Isomorphic chain complexes of Hamiltonian dynamics on tori

Von der Fakultät für Mathematik und Informatik
der Universität Leipzig
eingereichte

Dissertation

zur Erlangung des akademischen Grades

Doctor rerum naturalium
(Dr. rer. nat.)

im Fachgebiet

Mathematik

vorgelegt von

Diplom-Mathematiker Michael Hecht

geboren am 03.12 1980 in Zeitz

Die Annahme der Dissertation wurde empfohlen von:

1. Prof. Dr. Matthias Schwarz Universität Leipzig
2. Prof. Dr. Pietro Majer Universität Pisa

Die Verleihung des akademischen Grades erfolgt mit Bestehen der Verteidigung
am 17.07. 2013 mit dem Gesamtprädikat *magna cum laude*

Bibliographische Daten

Hecht, Michael
Isomorphic chain complexes of Hamiltonian dysnamics on tori
Dissertation, Universität Leipzig, 2012
81 Seiten, 37 Literaturangaben
Keywords: Floer homology, Morse homology, chain isomorphism

Abstract

In this thesis we construct for a given smooth, generic Hamiltonian $H : \mathbb{S}^1 \times \mathbb{T}^{2n} \longrightarrow \mathbb{R}$ on the torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ a chain-isomorphism $\Phi_* : (C_*(H), \partial_*^M) \longrightarrow (C_*(H), \partial_*^F)$ between the Morse-complex of the Hamiltonian action A_H on the free loop space of the torus $\Lambda_0(\mathbb{T}^{2n})$ and the Floer-complex. Though both complexes are generated by the critical points of A_H, their boundary operators differ. Therefore the construction of Φ is based on counting the moduli spaces of hybrid-type solutions which involves stating a new non-Lagrangian boundary value problem for Cauchy-Riemann type operators not yet studied in Floer theory. The essential reason for the non-triviality of the result is that one has to develop new methods to prove that this new coupling condition leads to a well posed Fredholm problem and to the fact that the moduli spaces are precompact. It is crucial for the statement that the torus is compact, possesses trivial tangent bundle and an additive structure. We finally want to note that the problem is completely symmetric. So we also could construct an isomorphism $\Psi_* : (C_*(H), \partial_*^F) \longrightarrow (C_*(H), \partial_*^M)$.

Contents

CHAPTER 1

Introduction

For a generic, smooth Hamiltonian $H : \mathbb{S}^1 \times \mathbb{T}^{2n} \longrightarrow \mathbb{R}$ with only contractible periodic orbits we prove the following statement.

THEOREM 1. *The Morse complex of the Hamiltonian action A_H is chain isomorphic to the Floer complex of* (H, J_0).

To give a short summary observe that the component of contractible loops of the free loop space of the $2n$ - dimensional torus $(\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}, \omega_0)$ equipped with the standard symplectic structure $\omega_0 = \sum_{i=1}^n dy_i \wedge dx_i$ can be completed to a Hilbert manifold $\mathbb{M}$ with respect to the $H^{1/2}$-inner product. In this case $\mathbb{M}$ splits into $\mathbb{M} = \mathbb{T}^{2n} \times \mathbb{H}$, where $\mathbb{H}$ is a Hilbert space and each $x \in \mathbb{M}$ can be written as

$$x = [x_0] + \sum_{k \neq 0} e^{2\pi J_0 k t} x_k\,, \quad [x_0] \in \mathbb{T}^{2n}\,, x_k \in \mathbb{R}^{2n} \quad \forall\, k \in \mathbb{Z} \setminus \{0\}$$

where J_0 given by

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

denotes the special complex structure on $\mathbb{T}^{2n}$ such that $(\mathbb{T}^{2n}, \omega_0, J_0)$ becomes a Kähler manifold with

$$\omega_0(\cdot, J_0 \cdot) = \langle \cdot, \cdot \rangle\ .$$

For given 1-periodic Hamiltonian $H : \mathbb{S}^1 \times \mathbb{T}^{2n} \longrightarrow \mathbb{R}$ the **Hamiltonian action** is defined by

$$A_H : \mathbb{M} \longrightarrow \mathbb{R}\,, \quad A_H(x) = -\frac{1}{2}\Big(\|\,\mathbb{P}^+ x\,\|^2_{H^{1/2}(\mathbb{S}^1)} - \|\,\mathbb{P}^- x\,\|^2_{H^{1/2}(\mathbb{S}^1)}\Big) + \int_0^1 H(t, x(t))dt\,,$$

where $\mathbb{P}^\pm : \mathbb{M} \longrightarrow \mathbb{H}^\pm$, $\sum_{k \in \mathbb{Z}} e^{2\pi J_0 k t} x_k \mapsto \sum_{\pm k > 0} e^{2\pi J_0 k t} x_k$ denote the projections onto the positive or negative frequented loops. The critical points of A_H are precisely the 1-periodic contractible solutions $\mathcal{P}_0(H)$ of the Hamiltonian flow $\varphi_{X_H} : \mathbb{R} \times \mathbb{T}^{2n} \longrightarrow \mathbb{T}^{2n}$ determined by the equation

$$\frac{d}{dt}\varphi_{X_H} = X_H \circ \varphi_{X_H}\,, \quad \varphi_{X_H}(0, \cdot) = \mathrm{id}_{\mathbb{T}^{2n}}\,, \quad \text{with} \quad i_{X_H}\omega_0 = -dH\,.$$

Now there are several possibilities to develop a Morse theory for this action functional, see for instance [**11**], [**19**], [**27**], [**28**]. We want to focus on two other ideas. The first one goes back to the results of A. Abbondandolo and P. Majer published in [**5**] and [**6**] and is due to the fact that

A_H possesses a smooth gradient $\mathbb{X} := -\nabla_{1/2} A_H$ on $\mathbb{M}$ to build up a theory directly on the loop space. The second one is known as A. Floer's approach [**15**] where solutions of a type of Cauchy Riemann PDE's are considered, i.e., the solutions $u : \mathbb{R} \times \mathbb{S}^1 \longrightarrow \mathbb{T}^{2n}$ of the Floer equation

$$\overline{\partial}_{J,H}(u(s,t)) = \partial_s u(s,t) + J_t(u(s,t))(\partial_t u(s,t) - X_H(t, u(s,t))) = 0\,, \quad J_t \in \mathcal{J}\,,$$

where $\mathcal{J}$ denotes the set of all smooth and 1-periodic almost complex structures on $(\mathbb{T}^{2n}, \omega_0)$ which are compatible with ω_0. In our case we have to model these equations in the $W^{1,2}$-setup therefore we find it convenient to restrict ourselves to the case where $J = J_0$ is the constant standard complex structure. To achieve transversality we asssume that the Hamiltonian H was generically chosen. Hence we can count the solutions with finite energy which therefore converge to critical points of A_H at the asymptotics and define the Floer boundary operator

$$\partial_k^F : C_k(H) \longrightarrow C_{k-1}(H)\,,$$

between the free abelian groups generated by the critical points of A_H with Conley-Zehnder index $k \in \mathbb{Z}$. According to Floer's fundamental theorem the above setup becomes a complex, i.e., $\partial_{k-1}^F \circ \partial_k^F = 0$. The homology of this complex is called the **Floer homology** and it is known due to Floer's continuation theorem that this homology is independent of the chosen Hamiltonian H. In particular this asserts why we can assume that there are only contractible 1-periodic Hamiltonian orbits and can therefore restrict ourselves to the component of contractible loops of the free loop space $\Lambda_0(\mathbb{T}^{2n})$ as introduced. Furthermore the continuation property was used by A. Floer to prove that the Floer homology is chain isomorphic to the singular homology of the symplectic manifold, i.e., in our case the torus $\mathbb{T}^{2n}$. This fact is the main ingredient in Floer's proof of the **Arnold conjecture** in the generic situation see again [**15**].

Coming back to the first approach the fact that A_H is strongly indefinite, i.e., possesses infinite Morse indices and co-indices for all singular points $x, y \in \operatorname{sing}(\mathbb{X})$ leads to the problem that the unstable and stable manifolds $\mathcal{W}^u(x), \mathcal{W}^s(x)$ are infinite dimensional submanifolds of $\mathbb{M}$. Nevertheless one can hope that the intersections $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ are finite dimensional. To prove such a result A. Abbondandolo and P. Majer require the existence of a subbundle $\mathcal{V} \subset T\mathbb{M} = \mathbb{M} \times \mathbb{E}$, $\mathbb{E} = \mathbb{R}^{2n} \times \mathbb{H}$ which in our situation can be chosen as $\mathcal{V} = \mathbb{M} \times (\mathbb{R}^n \times \mathbb{H}^+)$ such that the Morse vector field $\mathbb{X}$ satisfies two conditions, namely :

(C1) for every singular point $x \in \operatorname{sing}(\mathbb{X})$, the unstable eigenspace $E^u\,(D\mathbb{X}(x))$ of the Jacobian of $\mathbb{X}$ at x is a compact perturbation of $\mathcal{V}(x)$; meaning that the corresponding orthogonal projections $P_{E^u} := P_{E^u(D\mathbb{X}(x))}$ and P_V with $V = \mathbb{R}^n \times \mathbb{H}^+$ have compact difference, i.e., $P_{E^u} - P_V$ is a compact operator.

(C2) for all $p \in \mathbb{M}$ the operator $[D\mathbb{X}(p), P_V]$ is compact, where $[S, T] = ST - TS$, $\forall\, S, T \in \mathcal{L}(\mathbb{E})$ denotes the commutator.

Under condition **C1** and a strengthened, global version of **C2**, A. Abbondandolo and P. Majer could prove in [**5**] that if the intersections $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$, $x, y \in \operatorname{sing}(\mathbb{X})$ are transverse, then these are finite dimensional manifolds which are compact up to broken trajectories. Therefore we are almost in the same situation as in the finite dimensional case treated in [**33**] and can construct a Morse complex for the action functional A_H. That is to consider again the free abelian groups $C_k(H)$, generated by the critical points $x \in \operatorname{crit}(A_H)$ of A_H with Conley-Zehnder index $\mu(x) = k \in \mathbb{Z}$ and to define the boundary operator

$$\partial_k^M : C_k(H) \longrightarrow C_{k-1}(H)$$

in the usual way by counting the connected components of $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ with $\mu(x) = k$, $\mu(y) = k - 1$. The fact that $\partial_{k-1}^M \circ \partial_k^M = 0$ is again deep and proven in [**5**]. So we obtain a homology called the **Morse homology** of A_H. Though we have to perturb the gradient of A_H by a compact vector field $K : \mathbb{M} \longrightarrow \mathbb{E}$ to achieve transversality, the above homology is independent of the perturbation as well as of the particular choice of H. So again the theory can be treated in the situation of contractible critical points of A_H.

After introducing both complexes we now want to give a sketch of the construction of the isomorphism. The main ingredients are the **moduli spaces of hybrid type curves**. For given critical points $x^-, x^+ \in \operatorname{crit}(A_H)$, C^3- vector field X on $\mathbb{M}$ with globally defined flow and $Z = [0, +\infty) \times \mathbb{S}^1$ that are the spaces

$$\mathcal{M}_{\text{hyb}}(x^-, x^+, H, J_0, X) := \left\{ u \in W^{1,2}_{\text{loc}}(Z, \mathbb{T}^{2n}) \;\middle|\; \overline{\partial}_{J_0,H}(u) = 0\,, u(0,\cdot) \in \mathcal{W}^u_X(x^-)\,, u(+\infty,\cdot) = x^+ \right\}.$$

Though the solutions are smooth on the interior $(0, +\infty) \times \mathbb{S}^1$ the equation $\overline{\partial}_{J_0,H}(u) = 0$ is understood in the weak sense. The fact that the evaluation at zero is a well defined smooth submersion implies that the boundary condition $u(0,\cdot) \in \mathcal{W}^u_X(x^-)$ which says that the loop $u(0,\cdot) \in \mathbb{M}$ shall sit on the unstable manifold of x^- with respect to X is well posed. The new outcome of this thesis is that this non-Lagrangian boundary condition leads to a well posed Fredholm problem and to the fact that the moduli spaces are compact up to broken trajectories in the $W^{1,2}_{\text{loc}}$- sense as long as X is of the form $-\nabla_{1/2} A_H + K$ where $K : \mathbb{M} \longrightarrow \mathbb{E}$ plays the role of a compact perturbation.

Both results in this robust theory use new estimates for the linearized operator and the connecting curves and again the existence of the subbundle $\mathcal{V}$ which is admissible for X in the sense of conditions **C1** and **C2**. As an outlook one hopefully can extract these methods for the analog problem on more general manifolds as the torus $\mathbb{T}^{2n}$.

To finish the construction of the isomorphism we define the following map on the abelian groups

$$C_k(H) = \bigoplus_{\substack{x \in \mathcal{P}_0(H), \\ \mu(x) = k}} \mathbb{Z}_2 x\,.$$

$$\Phi_k : C_k(H) \longrightarrow C_k(H)\,, \quad x \mapsto \sum_{\mu(y)=k} \upsilon(x,y)y$$

where $\upsilon(x,y)$ is the sum of connected components of $\mathcal{M}_{\mathrm{hyb}}(x,y,H,J_0,X)$. By a standard argument using the gluing method Φ is a chain homomorphism. Finally, we order the critical points of A_H by their levels then Φ_k gets the form of an upper triangular matrix with 1's on the diagonal and therefore Φ is an isomorphism as claimed.

1.1. Acknowledgments

First of all I want to express my gratitude to my advisors, Prof. Matthias Schwarz and Prof. Alberto Abbondandolo for introducing me to Floer homology and Morse homology as well as for sharing their ideas with me, which are fundamental for this thesis. I was financially supported by the DFG through the Graduiertenkolleg "Analysis, Geometrie und ihre Verbindung zu den Naturwissenschaften ". The support was more than just financial, and I want to thank Prof. Hans-Bert Rademacher for his effort. I also was financially supported by the Max Planck Institute for Mathematics in the Sciences and I want to thank Prof. Stephan Luckhaus for his effort too. Special thanks goes to Frank Loebe and Hannes Straß for proofreading the manuscript. Finally I want to thank my family and my best friends who supported me in every way during the whole time.

CHAPTER 2

The Morse complex

In this chapter we construct the Morse complex for the Hamiltonian action functional A_H on the component of contractible loops of the free loop space $\Lambda_0(\mathbb{T}^{2n})$ of the torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$. To make this thesis better readable, proofs of already known statements - adapted to our particular setting - are explicitly given.

2.1. The analytical setting

This section is used to assert how to complete the component of contractible loops of the free loop space of the torus to a Hilbert manifold whose structure is induced by the special Sobolev-space $H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n})$. Most facts are already discussed in [**21**]. Therefore we restrict ourselves to the essential statements.

We consider the $2n$-dimensional torus $\mathbb{T}^{2n}$, $n \in \mathbb{N}$, as the quotient $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with additional standard symplectic structure $\omega_0 = \sum_{i=1}^{n} dy_i \wedge dx_i$ induced by the standard symplectic basis $\{\partial x_i, \partial y_i\}_{i=1,\dots,n}$ on $\mathbb{R}^{2n}$ and denote by

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the special complex structure on $\mathbb{T}^{2n}$ such that $(\mathbb{T}^{2n}, \omega_0, J_0)$ becomes a Kähler manifold with

$$\omega_0(\cdot, J_0\cdot) = \langle \cdot, \cdot \rangle \ .$$

We introduce a smooth, 1-periodic Hamiltonian $H : \mathbb{S}^1 \times \mathbb{T}^{2n} \longrightarrow \mathbb{R}$, its Hamiltonian vector field X_H defined by

$$i_{X_H}\omega_0 = -dH$$

and the corresponding smooth flow φ_{X_H}, which, since $\mathbb{T}^{2n}$ is compact, is globally defined by

$$\varphi_{X_H} : \mathbb{R} \times \mathbb{T}^{2n} \longrightarrow \mathbb{T}^{2n}\,, \quad \text{solves} \quad \frac{d}{dt}\varphi_{X_H}(t,p) = (X_H \circ \varphi_{X_H})(t,p)\,, \quad \varphi_{X_H}(0,\cdot) = \mathrm{id}_{\mathbb{T}^{2n}}\,.$$

It is well-known that the search of contractible 1-periodic orbits of φ_{X_H} can be reformulated in a variational problem as follows. Consider the **component of contractible loops of the free loop space of the torus**. That is

$$\Lambda_0(\mathbb{T}^{2n}) := \left\{ x \in C^\infty(\mathbb{S}^1, \mathbb{T}^{2n}) \;\middle|\; [x] = 0 \quad \text{in} \quad \pi_1(\mathbb{T}^{2n}) \right\}\,,$$

where $\pi_1(\mathbb{T}^{2n})$ denotes the first fundamental group. To introduce the **Hamiltonian action functional** on $\Lambda_0(\mathbb{T}^{2n})$ we lift the contractible loop $x \in \Lambda_0(\mathbb{T}^{2n})$ to a closed curve $\tilde{x} \in C^\infty(\mathbb{S}^1, \mathbb{R}^{2n})$ and lift H to a function $\tilde{H}$ that is 1-periodic in all variables on $\mathbb{S}^1 \times \mathbb{R}^{2n}$ and set

$$\tilde{A}_H(x) := \frac{1}{2}\int_0^1 \omega_0(\dot{\tilde{x}}(t), \tilde{x}(t))dt + \int_0^1 \tilde{H}(t, \tilde{x}(t))dt, \quad x \in \Lambda_0(\mathbb{T}^{2n}). \tag{2.1}$$

Now by partial integration, the first term is independent of the chosen lift of x and by the periodicity of $\tilde{H}$; so is the second. Hence $\tilde{A}_H$ is a well-defined functional and if there is no danger of confusion we will make no notational difference between x and its lift. Varying $\tilde{A}_H$ at $x \in \Lambda_0(\mathbb{T}^{2n})$ in direction of a vector field Y along x leads to

$$\delta\tilde{A}_H(x)[Y] = \int_0^1 \langle J_0\, \dot{x}(t) + \nabla H(x(t)), Y(t)\rangle\, dt\,,$$

which readily shows the one-to-one correspondence between critical points of $\tilde{A}_H$ and the contractible 1-periodic orbits of φ_{x_H}, which we denote by $\mathcal{P}_0(H)$.

We want to complete $\Lambda_0(\mathbb{T}^{2n})$ to a Hilbert manifold.

Definition 2.1.1. *An infinite dimensional C^k-* **Banach manifold** *is a paracompact Hausdorff topological space with a C^k-atlas whose charts taking values in an infinite dimensional Banach space E. A* **Hilbert manifold** *is always assumed to be a smooth separable Banach manifold modeled on a separable Hilbert space $(E, \langle\cdot,\cdot\rangle)$.*

Note that by the embedding theorem of Eells and Elworthy [**12**], each Hilbert manifold can be embedded into an open subset of E.

Since $\Lambda_0(\mathbb{R}^{2n}) \subset L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ each curve $x \in \Lambda_0(\mathbb{R}^{2n})$ can be written as a Fourier series with coefficients in $\mathbb{R}^{2n}$

$$x(t) = \sum_{k\in\mathbb{Z}} e^{2\pi k J_0 t} x_k\,, \quad x_k \in \mathbb{R}^{2n}\,.$$

Furthermore the $\mathbb{Z}^{2n}$-action acts only on the constants. Therefore we obtain by identifying those curves whose image on the torus is the same

$$x(t) = \big(x_0, \hat{x}(t)\big) := [x_0] + \sum_{k\neq 0} e^{2\pi k J_0 t} x_k\,, \quad [x_0] \in \mathbb{T}^{2n}\,, \quad x_k \in \mathbb{R}^{2n}\,, \quad \forall\, x \in \Lambda_0(\mathbb{T}^{2n})\,.$$

This special Fourier representation allows us to complete the component of contractible loops of the free loop space with respect to a Hilbert structure induced by the following **fractional Sobolev spaces**. For $s \geq 0$ we set

$$H^s(\mathbb{S}^1, \mathbb{R}^{2n}) = \left\{ x \in L^2(\mathbb{S}^1, \mathbb{R}^{2n}) \;\Big|\; \sum_{k\neq 0} |k|^{2s}|x_k|^2 < \infty \right\}.$$

The spaces $H^s(\mathbb{S}^1,\mathbb{R}^{2n})$ are Hilbert spaces with inner product and associated norm

$$\begin{aligned}\langle x,y\rangle_{H^s(\mathbb{S}^1)} &:= \langle x_0,y_0\rangle + 2\pi\sum_{k\in\mathbb{Z}}|k|^{2s}\,\langle x_k,y_k\rangle\\ \|\,x\,\|^2_{H^s(\mathbb{S}^1)} &= \langle x,x\rangle_{H^s(\mathbb{S}^1)}\end{aligned}$$

Especially we are interested in the case $s = 1/2$ and the orthogonal splitting

$$H^{1/2}(\mathbb{S}^1,\mathbb{R}^{2n}) = \mathbb{R}^{2n}\times\mathbb{H}^+\times\mathbb{H}^- =: \mathbb{E}\,,\quad \mathbb{H} := \mathbb{H}^+\times\mathbb{H}^- \tag{2.2}$$

into the the constants and the loops of positive or negative frequencies explicitly given by the corresponding orthogonal projections $\mathbb{P}^\pm : \mathbb{H}\longrightarrow\mathbb{H}^\pm$, $\mathbb{P}^0 : \mathbb{E}\longrightarrow\mathbb{R}^{2n}$ with

$$\mathbb{P}^\pm(x) = \sum_{0<\pm k} e^{2\pi kJ_0t}x_k\,,\quad \mathbb{P}^0(x) = x_0\,. \tag{2.3}$$

The space $\mathbb{H}^+$ is a special **Hardy-space** [**17**] on the unit circle, i.e., the space of limits of holomorphic curves on the open disc to the boundary and $\mathbb{H}^-$ is its anti-holomorphic counter-part. We set

$$\mathbb{M} := H^{1/2}(\mathbb{S}^1,\mathbb{T}^{2n}) = \mathbb{T}^{2n}\times\mathbb{H}\,, \tag{2.4}$$

then $\mathbb{M}$ carries the structure of a Hilbert manifold with trivial tangent bundle

$$T\,\mathbb{M} = \mathbb{M}\times\mathbb{E}\,.$$

We often abuse notation slightly by still writing $\mathbb{P}^\pm : \mathbb{M}\longrightarrow\mathbb{H}^\pm$, $\mathbb{P}^0 : \mathbb{M}\longrightarrow\mathbb{T}^{2n}$ for the maps $x = (x_0,\hat{x})\mapsto\mathbb{P}^\pm\hat{x}$ and $x = (x_0,\hat{x})\mapsto x_0$. Furthermore we can continuate $\tilde{A}_H$ to $\mathbb{M}$, which becomes

$$A_H(x) = -\frac{1}{2}\left(\|\,\mathbb{P}^+x\,\|^2_{H^{1/2}(\mathbb{S}^1)} - \|\,\mathbb{P}^-x\,\|^2_{H^{1/2}(\mathbb{S}^1)}\right) + \int_0^1 H(t,x(t))dt\,. \tag{2.5}$$

Before we go on we need a better understanding of the spaces $H^s(\mathbb{S}^1,\mathbb{R}^{2n})$. Therefore we follow H. Hofer and E. Zehnder [**21**] and give a brief summary of the essential properties.

For $t\ge s\ge 0$ the spaces decrease,

$$H^t(\mathbb{S}^1,\mathbb{R}^{2n})\subset H^s(\mathbb{S}^1,\mathbb{R}^{2n})\subset H^0(\mathbb{S}^1,\mathbb{R}^{2n}) = L^2(\mathbb{S}^1,\mathbb{R}^{2n})\,,$$

while the norms increase :

$$\|\,x\,\|^2_{H^t(\mathbb{S}^1)}\ge\|\,x\,\|^2_{H^s(\mathbb{S}^1)}\ge\|\,x\,\|^2_{L^2(\mathbb{S}^1)}\quad\text{for}\quad x\in H^t(\mathbb{S}^1,\mathbb{R}^{2n})\,.$$

In particular the inclusion maps $H^t(\mathbb{S}^1,\mathbb{R}^{2n})\longrightarrow H^s(\mathbb{S}^1,\mathbb{R}^{2n})$ are continuous for $t\ge s$; if in particular $t > s\ge 0$ then due to [**21**] they are moreover compact operators. Furthermore we obtain:

Proposition 2.1.2. ([**21**]) *Let $s > \frac{1}{2}$. If $x\in H^s(\mathbb{S}^1,\mathbb{R}^{2n})$ then $x\in C^0(\mathbb{S}^1,\mathbb{R}^{2n})$. Moreover there is a constant $c = c_s$ such that*

$$\|\,x\,\|_{C^0(\mathbb{S}^1)}\le c\|\,x\,\|_{H^s(\mathbb{S}^1)}\,,\quad\textit{for all}\quad x\in H^s(\mathbb{S}^1,\mathbb{R}^{2n})\,.$$

Since we choose the $H^{1/2}$-structure this leads us to the borderline case of embeddings into C^0. Nevertheless, since $\mathbb{T}^{2n}$ possesses a trivial tangent bundle, (2.4) provides us with a Hilbert manifold structure which will be sufficient for our purpose. For manifolds with non-trivial tangent bundle one needs pointwise control on the orbits and therefore at least a H^s, $s > 1/2$ setting to equip the loop space with a Banach manifold structure.

A central role in further discussions plays the inclusion

$$j : H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{S}^1, \mathbb{R}^{2n}) \tag{2.6}$$

and its adjoint $j^* : L^2(\mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n})$ defined as usual by

$$\langle j(x), y\rangle_{L^2(\mathbb{S}^1,\mathbb{R}^{2n})} = \langle x, j^*(y)\rangle_{H^{1/2}(\mathbb{S}^1,\mathbb{R}^{2n})} \,, \quad \text{for all} \quad x \in H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n}), y \in L^2(\mathbb{S}^1, \mathbb{R}^{2n}) \,.$$

As already mentioned j is compact and so is j^* as it is well-known. In particular we obtain even more.

Proposition 2.1.3. ([**21**]) *The adjoint factors*

$$j^* : L^2(\mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow H^1(\mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n}) \,,$$

i.e.,

$$j^*(L^2(\mathbb{S}^1, \mathbb{R}^{2n})) \subset H^1(\mathbb{S}^1, \mathbb{R}^{2n}) \quad \textit{and} \quad \| j^*(y) \|_{H^1(\mathbb{S}^1)} \le \| y \|_{L^2(\mathbb{S}^1)} \,.$$

Proof. Let $x \in H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n}) \subset L^2(\mathbb{S}^1, \mathbb{R}^{2n})$, $y \in L^2(\mathbb{S}^1, \mathbb{R}^{2n})$. Then

$$\sum_{k\in\mathbb{Z}} \langle x_k, y_k\rangle = \langle x_0, j^*(y)_0\rangle + 2\pi \sum_{k\in\mathbb{Z}} |k| \langle x_k, j^*(y)_k\rangle \,.$$

So we get the following formula for j^*

$$j^*(y) = y_0 + \sum_{k\neq 0} \frac{1}{2\pi|k|} e^{2\pi k J_0 t} y_k \,.$$

The estimate $\| j^*(y) \|_{H^1(\mathbb{S}^1)} \le \| y \|_{L^2(\mathbb{S}^1)}$ is now obvious. □

2.2. The $H^{1/2}$-action gradient

In this section we show that our choice of the $H^{1/2}$-setup matches perfectly for the Hamiltonian action A_H in the sense that its gradient with respect to the $H^{1/2}$-inner product is well defined and smooth.

We recall that by (2.5) the action A_H on $\mathbb{M}$ gets the special form

$$A_H(x) = -\frac{1}{2}\left(\| \mathbb{P}^+ x \|^2_{H^{1/2}(\mathbb{S}^1)} - \| \mathbb{P}^- x \|^2_{H^{1/2}(\mathbb{S}^1)}\right) + \int_0^1 H(t, x(t))dt \,, \quad x \in \mathbb{M} \,.$$

and consider the second part $b : \mathbb{M} \longrightarrow \mathbb{R}$

$$b(x) = \int_0^1 H(t, x(t))dt\,.$$

THEOREM 2.2.1. ([**21**]) *b belongs to $C^\infty(\mathbb{M}, \mathbb{R})$.*

PROOF. We lift all data to $\mathbb{R}^{2n}$ and for $k \in \mathbb{N}$ we write the k-th Taylor expansion of H as

$$H(t, x + y) = H(t, x) + \sum_{i=0}^{k} \frac{1}{i!} h_i(t, x)y^i + o_k(y)\,, \quad \text{for all} \quad x, y \in \mathbb{R}^{2n}\,.$$

where

$$h_i(t, x)y^i := \sum_{|\alpha|=i} \frac{i!}{\alpha!} D_x^\alpha H(t, x)y^\alpha\,.$$

Now assume that $x, y \in \mathbb{E}$, then integrating over t yields to

$$b(x + y) = \sum_{i=0}^{k} \frac{1}{i!} b_i(t, x)y^i + r_k(x, y)\,, \tag{2.7}$$

with

$$b_i(t, x)y^i = \int_0^1 h_i(t, x(t))y(t)^i dt\,,$$

which by Hölder and the uniform bound on $\| H \|_{C^k(\mathbb{R}\times\mathbb{S}^1)}$ is a bounded i-linear form on the Hilbert space $\mathbb{E}$, independent of the chosen lift, i.e., it depends uniform continuously on $x \in \mathbb{M}$. Moreover we obtain $|r_k(x, y)| \le c_k \| y \|^k_{H^{1/2}(\mathbb{S}^1)}$ for every k which implies by the converse to Taylor's Theorem, see for instance [**9**], that b is smooth and possesses the expected derivatives. □

Denote by $\hat{b}$ the extension of b to $L^2(\mathbb{S}^1, \mathbb{T}^{2n}) = L^2(\mathbb{S}^1, \mathbb{R}^{2n})/\mathbb{Z}^{2n}$, i.e.,

$$\hat{b}(\cdot) = \int_0^1 H(t, \cdot)dt : L^2(\mathbb{S}^1, \mathbb{T}^{2n}) \longrightarrow \mathbb{R}\,.$$

Then we conclude.

COROLLARY 2.2.2. *$A_H : \mathbb{M} \longrightarrow \mathbb{R}$ is a smooth functional. Let $\mathbb{P}^\pm$ be the orthogonal projections from* (2.3) *then the gradient of A_H with respect to the inner product on $\mathbb{M}$ is given by*

$$\nabla_{1/2} A_H(x) = -\mathbb{P}^+ x + \mathbb{P}^- x + j^* \nabla H(\cdot, x(\cdot))\,, \quad x \in \mathbb{M}\,. \tag{2.8}$$

Moreover $\nabla_{1/2} A_H$ is Lipschitz-continuous on $\mathbb{M}$ with uniform Lipschitz constant c_L and its Jacobian is given by

$$\nabla^2_{1/2} A_H(x) = -\mathbb{P}^+ + \mathbb{P}^- + j^* \nabla^2 H(\cdot, x(\cdot))\,, \quad x \in \mathbb{M}\,. \tag{2.9}$$

PROOF. We lift all data to $\mathbb{R}^{2n}$ and use the inclusion j from (2.6) and write $b(x) = \hat{b}(j(x))$ for all $x \in \mathbb{E}$. Moreover we deduce from (2.7)

$$\hat{b}(x+y) = \hat{b}(x) + \int_0^1 \langle \nabla H(t, x(t)), y(t) \rangle \, dt + r(x, y)$$

with $|r(x,y)| \leq \| H \|_{C^2(\mathbb{S}^1 \times \mathbb{R}^{2n})} \| y \|^2_{L^2(\mathbb{S}^1)}$. Since $\| H \|_{C^2(\mathbb{S}^1 \times \mathbb{R}^{2n})}$ is uniform bounded $\hat{b}$ is differentiable with derivative at x given by

$$d\hat{b}(x)[y] = \int_0^1 \langle \nabla H(t, x(t)), y(t) \rangle \, dt = \langle \nabla H(\cdot, x), y \rangle_{L^2(\mathbb{S}^1)} = \left\langle \nabla \hat{b}(x), y \right\rangle_{L^2(\mathbb{S}^1)}$$

The derivative of b is given by

$$db(x)[y] = d\hat{b}(j(x))[j(y)] = \left\langle \nabla \hat{b}(j(x)), j(y) \right\rangle_{L^2(\mathbb{S}^1)} = \left\langle j^* \nabla \hat{b}(j(x)), y \right\rangle_{H^{1/2}(\mathbb{S}^1)} .$$

Hence

$$\nabla_{1/2} b(x) = j^* \nabla \hat{b}(j(x)) = j^* \nabla H(\cdot, x) , \qquad \forall x \in \mathbb{M}$$

and we obtain the claimed formulas. The smoothness property is a direct consequence of Theorem 2.2.1 and the Lipschitz-continuity can be seen as follows. Let $x, y \in \mathbb{E}$ be two lifted loops such that their constant parts $x_0, y_0 \in \mathbb{R}^{2n}$ lie in the same fundamental domain. Then

$$\begin{aligned}
\| \nabla_{1/2} A_H(x) - \nabla_{1/2} A_H(y) \|_{H^{1/2}(\mathbb{S}^1)} &= \| (\mathbb{P}^+ - \mathbb{P}^-)(x - y) - (j^* \nabla H(x) - j^* \nabla H(y)) \|_{H^{1/2}(\mathbb{S}^1)} \\
&\leq \| x - y \|_{H^{1/2}(\mathbb{S}^1)} + \| \nabla H(x) - \nabla H(y) \|_{L^2(\mathbb{S}^1)} \\
&\leq \| x - y \|_{H^{1/2}(\mathbb{S}^1)} + \| H \|_{C^2(\mathbb{S}^1 \times \mathbb{R}^{2n})} \| x - y \|_{L^2(\mathbb{S}^1)} \\
&\leq (1 + \| H \|_{C^2(\mathbb{S}^1 \times \mathbb{R}^{2n})}) \| x - y \|_{H^{1/2}(\mathbb{S}^1)} ,
\end{aligned}$$

where $c_L := 1 + \| H \|_{C^2(\mathbb{S}^1 \times \mathbb{R}^{2n})}$ is by the periodicity of H uniformly bounded. □

We set $\mathbb{X} := -\nabla_{1/2} A_H$ then the previous result implies that the **negative gradient-flow** associated to $\mathbb{X}$ as usual as the solution of

$$\partial_s \varphi_{\mathbb{X}}(s, x) = (\mathbb{X} \circ \varphi_{\mathbb{X}})(s, x) , \quad \varphi_{\mathbb{X}}(0, x) = x , \quad \text{for all} \quad x \in \mathbb{M} , \tag{2.10}$$

is uniquely determined, smooth and globally defined.

