,0}(n)$ we have

$$\|p_0(n) - u_n\|_2 < 2^n\varepsilon.$$

Now consider $w_1 = \sum_{n\geq 0} u_{2n}p_1(2n+1)$, the other $w_i$'s can be treated similarly. From the above estimate we conclude

$$\begin{aligned}
\left\|\sum_{n\geq 0} p_1(2n+1) - \sum_{n\geq 0} u_{2n}p_1(2n+1)\right\|_2 &\leq \sum_{n\geq 0} \|(1-u_{2n})p_1(2n+1)\|_2 \\
&= \sum_{n\geq 0} \|(p_0(2n) - u_{2n})p_1(2n+1)\|_2 \\
&\leq \sum_{n\geq 0} \|p_0(2n) - u_{2n}\| \cdot \|p_1(2n+1)\|_2 \\
&\leq \sum_{n\geq 0} \min\left\{2, 2^{2n}\varepsilon\right\} \cdot 2^{-(2n+3)}.
\end{aligned}$$

That is, we have

$$\|1-u_i\|_2 = \left\| \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix} - \begin{pmatrix} 1&0&0\\0&w_1&0\\0&0&w_1^* \end{pmatrix} \right\|_2 < \sum_{n\geq 0} \min\left\{2, 2^{2n}\varepsilon\right\} \cdot 2^{-(2n+2)}.$$

It remains to show that for arbitrarily small $\delta > 0$ there exists $\varepsilon > 0$ such that $\|1-u\| < \varepsilon$ implies

$$\|1-u_i\|_2 < \delta.$$

Therefore we estimate the sum $\sum_{n\geq 0} \min\left\{2, 2^{2n}\varepsilon\right\} \cdot 2^{-(2n+2)}$. Let $k \in \mathbb{N}$ be suffiently large such that

$$\sum_{n\geq k} 2^{-(2n+2)} < \frac{\delta}{2}.$$

Then we choose $\varepsilon > 0$ small enough such that

$$\sum_{n\leq k} \min\left\{2, 2^{2n}\varepsilon\right\} \cdot 2^{-(2n+2)} < \frac{\delta}{2}.$$

This implies

$$\sum_{n\geq 0} \min\left\{2, 2^{2n}\varepsilon\right\} \cdot 2^{-(2n+2)} < \delta,$$

i.e. $\|1-u_i\|_2 < \delta$.

□

Let us conclude the this subsection with some remarks on similar results for different types of von Neumann algebras. Halmos and Kakutani proved the following result by a different method, see [HK 58, Theorem 1].

**Theorem 4.21** (Halmos-Kakutani)**.** *Every unitary operator on an infinite-dimensional Hilbert space $\mathcal{H}$ is the product of four symmetries on $\mathcal{H}$.*

They were able to show that on every Hilbert space there exists a unitary which is not the product of three symmetries, cf. [HK 58, Theorem 2]. The symmetries constructed by Halmos and Kakutani in their proof of Theorem 4.21 have infinite eigenspaces for 1 and $-1$. As those symmetries are conjugate by a unitary on $\mathcal{H}$, Theorem 4.21 also allows a reformulation: every unitary operator on $\mathcal{H}$ is the product of four conjugates of a symmetry having infinite-dimensional eigenspaces. Tsankov used this reformulation in his proof of automatic continuity of the unitary group of a separable infinite-dimensional Hilbert space [Ts 13, Theorem 1].

Fillmore [Fi 66] has generalized Theorem 4.21 to the case of properly infinite von Neumann algebras.

**Theorem 4.22.** *Every unitary operator in a properly infinite von Neumann algebra $\mathcal{M}$ is the product of at most four symmetries.*

These results suggest that the number 32 in Theorem 4.19 is not optimal. However, the proof of Theorem 4.22 cannot be adapted in a straight forward way.

*Remark.* In the finite-dimensional case, e.g. for the Lie groups $\mathrm{U}(n)$, $n \in \mathbb{N}$, we have the concept of a determinant. Every symmetry in $\mathrm{U}(n)$ has determinant $\pm 1$, while a unitary can have any determinant in $\mathbb{T}$. This implies that there cannot be a finite-dimensional analogue of the above theorem.
However, $\mathrm{U}(n)$ is generated by the symmetries and the scalar unitaries. Using this and [Pe 63, Theorem 1], one can show that every finite type I-factor is generated by its centre and its symmetries.

## 4.4 Generalized projective $s$-numbers

Throughout this section, let $\mathcal{M}$ denote a semifinite factor with faithful normal semifinite trace $\tau$, acting on a separable Hilbert space $\mathcal{H}$. Let $\mathrm{PU}(\mathcal{M})$ denote the projective unitary group of $\mathcal{M}$, i.e., $\mathrm{PU}(\mathcal{M}) = \mathrm{U}(\mathcal{M})/\mathcal{Z}(\mathrm{U}(\mathcal{M}))$, where $\mathcal{Z}(\mathrm{U}(\mathcal{M}))$ denotes the center of $\mathrm{U}(\mathcal{M})$. In particular, for factors we have $\mathcal{Z}(\mathrm{U}(\mathcal{M})) = \mathrm{U}(1)\cdot 1$.

In this section we develop the notion of generalized projective $s$-numbers and prove some useful properties of these. Some of these properties will be freely used in the following sections.

**Lemma 4.23.** Let $\mathcal{M}$ denote a semifinite factor. Let $x \in \mathcal{M}$. The function $\mu_t(1-\lambda x)$ is continuous in $\lambda \in \mathrm{U}(1)$ for all $t \geq 0$.

***Proof.*** Let $\varepsilon > 0$ be arbitrary. We claim that there exists $\delta > 0$ such that $\|\lambda_1 - \lambda_2\| < \delta$ for $\lambda_i \in \mathcal{Z}(\mathrm{U}(\mathcal{M}))$, $i = 1, 2$, implies $\|\mu_t(1-\lambda_1 x) - \mu_t(1-\lambda_2 x)\| < \varepsilon$. We may assume without loss of generality that $\inf_{\tau(1-p)\leq t} \|(1-\lambda_1 x)p\| \geq \inf_{\tau(1-q)\leq t} \|(1-\lambda_2 x)q\|$.

$$\begin{aligned}
|\mu_t(1-\lambda_1 x) - \mu_t(1-\lambda_2 x)| &= \left| \inf_{\tau(1-p)\leq t} \|(1-\lambda_1 x)p\| - \inf_{\tau(1-q)\leq t} \|(1-\lambda_2 x)q\| \right| \\
&\leq \|(1-\lambda_1 x)q_0\| - \|(1-\lambda_2 x)q_0\| \\
&\leq \|(1-\lambda_1 x)q_0 - (1-\lambda_2 x)q_0\| \\
&= \|\lambda_1 - \lambda_2\| \, \|q_0\| \\
&< \delta,
\end{aligned}$$

where $q_0$ is chosen such that it realizes $\inf_{\tau(1-q)\leq t} \|(1-\lambda_2 x)q\|$. Choosing $\delta = \varepsilon$, we are done. $\square$

**Definition 4.24.** Let $\mathcal{M}$ be a semifinite factor with faithful normal semifinite trace $\tau$. We define

$$\ell_t(x) := \inf_{\lambda \in \mathrm{U}(1)} \mu_t(1-\lambda x) \text{ for } t \geq 0,\ x \in \mathcal{M},$$

and call $\ell_t$ the **$t$-th generalized projective $s$-number of $x \in \mathcal{M}$.**
For a projection $p \in \mathrm{Proj}(\mathcal{M})$ we denote the restriction of $\ell_t$ to $p\mathcal{M}p$ by $\ell_t^{(p)}$, that is

$$\ell_t^{(p)}(x) = \inf_{\lambda \in \mathrm{U}(1)} \mu_t(p - \lambda pxp) \text{ for } t \geq 0.$$

We call the unique smallest number $s = s(x) \in [0, \infty]$ such that $\ell_t(x) \neq 0$ if and only if $t \in [0, s)$ the **projective rank of $x$.**

We choose the notation $\ell_t$ because it serves as a weaker notion of a length function in our context. Note that the infimum is attained Lemma 4.23. One can imagine $\ell_t(x)$, $x \in \mathcal{M}$, as a measure of the size of the spectrum of $x$ after cutting out a piece of size $t \geq 0$ (which reduces the size of the spectrum of $x$ as much as possible).

It follows immediately from the definition that $\ell_t(x) = \ell_t(\xi x)$ for all $\xi \in \mathcal{Z}(\mathrm{U}(\mathcal{M}))$ and $t \geq 0$. Observe that we have $\ell_t = 0$ for $t \geq \tau(1)$ by Lemma 2.22.
By Lemma 2.21(ii) we have $\ell_t(x) = \ell_t(x^*)$ for every $t \geq 0$ and $x \in \mathcal{M}$.

*Remark.* (i) The following sections deal with various unitary groups of functional analytic type. $\mathrm{II}_1$ factors are (by definition) always equipped with a unital trace and thus the generalized projective $s$-numbers may take nonzero values only for $t \in [0, 1)$. However, we usually equip the compact Lie groups $\mathrm{U}(n)$, $n \in \mathbb{N}$, with the unnormalized trace in order to count the generalized projective $s$-numbers from $0$ to $n-1$. This is just a matter of notational taste. When $\mathrm{U}(n)$ is embedded in $\mathrm{U}(\mathcal{M})$ we usually view the eigenvalues as constant functions on intervals $[i/n, (i+1)/n)$.

(ii) If working with general semifinite von Neumann algebra $\mathcal{M}$ one might define the generalized projective $s$-number by

$$\ell_t(x) := \inf_{\lambda \in \mathcal{Z}(\mathrm{U}(\mathcal{M}))} \mu_t(1-\lambda x) \text{ for } t \geq 0,\ x \in \mathcal{M}.$$

Using Lemma 2.21(vi), we conclude

$$\ell_t(gxg^*) = \inf_{\lambda\in \mathrm{U}(1)} \mu_t(g(1-\lambda x)g^*) \leq \inf_{\lambda\in \mathrm{U}(1)} \|g\|\,\|g^*\|\,\mu_t(1-\lambda x) = \ell_t(x)$$

for all $g \in \mathrm{PU}(\mathcal{M})$, $x \in \mathcal{M}$ and $t \geq 0$. Replacing now $x$ by $g^*xg$, we obtain that $\ell_t$ is invariant under conjugation, i.e.,

$$\ell_t(gxg^*) = \ell_t(x) \text{ for all } t \geq 0.$$

Now let $p \in \mathrm{Proj}(\mathcal{M})\setminus\{0\}$ and assume that $x \in \mathcal{M}$ commutes with $p$. Then we have

$$\ell_t^{(p)}(x) \leq \ell_t(x) \text{ for all } t \geq 0.$$

To see this, we use submultiplicativity of the operator norm. Fix $t \geq 0$. We have

$$\begin{aligned}
\ell_t^{(p)}(x) &= \inf_{\lambda\in \mathrm{U}(1)} \mu_t(p-\lambda pxp) \\
&= \inf_{\lambda\in \mathrm{U}(1)} \inf_{q\in \mathrm{Proj}(\mathcal{M}),\tau(1-q)\leq t} \|p(1-\lambda x)q\| \\
&\leq \inf_{\lambda\in \mathrm{U}(1)} \inf_{q\in \mathrm{Proj}(\mathcal{M}),\tau(1-q)\leq t} \|p\|\,\|(1-\lambda x)q\| \\
&= \inf_{\lambda\in \mathrm{U}(1)} \inf_{q\in \mathrm{Proj}(\mathcal{M}),\tau(1-q)\leq t} \|(1-\lambda x)q\| \\
&= \ell_t(x).
\end{aligned}$$

**Lemma 4.25.** $\ell_{s+t}(xy) \leq \ell_s(x) + \ell_t(y)$ for all $x, y \in \mathcal{M}$ and $s, t \geq 0$. In particular, $\ell_t$ is non-increasing in $t \geq 0$.

***Proof***. Since $\mathrm{U}(1)$ compact and since $\mu_t(1-\lambda x)$ is continuous in $\lambda \in \mathrm{U}(1)$, we can choose $\lambda_x, \lambda_y \in \mathrm{U}(1)$ such that $\ell_t(x) = \mu_t(1-\lambda_x x)$ and $\ell_t(y) = \mu(1-\lambda_y y)$. Using Lemma 2.21(i),(v), we obtain

$$\begin{aligned}
\ell_{s+t}(xy) &= \ell_{s+t}(\lambda_x x \lambda_y y) \\
&= \inf_{\lambda\in \mathcal{Z}(\mathrm{U}(\mathcal{M}))} \mu_{s+t}((1-\lambda\lambda_x x)\lambda_y y + (1-\lambda_y y)) \\
&\leq \inf_{\lambda\in \mathcal{Z}(\mathrm{U}(\mathcal{M}))} \mu_s((1-\lambda\lambda_x x)\lambda_y y) + \mu_t(1-\lambda_y y) \\
&= \inf_{\lambda\in \mathcal{Z}(\mathrm{U}(\mathcal{M}))} \mu_s(1-\lambda\lambda_x x) + \ell_t(y)
\end{aligned}$$

$$= \ell_s(x) + \ell_t(y).$$

To see that $\ell_t$ is non-increasing in $t$, let $y = 1$ and use that obviously $\ell_t(1) = 0$ for all $t \geq 0$ to obtain $\ell_{s+t}(x) \leq \ell_t(x)$ for all $t \geq 0$. □

**Lemma 4.26.** $\ell_t(x)$ is right continuous in $t \in [0, \infty]$, where $x \in \mathcal{M}$.

***Proof.*** Since $\mu_t$ is right continuous in $t \in [0, \infty]$, for all $\lambda \in \mathrm{U}(1)$ and all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mu_t(1 - \lambda x) - \mu_{t+\delta}(1 - \lambda x) < \varepsilon.$$

Now fix arbitrary $t \geq 0$ and $\varepsilon > 0$. By Lemma 4.23 for every $t \geq 0$ we can choose $\lambda \in \mathrm{U}(1)$ which realizes $\inf_{\lambda \in \mathrm{U}(1)} \mu_t(1 - \lambda x)$. We denote this element by $\lambda_t$. Moreover, for every $\lambda$ we can choose $\delta_\lambda > 0$ such that

$$\mu_t(1 - \lambda x) - \mu_{t+\delta_\lambda}(1 - \lambda x) < \varepsilon.$$

We claim that $\delta := \inf_{\lambda \in \mathrm{U}(1)} \delta_\lambda > 0$. Assume to the contrary that $\delta = 0$, i.e., there exists no $\widetilde{\delta} > 0$ such that $\mu_t(1 - \lambda x) - \mu_{t+\widetilde{\delta}}(1 - \lambda x) < \varepsilon$ for all $\lambda \in \mathrm{U}(1)$. Then there exist sequences $\{\lambda_n\}_{n \in \mathbb{N}}$ and $\{\delta_n\}_{n \in \mathbb{N}}$, $\delta_n := \delta_{\lambda_n}$, such that

- $\lambda_n \to \lambda$ and $\delta_n \to 0$ for $n \to \infty$,
- $\mu_t(1 - \lambda_n x) - \mu_{t+\delta_n}(1 - \lambda_n x) \geq \varepsilon$ for all $n$ greater than some $n_0 \in \mathbb{N}$ (by uniform continuity in $\lambda$, see Lemma 4.23, and right-continuity of $\mu_t$ in $t$).

On the other hand we have

$$\mu_t(1 - \lambda x) - \mu_{t+\delta_\lambda}(1 - \lambda x) < \varepsilon \tag{4.1}$$

with $\delta_\lambda > 0$. Thus there exists $n_1 \in \mathbb{N}$ such that $\delta_n < \delta_\lambda$ for all $n \geq n_1$. But this implies

$$\varepsilon > \mu_t(1 - \lambda_n x) - \mu_{t+\delta_\lambda}(1 - \lambda_n x) \geq \mu_t(1 - \lambda_n x) - \mu_{t+\delta_n}(1 - \lambda_n x) \geq \varepsilon,$$

whenever $n > \max\{n_0, n_1\}$ which is a contradiction to Inequality (4.1). Hence $\delta > 0$ and

$$\varepsilon > \mu_t(1 - \lambda_{t+\delta} x) - \mu_{t+\delta}(1 - \lambda_{t+\delta} x) = \mu_t(1 - \lambda_{t+\delta} x) - \ell_{t+\delta}(x) \geq \ell_t(x) - \ell_{t+\delta}(x).$$

Since $t \geq 0$ and $\varepsilon > 0$ were arbitrary, we are done. $\square$

Before coming to the easy direction in the main theorems on products of conjugates we collect the above proven properties of generalized projective $s$-numbers.

**Proposition 4.27.** Let $\mathcal{M}$ be a semifinite factor with faithful normal semifinite trace $\tau$. Let $x, y \in \mathcal{M}$ and $u \in \mathrm{U}(\mathcal{M})$.

(i) $\ell_t(x) = \ell_t(x^*)$ for all $t \geq 0$.

(ii) $\ell_t(x) = 0$ for all $t \geq \tau(1)$.

(iii) $\ell_t(uxu^*) = \ell_t(x)$ for all $t \geq 0$.

(iv) $\ell_t^{(p)}(x) \leq \ell_t(x)$ for all $t \geq 0$.

(v) $\ell_{s+t}(xy) \leq \ell_s(x) + \ell_t(y)$ for all $s, t \geq 0$.

(vi) $\ell_t$ is non-increasing in $t \geq 0$.

(vii) $\ell_t$ is right continuous in $t \geq 0$.

The following proposition is the easy direction in the main theorems on products of conjugates. The proof is a straight forward application of some properties of generalized projective $s$-numbers.

**Proposition 4.28.** If $u \in G := \mathrm{PU}(\mathcal{M})$ is a product of $k$ conjugates of $v \in G$ and $v^{-1}$, then $\ell_{k \cdot t}(u) \leq k \cdot \ell_t(v)$ for all $t \geq 0$.

***Proof.*** Let $t \geq 0$. By assumption we can write $u = g_1 v^{\varepsilon_1} g_1^* g_2 v^{\varepsilon_2} g_2^* \cdots g_k v^{\varepsilon_k} g_k^*$ for some $g_i \in G$ and $\varepsilon_i \in \{1, -1\}$, where $i = 1, \ldots, k$ . Using $\ell_t(gwg) = \ell_t(w)$ for all $g, w \in G$ and that $\ell_t(w) = \ell_t(w^*)$ for all $t \geq 0$, we deduce

$$\begin{aligned}
\ell_{kt}(u) &\leq \ell_t(g_1 v^{\varepsilon_1} g_1^*) + \ell_{(k-1)t}(g_2 v^{\varepsilon_2} g_2^* \cdots g_k v^{\varepsilon_k} g_k^*) \\
&\leq \ell_t(v^{\varepsilon_1}) + \ell_t(g_2 v^{\varepsilon_2} g_2^*) + \ell_{(k-2)t}(g_3 v^{\varepsilon_3} g_3^* \cdots g_k v^{\varepsilon_k} g_k^*) \\
&= \ell_t(v) + \ell_t(g_2 v^{\varepsilon_2} g_2^*) + \ell_{(k-2)t}(g_3 v^{\varepsilon_3} g_3^* \cdots g_k v^{\varepsilon_k} g_k^*) \\
&\vdots \\
&\leq k \cdot \ell_t(v),
\end{aligned}$$

which proves our claim. $\square$

The following Markov-type inequality turns out to be useful in the proof of Lemma 4.30.

**Lemma 4.29.** If $\tau(|x|) = \int_{[0,\tau(1)]} \mu_t(x)dt \leq \varepsilon$, then $\mu_t(x) \leq \varepsilon/t$ for all $t > 0$.

***Proof.*** Assume to the contrary that $\mu_{t_0}(x) > \varepsilon/t_0$ for some $t_0 > 0$. Since $\mu_t$ is non-increasing in $t$, this implies $\mu_t(x) > \varepsilon/t_0$ for all $t \in [0, t_0]$. Hence, by positivity of $\mu_t$,

$$\int_{[0,\tau(1)]} \mu_t(x)dt \geq \int_{[0,t_0]} \mu_t(x)dt > \int_{[0,t_0]} \frac{\varepsilon}{t_0}dt = \varepsilon,$$

a contradiction. □

The following lemma analyzes the behaviour of projective generalized $s$-numbers under approximation in the operator norm and in the 2-norm. It will be very useful in proving some of our main results.

**Lemma 4.30.** Let $\mathcal{M}$ be a semi-finite von Neumann algebra.

(i) Assume that $u, u', v, v'$ are elements of $\mathcal{M}$ satisfying $\|u - u'\|, \|v - v'\| < \varepsilon$ and $\ell_{kt}(u) \leq \max\{m\ell_t(v) - \delta,\ 0\}$ for all $t \geq 0$ and some $m, k \in \mathbb{N},\ \delta \geq 0$. Then

$$\ell_{kt}(u') \leq \max\left\{m\ell_t(v') - \delta + (m+1)\varepsilon,\ 0\right\} \qquad \text{for all } t \geq 0.$$

(ii) Assume that $u, u', v, v'$ are elements of $\mathcal{M}$ satisfying $\|u - u'\|_2, \|v - v'\|_2 < \varepsilon$ and $\ell_{k(t+\delta_0)}(u) \leq \max\{m\ell_t(v) - \delta,\ 0\}$ for all $t \geq 0$ and some $m, k \in \mathbb{N},\ \delta_0 > 0,\ \delta \geq 0$. Then

$$\ell_{4k(t+\delta_0)}(u') \leq \max\left\{m\ell_t(v') - \delta + \frac{m + 1/k}{t + \delta_0}\varepsilon,\ 0\right\} \qquad \text{for all } t \geq 0.$$

***Proof.*** (i) For $t \geq 0$ we obtain

$$\begin{aligned}
\ell_t(u') &= \inf_\lambda \mu_t(1 - \lambda(u' - u + u)) \\
&\leq \ell_{(n-1)t/n}(u) + \mu_{t/n}(u - u') \\
&\leq \max\left\{m\ell_{(n-1)t/nk}(v) - \delta + \mu_{t/n}(u - u'),\ 0\right\} \\
&\leq \max\left\{m\ell_{(n-1)^2t/n^2k}(v') - \delta + m\mu_{(n-1)t/n^2k}(v - v') + \mu_{t/n}(u - u'),\ 0\right\} \\
&\leq \max\left\{m\ell_{(n-1)^2t/n^2k}(v') - \delta + m\left\|v - v'\right\| + \left\|u - u'\right\|,\ 0\right\} \\
&< \max\left\{m\ell_{(n-1)^2t/n^2k}(v') - \delta + (m+1)\varepsilon,\ 0\right\}.
\end{aligned}$$

This implies

$$\ell_{n^2kt/(n-1)^2}(u') < \max\left\{m\ell_t(v') - \delta + (m+1)\varepsilon,\ 0\right\}.$$

By right-continuity of $\ell_t$, see Lemma 4.26, letting $n$ tend to $+\infty$, we arrive at

$$\ell_{kt}(u') \leq \max\left\{m\ell_t(v') - \delta + (m+1)\varepsilon,\ 0\right\}.$$

(ii) For $t \geq 0$ and $n \geq 2$, we conclude

$$\begin{aligned}\ell_t(u') &\leq \ell_{(n-1)t/n}(u) + \mu_{t/n}(u-u') \\ &\leq \max\left\{m\ell_{(n-1)t/nk-(n-1)\delta_0/n}(v) - \delta + \mu_{t/n}(u-u'),\ 0\right\} \\ &\leq \max\left\{m\ell_{(n-1)^2t/n^2k-(n-1)\delta_0/n}(v') - \delta + m\mu_{(n-1)t/n^2k}(v-v') + \mu_{t/n}(u-u'),\ 0\right\}.\end{aligned}$$

Hence

$$\begin{aligned}\ell_{n^2k(t+\delta_0)/(n-1)^2}(u') \leq &\max\left\{m\ell_{t+\delta_0(1-(n-1)/n)}(v') - \delta + m\mu_{(t+\delta_0)/(n-1)}(v-v')\right. \\ &\left. + \mu_{nk(t+\delta_0)/(n-1)^2}(u-u'),\ 0\right\},\end{aligned}$$

and using Lemma 4.29 (and that $\|u-u'\|_2 < \varepsilon$ implies $\|u-u'\|_1 < \varepsilon$) yields

$$\ell_{n^2k(t+\delta_0)/(n-1)^2}(u') \leq m\ell_t(v') - \delta + m \cdot \min\left\{\frac{(n-1)\varepsilon}{t+\delta_0}, 2\right\} + \min\left\{\frac{(n-1)\varepsilon}{k(t+\delta_0)}, 2\right\}.$$

Putting $n \coloneqq 2$ completes the proof. □

Let us conclude this section by proving that products of $\varepsilon$-thickened conjugacy classes of topological groups with compatible bi-invariant metric behave well under $\varepsilon$-thickening. This result will be needed for some of the main theorems, namely those which rely on finite-dimensional approximation.