LEMMA 2.2.3. (**[21]**) *Let $x \in \mathbb{M}$ be a critical point of A_H then $x \in C^\infty(\mathbb{S}^1, \mathbb{T}^{2n})$. Moreover, it solves the Hamilton equation*

$$\dot{x}(t) = J_0 \nabla H(t, x(t)) , \quad \forall t \in \mathbb{S}^1 .$$

PROOF. We write x and $\nabla H(\cdot, x)$ as Fourier series

$$x(t) = x_0 + \sum_{k \neq 0} e^{2\pi J_0 k t} x_k , \qquad \nabla H(t, x(t)) = \sum_{k \in \mathbb{Z}} e^{2\pi J_0 k t} h_k , \qquad x_0 \in \mathbb{T}^{2n} , x_k, h_k \in \mathbb{R}^{2n} .$$

Let $y \in H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n})$ then by assumption and (2.8) we obtain

$$\langle (\mathbb{P}^+ - \mathbb{P}^-) x, y \rangle_{H^{1/2}(\mathbb{S}^1)} = \langle j^* \nabla H(\cdot, x), y \rangle_{H^{1/2}(\mathbb{S}^1)} = \langle \nabla H(\cdot, x), y \rangle_{L^2(\mathbb{S}^1)} .$$

Choosing the test functions $y(t) = e^{2\pi J_0 kt} e_i$, where e_i, $i = 1, \dots, 2n$ denotes the standard basis of $\mathbb{R}^{2n}$ we find

$$2\pi k x_k = h_k\,, \quad k \in \mathbb{Z} \setminus \{0\}\,. \tag{2.11}$$

and $h_0 = 0$. Since $\nabla H(\cdot, x) \in L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ we obtain

$$\sum_{k \neq 0} |k|^2 |x_k|^2 < \infty$$

i.e., $x \in H^1(\mathbb{S}^1, \mathbb{T}^{2n})$. By Proposition 2.1.2 this implies that x and consequently $\nabla H(\cdot, x)$ belong to C^0. Using (2.11) we realize that for

$$z(t) = \int_0^t J_0 \nabla H(\tau, x(\tau)) d\tau = \int_0^t \dot{x}(\tau) d\tau$$

we have $z(t) = x(t) - x(0) \in C^1(\mathbb{S}^1, \mathbb{T}^{2n})$. Hence

$$\dot{x}(t) = J_0 \nabla H(t, x(t))\,.$$

Now $J_0 \nabla H(t, x(t))$ is of class C^1 therefore x is of class C^2. Iterating this argument leads to $x \in C^\infty(\mathbb{S}^1, \mathbb{T}^{2n})$. □

Definition 2.2.4. *Let X be a C^1-vector field on a Hilbert manifold M. We denote by* $\operatorname{sing}(X) := \left\{ x \in M \;\middle|\; X(x) = 0 \right\}$ *the set of all* **singular points** *of X. Then $x \in \operatorname{sing}(X)$ is called* **hyperbolic** *if and only if* $\operatorname{spec}(DX(x)) \cap i\mathbb{R} = \emptyset$. *$X$ is called a* **Morse vector field** *if all singular points are hyperbolic.*

If $DX(x) \in \mathcal{L}(E)$ is self-adjoint then the spectrum of $DX(x)$ splits into the disjoint sets

$$\operatorname{spec}^+(DX(x)) = \left\{ \lambda \in \operatorname{spec}(DX(x)) \;\middle|\; \lambda > 0 \right\}, \ \operatorname{spec}^-(DX(x)) = \left\{ \lambda \in \operatorname{spec}(DX(x)) \;\middle|\; \lambda < 0 \right\}.$$

and by the Spectral Theorem we have that $E = E^u \oplus E^s$ splits into the **positive (or unstable)** *and* **negative (or stable) eigenspaces** *of $DX(x)$. Often such operators are also called* **hyperbolic**. *A notion of non-self-adjoint hyperbolic operators can be found again in* [**6**].
If there is $f \in C^1(M, \mathbb{R})$ such that

$$Df(p)[X(p)] < 0\,, \quad \textit{for all } p \in M \setminus \operatorname{sing}(X)$$

then f is called a **Lyapunov function** *for X. If X is a Morse vector field then f is called* **non-degenerate** *if f is twice differentiable on* $\operatorname{sing}(X)$ *and $D^2 f(x)$, seen as a symmetric bounded bilinear form on E, is such that there is $\varepsilon > 0$ with $D^2 f(x)[\xi, \xi] \geq \varepsilon \|\xi\|^2$ for all $\xi \in E^s(x)$ and $D^2 f(x)[\xi, \xi] \leq -\varepsilon \|\xi\|^2$ for all $\xi \in E^u(x)$. The Morse vector field X is called* **gradient-like** *if it has a non-degenerate Lyapunov function.*

Clearly the set of critical points of f coincides with $\operatorname{sing}(X)$ if X is the positive or negative gradient of f. If X is a Morse vector field then $\operatorname{crit}(f) = \operatorname{sing}(X)$ even holds if f is just a Lyapunov function for X, see [**6**]. Moreover all **singular points are isolated** in this case. That is due to the fact that $DX(x)$ is invertible, combined with the inverse function theorem, see for example [**25**].

Proposition 2.2.5. *There is a residual set $\mathcal{H}_{\text{reg}} \subset C^\infty(\mathbb{S}^1 \times \mathbb{T}^{2n}, \mathbb{R})$ of Hamiltonians such that the negative $H^{1/2}$-gradient $\mathbb{X}$ of the Hamiltonian action A_H is a Morse vector field for every $H \in \mathcal{H}_{\text{reg}}$. In particular the set of all contractible critical points of A_H, denoted by $\mathcal{P}_0(H)$, is a finite set.*

This statement is based on the famous Theorem of Sard whose infinite dimensional version was proven by Smale in [**36**] we want to cite here.

Theorem 2.2.6. ([**36**]) **(Sard-Smale Theorem)** *Let X and Y be separable Banach spaces and $U \subset X$ be an open set. Suppose that $f : U \longrightarrow Y$ is a Fredholm map of class C^l, where*

$$l \geq \max\{1, \operatorname{ind}(f) + 1\} .$$

Then the set

$$Y_{\text{reg}}(f) := \left\{ y \in Y \;\middle|\; x \in U, f(x) = y \Longrightarrow \operatorname{Im}(df(x)) = Y \right\}$$

of regular values of f is residual in Y in the sense of Baire, i.e., a countable intersection of open an dense sets.

Now we prove Proposition 2.2.5.

Proof. By Lemma 2.2.3 all critical points $x \in \operatorname{crit}(A_H)$ are smooth and surely the same argument holds for the linearization $\nabla^2_{1/2} A_H(x)$, i.e., for all for $\xi \in H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n})$ there holds

$$\nabla^2_{1/2} A_H(x)\xi = 0 \quad \Longleftrightarrow \quad \xi \in C^\infty(\mathbb{S}^1, \mathbb{R}^{2n}) \quad \text{and} \quad \dot{\xi}(t) = J_0 \nabla^2 H(t, x(t))\xi(t)\,, \forall t \in \mathbb{S}^1 \,. \tag{2.12}$$

Saying that the condition on $D\mathbb{X}(x) = -\nabla^2_{1/2} A_H(x)$, to be hyperbolic is equivalent to the nondegeneracy condition formulated in Hamiltonian dynamics, i.e., the linearized Hamiltonian flow at x possesses no 1-periodic orbits meaning that there are no Floquet-multipliers 1. Now it is a well-known fact that this nondegeneracy condition is achievable for a residual set of Hamiltonians, see for instance [**20**]. If we finally choose a generic Hamiltonian $H \in \mathcal{H}_{\text{reg}}$, then the set of contractible critical points $\mathcal{P}_0(H)$ of A_H is a family of Hamiltonian orbits of a discrete set of points in $\mathbb{T}^{2n}$. Hence $\mathcal{P}_0(H)$ has to be finite due the compactness of $\mathbb{T}^{2n}$. □

2.3. Fredholm pairs and relative dimensions

In contrast to the case of a smooth Morse function on a finite dimensional manifold, as it will turn out, we have to deal with the **strongly indefinite** functional A_H defined on the Hilbert manifold $\mathbb{M}$, i.e., A_H possesses infinite Morse indices and co-indices. A. Abbondandolo an P. Majer developed methods in [**6**] which enable them to construct an infinite dimensional Morse theory for such functionals which we want to apply here.

We start with some facts about linear Cauchy-problems discussed in [**4**]. Let

$$A : [-\infty, \infty] \longrightarrow \mathcal{L}(E)$$

be a continuous path of bounded linear operators on a Banach space E, such that $A(\pm\infty)$ are hyperbolic. We denote by

$$X_A(s) : (-\infty, \infty) \longrightarrow \mathcal{L}(E)$$

the solution of the linear Cauchy-problem

$$\begin{cases} X_A'(s) &= A(s)X_A(s) \\ X_A(0) &= I \end{cases} .$$

Note that $X_A(s)$ is an isomorphism for every s. The inverse $Y(s) := X_A(s)^{-1}$ solves the linear Cauchy-problem

$$\begin{cases} Y(s) &= -Y(s)A(s) \\ Y(0) &= I \end{cases} .$$

We define the **linear, unstable** and **stable** subspaces of E by

$$\begin{aligned} W_A^u &= \left\{\xi \in E \;\Big|\; \lim_{s\to-\infty} X_A(s)\xi = 0\right\} \\ W_A^s &= \left\{\xi \in E \;\Big|\; \lim_{s\to+\infty} X_A(s)\xi = 0\right\} . \end{aligned} \tag{2.13}$$

If in particular $A \equiv L$ is constant and hyperbolic then $W_L^u = E^u(L)$ and $W_L^s = E^s(L)$ are direct complements, i.e., $W_L^u + W_L^s = E$ and $W_L^u \cap W_L^s = \{0\}$. In general one can show that $W_A^u \cong E^u(A(-\infty))$ and $W_A^s \cong E^s(A(+\infty))$. Indeed if A is sufficiently close to $A(+\infty)$ in the L^∞-norm, then W_A^s is the graph of a bounded operator from $E^s(A(+\infty))$ to $E^u(A(+\infty))$. The statement for general asymptotically hyperbolic operators then follows from the identity

$$W_A^s = X_A(T)^{-1} W^s_{A(T+\cdot)}\,, \quad T \in \mathbb{R}\,. \tag{2.14}$$

For details, see [**4**].

Proposition 2.3.1. ([**6**]) *Denote by $C_0^k([0,+\infty), E)$ the space of all C^k-curves ξ which satisfy $\lim_{s\to+\infty} \xi^{(l)}(s) = 0$ for all $0 \le l \le k$ and let $A \in C^0([0,+\infty], \mathcal{L}(E))$ be such that $A(+\infty)$ is self-adjoint and hyperbolic.*

(*i*) *The bounded linear operator*

$$F_A^+ : C_0^1([0,+\infty), E) \longrightarrow C_0^0([0,+\infty), E)\,, \quad \xi \mapsto \xi' - A\xi\,,$$

is a left inverse. Moreover F_A^+ admits a right inverse R_A^+ such that

$$W_A^s + \left\{R_A^+(\eta)(0) \;\Big|\; \eta \in C_0^0([0,+\infty), E)\,, \eta(0) = 0\right\} = E \tag{2.15}$$

(*ii*) *The evaluation map*

$$\ker F_A^+ \longrightarrow E\,, \xi \mapsto \xi(0)$$

is a right inverse. Consequently if $A \in C^0([-\infty,0],\mathcal{L}(E))$ and $A(-\infty)$ is hyperbolic. Then

(iii)

$$F_A^- : C_0^1((-\infty,0],E) \longrightarrow C_0^0((-\infty,0]),E)\,,\quad \xi \mapsto \xi' - A\xi\,,$$

is a left inverse with right inverse R_A^- such that

$$W_A^u + \left\{R_A^-(\eta)(0) \;\middle|\; \eta \in C_0^0((-\infty,0],E)\,, \eta(0) = 0\right\} = E \tag{2.16}$$

(iv)

$$\ker F_A^- \longrightarrow E\,, \xi \mapsto \xi(0)$$

is a right inverse.

Proof. (i) Assume that the path $A(s) \equiv L$ is a constant hyperbolic operator. Consider the operator

$$K : \mathbb{R} \longrightarrow \mathcal{L}(E)\,,\quad K(s) = \begin{cases} e^{sL}P^s & , \quad s \geq 0 \\ -e^{sL}P^u & , \quad s \leq 0 \end{cases}\,,$$

which since L is hyperbolic indeed satisfies $\|K(s)\| \leq e^{-\lambda s}$ for $\lambda \in (0,\|L^{-1}\|)$. Let $\eta \in C_0^0([0,+\infty),E)$, we set

$$\xi(s) := (K * \eta)(s) = \int_0^\infty K(s-\tau)\eta(\tau)d\tau\,.$$

Then the estimate

$$\|\xi(s)\| \leq \|K\|_{L^1(\mathbb{R},\mathcal{L}(E))}\|\eta\|_{C^0([T,\infty),E)} + \|K\|_{L^1([s-T,s],\mathcal{L}(E))}\|\eta\|_{C^0([0,\infty),E)} \tag{2.17}$$

shows that $\xi \in C_0^0([0,+\infty),E)$. Furthermore ξ is continuously differentiable and solves

$$\xi'(s) - L\xi(s) = \eta(s)\,, \tag{2.18}$$

which implies that $\xi \in C_0^1([0,+\infty),E)$. Hence the operator

$$R_L^+ : C_0^0([0,+\infty),E) \longrightarrow C_0^1([0,+\infty),E)\,, \eta \longrightarrow K * \eta\,,$$

is a right inverse for F_L^+ which by (2.18) and (2.17) with $T = 0$ is indded a continuous linear operator. Now we consider the general case of a non-constant path. We set $A_T(s) = A(T+s)$ and obtain that $F_{A_T}^+ \longrightarrow F_L^+$ uniform in T with respect to the operator topology. Since the set of left inverses is open, there is $T > 0$ such that

$$F_{A_T}^+ : C_0^1([0,+\infty),E) \longrightarrow C_0^0([0,+\infty),E)$$

is a left inverse possessing a right inverse

$$R_{A_T}^+ : C_0^0([0,+\infty),E) \longrightarrow C_0^1([0,+\infty),E)\,,$$

which also converges uniform in T to R_L^+. Now we can define a right inverse for F_A^+ by setting $\xi(s) := R_A^+(\eta)(s)$ as the solution of the linear Cauchy-problem

$$\begin{cases} \xi'(s) - A(s)\xi(s) &= \eta(s)\,, \quad s \in [0,T] \\ \xi(T) &= R_{A_T}^+(\eta)(0) \end{cases} .$$

That R_A^+ is continuous is seen by the formula

$$R_A^+(\eta)(s) = X_A(s)\left(X_A(T)^{-1}R_{A_T}^+(\eta)(0) + \int_T^s X_A(\tau)^{-1}\eta(\tau)d\tau\right).$$

To prove (2.15) we first show that the asymptotic right inverse R_L^+ maps $\eta \in C_c^0((0,\infty),E)$ to a curve $\xi = R_L^+(\eta)$ with $\xi(0) \in E^u(L)$. Since $E^u(L)$ is a direct complement of $E^s(L) = W_L^s$ this implies (2.15). We choose a cut-off function β with small support in $(0,\infty)$ such that the operator

$$Q := \int_0^{+\infty} \beta(s)e^{-sL}ds \in \mathcal{L}(E)$$

is an isomorphism. Since Q preserves the splitting $E = E^u(L) \oplus E^s(L)$ we obtain by choosing $v \in E^u(L)$ and setting $\eta(s) := -\beta(s)Q^{-1}(v)$ that

$$R_L^+(\eta)(0) = \int_0^{+\infty} e^{-\tau L}P^u\beta(\tau)Q^{-1}(v)d\tau = \left(\int_0^{+\infty}\beta(\tau)e^{-\tau L}P^u d\tau\right)Q^{-1}(v) = v\,.$$

Hence R_L^+ satisfies (2.15) and so, since the space of surjective operators is open, does $R_{A_T}^+$ as long as T is chosen large enough. We show that in this case already R_A^+ satisfies (2.15). Indeed let $v \in E$ and $\eta \in C_0^0((0,\infty),E)$ such that

$$X_A(T)v \in R_{A_T}^+\eta(0) + W_{A_T}^s\,.$$

Since $\eta(0) = 0$ the curve

$$\rho(s) = \begin{cases} \eta(T-s) & , \quad s \geq T \\ 0 & , \quad 0 \leq s \leq T \end{cases}$$

belongs to $C_0^0([0,\infty),E)$ and $\kappa := R_A^+(\rho)$ solves $\kappa'(s) - A\kappa(s) = \rho(s) = 0$ on $[0,T]$. Hence

$$R_{A_T}^+(\eta)(0) = R_A^+(\rho)(T) = X_A(T)\kappa(0)$$

and therefore

$$X_A(T)v \in R_{A_T}^+\eta(0) + W_{A_T}^s \iff v \in \kappa(0) + X_A(T)^{-1}W_{A_T}^s = R_A^+(\rho)(0) + W_A^s$$

proving (i).

(ii) The kernel of F_A^+ is given by

$$\ker F_A^+ = \left\{X_A(s)v \;\middle|\; v \in W_A^s\right\},$$

so the operator

$$E \longrightarrow \ker F_A^+ \,, w \mapsto X_A(\cdot) P_{W_A^s} w \,,$$

where $P_{W_A^s}$ denotes the projection onto W_A^s, is a left inverse of the evaluation at 0,

$$\ker F_A^+ \longrightarrow E \,, \xi \mapsto \xi(0) \,.$$

(*iii*) and (*iv*) now follow analogous. □

We need a further statement of the operator F_A on the whole real line.

Lemma 2.3.2. ([**6**]) *Let $A : [-\infty, +\infty] \longrightarrow \mathcal{L}(E)$ be a path of bounded linear operators with hyperbolic asymptotics $A(\pm\infty)$. Denote by F_A the bounded linear operator*

$$F_A : C_0^1(\mathbb{R}, E) \longrightarrow C_0^0(\mathbb{R}, E) \,, \quad \xi \mapsto \xi' - A\xi \,.$$

Then

$$\ker F_A \cong W_A^u \cap W_A^s \quad \textit{and} \quad \operatorname{coker} F_A \cong E/(W_A^u + W_A^s) \,.$$

Proof. The kernel of F_A is the linear subspace

$$\ker F_A = \left\{ X_A(s)v \;\middle|\; v \in W_A^u \cap W_A^s \right\} \subset C_0^1(\mathbb{R}, E) \,.$$

which is therefore by the second statement of Proposition 2.3.1 isomorphic to $W_A^u \cap W_A^s$ via the evaluation map $X_A(s)v \mapsto X_A(0)v = v$. By the first statement the operators

$$\begin{aligned} F_A^+ : C_0^1([0, +\infty), E) &\longrightarrow C_0^0([0, +\infty), E) \,, \quad \xi \mapsto \xi' - A\xi \,, \\ F_A^- : C_0^1((-\infty, 0], E) &\longrightarrow C_0^0((-\infty, 0], E) \,, \quad \xi \mapsto \xi' - A\xi \,, \end{aligned}$$

have right inverses R_A^+ and R_A^-. If $\eta \in C_0^0(\mathbb{R}, E)$ and ξ solves $\xi' - A\xi = \eta$, then ξ is of the form

$$\begin{aligned} \xi(s) &= X_A(s)\Big(\xi(0) - R_A^+(\eta)(0)\Big) + R_A^+(\eta)(s) \,, \quad \text{for} \quad s \geq 0 \,, \\ \xi(s) &= X_A(s)\Big(\xi(0) - R_A^-(\eta)(0)\Big) + R_A^-(\eta)(s) \,, \quad \text{for} \quad s \leq 0 \,. \end{aligned}$$

If we require that $\xi \in C_0^1(\mathbb{R}, E)$, then $\xi(0)$ has to be such that

$$\xi(0) - R_A^+(\eta)(0) \in W_A^s \quad \text{and} \quad \xi(0) - R_A^-(\eta)(0) \in W_A^u \,.$$

In other words, η belongs to the range of F_A if and only if the affine subspaces $R_A^+(\eta)(0) + W_A^s$ and $R_A^-(\eta)(0) + W_A^u$ have a non-empty intersection, that is, if and only if $R_A^+(\eta)(0) - R_A^-(\eta)(0) \in W_A^s + W_A^u$. So the range of F_A is the linear subspace

$$\operatorname{ran} F_A = \left\{ \eta \in C_0^0(\mathbb{R}, E) \;\middle|\; R_A^+(\eta)(0) - R_A^-(\eta)(0) \in W_A^s + W_A^u \right\} \,.$$

By the remark in (2.14) the spaces W_A^s and W_A^u are closed and so is $\operatorname{ran} F_A$ by the continuity of R_A^+, R_A^-. Furthermore due to the properties (2.15) and (2.16) the operator

$$C_0^0(\mathbb{R}, E) \longrightarrow E/(W_A^s + W_A^u) \,, \quad \eta \mapsto [R_A^+(\eta)(0) - R_A^-(\eta)(0)] \,.$$

is onto and factors over $\operatorname{ran} F_A$, proving the isomorphism. □

Clearly, we want to show that the operator F_A is a Fredholm operator, but a priori there is no reason why $\ker F_A$ and $\operatorname{coker} F_A$ should be finite dimensional. In particular, this will not be true in general. To formulate the correct additional conditions, we use the notion of Fredholm pairs which we briefly introduce next. For a deeper presentation see [**7**].

We consider the **Hilbert Grassmannian** $\mathrm{Gr}(E)$ of a separable Hilbert space E, which is the set of all closed subspaces of E. For $V \in \mathrm{Gr}(E)$, we denote by P_V the orthogonal projection onto V. The distance between two subspaces $V, W \in \mathrm{Gr}(E)$ shall be defined by

$$\operatorname{dist}(V, W) := \| P_V - P_W \| .$$

Indeed $(\mathrm{Gr}(E), \operatorname{dist}(\cdot,\cdot))$ becomes a complete metric space with connected components

$$\mathrm{Gr}_{n,m}(E) = \left\{ V \in \mathrm{Gr}(E) \;\middle|\; \dim V = n\,,\ \operatorname{codim} V = m \right\}, \quad n, m \in \mathbb{N} \cup \{\infty\}\,,\ n + m = \infty\,.$$

DEFINITION 2.3.3. *A pair $(V, W) \in \mathrm{Gr}(E) \times \mathrm{Gr}(E)$ is called a* **Fredholm pair** *if $V \cap W$ is finite-dimensional and $V + W$ is closed and finite-codimensional. In this case the index*

$$\operatorname{ind}(V, W) := \dim (V \cap W) - \operatorname{codim} (V + W)$$

is called the **Fredholm index** *of (V, W).*

In particular an Operator $F : X \longrightarrow Y$ between Hilbert spaces is Fredholm if and only if $(\operatorname{graph} F, X \times \{0\})$ is a Fredholm pair in $\mathrm{Gr}(X \times Y) \times \mathrm{Gr}(X \times Y)$. The set of all Fredholm pairs, denoted by $\mathrm{Fp}(E)$, is an open subset of $\mathrm{Gr}(E) \times \mathrm{Gr}(E)$ and the Fredholm index is a continuous, i.e., locally constant function on it. For details see [**22**].

DEFINITION 2.3.4. *Let $V, W \in \mathrm{Gr}(E)$. Then V is called a* **compact perturbation** *of W if the operator $P_V - P_W$ is compact. In this case the pair $(V, W^{\perp})$ is Fredholm and its index is defined as the* **relative dimension** *of V with respect to W, i.e.,*

$$\dim(V, W) := \operatorname{ind}(V, W^{\perp}) = \dim\left(V \cap W^{\perp}\right) - \dim\left(V^{\perp} \cap W\right). \tag{2.19}$$

If (V, W) is a Fredholm pair and Z is a compact perturbation of V, then (Z, W) is still a Fredholm pair and its index is given by

$$\operatorname{ind}(Z, W) = \dim(Z, V) + \operatorname{ind}(V, W)\,. \tag{2.20}$$

PROPOSITION 2.3.5. ([**3**]) *Let F_1 and F_2 be two self-adjoint Fredholm operators on the Hilbert space E with $F_1 - F_2$ compact. Then the positive and negative eigenspaces of F_1 are compact perturbations of the positive and negative eigenspaces of F_2, respectively.*

PROOF. The operator $p(F_1) - p(F_2)$ is compact for any polynomial p, for it belongs to the two-sided ideal spanned by $F_1 - F_2$. By density, $h(F_1) - h(F_2)$ is then compact for any continuous function on $\operatorname{spec}(F_1) \cup \operatorname{spec}(F_2) \subset \mathbb{R}$. Since F_1 and F_2 are self-adjoint, 0 is not an accumulation point of $\operatorname{spec}(F_1) \cup \operatorname{spec}(F_2)$. Hence $\chi_{\mathbb{R}^+}$ and $\chi_{\mathbb{R}^-}$ are continuous functions on $\operatorname{spec}(F_1) \cup \operatorname{spec}(F_2)$. Then $P_{E^+(F_1)} - P_{E^+(F_2)} = \chi_{\mathbb{R}^+}(F_1) - \chi_{\mathbb{R}^+}(F_2)$ and $P_{E^-(F_1)} - P_{E^-(F_2)} = \chi_{\mathbb{R}^-}(F_1) - \chi_{\mathbb{R}^-}(F_2)$ are compact operators as claimed. □

No we are able to study our non-linear situation, using the introduced concepts for linear Cauchy-problems, as follows.

Definition 2.3.6. *Let $x \in M$ be a singular point of a C^1-vector field X on a Banach manifold M with globally defined flow φ_X. Then we define the* **unstable** *and* **stable manifolds** *of x by*

$$\begin{aligned} \mathcal{W}^u(x) &= \left\{ p \in M \mid \lim_{s \to -\infty} \varphi_X(s,p) = x \right\} \\ \mathcal{W}^s(x) &= \left\{ p \in M \mid \lim_{s \to +\infty} \varphi_X(s,p) = x \right\}, \end{aligned}$$

The generalization of the usual **stable / unstable manifold theorem** in this situation is explicitly proven in [**6**]. We only mention it :

Theorem 2.3.7. ([**6**]) *Let $x \in M$ be a hyperbolic singular point of a C^k-vector field X, $k \geq 1$, on a Banach manifold M with globally defined flow. Then $\mathcal{W}^u(x)$ and $\mathcal{W}^s(x)$ are the images of injective C^k-immersions of manifolds which are homeomorphic to the unstable and stable eigenspaces E^u_x and E^s_x of the Jacobien of X at x, respectively.*

If M is a Hilbert manifold then $\mathcal{W}^u(x)$ and $\mathcal{W}^s(x)$ are actually images of C^k-injective immersions of E^u_x and E^s_x, respectively. If in addition X possesses a Lyapunov function f, non-degenerate in x, then these immersions are actually embeddings.

Let us recall some basic definitions and facts appearing in a Banach manifold setting. In [**9**] these concepts are described in detail.

Two linear subspaces V, W of a Banch space E **intersect transversal** if $V + W = E$ and $V \cap W$ is complemented in E. Two C^1- submanifolds M_1, M_2 of a Banach manifold M are called **transverse** if the closed subspaces T_pM_1 and T_pM_2 are transverse for all $p \in M_1 \cap M_2$.

Let $\phi : M \longrightarrow N$ be a C^k-map, $k \geq 1$, between Banach manifolds. Then $q \in N$ is called a **regular value** for ϕ if for all $p \in \phi^{-1}(q)$ the differential $D\phi(p) : T_pM \longrightarrow T_qN$ is a left inverse, i.e., if it is onto and possesses a complemented kernel. In this case the implicit function theorem holds and $\phi^{-1}(q)$ is a submanifold of M of class C^k. ϕ is called a **Fredholm map** if its differential at every point is a Fredholm operator. If in addition the index of the differential is constant, which is the case if for instance M is connected, this integer can be defined as the **Fredholm index** of ϕ. If q is a regular value of ϕ then $\phi^{-1}(q)$ is actually a finite dimensional C^k-submanifold possessing the dimension of its index.

Though in general $\mathcal{W}^u(x)$ and $\mathcal{W}^s(x)$ are infinite-dimensional manifolds, one can hope that for a generic choice of vector fields X or metrics on M their intersection is a **finite-dimensional** manifold. A. Abbondandolo and P. Majer formulated two conditions which guarantee that we are in this situation. We give a version of these conditions matching our setup.

Definition 2.3.8. *Let X be a C^1- Morse vector field on the Hilbert manifold $\mathbb{M} = H^{1/2}(\mathbb{S}^1, \mathbb{T}^{2n})$, $V \subset \mathbb{E}$ a closed subspace and $\mathcal{V} = \mathbb{M} \times V$ a constant subbundle of $T\,\mathbb{M}$. Then $\mathcal{V}$ is* **admissible** *for X if*

(C1) *for every singular point $x \in \operatorname{sing}(X)$, the unstable eigenspace $E^u\,(DX(x))$ of the Jacobian of X at x is a compact perturbation of V;*

(C2) *for all $p \in M$ the operator $[DX(p), P_V]$ is compact, where $[S, T] = ST - TS$, $\forall\, S, T \in \mathcal{L}(\mathbb{E})$ denotes the commutator and P_V the projection onto V.*

By **C1** we can define the **relative Morse index** of $x \in \operatorname{sing}(X)$ relative to $\mathcal{V}$ by

$$m(x, \mathcal{V}) := \dim\Big(E^u(DX(x)), V\Big). \tag{2.21}$$

Furthermore the subbundle $\mathcal{V}$ is φ_X-invariant, i.e., $D_2\varphi_X(s, p)V = V$ for all $(s, p) \in \mathbb{R} \times \mathbb{M}$ if and only if $[DX(p), P_V] = 0$. Hence **C2** is equivalent to the fact that $\mathcal{V}$ is **essentially** φ_X-invariant in the sense that $D_2\varphi_X(s, p)V$ is a compact perturbation of V for all $(s, p) \in \mathbb{R} \times \mathbb{M}$.

Lemma 2.3.9. *Let $H \in \mathcal{H}_{\text{reg}}$ be a generic Hamiltonian in the sense of Proposition 2.2.5 and $\mathbb{X} = -\nabla_{1/2} A_H$ which is therefore a smooth Morse vector field on $\mathbb{M}$. Then the subbundle*

$$\mathbb{M} \times (\mathbb{R}^n \times \mathbb{H}^+) \subset T\,\mathbb{M}$$

is admissible for $\mathbb{X}$.

Proof. Since $\mathbb{X}$ is assumed to be Morse, the operator $D\mathbb{X}(x)$ is hyperbolic for all $x \in \operatorname{sing}(\mathbb{X})$. Hence its spectrum splits into the disjoint sets $\operatorname{spec}^+(D\mathbb{X}(x))$ and $\operatorname{spec}^-(D\mathbb{X}(x))$ of positive and negative eigenvalues. Furthermore $D\mathbb{X}(x)$ satisfies the estimate

$$\begin{aligned}
\|\xi\|_{H^{1/2}(\mathbb{S}^1)} &= \|\mathbb{P}^0\xi + \mathbb{P}^+\xi - \mathbb{P}^-\xi - j^*\nabla^2 H(\cdot, x(\cdot))\xi + j^*\nabla^2 H(\cdot, x(\cdot))\xi\|_{H^{1/2}(\mathbb{S}^1)} \\
&\le \|D\mathbb{X}(x)\xi\|_{H^{1/2}(\mathbb{S}^1)} + \|j^*\mathbb{P}^0\xi + j^*\nabla^2 H(\cdot, x(\cdot)\xi)\|_{H^{1/2}(\mathbb{S}^1)} \\
&\le \|D\mathbb{X}(x)\xi\|_{H^{1/2}(\mathbb{S}^1)} + (1 + \|H\|_{C^2(\mathbb{S}^1\times\mathbb{T}^{2n})})\|\xi\|_{L^2(\mathbb{S}^1)}
\end{aligned}$$

for all $\xi \in \mathbb{E}$. Now recall that the inclusion $j : H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ from (2.6) is compact. Hence the above estimate implies that $D\mathbb{X}(x)$ is a semi-Fredholm operator as it is well-known in Fredholm analysis, see for instance [**33**]. Since $D\mathbb{X}(x)$ is moreover self-adjoint, kernel and co-kernel coincide and therefore $D\mathbb{X}(x)$ is indeed Fredholm of index zero. In particular 0 is not an accumulation point in $\operatorname{spec}(D\mathbb{X}(x))$. So setting $S_1 = D\mathbb{X}(p)\mathbb{P}^+$ and $S_2 = \mathbb{P}^+ D\mathbb{X}(p)$ for $p \in \mathbb{M}$ we have that

$$[D\mathbb{X}(p), \mathbb{P}^+] = S_1 - S_2 = j^*\nabla^2 H(\cdot, p)\mathbb{P}^+ - \mathbb{P}^+ j^*\nabla^2 H(\cdot, p)$$

is compact, which shows **C2**. For $p = x$ we can apply Proposition 2.3.5 to $F_1 = D\mathbb{X}(x)$, $F_2 = \mathbb{P}^+ - \mathbb{P}^-$ and obtain that the positive eigenspace $E^u(D\mathbb{X}(x))$ is a compact perturbation of $\mathbb{H}^+$

and therefore of $\mathbb{R}^n \times \mathbb{H}^+$ proving **C1** .

$\square$

Note that since $I - \mathbb{P}^-$ differs from $\mathbb{P}^+$ by an operator of finite rank, we have that $[D\mathbb{X}(p), \mathbb{P}^-]$ is also compact for all $p \in \mathbb{M}$. In particular this shows that $E^s(D\mathbb{X}(x))$ is a compact perturbation of $\mathbb{H}^-$ for all $x \in \mathrm{sing}(\mathbb{X})$. Since both spaces $\mathbb{H}^+$ and $\mathbb{H}^-$ are infinite dimensional this implies that $E^u(D\mathbb{X}(x))$ and $E^s(D\mathbb{X}(x))$ are infinite dimensional for any $x \in \mathrm{sing}(\mathbb{X})$, which in particular shows the **strongly indefinite character** of A_H, i.e., the usual Morse indices and co-indices are infinite for all singular points implying that by Theorem 2.3.7 the stable and unstable manifolds are infinite dimensional. Further crucial consequences of the two conditions are the following.