**Lemma 4.31.** Let $G$ be a topological group equipped with a compatible bi-invariant metric $d$. Let $\varepsilon > 0$. Then $(((g^G)_\varepsilon)^n)_\varepsilon \subseteq ((g^G)^n)_{(n+1)\varepsilon}$ for all $n \in \mathbb{N}$.

***Proof.*** Let $h \in ((g_\varepsilon^G)^n)_\varepsilon$ and assume that $g_{i,\varepsilon}$ for $i = 1, \ldots, n$, are elements of $g_\varepsilon^G$ satisfying $d(h, g_{1,\varepsilon} \cdots g_{n,\varepsilon}) < \varepsilon$. Then there are elements $g_1, \ldots, g_n \in g^G$ such that

$d(g_i, g_{i,\varepsilon}) < \varepsilon$. Using the bi-invariance of $d$, we obtain

$$\begin{aligned} d(h, g_1 \cdots g_n) &\leq d(h, g_{1,\varepsilon} \cdots g_{n,\varepsilon}) + d(g_{1,\varepsilon} \cdots g_{n,\varepsilon}, g_1 g_{2,\varepsilon} \cdots g_{n,\varepsilon}) \\ &\quad + \ldots + d(g_1 \cdots g_{n-1} g_{n,\varepsilon}, g_1 \cdots g_n) \\ &< \varepsilon + d(g_{1,\varepsilon}, g_1) + \ldots + d(g_{n,\varepsilon}, g_n) \\ &< (n+1)\varepsilon, \end{aligned}$$

which shows that $h \in ((g^G)^n)_{(n+1)\varepsilon}$. □

## 4.5 Bounded normal generation for type $\mathrm{I_n}$ factors

Property (BNG) for compact connected simple Lie groups (e.g. the projective unitary group $\mathrm{PU}(n)$) has been settled quantitatively in [NS 12]. We repair the rank-independent result [ST 14, Lemma 4.15] for $\mathrm{PU}(n)$ and clarify in Proposition 4.32 why this is necessary. Some arguments are borrowed from these articles but our path focuses on the $\mathrm{PU}(n)$-case and our version of [ST 14, Lemma 4.15] as well as its proof differ considerably. Recall that every type $I_n$ factor is isomorphic to the matrix algebra $\mathcal{M}_{n\times n}(\mathbb{C})$.

In this section we fix the following notation. Let $T$ denote the maximal torus of diagonal entries in $\mathrm{U}(n)$, $2 \leq n \in \mathbb{N}$, i.e.,

$$T = \left\{ \operatorname{diag}(e^{\mathrm{i}\,\theta_0}, \ldots, e^{\mathrm{i}\,\theta_{n-1}}) \mid \theta_i \in [0, 2\pi),\ i = 0, 1, \ldots, n-1 \right\}.$$

Decompose $T$ into $n-1$ subgroups $T_j$, $j = 0, \ldots, n-1$, which are defined as follows.

$$\begin{aligned} &T_0 \coloneqq \mathcal{Z}(\mathrm{U}(n)), \\ &T_j \coloneqq \{\operatorname{diag}(1, \ldots, 1, \lambda, \ldots, \lambda) \mid \lambda \in \mathrm{U}(1)\}, \\ &\qquad \text{where } \lambda \text{ is on the positions } j+1, \ldots, n-1. \end{aligned}$$

Observe that every element $u = \operatorname{diag}(\lambda_0, \ldots, \lambda_{n-1}) \in T$ can be decomposed into the product of commuting factors $u = u_0 \cdot \ldots \cdot u_{n-1}$, where $u_i \in T_i$. Here,

$$\begin{aligned} u_0 &= \operatorname{diag}(\lambda_0, \lambda_0, \ldots, \lambda_0), \\ u_i &= \operatorname{diag}(1, \ldots, 1, \lambda_i \overline{\lambda}_{i-1}, \lambda_i \overline{\lambda}_{i-1}, \ldots, \lambda_i \overline{\lambda}_{i-1}) \quad \text{for } i \geq 1. \end{aligned}$$

We call this decomposition the **one-parameter torus decomposition of** $u$. Let us point out that when working in $\mathrm{PU}(n)$, the factor $u_0$ in the decomposition $u = u_0 \cdot \ldots \cdot u_{n-1}$ actually can be left out since $u_0$ is central.

We will use another decomposition for $u \in \mathrm{SU}(n)$ (respectively $u \in \mathrm{PU}(n)$). For $j = 0, \ldots, n-2$ let $S_j$, denote the subgroup of $\mathrm{U}(n)$ of matrices of the form

$$\begin{pmatrix} 1 & & 0 \\ & \mathrm{SU}(2) & \\ 0 & & 1 \end{pmatrix},$$

where the $\mathrm{SU}(2)$-copy sits at the entries $(j+1, j+1), (j+2, j+1), (j+1, j+2)$ and $(j+2, j+2)$. Then $u$ can be decomposed into factors $u_i \in S_i$, $i = 0, \ldots, n-2$, where

$$\begin{aligned} u_0 &= \mathrm{diag}(\lambda_0, \overline{\lambda}_0, 1, \ldots, 1), \\ u_i &= \mathrm{diag}(1, \ldots, 1, \lambda_0 \cdot \ldots \cdot \lambda_i, \overline{\lambda}_0 \cdot \ldots \cdot \overline{\lambda}_i, 1, \ldots, 1). \end{aligned}$$

This decomposition is called the $\mathrm{SU}(2)$ **product decomposition**. Note that the factors in the $\mathrm{SU}(2)$ product decomposition mutually commute.

We will need both of the above introduced decompositions in order to get the desired rank-independent result. Actually the error that is hidden in [ST 14, Lemma 4.15] stems from an incorrect use of these decompositions. To see that [ST 14, Lemma 4.15] is wrong, we provide the following result.

**Proposition 4.32.** Let $u = \mathrm{diag}(\lambda^{-n-1}, \lambda, \lambda, \ldots, \lambda)$ and $v = \mathrm{diag}(\mu^{-n-1}, \mu, \mu, \ldots, \mu)$ be nontrivial elements in $G := \mathrm{PU}(n)$. Assume that $\arg(\lambda)/\arg(\mu)$ is irrational.
If $u \in (v^G \cup v^{-G})^k$, then $k \geq n-1$.

***Proof***. Assume that $u \in (v^G \cup v^{-G})^k$, i.e. $u = g_1 v^{\varepsilon_1} g_1^{-1} \cdot \ldots \cdot g_k v^{\varepsilon_k} g_k^{-1}$ with $g_i \in G$, $\varepsilon_i \in \{1, -1\}$. Then there exists a lift of $u$ into $\mathrm{SU}(n)$ such that $zu$ is a product of $k$ conjugates of $z_0 v \in \mathrm{SU}(n)$ for some $z, z_0 \in \mathcal{Z}(\mathrm{U}(n))$ (actually $z, z_0 = 1$ if $u, v$ are written down as in the assumption). Hence $u' := \mu^{-k} u$ is a product of k conjugates of $v' := \mu^{-1} v$ in $\mathrm{U}(n)$. Now $\mu^{-1} v$ is a rank one perturbation of the identity and thus $u'$ is at most a rank $k$ perturbation of the identity in $\mathrm{U}(n)$. But $n-1$ diagonal entries of $u'$ are of the form $\lambda \mu^{-k}$, which are different from 1. Hence $1 - u'$ has rank at least $n-1$ and this implies $k \geq n-1$. $\square$

Let us now come to the first step in the proof of our rank-independent result. For convenience we repeat the proof of the following SU(2)-result from [NS 12].

**Lemma 4.33** (Nikolov-Segal)**.** Let $u = \begin{pmatrix} e^{\mathrm{i}\,\varphi} & 0 \\ 0 & e^{-\mathrm{i}\,\varphi} \end{pmatrix}$ and $v = \begin{pmatrix} e^{\mathrm{i}\,\theta} & 0 \\ 0 & e^{-\mathrm{i}\,\theta} \end{pmatrix}$ be noncentral elements in $G \coloneqq \mathrm{SU}(2)$. If $|\varphi| \leq m\,|\theta|$ for some even $m \in \mathbb{N}$, then $u \in (v^G)^m$.

***Proof.*** Consider the realization of SU(2) by unitary matrices $\left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1 \right\}$.

The conjugacy class of $v$ is uniquely determined by the normalized trace $\mathrm{tr}(v) = \cos\theta$ with $\theta \in [0, \pi]$. As $m$ is even, we have $(v^G)^m = ((-v)^G)^m$ and hence we may assume that $\theta \in [0, \pi/2]$.

If $w \in \mathrm{SU}(2)$ is diagonal with $\mathrm{tr}(w) = \cos\gamma$, then we can choose $v' \in v^{\mathrm{SU}(2)}$ such that $\mathrm{tr}(wv') = \cos\gamma_1$ for any $\gamma_1 \in [\gamma - \theta, \gamma + \theta]$, namely $v' \coloneqq \begin{pmatrix} \cos\theta + \mathrm{i}\sin\theta_1 & b \\ -\bar{b} & \cos\theta - \mathrm{i}\sin\theta_1 \end{pmatrix}$ for $\theta_1 \in [-\theta, \theta]$, where $|b|^2 = 1 - \cos^2\theta - \sin^2\theta_1 = \sin^2\theta - \sin^2\theta_1 \geq 0$.

Observe that multiplication of diagonal elements by $v$ adds the angle $\theta$ while multiplication with $v^{-1}$ subtracts the angle $\theta$. Thus, regarding the given inequality $|\varphi| \leq m\,|\theta|$, we can use $v$ and $v^{-1}$ in the first $m-1$ steps and a possibly nondiagonal element $v' \in v^{\mathrm{SU}(2)}$ to obtain an possibly nondiagonal element with the same trace as $u$. Using again that elements of the same trace are conjugate in SU(2), we conclude that $u$ is the product of $m$ conjugates of $v$. □

Note that in particular, $1 \in (u^{\mathrm{SU}(2)})^2$ for every $u \in \mathrm{SU}(2)$.

Let us now analyze how to use the above lemma on a single factor $u_i \in S_i$ in $\mathrm{PU}(n)$.

**Lemma 4.34.** Let $G \coloneqq \mathrm{PU}(n)$ with $n \geq 2$, $n \in \mathbb{N}$. Let $u = \mathrm{diag}(e^{\mathrm{i}\,\varphi_0}, \dots, e^{\mathrm{i}\,\varphi_{n-1}}), v = \mathrm{diag}(e^{\mathrm{i}\,\theta_0}, \dots, e^{\mathrm{i}\,\theta_{n-1}}) \in G$ and assume that $u_0 \cdot \ldots \cdot u_{n-1}$ with $u_i \in S_i$. If $|\varphi_1 + \ldots + \varphi_{i-1}| \leq m\,|\theta_{j-1} - \theta_j|$ for some $i, j \in \{1, \dots, n-1\}$ and even $m \in \mathbb{N}$ then

$$u_i \in (v^G \cup v^{-G})^{2m}.$$

***Proof.*** Write $v_0 \cdot \ldots \cdot v_{n-1},\ v_i \in T_i$ in its one-parameter torus decomposition. Let $g \in S_j$ be the permutation swapping the diagonal entries at the positions $j, j+1$. Then $[v, g] = [v_j, g] \in S_j$. Let $h \in G$ such that $u_i^h \in S_j$. Using the given inequality $|\varphi_1 + \ldots + \varphi_{i-1}| \leq m\,|\theta_{j-1} - \theta_j|$ (note that $\varphi_1 + \ldots + \varphi_{i-1}$ is the angle of $u_i$) and Lemma 4.33 we conclude

$$u_i \in h^{-1}([v_j, g]^{S_j} \cup [v_j, g]^{-S_j})^m h \subset (v^G \cup v^{-G})^{2m}.$$

This concludes the proof. □

The following result is a crucial point in simultaneous generation with the help of SU(2)-copies. Our proof differs from that of Stolz and Thom - in fact, the error was hidden in the proof of this result. An important point to notice is that we have to decompose the generating element $v$ in the following Lemma into elements of the one-parameter tori $T_i$ to generate simultaneously. But the generated element $u$ needs to be decomposed into elements of $S_j$.

**Lemma 4.35.** Let $G \coloneqq \mathrm{PU}(n)$, $n \geq 2$, $m \in \mathbb{N}$ even and $s \in \mathbb{N}_0$. Let

$$u = \operatorname{diag}(e^{\mathrm{i}\,\varphi_0}, \ldots, e^{\mathrm{i}\,\varphi_{n-1}}) = u_0 \cdot u_1 \cdot \ldots \cdot u_{n-1}$$

be the SU(2) product decomposition of $u$ and let

$$v = \operatorname{diag}(e^{\mathrm{i}\,\theta_0}, \ldots, e^{\mathrm{i}\,\theta_{n-1}}) = v_0 \cdot v_1 \cdot \ldots \cdot v_{n-1}$$

with $v_i \in T_i$ be the one-parameter torus decomposition of $v$. For $0 \leq k \leq s$ and $0 \leq l \leq s$ let $i_k$ and $j_l$ be elements of $\{0, \ldots, n-1\}$. If $|i_k - i_l|, |j_k - j_l| \geq 2$ for all $k \neq l$ and

$$|\varphi_0 + \varphi_1 + \ldots + \varphi_{i_k}| \leq m\, |\theta_{j_k} - \theta_{j_k+1}| \qquad \text{for } k, l = 0, \ldots, s.$$

Then

$$u_{i_1} \cdot u_{i_2} \cdot \ldots \cdot u_{i_s} \in \left(v^G \cup v^{-G}\right)^{2m}.$$

***Proof.*** Write $v = v_{j_1} \cdot \ldots \cdot v_{j_s} \cdot \widetilde{v}$ where $\widetilde{v}$ commutes with $S_{i_1}, \ldots, S_{i_s}$. Note that $S_{j_k}$ and $S_{j_l}$ commute elementwise for $k \neq l$. Moreover, $\widetilde{v}$ commutes with $S_{j_k}$ for all $k = 1, \ldots, s$. Thus we get

$$\left(v^{S_{j_1} \cdot \ldots \cdot S_{j_s}}\right)^m = \left(v_{j_1}^{S_{j_1}} \cdot \ldots \cdot v_{j_s}^{S_{j_s}} \cdot \widetilde{v}^{S_{j_1} \cdot \ldots \cdot S_{j_s}}\right)^m = \left(v_{j_1}^{S_{j_1}}\right)^m \cdot \ldots \cdot \left(v_{j_s}^{S_{j_s}}\right)^m \cdot \widetilde{v}^m.$$

Let $g_{j_k} \in S_{j_k}$ be a permutation switching positions $j_k$ and $j_{k+1}$ for $k = 0, \ldots, s$. Define

$$g \coloneqq g_{j_1} \cdot \ldots \cdot g_{j_s} \in S_{j_1} \cdot \ldots \cdot S_{j_k}.$$

Consider now the commutator $[v, g] = vgv^{-1}g^{-1} \in (v^G \cup v^{-G})^2$. Observe that $[v, g] \in$

$S_{j_1} \cdot \ldots \cdot S_{j_s}$. Let $h \in G$ be a permutation such that $S_{i_k}^h = S_{j_k}$ for all $k = 1, \ldots, s$. Using Lemma 4.33, we obtain $u_{i_k}^h \in \left([v_{j_k}, g_{j_k}]^{S_{j_k}} \cup [v_{j_k}, g_{j_k}]^{-S_{j_k}}\right)^m$ for all $k = 1, \ldots, s$, and hence

$$u_{i_1} \cdot \ldots \cdot u_{i_s} \in h^{-1} \left( \left( [v, g]^{S_{j_1} \cdot \ldots \cdot S_{j_s}} \right)^m \right) h \subset (v^G \cup v^{-G})^{2m}.$$

This completes the proof. □

In order to have a relation between projective $s$-numbers and angles, we need the following lemma.

**Lemma 4.36.** For all $\theta \in [-\pi, \pi]$, one has $|\theta|/2 \leq \sqrt{2(1 - \cos\theta)} \leq |\theta|$.

***Proof.*** By symmetry of cos, it suffices to check the case $\theta \in [0, \pi]$.

To see the first inequality, define $f(\theta) := 1 - \cos\theta - \theta^2/8$. We have $f'(\theta) = \sin\theta - \theta/4$ and $f'(0) = 0$. Since $f''(\theta) = \cos\theta - 1/4$ is monotone decreasing in the interval $[0, \pi]$, $f''$ has a unique zero in $[0, \pi]$ (note that $f''(0) = 3/4 > 0$). Hence $f'$ has a unique extreme point in $[0, \pi]$, which is a maximum in $[0, \pi]$. Thus $f$ has at most two zeroes in $[0, \pi]$, one of which is 0. We have $f(\pi) > 0$ and thus $f(\theta) \geq 0$ for all $\theta \in [0, \pi]$. This implies $|\theta|/2 \leq \sqrt{2(1 - \cos\theta)}$.

To prove the second inequality, observe that $\cos\theta = \sum_{j=0}^{\infty} (-1)^j \theta^{2j}/(2j)! \geq 1 - \theta^2/2$ and hence $\sqrt{2(1 - \cos\theta)} \leq \sqrt{2(1 - (1 - \theta^2/2))} \leq \theta$. □

The following definition is crucial in order to obtain estimates between projective $s$-numbers and eigenvalue differences, which in turn will be compared to angles.

**Definition 4.37.** Assume that $u \in G := \mathrm{U}(n)$, $2 \leq n \in \mathbb{N}$. Let us say that $\widetilde{u} = \mathrm{diag}(\lambda_0, \ldots, \lambda_{n-1}) \in u^G \cap T$ is **optimal** if

- $|\lambda_0 - \lambda_1| \geq |x_0 - x_1|$ for all $v = \mathrm{diag}(x_0, \ldots, x_{n-1}) \in u^G \cap T$;
- $|\lambda_i - \lambda_{i+1}| = |x_i - x_{i+1}|$ for all $i = 0, \ldots, k-1$ implies $|\lambda_k - \lambda_{k+1}| \geq |x_k - x_{k+1}|$.

This defines a lexicographic order on the eigenvalue differences, hence for every $u \in \mathrm{U}(n)$ respectively $\mathrm{PU}(n)$, there exists an optimal element $\widetilde{u}$. For two different optimal elements $\widetilde{u}, \widetilde{v}$, we have $|\lambda_i - \lambda_{i+1}| = |x_i - x_{i+1}|$ for all $i \in \{0, \ldots, n-2\}$.
For an optimal element $u$ there exists a permutation $\sigma \in S_X$, where $X = \{0, \ldots, n-2\}$ and $S_X$ denotes the group of permutations on $X$, such that

$$\left|\lambda_{\sigma(0)} - \lambda_{\sigma(0)+1}\right| \geq \left|\lambda_{\sigma(1)} - \lambda_{\sigma(1)+1}\right| \geq \ldots \geq \left|\lambda_{\sigma(n-2)} - \lambda_{\sigma(n-2)+1}\right|.$$

We call such a permutation the **permutation associated to** the optimal element $u$. Note that our definition of optimality slightly differs from the one given in [ST 14].

**Lemma 4.38.** Let $u = \operatorname{diag}(\lambda_0, \ldots, \lambda_{n-1}) \in T \subset \mathrm{U}(n)$ and $\sigma$ a permutation such that $\left|\lambda_{\sigma(i)} - \lambda_{\sigma(i)+1}\right|$ is monotone decreasing in $i = 0, \ldots, n-2$, where $n \geq 2$, $n \in \mathbb{N}$. Then

$$\frac{1}{2}\left|\lambda_{\sigma(2i)} - \lambda_{\sigma(2i)+1}\right| \leq \ell_i(u).$$

If $u$ is optimal with associated permutation $\sigma$ then

$$\frac{1}{2}\left|\lambda_{\sigma(2i)} - \lambda_{\sigma(2i)+1}\right| \leq \ell_i(u) \leq \left|\lambda_{\sigma(i)} - \lambda_{\sigma(i)+1}\right| \qquad \text{for all } i = 0, \ldots, n-2.$$

***Proof***. To prove the first inquality, let $z_0 = \operatorname{diag}(z, \ldots, z) \in \mathcal{Z}(\mathrm{U}(n))$ be arbitrary and fix a permutation $\tau \in S_Y$, $Y := \{0, \ldots, n-1\}$, such that $\left|z - \lambda_{\tau(0)}\right| \geq \left|z - \lambda_{\tau(1)}\right| \geq \ldots \geq \left|z - \lambda_{\tau(n-1)}\right|$. Assume to the contrary, that $\left|\lambda_{\sigma(2i)} - \lambda_{\sigma(2i)+1}\right| > 2\left|z - \lambda_{\tau(i)}\right|$. Hence

$$\left|z - \lambda_{\sigma(k)}\right| + \left|z - \lambda_{\sigma(k)+1}\right| \geq \left|\lambda_{\sigma(k)} - \lambda_{\sigma(k)+1}\right| > 2\left|z - \lambda_{\tau(i)}\right|$$

for all $k = 0, \ldots, 2i$ by the choice of $\sigma$. This implies $\sigma(k) \in \{\tau(0), \ldots, \tau(i-1)\}$ or $\sigma(k+1) \in \{\tau(0), \ldots, \tau(i-1)\}$. Since $\{\tau(0), \ldots, \tau(i-1)\}$ contains $i$ elements but the inequality holds for $2i+1$ elements by assumption, we arrive at a contradiction. Since $z_0$ was chosen arbitrarily, the first inequality follows.

Now assume that $u$ is optimal. To see the last inequality, let $\tau$ be a permutation such that $\left|\lambda_{n-1} - \lambda_{\tau(0)}\right| \geq \left|\lambda_{n-1} - \lambda_{\tau(1)}\right| \geq \ldots \geq \left|\lambda_{n-1} - \lambda_{\tau(n-2)}\right|$. By optimality of $u$, we have $|\lambda_i - \lambda_{i+1}| \geq |\lambda_i - \lambda_j|$ for all $j \geq i+1$. Observe that for all $\tau(i) = 0, \ldots, n-2$,

$$\ell_i(u) \leq \mu_i(\lambda_{n-1} - u) = \left|\lambda_{n-1} - \lambda_{\tau(i)}\right| \leq \left|\lambda_{\tau(i)} - \lambda_{\tau(i)+1}\right|,$$

while for $\tau(i) = n-1$, we get $\ell_i(u) = 0$. Thus for each $i$, $\ell_i(u)$ can be estimated from above by $\left|\lambda_{\tau(i)} - \lambda_{\tau(i)+1}\right|$. Since both $\left|\lambda_{\sigma(i)} - \lambda_{\sigma(i)+1}\right|$ and $\ell_i(u)$ are decreasing in $i$, we obtain

$$\ell_i(u) \leq \left|\lambda_{\sigma(i)} - \lambda_{\sigma(i)+1}\right|.$$

□

The above two lemmata imply the following important corollary which relates projective singular values and angles of elements in $\mathrm{U}(n)$.