Proposition 2.3.10. ([**6**]) *Assume that X is a C^1-Morse vector field defined on the Hilbert manifold $\mathbb{M} = \mathbb{T}^{2n} \times \mathbb{H}$, possesses a Lyapunov function f and an admissible constant subbundle $\mathcal{V} = \mathbb{M} \times V \subset \mathbb{M} \times \mathbb{E}$. Then for every $x \in \mathrm{sing}(X)$ there holds :*

(i) *for every $p \in \mathcal{W}^u(x)$, $T_p\mathcal{W}^u(x)$ is a compact perturbation of V with*

$$\dim(T_p\mathcal{W}^u(x), V) = m(x, \mathcal{V}) .$$

(ii) *for every $p \in \mathcal{W}^s(x)$, $\big(T_p\mathcal{W}^s(x), V\big)$ is a Fredholm pair, with*

$$\mathrm{ind}\,(T_p\mathcal{W}^s(x), V) = -m(x, \mathcal{V}) .$$

(iii) *Let $x, y \in \mathrm{sing}(X)$ and assume that $\mathcal{W}^u(x)$ and $\mathcal{W}^s(y)$ meet transversally. Then $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ is a submanifold of dimension*

$$\dim\,(\mathcal{W}^u(x) \cap \mathcal{W}^s(y)) = m(x, \mathcal{V}) - m(y, \mathcal{V}) .$$

Proof. Let $p \in \mathcal{W}^u(x)$ and let $v(s) = \varphi_X(s, p)$ be the orbit of p. By linearization, using the notation from (2.13), we obtain

$$T_p\mathcal{W}^u(x) = W^u_A \tag{2.22}$$

with $A(s) = DX(v(s))$. **C1** implies that $W := T_x\mathcal{W}^u(x) = E^u(A(-\infty))$ is a compact perturbation of V. By **C2**, the operator $[A(s), P_V]$ is compact for every s, and so is the operator $[A(s), P_W]$. Set

$$B(s) = A(s) - [A(s), P_W] .$$

Then $B(-\infty) = A(-\infty) = DX(x)$, $E^u(B(-\infty)) = W$,

$$P_{W^\perp}B(s) = P_{W^\perp}(A(s) - A(s)P_W) = P_{W^\perp}A(s)P_{W^\perp}$$

and therefore $B(s)W \subset W$ for every s. It is readily seen that these facts imply that

$$W^u_B = W .$$

Furthermore $B(s) - A(s)$ is compact for every s saying that W^u_A is a compact perturbation of $W^u_B = W$ and therefore of V. So we conclude by (2.22) that $T_p\mathcal{W}^u(x)$ is a compact perturbation of V for all $p \in \mathcal{W}^u(x)$. The dimension formula now follows by continuity and we obtain (i).

Since the set $\mathrm{Fp}(E)$ of all Fredholm pairs is open and the index is locally constant **C1** implies that the pair $(T_p\mathcal{W}^s(x), V)$ is Fredholm of index $-m(x,\mathcal{V})$ for all p in a neighborhood of x. Furthermore the tangent bundle $T\mathcal{W}^s(x)$ is invariant under the flow φ_X and by **C2** the subbundle $\mathcal{V}$ is essentially invariant, i.e., invariant up to compact perturbation. Hence $(T_p\mathcal{W}^s(x), V)$ is Fredholm of index $-m(x,\mathcal{V})$ for all $p \in \mathcal{W}^s(x)$ proving (ii).

To prove (iii) let

$$C^1_{x,y}(\mathbb{R},\mathbb{M}) = \left\{ v \in C^1(\mathbb{R},\mathbb{M}) \;\middle|\; \lim_{s\to-\infty} v(s) = x\,, \quad \lim_{s\to+\infty} v(s) = y\,, \quad \lim_{s\to\pm\infty} v'(s) = 0 \right\} \tag{2.23}$$

be the Banach manifold of all C^1-curves connecting x and y. We consider the map

$$f : C^1_{x,y} \longrightarrow C^0_0(\mathbb{R}, T\,\mathbb{M})\,, \quad v \mapsto v' - X(v)$$

and its linearization at v

$$F(v) : C^1_0(\mathbb{R},\mathbb{E}) \longrightarrow C^0_0(\mathbb{R},\mathbb{E})\,, \quad \xi \mapsto \xi' - DX(v)\xi\,.$$

We set $A(s) := DX(v(s))$, then by Lemma 2.3.2 the operator $F_A := F(v)$ has range

$$\operatorname{ran} F_A = \left\{ \eta \in C^0_0(\mathbb{R},\mathbb{E}) \;\middle|\; R^+_A(\eta)(0) - R^-_A(\eta)(0) \in W^u_A + W^s_A \right\} \cong W^u_A + W^s_A\,.$$

and kernel

$$\ker F_A = \left\{ X_A(s)w \;\middle|\; w \in W^u_A \cap W^s_A \right\} \cong W^u_A \cap W^s_A \subsetneq \mathbb{E}$$

Since we assume that $\mathcal{W}^u(x)$ and $\mathcal{W}^s(x)$ meet transversally, $W^s_A + W^u_A = \mathbb{E}$, for all $v \in f^{-1}(0)$ which implies that $\operatorname{coker} F_A = \{0\}$. In this case the dimension formula for the kernel follows from (i), (ii) and (2.20), i.e.,

$$\dim(W^u_A \cap W^s_A) = \operatorname{ind}(W^u_A, W^s_A) = \dim(W^u_A, V) + \operatorname{ind}(W^s_A, V) = m(x,\mathcal{V}) - m(y,\mathcal{V})\,.$$

Hence f is a Fredholm map with regular value 0 possessing the claimed index.

□

Note that even if the operator F_A is not onto we have that $\dim(W^u_A, V) + \operatorname{ind}(W^s_A, V)$ is finite by (i) and (ii) and so again by (2.20) there holds

$$\operatorname{ind} F_A = \operatorname{ind}(W^u_A, W^s_A) = \dim(W^u_A, V) + \operatorname{ind}(W^s_A, V) = m(x,\mathcal{V}) - m(y,\mathcal{V})\,. \tag{2.24}$$

2.4. Genericity of Morse-Smale

Though A. Abbondandolo and P. Majer established a transversality result by perturbing the metric on the Hilbert manifold M, see [**6**], in preparation for the transversality result for hybrid type curves in Chapter 4 we give an alternative result by perturbing $\mathbb{X} = -\nabla_{1/2}A_H$ directly by adding a further small compact vector field $K : \mathbb{M} \longrightarrow \mathbb{E}$.

Definition 2.4.1. *Let X be a Morse vector field on the Hilbert manifold $\mathbb{M}$ possessing a Lyapunov function f and an admissible constant subbundle $\mathcal{V}$. We will say that X satisfies the* **Morse-Smale condition** *up to order $k \in \mathbb{N}$ if for every pair of singular points $x, y \in \operatorname{sing}(X)$ with $m(x, \mathcal{V}) - m(y, \mathcal{V}) \leq k$ the intersection of the submanifolds $\mathcal{W}^u(x)$ and $\mathcal{W}^s(y)$ is transverse.*

Notice that the presence of a Lyapunov function implies that $\mathcal{W}^u(x) \cap \mathcal{W}^s(x) = \{x\}$ and such an intersection is always transverse. Furthermore the fact that $\varphi_X(s, \cdot)$ is a diffeomorphism implies that if $\mathcal{W}^u(x)$ and $\mathcal{W}^s(y)$ meet tranversally at some $p \in M$, they meet transversally at every point of the orbit of p.

By Proposition 2.2.5 we can assume that we have chosen H generically such that $\mathbb{X} = -\nabla_{1/2} A_H$ is a Morse vector field. Moreover by Lemma 2.3.9, $\mathbb{X}$ satisfies the conditions **C1** and **C2** with respect to the subbundle $\mathbb{M} \times (\mathbb{H}^+ \times \mathbb{R}^n)$. To achieve the transversality result we have to choose a good set of suitable perturbations. That is :

Let $\mathcal{K}(\mathbb{M}, \mathbb{E}) \subset C^3_b(\mathbb{M}, \mathbb{E})$ be the closed subspace of all C^3- vector fields which are compact and bounded on $\mathbb{M}$. We choose a function $\theta \in C^1(\mathbb{M}, \mathbb{R}^+)$ such that

(i) $\theta(x) = 0$ for all $x \in \operatorname{crit}(A_H)$

(ii) $\theta(p) > 0$ for all $p \in \mathbb{M} \setminus \operatorname{crit}(A_H)$.

(iii) $\theta(p) \leq \frac{1}{2} \| \nabla_{1/2} A_H(p) \|_{H^{1/2}(\mathbb{S}^1)}$ for all $p \in \mathbb{M}$.

and consider

$$\mathcal{K}_\theta := \left\{ K \in \mathcal{K}(\mathbb{M}, \mathbb{E}) \;\middle|\; \exists\, c > 0 \text{ such that} \| K(p) \|_{H^{1/2}(\mathbb{S}^1)} \leq c\theta(p)\,, \forall p \in \mathbb{M} \right\}, \tag{2.25}$$

which becomes a Banach space with the norm

$$\| K \|_\theta := \sup_{p \in \mathbb{M} \setminus \operatorname{sing}(X)} \frac{\| K(p) \|_{H^{1/2}(\mathbb{S}^1)}}{\theta(p)} + \| \nabla K \|_{C^2(\mathbb{M}, \mathbb{E})}\,.$$

By $\mathcal{K}_{\theta,1}$ we denote the open unit ball in $\mathcal{K}_\theta$, which as an open subset of a Banach space is a Banach manifold with trivial tangent bundle $\mathcal{K}_{\theta,1} \times \mathcal{K}_\theta$. We claim that $\mathcal{K}_{\theta,1}$ is a good set of perturbations.

Lemma 2.4.2. *Let $K \in \mathcal{K}_{\theta,1}$ and denote by $\mathbb{X}_K := -\nabla_{1/2} A_H + K$.*

(i) $\operatorname{sing}(\mathbb{X}_K) = \operatorname{crit}(A_H)$.

(ii) $D\mathbb{X}_K(x) = -D^2 A_H(x)$ *for all* $x \in \operatorname{crit}(A_H)$.

(iii) A_H *is a Lyapunov function for* $\mathbb{X}_K$.

(iv) $\mathcal{V} := \mathbb{M} \times (\mathbb{R}^n \times \mathbb{H}^+)$ *is admissible for* $\mathbb{X}_K$.

PROOF. Since θ and $d\theta$ vanish on $\mathrm{crit}(A_H)$, we have that K vanishes on $\mathrm{crit}(A_H)$ and obtain $\mathrm{crit}(A_H) \subset \mathrm{sing}(\mathbb{X}_K)$. Let $x \in \mathrm{crit}(A_H)$ and $y \in \mathbb{E}$, then we compute

$$\begin{aligned} \|DK(x)[hy]\|_{H^{1/2}(\mathbb{S}^1)} &= \|K(x+hy) - K(x) + o(h)\|_{H^{1/2}(\mathbb{S}^1)} = \|K(x+hy)\|_{H^{1/2}(\mathbb{S}^1)} + o(h) \\ &\leq \theta(x+hy) + o(h) = \theta(x) + o(h) = o(h)\,. \end{aligned}$$

Hence $DK(x) = 0$ proving (*ii*). To prove (*iii*) let $p \in \mathbb{M} \setminus \mathrm{crit}(A_H)$. Then there holds

$$\begin{aligned} DA_H(p)[\mathbb{X}_K] &= -\langle \mathbb{X}(p), \mathbb{X}(p) + K(p)\rangle_{H^{1/2}(\mathbb{S}^1)} \\ &\leq -\|\mathbb{X}(p)\|^2_{H^{1/2}(\mathbb{S}^1)} + \|\mathbb{X}(p)\|_{H^{1/2}(\mathbb{S}^1)}\|K(p)\|_{H^{1/2}(\mathbb{S}^1)} \\ &\leq -\frac{1}{2}\|\mathbb{X}(p)\|^2_{H^{1/2}(\mathbb{S}^1)} < 0 \end{aligned}$$

which implies furthermore that $\mathrm{sing}(\mathbb{X}_K) \subset \mathrm{crit}(A_H)$ proving (*i*). Finally (*i*) to (*iii*) imply that the dynamics of $\mathbb{X}_K$ do not qualitatively differ from those of $\mathbb{X}$. Together with the fact that K is compact this implies that Lemma 2.3.9 continues to hold for $\mathbb{X}_K$ and we obtain (*iv*).

□

The main result of this section is the following Theorem.

THEOREM 2.4.3. *There is a residual subset $\mathcal{K}_{\mathrm{reg}} \subset \mathcal{K}_{\theta,1}$ of compact maps $K : \mathbb{M} \longrightarrow \mathbb{E}$ such that the perturbed vector field $\mathbb{X}_K = -\nabla_{1/2} A_H + K$ fulfills the Morse-Smale condition up to order 2.*

The proof of the theorem needs some preparation. For $x \neq y \in \mathrm{crit}(A_H)$ we consider the map

$$\psi : C^1_{x,y}(\mathbb{R}, \mathbb{M}) \times \mathcal{K}_{\theta,1} \longrightarrow C^0_0(\mathbb{R}, \mathbb{E})\,, \quad v \mapsto v' - \mathbb{X}_K(v)\,, \tag{2.26}$$

where $C^1_{x,y}(\mathbb{R}, \mathbb{M})$ was defined in (2.23). The derivative with respect to the first variable is given by

$$D_1\psi(v, K) : C^1_0(\mathbb{R}, \mathbb{E}) \longrightarrow C^0_0(\mathbb{R}, \mathbb{E})\,, \quad \eta \mapsto \eta' - (D\mathbb{X}(v) + DK(v))\eta\,.$$

Since K is assumed to be bounded in $C^3(\mathbb{M}, \mathbb{E})$, $D_1\psi(v, K)$ is a bounded linear operator. Furthermore by our remark in (2.24) the operator $F_A = D_1\psi(v, K)$, with $A = D\mathbb{X}_K(v)$, is Fredholm of index $m(x, \mathcal{V}) - m(y, \mathcal{V})$, where $\mathcal{V} = \mathbb{M} \times (\mathbb{R}^n \times \mathbb{H}^+)$ and by Lemma 2.3.2 F_A is onto if and only if the unstable and stable manifolds $\mathcal{W}^u(x)$ and $\mathcal{W}^s(y)$ with respect to $\mathbb{X}_K$ meet transversally.

Deriving ψ with respect to the second variable yields in

$$D_2\psi(v, K) : \mathcal{K}_\theta \longrightarrow C^0_0(\mathbb{R}, \mathbb{E})\,, \quad \kappa \mapsto \kappa(v)\,,$$

which is due to the estimate $\|\kappa(v)\|_{C^0} \leq \max_{p \in v(\mathbb{R})} \theta(p)\|\kappa\|_\theta$ indeed a bounded linear operator.

The key point in the proof of the theorem is the following observation.

LEMMA 2.4.4. *Let $\mathcal{Z} = \psi^{-1}(0)$, then the operator*

$$D\psi(v, K) : C^1_0(\mathbb{R}, \mathbb{E}) \times \mathcal{K}_\theta \longrightarrow C^0_0(\mathbb{R}, \mathbb{E})\,, \quad (\eta, \kappa) \mapsto D_1\psi(v, K)[\eta] + D_2\psi(v, K)[\kappa] \tag{2.27}$$

is a left inverse for all $(v, K) \in \mathcal{Z}$. In particular $\mathcal{Z}$ is a Banach manifold.

The proof uses the following general linear statement.

Lemma 2.4.5. **([6])** *Let E, F, G be Banach spaces and let $A \in \mathcal{L}(E, G)$ possess finite codimensional range and complemented kernel. Then for every $B \in \mathcal{L}(F, G)$ the kernel of the operator*

$$C : E \times F \longrightarrow G\,, \quad C(e, f) = Ae - Bf\,,$$

is complemented in $E \times F$.

Proof. Let E_0 be the kernel of A, E_1 be a closed complement of E_0 in E, and P_0, P_1 be the associated projections. Let $G_1 := \operatorname{ran} A$, G_0 be a finite dimensional complement of G_1 in G and Q_0, Q_1 be the associated projections. Then A induces an isomorphism from E_1 to G_1 , whose inverse will be denoted by $T \in \mathcal{L}(G_1, E_1)$. The equation $C(e, f) = 0$ is equivalent to $AP_1 e = Bf$, which is equivalent to

$$\begin{cases} AP_1 e &= Q_1 Bf \\ Q_0 Bf &= 0 \end{cases}$$

and again equivalent to

$$\begin{cases} P_1 e &= TQ_1 Bf \\ Q_0 Bf &= 0 \end{cases} \tag{2.28}$$

Since $Q_0 B$ has finite rank, its kernel $F_0 := \ker Q_0 B$ has a finite dimensional complement F_1 in F. Hence by (2.28) the kernel of C is

$$\ker C = \Big\{(e_0 + TQ_1 Bf_0, f_0) \in E \times F \;\Big|\; (e_0, f_0) \in E_0 \times F_0\Big\}\,,$$

and the closed linear subspace $E_1 \times F_1$ is a complement of $\ker C$. □

Now we prove Lemma 2.4.4.

Proof. Since $D_1\psi(v, K)$ is Fredholm setting $A = D_1\psi(v, K)$ and $B = -D_2\psi(v, K)$ in Lemma 2.4.5 proves that $D\psi(v, K)$ possesses complemented kernel in $C^1_0(\mathbb{R}, \mathbb{M}) \times \mathcal{K}_\theta$ and it remains to show that $D\psi(v, K)$ is onto.

The range of $D\psi(v, K)$ contains the range of $D_1\psi(v, K)$ which is closed and finite codimensional. So it suffices to show that the range of $D_2\psi(v, K)$ is dense. If v is a constant flow line of φ_{X_K} then transversality is already achieved by the regularity of H so we may assume that $v : \mathbb{R} \longrightarrow \mathbb{M}$ is a non-constant flow line and therefore a C^1-embedding. Let $\rho \in C^0_0(\mathbb{R}, \mathbb{E})$ then $\rho \circ v^{-1} : v(\mathbb{R}) \subset \mathbb{M} \longrightarrow \mathbb{E}$ is bounded and maps bounded sets $U \subset v(\mathbb{R})$ to precompact sets. Hence we can find $\kappa \in \mathcal{K}_\theta$ such that $\|\, \kappa_{|v(\mathbb{R})} - \rho \circ v^{-1} \,\|_{C^0(v(\mathbb{R}),\mathbb{E})}$ becomes arbitrarily small. This proves the density and therefore the claim. □

Before we finally can prove Theorem 2.4.3 we need a further general result.

Proposition 2.4.6. **([6])** *Let E, F, G be Banach spaces and $A \in \mathcal{L}(E, F)$, $B \in \mathcal{L}(E, G)$ be left inverses. Then :*

(*i*) $A_{|\ker B}$ *is a left inverse if and only if* $B_{|\ker A}$ *is a left inverse.*

(*ii*) $A_{|\ker B}$ *is Fredholm if and only if* $B_{|\ker A}$ *is Fredholm, in which case the indices coincide.*

Proof. Denote by $R \in \mathcal{L}(F, E)$ and $S \in \mathcal{L}(G, E)$ the right inverses of A, B respectively.

(*i*) If $R_0 \in \mathcal{L}(F, \ker B)$ is a right inverse of $A_{|\ker B}$, i.e., a right inverse of A with range in $\ker B$, the map $S_0 := (I_E - R_0 A)S$ is a right inverse of B, being a perturbation of S by an operator with range in $\ker B$ and it takes values in $\ker A$ because

$$AS_0 = AS - AR_0AS = AS - I_F AS = 0 .$$

Therefore, S_0 is a right inverse of $B_{|\ker A}$.

(*ii*) The kernels of $A_{|\ker B}$ and $B_{|\ker A}$ coincide :

$$\ker A_{|\ker B} = \ker B_{|\ker A} = \ker A \cap \ker B .$$

Moreover $R : F \longrightarrow R(F)$ is an isomorphism and $I - RA$ is a projector onto $\ker A$. Therefore

$$\operatorname{coker} A_{|\ker B} = \frac{F}{\operatorname{ran} A_{|\ker B}} \cong \frac{RF}{\operatorname{ran} RA_{|\ker B}} \cong \frac{\ker A + RF}{\ker A + \operatorname{ran} RA_{|\ker B}} = \frac{E}{\ker A + \ker B} .$$

The analogous identity holds for $\operatorname{coker} B_{|\ker A}$ using the right inverse S which proves that each of the assertions in (*ii*) is equivalent to the fact that $(\ker A, \ker B)$ is a Fredholm pair with

$$\operatorname{ind}(\ker A, \ker B) = \operatorname{ind} A_{|\ker B} = \operatorname{ind} B_{|\ker A} .$$

□

The Proposition has an immediate consequence.

Corollary 2.4.7. ([**6**]) *Let M, N, O be Banach manifolds and $\phi \in C^1(M, N)$, $\psi \in C^1(M, O)$ be maps with regular values $p \in N$, $q \in O$. Then :*

(*i*) *p is a regular value of $\phi_{|\psi^{-1}(q)}$ if and only if q is a regular value for $\psi_{|\phi^{-1}(p)}$.*

(*ii*) *$\phi_{|\psi^{-1}(q)}$ is a Fredholm-map if and only if $\psi_{|\phi^{-1}(p)}$ is Fredholm-map, in which case the indices coincide.*

Now we can state the **proof of Theorem 2.4.3:**

Proof. Recall that by Lemma 2.4.4 $\mathcal{Z} = \psi^{-1}(0)$ with ψ from (2.26) is a Banach manifold. Let $\pi : C^1_{x,y}(\mathbb{R}, \mathbb{M}) \times \mathcal{K}_{\theta,1} \longrightarrow \mathcal{K}_{\theta,1}$ be the projection onto the second factor. We claim that $\pi_{|\mathcal{Z}}$ is Fredholm of index $m(x, \mathcal{V}) - m(y, \mathcal{V})$. Indeed by Corollary 2.4.7 applied to ψ from $M = C_{x,y}(\mathbb{R}, \mathbb{M}) \times \mathcal{K}_{\theta,1}$ to $O = C^0_0(\mathbb{R}, \mathbb{E})$ and $\phi = \pi$ from M to $N = \mathcal{K}_{\theta,1}$ we obtain the claimed result. Assume that $m(x, \mathcal{V}) - m(y, \mathcal{V}) \leq 2$ and denote by $\mathcal{K}_{\mathrm{reg}}(x, y) \subset \mathcal{K}_{\theta,1}$ the regular values of $\pi_{|\mathcal{Z}}$. Again by Corollary 2.4.7 this is the set such that

$$\psi : C^1_{x,y}(\mathbb{R}, \mathbb{M}) \times \mathcal{K}_{\mathrm{reg}}(x, y) \longrightarrow C^0_0(\mathbb{R}, \mathbb{E})$$

possesses regular value 0, i.e., $\mathcal{W}^u(x)$ and $\mathcal{W}^s(y)$ with respect to $\mathbb{X}_K$, $K \in \mathcal{K}_{\text{reg}}(x,y)$ meet transversally. Now we would like to apply the Sard-Smale Theorem 2.2.6 but we are in the delicate situation where the manifold $\mathcal{K}_{\theta,1}$ is not separable. In this case, due to [**18**] Theorem 2.2.6 still holds if $\pi_{|\mathcal{Z}}$ is σ **- proper**, i.e., there exists a countable family of sets $\{M_k\}_{k\in\mathbb{N}}$ which covers $\mathcal{Z}$ such that $\pi_{|M_k}$ is proper. We prove this additional property in Section 2.5 (see Lemma 2.5.11) and finish the proof now as follows. Since $\pi_{|\mathcal{Z}}$ is σ - proper and of class C^3 the Sard-Smale Theorem 2.2.6 tells us that $\mathcal{K}_{\text{reg}}(x,y)$ is residual in $\mathcal{K}_{\theta,1}$ and so is the set

$$\mathcal{K}_{\text{reg}} := \bigcap_{\substack{x,y\in\text{sing}(X)\,,\\ 0<m(x,\mathcal{V})-m(y,\mathcal{V})\le 2\,,}} \mathcal{K}_{\text{reg}}(x,y)\,.$$

□

2.5. Compact intersections

In this section we follow A. Abbondandolo and P. Majer in [**5**] to prove that as in the finite dimensional case, treated in [**33**], for singular points $x \neq y$ of the perturbed vector field $\mathbb{X}_K = -\nabla_{1/2}A_H + K$, with $K \in \mathcal{K}_{\theta,1}$ as in (2.25) the intersections of the unstable and stable manifolds $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ with respect to $\mathbb{X}_K$ are compact up to broken trajectories. The second crucial result of this section is that the unstable manifold is essential vertical in the sense that the projection of the sub-level sets of the unstable manifold onto $\mathbb{H}^-$ is precompact. This crucial fact allows us to prove the compactness result for the hybrid type curves in chapter 4. Here is the precise result :

THEOREM 2.5.1. *Let $\mathbb{X}_K = -\nabla_{1/2}A_H + K$, $K \in \mathcal{K}_{\theta,1}$ as in* (2.25)*, $x, y \in \text{crit}(A_H)$ and $\mathcal{W}^u(x), \mathcal{W}^s(y)$ be the unstable, stable manifolds of x, y with respect to $\mathbb{X}_K$ and $c, d \in \mathbb{R}$ be regular values of A_H with $c < A_H(x) < d$.*

(*i*) *The intersection $\mathcal{W}^u(x) \cap \mathcal{W}^s(y) \subset \mathbb{M}$ is precompact.*

(*ii*) *The sets $\mathbb{P}^+\big(\mathcal{W}^s(x) \cap \{A_H \le d\}\big)$ and $\mathbb{P}^-\big(\mathcal{W}^u(x) \cap \{A_H \ge c\}\big)$ are precompact.*

So (*ii*) says that $\mathcal{W}^u(x)$ is **essentially vertical**, i.e., vertical up to a precompact set, and $\mathcal{W}^s(x)$ is **essentially horizontal** with respect to the splitting $\mathbb{H}^- \oplus \mathbb{H}^+ = \mathbb{H}$. Before we prove the theorem we we want to state the compactness result for gradient-like trajectories.

DEFINITION 2.5.2. *Let X be a C^1- Morse vector field on a Banach manifold M and $x, y \in \text{sing}(X)$. A* **broken trajectory** *from x to y is a set*

$$S = S_1 \cup \cdots \cup S_k\,, \quad k \ge 1\,,$$

and S_i is the closure of a flow line from z_{i-1} to z_i, where $x = z_0 \neq z_1 \neq \cdots \neq z_{k-1} \neq z_k = y$ are all singular points.

When $k = 1$, a broken trajectory is just the closure of a genuine trajectory. If f is a Lyapunov function for X and S a broken trajectory the following must hold :

$$f(x) > f(z_1) > \cdots > f(z_{k-1}) > f(y)\,.$$

It is easy to check that a compact set $S \subset M$ is a broken trajectory from x to y if and only if

(*i*) $x, y \in S$,

(*ii*) S is φ_X invariant,

(*iii*) $S \cap \left\{p \in M \;\middle|\; f(p) = c\right\}$ consists of a single point if $c \in [f(x), f(y)]$ and is empty otherwise.

We recall that on a complete metric space (E, d) the metric

$$\mathrm{dist}(A, B) := \max\left\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right\}, \quad A, B \subset E$$

is called the **Hausdorff-distance**.

Proposition 2.5.3. (**[5]**) *Assume that the Morse vector field X on a Banach manifold M has a Lyapunov function f and that $x, y \in \mathrm{sing}(X)$ such that $\overline{(\mathcal{W}^u(x) \cap \mathcal{W}^s(y))}$ is compact. Let $p_n \in \mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ be a sequence and set $S_n := \varphi_X(\mathbb{R} \times \{p_n\}) \cup \{x, y\}$, where φ_X denotes the flow of X. Then S_n has a subsequence which converges to a broken flow line from x to y, in the Hausdorff-distance.*

Proof. The space of compact subspaces of a compact metric space is compact with respect to the Hausdorff-distance. Hence (S_n) has a subsequence (S'_n) which converges to a compact set $S \subset \overline{(\mathcal{W}^u(x) \cap \mathcal{W}^s(y))}$. Then $x, y \in S$ and since (S'_n) is φ_X invariant so is S. Since $S'_n \subset f^{-1}([f(y), f(x)])$, we obtain that the set

$$S \cap \left\{p \in M \;\middle|\; f(p) = c\right\} \tag{2.29}$$

is empty for every $c \notin [f(y), f(x)]$. Let $c \in [f(y), f(x)]$ then (S'_n) has a subsequence (S''_n) such that $(S''_n) \cap \left\{p \in M \;\middle|\; f(p) = c\right\}$ converges to some point in (2.29), which is then non-empty. Assume (2.29) contains two points p and q. Since (S_n) is a sequence of trajectories of φ_X we can find a sequence (p_n) converging to p, and numbers $s_n \in \mathbb{R}$ such that $\varphi_X(s_n, p_n) \to q$. By reversing if necessary the role of p and q we can assume that $s_n \geq 0$ for all $n \in \mathbb{N}$. Hence we deduce the convergence

$$\int_0^{s_n} Df(\varphi_X(\tau, p_n))[X(\varphi_X(\tau, p_n)]d\tau = f(\varphi_X(s_n, p_n)) - f(p_n) \longrightarrow f(q) - f(p) = 0$$

Since all singular points of X are isolated either $s_n \to 0$ or $\varphi_X([0, s_n] \times \{p_n\})$ converges to a singular point. In both cases, we obtain $p = q$. This shows that (2.29) consists of a single point. Hence S is a broken trajectory from x to y. □

COROLLARY 2.5.4. *Let the assumptions of Proposition 2.5.3 be fulfilled and assume furthermore that X satisfies the Morse-Smale condition up to order 1 and that the intersection of $\mathcal{W}^u(x)$ and $\mathcal{W}^s(y)$ is 1-dimensional, i.e., $m(x,\mathcal{V}) - m(y,\mathcal{V}) = 1$. Then $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ is already compact.*

PROOF. By our assumptions there are no constant trajectories in $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$, which implies that the $\mathbb{R}$-action

$$\mathbb{R} \times \mathcal{W}^u(x) \cap \mathcal{W}^s(y) \longrightarrow \mathcal{W}^u(x) \cap \mathcal{W}^s(y), \quad (s,p) \mapsto \varphi_X(s,p)$$

acts freely and properly on the 1-dimensional manifold $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$, which is therefore the union of orbits of a discrete set of points. Assume there is a sequence of orbits converging to a broken trajectory with at least on intermediate point z. Then by transversality $m(x,\mathcal{V}) - m(z,\mathcal{V}), m(z,\mathcal{V}) - m(y,\mathcal{V}) \geq 1$ and therefore $m(x,\mathcal{V}) - m(y,\mathcal{V}) \geq 2$ a contradiction. Hence $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ is compact and consists not of orbits of a discrete set of points but of orbits of finitely many points. □

We still have to prove Theorem 2.5.1 which needs a better understanding of the unstable and stable manifolds, especially locally near the singular points.

Assume that X is a C^k-Morse vector field, $k \geq 1$, defined on an open subset $U \subset E$ of a Banach space E, possesses globally defined flow φ_X and that $0 \in U$ is a singular point of X. Since X is Morse we have the splitting $E = E^u \oplus E^s$ into the positive and negative eigenspaces of $DX(0)$. Let $r > 0$ then we denote by E^u_r and E^s_r the closed balls of radius r in E^u,E^s respectively and by $E_r = E^u_r \times E^s_r$. In the language of Conley theory E_r is an **isolating block** of the singular point 0 and $\partial E^u_r \times E^s_r$ is its **exit set**. We set

$$\begin{aligned}
\mathcal{W}^u_{\mathrm{loc},r}(0) &:= \Big\{ v \in E_r \;\Big|\; \varphi_X((-\infty,0] \times \{v\}) \subset E_r \quad \text{and} \quad \lim_{s \to -\infty} \varphi_X(s,v) = 0 \Big\} \\
\mathcal{W}^s_{\mathrm{loc},r}(0) &:= \Big\{ v \in E_r \;\Big|\; \varphi_X([0,+\infty) \times \{v\}) \subset E_r \quad \text{and} \quad \lim_{s \to +\infty} \varphi_X(s,v) = 0 \Big\}
\end{aligned}$$

Then the following holds

THEOREM 2.5.5. **([6]) (Local (un)stable manifold Theorem)**. *For any $r > 0$ small enough there are C^k-maps*

$$\sigma^u : E^u_r \longrightarrow E^s_r, \quad \sigma^s : E^s_r \longrightarrow E^u_r$$

with $\sigma^u(0) = \sigma^s(0) = 0$ and $D\sigma^u(0) = D\sigma^s(0) = 0$ such that $\mathcal{W}^u_{\mathrm{loc},r}(0)$ is the graph of σ^u and similarly $\mathcal{W}^s_{\mathrm{loc},r}(0)$ is the graph of σ^s. Moreover

$$\mathcal{W}^u(0) \cap E_r = \mathcal{W}^u_{\mathrm{loc},r}(0) \quad \textit{and} \quad \mathcal{W}^s(0) \cap E_r = \mathcal{W}^s_{\mathrm{loc},r}(0)\,.$$

Note that by the properties $D\sigma^u(0) = D\sigma^s(0) = 0$ the local unstable /stable manifolds are tangent to E^u, E^s at 0 respectively.

Chronologically, of course one first proves this local result before one deduces the global version in Theorem 2.3.7. Alternatively to the proofs in [**6**] (Theorems 1.12 and 1.20) a proof can be found in [**35**] using the so called **graph transform method** which will be the essential tool in our proof of Theorem 2.5.1.

THEOREM 2.5.6. **([3],[35]) (graph transform method)** *For any* $r > 0$ *small enough there exists a continuous (nonlinear) semigroup*

$$\Gamma : [0, +\infty] \times \mathrm{Lip}_1(E^u_r, E^s_r) \longrightarrow \mathrm{Lip}_1(E^u_r, E^s_r)$$

such that for every 1*-Lipschitz continuous map* $\sigma \in \mathrm{Lip}_1(E^u_r, E^s_r)$ *there holds :*

(i) $\Gamma(0, \sigma) = \sigma$ *and* $\Gamma(\tau + s, \sigma) = \Gamma(\tau, \Gamma(s, \sigma))$ *for all* $\tau, s \in [0, +\infty]$.

(ii) *For every* $\tau \in [0, +\infty)$*, the restriction of* $\varphi_X(\tau, \cdot)$ *to* E_r *maps the graph of* σ *onto the graph of* $\Gamma(\tau, \sigma)$*. That is :*

$$\mathrm{graph}\,\Gamma(\tau, \sigma) = \left\{\varphi_X(\tau, v) \;\middle|\; v \in \mathrm{graph}\,\sigma \quad \textit{and} \quad \varphi_X([0, \tau] \times \{v\}) \subset E_r\right\}.$$

(iii) $\mathrm{graph}\,\Gamma(+\infty, \sigma) = \mathcal{W}^u_{\mathrm{loc},r}(0)$

(iv) *For any* $\alpha > 0$ *there exists* $r_0 \in (0, r]$ *and* $\tau_0 \geq 0$ *such that the restriction of* $\Gamma(\tau, \sigma)$ *to* $E^u_{r_0}$ *is in* $\mathrm{Lip}_\alpha(E^u_{r_0}, E^s_{r_0})$*, for any* $\tau \in [\tau_0, +\infty)$ *and any* $\sigma \in \mathrm{Lip}_1(E^u_r, E^s_r)$.