**Corollary 4.39.** Let $u = \operatorname{diag}(e^{\mathrm{i}\theta_0}, \ldots, e^{\mathrm{i}\theta_{n-1}})$, $v = \operatorname{diag}(e^{\mathrm{i}\gamma_0}, \ldots, e^{\mathrm{i}\gamma_{n-1}}) \in T \subset \mathrm{U}(n)$ be optimal with associated permutation $\sigma$, $\tau$. Then $\ell_{ki}(u) \leq m\ell_i(v)$ for all $i = 0, \ldots, n-1$ and some $k, m \in \mathbb{N}$ implies $\left|\theta_{\sigma(2ki)} - \theta_{\sigma(2ki)+1}\right| \leq 4m \left|\gamma_{\tau(i)} - \gamma_{\tau(i)+1}\right|$ for all $i = 0, \ldots, n-1$. Here we set $\theta_i = \gamma_i = 0$ for all $i \geq n$.

***Proof.*** We use the above two lemmata to prove this. First we conclude that

$$\left|e^{\mathrm{i}\theta_{\sigma(2kj)}} - e^{\mathrm{i}\theta_{\sigma(2kj)+1}}\right| \leq 2m \left|e^{\mathrm{i}\gamma_{\tau(j)}} - e^{\mathrm{i}\gamma_{\tau(j)+1}}\right|.$$

Using now the estimates

$$\begin{aligned} \left|1 - e^{\mathrm{i}(\theta_{\sigma(2kj)} - \theta_{\sigma(2kj)+1})}\right| &\leq \left|\theta_{\sigma(2kj)} - \theta_{\sigma(2kj)+1})\right|, \\ \left|1 - e^{\mathrm{i}(\gamma_{\tau(2kj)} - \gamma_{\tau(2kj)+1})}\right| &\geq \left|\gamma_{\tau(2kj)} - \gamma_{\tau(2kj)+1})\right|/2, \end{aligned}$$

we obtain the claimed inequality. □

We need the following combinatorial lemma to control sums of angles (occuring in the SU(2) product decomposition of an element in SU($n$) (respectively PU($n$)) rank-independently.

**Lemma 4.40.** Let $n \in \mathbb{N}$. Assume that $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ satisfy $\sum_{i=1}^n \alpha_i = 0$. Then there exists a permutation $\sigma \in S_n$ such that for every $k \in \{1, \ldots, n\}$ one has

$$\left|\sum_{i=1}^k \alpha_{\sigma(i)}\right| \leq \max_{i=1,\ldots,n} |\alpha_i|.$$

***Proof.*** Without loss of generality we have $\alpha_1 = \max_{i=1,\ldots,n} |\alpha_i| > 0$ and $\alpha_i \neq 0$ for all $i = 1, \ldots, n$. Moreover, we may assume that $\alpha_1 \geq \ldots \geq \alpha_l > 0$ and $\alpha_{l+1} \leq \ldots \leq \alpha_n < 0$ for some $l < n$. We now construct the permutation $\sigma \in S_n$. We let $\sigma(1) \coloneqq 1$ and $\sigma(2) = l + 1$. Then $\alpha_{\sigma(1)} + \alpha_{\sigma(2)} \geq 0$.

(1) Let $1 \leq j_1 \leq n$ be the (unique) smallest number such that

$$\alpha_1 + \alpha_{l+1} + \ldots + \alpha_{l+j_1} \geq 0.$$

Set $\sigma(1 + i) \coloneqq l + i$, where $i = 1, \ldots, j_1$.

(2) If there are no $\alpha_i$'s left, then we are done. Else we let $1 \leq j_2 \leq l$ be the unique

smallest number such that

$$\alpha_1 + \alpha_{l+1} + \ldots + \alpha_{l+j_1} + \alpha_2 + \ldots + \alpha_{1+j_2} \geq 0.$$

Put $\sigma(1 + j_1 + i) \coloneqq 1 + i$ for $i = 1, \ldots, j_2$.

We obviously have for $k \leq 1 + j_1 + j_2$

$$\left| \sum_{i=1}^{k} \alpha_{\sigma(i)} \right| \leq \max_{i=1,\ldots,n} |\alpha_i| = \alpha_{\sigma(1)}.$$

Proceed inductively interchanging steps (1) and (2) until $\sigma$ is defined on $\{1, \ldots, n\}$. This finishes the proof. $\square$

**Definition 4.41.** Let $u = \mathrm{diag}(e^{\mathrm{i}\,\theta_0}, \ldots, e^{\mathrm{i}\,\theta_{n-1}}) \in \mathrm{SU}(n)$ such that $\sum_{i=0}^{n-1} \theta_i = 0$. Let $\alpha \in S_n$ be as in Lemma 4.40. Then we say that the element $\mathrm{diag}(e^{\theta_{\alpha(0)}}, \ldots, e^{\mathrm{i}\,\theta_{\alpha(n-1)}})$ **angle sum optimal**. The permutation $\alpha$ is said to be **associated to** the angle sum optimal element $u$.

**Lemma 4.42.** Assume that $u = \mathrm{diag}(e^{\mathrm{i}\,\theta_0}, \ldots, e^{\mathrm{i}\,\theta_{n-1}}) \in \mathrm{SU}(n)$ with $\sum_{i=0}^{n-1} \theta_i = 0$. Then we have

$$2\ell_0(u) \geq \max_{i=0,\ldots,n-1} |\theta_i| \,.$$

***Proof.*** We may assume that $u$ is optimal and thus $|\theta_0 - \theta_1| \geq |\theta_i - \theta_j|$ for all $i, j = 0, \ldots, n-1$. Since $\sum_{i=0}^{n-1} \theta_i = 0$ we obtain

$$|\theta_0 - \theta_1| \geq \max_{i=0,\ldots,n-1} |\theta_i| \,.$$

Thus by Lemma 4.38 we have $2\ell_0(u) \geq \max_{i=0,\ldots,n-1} |\theta_i|$ as claimed. $\square$

For Lie group $\mathrm{PU}(n)$ we obtain the following rank-dependent result by successive application of Lemma 4.34.

**Theorem 4.43.** *Let $G \coloneqq \mathrm{PU}(n)$, $n \geq 2$, and assume that $u, v \in G \setminus \{1\}$ satisfy $\ell_0(u) \leq m\ell_0(v)$ for some $m \in \mathbb{N}$. Then*

$$u \in (v^G \cup v^{-G})^{8mn}.$$

***Proof.*** Without loss of generality,

$$u = \operatorname{diag}(e^{\mathrm{i}\,\theta_0}, \dots, e^{\mathrm{i}\,\theta_{n-1}}) = u_0 \cdot \ldots \cdot u_{n-2} \in S_0 \cdot \ldots \cdot S_{n-2}$$

is angle sum optimal with associated permutation $\alpha$ and

$$v = \operatorname{diag}(e^{\mathrm{i}\,\gamma_0}, \dots, e^{\mathrm{i}\,\gamma_{n-1}})$$

is optimal with associated permutation $\tau$. Since $\ell_0(u) \leq m\ell_0(v)$, we conclude from Corollary 4.39 and the definition of optimality that for all $i = 0, \dots, n-1$ we have

$$4m\left|\gamma_{\tau(0)} - \gamma_{\tau(0)+1}\right| \geq \max_{j,k=0,\dots,n-1} |\theta_j - \theta_k| \geq \left|\theta_{\alpha(0)} + \theta_{\alpha(1)} + \ldots + \theta_{\alpha(i)}\right|.$$

Now we can apply Lemma 4.34 for each $u_i$ and hence obtain

$$u_i \in (v^G \cup v^{-G})^{8m}.$$

Proceeding the same way for all $n-1$ terms $u_i$ we have

$$u \in (v^G \cup v^{-G})^{8m(n-1)} \subseteq (v^G \cup v^{-G})^{8mn}.$$

□

*Remark.* Theorem 4.43 can actually be sharpened in the sense that one does not need the conjugacy class of $v^{-1}$. To see this, observe that one may choose $n$ permutations $\pi_1, \dots, \pi_{n-1} \in G$ such that

$$v\pi_1 v\pi_1^{-1} \cdot \ldots \cdot \pi_{n-1} v \pi_{n-1}^{-1} = \operatorname{diag}(e^{\mathrm{i}\,\gamma_0} \cdot \ldots \cdot e^{\mathrm{i}\,\gamma_{n-1}}, \dots, e^{\mathrm{i}\,\gamma_0} \cdot \ldots \cdot e^{\mathrm{i}\,\gamma_{n-1}}) = 1.$$

Thus $1 \in (v^G)^n$, which implies $v^{-1} \in (v^G)^{n-1}$.

**Corollary 4.44.** Assume that $v \in G \coloneqq \mathrm{PU}(n)$ is nontrivial, where $n \geq 2$. Then for every $k \geq 16n/\ell_0(v)$ we have

$$G = (v^G \cup v^{-G})^k.$$

In particular, $\mathrm{PU}(n)$ has property (BNG).

***Proof.*** Since $v$ is nontrivial we have $\ell_0(v) > 0$. It is trivial that for any $u \in G$ one has

$\ell_0(u) \leq \frac{2}{\ell_0(v)}\ell_0(v) = 2$. Using Theorem 4.43 we conclude

$$u \in (v^G \cup v^{-G})^{8n\cdot\lceil 2/\ell_0(v)\rceil}.$$

From this we conclude that $G = (v^G \cup v^{-G})^{8n\cdot\lceil 2/\ell_0(v)\rceil}$ and in particular that $G$ has property (BNG). □

Now we come to the main result of this section. The main ingredient is Lemma 4.35.

**Theorem 4.45.** *Let $G \coloneqq \mathrm{PU}(n)$, where $n \geq 2$. Assume that $u, v \in G$ satisfy $\ell_0(u) \leq m\ell_t(v)$ for some $m \in \mathbb{N}$ and $t = 0, 1, \dots, s-1 \leq n-1$. Then*

$$u \in (v^G \cup v^{-G})^{16m\lceil (n-1)/s\rceil}.$$

***Proof.*** Since we are in $\mathrm{PU}(n)$ we may assume, multiplying with a central element if necessary, that the angle sums of $u$ and $v$ add up to 0. Without loss of generality $u$ is angle sum optimal and $v$ is optimal with associated permutation $\alpha$ and $\tau$ respectively. The first step is to generate most of $u = u_0 \cdot \ldots \cdot u_{n-2}$ (in the $\mathrm{SU}(2)$ product decomposition) simultaneously. Assume that $n-1$ is divisible by two (if not, the following works equally well for $n-2$ instead since we are generous with the number of conjugates). We split the set $A \coloneqq \{0, \dots, n-2\}$ of indices into two sets $A_i \subset A$ with cardinality $(n-1)/2$ and such that $|a-b| \geq 2$ for any distinct $a, b \in A_i$, $i = 1, 2$. Let $N$ denote the unique largest integer divisible by $s$ such that $N \leq \frac{n-1}{2}$. Further decompose each $A_i$ into $2N/s$ sets $A_{i,j}$ of cardinality $s/2$. Then $A_i \setminus \bigcup_{l=1,\dots,2N/s} A_{i,l}$ has at most $s-1$ elements.

By Corollary 4.39, for all $j = 0, \dots, s-1$, we have

$$\left|\sum_{i=0}^{j} \theta_{\alpha(i)}\right| \leq 4m\left|\gamma_{\tau(i)} - \gamma_{\tau(i+1)}\right|.$$

Let $B \coloneqq \bigcup_{i=1,2,3,\ l=1,\dots,N} A_{i,l}$ and observe that the cardinality of $A \setminus B$ is at most $s-1$. Applying now Lemma 4.35 to all $2 \cdot 2N/s$ sets $A_{i,l}$ we have

$$\prod_{j\in B} u_j \in (v^G \cup v^{-G})^{4m\cdot 4N/s}.$$

Using again Lemma 4.35 for the remaining factors of $u$ we obtain

$$\prod_{j\in A\setminus B} u_j \in (v^G \cup v^{-G})^{4m\cdot 2}.$$

Thus from $N \leq (n-1)/2$ and $(N+s/2)/s \leq (n-1+s)/(2s) \leq (n-1)/s$ we conclude

$$u \in (v^G \cup v^{-G})^{16mN/s+8m} \subseteq (v^G \cup v^{-G})^{16m\lceil (n-1)/s\rceil}.$$

□

Let us explain in what sense Theorem 4.45 is rank-independent. Retain the notations of Theorem 4.45. If $s = 1$, then the rank of $\mathrm{PU}(n)$ is clearly involved. However, if $s = (n-1)/p$ with $p \in \mathbb{N}$, then one needs at most $16mp$ conjugates to generate $u$, which is independent of the rank if, e.g., $p = 2$. This will be useful to prove the topological bounded normal generation property for projective unitary groups of $\mathrm{II}_1$ factors in Section 4.7.

## 4.6 Bounded normal generation for the Calkin algebra

Let $\mathcal{M}$ be a separable type $I_\infty$-factor. Then $\mathcal{M} = \mathcal{B}(\mathcal{H})$ for some infinite-dimensional separable Hilbert space $\mathcal{H}$ by Corollary V.1.28 in [Ta 03]. In this section, we consider $G := \mathrm{PU}(\mathcal{M})$ endowed with the strong operator topology. In this topology $G$ is a Polish group. However, we also need to consider Hilbert-Schmidt perturbations in this group in the proof of Theorem 4.51. The topology induced from the Hilbert-Schmidt norm does not even make $\mathrm{U}(\mathcal{M})$ respectively $\mathrm{PU}(\mathcal{M})$ a topological group.

We explain the notion used in Theorem 4.51. Let $u, v \in G$, $n \in \mathbb{N}$ and denote by $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm. The notion $u \in \overline{(v^G \cup v^{-G})^n}^{\|\cdot\|_{HS}}$ used in Theorem 4.51 means that $u$ lies in $(v^G \cup v^{-G})^n$ up to an arbitrarily $\|\cdot\|_{HS}$-small Hilbert-Schmidt perturbation.

There is an obstruction for the bounded normal generation property of $\mathrm{PU}(\mathcal{H})$ which we will now describe. Let $K$ denote the norm-ideal of compact operators $\mathcal{K}(\mathcal{H})$ on $\mathcal{H}$, endowed with the operator norm. We define $\mathrm{U}(\mathcal{H})_K$ as the group of unitary operators on a Hilbert space $\mathcal{H}$ such that $1-u$ is an element of $K$,

$$\mathrm{U}(\mathcal{H})_K \coloneqq \{u \in \mathrm{U}(\mathcal{H}) \mid 1-u \in K\}.$$

We endow $\mathrm{U}(\mathcal{H})_K$ with the topology given by the operator norm.

Obviously, $\mathrm{U}(\mathcal{H})_K$ is a normal subgroup of $\mathrm{U}(\mathcal{H})$ which contains every finite-dimensional unitary group $\mathrm{U}(n)$, $n \in \mathbb{N}$. The center of $\mathrm{U}(\mathcal{H})_K$ does not contain the circle rotation group $\mathrm{U}(1)$, in fact it only consists of the element 1. Since $\mathrm{U}(\mathcal{H})_K$ is naturally embedded in $\mathrm{U}(\mathcal{H})$, we consider the generalized projective $s$-numbers

$$\ell_t(u) = \inf_{\lambda \in \mathcal{Z}(\mathrm{U}(\mathcal{H}))} \mu_t(1-\lambda u) = \inf_{\lambda \in \mathrm{U}(1)} \mu_t(1-\lambda u).$$

Observe that for any $u \in \mathrm{U}(\mathcal{H})_K$ we have $\ell_t(u) \leq \mu_t(1-u) \to 0$ for $t \to \infty$ by compactness of $1-u$. In particular, for elements $u, v \in \mathrm{U}(\mathcal{H})_K$ there usually does not exist a number $m \in \mathbb{N}$ such that $\ell_0(u) \leq m\ell_t(v)$ for all $t \geq 0$. This is the obstruction for the bounded normal generation property of $\mathrm{PU}(\mathcal{H})$. Thus we can only hope for property (BNG) for the connected components of the projective unitary group of the Calkin algebra.

We embed $u \in \mathrm{U}(n)$ into $\mathrm{U}(\mathcal{H})_K$ in the usual way by $\mathrm{U}(n) \ni u \mapsto \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{U}(\mathcal{H})_K$. It is not hard to show that the unions $\bigcup_{n\in\mathbb{N}} \mathrm{U}(n)$ as well as $\bigcup_{n\in\mathbb{N}} \mathrm{SU}(n)$ are dense in $\mathrm{U}(\mathcal{H})_K$ in the uniform topology.

It is known that $\mathrm{U}(\mathcal{H})_K$ is topologically simple in the uniform topology. However, there is no topological uniform normal generator for $\mathrm{U}(\mathcal{H})_K$. Suppose the contrary and let $v$ be a topological uniform normal generator for $\mathrm{U}(\mathcal{H})_K$. Then one can replace the sequence of singular values of $1-v \in K$ with their square roots and obtain a corresponding element $u \in \mathrm{U}(\mathcal{H})_K$. But then there exists no $k \in \mathbb{N}$ such that $\ell_{kt}(u) \leq k\ell_t(v)$ for all $t \geq 0$, which contradicts Proposition 4.28.

Analogously to the finite-dimensional case we deal with two different decompositions. If $u = \operatorname{diag}(\lambda_0, \lambda_1, \ldots) \in \mathrm{U}(\mathcal{H})$ is a diagonal element, then we can again write $u = \prod_{j\in\mathbb{N}_0} u_j$ in the $\mathrm{SU}(2)$ product decomposition with $u_j$ defined as follows:

$$u_j = \operatorname{diag}(1, \ldots, 1, \lambda_0 \cdot \ldots \cdot \lambda_j, \overline{\lambda_0 \cdot \ldots \cdot \lambda_j}, 1, 1, \ldots),\ j \in \mathbb{N}_0.$$

Then we have

$$\prod_{j=0}^{n} u_j = \operatorname{diag}(\lambda_0, \ldots, \lambda_n, \overline{\lambda}_0 \cdot \ldots \cdot \overline{\lambda}_n, 1, 1, \ldots) \to_{n\to\infty} u$$

strongly (but not uniformly). We can also define the one-parameter torus decomposition $\prod_{j\in\mathbb{N}_0} \widetilde{u}_j$ of $u$ by setting

$$\widetilde{u}_0 = \operatorname{diag}(\lambda_0, \lambda_0, \ldots),\ \widetilde{u}_j = \operatorname{diag}(1, \ldots, 1, \overline{\lambda}_{j-1}\lambda_j, \overline{\lambda}_{j-1}\lambda_j, \ldots),\ j \in \mathbb{N}_0.$$

Then in the strong operator topology (but not in the uniform topology) we have

$$\prod_{j=0}^{n} \widetilde{u}_j = (\lambda_0, \lambda_1, \ldots, \lambda_{n-1}, \lambda_n, \lambda_n, \ldots) \to_{n\to\infty} u.$$

That is, when $\mathrm{U}(\mathcal{H})$ is endowed with the strong operator topology, then we have

$$u' = \prod_{j\in\mathbb{N}_0} \widetilde{u}_j = \prod_{j\in\mathbb{N}_0} u_j.$$

Let us prove an infinite-dimensional analogue of Lemma 4.35. As in the previous section we denote by $S_j$ a copy of $\mathrm{SU}(2)$ embedded in $\mathrm{U}(\mathcal{H})$ around the diagonal entries $j, j+1$.

**Lemma 4.46.** Consider $u = \operatorname{diag}(e^{\mathrm{i}\theta_0}, \ldots), v = \operatorname{diag}(e^{\mathrm{i}\gamma_0}, \ldots) \in G := \mathrm{U}(\mathcal{H})$, $G$ endowed with the strong operator topology. Assume that $\prod_{i\in\mathbb{N}_0} u_i$ is the $\mathrm{SU}(2)$ product decomposition of $u$ and $v = \prod_{i\in\mathbb{N}_0} v_i$ is the one-parameter torus decomposition. Let $I, J \subseteq \mathbb{N}_0$ be countable index sets such that $|i_k - i_l|, |j_k - j_l| > 1$ for all $k \neq l \in \mathbb{N}_0$, where $i_k, i_l \in I, j_k, j_l \in J$. Assume that $\left|\sum_{i=1}^{i_k} \theta_i\right| \leq m\,|\gamma_{j_k} - \gamma_{j_k+1}|$ for some even number $m \in \mathbb{N}$ and all $k, l \in \mathbb{N}_0$. Then

$$\prod_{i\in I} u_i \in \left(v^G \cup v^{-G}\right)^{2m}.$$

***Proof***. Write as above $u = \widetilde{u}\prod_{k=1}^{\infty} u_{i_k}$ where $\widetilde{u} = \prod_{l\in\mathbb{N}\setminus I} u_l$ and analogously $v = \widetilde{v}\prod_{l=1}^{\infty} v_{j_l}$. Note that $S_{j_k}$ and $S_{j_l}$ commute elementwise for $k \neq l$. Moreover, $\widetilde{v}$ commutes with $S_{j_k}$ for all $j_k \in J$. Thus we get

$$\left(v^{\prod_{j\in J} S_j}\right)^m = \left(\widetilde{v}^{\prod_{j\in J} S_j} \prod_{j\in J} v_j^{S_j}\right)^m = \widetilde{v}^m \prod_{j\in J} \left(v_j^{S_j}\right)^m.$$

Let $w \in \mathrm{U}(\mathcal{H})$ be an unitary operator such that $S_{i_k}^w = S_{j_k}$ for all $k \in \mathbb{N}_0$. Using Lemma 4.33 and the given inequality, we obtain $u_{i_k}^w \in \left(v_{j_k}^{S_{j_k}}\right)^m$ for all $k \in \mathbb{N}_0$, i.e.,

$$\prod_{i\in I} u_i \in \left(\widetilde{v}^{-m} \left(v^{\prod_{j\in J} S_j}\right)^m\right)^{w^*}.$$

Since

$$(\widetilde{v})^{-2} \in \widetilde{v}^{-2} \prod_{j\in J} \left(v_j^{S_j}\right)^{-2} = \left(v^{\prod_{j\in J} S_j}\right)^{-2},$$

we conclude

$$\prod_{i\in I} u_i \in \left(\left(v^{\prod_{j\in J} S_j}\right)^m \cdot \left(v^{-\prod_{j\in J} S_j}\right)^m\right)^{w^*} \subset w^* \left(v^G \cup v^{-G}\right)^{2m} w.$$

This finishes the proof. □

In Section 4.5 we defined for elements in $\mathrm{U}(n)$ the notions of optimality and angle-sum optimality in order to be able to apply the local $\mathrm{SU}(2)$-result of Nikolov and Segal, cf. Lemma 4.33, at several positions simultaneously. We want to apply the same strategy at infinitely many positions and hence need to transfer the just mentioned notions to the unitary group $\mathrm{U}(\mathcal{H})$.

Let us now adapt the concept of optimality. The main reason for the following more complicated definition is that a diagonal (unitary) operator might have infinitely many cluster points.

**Definition 4.47.** Assume that $u = \mathrm{diag}(e^{\mathrm{i}\,\theta_0}, e^{\mathrm{i}\,\theta_1}, \ldots) \in G := \mathrm{U}(\mathcal{H})$ and let $A \subseteq \mathbb{N}_0$ be an index set the form $\emptyset$, or $\{0, 1, \ldots, d\}$ for some $d \in \mathbb{N}_0$ or $\mathbb{N}_0$. Let $(\varepsilon_n)_{n\in\mathbb{N}_0}$ be a sequence of elements $\varepsilon_n \in [-1/2, 1/2]$ and $\varepsilon_n := 0$ for all $n \in \mathbb{N}_0 \setminus A$. We say that a diagonal operator $u' = \mathrm{diag}(e^{\mathrm{i}(\theta_0+\varepsilon_0)}, e^{\mathrm{i}(\theta_1+\varepsilon_1)}, \ldots)$ is $(\varepsilon_n)_{n\in A}$**-optimal** for $u$ if

- $|\varepsilon_n| \geq |\varepsilon_{n+1}|$;
- $\left|e^{\mathrm{i}(\theta_j+\varepsilon_j)} - e^{\mathrm{i}(\theta_{j+1}+\varepsilon_{j+1})}\right| = 2$ for all $j \in A$;
- if $v = \mathrm{diag}(e^{\mathrm{i}\,\gamma_0}, \ldots) \in u^G$ such that $\gamma_i = \theta_i$ for $i = 0, \ldots, n$ and $\left|e^{\mathrm{i}\,\theta_j} - e^{\mathrm{i}\,\theta_{j+1}}\right| =$

$\left|e^{\mathrm{i}\,\gamma_j} - e^{\mathrm{i}\,\gamma_{j+1}}\right|$ for all $j = 0, \dots, d, d+1, \dots, k-1$, then

$$\left|e^{\mathrm{i}\,\theta_k} - e^{\mathrm{i}\,\theta_{k+1}}\right| \geq \left|e^{\mathrm{i}\,\gamma_k} - e^{\mathrm{i}\,\gamma_{k+1}}\right|.$$

If $A = \emptyset$, then we also call $u'$ **optimal** for $u$.
Moreover, to an $(\varepsilon_n)_{n\in A}$-optimal element $u'$ we associate an injective map $\sigma : \mathbb{N}_0 \to \mathbb{N}_0$ with $u'$, such that the differences $\left|e^{\mathrm{i}(\theta_{\sigma(i)}+\varepsilon_{\sigma(i)})} - e^{\mathrm{i}(\theta_{\sigma(i)+1}+\varepsilon_{\sigma(i)+1})}\right|$ are decreasing.

The map $\sigma$ in the above definition can be inductively constructed as follows. We set $\sigma(0) = 0$. Suppose that $\sigma(i-1)$ is constructed. Then $\sigma(i)$ takes the value of the smallest index $j$ which satisfies

$$\begin{aligned}\left|e^{\mathrm{i}(\theta_{\sigma(l)}+\varepsilon_{\sigma(l)})} - e^{\mathrm{i}(\theta_{\sigma(l)+1}+\varepsilon_{\sigma(l)+1})}\right| &\geq \left|e^{\mathrm{i}(\theta_j+\varepsilon_j)} - e^{\mathrm{i}(\theta_{j+1}+\varepsilon_{j+1})}\right| \\ &\geq \left|e^{\mathrm{i}(\theta_k+\varepsilon_k)} - e^{\mathrm{i}(\theta_{k+1}+\varepsilon_{k+1})}\right|,\end{aligned}$$

for all $l \leq i-1$ and $k \notin \{\sigma(0), \dots, \sigma(i-1)\}$. There are cases where $\sigma$ cannot be constructed surjectively and hence not as an element of the infinite permutation group $S_\infty$. Namely if infinitely of the above differences are bigger than some others.

We need $(\varepsilon_n)_{n\in A}$-optimality in order to have Corollary 4.39. Note that for a diagonal element $u \in \mathrm{U}(\mathcal{H})$ there always exists an $(\varepsilon_n)_{n\in A}$-optimal element, the set $A$ and the possible values of each $\varepsilon_n$ depend however very much on the spectrum of $u$. Observe that an $(\varepsilon_n)_{n\in A}$-optimal element $u'$ as above lies in $(u^G)_{2|\varepsilon_0|}$ with respect to the operator norm, by continuity of $\left|1 - e^{\mathrm{i}\,\varepsilon_0}\right|$, where $G \coloneqq \mathrm{U}(\mathcal{H})$. The third condition in the above definition defines a lexicographic order starting from the $(d+1)$th entry.

**Definition 4.48.** Let $u \in \mathrm{U}(\mathcal{H})$. If $\lambda_1, \lambda_2$ lie in the spectrum of $u$, then we call the multiplicity of $\lambda_1$ minus the multiplicity of $\lambda_2$ the **relative** $(\lambda_1, \lambda_2)$**-multiplicity of** $u$. We denote this integer by $\nu_{\lambda_1,\lambda_2}$ or simply $\nu$ if $\lambda_1, \lambda_2$ are clear from the context. The multiplicity of a non-eigenvalue is always set to be zero. If the multiplicities of both $\lambda_1$ and $\lambda_2$ are infinite, then $\nu \coloneqq 0$. For notational convenience we always set $\varepsilon_{-n} \coloneqq 0$ for $n \in \mathbb{N}$ and $\varepsilon_0 \coloneqq 0$ if the relative $(\lambda_1, \lambda_2)$-multiplicity is less than 1. Moreover, we set $\varepsilon_n \coloneqq 0$ for all $n \geq \nu$.

For example, an element $u \in \mathrm{U}(\mathcal{H})_K$ might have $-1$ in its spectrum, and $1 \in \sigma(u)$ as a limit point or with lower multiplicity than $-1$. In any case, if the relative $(-1, 1)$

multiplicity is positive, then it is finite, by compactness of $1-u$.

Before being able to define an infinite-dimensional variant of angle-sum optimality, we need to imitate Lemma 4.40.

**Lemma 4.49.** Let $\{\alpha_n\}_{n\in\mathbb{N}_0}\subseteq\mathbb{R}$ be a sequence with infinitely many positive and infinitely many negative real numbers such that both the sum over all positive $\alpha_n$ and over all negative $\alpha_n$ diverge. Then there exists a permutation $\sigma\in S_\infty$ such that

$$\left|\sum_{i=0}^{n}\alpha_{\sigma(i)}\right|\leq\sup_{j\in\mathbb{N}_0}|\alpha_j|\quad\text{for every } n\in\mathbb{N}_0.$$

***Proof.*** The construction of $\sigma$ is analogous to that in Lemma 4.40. We divide $\mathbb{N}_0$ into three sets $A_1 := \{n\in\mathbb{N}_0\mid\alpha_n>0\}$, $A_2 := \{n\in\mathbb{N}_0\mid\alpha_n=0\}$ and $A_3 := \{n\in\mathbb{N}_0\mid\alpha_n<0\}$. Define $\sigma$ inductively. First put $\sigma(0)=\min_{n\in A_1}n$, $\sigma(1)=\min_{n\in A_2}n$ and $\sigma(2)=\min_{n\in A_3}n$. If $\sum_{i=0}^{2}\alpha_{\sigma(i)}>0$, then set $\sigma(2+i)=\min_{n\in A_3\setminus\{\sigma(2),\sigma(3),\ldots,\sigma(i+1)\}}n$ for $i=1,\ldots,j_1$ with $j_1$ the smallest number such that $\sum_{i=0}^{2+j_1}\alpha_{\sigma(i)}\leq 0$. Now let $\sigma(2+j_1+i)=\min_{n\in A_1\setminus\{\sigma(0),\sigma(2+j_1+1),\ldots,\sigma(2+j_1+i-1)\}}n$ for $i=1,\ldots,j_2$ with $j_2$ the smallest number such that $\sum_{i=0}^{2+j_1+j_2}\alpha_{\sigma(i)}>0$. Put $\sigma(2+j_1+j_2+1)=\min_{n\in A_2\setminus\{\sigma(0)\}}n$ and $\sigma(3+j_1+j_2+i)=\min_{n\in A_3\setminus\{\sigma(0),\ldots,\sigma(3+j_1+j_2+i-1)\}}n$ for $i=1,\ldots,j_3$ with $j_3$ the smallest number such that $\sum_{i=0}^{3+j_1+j_2+j_3}\alpha_{\sigma(i)}\leq 0$. Proceed by induction (alternating the above steps). □

Let $u\in\mathrm{PU}(\mathcal{H})\setminus\mathrm{U}(\mathcal{H})_K$, $u\neq 1$. Observe that there is $\lambda\in\mathrm{U}(1)$ such that $\lambda u$ has infinitely many positive and infinitely many negative angles. For example, if the spectrum $\sigma(u)$ of $u$ is $\{e^{\mathrm{i}\,\pi/2},1\}$, where $e^{\mathrm{i}\,\pi/2}$ and 1 have infinite multiplicities, then one can choose $\lambda=e^{-\mathrm{i}\,\pi/4}$ to obtain $\sigma(\lambda u)=\{e^{\mathrm{i}\,\pi/4},e^{-\mathrm{i}\,\pi/4}\}$.

**Definition 4.50.** Let $u=\mathrm{diag}(e^{\mathrm{i}\,\theta_0},e^{\mathrm{i}\,\theta_1},\ldots)\in\mathrm{PU}(\mathcal{H})\setminus\mathrm{U}(\mathcal{H})_K$, $u\neq 1$, and $\sigma\in S_\infty$ as in Lemma 4.49. Then we say that $\mathrm{diag}(e^{\mathrm{i}\,\theta_{\sigma(0)}},e^{\mathrm{i}\,\theta_{\sigma(1)}},\ldots)$ is **angle sum ordered with respect to** $u$.

Note that by Lemma 4.49 and the remark preceding the above definition, given any $u\in\mathrm{PU}(\mathcal{H})\setminus\mathrm{U}(\mathcal{H})_K$, $u\neq 1$, there exists an element in $u^G$, $G:=\mathrm{PU}(\mathcal{H})$, which is angle sum ordered with respect to $u$. In fact, there exist infinitely many such elements.

**Theorem 4.51.** *Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. Consider the projective unitary group $G:=\mathrm{PU}(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$, endowed with the strong operator* *topology. Assume that $u, v \in G \setminus \mathrm{U}(\mathcal{H})_K$ are nontrivial elements satisfying $\ell_0(u) \leq m\ell_t(v)$ for all $t \geq 0$ and some $m \in \mathbb{N}$. Then*

$$u \in \overline{(v^G \cup v^{-G})^{20m}}^{\|\cdot\|_{HS}}.$$

***Proof.*** Since $v \in \mathrm{PU}(\mathcal{H}) \setminus \mathrm{U}(\mathcal{H})_K$, $\ell_t(v)$ does not tend to zero as $t \to \infty$. Hence there exists $\delta > 0$ such that $\ell_0(u) \leq 2m\ell_t(v) - \delta$ for all $t \geq 0$. We choose $\varepsilon > 0$ such that $\varepsilon \leq \delta/(2m+2)$. Using a version of the noncommutative Weyl-von Neumann-type theorem by Voiculescu, see [Vo 79, Theorem 2.4], we obtain the existence of diagonal elements

$$u' = \mathrm{diag}(e^{\mathrm{i}\,\theta_0}, e^{\mathrm{i}\,\theta_1}, \ldots), \quad v' = \mathrm{diag}(e^{\mathrm{i}\,\gamma_0}, e^{\mathrm{i}\,\gamma_1}, \ldots)$$

and $g, h \in G$ such that $gu'g^{-1}$ and $hv'h^{-1}$ are $\varepsilon$-close to $u$ and $v$ in the Hilbert-Schmidt norm $\|\cdot\|_{HS}$ respectively. By Lemma 4.30 and since the Hilbert-Schmidt norm is always greater or equal than the operator norm, we have

$$\begin{aligned}\ell_0(u') &\leq 2m\ell_t(v') - \delta + 2(m+1)\varepsilon \\ &\leq 2m\ell_t(v') \qquad \text{for all } t \geq 0.\end{aligned}$$

We now describe how to get an $(\varepsilon_n)_{n\in A}$-optimal element for $v'$. Let $(c_n)_{n\in A''}$ denote the (possibly empty, finite or countably infinite) sequence of cluster points $c_n$ in the set $\{e^{\mathrm{i}\,\gamma_0}, e^{\mathrm{i}\,\gamma_1}, \ldots\}$ (starting from 1 in mathematically positive direction). Consider the corresponding sequence $(\nu_n)$ of relative $(\overline{c}_n, c_n)$ multiplicities and throw out the nonpositive ones. Here we set $\nu_n := 0$ if both $-c_n$ and $c_n$ have infinite multiplicity. We obtain a subsequence $(\nu_n)_{n\in A'}$ of $(\nu_n)_{n\in A''}$, where $A' \subseteq A''$. The index set $A$ in the definition of $(\varepsilon_n)_{n\in A}$-optimality is now empty if $A'$ is empty, and else $A = \{0, 1, \ldots, 2\sum_{n\in A'} \nu_n - 1\}$, in particular, $A = \mathbb{N}_0$ if $2\sum_{n\in A'} \nu_n$ is not finite.
Now we may choose arbitarily small $\varepsilon_n$, decreasing in absolute value, such that $\varepsilon > |\varepsilon_n| > 0$ for $n \in A$ and $|\varepsilon_n| \to 0$ and such that $v'' = \mathrm{diag}(e^{\mathrm{i}(\gamma_0+\varepsilon_0)}, \ldots)$ is $(\varepsilon_n)_{n\in A}$-optimal for $v'$ with associated permutation $\tau$, where we assume without loss of generality that the eigenvalues are ordered in such a way, that this is possible (follows from bi-invariance of the operator norm and renumbering if necessary).

Assume that $u'' \in (u')^G$ is angle sum ordered with respect to $u'$ (with corresponding permutation $\sigma$). Then we have $2\ell_0(u'') \geq \left|\sum_{i=0}^{n} \theta_{\sigma(i)}\right|$ for all $n \in \mathbb{N}_0$. Using Corollary

4.39 we obtain for all $n \in \mathbb{N}_0$ and $t \geq 0$

$$\left|\sum_{i=0}^{n} \theta_{\sigma(i)}\right| \leq 2\ell_0(u'') \leq 8m \left|\gamma_{\tau(t)} + \varepsilon_{\tau(t)} - \gamma_{\tau(t)+1} - \varepsilon_{\tau(t)+1}\right|.$$

Hence by the triangle inequality, assuming $\varepsilon_n$ to be sufficiently small,

$$\begin{aligned} \left|\sum_{i=0}^{n} \theta_{\sigma(i)}\right| &\leq 8m \left|\gamma_{\tau(t)} - \gamma_{\tau(t)+1}\right| + 8m \left|\varepsilon_{\tau(t)} - \varepsilon_{\tau(t)+1}\right| \\ &\leq 10m \left|\gamma_{\tau(t)} - \gamma_{\tau(t)+1}\right| \qquad \text{for all } t \geq 0,\ n \in \mathbb{N}_0. \end{aligned} \tag{4.2}$$

Before being able to generate $u''$ from $v''$, we need to decompose both elements in an appropriate way. Decompose $u'' = \prod_{i\in\mathbb{N}_0} u_i$ into its SU(2) product decomposition and $v'' = \prod_{i\in\mathbb{N}_0} v_i$ into its one-parameter torus decomposition. In order to generate infinitely many entries of $u''$ simultaneously, we partition $\mathbb{N}_0$ into disjoint sets $A_1$ and $A_2$, where $A_1 \coloneqq \{2n \mid n \in \mathbb{N}_0\}$ and $A_2 \coloneqq \{2n+1 \mid n \in \mathbb{N}_0\}$.
We use Lemma 4.46 and Inequality (4.2) to obtain

$$\prod_{j\in A_i} u_j \in \left((v'')^G \cup (v'')^{-G}\right)^{10m}, \quad i = 1, 2,$$

and thus

$$u'' \in \left((v'')^G \cup (v'')^{-G}\right)^{20m}.$$

Since $u'' \in (u')^G$, $v'' \in ((v')^G)_{\varepsilon_0} \subseteq ((v')^G)_\varepsilon$ and $u' \in (u^G)_{\varepsilon,\|\cdot\|_{HS}}$, $v' \in (v^G)_{\varepsilon,\|\cdot\|_{HS}}$, we get

$$u \in \left((u'')^G\right)_\varepsilon \subseteq \left(\left((v^G)_{2\varepsilon,\|\cdot\|_{HS}} \cup (v^{-G})_{2\varepsilon,\|\cdot\|_{HS}}\right)^{20m}\right)_{\varepsilon,\|\cdot\|_{HS}}.$$

Thus by Lemma 4.31 we have $u \in \left(\left(v^G \cup v^{-G}\right)^{20m}\right)_{(20m+1)2\varepsilon,\|\cdot\|_{HS}}$.
Letting $\varepsilon$ tend to zero, we arrive at

$$u \in \overline{(v^G \cup v^{-G})^{20m}}^{\|\cdot\|_{HS}}.$$

This completes the proof. $\square$

In Theorem 4.51 we actually do not need the condition $u \notin \mathrm{U}(\mathcal{H})_K$. The only problem that can arise for $u \in \mathrm{U}(\mathcal{H})_K$ is that the angle sum increases for every possible SU(2) decomposition of $u$. Hence the assumption has to be changed to $2 \leq m\ell_t(v)$ (instead of $\ell_0(u) \leq m\ell_t(v)$) for all $t \geq 0$. The proof works equally well. Thus one can interpret Theorem 4.51 as a criterion to be a topological uniform normal generator for $\mathrm{PU}(\mathcal{H})$, endowed with the strong operator topology.

*Remark.* Fong and Sourour showed in [FS 85] that every proper normal subgroup of $\mathrm{U}(\mathcal{H})$ is contained in the normal subgroup $\mathrm{U}(\mathcal{H})_{\mathcal{K}(\mathcal{H})}$ and that if $u \in \mathrm{U}(\mathcal{H}) \setminus \mathrm{U}(\mathcal{H})_{\mathcal{K}(\mathcal{H})}$, then $u$ is a product of a finite number of operators, each of which is unitarily equivalent. So Theorem 4.51 on the one hand can be considered weaker than the result of Fong and Sourour because it involves the Hilbert-Schmidt norm closure, but on the other hand we give quantitative estimates. Our version will allow us to prove the bounded normal generation property for the connected component of the identity of the projective unitary group of the Calkin algebra.

Let $\mathcal{H}$ be an infinite-dimensional separable (complex) Hilbert space. By $\mathcal{C}$ we denote the Calkin algebra on $\mathcal{H}$, i.e., $\mathcal{C} = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Moreover, we write $\mathrm{U}(\mathcal{C})$ for its unitary group and $\mathrm{PU}(\mathcal{C})$ for its projective unitary group. Each equivalence class in $\mathrm{U}(\mathcal{C})$ contains a diagonal element by the Weyl-von Neumann-Berg-Voiculescu Theorem, see [Vo 79, Theorem 2.4]. By [Mu 90, Theorem 4.1.6] $\mathcal{C}$ is a simple $C^*$-algebra (but not a von Neumann algebra). The (projective) unitary group of the Calkin algebra is not connected (recall that in contrast, the unitary group of a von Neumann algebra is always connected in the uniform topology and hence also in the strong operator topology, see [KR 86, Exercise 5.7.24(ii)]). Its connected components are characterized by the Fredholm index. Since we want to use Theorem 4.51 we need to ensure that the elements of consideration in $\mathrm{PU}(\mathcal{C})$ can be lifted to $\mathrm{U}(\mathcal{H})$. This lift exists precisely if the elements have Fredholm index 0, that is, they are in the connected component of the identity.

The essential norm $\|\cdot\|_{\mathrm{ess}}$ on $\mathcal{C}$ is the norm defined by

$$\|x\|_{\mathrm{ess}} \coloneqq \inf_{y \in \mathcal{K}(\mathcal{H})} \|x - y\|, \qquad \text{where } x, y \in \mathcal{C}.$$

For $u \in \mathrm{PU}(\mathcal{C})$ we let

$$\ell_{\mathrm{ess}}(u) \coloneqq \inf_{\lambda \in \mathrm{U}(1)} \|1 - \lambda u\|_{\mathrm{ess}}.$$

**Theorem 4.52.** *Let $G$ denote connected component of the identity of* $\mathrm{PU}(\mathcal{C})$. *Assume* *that $u, v \in G \setminus \{1\}$ satisfy $\ell_{\text{ess}}(u) \leq m\ell_{\text{ess}}(v)$. Then*

$$u \in (v^G \cup v^{-G})^{20m}.$$

*In particular, if $\ell_{\text{ess}}(v) > 0$, then*

$$G = (v^G \cup v^{-G})^m,$$

*for every $m \geq 40/\ell_{\text{ess}}(v)$. That is, $G$ has property (BNG).*

***Proof.*** Let $H := \mathrm{PU}(\mathcal{H})$. Since $u$ and $v$ are of Fredholm index 0, there exists a lift into $H$. We denote the corresponding elements by $u'$ and $v'$. Since the eigenvalues of $u, v$ have infinite multiplicity, we have $\ell_t(u') \leq m\ell_t(v')$ for all $t \geq 0$. Hence by Theorem 4.51 we have

$$u' \in \overline{(v'^H \cup v'^{-H})^{20m}}^{\|\cdot\|_{HS}}.$$

We may pass back to $\mathrm{PU}(\mathcal{C})$ by using the quotient map, so that we obtain

$$u \in (v^G \cup v^{-G})^{20m},$$

as claimed.

To see the second claim, note that if $\ell_{\text{ess}}(v) > 0$, then trivially

$$\ell_{\text{ess}}(u) \leq \frac{2}{\ell_{\text{ess}}(v)} \ell_{\text{ess}}(v).$$

This holds for arbitrary $u \in G$, i.e. $G$ has property (BNG). □

The following corollary was also found by Fong and Sourour in [FS 85].

**Corollary 4.53.** The connected component $G$ of the identity in $\mathrm{PU}(\mathcal{C})$ is algebraically simple.

Fong and Sourour actually showed more, see [FS 85, Theorem 3]: the normal subgroups of $\mathrm{U}(\mathcal{C})$ are its center and the groups

$$N_n := \{u \in \mathrm{U}(\mathcal{C}) \mid n \text{ divides the Fredholm index of } u\}, \qquad n \in \mathbb{N}_0.$$

## 4.7 Topological bounded normal generation for $\mathrm{II}_1$ factors

This section deals with property (topBNG) for projective unitary groups of $\mathrm{II}_1$ factors, endowed with the strong operator topology. This section is mainly included because our proof of property (topBNG) is easier to understand than our proof of property (BNG) which is treated in the next section.

The strategy of the proof of property (topBNG) is to approximate both $u$ and $v$ (arbitarily close) with elements having finite spectrum and rational weights and then map them to the same element in $\mathrm{PU}(n)$ via partial isometries. This allows us to use Theorem 4.45. Letting the approximation be finer and finer and using Lemma 4.31 we conclude that $u$ is in the strong closure of a product of conjugates of $v$ and $v^*$.

Our first step is to prove the following approximation result.