(v) *If* $U \subset E^u_r$ *is open and* $\sigma \in \mathrm{Lip}_1(U, E^s_r)$ *is such that* $\mathrm{graph}\,\sigma \cap \mathcal{W}^s_{\mathrm{loc},r}(0) \neq \emptyset$ *then there exists* $\tau \geq 0$ *and* $\sigma' \in \mathrm{Lip}_1(E^u_r, E^s_r)$ *such that the restriction of* $\varphi_X(\tau, \cdot)$ *to* E_r *maps the graph of* σ *onto the graph of* σ'*, i.e.,*

$$\mathrm{graph}\,\sigma' = \left\{\varphi_X(\tau, v) \;\middle|\; v \in \mathrm{graph}\,\sigma \quad \textit{and} \quad \varphi_X([0, \tau] \times \{v\}) \subset E_r\right\}.$$

Especially statement (i) of the next lemma will be useful later on.

LEMMA 2.5.7. **([5])** *Let* f *be a Lyapunov function for* X *on* E*, nondegenerate in* 0*,* $r > 0$*,* $(v_n) \subset E$ *be a sequence converging to* 0 *and* $(s_n) \subset [0, \infty)$ *be a sequence such that* $\varphi_X([0, s_n] \times \{v_n\}) \subset E_r$ *and* $\varphi_X(s_n, v_n) \in \partial E_r$*. Then if* r *is small enough there holds :*

(i) $\mathrm{dist}\left(\varphi_X(s_n, v_n), \mathcal{W}^u_{\mathrm{loc},r}(0) \cap \partial E_r\right) \longrightarrow 0 \quad \textit{for} \quad n \to \infty.$

(ii) $\limsup_{n\to\infty} f(\varphi_X(s_n, v_n)) < f(0)$

(iii) *There exists* $0 < r' < r$ *such that*

$$\sup\left\{f(v) \;\middle|\; v \in \partial E_r\,,\ \exists\, T < 0 \textit{ s.t. } \varphi_X(T, v) \in E_{r'} \textit{ with } \varphi_X([T, 0] \times \{v\}) \subset E_r\right\} < \inf\left\{f(v) \;\middle|\; v \in E_{r'}\right\}$$

PROOF. (i): If the vector field is linear, $X(v) = Lv$, then actually for any sequence $(v_n) \subset E$ converging to 0 and $(s_n) \in [0, \infty)$ there holds

$$\lim_{n\to\infty} \mathrm{dist}\left(e^{s_n L} v_n, E^u(L)\right) = 0\,. \tag{2.30}$$

By the Grobmann-Hartmann-Theorem, see [**6**], if $r_1 > 0$ is small enough the local flow of φ_X restricted to E_{r_1} is conjugated to its linearization

$$(s, v) \mapsto e^{sDX(0)}v$$

by a bi-uniformly continuous homeomorphism. By Theorem 2.5.5, we may also assume that r_1 is so small such that $\mathcal{W}^u_{\mathrm{loc},r_1}(0)$ is a graph of a uniformly continuous map $\sigma^u : E^u_{r_1} \longrightarrow E^s_{r_1}$. Let $r < r_1$ and set $w_n := \varphi_X(s_n, v_n) \in \partial E_r$, with $v_n \to 0$ and $s_n \geq 0$. By (2.30) and by the uniform continuity of the conjugacy, there exists $(w_n') \subset \mathcal{W}^u_{\mathrm{loc},r_1}(0)$ such that $\| w_n - w_n' \|$ is infinitesimal. Setting $w_n'' = (P^u w_n, \sigma^u(P^u w_n)) \in \mathcal{W}^u_{\mathrm{loc},r}(0) \cap \partial E_r$, by the uniform continuity of σ^u we compute

$$\begin{aligned} \mathrm{dist}\Big(w_n, \mathcal{W}^u_{\mathrm{loc},r}(0) \cap \partial E_r\Big) &\leq \| w_n - w_n'' \| \\ &\leq \| w_n - w_n' \| + \| P^u w_n' - P^u w_n'' \| + \| P^s w_n' - P^s w_n'' \| \\ &= \| w_n - w_n' \| + \| P^u w_n' - P^u w_n \| + \| \sigma^u(P^u w_n') - \sigma^u(P^u w_n) \| \end{aligned}$$

proving (i). Since the local unstable manifold is tangent to E^u at 0 by $o(r)$ considerations we obtain

$$\sup\Big\{ f(v) \;\Big|\; v \in \mathcal{W}^u_{loc,r}(0) \Big\} < f(0)$$

if $r > 0$ is small enough. Since f is uniformly continuous on E_r for small r, (ii) follows from (i). Finally (iii) is an immediate consequence of (ii), arguing by contradiction. □

Now we turn again to our concrete situation and establish a further statement about the behavior of essential vertical / horizontal sets under transport by the flow.

Lemma 2.5.8. *Let $B \subset \mathbb{M}$ be a bounded set, $\mathbb{X}_K = -\nabla_{1/2} A_H + K$, $K \in \mathcal{K}_{\theta,1}$ as* (2.25)*. Then for $T > 0$ there holds :*

- (i) *$\varphi_{\mathbb{X}_K}([-T, T] \times B)$ is bounded.*
- (ii) *If one of the sets $\mathbb{P}^- B$ or $\mathbb{P}^+ B$ is precompact then the sets $\mathbb{P}^- \varphi_{\mathbb{X}_K}([-T, T] \times B)$ and $\mathbb{P}^+ \varphi_{\mathbb{X}_K}([-T, T] \times B)$ are precompact, respectively.*

Proof. Let $p \in B$ and denote by $v(s) = \varphi_{\mathbb{X}_K}(s, b)$ the trajectory of p. Then v solves

$$\left\{ \begin{array}{rcl} v'(s) - \Big(\mathbb{P}^+ v(s) - \mathbb{P}^- v(s) - j^* \nabla H(\cdot, v(s)) + K\big(v(s)\big)\Big) &=& 0 \\ v(0) &=& p \end{array} \right. \qquad \text{for all} \quad s \in [-T, T]$$

We write $p = (p_0, \hat{p})$, $v(s) = (v_0(s), \hat{v}(s)) \in \mathbb{T}^{2n} \times \mathbb{H} = \mathbb{M}$ then there holds

$$\hat{v}(s) = e^{s(\mathbb{P}^+ - \mathbb{P}^-)} \left[\hat{p} + \int_0^s e^{-\tau(\mathbb{P}^+ - \mathbb{P}^-)} \Big(K\big(v(\tau)\big) - j^* \nabla H(\cdot, v) \Big) d\tau \right].$$

By our assumptions on H and K we have that $\int_0^{(\cdot)} e^{-\tau L} K(\cdot) d\tau \in C^0(\mathbb{R}, \mathcal{L}(\mathbb{H}, \mathbb{H}))$ are paths of uniform bounded linear operators on $\mathbb{H}$ as long as s is bounded, which is certainly true. Furthermore K and $j^* \nabla H$ are bounded on $\mathbb{M}$ which shows (i).

To prove (*ii*) let (s_n, p_n) be a sequence in $[-T,T] \times B$ and assume that $\mathbb{P}^- B$ is precompact. Then there exists a subsequence, which we can assume to be (s_n, p_n) itself, such that $(s_n, \mathbb{P}^- p_n)$ converges to some $(\tau_0, \rho_0) \in [-T,T] \times \overline{\mathbb{P}^- B}$. If $\tau_0 = 0$ then $\varphi_{\mathbb{X}_K}(s_n, p_n) \to \rho$ and there is nothing to prove, so we assume that $s_n \neq 0$ for all $n \in \mathbb{N}$ and set $\mathbb{K} = K - j^* \nabla H$. Then there holds

$$\mathbb{P}^- \varphi_{\mathbb{X}_K}(s_n, p_n) = e^{s_n(\mathbb{P}^+ - \mathbb{P}^-)} \mathbb{P}^- p_n + s_n \mathbb{P}^- e^{s_n(\mathbb{P}^+ - \mathbb{P}^-)} \cdot \left(\frac{1}{s_n} \int_0^{s_n} e^{-\tau(\mathbb{P}^+ - \mathbb{P}^-)} \mathbb{K}(\varphi_{\mathbb{X}_K}(\tau, p_n)) d\tau \right).$$

Now $e^{s_n(\mathbb{P}^+ - \mathbb{P}^-)} \mathbb{P}^- p_n \longrightarrow e^{-\tau_0 \mathbb{P}^-} \rho_0 \in \mathbb{H}$, $s_n \mathbb{P}^- e^{s_n(\mathbb{P}^+ - \mathbb{P}^-)} \longrightarrow \tau_0 e^{-\tau_0 \mathbb{P}^-} \in \mathcal{L}(\mathbb{H}, \mathbb{H})$ and since $\mathbb{K}$ is compact the operator

$$\mathbb{K}_0 := e^{-(\cdot)(\mathbb{P}^+ - \mathbb{P}^-)} \mathbb{K}(\varphi_{\mathbb{X}_K}(\cdot, \cdot)) : [-T,T] \times \mathbb{M} \longrightarrow \mathbb{E}$$

is compact by (*i*). Since $\frac{1}{s_n} \int_0^{s_n} e^{-\tau(\mathbb{P}^+ - \mathbb{P}^-)} \mathbb{K}(\varphi_{\mathbb{X}}(\tau, p_n)) d\tau$ is an element in the convex hull of $\mathbb{K}_0([-T,T] \times B)$ which is precompact, we can assume by taking a further subsequence if necessary that

$$\frac{1}{s_n} \int_0^{s_n} e^{-\tau(\mathbb{P}^+ - \mathbb{P}^-)} \mathbb{K}(\varphi_{\mathbb{X}}(\tau, p_n)) d\tau \longrightarrow q_0 \in \overline{\operatorname{conv}(\mathbb{K}_0([-T,T] \times B))} \subset \mathbb{E},$$

i.e., $\varphi_{\mathbb{X}_K}(s_n, p_n) \longrightarrow e^{\tau_0(\mathbb{P}^+ - \mathbb{P}^-)} \rho_0 + \tau_0 e^{-\tau_0 \mathbb{P}^-} q_0 \in \mathbb{H}$ which shows (*ii*) under the assumption that $\mathbb{P}^- B$ is precompact. The proof for the remaining case is completely analogous. □

We recall the definition of the **Palais-Smale condition**.

Definition 2.5.9. *Let X be a C^1 vector field on a Hilbert manifold M, and $f \in C^1(M, \mathbb{R})$ be a Lyapunov function for X. A (PS) sequence for (X, f) is a sequence $(p_n) \subset M$ with $f(p_n)$ bounded and*

$$Df(p_n)[X(p_n)] \longrightarrow 0.$$

The pair (X, f) satisfies the (PS) condition if every (PS) sequence is compact. We shall say that X satisfies (PS) if (X, f) satisfies (PS) for some Lyapunov function f.

If X is the negative gradient of a function f with respect to some Riemannian metric on M, one finds the usual notion of a (PS) sequence : If $f(p_n)$ is bounded and $\| X(p_n) \| \longrightarrow 0$ then (p_n) possesses a convergent subsequence. Since f is C^1, the limit of such a sequence is a critical point of f.

Genesis of (PS) sequences : Assume that the flow φ_X of X is globally defined and that $(s_n, p_n) \subset \mathbb{R} \times M$ is a sequence with $s_n \longrightarrow \infty$ and $f(p_n)$, $f(\varphi_X(s_n, p_n))$ are bounded. Then by the mean value theorem there is a sequence $\tau_n \in (0, s_n)$ with

$$\left\| f(p_n) - f(\varphi_X(s_n, p_n)) \right\| = \left\| Df(\varphi_X(\tau_n, p_n))[X(\varphi_X(\tau_n, p_n))] \right\| \cdot s_n. \tag{2.31}$$

Since the left side is bounded $(\varphi_X(\tau_n, p_n))$ has to be a (PS) sequence for (X, f). If vice versa (s_n, p_n) is a sequence such that $\varphi_X([0, s_n], p_n)$ is bounded away from the singular points of X then (s_n) has to be bounded.

Lemma 2.5.10. *Let* $\mathbb{X}_K = -\nabla_{1/2} A_H + K$, $K \in \mathcal{K}_{\theta,1}$ *be as in* (2.25) *possessing Lyapunov function* A_H *and* $(p_n) \subset \mathbb{M}$ *be a sequence with* $\mathbb{X}_K(p_n) \longrightarrow 0$. *Then the sequence* (p_n) *is compact. In particular* $(\mathbb{X}_K, A_H)$ *satisfies the (PS) condition.*

Proof. Assume that $(p_n) = ((p_0)_n, \hat{p}_n) \subset \mathbb{T}^{2n} \times \mathbb{H} = \mathbb{M}$ is bounded then $K(p_n) - j^*\nabla H(\cdot, p_n)$ possesses a convergent subsequence due to the compactness of K and $j^*\nabla H$. Now

$$(\mathbb{P}^+ - \mathbb{P}^-)\hat{p}_n + K(p_n) - j^*\nabla H(\cdot, p_n) \longrightarrow 0\,.$$

Hence $(\mathbb{P}^+ - \mathbb{P}^-)\hat{p}_n$ also possesses a convergent subsequence. Furthermore $(\mathbb{P}^+ - \mathbb{P}^-)$ is invertible on $\mathbb{H}$ which implies that $(\hat{p}_n)$ is compact and so is (p_n) due the compactness of $\mathbb{T}^{2n}$. Moreover

$$A_H(p_n) = -\frac{1}{2}\Big(\| \mathbb{P}^+\hat{p}_n \|^2_{H^{1/2}(\mathbb{S}^1)} - \| \mathbb{P}^-\hat{p}_n \|^2_{H^{1/2}(\mathbb{S}^1)}\Big) + \int_0^1 H(t, p_n(t))dt$$

is bounded as long as $\hat{p}_n$ is bounded saying that (p_n) is a (PS) sequence for $(\mathbb{X}_K, A_H)$. Now assume p_n is unbounded, i.e., $\| \hat{p}_n \|_{H^{1/2}(\mathbb{S}^1)} \longrightarrow \infty$. We set

$$\hat{q}_n = \frac{\hat{p}_n}{\| \hat{p}_n \|_{H^{1/2}(\mathbb{S}^1)}}\,.$$

Since K and $j^*\nabla H$ are bounded on $\mathbb{M}$ we have that $(K(p_n) - j^*\nabla H(\cdot, p_n))/\| \hat{p}_n \|_{H^{1/2}(\mathbb{S}^1)} \longrightarrow 0$ and therefore

$$\frac{1}{\| \hat{p}_n \|_{H^{1/2}(\mathbb{S}^1)}}\Big((\mathbb{P}^+ - \mathbb{P}^-)\hat{p}_n + K(p_n) - j^*\nabla H(\cdot, p_n)\Big) = (\mathbb{P}^+ - \mathbb{P}^-)\hat{q}_n + \frac{K(p_n) - j^*\nabla H(\cdot, p_n)}{\| \hat{p}_n \|_{H^{1/2}(\mathbb{S}^1)}} \longrightarrow 0$$

implies that since $(\mathbb{P}^+ - \mathbb{P}^-)$ is invertible on $\mathbb{H}$ that $\hat{q}_n \longrightarrow 0$ contradicting $\| \hat{q}_n \|_{H^{1/2}(\mathbb{S}^1)} = 1$. Hence there are no singular points at infinity and we obtain the claim. □

Now we have all ingredients for the **proof of Theorem 2.5.1 :**

Proof. Statement (*i*) is already treated in [**5**] and is similarly to prove as (*ii*), we therefore concentrate on:

Step 1 (locally) : Assume we have chosen a chart $\phi : U(x) \longrightarrow \mathbb{E}_r \subset \mathbb{E}$ such that x corresponds to 0 and the neighborhood U of x is mapped onto an isolating block $\mathbb{E}_r$. Let $r > 0$ be small enough such that all local results of this section hold then in particular the restriction $\mathcal{W}^u(x) \cap U(x)$ corresponds to $\mathcal{W}^u(0)_{\mathrm{loc},r}$.

Since $\mathbb{X}_K$ satisfies condition **C1** with respect to the constant subbundle $\mathbb{M} \times (\mathbb{R}^n \times \mathbb{H}^+)$ by Lemma 2.3.9, $E^u(D\mathbb{X}_K(0))$ is a compact perturbation of $\mathbb{H}^+$ and by the splitting $\mathbb{E} = E^s(D\mathbb{X}_K(0)) \oplus E^u(D\mathbb{X}_K(0))$ we have that $E^s(D\mathbb{X}_K(0))$ is a compact perturbation of $\mathbb{H}^-$. Hence $\mathbb{P}^-\mathcal{W}^u_{\mathrm{loc},r}(0)$ is precompact if and only if $P^s\mathcal{W}^u_{\mathrm{loc},r}(0)$ is precompact, where P^s denotes the projector onto the negative eigenspace of $D\mathbb{X}_K(0)$. By Theorem 2.5.5 $\mathcal{W}^u_{\mathrm{loc},r}(0)$ is a graph of a map $\sigma^u : \mathbb{E}^u_r \longrightarrow \mathbb{E}^s_r$. So $\mathbb{P}^-\mathcal{W}^u_{\mathrm{loc},r}(0)$ is precompact if and only if σ^u is a compact map.

Let $\sigma_0 : \mathbb{E}^u_r \longrightarrow \mathbb{E}^s_r$ be a 1-Lipschitz map. By the graph transform method Theorem 2.5.6, for every $\tau > 0$ the set

$$\Big\{\varphi_{\mathbb{X}_K}(\tau, v) \;\Big|\; v \in \operatorname{graph}\sigma_0 \quad \text{and} \quad \varphi_{\mathbb{X}_K}([0,\tau]\times\{v\}) \subset E_r\Big\} \tag{2.32}$$

is a graph of a 1-Lipschitz map $\sigma_\tau : \mathbb{E}^u_r \longrightarrow \mathbb{E}^s_r$ and σ_τ converges uniformly to σ^u for $\tau \to +\infty$. If σ_0 is compact, for example $\sigma_0 = 0$, then (2.32) is essentially $\varphi_{\mathbb{X}_K}$ invariant for every $\tau > 0$ in the sense of Lemma 2.5.8. Hence σ_τ is a compact map for every $\tau > 0$. By the uniform convergence we obtain that σ^u is a compact map, which is equivalent to the fact that $\mathbb{P}^-\mathcal{W}^u(x) \cap U(x)$ is precompact.

Step 2 (globally) : Recall that $\mathbb{X}_K$ satisfies (PS), which by our remark in (2.31) says that if $(s_n, p_n) \subset \mathbb{M}\times\mathbb{R}$ is a sequence such that $\|\mathbb{X}_K(\varphi_{\mathbb{X}_K}([0,s_n]\times\{p_n\}))\| > \varepsilon > 0$, for all $n \in \mathbb{N}$, is bounded away from $\operatorname{sing}(\mathbb{X}_K)$. Then the sequence (s_n) has to be bounded.

Let c be a regular value of A_H and assume that there is no critical value in $(c, A_H(x))$. Let $(p_n) \subset \mathcal{W}^u(x) \cap \{A_H \geq c\}$ be a sequence, then we can choose times $(s_n) \subset (-\infty, 0]$ such that

$$\varphi_{\mathbb{X}_K}(s_n, p_n) \in \partial(\mathcal{W}^u(x) \cap U(x))\,.$$

Hence $\varphi_{\mathbb{X}_K}(s_n, p_n) \in \mathcal{W}^u_{\mathrm{loc},r}(0)$ and so the sequence $\mathbb{P}^-\varphi_{\mathbb{X}_K}(s_n, p_n)$ is compact. Since by assumption the trajectories $\varphi_{\mathbb{X}_K}(s, p_n)$, $s \leq 0$ of (p_n) are bounded away from $\operatorname{sing}(\mathbb{X})$ the sequence (s_n) is bounded and therefore by Lemma 2.5.8 already the sequence $\mathbb{P}^-(p_n)$ is compact.

Now assume there is $y \in \operatorname{sing}(\mathbb{X}_K)$ with $A_H(y) = c_1$ and $c_1 \in (c, A_H(x))$ is the only critical value. Then either the orbits of (p_n) are still bounded away from $\operatorname{sing}(\mathbb{X}_K)$ and we proceed as before or there is a subsequence (p'_n) such that $\varphi_{\mathbb{X}_K}(s, p'_n) \longrightarrow y$ for $s \to -\infty$. Then we can find $(s'_n) \subset (-\infty, 0]$ such that

$$\varphi_{\mathbb{X}_K}(s'_n, p'_n) \in \partial(\mathcal{W}^u(y) \cap V(y))\,,$$

where we assume again that we have chosen a chart such that y corresponds to 0 and $V(y)$ to $\mathbb{E}_r(0)$. By (i) of Lemma 2.5.7 $\operatorname{dist}(\varphi_{\mathbb{X}_K}(s'_n, p'_n), \mathcal{W}^u_{\mathrm{loc},r} \cap \partial\mathbb{E}_r) \longrightarrow 0$. Hence we can take a sequence $q_n \in \mathcal{W}^u_{\mathrm{loc},r}(y)$ such that $\|\mathbb{P}^-(\varphi_{\mathbb{X}_K}(s'_n, p'_n) - q_n)\|_{H^{1/2}(\mathbb{S}^1)}$ becomes infinitesimal. Now the sequence $\mathbb{P}^-q_n$ is compact and so is the sequence $(\mathbb{P}^-\varphi_{\mathbb{X}_K}(s'_n, p'_n))$. Furthermore

$$\|\mathbb{X}_K(\varphi_{\mathbb{X}_K}(s'_n, p'_n))\|_{H^{1/2}(\mathbb{S}^1)} > 0$$

is bounded away from 0 and therefore the sequence (s'_n) is bounded and so again by Lemma 2.5.8 we have that $\mathbb{P}^-(p'_n)$ is compact. Since there are only finitely many singular points we conclude (*ii*) by an induction argument and by considering $-\mathbb{X}_K$ instead of $\mathbb{X}_K$ to prove the statement for the stable manifold. □

As promised we prove the additional property of σ **- properness** used to prove the transversality result in Theorem 2.4.3. A proof in a more general setup will come up in [**2**].

LEMMA 2.5.11. *Let $x, y \in \mathcal{P}_0(H)$ and ψ be the map from* (2.26)*. Consider furthermore the Banach manifold $\mathcal{Z} = \psi^{-1}(0) \subset C^1_{x,y}(\mathbb{R}, \mathbb{M}) \times \mathcal{K}_{\theta,1}$ and denote by $\pi : C^1_{x,y}(\mathbb{R}, \mathbb{M}) \times \mathcal{K}_{\theta,1} \longrightarrow \mathcal{K}_{\theta,1}$ the projection to the second factor. Then the restriction of π to $\mathcal{Z}$ is σ - proper.*

PROOF. We have to show that there exists a countable family of sets $\{M_k\}_{k\in\mathbb{N}}$ which cover $\mathcal{Z}$ such that $\pi_{|M_k}$ is proper. The main problem which occurs is that there can be breaking. The M_k are chosen in a way such that breaking does not happen. That is, for $\varepsilon > 0$ we choose balls $B_\varepsilon(x)$ and $B_\varepsilon(x)$ of radius ε centered at x and y and for $N_1, N_2 \in \mathbb{N}$ we denote by $M_{N_1,N_2,\varepsilon}$ the set which consits of those $(v, K) \in \mathcal{Z}$ such that

(*i*) $v(s) \in B_\varepsilon(x)$ for all $s \leq -N_1$
(*ii*) $v(s) \in B_\varepsilon(y)$ for all $s \geq N_1$
(*iii*) $\| v'(s) \|_{H^{1/2}(\mathbb{S}^1)} \geq \frac{1}{N_2}$ for all $s \in [-N_1, N_1]$

Obviously for each fixed ε we have that

$$\mathcal{Z} = \bigcup_{(N_1,N_2)\in\mathbb{N}^2} M_{N_1,N_2,\varepsilon}\,.$$

Furthermore we can assume that ε was chosen small enough such that the balls at x and y contain no other critical points than x or y respectively. Now let $(v_n, K_n) \in M_{N_1,N_2,\varepsilon}$ and assume that K_n converges to $K \in \pi(M_{N_1,N_2,\varepsilon}) \subset \mathcal{K}_{\theta,1}$, then we claim that a subsequence of v_n converges to some $v \in C^1_{x,y}$. To see this let

$$\mathbb{W}_{N_1,N_2,\varepsilon} = \left(\mathcal{W}^u_{\mathbb{X}_K}(x) \cap \mathcal{W}^s_{\mathbb{X}_K}(y)\right) \cap \left\{u(s) \;\middle|\; (u, K) \in M_{N_1,N_2,\varepsilon}\,,\, s \in \mathbb{R}\right\}$$

where $\mathcal{W}^u_{\mathbb{X}_K}(x), \mathcal{W}^s_{\mathbb{X}_K}(y)$ denote the unstable / stable manifolds of x and y with respect to $\mathbb{X}_K = \mathbb{X}+K$. Since the set $M_{N_1,N_2,\varepsilon}$ is closed and $\mathcal{W}^u_{\mathbb{X}_K}(x)\cap\mathcal{W}^s_{\mathbb{X}_K}(y)$ is precompact due to Theorem 2.5.1 we have that $\mathbb{W}_{N_1,N_2,\varepsilon}$ is compact. Now each v_n solves $v_n'(s) = \mathbb{X}_{K_n}(v_n(s))$ for all $s \in [-N_1, N_2]$ and

$$v_n(s) \in B_\varepsilon(x) \quad \text{for} \quad s \leq -N_1 \quad v_n(s) \in B_\varepsilon(y) \quad \text{for} \quad s \geq N_1\,.$$

So by the continuous dependence of ODE's on the initial data (see for instance [**23**]), we have that $\operatorname{dist}(v_n, \mathbb{W}_{N_1,N_2,\varepsilon})$ becomes arbitrarily small as long as ε is chosen small enough and n large enough such that K_n becomes close enough to K. In other words we can find a sequence of trajectories $w_n \in$ of $\in C^1_{x,y}(\mathbb{R}, \mathbb{M})$ with respect to $\mathbb{X}_K$ with trace in $\mathbb{W}_{N_1,N_2,\varepsilon}$ such that $\operatorname{dist}(v_n, w_n)$ becomes infinitesimal. Since the the sequence w_n is compact we can choose a subsequence v_{n_k} which converges in C^0 to a curve $v \in C^0_{x,y}(\mathbb{R}, \mathbb{M})$. Since all curves v_{n_k} are trajectories of a C^3-vector field we finally obtain the claimed convergence in $C^1_{x,y}(\mathbb{R}, \mathbb{M})$. □

2.6. The boundary operators

By the previous sections we can assume that the perturbed vector field $\mathbb{X}_K = -\nabla_{1/2} A_H + K$, $K \in \mathcal{K}_{\text{reg}}$, is a C^3- Morse vector field and satisfies the Morse-Smale condition up to order 2. Furthermore the functional A_H is strongly indefinite, i.e., the usual Morse indices are infinite and so is the dimension of the spaces $T_x\mathcal{W}^u(x)$, $x \in \text{crit}(A_H)$. Hence $\mathcal{W}^u(x)$ cannot be oriented and we cannot proceed as in the case of finite Morse indices where an chosen orientation of all $\mathcal{W}^u(x)$ induces an orientation of each transverse intersection of stable and unstable manifolds, see [**33**]. The right objects to orient in our situation are the Fredholm pairs $(T_x\mathcal{W}^s(x), \mathcal{V}(x))$, where $\mathcal{V} = \mathbb{M} \times (\mathbb{R}^n \times \mathbb{H}^+)$. That is to choose a finite-dimensional subspace Z with $Z + T_x\mathcal{W}^s(x) + \mathcal{V}(x) = \mathbb{E}$ and $Z \cap T_x\mathcal{W}^u(x) = 0$. Then an orientation of Z defines an orientation of the Fredholm pair. Finally, using the determinant bundle of Fredholm pairs these orientations can be used to orient $T_p(\mathcal{W}^u(x) \cap \mathcal{W}^s(y))$ in a coherent way. A detailed discussion of this approach can be found in [**5**].

If in particular $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ is 1-dimensional and oriented then each connected component W of $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ is a flow line and we can associate a **sign**

$$\epsilon(W) = \pm 1$$

to W, depending on whether $\mathbb{X}_K(p)$ is positively or negatively oriented in T_pW, $p \in W$.

The **Morse complex** for the action functional A_H on the Hilbert manifold $\mathbb{M}$ is now constructed in the usual way. For any $k \in \mathbb{Z}$ we denote by $\text{crit}_k(A_H)$ the critical points x with relative index $m(x, \mathcal{V}) = k$. We choose a generic compact vector field $K \in \mathcal{K}_{\text{reg}}$ such that $\mathbb{X}_K = -\nabla_{1/2} A_H + K$ satisfies the Morse - Smale property up to order 2. For every critical point we fix an orientation of $(T_x\mathcal{W}^s(x), \mathcal{V}(x))$, where $\mathcal{V} = \mathbb{M} \times (\mathbb{R}^n \times \mathbb{H}^+)$, which induces an orientation for all intersections $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ with respect to $\mathbb{X}_K$ and $m(x, \mathcal{V}) - m(y, \mathcal{V}) = 1$. Let $C_k(H)$ be the free abelian group generated by $\text{crit}_k(A_H)$ and r be the number of connected components of $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$. We set

$$\rho(x, y) := \sum_{i=1}^{r} \epsilon(W_i)$$

and define the homomorphisms $\partial_k^M : C_k(H) \longrightarrow C_{k-1}(H)$ generator wise by

$$\partial_k^M x = \sum_{y \in \text{crit}_{k-1}(A_H)} \rho(x, y) y\,, \quad \text{for all} \quad x \in \text{crit}_k(A_H)\,,$$

which is a finite sum for each $x \in \text{crit}_k(A_H)$ due to Corollary 2.5.4. Now a gluing statement, that is proved in detail in [**5**] by using again the graph transform method, implies that if $x, z \in \text{sing}(X)$ with $m(x, \mathcal{V}) - m(z, \mathcal{V}) = 2$ and $\mathcal{S}(x, z)$ denoting the set of broken trajectories from x to z with one intermediate critical point y, which by transversality has index $m(y, \mathcal{V}) = m(z, \mathcal{V}) + 1$, then there is an involution

$$i : \mathcal{S}(x,z) \longrightarrow \mathcal{S}(x,z)\,, \quad \overline{W_1 \cup W_2} \mapsto \overline{W_1' \cup W_2'}$$

without fixed points fulfilling

$$\epsilon(W_1)\epsilon(W_2) = -\epsilon(W_1')\epsilon(W_2')\,,$$

where W_1, W_2, W_1', W_2' are connected components of $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ and $\mathcal{W}^u(y) \cap \mathcal{W}^s(z)$ respectively. Therefore

$$\sum_{y \in \mathrm{crit}_{k-1}(A_H)} \rho(x,y)\rho(y,z) = \sum_{\overline{W_1 \cup W_2} \in \mathcal{S}(x,z)} \epsilon(W_1)\epsilon(W_2)\,,$$

must be zero which implies that $\partial_{k-1}^M \circ \partial_k^M = 0$ holds for all $k \in \mathbb{Z}$.

In other words, $\{C_k(H), \partial_k\}_{k \in \mathbb{Z}}$ is a complex which we call the **Morse complex** of A_H. Clearly, the construction depends on the chosen subbundle $\mathcal{V} = \mathbb{M} \times (\mathbb{R}^n \times \mathbb{H}^+)$ and on the chosen orientations of $(T_x\mathcal{W}^s(x), \mathcal{V}(x))$. If we choose an other subbundle that is a compact perturbation of $\mathcal{V}$, then this produces an isomorphic chain complex by shifting the indices, equal to the dimension of the compact perturbation. If we change the orientations, we obtain another isomorphic chain complex, where the isomorphism is an involution. If we restrict ourselves to the case of $\mathbb{Z}_2$-coefficients, we still obtain a complex. In this case the notion of orientations is not reqired and we will focus on this case at the end of chapter 4.

The homology of this complex given by

$$HM_k := \frac{\ker \partial_k^M}{\operatorname{ran} \partial_{k+1}^M}$$

is called the **Morse homology** of $\mathbb{M}$ with respect to $(A_H, \mathbb{X}_K)$. If we replace $\mathbb{X}_K$ by another vector field $\mathbb{X}_{K'}$, $K, K' \in \mathcal{K}_{\mathrm{reg}}$, which therefore still satisfies conditions **C1** and **C2** with respect to the same bundle $\mathcal{V}$ yields in an isomorphic chain complex. The argument is similar to the one in Theorem 2.26 in [**6**]. Thus the homology is independent of the chosen perturbation K and can therefore be denoted by $HM_*(A_H)$. By the same argument the homology is independent of the particular choice of H, which asserts why we could expect all critical points of A_H to be contractible.

CHAPTER 3

The Floer complex in the $W^{1,2}$-setup

The construction of the Floer complex on a **compact symplectic** and **monotone manifold** was the essential tool of Floer's proof of the **Arnold conjecture** and we refer to [**15**] for further details. Since we use the $W^{1,2}$-setup instead of a $W^{1,p}, p > 2$ setup we find it convenient to restrict ourselves to the case of a constant, almost complex structure J_0 and a generically chosen Hamiltonian H, such that we achieve transversality. In this situation we can prove an elliptic regularity statement which would enable us to show that the moduli spaces are compact in the usual C^∞_{loc}- Gromov sense. However, as a preparation for the next chapter we prove an alternative $W^{1,2}_{\mathrm{loc}}$-compactness statement.