**Proposition 4.54.** Assume that $u \in \mathrm{U}(\mathcal{M})$ and $\varepsilon > 0$. There exists an element $u' \in \mathrm{U}(\mathcal{M})$ having finite spectrum and corresponding spectral projections of rational trace such that

$$\left\|u - u'\right\|_2 < \varepsilon.$$

***Proof.*** Choose pairwise distinct elements $\lambda_1, \ldots, \lambda_n \in \mathrm{U}(1)$, $n \geq 2$, such that for every $\lambda \in \sigma(u)$ there exists $i \in \{1, \ldots, n\}$ such that $|\lambda - \lambda_i| < \varepsilon/4$ and $\arg(\lambda_i) < \arg(\lambda_{i+1})$ mod $2\pi$. Denote by $p_u$ the spectral measure of $u$ and define

$$p_i := p_u(\{\lambda \mid \arg(\lambda) \in [\arg(\lambda_i), \arg(\lambda_{i+1}))\})$$

for $i = 1, \ldots, n-1$ and $p_n := p_u([\lambda_n, \lambda_1)) = 1 - \sum_{i=1}^{n-1} p_i$. If $p_i$ has rational trace we set $q_i := p_i$. Without loss of generality we may assume that $p_i \neq 0$ and $\tau(p_i) \in \mathbb{R} \setminus \mathbb{Q}$. For $i = 1, \ldots, n-1$ let $q_i$ be a subprojection of $p_i$ with rational trace and such that $\|p_i - q_i\|_2 < \varepsilon/(n-1)$. Set $q_n := 1 - \sum_{i=1}^{n-1} q_i$ and observe that $\tau(q_n) \in \mathbb{Q}$ and $p_n$ is a subprojection of $q_n$ such that

$$\|p_n - q_n\|_2 = \left\|1 - \sum_{i=1}^{n-1} p_i - (1 - \sum_{i=1}^{n-1} q_i)\right\|_2 \leq \sum_{i=1}^{n-1} \|p_i - q_i\|_2 < \varepsilon.$$

Now set $u' := \sum_{i=1}^{n} \lambda_i q_i$. From the inequality $\|xy\|_1 \leq \|x\| \cdot \|y\|_2$ for $x, y \in \mathcal{M}$ and

Proposition 2.18(ii) we conclude

$$\|xy\|_1 \leq \|xy\|_2 \leq \|x\| \cdot \|y\|_2 \leq \|x\| \cdot 2\,\|y\|_1 .$$

Hence we obtain

$$\begin{aligned}\left\|u - u'\right\|_1 &= \left\|\sum_{i=1}^{n}(uq_i - \lambda_i q_i)\right\|_1 \\ &\leq \sum_{i=1}^{n} \|uq_i - \lambda_i q_i\|_1 \\ &\leq \sum_{i=1}^{n} \|uq_i - \lambda_i q_i\| \cdot 2\,\|q_i\|_1 \\ &< \sum_{i=1}^{n} \frac{\varepsilon}{4} \cdot 2\,\|q_i\|_1 \\ &= \frac{\varepsilon}{2}.\end{aligned}$$

By Proposition 2.18(ii) we thus have

$$\left\|u - u'\right\|_2 \leq 2 \cdot \left\|u - u'\right\|_1 < \varepsilon,$$

as desired. ☐

*Remark.* Using similar arguments as in the proof of Proposition 4.54 one can show that if the connected components of the spectrum of $u \in \mathrm{U}(\mathcal{M})$ have rational weight, then for arbitrary $\varepsilon > 0$ there exists $u'$ with finite spectrum and spectral projections of rational trace such that

$$\left\|u - u'\right\| < \varepsilon.$$

The proof of Theorem 4.57 uses the following technical lemma, which allows us estimate singular values for sufficiently close 2-norm approximations of a given element in a $\mathrm{II}_1$ factor $\mathcal{M}$.

**Lemma 4.55.** Let $x, x' \in \mathcal{M}$. There exists $\varepsilon > 0$ dependent only on $x$ such that if $\|x - x'\|_2 < \varepsilon$, then

$$\ell_{2t}(x) \leq 2\ell_t(x') \text{ for all } t \geq 0.$$

***Proof.*** Let $s$ denote the projective rank of $x$. Put $\delta := \ell_{3s/4}(x)/2$. Assuming $\varepsilon$

to be small enough we have a sufficiently fine 2-norm approximation $x'$ of $x$ such that $\ell_{s/2}(x') \geq 2\delta > 0$. Right continuity (see Lemma 4.26) implies that there exists $\delta_0 \in (0, s/2]$ such that

$$\ell_0(x) - \ell_t(x) \leq \delta \quad \text{ for all } t \in [0, \delta_0].$$

Thus for all $t \in [0, \delta_0]$ we conclude

$$\begin{aligned}\ell_0(x) &\leq \ell_{\delta_0}(x) + \delta \\ &\leq \ell_{\delta_0/2}(x') + \mu_{\delta_0/2}(x - x') + \delta \\ &\leq \ell_t(x') + \frac{2\varepsilon}{\delta_0} + \delta.\end{aligned}$$

Thus if $\varepsilon$ is small enough, we have $2\varepsilon/\delta_0 < \delta$ and hence

$$\ell_t(x) \leq \ell_t(x') + 2\delta \leq \ell_t(x') + \ell_{s/2}(x') \leq 2\ell_t(x') \quad \text{ for all } t \in [0, \delta_0].$$

For $t \in [\delta_0, s/2]$ we obtain

$$\ell_{2t}(x) \leq \ell_t(x') + \mu_t(x - x') \leq \ell_t(x') + \frac{\varepsilon}{\delta_0} \leq 2\ell_t(x').$$

Thus for all $t \geq 0$

$$\ell_{2t}(x) \leq 2\ell_t(x'),$$

as claimed. $\square$

Assume now that $u, v \in G \coloneqq \mathrm{PU}(\mathcal{M})$ satisfy $\ell_0(u) \leq m\ell_t(v)$ for all $t \in [0, s]$ and some $m \in \mathbb{N}$. We want to show that under these circumstances we have

$$u \in \overline{(v^G \cup v^{-G})^{32m\lceil 1/s \rceil}}^{\|\cdot\|_2}.$$

Let $\varepsilon > 0$ be arbitrarily small. By Proposition 4.54 there exist elements $u', v' \in \mathrm{U}(\mathcal{M})$ such that $\|u - u'\|_2\,,\ \|v - v'\|_2 < \varepsilon$ and $u' = \sum_{i=1}^{n} \lambda_i p_i,\ v' = \sum_{j=1}^{m} \zeta_j q_j$, where $\lambda_i, \zeta_j \in \mathrm{U}(1)$ and $p_i, q_j \in \mathrm{Proj}(\mathcal{M})$ satisfy $\tau(p_i) = r_i/s_i,\ \tau(q_j) = r_{j+n}/s_{j+n}$ for some $r_k, s_k \in \mathbb{N} \setminus \{0\}$.
Let $s_0$ denote the least common multiple of $s_1, \ldots, s_{n+m}$. Take subprojections $p_i'$ of the

$p_i$ and $q_j'$ of the $q_j$ such that $\tau(p_i') = \tau(q_j') = 1/s_0$ and

$$u' = \sum_{i=1}^{s_0} \lambda_i' p_i', \quad v' = \sum_{j=1}^{s_0} \zeta_j' q_j',$$

where multiplicities are taken into account. We need the following easy lemma.

**Lemma 4.56.** Let $p_i'$ and $q_i'$ be as defined above. There exists $w \in \mathrm{U}(\mathcal{M})$ such that $wp_i'w^* = q_i'$ for all $i = 1, \ldots, s_0$.

***Proof***. Since $p_i' \sim q_j'$ for all $i, j = 1, \ldots, s_0$, we may choose partial isometries $x_i$ (respectively $y_i$) with initial projection $p_i'$ (respectively $q_i'$) and final projection $p_1'$. Let $w := \sum_{i=1}^k y_i^* x_i$. We claim that $w$ is unitary.

$$w^*w = \sum_i x_i y_i y_i^* x_i + \sum_{i \neq j} x_i^* y_i y_j^* x_j = \sum_i x_i^* p_1 x_i = \sum_i x_i^* x_i x_i^* x_i = \sum_i p_i = 1,$$

and analogously $ww^* = 1$. □

Using the above lemma, we obtain that

$$wu'w^* = \sum_{i=1}^{s_0} \lambda_i' q_i', \quad v' = \sum_{i=1}^{s_0} \zeta_i' q_i'.$$

We assume that $u'$ is such that we have $\ell_0(u') \leq \ell_0(u)$. Note that this is always possible by choosing the eigenvalues of the approximating element $u'$ such that

$$\sup_{\lambda, \zeta \in \sigma(u')} |\lambda - \zeta| \leq \sup_{\lambda, \zeta \in \sigma(u)} |\lambda - \zeta| .$$

Using Lemma 4.55 for $v, v'$ and assuming $\varepsilon$ to be sufficiently small we obtain for all $t \in [0, s/2]$ that

$$\ell_0(wu'w^*) \ = \ell_0(u') \leq \ell_0(u) \leq m\ell_{2t}(v) < 2m\ell_t(v'). \tag{4.3}$$

We may assume that $s$ is rational. Indeed, if $s$ is irrational, using right continuity and the fact that the inequality $\ell_{2t}(v) < 2\ell_t(v')$ for $t \in [0, s/2]$ is strict, we replace $s$ by some rational $\tilde{s} > s$ such that $\ell_{2t}(v) \leq 2\ell_t(v')$ for all $t \in [0, \tilde{s}/2]$. Replacing $s_0$ by a multiple of $s_0$ if necessary, we may also assume that $s(s_0 - 1)/2 \in \mathbb{N}$. Using Theorem

4.45 for the elements $u', v'$ with Inequality (4.3) we conclude that

$$wu'w^* \in (v^G \cup v^{-G})^{16m\lceil (s_0-1)/(s/2)(s_0-1)\rceil} = (v^G \cup v^{-G})^{32m\lceil 1/s\rceil}.$$

Hence

$$u \in (((v^G)_\varepsilon \cup (v^{-G})_\varepsilon)^{32m\lceil 1/s\rceil})_\varepsilon,$$

where $G \coloneqq \mathrm{PU}(\mathcal{M})$. Using Lemma 4.31, we obtain

$$u \in ((v^G \cup v^{-G})^{32m\lceil 1/s\rceil})_{(32m\lceil 1/s\rceil+1)\varepsilon}.$$

Now letting $\varepsilon$ tend to zero, i.e. approximating both $u$ and $v$ finer and finer in the 2-norm, we obtain

$$u \in \overline{(v^G \cup v^{-G})^{32m\lceil 1/s\rceil}}^{\|\cdot\|_2}.$$

Summarizing the above discussion, we have proven the following theorem.

**Theorem 4.57.** *Let $\mathcal{M}$ be a $\mathrm{II}_1$-factor. Assume that $u, v \in G \coloneqq \mathrm{PU}(\mathcal{M})$ satisfy $\ell_0(u) \leq m\ell_t(v)$ for all $t \in [0, s]$ and some $m \in \mathbb{N}$. Then*

$$u \in \overline{(v^G \cup v^{-G})^{32m\lceil 1/s\rceil}}^{\|\cdot\|_2}.$$

*If both $u$ and $v$ have finite spectrum and rational spectral weights, then*

$$u \in (v^G \cup v^{-G})^{32m\lceil 1/s\rceil}.$$

**Lemma 4.58.** Let $\mathcal{M}$ be a finite von Neumann algebra with normalized trace. For any two elements $u, v \in \mathrm{PU}(\mathcal{M}) \setminus \{1\}$ there exist $k, m \in \mathbb{N}$ such that $\ell_{kt}(u) \leq \max\{m\ell_t(v) - \delta,\ 0\}$ for all $t \geq 0$ and some $\delta \geq 0$. In particular, for every $v \in \mathrm{PU}(\mathcal{M}) \setminus \{1\}$, there exist $k, m \in \mathbb{N}$ such that $\ell_{kt}(u) \leq \max\{m\ell_t(v) - \delta,\ 0\}$ for all $t \geq 0$ and for all $u \in \mathrm{PU}(\mathcal{M})$.

***Proof***. Since $v \neq 1$, we have $\ell_t(v)$ at least for $t = 0$. Right continuity of $\mu_t$ in $t$ implies that there exists an interval $[0, \delta_0)$ such that $\ell_t(v) \neq 0$ for all $t \in [0, \delta_0)$. There exist $\delta_1, \varepsilon > 0$ such that $\ell_t(v) \geq \varepsilon$ for all $t \in [0, \delta_1]$. Since $\ell_t(u) \leq 2$ for all $t \geq 0$, there exists $m \in \mathbb{N}$ such that $m\varepsilon \geq 2 + \delta$. Clearly, there also exists $k \in \mathbb{N}$ such that $k\delta_1 > 1$ (recall that $\ell_t \equiv 0$ for all $t > \tau(1)$). That is, we have $\ell_{kt}(u) \leq \max\{m\ell_t(v) - \delta,\ 0\}$ for all

$t \geq 0$. □

**Theorem 4.59.** *The projective unitary group* $G$ *of a* $\mathrm{II}_1$*-factor, endowed with the strong operator topology, has property (topBNG).*

***Proof.*** Let $v \in G \setminus \{1\}$ be arbitrary. By Lemma 4.26 there exist $\delta > 0$ and $\varepsilon > 0$ such that

$$\ell_t(v) \geq \varepsilon \text{ for all } t \in [0, \delta].$$

Let $u \in G$ be arbitrary. Since $\ell_t(u) \leq 2$ for all $t \geq 0$ and $\ell_t(u) = 0$ for all $t \geq 1$ we have

$$\ell_0(u) \leq \lceil 2/\varepsilon \rceil \ell_t(v) \quad \text{ for all } t \in [0, \delta].$$

Using Theorem 4.57 we conclude

$$u \in \overline{\left((v^G \cup v^{-G})^{32\lceil 2/\varepsilon\rceil \cdot \lceil 1/\delta \rceil}\right)}^{\|\cdot\|_2}.$$

Since $u$ was an arbitrary element, this implies that $G$ has property (topBNG). □

An immediate corollary of Theorem 4.59 is the topological simplicity of the projective unitary group in the strong operator topology. This was discovered for the uniform topology by Kadison in [Ka 52].

**Corollary 4.60.** The projective unitary group of a $\mathrm{II}_1$ factor is topologically simple in the strong operator topology.

We remark here that Theorem 4.59 implies that the projective unitary group of a metric ultraproduct of a $\mathrm{II}_1$ factor has property (BNG). However, in the next section we will prove that the projective unitary group of any separable $\mathrm{II}_1$ factor has the bounded normal generation property.

## 4.8 Bounded normal generation for $\mathrm{II}_1$ factors

This section deals with the proof of the bounded normal generation property of projective unitary groups of $\mathrm{II}_1$ factors. The proof of this algebraic property is considerably more complicated than that of its topological counterpart in the previous section.
In this section $\mathcal{M}$ will always denote a separable $\mathrm{II}_1$ factor.

The following result is a first observation on the spectral behaviour under taking appropriate commutators.

**Lemma 4.61.** For every $u \in \mathrm{U}(\mathcal{M})$ there exists $v \in \mathrm{U}(\mathcal{M})$ such that

$$\|1 - uvu^*v^*\|_2 \geq \inf_{\lambda \in \mathrm{U}(1)} \|1 - \lambda u\|_2 .$$

***Proof.*** Apply [Po 81, Lemma 2.3] (see also [Ta 03, Lemma XIV.5.6]) to the element $u - \tau(u)$ in order to obtain for arbitrary $\varepsilon > 0$ the existence of $v \in \mathrm{U}(\mathcal{M})$ such that

$$\|v(u - \tau(u))v^* - (u - \tau(u))\|_2^2 = \|v - uvu^*\|_2^2 \geq (2 - \varepsilon) \|u - \tau(u)\|_2^2 .$$

For $\varepsilon > 0$ sufficiently small, we obtain

$$(2 - \varepsilon) \|u - \tau(u)\|_2^2 \geq \inf_{\lambda \in \mathrm{U}(1)} \|1 - \lambda u\|_2^2 ,$$

as claimed. □

However, Lemma 4.61 does not reveal information about the generalized projective $s$-numbers of the commutator. It is much harder to keep track of that information under commutators. We now construct for a given unitary $u$ another unitary $v$ such that the commutator $[u, v]$ retains much of the spectral information of $u$. On the one hand this result is crucial for our proof of property (BNG) in the $\mathrm{II}_1$ factor case, on the other hand it is of independent interest since it allows to consider commutators instead of the original element without qualitatively changing the (projective) spectral information.

**Proposition 4.62.** Let $\mathcal{M}$ be a $\mathrm{II}_1$ factor. For every $u \in \mathrm{U}(\mathcal{M})$ there exist $v \in \mathrm{U}(\mathcal{M})$ such that

$$\ell_{24t}(u) \leq 4\ell_t([u, v]) \text{ for all } t \geq 0.$$

***Proof.*** If $u \in \mathcal{Z}(\mathrm{U}(\mathcal{M}))$, then the claim is trivial. So assume that $u$ is noncentral. Right continuity of $\ell_t$ in $t$ implies that there exists $s \in (0, 1]$ such that $\ell_t(u) > 0$ for all $t \in [0, s)$ and $\ell_t(u) = 0$ for $t \geq s$. For $\delta := \ell_{s/2}(u) > 0$ we obtain

$$\ell_{2t}(u) \leq 2\ell_t(u) - \delta \text{ for all } t \in [0, s/2).$$

Using the right continuity of $\ell_t$ once again we get the existence of $\delta_0 > 0$ satisfying

$$\ell_0(u) - \ell_{24\delta_0}(u) \leq \frac{\delta}{2}$$

and thus

$$\ell_{24t}(u) - \ell_{24\delta_0}(u) \leq \frac{\delta}{2} \text{ for all } t \in [0, \delta_0). \tag{4.4}$$

Let $\varepsilon > 0$ such that $\varepsilon \leq \delta\delta_0/40$. By Proposition 4.54 we can find $u'$ such that $\|u - u'\|_2 < \varepsilon$ and $u' = \sum_{i=0}^{n-1} \lambda_i p_i$ with orthogonal projections $p_i$ and $\tau(p_i) = 1/n$ for $i = 0, \dots, n-1$. Relabelling if necessary, we may assume that $\operatorname{diag}(\lambda_0, \dots, \lambda_{n-1})$ is optimal with associated permutation $\pi$. Taking orthogonal subprojections $p_{i,1}, p_{i,2}$ of $p_i$ with trace $1/(2n)$, the multiplicities of the eigenvalues are divisible by 2. We write $\lambda_{i,j}$ for the eigenvalue corresponding to $p_{i,j}$, $j = 1, 2$. Note that $\lambda_{i,j} = \lambda_i$ for $j = 1, 2$ and all $i = 0, \dots, n-1$. We consider $\lambda_{t,j} = \lambda_t$ as a right-continuous function in $t \in [0, \infty)$ which is constant on the intervals $[0, 1), \dots, [n-1, n)$, $\lambda_t = \lambda_i$ for $t \in [i, i+1)$, and zero for $t \geq n$. We now construct a permutation $\sigma \in S_X$, $X = \{0, \dots, n-1\} \times \{1, 2\}$, as follows. Set

$$\lambda_{\sigma(i,1)} \coloneqq \lambda_{i+1,1}, \qquad \lambda_{\sigma(n-1,1)} \coloneqq \lambda_{0,1}, \qquad \lambda_{\sigma(i,2)} \coloneqq \lambda_{i,2}.$$

For notational convenience we can consider $\sigma(t, j)$ as a right-continuous function in $t \in [0, \infty)$ which is zero if $t \geq n$ and constant on the intervals $[i, i+1)$, $i = 0, \dots, n$.

Let $v \in \mathrm{U}(\mathcal{M})$ be such that $vp_{i,j}v^* = p_{\sigma^{-1}(i,j)}$. Then

$$[u', v] = \sum_{i=0,\dots,n-1,\ j=1,2} \lambda_{i,j} p_{i,j} \sum_{k=0,\dots,n-1,\ l=1,2} \overline{\lambda}_{k,l} p_{\sigma^{-1}(k,l)} = \sum_{i,j} \lambda_{i,j} \overline{\lambda}_{\sigma(i,j)} p_{i,j}.$$

Put $\widetilde{\lambda}_{i,j} \coloneqq \lambda_{i,j}\overline{\lambda}_{\sigma(i,j)}$ and observe that $\widetilde{\lambda}_{i,2} = 1$. We want to have an inequality between the projective singular values of $[u', v]$ and $u'$. It is clear that $\ell_t([u', v])$ is constant in the intervals $[i/n, (i+1)/n)$, $i = 0, \dots, n-1$, and zero for $t \geq 1$. Therefore we use Lemma 4.38 (recall that $\tau$ is normalized). We have

$$\frac{1}{2} \left|\lambda_{\pi(2tn)} - \lambda_{\pi(2tn)+1}\right| \leq \ell_t(u') \leq \left|\lambda_{\pi(tn)} - \lambda_{\pi(tn)+1}\right| \text{ for } t \geq 0.$$

Let $\widetilde{\sigma}$ be a permutation on $\{0, \dots, n-1\}$ such that $\left|\widetilde{\lambda}_{\widetilde{\sigma}(i),1} - \widetilde{\lambda}_{\widetilde{\sigma}(i),2}\right| \geq \left|\widetilde{\lambda}_{\widetilde{\sigma}(j),1} - \widetilde{\lambda}_{\widetilde{\sigma}(j),2}\right|$

for $i \leq j$. Analogously to $\lambda_t$ and $\sigma(t,j)$ we view $\widetilde{\sigma}(t)$ as a right-continuous function in $t \in [0,\infty)$ being constant on the intervals $[i,i+1)$. We claim that for $t \in \{0, 1/n, \ldots, (n-1)/n\}$ we have

$$\ell_t([u',v]) \geq \frac{1}{2}\left|\widetilde{\lambda}_{\widetilde{\sigma}(2tn),1} - \widetilde{\lambda}_{\widetilde{\sigma}(2tn),2}\right|. \tag{4.5}$$

Assume the contrary that there exists $t \in \{0, 1/n, \ldots, (n-1)/n\}$ such that the above inequality does not hold. We conclude analogously to Lemma 4.38. Let $z \in \mathrm{U}(1)$ be arbitrary and $\eta$ be a permutation of $\{0, \ldots, n-1\}$ such that

$$\left|z - \widetilde{\lambda}_{\eta(0),1}\right| \geq \left|z - \widetilde{\lambda}_{\eta(1),1}\right| \geq \ldots \geq \left|z - \widetilde{\lambda}_{\eta(n-1),1}\right|.$$

Thus by assumption $\left|\widetilde{\lambda}_{\widetilde{\sigma}(2i),1} - \widetilde{\lambda}_{\widetilde{\sigma}(2i),2}\right| > 2\left|z - \widetilde{\lambda}_{\eta(i),1}\right|$ for some $i \in \{0, \ldots, n-1\}$. Hence for $k = 0, \ldots, 2i$ we have

$$\left|z - \widetilde{\lambda}_{\widetilde{\sigma}(j),1}\right| + \left|z - \widetilde{\lambda}_{\widetilde{\sigma}(j),2}\right| \geq \left|\widetilde{\lambda}_{\widetilde{\sigma}(k),1} - \widetilde{\lambda}_{\widetilde{\sigma}(k),2}\right| > 2\left|z - \widetilde{\lambda}_{\eta(i),1}\right|,$$

that is, $\widetilde{\sigma}(j)$ or $\widetilde{\sigma}(j)$ lies in $\{\eta(0), \ldots, \eta(i-1)\}$ (which has only $i$ elements). Since the inequality holds for $2i+1$ elements we conclude a contradiction. Since $z \in \mathrm{U}(1)$ was arbitrary, Inequality (4.5) follows.