3.1. The setup

By Proposition 2.2.5 we can assume we have chosen smooth, non-degenerate Hamiltonian $H \in \mathcal{H}_{\mathrm{reg}} \subset C^\infty(\mathbb{S}^1 \times \mathbb{T}^{2n}, \mathbb{R})$ on the $2n$-dimensional torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, $n \in \mathbb{N}$ with associated Hamiltonian vector field X_H defined with respect to the standard symplectic structure $\omega_0 = \sum_{i=1}^{n} dy_i \wedge dx_i$. Moreover we denote by

$$J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.1}$$

the special complex structure on $\mathbb{T}^{2n}$ such that $(\mathbb{T}^{2n}, \omega_0, J_0)$ becomes a Kähler manifold with

$$\omega_0(\cdot, J_0 \cdot) = \langle \cdot, \cdot \rangle \ .$$

Instead of studying gradient-like trajectories of the Hamiltonian action

$$A_H : \mathbb{M} \longrightarrow \mathbb{R}\,, \quad A_H(x) = -\frac{1}{2}\left(\| \mathbb{P}^+ x \|^2_{H^{1/2}(\mathbb{S}^1)} - \| \mathbb{P}^- x \|^2_{H^{1/2}(\mathbb{S}^1)}\right) + \int_0^1 H(t, x(t))dt\,,$$

on $\mathbb{M} = H^{1/2}(\mathbb{S}^1, \mathbb{T}^{2n})$, which by Lemma 2.2.3 possesses the same smooth and contractible critical points, denoted by $\mathcal{P}_0(H)$, as its restriction to component of contractible loops of the free loop space $\Lambda_0(\mathbb{T}^{2n})$, introduced in (2.1), we are now interested in solutions of a Cauchy-Riemann type partial differential equation. As it will turn out, the solutions $u = u(s,t) : \mathbb{R} \times \mathbb{S}^1 \longrightarrow \mathbb{T}^{2n}$ of the **Floer-equation** which, since the tangent bundle of $\mathbb{T}^{2n}$ is trivial, $\mathrm{T}\mathbb{T}^{2n} \cong \mathbb{T}^{2n} \times \mathbb{R}^{2n}$, can be written as

$$\overline{\partial}_{J_0,H}(u(s,t)) = \partial_s u(s,t) + J_0(\partial_t u(s,t) - X_H(t, u(s,t))) = 0\,, \tag{3.2}$$

build a set of countable moduli spaces. To give a precise definition of those moduli spaces we first have to make some remarks on the chosen analytical setting. As usual, for domain $\Omega \subset \mathbb{R}^2$, we denote by $W^{k,p}(\Omega, \mathbb{R}^{2n})$ the Sobolev-spaces of all L^p-curves possessing weak derivatives up to order k, and denote by

$$\| u \|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \| D^\alpha u \|^p_{L^p(\Omega)} \right)^{1/p} , \quad \alpha \in \mathbb{N}^2$$

the usual norm. By $C^k_0(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$ we denote the Banach space of all C^k-curves u with

$$\lim_{s \to \pm\infty} D^\alpha u(s,t) = 0 , \quad \text{for all} \quad t \in \mathbb{S}^1 , \alpha \in \mathbb{N}^2 , \quad \text{with} \quad |\alpha| \le k .$$

Let $u \in W^{1,2}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$, then observe that, since

$$W^{1,2}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n}) = L^2(\mathbb{R}, H^1(\mathbb{S}^1, \mathbb{R}^{2n})) \cap H^1(\mathbb{R}, L^2(\mathbb{S}^1, \mathbb{R}^{2n}))$$

the Fourier representation of u is given by

$$u(s,t) = u_0(s) + \sum_{k \neq 0} e^{2\pi k J_0 t} u_k(s) , \quad u_k \in H^1(\mathbb{R}, \mathbb{R}^{2n}) .$$

As a consequence of the Sobolev embedding theorem, see [**10**], similarly to Proposition 2.1.2, all u_k belong already to $C^0_0(\mathbb{R}, \mathbb{R}^{2n})$ and satisfy

$$|u_0(s)|^2 + 4\pi^2 \sum_{k \in \mathbb{Z}} |k|^2 |u_k(s)|^2 < \infty \tag{3.3}$$

almost everywhere. Now we identify those curves whose image on the torus is the same and denote by

$$W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{S}^1, \mathbb{T}^{2n}) := W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n}) / \mathbb{Z}^{2n} .$$

Consider the evaluation map

$$\widetilde{\text{ev}}_0 : C^\infty_0([0,\infty) \times \mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow C^\infty(\mathbb{S}^1, \mathbb{R}^{2n}) , \quad u(\cdot,\cdot) \mapsto u(0,\cdot) .$$

Then there holds:

LEMMA 3.1.1. *The continuation of* $\widetilde{\text{ev}}_0$ *to*

$$\text{ev}_0 : W^{1,2}([0,\infty) \times \mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow \mathbb{E} , \quad u(\cdot,\cdot) \mapsto u(0,\cdot) , \quad \mathbb{E} = H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n})$$

is a well defined smooth submersion. In particular there holds

$$\| \text{ev}_0(u) \|_{H^{1/2}(\mathbb{S}^1)} \le \sqrt{2} \| u \|_{W^{1,2}([0,\infty) \times \mathbb{S}^1)} .$$

Though one can find several generalizations of this statement, for example in [**32**] or [**37**], we give a short proof for our particular situation.

Proof. Let $v \in C_0^\infty([0,\infty) \times \mathbb{S}^1, \mathbb{R}^{2n})$ and $v = \sum_{k\in\mathbb{Z}} e^{2\pi kJt} v_k(s)$ be its Fourier series. Observe that

$$\langle x, y\rangle_{H^{1/2}(\mathbb{S}^1)} = \langle x_0, y_0\rangle + \langle x, J_0(\partial_t \mathbb{P}^- y - \partial_t \mathbb{P}^+ y)\rangle_{L^2(\mathbb{S}^1)} \tag{3.4}$$

for all $x, y \in C^\infty(\mathbb{S}^1, \mathbb{R}^{2n})$, where $\mathbb{P}^\pm$ denote the projections from (2.3). Hence

$$\begin{aligned} \| v(0,\cdot)\|^2_{H^{1/2}(\mathbb{S}^1)} &= \Big| \int_0^\infty \partial_s \| v(s,\cdot)\|^2_{H^{1/2}(\mathbb{S}^1)} ds\Big| \\ &= 2\Big|\int_0^\infty \langle \partial_s v, v\rangle_{H^{1/2}(\mathbb{S}^1)}\, ds\Big| \\ &= 2\Big|\int_0^\infty\int_0^1 \Big(\langle \partial_s v_0, v_0\rangle + \langle \partial_s v, J_0(\partial_t\mathbb{P}^- v - \partial_t \mathbb{P}^+ v)\rangle\Big) dt ds\Big| \\ &\leq 2\| v\|^2_{W^{1,2}([0,\infty)\times\mathbb{S}^1,\mathbb{T}^{2n})} . \end{aligned}$$

By density the estimate has to hold for all $u \in W^{1,2}([0,\infty)\times\mathbb{S}^1, \mathbb{R}^{2n})$ and enables us to continuate $\widetilde{\mathrm{ev}}_0$ to ev_0. Moreover, ev_0 is a linear operator which shows continuity and the existence of all Fréchet derivatives. Now if $x = \sum_{k\in\mathbb{Z}} e^{2\pi kJ_0 t} x_k \in \mathbb{E}$ then its readily seen, that for instance, the curve

$$\xi(s,t) = \sum_{k\in\mathbb{Z}} e^{2\pi kJ_0 t} e^{-s|k|} x_k$$

belongs to $W^{1,2}([0,\infty)\times\mathbb{S}^1, \mathbb{R}^{2n})$ and is therefore a pre-image under $D\,\mathrm{ev}_0$ of x. □

Now we can define the **moduli spaces of Floer-cylinders**. That is :

Definition 3.1.2. *Let $H \in \mathcal{H}_{\mathrm{reg}}$ and $x^-, x^+ \in \mathcal{P}_0(H)$ then we consider the following set*

$$\mathcal{M}_F(x^-, x^+, H, J_0) := \Big\{ u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{S}^1, \mathbb{T}^{2n}) \;\Big|\; \overline{\partial}_{J_0,H}(u) = 0 \textit{ with } \lim_{s\to\pm\infty} u(s,\cdot) = x^\pm \Big\}\,,$$

with $\overline{\partial}_{J_0,H}$ from (3.2). *Surely $\overline{\partial}_{J_0,H}(u) = 0$ shall be understood in the weak sense and the convergence to the orbits at infinity shall be understood as follows. Let $\Omega_T := (-T,T)\times\mathbb{S}^1$ and*

$$u(s,t) = [u_0(s)] + \sum_{k\neq 0} e^{2\pi kJ_0 t} u_k(s,t)\,,$$

where $[u_0] \in C^0((-T,T), \mathbb{T}^{2n})$, $u_k \in C^0((-T,T), \mathbb{R}^{2n})$, $|k| \geq 1$, be the Fourier representation of the restriction of u to $\Omega_T \subset \mathbb{R}\times\mathbb{S}^1$. Let $c_{x^-,x^+} \in C^\infty(\mathbb{R}\times\mathbb{S}^1, \mathbb{T}^{2n})$ be a reference cylinder such that

$$c_{x^-,x^+}(s,\cdot) = \begin{cases} x^-\,, & \textit{for} \quad s \leq -1 \\ x^+\,, & \textit{for} \quad s \geq 1 \end{cases}. \tag{3.5}$$

Then using the additive structure of $\mathbb{T}^{2n}$ and the evaluation map from Lemma 3.1.1 enables us to set $\lim_{s\to\pm\infty} u(s,\cdot) = x^\pm$ if and only if

$$\lim_{s\to\pm\infty} (\mathbb{P}^0)^\perp\, \mathrm{ev}_s(u - c_{x^-,x^+}) = 0 \quad \textit{in } \mathbb{H}\,, \tag{3.6}$$

where $\mathbb{P}^0$ *denotes the restriction to the constant loops defined in* (2.3) *and*

$$\lim_{s\to\pm\infty} \mathbb{P}^0 \operatorname{ev}_s(u - c_{x^-,x^+}) = 0 \quad \textit{on } \mathbb{T}^{2n}\,.$$

Next is to be shown that the solutions $u \in \mathcal{M}_F(x^-, x^+, H, J_0)$ are actually smooth, therefore we recall the inner regularity property of the Laplace operator, which is proven by using the Calderon-Zygmund inequality. For details see [**25**] (Appendix B).

THEOREM 3.1.3. ([**25**]) *Let* $1 < p < \infty$*,* $k \geq 0$ *be an integer, and* $\Omega \subset \mathbb{R}^n$ *be an open domain. If* $u \in L^1_{\mathrm{loc}}(\Omega)$ *is a weak solution of*

$$\Delta u = f\,, \quad f \in W^{k,p}_{\mathrm{loc}}(\Omega)\,,$$

then $u \in W^{k+2,p}_{\mathrm{loc}}(\Omega)$*. Moreover, for every bounded subset* Ω' *whose closure is contained in* Ω *there exists a constant* $c = c(k, p, n, \Omega', \Omega) > 0$ *such that, for every* $u \in C^\infty(\overline{\Omega})$*,*

$$\| u \|_{W^{k+2,p}(\Omega')} \leq c\Big(\| \Delta u \|_{W^{k,p}(\Omega)} + \| u \|_{L^p(\Omega)}\Big)\,.$$

LEMMA 3.1.4. **(local regularity)** *Let* $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{S}^1, \mathbb{T}^{2n})$ *be a weak solution of* $\overline{\partial}_{J_0,H}(u) = 0$ *on every open and bounded domain* $\Omega \subset \mathbb{R} \times \mathbb{S}^1$*. Then* $u \in C^\infty(\mathbb{R} \times \mathbb{S}^1, \mathbb{T}^{2n})$*.*

PROOF. Since $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{S}^1, \mathbb{T}^{2n})$ and $H \in C^\infty(\mathbb{S}^1, \mathbb{T}^{2n})$ we have that $J_0 X_H(u)$ is of class $W^{1,2}_{\mathrm{loc}}$. Denote by $\partial_{J_0} = \partial_s - J_0(\cdot)\partial_t$ and by $\overline{\partial}_{J_0} = \partial_s + J_0\partial_t$, then u solves

$$\Delta u = \partial_{J_0} \circ \overline{\partial}_{J_0}(u) = \partial_{J_0} \circ (J_0(u) X_H(u)) =: f \in L^2_{\mathrm{loc}}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$$

in the weak sense on every open and bounded domain $\Omega \subset \mathbb{R} \times \mathbb{R}/\mathbb{Z}$. Since every L^2_{loc}-curve is certainly of class L^1_{loc}, any lift of u to a curve in $W^{1,2}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$ satisfies the conditions of Theorem 3.1.3 and is therefore a $W^{2,2}_{\mathrm{loc}}$-curve, and so is u. This implies that $J_0 X_H(u)$ is of class $W^{1,2}$ and therefore u is of class $W^{3,2}$. Iterating this argument shows that u is a $W^{\infty,2}_{\mathrm{loc}}$-curve and therefore by Sobolev embedding already of class C^∞. □

DEFINITION 3.1.5. *Let* $u : \mathbb{R} \times \mathbb{S}^1 \longrightarrow \mathbb{T}^{2n}$ *be a smooth solution of* (3.2) *then we define the* **energy** *of u by*

$$E(u) := \frac{1}{2} \int_{-\infty}^{\infty} \int_0^1 \left(\left| \frac{\partial u}{\partial s} \right|^2 + \left| \frac{\partial u}{\partial t} - X_H(u) \right|^2 \right) ds dt\,. \tag{3.7}$$

For $T > 0$ we compute

$$\begin{aligned}
\int_{-T}^{T} \int_0^1 |\partial_s u(s,t)|^2 dt ds &= -\int_{-T}^{T} \int_0^1 \langle \partial_s u(s,t), J_0(\partial_t u(s,t) - X_H(u(s,t))) \rangle \, dt ds \\
&= -\int_{-T}^{T} \partial_s A_H(u(s,\cdot)) ds \\
&= A_H(u(-T,\cdot) - A_H(u(T,\cdot)) \,.
\end{aligned} \tag{3.8}$$

Now recall that the action A_H is smooth on $\mathbb{M}$ and therefore

$$E(u) = \lim_{T\to-\infty} A_H(\mathrm{ev}_T(u)) - \lim_{T\to+\infty} A_H(\mathrm{ev}_T(u)) = A_H(x^-) - A_H(x^+) < \infty\,. \tag{3.9}$$

In particular, this shows that the curves u with vanishing energy are exactly the constant ones, i.e., $u(s,\cdot) = x$ for all $s \in \mathbb{R}$ and some $x \in \mathcal{P}_0(H)$. Furthermore Lemma 3.1.4 implies that the following well-known result, explicitly proven in [**30**], holds for every $u \in \mathcal{M}_F(x^-, x^+, H, J_0)$.

Proposition 3.1.6. **([30])** *Let (M,ω) be a compact symplectic manifold $H : M \longrightarrow \mathbb{R}$ be a non-degenerate Hamiltonian and $u : \mathbb{R}\times\mathbb{S}^1 \longrightarrow M$ be a smooth solution of* (3.2)*, then the following are equivalent :*

(i) $E(u) < \infty$

(ii) *There exist periodic solutions $x^-, x^+ \in \mathcal{P}(H)$ such that*

$$\lim_{s\to\pm\infty} u(s,\cdot) = x^\pm$$

and $\lim_{s\to\pm\infty} \partial_s(u,\cdot) = 0$, where both limits are taken uniformly in t.

(iii) *For every u there exists constants $c, \delta > 0$ such that*

$$|\partial_s u(s,t)| \le c e^{-\delta|s|}$$

for all $(s,t) \in \mathbb{R}\times\mathbb{S}^1$.

Thus, in particular our choice of the convergence behavior at infinity turns out to be equivalent to the classical uniform convergence in (ii). The next result states that the moduli spaces are affine translations of subsets of $W^{1,2}(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n})$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{S}^1, \mathbb{T}^{2n})$, which will be crucial especially for our compactness statement in Theorem 3.3.3.

Proposition 3.1.7. *Let c_{x^-,x^+} be as in* (3.5)*. Then the non-linear operator*

$$\Phi_{c_{x^-,x^+}} : W^{1,2}(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n})\,, \quad w \mapsto \overline{\partial}_{J_0,H}(w + c_{x^-,x^+})\,. \tag{3.10}$$

is well defined, smooth and there holds

$$\mathcal{M}_F(x^-, x^+, H, J_0) = c_{x^-,x^+} + \Phi^{-1}_{c_{x^-,x^+}}(0) \subset W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{S}^1, \mathbb{T}^{2n})\,. \tag{3.11}$$

Proof. We show that $\Phi_{c_{x^-,x^+}}$ is well defined. For that we use the estimate

$$\|\,\Phi_{c_{x^-,x^+}}(w)\,\|^2_{L^2(\mathbb{R}\times\mathbb{S}^1)} \le \|\,\partial_s(w + c_{x^-,x^+})\,\|^2_{L^2(\mathbb{R}\times\mathbb{S}^1)} + \|\,\partial_t(w + c_{x^-,x^+}) - X_H(w + c_{x^-,x^+})\,\|^2_{L^2(\mathbb{R}\times\mathbb{S}^1)}$$

and show that both terms are bounded. By our assumptions on c_{x^-,x^+}, we have that for $T > 1$ and $Z_T = [T,+\infty)\times\mathbb{S}^1$ there holds

$$\|\,\partial_s(w + c_{x^-,x^+})\,\|_{L^2(Z_T)} = \|\,\partial_s w\,\|_{L^2(Z_T)} \le \|\,w\,\|_{W^{1,2}(\mathbb{R}\times\mathbb{S}^1)}\,.$$

Now recall that $X_H = J_0 \nabla H$; by the mean value theorem we obtain

$$\begin{aligned} \| \partial_t w + \partial_t c_{x^-,x^+} - X_H(w + c_{x^-,x^+}) \|_{L^2(Z_T)} &= \| \partial_t w + \partial_t x^+ - X_H(w + x^+) \|_{L^2(Z_T)} \\ &\leq \| \partial_t w \|_{L^2} + \| H \|_{C^2(\mathbb{S}^1 \times \mathbb{T}^{2n})} \| w \|_{L^2(Z_T)} . \end{aligned}$$

The analog estimates for $T < -1$ show that both terms are bounded at the cylindrical ends and therefore on the whole cylinder. Smoothness can be seen similarly to the proof of Theorem 2.2.1. Denote by $h : W^{1,2}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow \mathbb{R}$ the map

$$h(w) = \int_{-\infty}^{\infty} \int_0^1 H(t, w + c_{x^-,x^+}) dt ds$$

and write the k-th Taylor expansion of H as

$$H(t, x + y) = H(t, x) + \sum_{i=1}^{k} \frac{1}{i!} H_i(t, x) y^i + o_k(y) , \quad \text{for all} \quad x, y \in \mathbb{R}^{2n} .$$

with

$$H_i(t, x) y^i = \sum_{|\alpha| = i} \frac{i!}{\alpha!} D^\alpha H(t, x) y^\alpha .$$

We insert $x = w + c_{x^-,x^+}$, $w \in W^{1,2}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$ and $y = \xi \in W^{1,2}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$ and integrate over $\mathbb{R} \times \mathbb{S}^1$ which leads to

$$h(w + \xi) = \int_{-\infty}^{\infty} \int_0^1 H(t, w + c_{x^-,x^+} + \xi) dt ds = \sum_{i=0}^{k} \frac{1}{i!} h_i(w + c_{x^-,x^+}) \xi^i + r_k(w, \xi)$$

with

$$h_i(w + c_{x^-,x^+}) \xi^i = \int_{-\infty}^{\infty} \int_0^1 H_i(t, w + c_{x^-,x^+}) \xi^i dt ds .$$

Now by Hölder and the uniform bound on $\| H \|_{C^k(\mathbb{S}^1 \times \mathbb{T}^{2n})}$ we have that h_i is a bounded i-linear form on $W^{1,2}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$ that depends uniform continuously on w. Moreover we obtain $|r_k(w, \xi)| \leq c_k \| \xi \|^k_{W^{1,2}(\mathbb{R} \times \mathbb{S}^1)}$ for every k, which by the converse to Taylor's Theorem, [**9**], implies that h is smooth and the derivatives are as expected given by

$$D^i h = h_i .$$

Finally, since the linear operator $\overline{\partial}_{J_0} : W^{1,2}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$ is smooth we have that $\Phi_{c_{x^-,x^+}}$ is smooth.

It remains to prove (3.11). To this end let $u \in \mathcal{M}_F(x^-, x^+, H, J_0)$ and set $w = u - c_{x^-,x^+}$. Recall that by Lemma 3.1.4 u and c_{x^-,x^+} are smooth and so is w. By the decay in (*iii*) of Proposition 3.1.7 this leads to $\partial_s w \in L^2(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$. With $w(s,t) = -\int_s^\infty \partial_\tau w(\tau, t) d\tau$ we use again (*iii*) and observe that at the cylindrical ends, i.e., for $R >> 1$ sufficient large there holds

$$\begin{aligned}\int_R^\infty \int_0^1 |w(s,t)|^2 dtds &= \int_R^\infty \int_0^1 \Big| \int_s^\infty \partial_\tau w(\tau,t)d\tau \Big|^2 dtds \leq \int_R^\infty \int_0^1 \left(\int_s^\infty |\partial_\tau w(\tau,t)|d\tau \right)^2 dtds \\ &\leq \frac{c^2}{\delta^2} \int_R^\infty e^{-2|s|\delta} ds < \infty\,,\end{aligned}$$

and analogous on the other end. Hence $w \in L^2(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n})$.

Now for all $(s,t) \in \mathbb{R}\times\mathbb{S}^1$ with $|s| > 1$ we get

$$\begin{aligned}|\partial_t w(s,t)| &= \Big|\partial_s u(s,t) + J_0\Big(\partial_t x^\pm(t) - X_H(w(s,t) + x^\pm(t))\Big)\Big| \\ &\leq |\partial_s u(s,t)| + \|H\|_{C^2(\mathbb{S}^1\times\mathbb{T}^{2n})} |w(s,t)|\ .\end{aligned}$$

Hence $\partial_t w \in L^2(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n})$ and every $u \in \mathcal{M}_F(x^-, x^+, H, J_0)$ can be written as

$$u = w + c_{x^-,x^+}\,,$$

for some $w \in \Phi^{-1}_{c_{x^-,x^+}}(0) \subset W^{1,2}(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n})$. □

3.2. Fredholm-problem and Transversality

In this section we prove that for 1-periodic solutions $x^-, x^+ \in \mathcal{P}_0(H)$ the associated maps $\Phi_{c_{x^-,x^+}}$ are Fredholm, i.e., their differentials are Fredholm operators. By the results of A. Floer, H. Hofer and D. Salamon in [**16**], for a generic set of smooth Hamiltonians H, zero will be a regular value for $\Phi_{c_{x^-,x^+}}$ as long as $x^- \neq x^+$, which by the implicit function theorem [**25**] implies that the moduli spaces are affine translations of smooth submanifolds of $W^{1,2}(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n})$ in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{S}^1, \mathbb{T}^{2n})$.

Consider the operator $F : W^{1,2}(\mathbb{R}\times\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R}\times\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$ defined by

$$F\xi = \partial_s\xi + J_0\partial_t\xi + S\xi\,, \quad J_0 \in \mathcal{J} \quad \text{as in (3.1)}\,, \tag{3.12}$$

where $S \in C^0(\mathbb{R}^2, \mathbb{R}^{2n}\times\mathbb{R}^{2n})$ is a continuous matrix valued function on $\mathbb{R}^2$ satisfying

$$S(s,t) = S(s,t)^T = S(s,t+1)\,, \quad \text{for all} \quad (s,t) \in \mathbb{R}^2\,.$$

Furthermore we assume that $S(s,t)$ converges uniformly in t , as s sends to $\pm\infty$ and we set

$$S^\pm(t) := \lim_{s\to\pm\infty} S(s,t)\,. \tag{3.13}$$

Now we use the standard complex structure J_0 to define paths $\Psi^\pm : [0,1] \longrightarrow \mathrm{Sp}(2n,\mathbb{R})$ in the symplectic group by

$$\partial_t\Psi^\pm(t) = J_0 S^\pm(t)\Psi^\pm(t)\,, \quad \Psi^\pm(0) = \mathrm{Id}\ .$$

If

$$\Psi^\pm \in P^* := \Big\{\gamma \in C^0([0,1], \mathrm{Sp}(2n,\mathbb{R})) \;\Big|\; \gamma(0) = I\,, \quad \det(I - \gamma(1)) \neq 0\Big\}\,, \tag{3.14}$$

then the **Conley-Zehnder index** $\mu : P^* \longrightarrow \mathbb{Z}$ is well defined. We will not give a further discussion here, for details see [**26**],[**29**],[**31**]. For a given nondegenerate Hamiltonian H and $x \in \mathcal{P}_0(H)$ we shortly denote by $\mu(x)$ the index of the associated symplectic path $J_0\nabla^2 H(\cdot, x(\cdot))$.

For the next well known statement we refer to [**31**], alternative proofs can be found for instance in [**14**],[**24**], [**34**].

Theorem 3.2.1. ([**31**]) *Assume that $S(s,t) = S(s,t)^T$ fulfills* (3.13) *and that $\Psi^\pm \in P^*$. Then F is a Fredholm operator with index*

$$\operatorname{ind}(F) = \mu(\Psi^-) - \mu(\Psi^+)\,.$$

Corollary 3.2.2. *Let $x^-, x^+ \in \mathcal{P}_0(H)$ be two 1-periodic solutions, $w \in W^{1,2}(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n})$, c_{x^-,x^+} as in* (3.5) *and $H \in \mathcal{H}_{\text{reg}}$. Then the linearized operator*

$$D\Phi_{c_{x^-,x^+}}(w) : W^{1,2}(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n}) \tag{3.15}$$

$$D\Phi_{c_{x^-,x^+}}(w)\xi = \partial_s\xi + J_0\partial_t\xi + \nabla^2 H(\cdot, w + c_{x^-,x^+})\xi$$

is a Fredholm operator with index

$$\operatorname{ind}\left(D\Phi_{c_{x^-,x^+}}(w)\right) = \mu(x^-) - \mu(x^+)\,.$$

Proof. Let $w \in W^{1,2}(\mathbb{R}\times\mathbb{S}^1, \mathbb{R}^{2n})$, then we can approximate w by a smooth curve v and consider the operator

$$\widetilde{D\Phi}_{c_{x^-,x^+}}(v)\xi = \partial_s\xi + J_0(v + c_{x^-,x^+})\partial_t\xi + \nabla^2 H(\cdot, v + c_{x^-,x^+})\xi\,.$$

Since $H \in \mathcal{H}_{\text{reg}}$ the operator F in (3.12) with $S = \nabla^2 H(\cdot, v + c_{x^-,x^+})$ satisfies condition (3.14) and coincides with $\widetilde{D\Phi}_{c_{x^-,x^+}}(v)$. Now F satisfies the assumptions of Theorem 3.2.1 and is therefore Fredholm of index $\mu_1(x^-) - \mu_1(x^+)$. Since $\Phi_{c_{x^-,x^+}}$ is smooth, the set of Fredholm operators is open with respect to the uniform operator topology, and the index is locally constant, see [**25**], we have that $D\Phi_{c_{x^-,x^+}}(w)$ is also Fredholm, possessing the same index as F. □

Now we cite the announced transversality result stated in [**16**]. Therefore let (M,ω) be a compact symplectic manifold and $\phi : M \longrightarrow M$ be a symplecteomorphism. By $\mathcal{J}_\phi(M,\omega)$ we denote the space of all smooth t-dependent almost complex structures $J = J_t : \mathbb{R} \longrightarrow \operatorname{End}(TM)$ compatible with ω and satisfying $J_t = \phi^* J_{t+1}$, and by $C^\infty_\phi(M)$ the space of all smooth functions $H : \mathbb{R}\times M \longrightarrow \mathbb{R}$ such that $H(t,\cdot) = H(t+1,\cdot)\circ\phi$. Furthermore, for a given $H_0 \in C^\infty_\phi(M)$ possessing only non-degenerate 1-periodic solutions $x \in \mathcal{P}(H_0)$, denote by $C^\infty_\phi(M, H_0)$ the space of all Hamiltonians which coincide with H_0 on $\mathcal{P}(H_0)$ up to second order and by φ_{H_0} the time 1-map of the Hamiltonian flow of H_0.

Finally we say that the **pair** $(H,J) \in C^\infty_\phi(M)\times\mathcal{J}_\phi(M,\omega)$ is **regular** if and only if there are no degenerate 1-periodic solutions and the associated operator

$$F_{J,H}(u) : W^{1,p}_\phi(u^*TM) \longrightarrow L^p_\phi(u^*TM) \tag{3.16}$$

defined with respect to the Levi-Civita connection by

$$F_{J,H}(u)\xi = \nabla_s\xi + \nabla_\xi J_t(u)\nabla_t u + J_t(u)\nabla_t\xi + \nabla_\xi\nabla H(\cdot, u)$$

is onto for every $u \in \mathcal{M}_F(x^-, x^+, H, J)$ with $x^- \neq x^+$. Then the statement is :

THEOREM 3.2.3. (**[16]**) *Let (M, ω) be a compact symplectic manifold and $\phi : M \longrightarrow M$ a symplecteomorphism.*

(i) *Let $H \in C^\infty_\phi(M)$ and assume that the fixed points of φ_H are non-degenerate. Then the set*

$$\mathcal{J}_{\text{reg}} = \mathcal{J}_{\text{reg}}(M, \omega, \phi, H) = \left\{ J \in \mathcal{J}_\phi(M, \omega) \;\middle|\; (H, J) \quad \textit{is a regular pair} \right\}$$

is of second category in $\mathcal{J}_\phi(M, \omega)$.

(ii) *Let $J \in \mathcal{J}_\phi(M, \omega)$ and $H_0 \in C^\infty_\phi(M)$ and assume that the fixed points of φ_{H_0} are non-degenerate. Then the set*

$$\mathcal{H}_{\text{reg}} = \mathcal{H}_{\text{reg}}(M, \omega, \phi, H_0) = \left\{ H \in C^\infty_\phi(M) \;\middle|\; (H, J) \quad \textit{is a regular pair} \right\}$$

is of second category in $C^\infty_\phi(M, H_0)$.

Clearly, the proof uses the Sard-Smale-Theorem 2.2.6, which demands a Banach manifold setting. For this reason the theorem is proven with respect to the $W^{1,p}$, $p > 2$, setup to assure that the spaces $W^{1,p}(\mathbb{R} \times \mathbb{S}^1, M)$ are Banach manifolds, which would not hold if one loses the embedding into $C^0(\mathbb{R} \times \mathbb{S}^1, M)$. In the special case where M possesses a trivial tangent bundle, the spaces $W^{1,2}(\mathbb{R} \times \mathbb{S}^1, M)$ and $L^2(\mathbb{R} \times \mathbb{S}^1, M)$ are Hilbert manifolds. Thus the above statement is not sensitive to the choice of the functional spaces and therefore holds also in our situation.

COROLLARY 3.2.4. *Let $H_0 \in \mathcal{H}_{\text{reg}} \subset C^\infty(\mathbb{S}^1 \times \mathbb{T}^{2n}, \mathbb{R})$ be a Hamiltonian such that all 1-periodic solutions $x \in \mathcal{P}_0(H)$ are contractible and non-degenerate. Then there exists a residual set $\mathcal{H}_{\text{reg}}(H_0) \subset C^\infty(\mathbb{S}^1 \times \mathbb{T}^{2n}, H_0)$ of Hamiltonians that coincide with H_0 on $\mathcal{P}_0(H)$ up to second order such that for all $x^-, x^+ \in \mathcal{P}_0(H_0)$ with $x^- \neq x^+$ the moduli spaces $\mathcal{M}_F(x^-, x^+, H, J_0)$ are smooth manifolds of dimension*

$$\dim\left(\mathcal{M}_F(x^-, x^+, H, J_0)\right) = \mu(x^-) - \mu(x^+)\,.$$

PROOF. We set $M = \mathbb{T}^{2n}$, $\phi = \text{id}_{\mathbb{T}^{2n}}$. Since $H_0 \in \mathcal{H}_{\text{reg}}$ the fixed points of φ_H are non-degenerate and by Theorem 3.2.3, (i) we have that the operator $F_{J_0,H}(u)$ in (3.16) is onto for a residual set $\mathcal{H}_{\text{reg}}(H_0)$ of Hamiltonians that coincide with H_0 on $\mathcal{P}_0(H)$ up to second order. Now the operator $D\Phi_{c_{x^-,x^+}}(w)$ in (3.15) with $H \in \mathcal{H}_{\text{reg}}(H_0)$ and $w = u - c_{x^-,x^+}$ coincides with $F_{J_0,H}(u)$ and is Fredholm of the right index due to Corollary 3.2.2. Hence we conclude the proof by the implicit function theorem and Proposition 3.1.7. □

Finally we have to treat the case when $x^- = x^+$. That is :

LEMMA 3.2.5. *Assume we have chosen a regular pair* (H, J_0) *and* $x \in \mathcal{P}_0(H)$. *Then the moduli space* $\mathcal{M}_F(x, x, H, J_0)$ *is a zero dimensional manifold which consists only of the constant solution. Moreover the linearized operator*

$$D\Phi_{c_{x,x}}(w)\xi = \partial_s \xi + J_0 \partial_t \xi + \nabla^2 H(\cdot, w + c_{x,x})\xi\,, \quad c_{x,x}(s, \cdot) = x \quad \forall s \in \mathbb{R}\,,$$

with $w = x - c_{x,x} = 0 \in W^{1,2}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$*, is an isomorphism.*

PROOF. Let $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{S}^1, \mathbb{T}^{2n})$ be a solution connecting x with itself. Then we have that

$$E(u) = \int_{-\infty}^{\infty} ||\,\partial_s u(s, \cdot)\,||^2_{L^2(\mathbb{S}^1)} ds = A_H(x) - A_H(x) = 0$$

which implies that u is constant as well proving that $\mathcal{M}_F(x, x, H, J_0)$ consists only of the constant solution and is therefore a submanifold of $W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{S}^1, \mathbb{T}^{2n})$. Now $D\Phi_{c_{x,x}}(w)$ is Fredholm of index zero due to Corollary 3.2.2. Let $\xi \in \ker D\Phi_{c_{x,x}}(w)$ then we have that

$$\xi(s, t) = e^{-(J_0\partial_t + S)s}\xi(0, t)\,, \quad \text{with} \quad S(t) = \nabla^2 H(t, x(t))\,.$$

Using the fact that $\xi(s, t) \longrightarrow 0$ for $s \longrightarrow \pm\infty$ we observe that $\xi = 0$. Hence the kernel of $D\Phi_{c_{x,x}}(w)$ is trivial implying that $D\Phi_{c_{x,x}}(w)$ is an isomorphism which proves the claimed result. $\square$

3.3. Compactness

As seen in section 3.1, our choice of the $W^{1,2}$-setup does not touch the fact that the moduli spaces consist of smooth curves. For this reason the classical elliptic bootstrapping methods for elliptic operators can be applied and the usual C^∞_{loc}-Gromov-compactness results hold. In preparation for the compactness statements in chapter 4 we give an alternative proof.

Since we are in the special situation of the torus the following **crucial formulas** can be applied.