We conclude

$$\begin{aligned}
\ell_t([u',v]) &\geq \frac{1}{2}\left|\widetilde{\lambda}_{\widetilde{\sigma}(2t),1} - \widetilde{\lambda}_{\widetilde{\sigma}(2t),2}\right| \\
&= \frac{1}{2}\left|\lambda_{\widetilde{\sigma}(2t),1}\overline{\lambda}_{\widetilde{\sigma}(2t)+1,1} - 1\right| \\
&= \frac{1}{2}\left|\lambda_{\widetilde{\sigma}(2t)} - \lambda_{\widetilde{\sigma}(2t)+1}\right| \\
&\geq \frac{1}{2}\left|\lambda_{\pi(2t)} - \lambda_{\pi(2t)+1}\right|
\end{aligned}$$

for all $t \in [0,(n-1)/n)$ except possibly $t \geq (n-1)/2n$ - namely if $\widetilde{\sigma}(n-1) = n-1$ (thus $\sigma(\widetilde{\sigma}(n-1)+1) = 1$) and $|\lambda_{n-1} - \lambda_1| < \left|\lambda_{\pi(n-1)} - \lambda_{\pi(n-1)+1}\right|$. Hence $2\ell_t([u',v]) \geq \left|\lambda_{\pi(3tn)} - \lambda_{\pi(3tn)+1}\right|$ and

$$\ell_{3t}(u') \leq 2\ell_t([u',v]) \text{ for all } t \geq 0. \tag{4.6}$$

The following calculation uses the fact that $\|xy\|_2 \le \|x\| \cdot \|y\|_2$ for any $x, y \in \mathcal{M}$.

$$\begin{aligned}\left\|[u,v]-[u',v]\right\|_2 &\le \|v^*\| \cdot \left\|uvu^* - uvu'^* + uvu'^* - u'vu'^*\right\|_2 \\ &\le \|uv\| \cdot \left\|u^* - u'^*\right\|_2 + \left\|vu'^*\right\| \cdot \left\|u - u'\right\|_2 \\ &< 2\varepsilon.\end{aligned}$$

Using Lemma 4.25 we have the following estimates for every $t > 0$:

$$\begin{aligned}\ell_t(u) &= \inf_\lambda \mu_t(1 - \lambda(u - u' + u')) \\ &\le \ell_{t/2}(u') + \mu_{t/2}(u - u').\end{aligned}$$

From Lemma 4.29 and the inequality $\|\cdot\|_1 \le \|\cdot\|_2$ we further conclude

$$\begin{aligned}\ell_t(u) &\le \ell_{t/2}(u') + 2\varepsilon/t \\ &\overset{(4.6)}{\le} 2\ell_{t/6}([u',v]) + 2\varepsilon/t.\end{aligned} \tag{4.7}$$

The same calculation with $u$ replaced by $[u',v]$ and $u'$ replaced by $[u,v]$ shows that

$$\ell_t([u',v]) \le \ell_{t/2}([u,v]) + 2 \cdot 2\varepsilon/t. \tag{4.8}$$

Combining Inequalities (4.7) and (4.8) we get

$$\ell_t(u) \le 2\ell_{t/12}([u,v]) + \min\{10\varepsilon/t, 2\} \qquad \text{for all } t \ge 0. \tag{4.9}$$

From the inequality $\ell_{2t}(u) \le 2\ell_t(u) - \delta$ for all $t \in [0, s/2)$ and the above estimates we conclude for $t \in [0, s/24)$ that

$$\begin{aligned}\ell_{24t}(u) &\le 2\ell_{12t}(u) - \delta \\ &\overset{(4.9)}{\le} 4\ell_t([u,v]) + \frac{20\varepsilon}{t} - \delta.\end{aligned}$$

Using Equation (4.4) we obtain from the above inequality that for all $t \in [0, \delta_0)$

$$\begin{aligned}\ell_{24t}(u) &\le \ell_{24\delta_0}(u) + \frac{\delta}{2} \\ &\le 4\ell_{\delta_0}([u,v]) + \frac{20\varepsilon}{\delta_0} - \delta + \frac{\delta}{2}\end{aligned}$$

$$\begin{aligned} &\le 4\ell_t([u,v]) + \frac{20\varepsilon}{\delta_0} - \frac{\delta}{2} \\ &\le 4\ell_t([u,v]). \end{aligned}$$

If $s/24 > t \ge \delta_0$ we have

$$\begin{aligned} \ell_{24t}(u) &\le 4\ell_t([u,v]) + \frac{20\varepsilon}{t} - \delta \\ &\le 4\ell_t([u,v]) + \frac{20\varepsilon}{\delta_0} - \delta \\ &\le 4\ell_t([u,v]). \end{aligned}$$

Since $\ell_{24t}(u) = 0$ for all $t \ge s/24$ we can summarize our estimates to

$$\ell_{24t}(u) \le 4\ell_t([u,v]) \text{ for all } t \ge 0,$$

which concludes the proof. □

*Remark.* The proof of Proposition 4.62 shows that $v$ can be chosen such that it has finite spectrum and rational spectral weights. If $u$ has finite spectrum and rational spectral weights itself, then $v$ can be defined on subprojections of the spectral projections of $u$.

We need the following Borel measurable version of Lemma 4.33.

**Lemma 4.63.** Let $(X,\nu)$ be a Borel measure space and let $u = \left(\begin{smallmatrix} e^{\mathrm{i}\varphi} & 0 \\ 0 & e^{-\mathrm{i}\varphi}\end{smallmatrix}\right)$, $v = \left(\begin{smallmatrix} e^{\mathrm{i}\theta} & 0 \\ 0 & e^{-\mathrm{i}\theta}\end{smallmatrix}\right) \in G \coloneqq \mathrm{U}(M_{2\times 2}(\mathbb{C}) \otimes L^\infty(X,\nu))$ be nontrivial elements. If $|\varphi(x)| \le m\,|\theta(x)|$ for some even $m \in \mathbb{N}$ and $\nu$-almost every $x \in X$, then $u \in (v^G)^m$.

***Proof.*** The proof follows closely the proof of Lemma 4.33, but we need to ensure that the steps are Borel. This will be clear from the construction.

Observe that multiplication of diagonal elements by $v(x)$ adds the angle $\theta(x)$ while multiplication with $v^{-1}(x)$ subtracts the angle $\theta(x)$. If $w(x) \in \mathrm{SU}(2)$ is diagonal with $\mathrm{tr}(w(x)) = \cos\gamma(x)$, then we can choose $v'(x) \in v(x)^{\mathrm{SU}(2)}$ such that $\mathrm{tr}(w(x)v'(x)) = \cos\gamma_1(x)$ for any $\gamma_1(x) \in [\gamma(x)-\theta(x), \gamma(x)+\theta(x)]$, namely

$$v'(x) \coloneqq \left(\begin{smallmatrix} \cos\theta(x)+\mathrm{i}\sin\theta_1(x) & b(x) \\ -\bar{b}(x) & \cos\theta(x)-\mathrm{i}\sin\theta_1(x)\end{smallmatrix}\right)$$

for $\theta_1(x) \in [-\theta(x), \theta(x)]$, where $|b(x)|^2 = 1 - \cos^2\theta(x) - \sin^2\theta_1(x) = \sin^2\theta(x) - \sin^2\theta_1(x) \ge 0$.

Assume that $\varphi(x)$ and $\theta(x)$ have the same sign for $\nu$-almost all $x \in X$ (else one needs to replace $v$ by $v^*$ in the following). Multiply $v(x)$ $n \in \{1, \ldots, m-1\}$ times by itself until either $\varphi(x) \leq n\theta(x)$ or $\varphi(x) \geq (m-1)\theta(x)$. In the second case, multiplying $v^{m-1}(x)$ by the right element $v'(x)$ one obtains $u(x) = v^{m-1}(x) \cdot v'(x)$. In the first case, if $n = m-1$ then we also get $u(x) = v^{m-1}(x) \cdot v'(x)$. If $n < m-1$ then we multiply interchangingly by $v^*(x)$ and $v(x)$ until one step is left. The last step is to use the conjugate $v'(x)$ of $v(x)$ to obtain $u(x) = v^n(x)v^*(x)v(x) \cdot \ldots \cdot v^*(x)v(x) \cdot v'(x)$. This gives an algorithm which determines in finitely many steps and divides $X$ into Borel sets in each step. $\square$

Before proving the main result of this section we want to outline the strategy of the proof. We want to generate an element $u \in \mathrm{PU}(\mathcal{M})$ having finite spectrum and rational weights with an arbitrary element $v \in \mathrm{PU}(\mathcal{M})$ under the assumption of an inequality between their projective $s$-numbers. Our first step is to map $v$ via an isomorphism into $2 \times 2$ matrices over $p\mathcal{M}p$, $\tau(p) = 1/2$, such that they have diagonal form. Then $v = \left(\begin{smallmatrix} v_0 & 0 \\ 0 & v_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} v_0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & v_1 \end{smallmatrix}\right)$. Using Proposition 4.62 we can ensure that the projective singular values of $[v_0, w_0]$, where $w = \left(\begin{smallmatrix} w_0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, are still comparable with those of the original element $v$. We then use two conjugates of $[v,w]g[v,w]^{-1}g^{-1}$ to construct a unitary $v'$ which has finite spectrum and rational spectral weights, where $g$ permutes the diagonal entries of the $2 \times 2$ matrix $[v,w]$. Using now Theorem 4.57 we can generate $u$ with $v'$.

**Theorem 4.64.** *Let $\mathcal{M}$ be a separable $\mathrm{II}_1$-factor and $u, v \in G := \mathrm{PU}(\mathcal{M})$. Assume that $u$ has finite spectrum and rational spectral weights. If $\ell_0(u) \leq m\ell_t(v)$ for all $t \in [0,s]$ and some $m \in \mathbb{N}$, then*

$$u \in (v^G \cup v^{-G})^{24576m\lceil 1/s \rceil}.$$

***Proof.*** First note that for $\delta := \ell_s(v) > 0$ we have

$$\ell_0(u) \leq m(2\ell_t(v) - \delta) \quad \text{for all } t \in [0,s].$$

Put $\varepsilon := \delta/4$. Let $\zeta_1, \ldots, \zeta_n$ be $n$ roots of unity with $\arg(\zeta_i) < \arg(\zeta_{i+1})$ such that for every $\lambda \in \sigma(v)$ there is an $i \in \{1, \ldots, n\}$ such that $|\lambda - \zeta_i| < \varepsilon$. We may assume that there exists no $\zeta_i$ satisfying $|\lambda - \zeta_i| > \varepsilon$ for all $\lambda \in \sigma(v)$. Denote by $p_i$ the spectral projection of $v$ corresponding to the set $\left\{e^{\mathrm{i}\varphi} \mid \varphi \in [\arg(\xi_i), \arg(\xi_{i+1}))\right\}$, where

$\zeta_{n+1} := \zeta_1$ and $i \in \{1, \ldots, n\}$. Define $f(v) = \sum_{i=1}^{n'} \zeta_i p_i$. It follows that

$$\|v - f(v)\| < \varepsilon.$$

Now take subprojections $p_i'$ of $p_i$ with $\tau(p_i') = \frac{1}{2}\tau(p_i)$. Let $p := \sum_{i=0}^{n'} p_i'$. Then $\tau(p) = 1/2$ and $p$ commutes with $v$.

Denote in the following by $\ell_t^{(p)}$ the restriction of $\ell_t$ to $p\mathcal{M}p$, i.e., $\ell_t^{(p)}(x) = \inf_\lambda \mu_t(p - \lambda pxp)$ for $x \in \mathcal{M}$. We conclude from Lemma 4.30 that

$$\ell_{2t}(v) \le \ell_{2t}(f(v)) + \varepsilon = \ell_t^{(p)}(f(v)) + \varepsilon \text{ for every } t \ge 0.$$

Since we also have $\|f(v)p - vp\| < \varepsilon$ we obtain $\ell_t^{(p)}(f(v)) \le \ell_t^{(p)}(v) + \varepsilon$ for all $t \ge 0$ and thus

$$\ell_{2t}(v) \le \ell_t^{(p)}(v) + 2\varepsilon \text{ for every } t \ge 0. \tag{4.10}$$

We have $v \cong \left(\begin{smallmatrix} v_0 & 0 \\ 0 & v_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} v_0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & v_1 \end{smallmatrix}\right) \in \mathrm{U}(p\mathcal{M}p \otimes M_{2\times 2}(\mathbb{C}))$ for $v_0 := vp$ and some $v_1 \in \mathrm{U}(p\mathcal{M}p)$. By Proposition 4.62 applied to the algebra $p\mathcal{M}p$ there exists $w = \left(\begin{smallmatrix} w_0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{U}(p\mathcal{M}p \otimes M_{2\times 2}(\mathbb{C}))$ such that

$$\ell_{24t}^{(p)}(v) \le 4\ell_t^{(p)}([v, w]) \text{ for all } t \ge 0,$$

where

$$[v, w] = \begin{pmatrix} v_0 w_0 v_0^* w_0^* & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $g \in \mathrm{U}(p\mathcal{M}p \otimes M_{2\times 2}(\mathbb{C}))$ be such that

$$g[v, w]^{-1} g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & (v_0 w_0 v_0^* w_0^*)^{-1} \end{pmatrix}.$$

Then under the identification of $G$ with its image under the isomorphism $\mathcal{M} \to p\mathcal{M}p \otimes M_{2\times 2}(\mathbb{C})$ we have

$$\widetilde{v} := [v, w] g [v, w]^{-1} g^{-1} \in (v^G \cup v^{-G})^4.$$

In particular,

$$\ell_{24t}^{(p)}(v) \le 4\ell_t^{(p)}([v, w]) = 4\ell_t^{(p)}(\widetilde{v}) \text{ for all } t \ge 0. \tag{4.11}$$

By Theorem II.6.1 in [Di 81] we can decompose $L^\infty(\sigma(\widetilde{v}),\nu)$ into a direct integral such that $\widetilde{v}$ is represented as $\int_{\sigma(\widetilde{v})}^{\oplus} \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right) d\nu(\lambda)$.

Now we can use Lemma 4.63 to generate an element $v'$ with discrete spectrum and rational spectral weights such that $\ell_t(v') + \varepsilon \geq \ell_t(\widetilde{v})$ for all $t \geq 0$ and

$$v' \in (\widetilde{v}^G \cup \widetilde{v}^{-G})^2 \subseteq (v^G \cup v^{-G})^8.$$

In the following, we describe how to generate such an element explicitly.

- First note that right continuity of $\ell_t$ in $t \geq 0$ implies that there exists $\delta_0 \in (0,1)$ such that $\ell_0(\widetilde{v}) \leq \ell_{\delta_0}(\widetilde{v}) + \delta/4$. Let $\varepsilon_0 \in (0,1)$ be such that $\varepsilon_0 < \delta_0\delta/24$ and such that there exists $\lambda \in \sigma(\widetilde{v})$ with $|1-\lambda| > \varepsilon_0$.
- Let $\lambda_0 \coloneqq 1$ and $\lambda_1, \ldots, \lambda_n \in \mathrm{U}(1)$, $n \in \mathbb{N}$, such that

  1. $|1-\lambda_i| \geq \varepsilon_0$ for all $i \in \{1,\ldots,n\}$ and $|1-\lambda_i| = \varepsilon_0$ for $i = 1, n$,
  2. for every $\lambda \in \sigma([v,w])$ with $|1-\lambda| \geq \varepsilon_0$ there exists $i \in \{1,\ldots,n\}$ such that
  $$|\lambda - \lambda_i| < \varepsilon_0,$$
  3. $|\varphi_i| < |\varphi_{i+1}| \leq 2\,|\varphi_i|$ for all $i \in \{1,\ldots,n-1\}$, where $\varphi_i = \arg(\lambda_i)$.

- Denote the subprojections of the spectral projections of $\widetilde{v}$ corresponding to the parts
$$(\varphi_1/2, \varphi_1], (\varphi_1,\varphi_2],\ (\varphi_2,\varphi_3], \ldots, (\varphi_{n-1},\varphi_n],\ (\varphi_n, \varphi_1/2]$$
by $p_1,\ldots,p_n,p_0$. Then $\sum_{i=0}^n p_i = 1$. Without loss of generality all these projections are nontrivial (else we can leave out some parts and renumber). Let $q_i \precsim p_i$ for $i = 0,\ldots,n$ be subprojections of rational trace such that $\tau(p_i - q_i) < \varepsilon_0/n$.
- Using Lemma 4.63 we can generate
$$v' = \sum_{i=0}^n \lambda_i q_i + \widetilde{q}$$
in two steps, where $\widetilde{q} \coloneqq 1 - \sum_{i=0}^n q_i$, $\tau(\widetilde{q}) \leq 1 - (1 - n\cdot\varepsilon_0/n) = \varepsilon_0$.

We have generated a unitary with finite spectrum and rational spectral weights. The inequality

$$\|xy\|_1 \leq \|x\| \cdot \|y\|_1$$

allows us to conclude

$$\begin{aligned}\left\|v' - \widetilde{v}\right\|_1 &\leq \sum_{i=0}^{n} \left\|(v' - \widetilde{v})q_i\right\|_1 + \left\|(v' - \widetilde{v})\widetilde{q}\right\|_1 \\ &\leq \sum_{i=0}^{n} \left\|(v' - \widetilde{v})q_i\right\| \cdot \|q_i\|_1 + \left\|v' - \widetilde{v}\right\| \cdot \|\widetilde{q}\|_1 \\ &< \varepsilon_0 \cdot \sum_{i=0}^{n} \|q_i\|_1 + 2\,\|\widetilde{q}\|_1 \\ &\leq 3\varepsilon_0.\end{aligned}$$

Thus for $t \in [0, \delta_0/2)$ we conclude

$$\begin{aligned}\ell_{2t}(\widetilde{v}) &\leq \ell_{\delta_0}(\widetilde{v}) + \delta/4 \\ &\leq \ell_{\delta_0/2}(v') + \frac{6\varepsilon_0}{\delta_0} + \delta/4 \\ &< \ell_{\delta_0/2}(v') + \delta/2.\end{aligned}$$

For $t \geq \delta_0/2$ we obtain

$$\ell_{2t}(\widetilde{v}) \leq \ell_t(v') + \frac{6\varepsilon_0}{\delta_0} \leq \ell_t(v') + \delta/4,$$

so that we have

$$\ell_{2t}(\widetilde{v}) \leq \ell_t(v') + \delta/2 \text{ for all } t \geq 0, \tag{4.12}$$

as well as

$$\ell_{2t}^{(p)}(\widetilde{v}) \leq \ell_t^{(p)}(v') + \delta/2 \text{ for all } t \geq 0.$$

From $\ell_0(u) \leq m(2\ell_t(v) - \delta)$ for all $t \in [0, s]$ and from Equation (4.12) we conclude for all $t \in [0, s]$ that

$$\begin{aligned}\ell_0(u) &\leq m(2\ell_t(v) - \delta) \\ &\overset{(4.10)}{\leq} m(2\ell_{t/2}^{(p)}(v) + 2\varepsilon - \delta) \\ &\overset{(4.11)}{\leq} m(8\ell_{t/48}^{(p)}(\widetilde{v}) + 2\varepsilon - \delta) \\ &\overset{(4.12)}{\leq} m(8\ell_{t/96}^{(p)}(v') + \delta/2 + 2\varepsilon - \delta)\end{aligned}$$

$$\begin{aligned} &\leq 8m\ell^{(p)}_{t/96}(v') \\ &\leq 8m\ell_{t/96}(v'). \end{aligned}$$

Summarizing these estimates we have

$$\ell_0(u) \leq 8m\ell_t^{(p)}(v') \leq 8m\ell_t(v') \quad \text{ for all } t \in [0, s/96]. \tag{4.13}$$

Since $u$ has finite spectrum and rational weights we can use Theorem 4.57 to obtain:

$$\begin{aligned} u \in ((v')^G \cup (v')^{-G})^{32m\lceil 96/s\rceil} &\subseteq ((v')^G \cup (v')^{-G})^{3072m\lceil 1/s\rceil} \\ &\subseteq (v^G \cup v^{-G})^{24576m\lceil 1/s\rceil}. \end{aligned}$$

This concludes the proof. □

In Theorem 4.64 we required the element $u$ to have finite spectrum and rational spectral weights. So in particular, we can generate any symmetry of trace 0. To prove that $\mathrm{PU}(\mathcal{M})$ has property (BNG) it then suffices then to combine Theorem 4.19 and Theorem 4.64.

**Theorem 4.65.** *The projective unitary group of a separable* $\mathrm{II}_1$ *factor has property (BNG).*

***Proof.*** Let $v \in G := \mathrm{PU}(\mathcal{M}) \setminus \{1\}$ be arbitrary and denote by $s$ its projective rank. Let $w$ be a symmetry of trace 0. By Lemma 4.58 there exist $m \in \mathbb{N}$ such that $\ell_0(w) \leq m\ell_t(v)$ for all $t \in [0, s/2]$. Using Theorem 4.64 we obtain

$$w \in (v^G \cup v^{-G})^{24576m\lceil 1/s\rceil}.$$

Using now Theorem 4.19 we obtain

$$u \in (w^G \cup w^{-G})^{32}$$

for any $u \in G$. That is,

$$G = (v^G \cup v^{-G})^{786432m\lceil 1/s\rceil}.$$

This finishes the proof. □

Theorem 4.65 easily implies the algebraic simplicity of $\mathrm{PU}(\mathcal{M})$ which was first discovered by de la Harpe - see the main theorem in [Ha 79].

**Corollary 4.66.** The projective unitary group of a $\mathrm{II}_1$ factor is simple.

We now present a formulation of Theorem 4.65 with a suitable normal generation function. For $x \in \mathcal{M}$ we define

$$L(x) := \int_{t\in[0,1]} \ell_t(x)dt.$$

**Corollary 4.67.** Let $G$ denote the projective unitary group of a separable $\mathrm{II}_1$ factor. For some constant $C \in \mathbb{N}$ the function $f : G \setminus \{1\} \to \mathbb{N}$ given by

$$f(v) := \begin{cases} C \cdot \lceil -\ln(L(v)/2)/L(v)\rceil, & \text{if } L(v) \leq 1/3, \\ C, & \text{if } L(v) > 1/3, \end{cases}$$

defines a normal generation function for $G$. That is,

$$G = \left(v^G \cup v^{-G}\right)^k$$

for every $k \geq f(v),\ v \in G \setminus \{1\}$.

***Proof.*** Observe that $\ell_{.}(v)/2 : [0,1] \to [0,1]$ is non-zero and monotone decreasing. Assume that $L(v) \leq 1/3$. From [Th 14, Lemma 2] we conclude that there exists some $t_0 \in [0,1]$ such that

$$t_0\ell_{t_0}(v) \geq \frac{L(v)/2}{-4\ln(L(v)/2)}.$$

As in the proof of Theorem 4.65 we conclude that

$$G = (v^G \cup v^{-G})^{786432\cdot\lceil 1/\ell_{t_0}(v)\rceil\cdot\lceil 1/x_0\rceil} \subseteq (v^G \cup v^{-G})^{786432\cdot\lceil(-8\ln(L(v)/2))/L(v)\rceil}.$$

Now if $L(v) > 1/3$, then (for $t_0 = 1/6$) we even have $\ell_{1/6}(v)/6 \geq 1/36$. Assume to the contrary that $\ell_{1/6}(v) < 1/6$, then we would have

$$L(v) \leq \int_{[0,1/6]} 1dt + \int_{[1/6,1]} \frac{1}{6}dt = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} < \frac{1}{3},$$

a contradiction to $L(v) > 1/3$. It follows that there is a constant $C \in \mathbb{N}$ such that the function $f : G \setminus \{1\} \to \mathbb{N}$ defined by

$$f(v) := \begin{cases} C \cdot \lceil -\ln(L(v)/2)/L(v) \rceil, & \text{if } L(v) \leq 1/3, \\ C, & \text{if } L(v) > 1/3, \end{cases}$$

is a normal generation function. $\square$

# 5 Invariant Automatic Continuity

The aim of this chapter is to prove that every homomorphism from the group

- $\mathrm{PU}(n)$, $n \in \mathbb{N}$, endowed with the norm topology,
- $\mathrm{PU}(\mathcal{M})$, $\mathcal{M}$ a separable $\mathrm{II}_1$-factor, endowed with the strong operator topology,

into any separable SIN group is continuous.
In general we say that a Polish group $G$ has automatic continuity if every homomorphism of $G$ into any other separable topological group is continuous. It is known that $\mathrm{PU}(n)$ does not have automatic continuity.

Another goal is to prove the uniqueness of the Polish group topology of the projective unitary group of a separable $\mathrm{II}_1$ factor. To the author's knowledge this was previously unknown even for the hyperfinite $\mathrm{II}_1$ factor.

Throughout this chapter $\mathrm{II}_1$ factors are assumed to be separable.

In the following introduction on automatic continuity we follow to some extent Rosendal's excellent survey [Ro 09b] on the subject.

The question of automatic continuity goes back to A. L. Cauchy, who analyzed the question whether every function $\pi : \mathbb{R} \to \mathbb{R}$ satisfying the equation $\pi(x + y) = \pi(x) + \pi(y)$, $x, y \in \mathbb{R}$, is of the form $\pi(x) = rx$ for some fixed $r \in \mathbb{R}$. He proved that any continuous solution is of this form. So in modern terminology his question asks if every endomorphism of the additive group of the reals is continuous. Using the axiom of choice one can show that there are discontinuous homomorphisms. In fact, Cauchy's question drew a lot of attention around the beginning of the 20th century. Several mathematicians attempted to find additional assumptions on the function $\pi$ which imply that the solution is continuous. Successful attempts were provided around 1920, e.g., by M. Fréchet (any Lebesgue measurable solution is continuous), S. Banach, W. Sierpiński and H. Steinhaus. A result of Steinhaus was extended in the 1930's by A. Weil to all locally compact groups.

A very general form of the question is:

When is a homomorphism $\pi : G \to H$ between Polish groups continuous?

One the one hand, the motivation to study this question is intrinsic, since it is the study of close connections between algebraic and topological structure of Polish groups. On the other hand, there are connections to many other fields, such as operator algebras, ergodic theory, geometry and model theory and dynamics of large Polish groups.

We note that the general form of this question is nontrivial, i.e. there are many discontinuous homomorphisms between Polish groups - see [Ro 09b]. For example some matrix groups such as $\mathrm{SO}(3, \mathbb{R})$ embed discontinuously into the group $\mathrm{S}_\infty$ of all permutations on $\mathbb{N}$ (this is [Ro 09b, Example 1.5], it follows from results of R. R. Kallman [Ka 00] and S. Thomas [Th 99]).

One of the first general results on automatic continuity in group theory can be found in Pettis' article [Pe 50] (cf. also the book of Kechris [Ke 95, Theorem 9.10]). To state the result let us recall some definitions. A subset $A$ of a Polish space $X$ has the **Baire property** if there is an open set $B \subseteq X$ such that the symmetric difference

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$

is meagre. We say that a map $\pi : X \to Y$ between Polish spaces is **Baire measurable** if $\pi^{-1}(V)$ has the Baire property for every open set $V \subseteq Y$.

**Theorem 5.1** (Pettis)**.** *Any Baire measurable homomorphism between Polish groups is continuous.*

Christensen has archieved a general measurable automatic continuity result in [Ch 71]. Before stating it, we repeat some notion. A subset $A$ of a Polish space $X$ is called **universally measurable** if for any Borel probability measure $\nu$ on $X$, $A$ differs from a Borel set by a set of $\nu$-measure zero. A map $\pi : X \to Y$ between Polish spaces is **universally measurable** if $\pi^{-1}(V)$ is universally measurable for every open set $V \subseteq Y$.

**Theorem 5.2** (Christensen)**.** *Suppose that $\pi : G \to H$ is a universally measurable homomorphism from a Polish group $G$ to a Polish group $H$, where $H$ admits a bi-invariant metric compatible with its topology. Then $\pi$ is continuous.*

Another early general automatic continuity result was proved by Dudley in [Du 61]. Recall that a **norm** on a group $G$ is a function $\|\cdot\| : G \to \mathbb{N}$ such that

(i) $\|gf\| \leq \|g\| + \|f\|$;

(ii) $\|1_G\| = 0$;

(iii) $\|g\| = \|g^{-1}\|$;

(iv) $\|g^n\| \geq \max\{n, \|g\|\}$ for all $g \neq 1_G$.

Examples of normed groups are free groups with word length function.

**Theorem 5.3** (Dudley). *Any homomorphism from a Polish group $G$ into a normed group $H$ equipped with the discrete topology is continuous.*

Slutsky has recently generalized this theorem in [Sl 13] to homomorphisms into free products. Kechris and Rosendal [KR 07, Theorem 1.10] have shown the following general result.

**Theorem 5.4** (Kechris-Rosendal). *Any homomorphism from a Polish group with ample generics into any separable topological group is continuous.*

A Polish group $G$ has ample generics if for each $n \in \mathbb{N}$ there is a comeager orbit for the diagonal conjugacy action of $G$ on $G^n$:

$$g \cdot (g_1, \ldots, g_n) = (gg_1g^{-1}, \ldots, gg_ng^{-1}).$$

This notion has been generalized in [KR 07] from the notion of ample generics introduced in [HHLS 93] (for the purpose of studying the small index property of some automorphism groups). Groups having ample generics also have the small index property by [KR 07, Theorem 1.6], i.e., any subgroup of index less than $2^{\aleph_0}$ is open. An important example for a group having ample generics is the group $\mathrm{S}_\infty$. The authors prove in [KR 07] that the group of Haar measure-preserving homeomorphisms of the Cantor space and the group of Lipschitz homeomorphisms of the Baire space have ample generics.

It is very rare that a group has ample generics. For example, (projective) unitary groups of $\mathrm{II}_1$ factors do not have ample generics - even more they do not have comeager conjugacy classes (which can be seen from the bi-invariance of the trace).

In [RS 07] Rosendal and Solecki develop a more general framework for groups having automatic continuity.

**Definition 5.5.** A topological group $G$ is **Steinhaus** (**with exponent** $k$) if there exists an element $k \in \mathbb{N}$ such that $W^k$ contains an open neighbourhood of $1_G$ for any symmetric countably syndetic set $W \subseteq G$ (see Definition 5.8).

In Proposition 2 of [RS 07] the authors can show the following.

**Proposition 5.6** (Rosendal-Solecki)**.** Every homomorphism from Steinhaus topological group into any separable topological group is continuous.

For example, topological groups with ample generics are Steinhaus with exponent 10 (see [KR 07, Lemma 6.15]). Rosendal and Solecki show that the group $\mathrm{Aut}(\mathbb{Q}, <)$ of order-preserving bijections of the rationals and several homeomorphism groups are Steinhaus. Their proofs crucially use the existence of comeager conjugacy classes (the group $\mathrm{Homeo}_+(S^1)$ of orientation preserving homeomorphisms on the unit circle $S^1$ only has meager conjugacy classes, but the proof heavily uses that the group $\mathrm{Homeo}_+(\mathbb{R})$ of increasing homeomorphisms of $\mathbb{R}$ is Steinhaus, which in turn relies on the existence of comeager conjugacy classes). This indicates that we need some new ideas to show an automatic continuity result for unitary groups of $\mathrm{II}_1$ factors.

Recently Tsankov has obtained in [Ts 13] the result which motivated us to check if unitary groups of $\mathrm{II}_1$ factors, endowed with the strong operator topology, have an automatic continuity property.

**Theorem 5.7** (Tsankov)**.** *Every homomorphism from the unitary group* $\mathrm{U}(\mathcal{H})$ *on a separable infinite-dimensional Hilbert space* $\mathcal{H}$*, endowed with the strong operator topology, into any separable topological group is continuous.*

The proof shows the Steinhaus property for $\mathrm{U}(\mathcal{H})$ and relies on the work of Ben-Yaacov, Berenstein and Melleray in [BYBM 13] - namely it suffices to show that some fixed power of a countably syndetic set contains an open set in the uniform topology (instead of the strong operator topology). In fact, this result motivated us to check if the (projective) unitary group $\mathrm{U}(\mathcal{M})$ of a $\mathrm{II}_1$ factor $\mathcal{M}$, endowed with the strong operator topology, also has an automatic continuity property. We tried to adapt the proof of Tsankov - showing that there exists a fixed power $n \in \mathbb{N}$ such that $W^n$ contains an open neighborhood of the identity for any countably syndetic set $W$. We managed to transfer the setting and to prove (with the help of our modified version of Broise's result, Theorem 4.19) that some fixed power $k \in \mathbb{N}$ of any countably syndetic set contains a "small unitary group", i.e. $\mathrm{U}(p\mathcal{M}p) \oplus (1-p) \subseteq W^k$ for some nonzero $p \in \mathrm{Proj}(\mathcal{M})$. Unfortunately we were not able to control the trace of $p$ - thus further steps might lead to a dependence of the power $n$ on the set $W$.

It is worth mentioning that Sabok has found in [Sa 13] a more general result on automatic continuity which implies Tsankov's result as well as, e.g., that the isometry group of the Urysohn space has the automatic continuity property. More precisely, he has shown that the automatic continuity property holds for automorphism groups of homogeneous complete metric structures that have locally finite automorphisms, the extension property and admit islotated sequences, cf. [Sa 13] for definitions. The proof relies on the work of Kechris and Rosendal [KR 07] and Rosendal and Solecki [RS 07]. However, this still does not resolve the $\mathrm{II}_1$ case.

Let us close this introduction by mentioning that appearances of the phenomenon of automatic continuity can also be found in the theory of $C^*$-algebras and Banach algebras. We list some examples.

- Any algebra homomorphism from an abelian unital Banach algebra $\mathcal{A}$ into $\mathbb{C}$ is continuous. This can be easily seen. If $I$ is a modular maximal ideal of $\mathcal{A}$ then $\mathcal{A}/I$ is isomorphic to $\mathbb{C}$ by [Mu 90, Lemma 1.3.2]. Hence the projection $\mathcal{A} \to \mathcal{A}/I \cong \mathbb{C}$ is a continuous isomorphism.
- Any isomorphism of a $C^*$-algebra onto another $C^*$-algebra is continuous, see [Ta 03, Corollary I.5.6].
- Any derivation on a $C^*$-algebra is norm-continuous, see [Sa 60].

Dales has written an extensive monograph on automatic continuity in the context of Banach algebras, see [Da 00]. However, the techniques to analyze automatic continuity for Banach algebras are quite different from those used to study automatic continuity for groups.

## 5.1 Invariant Automatic continuity

Our strategy to prove automatic continuity of projective unitary groups of $\mathrm{II}_1$ factors (endowed with the strong operator topology) differs greatly from the ones mentioned in the introduction of this chapter. One reason is the lack of comeager conjugacy classes. The main ingredients in our proof are Theorem 4.65 and Propositon 5.18 which ensures that a fixed power of any conjugacy-invariant countably syndetic set contains a neighborhood of the identity. The rest of our proof is an adaption of Proposition 5.6 (cf. [RS 07, Proposition 2]).

Recall that for a pseudometric space $(X, d)$ and $x \in X$ we let

$$B_r^d(x) := \{y \in X \mid d(x, y) \leq r\}.$$

In a semifinite von Neumann algebra $\mathcal{M}$ with faithful semifinite normal trace $\tau$, one can measure the size of the support of an element $x \in \mathcal{M}$ as follows. We define

$$[x] := \inf\left\{\tau(p) + \tau(q) \mid p, q \in \mathrm{Proj}(\mathcal{M}),\ p^{\perp}xq^{\perp} = 0\right\}.$$

We observe that $[x]$ equals the trace of the support projection $s = s(x)$ of $x$. It is clear that $\tau(s) \geq [x]$. We check that $[x] \geq \tau(s)$. Obviously we have $p^{\perp}x(1-p)^{\perp} = 0$ for any spectral projection of $x$. Thus if $p^{\perp}xq^{\perp} = 0$ then $\tau(p^{\perp}) + \tau(s) + \tau(q^{\perp}) \leq 2$. It follows that $\tau(p) + \tau(q) \geq \tau(s)$ and so we have $[x] \geq \tau(s)$.

[Th 08, Lemma 2.1] implies that

$$d_r(x, y) := [x - y]$$

satisfies the triangle inequality and thus defines a pseudometric on $\mathcal{M}$. It is actually a metric on $\mathcal{M}$: if $d_r(x, y) = 0$ then $p^{\perp} = q^{\perp} = 1$ and hence $p^{\perp}(x-y)q^{\perp} = x - y = 0$ implies $x = y$. (It is in general not a metric on $\mathcal{M}$-bimodules!) Following Thom [Th 08, Section 2.1] we call $d_r$ the **rank metric**.

The work of Rosendal and Solecki in [RS 07] shows that the right sets to concider in order to get an highly abstract automatic continuity result are so called countably syndetic sets.

**Definition 5.8.** Let $W$ be a subset of a group $G$. We say that $W$ is **symmetric** if $W = W^{-1}$. A symmetric set $W$ is called **countably syndetic** if there exist countably many elements $g_n \in G$, $n \in \mathbb{N}$, such that $G = \bigcup_{n\in\mathbb{N}} g_n W$.

Note that for a countably syndetic set $W$, there exists some $n \in \mathbb{N}$ such that $g_n W$ contains the identity element of $G$ (and hence $g_n^{-1}, g_n \in W$). An example of a countably syndetic set in a separable topological group is any nonempty open symmetric set.

To ensure that every countably set in the projective unitary group of a $\mathrm{II}_1$ factor contains "well-behaved" elements, we need the following proposition.

**Proposition 5.9.** (i) The (projective) unitary group of a semifinite von Neumann algebra (not of type $I_n$, $n \in \mathbb{N}$) is not separable in the uniform topology.
(ii) The (projective) unitary group of a semifinite von Neumann algebra is not separable in the topology induced by the rank metric.

***Proof.*** (i) This is well-known. One can prove it directly or use that $\mathcal{M}$ contains an uniformly inseparable abelian von Neumann algebra and then use Proposition 2.9 to conclude that the unitary group is also inseparable.
(ii) Set $u_\varphi := e^{\mathrm{i}\varphi}p + e^{\mathrm{i}\,2\varphi}p^\perp$, where $p \in \mathrm{Proj}(\mathcal{M})$ satisfies $\tau(p) = 1/2$ and $\varphi \in [0, \pi/4]$. Then $\left\{B^{d_r}_{1/4}(u_\varphi)\right\}_{\varphi\in[0,\pi/4]}$ defines an uncountable family of disjoint open sets in $\mathrm{U}(\mathcal{M})$ as well as $\mathrm{PU}(\mathcal{M})$. Hence $\mathrm{U}(\mathcal{M})$ and $\mathrm{PU}(\mathcal{M})$ are not separable in the topology induced by $d_r$. $\square$

Proposition 5.11 will ensure that for every countably syndetic set $W$ in $\mathrm{PU}(\mathcal{M})$, $W^2$ contains elements of some suitable length in the above two inseparable topologies. In order to prove this, we need the following lemma.

**Lemma 5.10.** Suppose that $(X, d)$ is an inseparable space equipped with pseudometric $d$. Then there exists $\varepsilon > 0$ such that for every countable subset $A$ of $X$ there exists $x \in X$ with $d(x, A) \geq \varepsilon$.

***Proof.*** Suppose there exists no such $\varepsilon$. Then there exists a sequence $\{A_n\}_{n\in\mathbb{N}}$ of countable subsets of $X$ and a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\varepsilon_n \to 0$ for $n \to \infty$ such that for every $x \in X$ we have $\varepsilon_n > d(x, A_n)$. But then $d(x, \bigcup_{n\in\mathbb{N}} A_n) = 0$ for all $x \in X$. Thus $\bigcup_{n\in\mathbb{N}} A_n$ forms a countable dense set in $X$, which is a contradiction. $\square$

**Proposition 5.11.** Let $G$ be an inseparable topological group with compatible pseudometric $d$. There exists $\varepsilon > 0$ such that for every countably syndetic set $W \subseteq G$, $W^2$ contains an element $u$ satisfying $d(1, u) > \varepsilon$.

***Proof.*** For the moment, let $\varepsilon > 0$ be arbitrary. Recall that an $\varepsilon$-separated set $V \subseteq G$ is a set such that every pair of distinct points $u, v \in V$ has distance $d(u, v) > \varepsilon$. Zorn's lemma implies that there exists a maximal $\varepsilon$-separated set $V_\varepsilon$. Observe that $V_\varepsilon$ is $\varepsilon$-dense in $G$ by maximality (the existence of a point $u \in G \setminus V_\varepsilon$ such that $d(u, v) > \varepsilon$ for all $v \in V_\varepsilon$ obviously contradicts maximality of $V_\varepsilon$).

We conclude from Lemma 5.10 that there exists $\varepsilon > 0$ such that $V_\varepsilon$ is uncountable. We may assume that $1 \in V_\varepsilon$. Since $W$ is countably syndetic, there exists a sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq G$ such that $G = \bigcup_{n\in\mathbb{N}} g_n W$. In particular we have $V_\varepsilon = \bigcup_{n\in\mathbb{N}} V_\varepsilon \cap g_n W$. The pigeonhole principle implies that there exists $m \in \mathbb{N}$ such that

$$|V_\varepsilon \cap g_m W| \geq 2.$$

(Actually it implies that there is an intersection having uncountably many elements.) Let $u, v \in V_\varepsilon \cap g_m W$. Since $W^2 = (g_m W)^{-1}(g_m W)$ and $1 \in W^2$ we either have $d(1,u) > \varepsilon$ or $d(1,v) > \varepsilon$. This completes the proof. $\square$

Let us come to the main definition of this chapter.

**Definition 5.12.** Let $G$ be a topological group. If every homomorphism from $G$ to any separable SIN group is continous, then we say that $G$ has the **invariant automatic continuity property**.

Closely related to invariant automatic continuity we define an invariant version of the Steinhaus property.

**Definition 5.13.** A topological group $G$ has the **invariant Steinhaus property** (**with exponent** $k$) if there exists an element $k \in \mathbb{N}$ such that $W^k$ contains an open neighbourhood of $1_G$ for any symmetric conjugacy-invariant countably syndetic set $W \subseteq G$.

Following closely the proof of [RS 07, Proposition 2] we obtain the invariant automatic continuity for groups having the invariant Steinhaus property. In order to see that, we first recall following well-known result.

**Lemma 5.14.** Let $\pi : G \to H$ be a homomorphism between topological groups. If $\pi$ is continuous at the neutral element $1_G$ of $G$, then $\pi$ is continuous at every point $g \in G$.

**Proposition 5.15.** Let $G$ be a topological group with the invariant Steinhaus property. Then $G$ has the invariant automatic continuity property.

***Proof.*** Let $\pi : G \to H$ be a homomorphism into a separable SIN group $H$. Assume that $G$ has the invariant Steinhaus property with exponent $k$. By Lemma 5.14 it suffices to show that $\pi$ is continuous at $1_G$. Suppose that $U \subseteq H$ is an open neighbourhood of

$1_H$. Since $H$ is SIN we can find a conjugacy-invariant symmetric open set $V$ satisfying $1_H \in V \subseteq V^{2k} \subseteq U \subseteq H$. By separability of $H$, $V$ covers $H$ by countably many translates $\{h_nV\}_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$ such that $h_nV \cap \pi(G) \neq \emptyset$, choose $g_n \in G$ such that $\pi(g_n) \in h_nV$. Thus $h_nV \subseteq \pi(g_n)V^{-1}V = \pi(g_n)V^2$ and $\pi(g_n)V^2$ cover $\pi(G)$.

For fixed $g \in G$, choose $n \in \mathbb{N}$ such that $\pi(g) \in \pi(g_n)V^2$. Then $\pi(g_n^{-1}g) \in V^2$, thus $g_n^{-1}g \in \pi^{-1}(V^2)$ and hence $g_n\pi^{-1}(V^2)$ cover $G$. Moreover, since $H$ is SIN we obtain $xg_n^{-1}gx^{-1} \in \pi^{-1}(V^2)$ for every $x \in G$. It follows that $W := \pi^{-1}(V^2)$ is symmetric, countably syndetic and conjugacy invariant in $G$.

Since $G$ has the invariant Steinhaus property, $W^k$ contains an open neighborhood of the identity. Hence, $\pi(W^{2k}) \subseteq V^{2k} \subseteq U$, and we obtain $1_G \in \mathrm{Int}(\pi^{-1}(U))$, that is, $\pi$ is continuous at $1_G$. □

Let us verify the invariant Steinhaus property for finite-dimensional projective unitary groups.

**Proposition 5.16.** The projective unitary group $\mathrm{PU}(n)$, endowed with the norm topology, where $n \in \mathbb{N}$, has the invariant Steinhaus property with exponent $32n$.

***Proof.*** The case $n = 1$ is trivial and so we assume $n \geq 2$. Put $G := \mathrm{PU}(n)$ and let $W \subseteq G$ be a symmetric conjugacy-invariant countably syndetic set. By Propositions 5.9 and 5.11 there exists $v \in W^2$ such that $d_r(1, v) > \varepsilon$. Thus, $d_r(1, v) \geq 1$ and $\delta := \ell_0(v) > 0$. We use $v$ to generate a $\delta$-neighborhood of the identity in the operator norm. So consider an arbitrary element $u \in G$ satisfying $\ell_0(u) \leq \delta$. From Theorem 4.45 we then conclude

$$u \in (v^G \cup v^{-G})^{16n}.$$

Since $u \in B_\delta^{\|\cdot\|}(1)$ was arbitrary, this shows that $G$ has the invariant Steinhaus property with exponent $32n$. □

*Remark.* Basically the same proof as above shows that $\mathrm{SU}(n)$ also has the invariant Steinhaus property. The additional obstruction coming with $\mathrm{SU}(n)$ is that it has a nontrivial center. However, the center is finite and thus one can generate a small $\delta$-neighborhood of the identity with $\delta > 0$ and $\delta < \min_{\lambda\in\mathcal{Z}(\mathrm{SU}(n))\setminus\{1\}} \|1 - \lambda\|$.

Propositions 5.15 and 5.16 together with the previous remark imply the following.

**Theorem 5.17.** $\mathrm{PU}(n)$ *and* $\mathrm{SU}(n)$, *endowed with the norm topology, where* $n \in \mathbb{N}$, *have the invariant automatic continuity property.*

We feel obliged to explain that the compact connected Lie groups $\mathrm{PU}(n)$ and $\mathrm{SU}(n)$ do not have the automatic continuity property, i.e. there is a need for an extra condition on the class of target groups (also it is not clear if SIN groups form the most general such class). We follow [Ro 09b, Example 1.5] to show this. Let $G \coloneqq \mathrm{PU}(n)$. It is enough to show that it there is a discontinuous embedding into the Polish group $\mathrm{S}_\infty$ of all permutations of $\mathbb{N}$. From [HHM 14, Theorem 2.3] we deduce the existence of a non-open subgroup $H \subseteq G$ of countably infinite index. Thus the set $G/H$ of left cosets is countable and we view $\mathrm{S}_\infty$ as the group $\mathrm{Sym}(G/H)$ of all permutations of $G/H$. We define a group homomorphism

$$\pi : G \to \mathrm{Sym}(G/H), \quad \pi(g) = L_g,$$

where $L_f(gH) = fgH$, $f \in G$, is the left multiplication. Note that [Ka 00, Theorem 1] (see also [Ka 00, Corollaries 9 and 10]) tells us that $\pi$ is injective. We claim that $\pi$ is discontinuous. Obviously $L_f(1H) = 1H$ if and only if $f \in H$. Thus

$$\pi^{-1}\left(\{\sigma \in \mathrm{Sym}(G/H) \mid \sigma(1H) = 1H\}\right) = H,$$

which is not open by definition of $H$. Hence $\pi$ is discontinuous.

Now we come to the core in our proof of the invariant automatic continuity property of projective unitary groups of separable $\mathrm{II}_1$ factors. A major difficulty in the proof stems from the fact that we could prove Theorem 4.64 in this quantitative version only if the element that one wants to generate has finite spectrum and rational spectral weights. Many of the techniques and results developed in Chapter 4 are needed.

**Proposition 5.18.** The projective unitary group $\mathrm{PU}(\mathcal{M})$ of a separable $\mathrm{II}_1$ factor $\mathcal{M}$, endowed with the strong operator topology, has the invariant Steinhaus property.

***Proof.*** Let $W \subseteq G \coloneqq \mathrm{PU}(\mathcal{M})$ be a symmetric conjugacy-invariant countably syndetic set. We have to show that there exists a fixed $k \in \mathbb{N}$ such that $W^k$ contains a neighborhood of the identity. By Proposition 5.9 and Proposition 5.11 there exist

- $\varepsilon_1 > 0$ independent of $W$ and $u \in W^2$ with $\|1 - \lambda u\| > \varepsilon_1$ for all $\lambda \in \mathrm{U}(1)$,
- $\varepsilon_2 > 0$ independent of $W$ and $v \in W^2$ with $\ell_t(v) \neq 0$ for all $t \in [0, \varepsilon_2]$.