LEMMA 3.3.1. *Let* $T > 0$, $\Omega_T = [-T, T] \times \mathbb{S}^1$ *and let* $u \in W^{1,2}(\Omega_T, \mathbb{R}^{2n})$. *Denote furthermore by* $\overline{\partial}_{J_0} = \partial_s + J_0\partial_t$, $\partial_{J_0} = \partial_s - J_0\partial_t$ *with* J_0 *defined in* (3.1) *and by* $\mathbb{P}^\pm$ *the projections as in* (2.3). *Then*

$$\begin{aligned} ||\,\overline{\partial}_{J_0} u\,||^2_{L^2(\Omega_T)} &= ||\,\partial_s u\,||^2_{L^2(\Omega_T)} + ||\,\partial_t u\,||^2_{L^2(\Omega_T)} + ||\,\mathbb{P}^- u(T, \cdot)\,||^2_{H^{1/2}(\mathbb{S}^1)} - ||\,\mathbb{P}^+ u(T, \cdot)\,||^2_{H^{1/2}(\mathbb{S}^1)} \\ &- ||\,\mathbb{P}^- u(-T, \cdot)\,||^2_{H^{1/2}(\mathbb{S}^1)} + ||\,\mathbb{P}^+ u(-T, \cdot)\,||^2_{H^{1/2}(\mathbb{S}^1)}\,, \end{aligned} \tag{3.17}$$

$$\begin{aligned} ||\,\partial_{J_0} u\,||^2_{L^2(\Omega_T)} &= ||\,\partial_s u\,||^2_{L^2(\Omega_T)} + ||\,\partial_t u\,||^2_{L^2(\Omega_T)} + ||\,\mathbb{P}^+ u(T, \cdot)\,||^2_{H^{1/2}(\mathbb{S}^1)} - ||\,\mathbb{P}^- u(T, \cdot)\,||^2_{H^{1/2}(\mathbb{S}^1)} \\ &- ||\,\mathbb{P}^+ u(-T, \cdot)\,||^2_{H^{1/2}(\mathbb{S}^1)} + ||\,\mathbb{P}^- u(-T, \cdot)\,||^2_{H^{1/2}(\mathbb{S}^1)}\,. \end{aligned} \tag{3.18}$$

PROOF. By Lemma 3.1.1 for any $s_0 \in [-T, T]$ the evaluation $u(s_0, \cdot)$ of u is a loop in $\mathbb{M}$ that can be represented with respect to J_0 by

$$u(s_0, t) = [u_0(s)] + \sum_{k \neq 0} e^{2\pi k J_0 t} u_k(s_0), \quad [u_0(s)] \in \mathbb{T}^n, u_k(s_0) \in \mathbb{R}^{2n}$$

Hence

$$\begin{aligned}
2\langle \partial_s u, J_0 \partial_t u \rangle_{L^2(\Omega_T)} &= 2\int_{-T}^{T} \Big(\langle \partial_s u(s,\cdot), \mathbb{P}^- u(s,\cdot) \rangle_{H^{1/2}(\mathbb{S}^1)} - \langle \partial_s u(s,\cdot), \mathbb{P}^+ u(s,\cdot) \rangle_{H^{1/2}(\mathbb{S}^1)} \Big) ds \\
&= \int_{-T}^{T} \Big(\partial_s \| \mathbb{P}^- u(s,\cdot) \|^2_{H^{1/2}(\mathbb{S}^1)} - \partial_s \| \mathbb{P}^+ u(s,\cdot) \|^2_{H^{1/2}(\mathbb{S}^1)} \Big) ds \\
&= \| \mathbb{P}^- u(T,\cdot) \|^2_{H^{1/2}(\mathbb{S}^1)} - \| \mathbb{P}^+ u(T,\cdot) \|^2_{H^{1/2}(\mathbb{S}^1)} \\
&- \| \mathbb{P}^- u(-T,\cdot) \|^2_{H^{1/2}(\mathbb{S}^1)} + \| \mathbb{P}^+ u(-T,\cdot) \|^2_{H^{1/2}(\mathbb{S}^1)}
\end{aligned}$$

and we conclude the proof by the identities

$$\begin{aligned}
\| \overline{\partial}_{J_0} u \|^2_{L^2(\Omega_T)} &= \| \partial_s u \|^2_{L^2(\Omega_T)} + \| \partial_t u \|^2_{L^2(\Omega_T)} + \langle \partial_s u, J_0 \partial_t u \rangle_{L^2(\Omega_T)} \\
\| \partial_{J_0} u \|^2_{L^2(\Omega_T)} &= \| \partial_s u \|^2_{L^2(\Omega_T)} + \| \partial_t u \|^2_{L^2(\Omega_T)} - \langle \partial_s u, J_0 \partial_t u \rangle_{L^2(\Omega_T)} .
\end{aligned}$$

□

Recall that, by Proposition 3.1.7 for any $x^-, x^+ \in \mathcal{P}_0(H)$ the moduli space is the affine subset

$$\Phi^{-1}_{c_{x^-,x^+}}(0) + c_{x^-,x^+} = \mathcal{M}_F(x^-, x^+, H, J_0) \subset W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times \mathbb{S}^1, \mathbb{T}^{2n}) . \tag{3.19}$$

We aim at showing that $\Phi^{-1}_{c_{x^-,x^+}}(0)$ is a $W^{1,2}_{\mathrm{loc}}$- precompact set. For this we need the following result.

LEMMA 3.3.2. *Let $x^-, x^+ \in \mathcal{P}_0(H)$, $w \in \Phi^{-1}_{c_{x^-,x^+}}(0)$ and denote by $\mathbb{P}^0 w(s,\cdot) \in H^1(\mathbb{R}, \mathbb{R}^{2n})$ the constant part of w. Then $\mathbb{P}^0 w \in C^0(\mathbb{R}, \mathbb{R}^{2n})$ and there is a constant $C = C(x^-, x^+) > 0$ such that*

$$\| \mathbb{P}^0 w \|_{C^0(\mathbb{R})} \le C \quad \textit{for all} \quad w \in \Phi^{-1}_{c_{x^-,x^+}}(0) .$$

PROOF. That $\mathbb{P}^0 w \in C^0(\mathbb{R}, \mathbb{R}^{2n})$ is already mentioned in our remark in (3.3). To prove the existence of C we argue by contradiction. Assume there is a sequence $w_n \in \Phi^{-1}_{c_{x^-,x^+}}(0)$ such that

$$\lambda_n := \| \mathbb{P}^0 w_n \|_{C^0(\mathbb{R})} \longrightarrow \infty \quad \text{for} \quad n \longrightarrow \infty .$$

We set $\tilde{w}_n = \frac{1}{\lambda_n} \mathbb{P}^0 w_n$, then $\tilde{w}_n$ builds a sequence of continuous loops $\tilde{w}_n : \mathbb{R} \cup \{\infty\} \longrightarrow \mathbb{R}^{2n}$ starting at 0 with $\| \tilde{w}_n \|_{C^0(\mathbb{R} \cup \{\infty\})} = 1$ for all $n \in \mathbb{N}$. Now each $\tilde{w}_n$ solves the equation

$$\partial_s \tilde{w}_n = -\frac{1}{\lambda_n} \mathbb{P}^0 (\nabla H(w_n + c_{x^-,x^+}) + \partial_s c_{x^-,x^+}) \tag{3.20}$$

in the weak sense. By a standard bootstrapping argument each $\tilde{w}_n$ is actually a strong solution, and by (3.20) we obtain even more:

$$\lim_{n \to \infty} \| \partial_s \tilde{w}_n \|_{C^0(\mathbb{R} \cup \{\infty\})} = 0 .$$

Due to the Arzelà-Ascoli Theorem the sequence $\tilde{w}_n$ is compact. Thus there is a subsequence $\tilde{w}_{n_k}$ which converges uniformly on $\mathbb{R} \cup \{\infty\}$ to a constant loop $\tilde{w}_\infty$ with $\| \tilde{w}_\infty \|_{C^0(\mathbb{R}\cup\{\infty\})} = 1$ and $\tilde{w}_\infty(\infty) = 0$, a contradiction. Hence the sequence λ_n is bounded, which proves the claim. □

THEOREM 3.3.3. *Let $x^-, x^+ \in \mathcal{P}_0(H)$, $H \in \mathcal{H}_{\text{reg}}$, then the moduli space $\mathcal{M}_F(x^-, x^+, H, J_0)$ is a $W^{1,2}_{\text{loc}}$-precompact set.*

PROOF. We show that $\Phi^{-1}_{c_{x^-,x^+}}(0)$ is a $W^{1,2}_{\text{loc}}$-precompact set. Then the claim follows immediately from (3.19).

Step 1: Let $T > 0$, $\Omega_T = [-T, T] \times \mathbb{S}^1$ and $w \in \Phi^{-1}_{c_{x^-,x^+}}(0)$. We show that $\| w \|_{W^{1,2}(\Omega_T)}$ is uniformly bounded by terms of the energy and of the reference cylinder c_{x^-,x^+}. Set $u = w + c_{x^-,x^+}$ then we compute

$$\| \partial_s w \|_{L^2(\Omega_T)} \leq \| \partial_s u \|_{L^2(\Omega_T)} + \| \partial_s c_{x^-,x^+} \|_{L^2(\Omega_T)} \leq \sqrt{E(u)} + \| \partial_s c_{x^-,x^+} \|_{L^2(\Omega_T)} =: c_0(x^-, x^+, c_{x^-,x^+})\,.$$

and

$$\begin{aligned} \| \partial_t w \|_{L^2(\Omega_T)} &\leq \| \partial_t u - X_H(u) \|_{L^2(\Omega_T)} + \| X_H(u) \|_{L^2(\Omega_T)} + \| \partial_t c_{x^-,x^+} \|_{L^2(\Omega_T)} \\ &\leq \sqrt{E(u)} + 2T \| H \|_{C^1(\mathbb{S}^1\times\mathbb{T}^{2n})} + \| \partial_t c_{x^-,x^+} \|_{L^2(\Omega_T)} =: c_1(x^-, x^+, T, H, c_{x^-,x^+})\,. \end{aligned}$$

Observe that

$$\| (\mathbb{P}^0)^\perp w \|^2_{L^2(\Omega_T)} = \int_{-T}^{T} \sum_{k\neq 0} |w_k(s)|^2 ds \leq 4\pi^2 \int_{-T}^{T} \sum_{k\neq 0} |k|^2 |w_k(s)|^2 ds = \| \partial_t w \|^2_{L^2(\Omega_T)} \leq c_1^2\,,$$

where $\mathbb{P}^0$ denotes the restriction to the 0-order coefficients. Hence the t-derivative already bounds the non-constant part of w. We denote by $w_0 = \mathbb{P}^0 w$ the constant part of w then by Lemma 3.3.2 there is a constant $C > 0$ such that $|w_0(0)| \leq C$. So we obtain by writing

$$w_0(s) = \int_0^s \partial_\tau w_0(\tau) d\tau + w_0(0)$$

and by Hölder that

$$\begin{aligned}
\| w_0 \|^2_{L^2(\Omega_T)} &= \int_{-T}^{T}\int_0^1 \Big| \int_0^s \partial_\tau w_0(\tau) d\tau + w(0) \Big|^2 dt ds \\
&\leq 2 \int_{-T}^{T} \left[|w(0)|^2 + \left(\int_0^s |\partial_\tau w_0(\tau)| d\tau \right)^2 \right] ds \\
&\leq 4TC^2 + 2T \int_{-T}^{T}\int_0^s |\partial_\tau w_0(\tau)|^2 d\tau ds \\
&\leq 4TC^2 + 2T \int_{-T}^{T} \| \partial_s w \|^2_{L^2(\Omega_T)} ds \\
&\leq 4TC^2 + 4T^2 c_0^2
\end{aligned}$$

Hence there is a constant $c_2 = c_2(x^-, x^+, T, H, c_{x^-,x^+}, n)$ bounding $\| w \|_{L^2(\Omega_T)}$. Setting $c = (c_0, c_1, c_2)$ we obtain $\| w \|_{W^{1,2}(\Omega_T)} \leq |c|$ and we are done with step 1.

Step 2: Let $T > T' > 0$, $w_1, w_2 \in \Phi^{-1}_{c_{x^-,x^+}}(0)$, set $\delta w = w_1 - w_2$ and $u_i = w_i + c_{x^-,x^+}$, $i = 1, 2$. We claim there is a constant $C_1 = C_1(T, T', H) > 0$ such that

$$\| \partial_s \delta w \|^2_{L^2(\Omega_{T'})} + \| \partial_t \delta w \|^2_{L^2(\Omega_{T'})} \leq C_1 \Big(\| \overline{\partial}_{J_0}(u_1) - \overline{\partial}_{J_0}(u_2) \|^2_{L^2(\Omega_T)} + \| \delta w \|^2_{L^2(\Omega_T)} \Big). \tag{3.21}$$

Assume that (3.21) holds. Since $\overline{\partial}_{J_0,H}(u_i) = 0$ we obtain by the mean value theorem that

$$\begin{aligned}
\| \overline{\partial}_{J_0}(u_1) - \overline{\partial}_{J_0}(u_2) \|_{L^2(\Omega_T)} &= \| J_0(X_H(u_1) - X_H(u_2)) \|_{L^2(\Omega_T)} \\
&\leq \| H \|_{C^2(\mathbb{S}^1 \times \mathbb{T}^{2n})} \| \delta w \|_{L^2(\Omega_T)} .
\end{aligned}$$

So we can find another constant $C_2 = C_2(T, T', H) > 0$ such that

$$\| \delta w \|_{W^{1,2}(\Omega_{T'})} \leq C_2 \| \delta w \|_{L^2(\Omega_T)} \tag{3.22}$$

By step 1 the restriction of any sequence $(w_i) \in \Phi^{-1}_{c_{x^-,x^+}}(0)$ to Ω_T is uniformly bounded. Since the embedding of $W^{1,2}(\Omega_T, \mathbb{R}^{2n})$ into $L^2(\Omega_T, \mathbb{R}^{2n})$ is compact, (w_i) possesses a subsequence convergent in $L^2(\Omega_T, \mathbb{R}^{2n})$, which by (3.22) has to converge already in $W^{1,2}(\Omega_{T'}, \mathbb{R}^{2n})$. Thus $\Phi^{-1}_{c_{x^-,x^+}}(0)$ is $W^{1,2}_{\text{loc}}$-precompact and so is $\mathcal{M}_F(x^-, x^+, H, J_0)$. It remains to prove (3.21). For this purpose let β be a smooth function such that

$$\beta(s) = \begin{cases} 1 & , \quad |s| \leq T' \\ 0 & , \quad s \geq T' + |T - T'|/2 \\ 0 & , \quad s \leq -T' - |T - T'|/2 \end{cases} .$$

Now we apply the formula (3.17) of Lemma 3.3.1 to $\beta\delta w$ and obtain

$$\begin{aligned}
\|\partial_s\delta w\|^2_{L^2(\Omega_{T'})} + \|\partial_t\delta w\|^2_{L^2(\Omega_{T'})} &\le \|\partial_s\beta\delta w\|^2_{L^2(\Omega_T)} + \|\partial_t\beta\delta w\|^2_{L^2(\Omega_T)} \\
&= \|\overline{\partial}_{J_0}(\beta\delta w)\|^2_{L^2(\Omega_T)} \\
&\le 2(1+\|\beta'\|^2_{C^0(\mathbb{R})})\Big(\|(\overline{\partial}_{J_0}(u_1)-\overline{\partial}_{J_0}(u_2))\|^2_{L^2(\Omega_T)} + \|\delta w\|^2_{L^2(\Omega_T)}\Big),
\end{aligned}$$

which proves (3.21) and we conclude the proof. □

Next we want to prove the classical compactification picture saying that the moduli spaces are compact up to broken trajectories. Therefore we need to establish a certain version of the Palais-Smale condition. That is

Lemma 3.3.4. *Let $H \in \mathcal{H}_{\text{reg}}$ be a smooth 1-periodic Hamiltonian possessing only non-degenerate periodic solutions $x \in \mathcal{P}_0(H)$. Consider the action*

$$A_H(p) = \frac{1}{2}\int_0^1 \omega_0(\dot p(t), p(t))dt + \int_0^1 H(t, p(t))dt\,, \qquad p \in H^1(\mathbb{S}^1, \mathbb{T}^{2n})$$

and the non-linear operator

$$-\nabla_{L^2}A_H : H^1(\mathbb{S}^1, \mathbb{T}^{2n}) \longrightarrow L^2(\mathbb{S}^1, \mathbb{R}^{2n})\,, \quad -\nabla_{L^2}A_H(p) = -J_0(\dot p - X_H(p))\,.$$

Then $(A_H, -\nabla_{L^2}A_H)$ is (PS) in the sense that any sequence $(p_n) \subset H^1(\mathbb{S}^1, \mathbb{T}^{2n})$ satisfying

$$\|\nabla_{L^2}A_H(p_n)\|_{L^2(\mathbb{S}^1)} \longrightarrow 0\,, \quad \textit{for} \quad n \longrightarrow \infty\,, \tag{3.23}$$

is compact, while $(A_H(p_n))$ is bounded.

Proof. Let $(p_n) \subset H^1(\mathbb{S}^1, \mathbb{T}^{2n})$ be a sequence such that (3.23) holds, i.e.,

$$\lim_{n\to\infty}\langle\nabla_{L^2}A_H(p_n), v\rangle_{L^2(\mathbb{S}^1)} = 0\,, \quad \text{for all} \quad v \in L^2(\mathbb{S}^1, \mathbb{R}^{2n})\,.$$

Recall that $j^* : L^2(\mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n})$ denotes the compact adjoint operator of the embedding $j : H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ introduced in (2.6), thus

$$\lim_{n\to\infty}\langle\nabla_{L^2}A_H(p_n), j(v)\rangle_{L^2(\mathbb{S}^1)} = \lim_{n\to\infty}\langle j^*\nabla_{L^2}A_H(p_n), v\rangle_{H^{1/2}(\mathbb{S}^1)} = 0$$

holds for all $v \in H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n})$, where the operator

$$-j^*\nabla_{L^2}A_H = -\nabla_{1/2}A_H : H^{1/2}(\mathbb{S}^1, \mathbb{T}^{2n}) \longrightarrow H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n})\,,$$

satisfies the (PS) condition by Lemma 2.5.10. Hence (p_n) is a (PS) sequence for $(-\nabla_{1/2}A_H, A_H)$ in the sense of Definition 2.5.9, saying that $(A_H(p_n))$ is bounded and (p_n) is a compact sequence in $H^{1/2}(\mathbb{S}^1, \mathbb{T}^{2n})$. Let $\varepsilon > 0$, (p_{n_k}) be a convergent subsequence and $k_0 \in \mathbb{N}$ be large enough such that

$$\|\nabla_{L^2}A_H(p_{n_{k_1}}) - \nabla_{L^2}A_H(p_{n_{k_2}})\|_{L^2(\mathbb{S}^1)} \le \varepsilon\,, \quad \|p_{n_{k_1}} - p_{n_{k_2}}\|_{H^{1/2}(\mathbb{S}^1)} \le \varepsilon\,, \quad \forall\, k_1, k_2 \ge k_0\,.$$

Then the estimate

$$\begin{aligned}\|\dot p_{n_{k_1}} - \dot p_{n_{k_2}}\|_{L^2(\mathbb{S}^1)} &\le \|\nabla_{L^2} A_H(p_{n_{k_1}}) - \nabla_{L^2} A_H(p_{n_{k_2}})\|_{L^2(\mathbb{S}^1)} + \|X_H(p_{n_{k_1}}) - X_H(p_{n_{k_2}})\|_{L^2(\mathbb{S}^1)}\\ &\le \varepsilon + \|H\|_{C^2(\mathbb{S}^1\times\mathbb{T}^{2n})}\|p_{n_{k_1}} - p_{n_{k_2}}\|_{L^2(\mathbb{S}^1)}\\ &\le (1+\|H\|_{C^2(\mathbb{S}^1\times\mathbb{T}^{2n})})\varepsilon,\end{aligned}$$

shows that the derivatives build a Cauchy sequence in $L^2(\mathbb{S}^1,\mathbb{R}^{2n})$, which implies that (p_{n_k}) converges in $H^1(\mathbb{S}^1,\mathbb{R}^{2n})$, i.e., (p_n) is compact in $H^1(\mathbb{S}^1,\mathbb{T}^{2n})$. □

The Lemma has a crucial consequence using the following definition.

Definition 3.3.5. *Let $x^-, x^+ \in \mathcal{P}_0(H)$, $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{S}^1,\mathbb{T}^{2n})$ and $\hat u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{S}^1,\mathbb{T}^{2n})/\mathbb{R}$ be the equivalence class of u under the $\mathbb{R}$-action*

$$\mathbb{R}\times W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{S}^1,\mathbb{T}^{2n}) \longrightarrow W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{S}^1,\mathbb{T}^{2n}), \quad (\tau,u)\mapsto u(\cdot+\tau,\cdot).$$

Then $\hat u$ is called a **broken trajectory** *of order r connecting x^- and x^+ if and only if there are finitely many 1-periodic solutions $x_i \in \mathcal{P}_0(H)$, $i = 0,\dots,r$ and associated reparametrization times $\tau_i \in \mathbb{R}$ such that*

$$u(\cdot+\tau_i,\cdot) \in \mathcal{M}_F(x_i, x_{i+1}, H, J_0), \quad \textit{for all} \quad i = 0,\dots,r, \quad x_0 = x^-,\, x_{r+1} = x^+.$$

Proposition 3.3.6. *Let (H, J_0) be a regular pair and let furthermore $x^-, x^+ \in \mathcal{P}_0(H)$ with $x^- \neq x^+$. Then the quotient under the $\mathbb{R}$-action denoted by*

$$\widehat{\mathcal{M}}_F(x^-,x^+,H,J_0) := \mathcal{M}_F(x^-,x^+,H,J_0)/\mathbb{R}$$

is a $\mu_1(x^-) - \mu_1(x^+) - 1$ dimensional $W^{1,2}_{\mathrm{loc}}$-precompact manifold in the sense that for any sequence $(\hat u_n)_{n\in\mathbb{N}}$ in $\widehat{\mathcal{M}}_F(x^-,x^+,H,J_0)$ there exists a subsequence $(\hat u_{n_k})_{k\in\mathbb{N}}$ converging to a broken trajectory.

Proof. Let $u \in \mathcal{M}_F(x^-,x^+,H,J_0)$, $\tau\in\mathbb{R}$, $\tau\neq 0$, then $u(\cdot+\tau,\cdot) = u(\cdot,\cdot)$ if and only if u is constant in s. Since $x^-\neq x^+$ there are no constant curves. Hence $\mathbb{R}$ acts freely on $\mathcal{M}_F(x^-,x^+,H,J_0)$. Moreover, since the norm is $\mathbb{R}$-equivariant in the sense that for any lift $\tilde u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{S}^1,\mathbb{R}^{2n})$ of $u\in\mathcal{M}_F(x^-,x^+,H,J_0)$ it holds that

$$\|\tilde u(\cdot,\cdot)\|_{W^{1,2}(\Omega+(\tau,0),\mathbb{R}^{2n})} = \|\tilde u(\cdot+\tau,\cdot)\|_{W^{1,2}(\Omega,\mathbb{R}^{2n})}$$

for any open and bounded domain $\Omega\subset\mathbb{R}\times\mathbb{S}^1$. We have that the action is smooth and deduce by Theorem 3.3.3 that the quotient $\widehat{\mathcal{M}}_F(x^-,x^+,H,J_0)$ is indeed a precompact manifold of the claimed dimension.

Thus, if $(\hat u_n)\subset\widehat{\mathcal{M}}_F(x^-,x^+,H,J_0)$ there is a subsequence $(\hat u_{n_k})$ with $\lim_{n\to\infty}\hat u_{n_k} = \hat u_\infty$ in $W^{1,2}_{\mathrm{loc}}$. We have to show that $\hat u_\infty$ is a broken trajectory therefore we follow M. Schwarz in [**33**] and adapt the argumentation there to our situation. Consider the non-linear operator

$$\overline{\partial}_{J_0,H} = \partial_s + J_0(\partial_t - X_H(\cdot)) : W^{1,2}_{\mathrm{loc}}(\mathbb{R}\times\mathbb{S}^1,\mathbb{T}^{2n}) \longrightarrow L^2_{\mathrm{loc}}(\mathbb{R}\times\mathbb{S}^1,\mathbb{R}^{2n})$$

Let $\Omega \subset \mathbb{R} \times \mathbb{S}^1$ be an open and bounded domain and $u_1, u_2 \in W^{1,2}_{\text{loc}}(\mathbb{R} \times \mathbb{S}^1, \mathbb{T}^{2n})$ then the estimate

$$\begin{aligned} \| \overline{\partial}_{J_0,H}(u_1) - \overline{\partial}_{J_0,H}(u_2) \|_{L^2(\Omega)} &\leq \| \partial_s(u_1 - u_2)) \|_{L^2(\Omega)} + \| \partial_t(u_1 - u_2) \|_{L^2(\Omega)} \\ &+ \| X_H(u_1) - X_H(u_2) \|_{L^2(\Omega)} \\ &\leq \Big(1 + \| H \|_{C^2(\mathbb{S}^1 \times \mathbb{T}^{2n})}\Big) \| u_1 - u_2 \|_{W^{1,2}(\Omega)} \end{aligned}$$

shows that $\overline{\partial}_{J_0,H}$ is continuous. Hence, if $u_{n_k} \in \mathcal{M}_F(x^-, x^+, H, J_0)$ with $u_{n_k} \longrightarrow u_\infty$ in $W^{1,2}_{\text{loc}}$ then u_∞ solves $\overline{\partial}_{J_0,H}(u_\infty) = 0$ and is therefore smooth by Lemma 3.1.4. By our observation in (3.8) and the $W^{1,2}_{\text{loc}}$-convergence we obtain for all $T_1, T_2 \in \mathbb{R}$ with $T_1 \leq T_2$

$$\begin{aligned} A_H(x^-) - A_H(x^+) &\geq \lim_{k \to \infty} \int_{T_1}^{T_2} \int_0^1 |\partial_\tau u_{n_k}(\tau, t)|^2 dt d\tau = \int_{T_1}^{T_2} \int_0^1 |\partial_\tau u_\infty(\tau, t)|^2 dt d\tau \\ &= A_H(u_\infty(T_1, \cdot)) - A_H(u_\infty(T_2, \cdot)) \geq 0 \,. \end{aligned} \tag{3.24}$$

Hence $E(u_\infty) \leq A_H(x^-) - A_H(x^+)$ and by Proposition 3.1.6 we deduce that $\lim_{s \to \pm\infty} \partial_s u_\infty(s, t) = 0$ uniformly in t which is equivalent to the fact that $u_\infty \in \mathcal{M}_F(y^-, y^+, H, J_0)$ for some $y^\pm \in \mathcal{P}_0(H)$. Furthermore (3.24) implies that the action is monotone decreasing on each curve $u \in \mathcal{M}_F(x^-, x^+, H, J_0)$. Since by Lemma 3.1.1 the evaluation $\text{ev}_T : W^{1,2}([T_1, T_2] \times \mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow \mathbb{E}$, $T_i \in \mathbb{R}$, $i = 1, 2$, $T \in [T_1, T_2]$ is smooth, we obtain furthermore

$$\lim_{k \to \infty} A_H(u_{n_k}(T, \cdot)) = A_H(u_\infty(T, \cdot)) \,, \tag{3.25}$$

by the continuity of A_H on $\mathbb{M}$. Now $A_H(u_{n_k}(T, \cdot)) \in [A_H(x^-), A_H(x^+)]$ for all $T \in \mathbb{R}$ implies that

$$A_H(y^\pm) \in [A_H(x^-), A_H(x^+)] \,.$$

Observe that if $z_1, z_2 \in \mathcal{P}_0(H)$ are such that $A_H(z_1) = A_H(z_2)$ then for any connecting curve u with $\overline{\partial}_{J_0,H}(u) = 0$ we have that

$$E(u) = \int_{-\infty}^{\infty} \| \partial_s u(s, \cdot) \|^2_{L^2(\mathbb{S}^1)} ds = 0 \,.$$

Hence u is constant and therefore $z_1 = z_2$. Consequently we can continue the proof by induction on the critical levels in $[A_H(x^-), A_H(x^+)]$.

Thus either $y^\pm = x^\pm$ and we are done, or without loss of generality $x^- = y^-$ and

$$u_{n_k} \longrightarrow u_\infty \in \mathcal{M}_F(x^-, y^+, H, J_0) \,, \quad \text{in } W^{1,2}_{\text{loc}} \text{ with } \quad A_H(x^+) < A_H(y^+) \,. \tag{3.26}$$

Let $c \in (A_H(y^+), A_H(x^+))$ be a regular value and $(\tau_k) \subset \mathbb{R}$ be a sequence of reparametrization times such that $A_H(u_{n_k}(\tau_k, \cdot)) = c$, for all $k \in \mathbb{N}$. Then necessarily we have that $\tau_k \longrightarrow +\infty$. Otherwise due to (3.26) there would be $T \in \mathbb{R}$, $k_0 \in \mathbb{N}$ with $\tau_k \leq T$ for all $k \geq k_0$ and $A_H(u_{n_k}(T, \cdot)) > A_H(y^+)$ contradicting $A_H(u_{n_k}(\tau_k, \cdot)) = c$.

Let $s \in \mathbb{R}$, then, by the mean value theorem there exist sequences $\tau_k^+ = \tau_k^+(s) \in (\tau_k, \tau_k + s)$, $\tau_k^- = \tau_k^-(s) \in (\tau_k + s, \tau_k)$ depending on whether $s > 0$ or $s < 0$, such that

$$A_H(u_{n_k}(\tau_k, \cdot)) - A_H(u_{n_k}(\tau_k + s, \cdot)) \quad = \quad \| \nabla_{L^2} A_H(u_{n_k}(\tau_k^\pm, \cdot)) \|^2_{L^2(\mathbb{S}^1)} \cdot s \,.$$

Since the left-hand side is bounded Lemma 3.3.4 implies that for $s \longrightarrow \pm\infty$ the sequence $(u_{n_k}(\tau_k^\pm, \cdot))$ is a (PS) sequence for $\nabla_{L^2} A_H$, i.e., by taking a further subsequence if necessary, we can assume that

$$\lim_{k\to\infty} u_{n_k}(\tau_k^\pm, \cdot) = z^\pm \in \mathcal{P}_0(H)\,.$$

Since $\tau_k \longrightarrow +\infty$, for each $T > 0$ we can find $k_0 = k_0(T)$ such that

$$A_H(y^+) > A_H(u_{n_k}(-T + \tau_k, \cdot)) > A_H(u_{n_k}(\tau_k, \cdot)) = c \quad \forall\, k \geq k_0\,.$$

Hence, for $s \in \mathbb{R}$ and k large enough, we get the estimates

$$A_H(y^+) \geq A_H(u_{n_k}(\tau_k^-(s), \cdot)) \geq c > A_H(x^+)\,, \quad A_H(y^+) > c \geq A_H(u_{n_k}(\tau_k^+(s), \cdot)) \geq A_H(x^+)\,,$$

which implies that $z^+ = x^+$ or $A_H(z^+) \in (c, A_H(x^+))$ and $z^- = y^+$ or $A_H(z^-) \in (A_H(y^+), c)$. If possible we can iterate the above argument for regular values $c_1 \in (A_H(y^+), A_H(z^-))$, $c_2 \in (A_H(z^+), A_H(x^+))$ and conclude by induction and the fact that there are only finitely many critical values in $(A_H(x^+), A_H(x^-))$ that $\hat{u}_\infty$ is a broken trajectory connecting x^- and x^+ of finite order.

□

By precisely the same argument as used for Corollary 2.5.4 we obtain the analogous conclusion.

Corollary 3.3.7. *Let (H, J_0) be a regular pair and $x^-, x^+ \in \mathcal{P}_0(H)$ such that $\mu(x^-) - \mu(x^+) = 1$. Then $\widehat{\mathcal{M}}_F(x^-, x^+, H, J_0)$ is already compact and therefore a finite set of points.*

3.4. The Floer complex

In this section we follow [**30**] to give a short summary of the construction of the Floer - complex. The main ingredients are the compactness and transversality results proven in the previous sections and Floer's gluing theorem.

Let $H \in \mathcal{H}_{\text{reg}}$ be given, then by Corollary 3.2.4 we can assume that up to a small perturbation which leaves the periodic orbits fixed, H is such that (H, J_0) is a regular pair. Denote by $C_k(H)$ the free Abelian group generated by the elements $x \in \mathcal{P}_0(H)$ with Maslov index $\mu(x) = k \in \mathbb{Z}$, i.e.,

$$C_k(H) = \bigoplus_{\substack{x \in \mathcal{P}_0(H),\\ \mu(x) = k}} \mathbb{Z}x\,.$$

By the results of A. Floer and H. Hofer in [**13**] the moduli spaces $\mathcal{M}_F(x^-, x^+, H, J_0)$ can be oriented in a way that is coherent with gluing. That means for the one dimensional moduli spaces one can define a number $\varepsilon(u) \in \{-1, +1\}$ for all $u \in \mathcal{M}_F(x^-, x^+, H, J_0)$, $\mu(x^-) - \mu(x^+) = 1$

by comparing this orientation with the obvious flow orientation. The boundary operator is then defined generator wise as

$$\partial_k^F(H,J_0) : C_k(H) \longrightarrow C_{k-1}(H)\,, \quad \partial_k^F(H,J_0)x = \sum_{\substack{y\in\mathcal{P}_0(H)\,,\\ \mu(x)-\mu(y)=1}} \nu(x,y)y\,,$$

where

$$\nu(x,y) = \sum_{\hat{u}\in\widehat{\mathcal{M}_F}(x,y,H,J_0)} \varepsilon(u)\,,$$

which is a finite and well defined sum due to compactness and transversality. The fact that indeed $\partial_{k-1}^F(H,J_0)\circ\partial_k^F(H,J_0) = 0$ is a consequence of Floer's gluing theorem stated in [**13**]. Therefore the above setup describes a complex which we call the **Floer - complex** of (H,J_0). The homology of such a complex is called the **Floer homology** of (H,J_0) :

$$HF_k := \frac{\ker\partial_k^F(H,J_0)}{\operatorname{ran}\partial_{k+1}^F(H,J_0)}$$

Due to [**15**] this homology is known to be chain-isomorphic to the singular homology of the torus. A consequence of this fact was Floer's proof of the Arnold conjecture. A more detailed discussion can also be found in [**21**].

If we restrict us to the case of $\mathbb{Z}_2$ coefficients, i.e.,

$$C_k(H) = \bigoplus_{\substack{x\in\mathcal{P}_0(H)\,,\\ \mu(x)=k}} \mathbb{Z}_2 x\,.$$

then the above setup still describes a complex. The notion of orientations is not needed in this case and we will use this fact to construct the chain-isomorphism at this level at the end of this thesis.

CHAPTER 4

The chain isomorphism

The construction of the isomorphism is based on considering the **moduli spaces of hybrid type curves**. This involves a **new boundary value problem** for a type of Cauchy - Riemann equations. To handle this situation and develop new proofs of the Fredholm property and the compactness statements are aims of this chapter.

4.1. Moduli spaces of hybrid type curves

We still consider the torus $(\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}, \omega_0)$ with standard symplectic structure induced from $\mathbb{R}^{2n}$, a smooth 1-periodic nondegenerate Hamiltonian function $H \in \mathcal{H}_{\text{reg}}$ in the sense of Proposition 2.2.5 and its Hamiltonian vector field X_H. J_0 still denotes the special complex structure from (3.1) and by $Z := [0, \infty) \times \mathbb{S}^1$ we denote the half-cylinder. Since we must perturb the vector field $\mathbb{X} = -\nabla_{1/2} A_H$ to achieve transversality we formulate our results in a more general setting. That is to consider a C^3- Morse vector field X with globally defined flow, such that A_H is a Lyapunov function for X and the subbundle $\mathcal{V} = \mathbb{M} \times (\mathbb{R}^n \times \mathbb{H}^+)$ is admissible for X according to Definition 2.3.8.