Let $\varepsilon$ denote the minimum of $\varepsilon_1$, $\varepsilon_2$. By right continuity of $\ell_t$ in $t$, see Lemma 4.26, there exist $\delta > 0$ such that

$$\ell_t(u) \geq \varepsilon \text{ for all } t \in [0, \delta], \qquad \ell_t(v) \geq \delta \text{ for all } t \in [0, \varepsilon].$$

To generate an arbitrarily small neighborhood of the identity in the strong operator topology we need several steps.

(1) First we use $u$ and $v$ to generate elements $w$ with $\|1 - w\|_2 \leq \delta^2/2$ which are of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & w_0 & 0 \\ 0 & 0 & w_0^* \end{pmatrix} \tag{5.1}$$

in some $(\mathrm{U}(p_0\mathcal{M}p_0) \otimes M_{3\times3}(\mathbb{C}))/(\mathbb{C} \cdot 1) \cong G$, where $\tau(p_0) = 1/3$. Let $p$ be a projection commuting with $w$ such that

$$\left\|1 - p - wp^\perp\right\| < \delta \quad \text{and} \quad \tau(p) = \delta.$$

Decompose $w = w_1 w_2$ with $w_1 := wp + p^\perp$ and $w_2 := p + p^\perp w$. Hence $\ell_0(w_1) \leq \frac{2}{\varepsilon}\ell_t(u)$ for all $t \in [0, \delta]$. Using Theorem 4.64 we can generate a symmetry $s$ of trace 0 in $\mathrm{U}(p\mathcal{M}p)$ with $u$, namely we obtain $s \in (u^G \cup u^{-G})^{c\lceil 1/\varepsilon \rceil}$ for some constant $c \in \mathbb{N}$ (e.g. $c = 24576$). Theorem 4.19 allows us to conclude that

$$w_1 \in (s^G \cup s^{-G})^{32} \subseteq (u^G \cup u^{-G})^{c\lceil 1/\varepsilon \rceil}.$$

It remains to generate $w_2$. By Lemma 4.29 (together with Proposition 2.18) we have $\ell_0(w_2) \leq 2\delta^2/2\delta = \delta \leq \ell_t(v)$ for all $t \in [0, \varepsilon]$. Suitable approximation of $v$ in the operator norm, as in the beginning of the proof of Theorem 4.64, allows us to find a projection $p \in \mathcal{M}$ which commutes with $v$, is equivalent to $p_0$ and such that for $\varepsilon' = \delta/8$ we have

$$\ell_{3t}(v) \leq \ell_t^{(p)}(v) + 2\varepsilon'.$$

Now view $v$ as a (diagonal) element of $\mathrm{U}(p\mathcal{M}p \otimes M_{3\times3}(\mathbb{C}))$. Using Proposition 4.62 we can find an element $v' = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & v_0' & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \in \mathrm{U}(p\mathcal{M}p \otimes M_{3\times3}(\mathbb{C}))$ such that

$\ell^{(p)}_{24t}(v) \leq 4\ell^{(p)}_t([v,v'])$ for all $t \geq 0$. Let $g \in \mathrm{U}(p\mathcal{M}p \otimes M_{3\times 3}(\mathbb{C}))$ be a unitary permuting the second and third diagonal entry. Consider the element

$$\widetilde{v} \coloneqq [v,v']g[v,v']^{-1}g^{-1} \in (v^G \cup v^{-G})^4,$$

and observe that $\widetilde{v}$ satisfies

$$\ell^{(p)}_{24t}(v) \leq 4\ell^{(p)}_t([v,v']) \leq 4\ell^{(p)}_t(\widetilde{v}) \quad \text{for all } t \geq 0.$$

Thus we have

$$\ell_0(w_2) \leq \delta \leq \ell_t(v) \leq \ell^{(p)}_{t/3}(v) + 2\varepsilon_0 \leq 4\ell^{(p)}_{t/72}(\widetilde{v}) + \frac{\delta}{4} \quad \text{for all } t \in [0,\varepsilon].$$

As in the proof of Theorem 4.64 (restricting our attention to the lower $2\times 2$ part) we generate an element $v'' \in (\widetilde{v}^G \cup \widetilde{v}^{-G})^2 \subseteq (v^G \cup v^{-G})^8$ that has finite spectrum and rational weights such that

$$\delta \leq 4\ell^{(p)}_{t/72}(\widetilde{v}) + \frac{\delta}{4} \leq 4\ell^{(p)}_{t/72}(v'') + \frac{3\delta}{4} \quad \text{for all } t \in [0,\varepsilon].$$

In particular, $4\ell^{(p)}_t(v'') \geq \delta/4$ for all $t \in [0,\varepsilon/72]$. Hence

$$\ell_0(w_2) \leq 16\ell^{(p)}_t(v'') \quad \text{for all } t \in [0,\varepsilon/72]. \tag{5.2}$$

We restrict our attention to the lower $2 \times 2$ subalgebra $q\mathcal{M}q$ in (5.1) and pass to the direct integral $M_{2\times 2}(L^\infty(\sigma(qw_2)),\nu)$, where $qw_2 = \int_{\lambda\in\sigma(qw_2)} \left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right) d\nu(\lambda)$ (note that $q$ commutes with $w_2$). Let $p'$ denote the projection that cuts $v''$ down to the lower $2\times 2$ part. This allows us to conjugate $p'v''$ into $M_{2\times 2}(L^\infty(\sigma(qw_2)),\nu)$. Recall that Corollary 4.39 gives us a relation between the projective $s$-numbers and the angles of the eigenvalues (note that $v''$ has finite spectrum and for $w_2$ we only need the estimate for the 0-th projective $s$-number since $\ell_t(\cdot)$ is decreasing in $t$). We apply Lemma 4.63 with the relation (5.2) to generate $q'w_2$ for a sub-projection $q' \leq q$, $\tau(q') = eps/72$ (and 1's everywhere else). Thus using Lemma 4.63 on at most $\lceil 72/\varepsilon\rceil$ parts (where relation (5.2) holds) we obtain

$$w_2 \in (v''^G \cup v''^{-G})^{4\cdot 16\cdot\lceil 72/\varepsilon\rceil} \subseteq (v^G \cup v^{-G})^{8\cdot 64\cdot\lceil 72/\varepsilon\rceil},$$

(the factor 4 comes from Corollary 4.39).

We conclude that

$$w = w_1 w_2 \in W^{c\lceil 1/\varepsilon \rceil}$$

for some constant $c \in \mathbb{N}$ (which is independent of $\delta$).

(2) Assume that $w$ such that $\|1 - w\|_2 \leq \delta^2$ has finite spectrum and rational weights. This case follows in the same way as in the first step. Namely one decomposes $w = w_1 w_2$ and generates $w_1$ with the element $u$ (which has uniformly big projective $s$-numbers) and $w_2$ with the element $v$ (which has uniformly many nontrivial projective $s$-numbers). This leads us again to $w \in W^{c\lceil 1/\varepsilon \rceil}$ for some constant $n \in \mathbb{N}$ (independent of $\delta$).

(3) Assume that $w \in \mathrm{B}_{\varepsilon_0}^{\|\cdot\|}(1) \subseteq \mathrm{U}(\mathcal{M})$ for some $\varepsilon_0 \in (0, \delta^2)$ small enough such that using Corollary 4.20 we can decompose $w$ into a product $w_1 \cdot \ldots \cdot w_8$ of elements $w_i \in \mathrm{U}(\mathcal{M})$ of the form (5.1) satisfying

$$\|1 - w_i\|_2 < \delta, \qquad i = 1, \ldots, 8.$$

Note that $\varepsilon_0$ depends on the countably syndetic set $W$. Using the first step, we obtain

$$w = w_1 \ldots w_8 \in W^{8c\lceil 1/\varepsilon \rceil}.$$

Thus we can generate an $\varepsilon_0$-neighborhood in the operator norm in $8c\lceil 1/\varepsilon \rceil$ steps.

(4) Now let $w \in \mathrm{B}_{\varepsilon_0}^{\|\cdot\|_2}(1)$ be arbitrary. Approximate $w$ by an element $w'$ with finite spectrum in the operator norm, such that $\|w - w'\| = \|1 - ww'^*\| < \varepsilon_1$. From the third step we conclude that $ww'^* \in W^{8c\lceil 1/\varepsilon \rceil}$. It remains to show that $w'$ can be generated from elements in $W^{C\lceil 1/\varepsilon \rceil}$ for some constant $C \in \mathbb{N}$. Therefore, using Proposition 4.54, we approximate $w'$ with an element $w''$ that has finite spectrum and rational spectral weights such that

$$\left\|w' - w''\right\|_2 \leq \varepsilon_1 \quad \text{ and } \quad d_r(1, w'w''^*) \leq \delta.$$

The second step allows us to conclude $w'' \in W^{c\lceil 1/\varepsilon \rceil}$ for some constant $c \in \mathbb{N}$. We only have to generate the element $w'w''^*$ of small rank. It is clear that $\ell_t(w'w''^*) = 0$ for all $t > \delta$. Let $q$ denote the projection witnessing nontriviality of $w'w''^*$ and observe that $\tau(q) \leq \delta$. As in the first step, we use $u$ to generate a symmetry $s$ of trace 0 in $q\mathcal{M}q$ such that

$$s \in W^{c\lceil 1/\varepsilon \rceil}$$

for some constant $c \in \mathbb{N}$. From Theorem 4.19 we conclude that

$$w'w''^* \in W^{32\cdot c\lceil 1/\varepsilon \rceil}.$$

Summarizing the above three steps, we have shown that there exists a constant $C \in \mathbb{N}$ (independent of $\delta$ and $\varepsilon_0$) such that $W^{C\lceil 1/\varepsilon \rceil}$ contains a neighborhood of the identity in the strong operator topology. This shows that $\mathrm{PU}(\mathcal{M})$ has the invariant Steinhaus property. □

Actually the proof of Proposition 5.18 will allow us to conclude the uniqueness of the Polish group topology of $\mathrm{PU}(\mathcal{M})$ in Section 5.2.

We can now conclude the main theorem in this section from Proposition 5.15 and Proposition 5.18.

**Theorem 5.19.** *The projective unitary group of a separable* $\mathrm{II}_1$ *factor, endowed with the strong operator topology, has the invariant automatic continuity property.*

*Remark.* We mention that [Sa 13, Theorem 7.3] implies that homomorphisms from the unitary group of a separable infinite-dimensional Hilbert space (endowed with the strong operator topology, which is non-bi-invariant and Polish) into SIN groups are trivial.

Our strategy to obtain Theorem 5.19 mainly used that we could ensure the existence of elements of a certain size in a fixed power of every conjugacy-invariant countably syndetic set. The author hopes that this strategy leads to more new examples of groups having the invariant automatic continuity property.

## 5.2 Uniqueness of the Polish group topology

As an easy application of Theorem 5.19 we show that $\mathrm{PU}(\mathcal{M})$ has a unique Polish SIN group topology. The proof is actually valid for any separable topological group with the invariant automatic continuity property. In particular, $\mathrm{PU}(n)$ and $\mathrm{SU}(n)$ carry a unique Polish SIN group topology. In the case $\mathrm{PU}(n)$ it is already known that it has a unique Polish group topology, see [GP 08, Theorem 11].

**Proposition 5.20.** The projective unitary group $G$ of a separable $\mathrm{II}_1$ factor carries a unique Polish SIN group topology.

***Proof.*** Assume that $\mathcal{T}$ is another topology on $G$ which makes it a Polish group with a bi-invariant metric. Let $\mathrm{id} : (G, SOT) \Rightarrow (G, \mathcal{T})$ denote the identity homomorphism. Since $(G, \mathcal{T})$ is a separable SIN group, id is continuous by Theorem 5.19. So it is a continuous bijection and hence a homeomorphism by [BK 96, Theorem 1.2.6]. It follows that $\mathcal{T}$ and the strong operator topology coincide: on the one hand $\mathrm{id}^{-1}(O) = O$ is open in the strong operator topology for any open $O \subseteq (G, \mathcal{T})$ by continuity. On the other hand whenever $O \subseteq (G, SOT)$ is open, then $\mathrm{id}(O) = O$ is open in $\mathcal{T}$ since id is a homeomorphism. $\square$

Now we will make use of the proof of Proposition 5.18 to conclude the uniqueness of the Polish group topology on $\mathrm{PU}(\mathcal{M})$ for any separable $\mathrm{II}_1$ factor. For this purpose, we need [GP 08, Theorem 8]. We first clarify some notations. Let $G$ be a group. An **identity set** in $G$ is a subset of $G$ of the form

$$\{g \in G \mid w(g; u_1, \ldots, u_m) = 1\},$$

where $w$ denotes a free word in $G$ (i.e. without consecutive symbols of the form $gg^{-1}$ or $g^{-1}g$) and $u_1, \ldots, u_m \in G$. By definition identity sets can be viewed as inverse images of 1 under the maps $w(\cdot; u_1, \ldots, u_m)$.

A **verbal set** is a subset of $G$ of the form

$$\{w(g_1, \ldots, g_n; u_1, \ldots, u_m) \mid g_1, \ldots, g_n \in G\},$$

where $w$ is a free word and $u_1, \ldots, u_m \in G$. Verbal sets are forward images under the maps $w$.

An example of an identity set is the centralizer $\left\{g \in G \mid gug^{-1}u^{-1} = 1\right\}$ of an element $u \in G$. For us the most important example of a verbal set is the conjugacy class $\left\{gug^{-1} \mid g \in G\right\}$ of an element $u \in G$.

We say that a collection $\mathcal{N}$ of a topological space $X$ is a **network** if for every $x \in V$ with $V$ open in $X$, there exists $N \in \mathcal{N}$ such that $x \in N \subseteq V$.

We can now state [GP 08, Theorem 8].

**Theorem 5.21** (Gartside-Pejić). *Every Polish group that has a countable network of sets from the $\sigma$-algebra generated by identity sets and verbal sets has a unique Polish group topology.*

Here is the main result of this section - it is based on the proof of Proposition 5.18 and Theorem 5.21.

**Theorem 5.22.** *The projective unitary group $G$ of a separable $\mathrm{II}_1$ factor has a unique Polish group topology.*

***Proof.*** First recall that $G$ is a Polish group in the strong operator topology. We construct a countable network for $G$. For $n \in \mathbb{N}$ we let $\varepsilon_n := 1/n$ and $\delta = \delta(n) < \frac{1}{C\lceil 1/\varepsilon_n^2 \rceil} = \frac{1}{Cn^2}$, where $C \in \mathbb{N}$ is the universal constant coming from the proof of Proposition 5.18. Now choose $u, v \in G$ (only dependent on $n$) such that $\|1-u\|_2 < \delta, \|1-v\|_2 < \delta$ and

$$\ell_t(u) \geq \varepsilon_n \text{ for all } t \in [0,\delta], \qquad \ell_t(v) \geq \delta \text{ for all } t \in [0,\varepsilon_n].$$

Using the proof of Proposition 5.18 we conclude the existence of $\delta_0 = \delta_0(n) \in (0,\delta)$ (independent of $u$ and $v$) such that

$$\mathrm{B}_{\delta_0}^{\|\cdot\|_2}(1) \subseteq N_{\varepsilon_n} := \left(u^G \cup u^{-G} \cup v^G \cup v^{-G}\right)^{C\lceil 1/\varepsilon_n \rceil}.$$

However, we have $N_{\varepsilon_n} \subseteq \mathrm{B}_{1/n}^{\|\cdot\|_2}(1)$, since for every $x \in N_{\varepsilon_n}$

$$\|1-x\|_2 \leq C\lceil 1/\varepsilon_n \rceil \delta \leq 1/n.$$

Fix a countable dense subset $D \subseteq G$. We claim that

$$\mathcal{N} := \{xN_{\varepsilon_n} \mid x \in D, n \in \mathbb{N}\}$$

forms a countable network for $G$. First of all, $\mathcal{N}$ is formed by the $\sigma$-algebra generated from verbal sets

Now let $w \in V$ with $V \subseteq G$ open. Since $V$ is open, we can find $\varepsilon > 0$ such that $\mathrm{B}_{\varepsilon}^{\|\cdot\|_2}(w) \subseteq V$. Let $n \in \mathbb{N}$ such that $\varepsilon_n = 1/n < \varepsilon/2$. By denseness of $D$ we can choose $v_0 \in D$ such that $\|v_0 - w\|_2 \leq \delta_0$. Then we have (note that $\delta_0 = \delta_0(n) < \varepsilon_n < \varepsilon/2$):

$$
\begin{aligned}
w \in v_0 \mathrm{B}_{\delta_0}^{\|\cdot\|_2}(1) &\subseteq v_0 N_{\varepsilon_n} \\
&\subseteq v_0 \mathrm{B}_{1/n}^{\|\cdot\|_2}(1) \\
&\subseteq v_0 \mathrm{B}_{\varepsilon/2}^{\|\cdot\|_2}(1) \\
&\subseteq w \mathrm{B}_{\delta_0}^{\|\cdot\|_2}(1) \mathrm{B}_{\varepsilon/2}^{\|\cdot\|_2}(1) \\
&\subseteq w \mathrm{B}_{\varepsilon}^{\|\cdot\|_2}(1) \\
&= \mathrm{B}_{\varepsilon}^{\|\cdot\|_2}(w) \\
&\subseteq V.
\end{aligned}
$$

That is, for arbitrary $w \in V$, $V$ open in $G$, we find a set $N \in \mathcal{N}$ such that $w \in N \subseteq V$, i.e., $\mathcal{N}$ is a network. Since $D$ and $\mathbb{N}$ are countable, $\mathcal{N}$ is countable. Now from Theorem 5.21 we conclude that $G$ has a unique Polish group topology. □

In a similar fashion we obtain the (already known) uniqueness of the Polish group topology on $\mathrm{PU}(n)$.

**Theorem 5.23.** $\mathrm{PU}(n)$ *has a unique Polish group topology for every* $n \in \mathbb{N}$.

As a consequence of Theorem 5.22 we obtain the following further automatic continuity results, which are equivalent to the uniqueness of the Polish group topology by [Pe 07, Lemma 10, Lemma 13]. Of course, the results in Corollary 5.24 also hold true for $\mathrm{PU}(n)$, $n \in \mathbb{N}$.

**Corollary 5.24.** Let $\mathcal{M}$ denote a separable $\mathrm{II}_1$ factor and let $G$ be its projective unitary group.

(i) Every isomorphism from $G$ to a Polish group is continuous.

(ii) Every epimorphism from a Polish group to $G$ with closed kernel is continuous.

# 6 Outlook

Here we discuss some of the open problems that came out of our work or seem to be closely linked.

In Chapter 4 we proved many results on products on conjugates in various unitary groups of functional analytic type. Whenever possible, we concluded property (BNG) or property (topBNG). Our methods seem sufficiently general to be applicable for many other large (non-locally compact) groups.

We have defined projective generalized $s$-numbers for semifinite factors. Type III factors do not admit a semifinite trace, but they behave to some extent similar to Calkin algebras. This leads us to expect the following question to have a positive answer.

Does the projective unitary group of a type III factor have property (BNG)?

We intend to further pursue this question in the future.

It is likely that the closure condition in our theorem on products of conjugates in $\mathrm{U}(\mathcal{H})$ (see Theorem 4.51) can be omitted. Then it would not be surprising to have an algebraic criterion for products of conjugates in unitary groups of $\mathrm{II}_\infty$ factors (which are tensor products of $\mathrm{I}_\infty$ and $\mathrm{II}_1$ factors).

Necessary and sufficient criteria for products of conjugates are interesting in any noncommutative non-compact group. We did not yet deeply study what the complete absence of a (topological) uniform normal generator means for the structure of a group. Note that we have already observed that the group $\mathrm{PU}(\mathcal{H})$ does have uniform normal generators (namely symmetries with two infinite eigenspaces), although it is not simple as $\mathrm{U}(\mathcal{H})_{\mathcal{K}(\mathcal{H})}$ is a nontrivial normal subgroup.

A group $G$ has the strong uncountable cofinality if for every increasing sequence of symmetric sets $W_0 \subseteq W_1 \subseteq \ldots \subseteq G = \bigcup_n W_n$ there exist $k, m \in \mathbb{N}$ such that $W_m^k = G$.

If a group has propety (BNG) then the above condition is obviously satisfied whenever some $W_n$ contains a nontrivial conjugacy class. It would be interesting to see how property (BNG) compares in depth to the strong uncountable cofinality and the topological Bergman property, see e.g. [Ro 09a]. In particular, it is unknown whether the (projective) unitary group of $\mathrm{II}_1$ factor has the strong uncountable cofinality. In the future we want to figure out if this is an application of our results on property (BNG).

We proved in Chapter 5 that the projective unitary group $\mathrm{PU}(\mathcal{M})$ of a separable $\mathrm{II}_1$ factor $\mathcal{M}$ has the invariant automatic continuity property. It is still an open question if $\mathrm{PU}(\mathcal{M})$ has the automatic continuity property. Probably some of our ideas can be used to prove (or disprove) this - for example, if one can show the strong uncountable cofinality and apply methods developed in the previous chapters.

# Index of Symbols

| | |
|---|---|
| $(A)_\varepsilon$ | $\varepsilon$-neighborhood of the set $A$ |
| $\mathcal{B}(\mathcal{H})$ | algebra of bounded operators on $\mathcal{H}$ |
| $\mathbb{C}$ | complex numbers |
| $\mathcal{C}$ | Calkin algebra |
| $d$ | metric |
| $G$ | group |
| $g^G$ | conjugacy class of $g \in G$ |
| $G/H$ | quotient group $G$ modulo $H$ |
| $\mathcal{H}$ | Hilbert space |
| $K$ | norm ideal |
| $\mathcal{K}(\mathcal{H})$ | compact operators |
| $\mathcal{M}$ | von Neumann algebra, $\mathrm{II}_1$ factor |
| $M_{n\times n}(\mathbb{C})$ | algebra of $n \times n$ matrices |
| $\mathbb{N}$ | positive integers, $1, 2, \ldots$ |
| $\mathbb{N}_0$ | $\mathbb{N} \cup \{0\}$ |
| $\\|\cdot\\|$ | operator norm |
| $\\|\cdot\\|_1$ | 1-norm defined by the trace $\tau$ |
| $\\|\cdot\\|$ | 2-norm defined by the trace $\tau$ |
| $\\|\cdot\\|_{HS}$ | Hilbert-Schmidt norm |
| $\\|\cdot\\|_K$ | norm of the norm ideal $K$ |
| $\mathrm{Proj}(\mathcal{M})$ | set of projections of $\mathcal{M}$ |
| $\mathrm{PU}(\mathcal{C})$ | projective unitary group of the Calkin algebra |
| $\mathrm{PU}(\mathcal{H})$ | projective unitary group on $\mathcal{H}$ |
| $\mathrm{PU}(\mathcal{M})$ | projective unitary group of $\mathcal{M}$ |
| $\mathrm{PU}(n)$ | projective unitary group of $n \times n$ matrices |
| $\mathbb{Q}$ | rational numbers |

| | |
|---|---|
| $\mathbb{R}$ | real numbers |
| $\mathbb{R}^+$ | positive real numbers, i.e. $(0, \infty)$ |
| $\mathbb{R}_0^+$ | $\mathbb{R}^+ \cup \{0\}$ |
| $\langle \cdot, \cdot \rangle$ | scalar product |
| $\mathrm{SU}(n)$ | special unitary group of $n \times n$ matrices |
| $\mathbb{T}$ | one-dimensional torus, $\mathrm{U}(1)$ |
| $\tau$ | trace on a semifinite von Neumann algebra |
| $\mathrm{U}(\mathcal{H})$ | group of unitary operators on $\mathcal{H}$ |
| $\mathrm{U}(\mathcal{H})_K$ | group of unitary operators of $K$-perturbations from the identity on $\mathcal{H}$ |
| $\mathrm{U}(\mathcal{M})$ | unitary group of $\mathcal{M}$ |
| $\mathrm{U}(n)$ | unitary group of $n \times n$ matrices |
| $\mathbb{Z}$ | integers |

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# Index

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