Definition 4.1.1. *Given Hamiltonian $H \in \mathcal{H}_{\text{reg}}$, C^3- Morse vector field X on $\mathbb{M} = H^{1/2}(\mathbb{S}^1, \mathbb{T}^{2n})$ with globally defined flow such that A_H is a Lyapunov function for X then* $\operatorname{sing}(X) = \operatorname{crit}(A_H)$ *and for contractible 1-periodic orbits $x^\pm \in \operatorname{sing}(X)$ we define the* **moduli-spaces of hybrid type curves** *by*

$$\mathcal{M}_{\text{hyb}}(x^-, x^+, H, J_0, X) := \left\{ u \in W^{1,2}_{\text{loc}}(Z, \mathbb{T}^{2n}) \;\middle|\; \overline{\partial}_{J_0,H}(u) = 0\,, u(0,\cdot) \in \mathcal{W}^u_X(x^-)\,, u(+\infty,\cdot) = x^+ \right\},$$

where $\mathcal{W}^u_X(x^-)$ denotes the unstable manifold of x^- with respect to X and $\overline{\partial}_{J_0,H}(u) = 0$ shall be understood in the weak sense.

To concretize the boundary conditions we recall that by Lemma 3.1.1 the evaluation $u \mapsto u(s,\cdot)$ is smooth in u and therefore the condition $u(0,\cdot) \in \mathcal{W}^u_X(x^-)$ is well-posed. Similarly to Definition 3.1.2 we choose a smooth reference cylinder $c_{x^+} \in C^\infty([0,\infty) \times \mathbb{S}^1, \mathbb{T}^{2n})$ satisfying

$$c_{x^+}(s,\cdot) = \begin{cases} [0]\,, & \textit{for} \quad s \le 1 \\ x^+\,, & \textit{for} \quad s \ge 2 \end{cases} \tag{4.1}$$

and recall that, by our observation in (3.1) *each* $u \in W^{1,2}_{\mathrm{loc}}(Z, \mathbb{T}^{2n})$ *can be written as*

$$u(s,t) = [u_0(s)] + \sum_{k \neq 0} e^{2\pi k J_0 t} u_k(s)\,,$$

with continuous on s dependent Fourier-coefficients. So we set $\lim\limits_{s \to +\infty} u(s,\cdot) = x^+$ *if and only if*

$$\begin{aligned} \lim_{s \to +\infty} (\mathbb{P}^0)^{\perp}\, \mathrm{ev}_s(u - c_{x^+}) &= 0 \quad \textit{in} \quad \mathbb{H}\,, \\ \lim_{s \to +\infty} \mathbb{P}^0\, \mathrm{ev}_s(u - c_{x^+}) &= 0 \quad \textit{on} \quad \mathbb{T}^{2n}\,, \end{aligned}$$

where $\mathbb{P}^0$ *denotes the restriction to the constant loops defined in* (2.3).

We mark out some crucial facts :

(i) By the inner regularity property, see Lemma 3.1.4, the curves $u \in \mathcal{M}_{\mathrm{hyb}}(x^-, x^+, H, J_0, X)$ turn out to be smooth on $(0,\infty) \times \mathbb{S}^1$. Since the evaluation map is smooth in u but not even continuous in s, the loop $u(0,\cdot)$ needs not be continuous it can be any arbitrary element of $\mathbb{M}$.

(ii) Let $T > 0$, $u \in \mathcal{M}_{\mathrm{hyb}}(x^-, x^+, H, J_0, X)$ then we compute

$$\begin{aligned} \int_{1/T}^{T}\int_0^1 |\partial_s u(s,t)|^2 dtds &= -\int_{1/T}^{T}\int_0^1 \langle \partial_s u(s,t), J(\partial_t u(s,t) - X_H(u(s,t)))\rangle\, dtds \\ &= -\int_{1/T}^{T}\int_0^1 \omega_0\big(\partial_s u(s,t), \partial_t u(s,t) - X_H(u(s,t))\big) dtds \\ &= -\int_{1/T}^{T} \partial_s A_H(u(s,\cdot))ds \\ &= A_H(u(1/T,\cdot)) - A_H(u(T,\cdot)) < \infty\,. \end{aligned}$$

By continuity of A_H on $\mathbb{M}$ we get $E(u) = A_H(u(0,\cdot)) - A_H(x^+)$. If, furthermore, A_H is a Lyapunov function for X, then $A_H(u(0,\cdot)) \leq A_H(x^-)$ holds.

(iii) Proposition 3.1.6 implies that $\lim\limits_{s\to\infty} u(s,\cdot) = x^+$ and $\lim\limits_{s\to\infty} \partial_s(u,\cdot) = 0$ uniformly in t. Furthermore, for $T > 0$ large enough and every u, there exist constants $c, \delta > 0$ such that

$$|\partial_s u(s,t)| \leq c e^{-\delta |s|}$$

for all $(s,t) \in [T,\infty) \times \mathbb{S}^1$.

As in the previous chapter we want to understand the moduli spaces as the affine translation by the reference cylinder c_{x^+} of the zero set of a certain map. In this regard we need the following statement.

LEMMA 4.1.2. *Let H be a nondegenerate Hamiltonian, X be a C^k- Morse vector field, $k \geq 1$, on $\mathbb{M}$ with globally defined flow, such that the Hamiltonian action A_H is a Lyapunov function for X and furthermore $\mathcal{V} = \mathbb{M} \times (\mathbb{R}^n \times \mathbb{H}^+)$ is admissible for X as in Definition 2.3.8. Then the set*

$$W^{1,2}_{\mathcal{W}^u_X(x)}(Z, \mathbb{R}^{2n}) := \left\{ w \in W^{1,2}(Z, \mathbb{R}^{2n}) \;\middle|\; [0] + w(0, \cdot) \in \mathcal{W}^u_X(x) \right\}, \quad [0] \in \mathbb{T}^{2n}\ , x \in \operatorname{sing}(X)$$

is a C^k- Hilbert submanifold of $W^{1,2}(Z, \mathbb{R}^{2n})$ with inner product $\langle \cdot, \cdot \rangle_{W^{1,2}(Z)}$.

PROOF. Since we are in a Hilbert manifold setting and A_H is a Lyapunov function for X by Theorem 2.3.7, there is a C^k-embedding

$$F^u : E^u \longrightarrow \mathbb{M}, \quad \text{with} \quad F^u(E^u) = \mathcal{W}^u_X(x),$$

where E^u denotes the unstable eigenspaces of $DX(x)$. We consider the map

$$\lambda : W^{1,2}(Z, \mathbb{R}^{2n}) \times E^u \longrightarrow \mathbb{M}, \quad (w, v) \mapsto w(0) - F^u(v).$$

Observe that

$$\lambda^{-1}([0]) = \left\{ (w, v) \in W^{1,2}(Z, \mathbb{R}^{2n}) \times E^u \;\middle|\; w(0) - F^u(v) = [0] \right\}. \tag{4.2}$$

The derivative at (w, v) is given by

$$D\lambda(w, v) : W^{1,2}(Z, \mathbb{R}^{2n}) \times E^u \longrightarrow H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n}), \quad (\xi, \eta) \mapsto \xi(0, \cdot) - DF^u(v)\eta,$$

which is obviously onto. Furthermore set $p = F^u(v)$ then

$$\ker D\lambda(w, v) \cong W^{1,2}_{T_p\mathcal{W}^u_X(x)}(Z, \mathbb{R}^{2n})$$

is complemented in $W^{1,2}(Z, \mathbb{R}^{2n}) \times E^u$ if $T_p\mathcal{W}^u_X(x)$ is complemented in $\mathbb{E}$, which is certainly true since $\mathcal{V}$ is admissible for X and therefore $T_p\mathcal{W}^u_X(x)$ is a compact perturbation of $\mathcal{V}$ which is complemented by the infinite dimensional space $\mathbb{R}^n \times \mathbb{H}^-$. Hence, $\lambda^{-1}([0])$ is a C^k- Hilbert manifold with inner product. Let

$$\pi : W^{1,2}(Z, \mathbb{R}^{2n}) \times E^u \longrightarrow W^{1,2}(Z, \mathbb{R}^{2n})$$

be the projection onto the first factor, then, by (4.2), the restriction of π to $\lambda^{-1}([0])$ is a C^k-embedding into $W^{1,2}(Z, \mathbb{R}^{2n})$ with $\pi(\lambda^{-1}([0])) = W^{1,2}_{\mathcal{W}^u_X(x)}(Z, \mathbb{R}^{2n})$, which proves the claim. □

PROPOSITION 4.1.3. *Let the assumptions of Lemma 4.1.2 be fulfilled then the non-linear operator*

$$\Theta_{x^-, c_{x^+}} : W^{1,2}_{\mathcal{W}^u_X(x^-)}(Z, \mathbb{R}^{2n}) \longrightarrow L^2(Z, \mathbb{R}^{2n}), \quad w \mapsto \overline{\partial}_{J_0, H}(w + c_{x^+}).$$

is well defined, of class C^k and

$$\mathcal{M}_{\text{hyb}}(x^-, x^+, H, J_0, X) = c_{x^+} + \Theta^{-1}_{x^-, c_{x^+}}(0) \subset W^{1,2}_{\text{loc}}(Z, \mathbb{T}^{2n}).$$

PROOF. The proof is completely analogous to the proof of Proposition 3.1.7. □

4.2. Fredholm problem and Transversality

In this chapter we prove that if the Morse vector field X on $\mathbb{M}$ is of the form $\mathbb{X}_K = -\nabla_{1/2}A_H + K$ with $K \in \mathcal{K}_{\theta,r}$, $r > 0$, from (2.25) and is sufficient close to $-\nabla_{1/2}A_H$, then for all 1-periodic solutions $x^-, x^+ \in \mathcal{P}_0(H)$ the associated maps $\Theta_{x^-,c_{x^+}}$ are Fredholm. Furthermore, we extend our transversality result in chapter 2 in the sense that there is a residual set of compact maps $\widehat{\mathcal{K}}_{\mathrm{reg}} \subset \mathcal{K}_{\theta,r}$ such that 0 becomes a regular value for $\Theta_{x^-,c_{x^+}}$ and $\mathbb{X}_K$ satisfies the Morse-Smale condition up to order 2. So both, the intersections of stable and unstable manifolds with respect to $\mathbb{X}_K$ and the moduli spaces of hybrid type curves are manifolds. If, in addition, (H, J_0) is a regular pair, then also the moduli spaces of Floer cylinders are manifolds, which shows that for generic data all three problems can be treated simultaneously.

Recall that $\mathcal{K}(\mathbb{M},\mathbb{E}) \subset C^3_b(\mathbb{M},\mathbb{E})$ denotes the closed subspace of all bounded and compact C^3-vector fields on $\mathbb{M}$. Furthermore for $\theta \in C^1(\mathbb{M},\mathbb{R}^+)$ with

(i) $\theta(x) = 0$ for all $x \in \mathrm{crit}(A_H)$

(ii) $\theta(p) > 0$ for all $p \in \mathbb{M} \setminus \mathrm{crit}(A_H)$.

(iii) $\theta(p) \leq \frac{1}{2}\| \nabla_{1/2}A_H(p) \|_{H^{1/2}(\mathbb{S}^1)}$ for all $p \in \mathbb{M}$.

we consider the Banach space

$$\mathcal{K}_\theta := \Big\{ K \in \mathcal{K}(\mathbb{M},\mathbb{E}) \ \Big|\ \exists\, c > 0 \text{ such that } \| K(p) \|_{H^{1/2}(\mathbb{S}^1)} \leq c\theta(p)\,, \forall p \in \mathbb{M} \Big\}$$

with norm

$$\| K \|_\theta := \sup_{p\in\mathbb{M}\setminus \mathrm{sing}(X)} \frac{\| K(p) \|_{H^{1/2}(\mathbb{S}^1)}}{\theta(p)} + \| \nabla K \|_{C^2(\mathbb{M},\mathbb{E})}\,.$$

Let $\mathcal{K}_{\theta,r}$, $r > 0$ be the open ball of radius r in $\mathcal{K}_\theta$, then in Lemma 2.4.2 we have shown that for $r = 1$ and $\mathbb{X}_K := -\nabla_{1/2}A_H + K$, $K \in \mathcal{K}_{\theta,r}$

(i) $\mathrm{sing}(\mathbb{X}_K) = \mathrm{crit}(A_H)$.

(ii) $D\mathbb{X}_K(x) = -D^2A_H(x)$ for all $x \in \mathrm{crit}(A_H)$.

(iii) A_H is a Lyapunov function for $\mathbb{X}_K$.

Thus the dynamics of $\mathbb{X}_K$ do not qualitatively differ from those of $-\nabla_{1/2}A_H$.

Theorem 4.2.1. *Let $H \in \mathcal{H}_{\mathrm{reg}}$, $r > 0$ and $K \in \mathcal{K}_{\theta,r}$. Then, if r was chosen sufficient small, for any $x^\pm \in \mathcal{P}_0(H)$ and any $w \in W^{1,2}_{\mathcal{W}^u_{\mathbb{X}_K}(x^-)}(Z,\mathbb{R}^{2n})$, the linearized operator*

$$D\Theta_{x^-,c_{x^+}}(w) : W^{1,2}_{T_p\mathcal{W}^u_{\mathbb{X}_K}(x^-)}(Z,\mathbb{R}^{2n}) \longrightarrow L^2(Z,\mathbb{R}^{2n}) \tag{4.3}$$

with

$$D\Theta_{x^-,c_{x^+}}(w)\eta = \partial_s\eta + J_0\partial_t\eta + \nabla^2 H(\cdot, w + c_{x^+})\eta$$

and $p := [0] + w(0,\cdot)$, is a Fredholm operator of index

$$\mathrm{ind}\, D\Theta_{x^-,c_{x^+}}(w) = m(x^-, \mathcal{V}) - \mu(x^+)\,,$$

where $m(x^-, \mathcal{V})$ denotes the relative Morse index of x^- relative to $\mathcal{V} = \mathbb{M} \times (\mathbb{R}^n \times \mathbb{H}^+)$, and $\mu(x^+)$ the Conley-Zehnder index of x^+.

The proof splits into several results. First we prove the Fredholm-property, then we compute the index in a special situation, and in the end we use a homotopy argument to obtain the desired statement.

As in section 3.2 we consider a continuous symmetric matrix valued function $S \in C^0(Z, \mathbb{R}^{2n} \times \mathbb{R}^{2n})$, and assume that

$$\lim_{s \to +\infty} S(s,t) = S^+(t) \in C^0(\mathbb{S}^1, \mathbb{R}^{2n} \times \mathbb{R}^{2n}) \tag{4.4}$$

uniformly in t and define a symplectic path $\Psi^+ \in C^1([0,1], \mathrm{Sp}(2n, \mathbb{R}))$ by

$$\partial_t \Psi^+(t) = J_0 S^+(t) \Psi^+(t), \quad \Psi^+(0) = \mathrm{Id}\ . \tag{4.5}$$

and recall that the Conley-Zehnder index $\mu(\Psi^+)$ is well defined if

$$\det(I - \Psi^+(1)) \neq 0\,. \tag{4.6}$$

Lemma 4.2.2. *Let $S \in C^0(Z, \mathbb{R}^{2n} \times \mathbb{R}^{2n})$ such that* $\det(I - \Psi^+(1)) \neq 0$ *holds for Ψ^+ from* (4.5). *Then the operator*

$$\overline{D}_{J_0} + S^+ = \partial_s + J_0 \partial_t + S^+ : W^{1,2}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$$

and its formal adjoint $D_{J_0} + S^+ = -\partial_s + J_0 \partial_t + S^+$ are isomorphisms. In particular there are constants $c_1, c_2 > 0$ such that

$$\|\xi\|_{W^{1,2}(\mathbb{R}\times\mathbb{S}^1)} \leq c_1 \|(\overline{D}_{J_0} + S^+)\xi\|_{L^2(\mathbb{R}\times\mathbb{S}^1)}\,, \quad \|\xi\|_{W^{1,2}(\mathbb{R}\times\mathbb{S}^1)} \leq c_2 \|(D_{J_0} + S^+)\xi\|_{L^2(\mathbb{R}\times\mathbb{S}^1)}\,. \tag{4.7}$$

for all $\xi \in W^{1,2}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$.

Detailed proofs can be found in [**34**] or [**30**]. We give a short sketch only.

Proof. Consider the operator $A = J_0 \partial_t + S^+ : W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{S}^1, \mathbb{R}^{2n})$. This is an unbounded, self-adjoint operator on the Hilbert space $L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ with domain $W^{1,2}(\mathbb{S}^1, \mathbb{R}^{2n})$ and compact resolvent. Hence the spectrum is discrete and by the nondegeneracy condition on S there is no eigenvalue 0 and A is invertible. Thus there is a splitting

$$L^2(\mathbb{S}^1, \mathbb{R}^{2n}) = E^u(A) \oplus E^s(A)$$

into the positive and negative eigenspaces of A with associated projections P^u, P^s. Denote by $A^u = A_{|E^u}$, $A^s = A_{|E^s}$ the restriction of A to its eigenspaces and define $K : \mathbb{R} \longrightarrow \mathcal{L}(L^2(\mathbb{S}^1, \mathbb{R}^{2n}))$ by

$$K(\tau) = \begin{cases} e^{-A^u s} P^u\,, & \text{for } \tau \geq 0 \\ -e^{-A^s s} P^s\,, & \text{for } \tau < 0 \end{cases}\,.$$

K is discontinuous at 0 and strongly continuous for $\tau \neq 0$. Moreover

$$\|K(\tau)\| \leq e^{-\delta|\tau|}\,, \quad \text{for} \quad \delta = \min\{\mathrm{spec}\,(A^u), \mathrm{spec}\,(-A^s)\}\ . \tag{4.8}$$

Now we define

$$T : L^2(\mathbb{R}, L^2(\mathbb{S}^1, \mathbb{R}^{2n})) \longrightarrow W^{1,2}(\mathbb{R}, L^2(\mathbb{S}^1, \mathbb{R}^{2n})) \cap L^2(\mathbb{R}, H^1(\mathbb{S}^1, \mathbb{R}^{2n})) = W^{1,2}(\mathbb{R} \times \mathbb{S}^1, \mathbb{R}^{2n})$$

by

$$T(\eta(s)) = \int_{-\infty}^{\infty} K(s-\tau)\eta(\tau)d\tau\,.$$

Obviously, (4.8) implies that T is a bounded operator. Using the splitting $E^u(A) \oplus E^s(A)$, a simple calculation shows that T is indeed the inverse of $\overline{D}_{J_0} + S^+$, which in particular proves (4.7) with $c_1 = \| T \|$. Finally we consider the operator $-A$ and prove the analogous statement for $D_{J_0} + S^+$. □

The most difficult part in the proof of the Fredholm property in Theorem 4.2.1 is to show that the cokernel is finite dimensional. Therefore we need a certain elliptic regularity result which replaces the already known result for Floer half cylinders with Lagrangian boundary condition, see [**25**] or [**34**]. The result is established in step 2 of the proof of the following statement using again the notion of compact perturbations introduced in section 2.3.

Proposition 4.2.3. *Let $W \subset \mathbb{E}$ be a compact perturbation of $\mathbb{R}^n \times \mathbb{H}^+$ and $S \in C^0(Z, \mathbb{R}^{2n} \times \mathbb{R}^{2n})$ be symmetric such that* (4.4) *and* (4.6) *hold. Then*

$$\overline{D}_W := \overline{D}_{J_0} + S : W^{1,2}_W(Z, \mathbb{R}^{2n}) \longrightarrow L^2(Z, \mathbb{R}^{2n})\,,$$

with $\overline{D}_{J_0} = \partial_s + J_0\partial_t$, is a Fredholm operator and its cokernel is given by

$$\operatorname{coker}(\overline{D}_W) = \left\{\rho \in W^{1,2}(Z, \mathbb{R}^{2n}) \;\middle|\; (D_{J_0} + S)\rho = 0\,, \quad \rho(0,\cdot) \in W^\perp \subset \mathbb{E}\right\}.$$

Proof. **Step 1:** By the formula in (3.17), we have that

$$\| \partial_s\xi \|^2_{L^2(Z)} + \| \partial_t\xi \|^2_{L^2(Z)} = \| \overline{D}_{J_0}\xi \|^2_{L^2(Z)} - \| \mathbb{P}^+\xi(0,\cdot) \|^2_{H^{1/2}(\mathbb{S}^1)} + \| \mathbb{P}^-\xi(0,\cdot) \|^2_{H^{1/2}(\mathbb{S}^1)}\,.$$

We recall that $\| S \|_{C^0(Z)} > 0$. Therefore we can choose a constant $c_1 \geq 1 + 1/\| S \|_{C^0(Z)}$ and obtain

$$\| \xi \|_{W^{1,2}(Z)} \leq c_1\Big(\| (\overline{D}_{J_0} + S)\xi \|_{L^2(Z)} + \| S \|_{C^0(Z)} \|\xi\|_{L^2(Z)} + \| \mathbb{P}^-\xi(0,\cdot) \|^2_{H^{1/2}(\mathbb{S}^1)}\Big)\,. \tag{4.9}$$

Since by Lemma 4.2.2 the operator $\overline{D}_{J_0} + S^+ : W^{1,2}(Z, \mathbb{R}^{2n}) \longrightarrow L^2(Z, \mathbb{R}^{2n}))$ is an isomorphism, there is $R > 0$ and $c_2 > 0$ such that

$$\| \xi \|_{W^{1,2}([R,+\infty)\times\mathbb{S}^1)} \leq c_2\| (\overline{D}_{J_0} + S)\xi \|_{L^2([R,+\infty)\times\mathbb{S}^1)}\,, \tag{4.10}$$

for all ξ with $\operatorname{supp}\xi \subset [R, \infty) \times \mathbb{S}^1$. We choose a smooth cut-off function β with

$$\beta(s) = \begin{cases} 1\,, & s \leq R \\ 0\,, & s \geq R+1 \end{cases}$$

use (4.9) for $\beta\xi$ and (4.10) for $(1-\beta)\xi$ and compute

$$\begin{aligned}
\|\xi\|_{W^{1,2}(Z)} &\leq \|\beta\xi\|_{W^{1,2}(Z)} + \|(1-\beta)\xi\|_{W^{1,2}(Z)} \\
&\leq c_1\Big(\|(\overline{D}_{J_0}+S)\beta\xi\|_{L^2(Z)} + \|S\|_{C^0(Z)}\|\beta\xi\|_{L^2(Z)} + \|\mathbb{P}^-\xi(0,\cdot)\|^2_{H^{1/2}(\mathbb{S}^1)}\Big) \\
&+ c_2\|(\overline{D}_{J_0}+S)(1-\beta)\xi\|_{L^2(Z)} \\
&\leq c_3\Big(\|(\overline{D}_{J_0}+S)\xi\|_{L^2(Z)} + \|\xi\|_{L^2([0,R+1]\times\mathbb{S}^1)} + \|\mathbb{P}^-\xi(0,\cdot)\|^2_{H^{1/2}(\mathbb{S}^1)}\Big). \qquad (4.11)
\end{aligned}$$

Since W is assumed to be a compact perturbation of $\mathbb{R}^n \times \mathbb{H}^+$, the operator $\mathbb{P}^- : W \longrightarrow \mathbb{H}^-$ is compact, furthermore the restriction $W^{1,2}(Z) \longrightarrow L^2([0,R+1]\times\mathbb{S}^1)$ is also compact. Hence (4.11) implies that $\overline{D}_{J_0}+S$ is a semi-Fredholm operator, i.e., it is a bounded operator with finite dimensional kernel and closed range, see [**33**] .

Step 2: We have to show that the cokernel is finite dimensional. Therefore let $\eta \in L^2(Z,\mathbb{R}^{2n})$ be an annihilator of the range of

$$\overline{D}_{J_0}+S : W^{1,2}_W(Z,\mathbb{R}^{2n}) \longrightarrow L^2(Z,\mathbb{R}^{2n}),$$

i.e., $\big\langle \overline{D}_{J_0}+S\xi,\eta\big\rangle_{L^2(Z)} = 0$ for all $\xi \in W^{1,2}_W(Z,\mathbb{R}^{2n}) = \big\{\rho \in W^{1,2}(Z,\mathbb{R}^{2n}) \;\big|\; \rho(0,\cdot) \in W\big\}$. Then η is a weak solution of $D_{J_0}\eta = -S\eta =: -f$ on Z, i.e., for all $\phi \in C^\infty_0(Z,\mathbb{R}^{2n})$ with $\phi(0,\cdot) \in W$

$$\langle \overline{D}_{J_0}\phi,\eta\rangle_{L^2(Z)} = -\langle\phi,f\rangle_{L^2(\Omega)}, \quad \text{with} \quad \|f\|_{L^2(Z)} \leq \|S\|_{C^0(Z)}\|\eta\|_{L^2(Z)}. \qquad (4.12)$$

We claim that η is of class $W^{1,2}$ and more precisely

$$\eta \in \big\{\rho \in W^{1,2}(Z,\mathbb{R}^{2n}) \;\big|\; (D_{J_0}+S)\rho = 0, \quad \rho(0,\cdot) \in W^\perp \subset H^{1/2}(\mathbb{S}^1,\mathbb{R}^{2n})\big\}.$$

We write $\eta(s,t) = \sum_{k\in\mathbb{Z}} e^{2\pi kJ_0t}\eta_k(s)$ and $f(s,t) = \sum_{k\in\mathbb{Z}} e^{2\pi kJ_0t}f_k(s)$ as Fourier series. Then the right candidates for the weak derivatives are

$$\partial_t\eta(s,t) = 2\pi\sum_{k\in\mathbb{Z}} J_0ke^{2\pi kJ_0t}\eta_k(s), \qquad \partial_s\eta(s,t) = \sum_{k\in\mathbb{Z}} e^{2\pi kJ_0t}f_k(s) - 2\pi\sum_{k\in\mathbb{Z}} ke^{2\pi kJ_0t}\eta_k(s).$$

We must show that they are bounded in the $W^{1,2}$-norm. Since W is assumed to be a compact perturbation of $\mathbb{R}^n\times\mathbb{H}^+$ the spaces $\mathbb{P}^-W$ and $\mathbb{P}^+W^\perp$ are finite dimensional. Let $N \in \mathbb{N}$ and denote by $\mathbb{P}^+_N$ the projection onto the positive frequencies bounded by N, i.e.,

$$\mathbb{P}^+_N v = \sum_{0<k\leq N} e^{2\pi kJt}v_k, \quad \text{for all} \quad v \in \mathbb{E}.$$

Then we can approximate $\mathbb{P}^+$ by $\mathbb{P}^+_N$ on $W^\perp$, i.e., for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\|(\mathbb{P}^+ - \mathbb{P}^+_N)q\|_{H^{1/2}(\mathbb{S}^1)} \leq \varepsilon, \quad \text{for all} \quad q \in W^\perp. \qquad (4.13)$$

Now assume that η is smooth, then (4.12) implies that $\eta(0,\cdot) \in W^\perp$ and, since $D_{J_0}\eta = -f$ by the formulas for the weak derivatives, we observe that

$$\eta_k(s) = e^{-2\pi ks}\left[\int_0^s e^{2\pi k\tau}f_k(\tau)d\tau + \eta_k(0)\right].$$

Hence $\eta_k(0) = e^{2\pi ks}\eta_k(s) - \int_0^s e^{2\pi k\tau} f_k(\tau)d\tau$ and for $T > 0$ we obtain

$$|\eta_k(0)| \leq e^{2\pi kT}|\eta_k(s)| + \sqrt{\frac{e^{4\pi kT}-1}{4\pi k}}\| f_k \|_{L^2([0,T])}, \quad \text{for} \quad 0 \leq s \leq T,$$

where we used Hölder's inequality for the second term. Hence there is $d_1 = d_1(N,T) > 0$ such that

$$\| \mathbb{P}_N^+ \eta(0,\cdot) \|^2_{H^{1/2}(\mathbb{S}^1)} = \sum_{0<k\leq N} 2\pi k|\eta_k(0)|^2 \leq d_1\Big(\sum_{0<k\leq N} |\eta_k(s)|^2 + \| f \|^2_{L^2(Z_T)}\Big), \quad \text{for} \quad 0 \leq s \leq T,$$

and $Z_T = [0,T] \times \mathbb{S}^1$. Integrating over $[0,T]$ leads to

$$T\| \mathbb{P}_N^+ \eta(0,\cdot) \|^2_{H^{1/2}(\mathbb{S}^1)} \leq d_1\Big(\| P_N^+ \eta \|^2_{L^2(Z_T)} + T\| f \|^2_{L^2(Z_T)}\Big).$$

Choose $\varepsilon = 1/2$ in (4.13) and $N = N(\varepsilon)$, then surely there is a constant $d_2 = d_2(N,T)$ such that

$$\| \mathbb{P}^+ \eta(0,\cdot)) \|^2_{H^{1/2}(\mathbb{S}^1)} \leq 2\| \mathbb{P}_N^+ \eta(0,\cdot)) \|^2_{H^{1/2}(\mathbb{S}^1)} \leq d_2\Big(\| \eta \|^2_{L^2(Z_T)} + \| f \|^2_{L^2(Z_T)}\Big).$$

Now by (3.18) and (4.12) we can find a further constant $d_3 = d_3(N,T,S)$ such that

$$\begin{aligned}
\| \partial_s \eta \|^2_{L^2(Z)} + \| \partial_t \eta \|^2_{L^2(Z)} &= \| D_{J_0}\eta \|^2_{L^2(Z)} + \| \mathbb{P}^+ \eta \|^2_{H^{1/2}(\mathbb{S}^1)} - \| \mathbb{P}^- \eta \|^2_{H^{1/2}(\mathbb{S}^1)} \\
&\leq \| f \|^2_{L^2(Z)} + d_2\Big(\| \eta \|^2_{L^2(Z_T)} + \| f \|^2_{L^2(Z_T)}\Big) \\
&\leq d_3\| \eta \|^2_{L^2(Z)}. \qquad (4.14)
\end{aligned}$$

By density the weak derivatives are bounded which shows that $\eta \in W^{1,2}(Z,\mathbb{R}^{2n})$ and we obtain the formula of $\ker(D_{J_0} + S)$ by (4.12). Finally we use the cut-off function β and (3.18) as in Step 1 to establish the estimate

$$\| \eta \|_{W^{1,2}(Z)} \leq d_4\Big(\| (D_{J_0} + S)\eta \|_{L^2(Z)} + \| \eta \|_{L^2([0,R+1]\times\mathbb{S}^1)} + \| \mathbb{P}^+ \eta \|_{H^{1/2}(\mathbb{S}^1)}\Big), \quad d_4 > 0,$$

which shows that the operator $D_{J_0} + S : W^{1,2}_{W^\perp}(Z,\mathbb{R}^{2n}) \longrightarrow L^2(Z,\mathbb{R}^{2n})$ is a semi-Fredholm operator whose finite dimensional kernel coincides with the cokernel of $\overline{D}_{J_0} + S$, which is therefore indeed a Fredholm operator. □

It remains to calculate the index therefore we reformulate the Fredholm-problem studied so far in an equivalent way. To do this, recall that we denote by $W_A^u, W_A^s \subset \mathbb{E}$ the linear unstable and stable space of the Cauchy-problem

$$X_A'(s) = AX_A(s), \quad X_A(0) = I \quad \text{with} \quad A \in C^0([-\infty,+\infty],\mathbb{E}),$$

and $A(\pm\infty)$ hyperbolic defined in (2.13), by $P_{W_A^u}, P_{W_A^s}$ the associated projections and finally by $C_0^k((-\infty,0],\mathbb{E})$, $k \in \mathbb{N}$ the space of all C^k-curves η with $\lim_{s\to-\infty} \eta^{(l)}(s) = 0$ for all $0 \leq l \leq k$.

Lemma 4.2.4. *Let $A : [-\infty, 0] \longrightarrow \mathcal{L}(\mathbb{E})$ be a continuous path of bounded linear operators such that $A(-\infty)$ is hyperbolic. Assume that W^u_A is a compact perturbation of $\mathbb{R}^n \times \mathbb{H}^+$ and let $S \in C^0(Z, \mathbb{R}^{2n} \times \mathbb{R}^{2n})$ be symmetric such that* (4.4) *and* (4.6) *hold. Then the operator*

$$\begin{aligned}\overline{D}_{\mathrm{hyb}} : C^1_0((-\infty,0],\mathbb{E}) \times W^{1,2}(Z,\mathbb{R}^{2n}) &\longrightarrow C^0_0((-\infty,0],\mathbb{E}) \times L^2(Z,\mathbb{R}^{2n}) \times \mathbb{E}\\ (\eta,\xi) &\mapsto \Big(\eta' - A\eta, (\overline{D}_{J_0} + S)\xi, \eta(0) - \xi(0,\cdot)\Big),\end{aligned}$$

with $\overline{D}_{J_0} = \partial_s + J_0\partial_t$, is Fredholm and its index coincides with the index of the operator

$$\overline{D}_W := \overline{D}_{J_0} + S : W^{1,2}_W(Z,\mathbb{R}^{2n}) \longrightarrow L^2(Z,\mathbb{R}^{2n})$$

with $W = W^u_A$.

Proof. We show that the appearing kernels and cokernels are isomorphic, respectively. If $(\eta,\xi) \in \ker \overline{D}_{\mathrm{hyb}}$ then $\eta' - A\eta = 0$ and $\lim_{s\to-\infty} \eta(s) = 0$ imply that $\eta(0) \in W^u_A$. Furthermore, η is uniquely determined by $\eta(0) = \xi(0,\cdot)$. Hence

$$\ker(\overline{D}_{\mathrm{hyb}}) \cong \Big\{\xi \in W^{1,2}(Z,\mathbb{R}^{2n}) \;\Big|\; (\overline{D}_{J_0} + S)\xi = 0\,, \quad \xi(0,\cdot) \in W\Big\} = \ker(\overline{D}_W)\,.$$

By Proposition 2.3.1, the operator $F^-_A : C^1_0((-\infty,0],\mathbb{E}) \longrightarrow C^0_0((-\infty,0],\mathbb{E})$, $\eta \mapsto \eta' - A\eta$ is a left inverse with right inverse R^-_A such that

$$W^u_A + \Big\{R^-_A(\kappa)(0) \;\Big|\; \kappa \in C^0_0((-\infty,0],\mathbb{E})\,, \kappa(0) = 0\Big\} = \mathbb{E}\,,$$

and possesses $\ker(F^-_A) = \Big\{X_A(s)v \;\Big|\; v \in W\Big\}$, $s \in \mathbb{R}$ which readily implies the inclusions

$$\{0\} \times \operatorname{coker}(\overline{D}_W) \times \{0\} \subset \operatorname{coker}(\overline{D}_{\mathrm{hyb}}) \tag{4.15}$$

and

$$\Big\{\kappa \in C^0_0((-\infty,0],\mathbb{E}) \;\Big|\; \kappa(0) = 0\Big\} \times \operatorname{ran}(\overline{D}_W) \times \mathbb{E} \subset \operatorname{ran}(\overline{D}_{\mathrm{hyb}})\,.$$

Therefore it remains to consider the case where (κ,ρ,p) are in the complement of $\{0\}\times\operatorname{coker}(\overline{D}_W)\times\{0\}$ with $\kappa(0) \neq 0$ and $\operatorname{supp}\kappa \subset (-\infty,0]$ is compact. Then we can find $\xi \in W^{1,2}_W(Z,\mathbb{R}^{2n})$ with $\overline{D}_W\xi = \rho$ and set

$$\eta(s) = X_A(s)P_W\Big(\int_0^s X_A(\tau)^{-1}\kappa(\tau)d\tau + p + \xi(0,\cdot)\Big) - X_{-A}(s)P_{W^\perp}\Big(\int_s^0 X_{-A}(\tau)^{-1}\kappa(\tau)d\tau - p\Big)\,.$$

Since κ possesses compact support, η belongs to $C^1_0((-\infty,0],\mathbb{E})$ and therefore $\overline{D}_{\mathrm{hyb}}(\eta,\xi) = (\kappa,\rho,p)$ implies that (4.15) is in fact an equivalence, and we obtain $\operatorname{coker}(\overline{D}_W) \cong \operatorname{coker}(\overline{D}_{\mathrm{hyb}})$, concluding the proof. □

Before we compute the Fredholm index in a special situation recall that $j^* : L^2(\mathbb{S}^1,\mathbb{R}^{2n}) \longrightarrow H^{1/2}(\mathbb{S}^1,\mathbb{R}^{2n})$ denotes the adjoint of the embedding $j : H^{1/2}(\mathbb{S}^1,\mathbb{R}^{2n}) \longrightarrow L^2(\mathbb{S}^1,\mathbb{R}^{2n})$ from (2.6). Furthermore note that the Conley-Zehnder index with respect to

$$J_0 = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} \quad \text{as in} \quad (3.1)$$

of the symplectic path $\Psi(t) = e^{J_0 S_\lambda t} \in C^0([0,1], \mathrm{Sp}(2n,\mathbb{R}))$ with $S_\lambda = \lambda I$, $\lambda \in \mathbb{R}$, is given by

$$\mu(\Psi) = -2n\left\lfloor \frac{\lambda}{2\pi} \right\rfloor - n\,. \tag{4.16}$$

Explicit calculations can be found in [**31**].

LEMMA 4.2.5. *Let $a, b \in \mathbb{R}$, $a, b \notin 2\pi\mathbb{Z}$ and $S_a := aI$, $S_b := bI \in \mathrm{Gl}(2n,\mathbb{R})$. Denote by $W = W^u_{S_a} \subset H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n})$ the linear unstable space of the Cauchy-problem*

$$X_A'(s) = AX_A(s)\,, \quad X_A(0) = I \quad \textit{with} \quad A = \mathbb{P}^+ - \mathbb{P}^- - j^* S_a$$

and by $\Psi_b \in C^1([0,1], \mathrm{Sp}(2n))$ the symplectic path defined in (4.5) *with $S = S_b$. Then the linear operator*

$$\begin{aligned} \overline{D}_{a,b} : C^1((-\infty,0],\mathbb{E}) \times W^{1,2}(Z,\mathbb{R}^{2n}) \quad &\longrightarrow \quad C^0((-\infty,0],\mathbb{E}) \times L^2(Z,\mathbb{R}^{2n}) \times \mathbb{E} \\ (\eta,\xi) \quad &\mapsto \quad \Big(\eta' - (\mathbb{P}^+ - \mathbb{P}^- - j^*S_a)\eta, (\overline{D}_{J_0} + S_b)\xi, \eta(0) - \xi(0,\cdot)\Big) \end{aligned}$$

is Fredholm of finite index

$$\dim(W^u_{S_a}, \mathbb{R}^n \times \mathbb{H}^+) - \mu(\Psi_b)\,,$$

where $\dim(W^u_{S_a}, \mathbb{R}^n \times \mathbb{H}^+) = \dim\left(W^u_{S_a} \cap (\mathbb{R}^n \times \mathbb{H}^+)^\perp\right) - \dim\left((W^u_{S_a})^\perp \cap (\mathbb{R}^n \times \mathbb{H}^+)\right)$ *denotes the relative dimension of $W^u_{S_a}$ with respect to $\mathbb{R}^n \times \mathbb{H}^+$ introduced in* (2.19). *If, in particular, $a = b$, then $\overline{D}_{a,b}$ is an isomorphism.*

PROOF. In this particular situation we obviously have that

$$X_A(s) = e^{(\mathbb{P}^+ - \mathbb{P}^- - j^* S_a)s}\,, \quad \forall\, s \in \mathbb{R}\,.$$

Recall that, by Proposition 2.1.3, for all $q \in L^2(\mathbb{S}^1, \mathbb{R}^{2n})$ we have $j^*(q)(t) = q_0 + \sum_{k\neq 0} e^{2\pi k J_0 t} \frac{1}{2\pi|k|} q_k$. Since the matrices S_a, S_b possess no null-direction, $\lim_{s\to-\infty} X_A(s)p = 0$ holds if and only if

$$p \in \left\{ p \in H^{1/2}(\mathbb{S}^1, \mathbb{R}^{2n}) \;\middle|\; p(t) = \sum_{k \geq K} e^{2\pi k J_0 t} p_k\,, \quad K = \left\lfloor \frac{a}{2\pi} \right\rfloor + 1 \right\} = W^u_{S_a}\,,$$

where $\lfloor x \rfloor := \max\left\{k \in \mathbb{Z} \;\middle|\; k \leq x\right\}$, $x \in \mathbb{R}$, denotes the Gauß-bracket. So $W^u_{S_a}$ is a compact perturbation of $\mathbb{R}^n \times \mathbb{H}^+$ with

$$\dim(W^u_{S_a}, \mathbb{R}^n \times \mathbb{H}^+) = -2n\left\lfloor \frac{a}{2\pi} \right\rfloor - n\,. \tag{4.17}$$

Since Ψ_a, Ψ_b satisfy (4.6) if and only if $a, b \notin 2\pi\mathbb{Z}$, we can apply Proposition 4.2.3 and Lemma 4.2.4 and obtain that $\overline{D}_{a,b}$ is Fredholm of finite index. Let $0 \neq (\eta, \xi) \in \ker(\overline{D}_{a,b})$, then by Lemma 4.2.4 we have that the kernel is isomorphic to

$$\left\{ \xi \in W^{1,2}(Z, \mathbb{R}^{2n}) \;\middle|\; (\overline{D}_{J_0} + S_b)\xi = 0\,, \quad \xi(0,\cdot) \in W^u_{S_a} \right\}\,.$$

Therefore we can use the Fourier representation $\xi(s,t) = \xi_0(s) + \sum_{k\in\mathbb{Z}} e^{2\pi k J_0 t} \xi_k(s)$ to observe that $\dot{\xi}_k(s) - (2\pi k - b)\xi_k(s) = 0$. Hence

$$\xi_k(s) = e^{(2\pi k - b)s} \xi_k(0)\,.$$

Now, since $\xi_k(s) \longrightarrow 0$ for $s \to +\infty$ and $\xi(0,\cdot) \in W^u_{S_a}$ must hold, we obtain

$$\left\lfloor \frac{a}{2\pi} \right\rfloor + 1 \le k \le \left\lfloor \frac{b}{2\pi} \right\rfloor . \tag{4.18}$$

On the other hand, by Proposition 4.2.3 and Lemma 4.2.4, the cokernel of $\overline{D}_{a,b}$ is canonically isomorphic to

$$\left\{ \rho \in W^{1,2}(Z,\mathbb{R}^{2n}) \;\middle|\; (D_{J_0} + S_b)\rho = 0\,, \quad \rho(0,\cdot) \in (W^u_{S_a})^{\perp} \right\} .$$

If $0 \neq \rho$, the Fourier-coefficients satisfy $\dot{\rho}_k(s) + (2\pi k - b)\rho_k(s) = 0$. Hence

$$\rho_k(s) = e^{-(2\pi k - b)s}\rho_k(0)$$

and we must require that

$$\left\lfloor \frac{b}{2\pi} \right\rfloor + 1 \le k \le \left\lfloor \frac{a}{2\pi} \right\rfloor \tag{4.19}$$

to assure that $\rho_k(s) \longrightarrow 0$ for $s \to +\infty$. So if $a = b$, none of the conditions (4.18) or (4.19) can be fulfilled, which implies that kernel and cokernel are trivial and therefore $\overline{D}_{a,b}$ is an isomorphism in this case. If a, b are such that either (4.18) or (4.19) holds, then we obtain, by (4.16) and (4.17),

$$\mathrm{ind}(\overline{D}_{a,b}) = -2n\left\lfloor \frac{a}{2\pi} \right\rfloor + 2n\left\lfloor \frac{b}{2\pi} \right\rfloor = \dim(W^u_{S_a}, \mathbb{R}^n \times \mathbb{H}^+) - \mu(\Psi_b)\,.$$

□

Proof of Theorem 4.2.1:

Proof. We approximate $w \in W^{1,2}_{\mathcal{W}^u_{\mathbb{X}_K}(x^-)}(Z,\mathbb{R}^{2n})$ by a curve $\tilde{w}$ that is smooth in the interior $(0,\infty) \times \mathbb{S}^1$ and satisfies $\tilde{w}(0,\cdot) = w(0,\cdot)$, and we consider the operator

$$\widetilde{D\Theta}_{x^-,c_{x^+}}(\tilde{w})\xi = \partial_s \xi + J_0 \partial_t \xi + \nabla^2 H(\cdot, \tilde{w} + c_{x^+})\xi\,.$$

In this situation (4.4) and (4.6) are satisfied, and moreover, by Lemma 2.4.2, we have that $W := T_p\mathcal{W}^u_{\mathbb{X}_K}(x^-)$ with $p = w(0,\cdot)$ is a compact perturbation of $\mathbb{R}^n \times \mathbb{H}^+$. Hence all assumptions of Proposition 4.2.3 hold and therefore

$$\overline{D}_{J_0} + S \,:\, W^{1,2}_W(Z,\mathbb{R}^{2n}) \longrightarrow L^2(Z,\mathbb{R}^{2n})\,, \quad S = \nabla^2 H(\cdot, \tilde{w} + c_{x^+})$$

is a Fredholm operator, which implies that $\widetilde{D\Theta}_{x^-,c_{x^+}}(\tilde{w})$ is also Fredholm of equal index. Moreover, $\Theta_{x^-,c_{x^+}}$ is smooth, therefore $D\Theta_{x^-,c_{x^+}}(w)$ is also Fredholm possessing the same index as $\overline{D}_{J_0} + S$. By Lemma 4.2.4, the Fredholm operator $\overline{D}_{\mathrm{hyb}}$ with $A = D\mathbb{X}_K(v)$ for the trajectory v of $\varphi_{\mathbb{X}_K}$ with $v(0) = w(0,\cdot) \in \mathcal{W}^u_{\mathbb{X}_K}(x^-)$ possesses the same index as $D\Theta_{x^-,c_{x^+}}(w)$. Therefore, we can apply Lemma 4.2.5 to compute the index.

Observe that if we choose $A' = D\mathbb{X}(v)$ instead of $A = D\mathbb{X}_K(v)$, then this changes the operator $\overline{D}_{\mathrm{hyb}}$ only by a small perturbation. Hence the index does not change as long as we have chosen $r > 0$ small enough. Now set $S_1 = \nabla^2 H(\cdot, v)$, $S_2 = \nabla^2 H(\cdot, \tilde{w} + c_{x^+})$, then surely we can find a homotopy sending S_1 to $S_1^- = \nabla^2 H(\cdot, x^-)$ and S_2 to $S_2^+ = \nabla^2 H(\cdot, x^+)$. Since x^-, x^+ are nondegenerate, the Conley-Zehnder indices of the symplectic paths induced by S_1^- and S_2^+ are well defined. Moreover, the indices of two symplectic paths coincide if and only if they are homotopic. Thus we can find numbers $a, b \in \mathbb{R}$ and a homotopy from S_1^- to $S_a = aI$ and S_2^+ to $S_b = bI$ without changing the indices, which, by Lemma 4.2.5 and the fact that the Fredholm index is a locally constant function proves that the operator

$$\overline{D}_{\mathrm{hyb}}(\eta, \xi) = \Big(\eta' - (\mathbb{P}^+ - \mathbb{P}^- + S_1)\eta, (\overline{D}_{J_0} + S_2)\xi, \eta(0) - \xi(0, \cdot)\Big),$$

possesses the claimed index formula and so does $D\Theta_{x^-, c_{x^+}}(w)$. □

In particular the above argumentation shows that $m(x, \mathbb{R}^n \times \mathbb{H}^+) = \mu(x)$ for non degenerate singular points, which was already known in [**1**]. Nevertheless, we decided to follow [**8**] and calculate the index directly in this comfortable situation.

In order to prove the transversality result for the hybrid type curves, we apply the following well-known statement.

Theorem 4.2.6. ([**21**]) **(Carleman similarity principle)** *Denote by* $D = \Big\{z \in \mathbb{C}^n \;\Big|\; |z| \leq 1\Big\}$ *the unit disc and let* $S = S(z) \in L^\infty(D, \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n))$ *be a* $z = s + it \in \mathbb{C}$ *-dependent* $\mathbb{R}$*-linear map and* $J \in C^\infty(\mathbb{C}^n, \mathcal{L}_{\mathbb{R}}(\mathbb{C}^n))$ *be a z-dependent complex structure on* $\mathbb{C}^n$*. Assume that* $\varphi : \mathring{D} \longrightarrow \mathbb{C}^n$ *is a solution of*

$$\partial_s\vartheta + J(z)\partial_t\vartheta + A(z)\vartheta = 0\,, \quad \text{and} \quad \vartheta(0) = 0\,.$$

Then there is $0 < \delta < 1$*, a holomorphic map* $\psi : D_\varepsilon = \Big\{z \in \mathbb{C}^n \;\Big|\; |z| \leq \delta\Big\} \longrightarrow \mathbb{C}^n$ *and a continuous map* $\Gamma : D_\delta \longrightarrow Gl_{\mathbb{R}}(\mathbb{C}^n)$,

$$\Gamma \in \bigcap_{2<p<\infty} W^{1,p}\Big(D_\delta, Gl_{\mathbb{R}}(\mathbb{C}^n)\Big),$$

satisfying on D_δ*,*

$$J(z)\Gamma(z) = \Gamma(z)i\,, \quad \text{and} \quad \varphi(z) = \Gamma(z)\psi(z)\,.$$

So, if in particular, $\varphi = 0$ on a non-discrete subset of $\mathring{D}_\delta$, the same will hold for ψ, which, since it is holomorphic, will therefore vanish identically, which implies that $\varphi \equiv 0$ on D_δ.

Theorem 4.2.7. *Let* $H \in \mathcal{H}_{\mathrm{reg}}$*,* $r > 0$*. If r was chosen small enough there exists a residual set* $\widehat{\mathcal{K}_{\mathrm{reg}}} \subset \mathcal{K}_{\theta,r}$ *of compact vector fields K such that for all* $x^-, x^+ \in \mathcal{P}_0(H)$ *with* $x^- \neq x^+$ *and* $m(x^-, \mathcal{V}) - \mu(x^+) \leq 2$*,* $\mathcal{V} = \mathbb{R}^n \times \mathbb{H}^+$*, and any* $u \in \mathcal{M}_{\mathrm{hyb}}(x^-, x^+, H, J_0, \mathbb{X}_K)$ *the operator*

$$D\Theta_{x^-, c_{x^+}}(w) : W^{1,2}_{T_p\mathcal{W}^u_{\mathbb{X}_K}(x^-)}(Z, \mathbb{R}^{2n}) \longrightarrow L^2(Z, \mathbb{R}^{2n})$$

from (4.3) *with* $w = u - c_{x^+} \in W^{1,2}_{\mathcal{W}^u_{\mathbb{X}_K}(x^-)}(Z, \mathbb{T}^{2n})$*,* $p = u(0, \cdot) \in \mathcal{W}^u_{\mathbb{X}_K}(x^-)$ *and* c_{x^+} *as in* (4.1)*, is onto and the vector field* $\mathbb{X}_K = -\nabla_{1/2}A_H + K$ *satisfies the Morse-Smale condition up to order* 2.

Proof. We denote by $C_{x^-}((-\infty, 0], \mathbb{M})$ the Banach manifold of all C^1-curves v satisfying $\lim_{s\to-\infty} v(s) = x^-$, $\lim_{s\to-\infty} v'(s) = 0$ set

$$M = C_{x^-}((-\infty, 0], \mathbb{M}) \times W^{1,2}(Z, \mathbb{R}^{2n}) \times \mathcal{K}_{\theta,r}\,, N = \mathcal{K}_{\theta,r}\,, O = C^0_0((-\infty, 0], \mathbb{E}) \times L^2(Z, \mathbb{R}^{2n}) \times \mathbb{M}$$

and consider the map

$$\sigma : M \longrightarrow O\,, \quad \sigma(v,w,K) = \Big(v' - \mathbb{X}_K(v), \overline{\partial}_{J_0,H}(w + c_{x^+}), v(0) - ([0] + w(0,\cdot))\Big)\,.$$

Step 1 : We claim that 0 is a regular value of σ. Indeed, the linearization at some point $(v,w,K) \in \sigma^{-1}(0)$ is given by

$$D\sigma(v,w,K) : C^1_0((-\infty,0],\mathbb{E}) \times W^{1,2}(Z,\mathbb{R}^{2n}) \times \mathcal{K}_\theta \longrightarrow C^0_0((-\infty,0],\mathbb{E}) \times L^2(Z,\mathbb{R}^{2n}) \times \mathbb{E}$$
$$D\sigma(v,w,K)[\eta,\xi,L] = \Big(\eta' - D\mathbb{X}_K(v)\eta + L(v), F_{J_0,H}(w + c_{x^+})\xi, \eta(0) - \xi(0,\cdot)\Big)$$

with

$$F_{J_0,H}(w + c_{x^+})\xi = \partial_s\xi + J_0\partial_t\xi + \nabla^2 H(\cdot, w + c_{x^+})\xi\,.$$

We show that $D\sigma(v,w,K)$ is a left inverse. Note that the operator $D_1\sigma(v,w,k) + D_2\sigma(v,w,k)$ coincides with the operator $\overline{D}_{\mathrm{hyb}}$ with $S = \nabla^2 H(\cdot,u)$, which, due to Lemma 4.2.4 and Theorem 4.2.1, is Fredholm of index $m(x^-\mathcal{V}) - \mu(x^+)$ if r was chosen small enough. Thus, by Lemma 2.4.5, $D\sigma(v,w,k)$ has complemented kernel and it remains to show that $D\sigma(v,w,k)$ is onto. We set $W = W^u_A\,, A = D\mathbb{X}_K(v)$ and use the proven equivalence of (4.15) to observe that

$$\operatorname{coker}\big(D\sigma(v,w,K)\big) \subset \{0\} \times \operatorname{coker}(\overline{D}_W) \times \{0\} = \operatorname{coker}(\overline{D}_{\mathrm{hyb}})\,. \tag{4.20}$$

Since $\operatorname{coker}(\overline{D}_W)$ is finite dimensional we can choose a basis $\{\rho_i\}_{i=1,\dots,N}$ and consider the problem

$$\begin{cases} \zeta' - A\zeta + L(v) &= \varrho \\ \zeta(0) &= \rho_i(0,\cdot)\,, \quad 1 \le i \le N\,, \end{cases} \tag{4.21}$$

for given $\varrho \in C^0_0((-\infty,0],\mathbb{E})$. Now, by Proposition 2.3.1, we have that $F^-_A\zeta = \zeta' - A\zeta$ is a left inverse and therefore we can choose $\tilde{\zeta} \in C^1_0((-\infty,0],\mathbb{E})$ such that $\tilde{\zeta}' - A\tilde{\zeta} = \varrho$. Assume that $(v,w,K) \in \sigma^{-1}(0)$ and v is a constant flow line of $\mathbb{X}_K$, then $u(0,\cdot) = x^-$ holds, i.e., u runs into a critical point of A_H in finite time. Though it seems obvious, it is a nontrivial fact that u is constant in this case, see [**21**] (Lemma 2 in 6.4). By assumption we have that $x^- \neq x^+$ and therefore this case cannot occur implying that v is a non-constant flow line, i.e., a C^1-embedding and $v^{-1} : v(\mathbb{R}) \subset \mathbb{M} \longrightarrow (-\infty,0]$ is a compact map. Accordingly, for each $\zeta_i \in C^1_0((-\infty,0],\mathbb{E})$ that possesses compact support and satisfies $\zeta_i(0) = \rho_i(0,\cdot) - \tilde{\zeta}(0)$, we can choose $L_i \in \mathcal{K}_\theta$ such that

$$(\zeta_i' - A\zeta_i) \circ (v^{-1})(p) + L_i(p) = 0\,, \quad \forall\, p \in v(\mathbb{R}) \subset \mathbb{M}\,.$$

Hence we can solve (4.21) and therefore (4.20) actually becomes

$$\operatorname{coker}\big(D\sigma(v,w,K)\big) \subset \{0\} \times \operatorname{coker}(\overline{D}_{\mathbb{E}}) \times \{0\}$$

with

$$\operatorname{coker}(\overline{D}_{\mathbb{E}}) = \Big\{\rho \in W^{1,2}(Z,\mathbb{R}^{2n}) \;\Big|\; \partial_s\rho - J_0\partial_t\rho - \nabla^2 H(\cdot,u)\rho = 0\,, \quad \rho(0,\cdot) = 0\Big\}\,.$$

Since $\rho(0,\cdot) = 0$ is obviously a totally real boundary condition, we can use Schwarz-reflection and a standard elliptic regularity argument, see [**25**], to obtain that all $\rho \in \operatorname{coker}(\overline{D}_{\mathbb{E}})$ are smooth. Furthermore, we can apply the Carleman similarity principle (Theorem 4.2.6) to the reflected curves which therefore all vanish identically on an open neighborhood U of $\{0\} \times \mathbb{S}^1 \subset \mathbb{R} \times \mathbb{S}^1$, implying that $\rho \equiv 0$ holds for all $\rho \in \operatorname{coker}(\overline{D}_{\mathbb{E}})$ on some open neighborhood V of $\{0\} \times \mathbb{S}^1 \subset Z$. Thus for $\rho \in \operatorname{coker}(\overline{D}_{\mathbb{E}})$ the set

$$\Sigma = \Big\{(s,t) \in Z \;\Big|\; \rho(s,t) = 0 \text{ and } \exists\,(s_m,t_m) \to (s,t)\,,(s_m,t_m) \neq (s,t) \text{ satisfying } \rho(s_m,t_m) = 0\Big\}$$

is closed and, by our previous discussion, non-empty. If $(s_0,t_0) \in \Sigma$ we can apply the previous argument once again and deduce that (s_0,t_0) is an interior point of Σ. Hence $\Sigma = Z$ and consequently $\operatorname{coker}\left(D\sigma(v,w,k)\right) = \{0\}$ for all $(v,w,k) \in \sigma^{-1}(0)$, proving Step 1.

Step 2 : By step 1 we have that $\mathcal{M} := \sigma^{-1}(0)$ is a Banach manifold. Consider the projector

$$\tau : C^1_{x^-}((-\infty,0],\mathbb{M}) \times W^{1,2}_{x^+}(Z,\mathbb{T}^{2n}) \times \mathcal{K}_{\theta,r} \longrightarrow \mathcal{K}_{\theta,r}\,.$$

In annalogy to the proof of Theorem 2.4.3, we claim that the restriction of τ to $\sigma^{-1}(0)$ is Fredholm of index $m(x^-,\mathbb{R}^n \times \mathbb{H}^+) - \mu(x^+)$. Everything follows by Corollary 2.4.7 with M,N,O as already set and $\psi = \sigma$, $\phi = \tau$. Denote by $\widehat{\mathcal{K}}_{\text{reg}}(x^-,x^+) \subset \mathcal{K}_{\theta,r}$ the regular values of τ, then again by Corollary 2.4.7 this is the set such that $D_1\sigma(v,w,K) + D_2\sigma(v,w,K)$ is a left inverse for all $(v,w,K) \in \pi^{-1}(\widehat{\mathcal{K}}_{\text{reg}}(x^-,x^+))$ and, as a consequence of Lemma 4.2.4, so is the operator $D\Theta_{x^-,c_{x^+}}(w)$ for small r and all $w \in \Theta^{-1}_{x^-,c_{x^+}}(0)$. Now, by a similar argument as in Lemma 2.5.11, combining the results of section 4.3, one can show that the map $\tau_{|\mathcal{M}}$ is σ - proper. Hence the Sard-Smale Theorem 2.2.6 applies and tells us that $\widehat{\mathcal{K}}_{\text{reg}}(x^-,x^+)$ is residual in $\mathcal{K}_{\theta,r}$ and so is the set

$$\widehat{\mathcal{K}}_{\text{reg}} := \bigcap_{\substack{x^-,x^+ \in \mathcal{P}_0(H)\,, \\ 0 < m(x^-,\mathcal{V}) - \mu(x^+) \leq 2}} \widehat{\mathcal{K}}_{\text{reg}}(x^-,x^+) \cap \mathcal{K}_{\text{reg}}\,,$$

where $\mathcal{K}_{\text{reg}}$ is the residual set of compact vector fields K from Theorem 2.4.3, such that $\mathbb{X}_K$ satisfies the Morse-Smale condition up to order 2. Hence $\widehat{\mathcal{K}}_{\text{reg}}$ has the required properties and we conclude the proof. $\square$

For now we have not treated the case of $x^- = x^+$. This is done next.

Lemma 4.2.8. *Assume we have chosen a regular pair (H,J_0) and $K \in \widehat{\mathcal{K}}_{\text{reg}}$. Let $x \in \mathcal{P}_0(H)$, then the moduli space $\mathcal{M}_{\text{hyb}}(x,x,H,J_0,\mathbb{X}_K)$ is a zero dimensional manifold which consists of the constant solution only. Moreover, the linearized operator*

$$D\Theta_{x^-,c_{x^+}}(w) : W^{1,2}_{T_x\mathcal{W}^u_{\mathbb{X}_K}(x)}(Z,\mathbb{R}^{2n}) \longrightarrow L^2(Z,\mathbb{R}^{2n})$$

as in (4.3) *with $w = x - c_x \in W^{1,2}_{\mathcal{W}^u_{\mathbb{X}_K}(x)}(Z,\mathbb{R}^{2n})$ and c_x as in* (4.1)*, is an isomorphism.*

Proof. Let $(v,u) \in C^1_x((-\infty,0],\mathbb{M}) \times W^{1,2}_{\text{loc}}(Z,\mathbb{T}^{2n})$ be a hybrid type solution connecting x with itself, i.e.,

$$v' - \mathbb{X}_K(v) = 0\,, \quad \overline{\partial}_{J_0,H}(u) = 0\,, \quad v(0) = u(0,\cdot)\,, \quad \text{and} \quad v(-\infty) = x\,, \quad u(+\infty,\cdot) = x\,.$$

Since A_H is a Lyapunov function for $\mathbb{X}_K$ we observe $A_H(x) \geq A_H(v(0))$. On the other hand we have that

$$E(u) = \int_0^\infty \| \partial_s u(s,\cdot) \|^2_{L^2(\mathbb{S}^1)} ds = A_H(u(0,\cdot)) - A_H(x) \geq 0$$

Hence $A_H(v(0)) = A_H(x)$, which implies that $v(0) \in \operatorname{sing}(\mathbb{X}_K)$ and it possesses the same level as x, i.e., $v(0) = x$, and therefore we have that v is constant which implies that since $E(u) = 0$ we have that u is constant as well it proves that $\mathcal{M}_{\text{hyb}}(x, x, H, J_0, \mathbb{X}_K)$ consists only of the constant solution and is therefore a submanifold of $W^{1,2}_{\text{loc}}(Z, \mathbb{T}^{2n})$. Now, since DK vanishes on $\mathcal{P}_0(H)$, the linearization at (v, u) is of the form

$$\begin{aligned} \overline{D}_{\text{hyb}}(v,u) \quad &: \quad C^1_0((-\infty,0],\mathbb{E}) \times W^{1,2}(Z,\mathbb{R}^{2n}) \longrightarrow C^0_0((-\infty,0],\mathbb{E}) \times L^2(Z,\mathbb{R}^{2n}) \times \mathbb{E} \\ (\eta,\xi) \quad &\mapsto \quad \left(\eta' - (\mathbb{P}^+ - \mathbb{P}^- - j^*S)\eta, \partial_s\xi + J_0\partial_t\xi + S\xi, \eta(0) - \xi(0,\cdot)\right), \end{aligned}$$

with $S = \nabla^2 H(\cdot, x)$. Then, by Lemma 4.2.4, this is a Fredholm operator and, by Lemma 4.2.5, $\overline{D}_{\text{hyb}}(v, u)$ possesses index 0. Now let $(\eta, \xi) \in \ker \overline{D}_{\text{hyb}}(v, u)$ and assume that $\eta(0) = p = \xi(0, \cdot)$ is an eigenvector of S with respect to the eigenvalue $\lambda \in \mathbb{R}$. Then we have that

$$\eta(s) = e^{(\mathbb{P}^+ - \mathbb{P}^- - j^*\lambda I)s} p\,.$$

Using the Fourier representation $\eta(s) = e^{(\mathbb{P}^+-\mathbb{P}^- - j^*\lambda I)s} \sum_{k\in\mathbb{Z}} e^{2\pi J_0 kt} p_k$ and the fact that $\eta(s) \longrightarrow 0$ for $s \longrightarrow -\infty$, we observe that

$$k \geq \left\lfloor \frac{\lambda}{2\pi} \right\rfloor + 1\,,$$

where $\lfloor\cdot\rfloor$ denotes the Gauß-bracket. On the other hand we have that

$$\xi(s,t) = e^{-(J_0\partial_t + \lambda I)s} p(t)\,.$$

Using again the Fourier representation $\xi(s, t) = e^{-(J_0\partial_t+\lambda I)s} \sum_{k\in\mathbb{Z}} e^{2\pi J_0 kt} p_k$ and the fact that $\xi(s, t) \longrightarrow 0$ for $s \longrightarrow +\infty$, we observe that

$$k \leq \left\lfloor \frac{\lambda}{2\pi} \right\rfloor\,.$$

Hence the kernel of $\overline{D}_{\text{hyb}}(v, u)$ is trivial, which implies that $\overline{D}_{\text{hyb}}(v, u)$ is an isomorphism which further proves the claimed result due to Lemma 4.2.4. □

By the implicit function theorem we obtain the following corollary.

Corollary 4.2.9. *Let the assumptions of Theorem 4.2.7 be fulfilled and $K \in \widehat{\mathcal{K}}_{\text{reg}}$. Then, even if $x^- = x^+$, we have that the moduli spaces $\mathcal{M}_{\text{hyb}}(x^-, x^+, H, J_0, \mathbb{X}_K)$ are C^3-manifolds of dimension $m(x^-, \mathcal{V}) - \mu(x^+)$.*

4.3. Compactness

In this section we generalize the already given $W^{1,2}_{\text{loc}}$-compactness statement for the moduli spaces of Floer-cylinders to the case of moduli spaces of hybrid type curves. The compactness results of chapter 2 and 3 are used to handle our non-Lagrangian boundary condition.

As in Lemma 3.3.2 we use an indirect argument to achieve a uniform bound on the curves of constant loops.

LEMMA 4.3.1. *Let $x^-, x^+ \in \mathcal{P}_0(H)$, $K \in \mathcal{K}_{\theta,1}$ and recall that $\Theta_{x^-,c_{x^+}}$ denotes the non linear operator*

$$\Theta_{x^-,c_{x^+}} : W^{1,2}_{\mathcal{W}^u_{\mathbb{X}_K}(x^-)}(Z,\mathbb{R}^{2n}) \longrightarrow L^2(Z,\mathbb{R}^{2n}), \quad w \mapsto \bar{\partial}_{J_0,H}(w + c_{x^+}).$$

Let $w \in \Theta^{-1}_{x^-,c_{x^+}}(0)$ and denote by $\mathbb{P}^0 w(s,\cdot) \in H^1([0,\infty),\mathbb{R}^{2n})$ the constant part of w. Then $\mathbb{P}^0 w \in C^0([0,\infty),\mathbb{R}^{2n})$ and there is a constant $C = C(x^-,x^+) > 0$ such that

$$\|\mathbb{P}^0 w\|_{C^0([0,\infty)} \leq C \quad \text{for all} \quad w \in \Theta^{-1}_{x^-,c_{x^+}}(0).$$

PROOF. We choose a sequence $w_n \in \Theta^{-1}_{x^-,c_{x^+}}(0)$ and denote by $v_n \in C^1_{x^-}((-\infty,0],\mathbb{M})$ the sequence of half trajectories connecting x^- with $w_n(0,\cdot)$, i.e.,

$$v_n' - \mathbb{X}_K(v_n) = 0 \quad \text{and} \quad v_n(-\infty) = x^-, \quad v_n(0) = [0] + w_n(0,\cdot).$$

Let $c_{x^-} \in C^\infty_{x^-}((-\infty,0],\mathbb{M})$ be such that

$$c_{x^-}(s) = \begin{cases} [0], & \text{for} \quad s \geq -1 \\ x^-, & \text{for} \quad s \leq -2 \end{cases}$$

Let $u_n \in C^1_0((-\infty,0],\mathbb{E})$ be the sequence such that $v_n = u_n + c_{x^-}$. Then we set

$$p_n(s) = \begin{cases} \mathbb{P}^0 u_n(s), & \text{for} \quad s \leq 0 \\ \mathbb{P}^0 w_n(s,t), & \text{for} \quad s > 0. \end{cases}$$

Since $p_n(\pm\infty) = 0$, the p_n's are a sequence of continuous loops $p_n : \mathbb{R} \cup \{\infty\} \longrightarrow \mathbb{R}^{2n}$ starting at 0. Now assume that

$$\lambda_n := \|p_n\|_{C^0(\mathbb{R})} \longrightarrow \infty \quad \text{for} \quad n \longrightarrow \infty.$$

We set $\tilde{p}_n := \frac{1}{\lambda_n} p_n$, then still $\lim_{s\to\pm\infty} \tilde{p}_n(s) = 0$ holds. Thus we have a sequence of continuous loops $\tilde{p}_n : \mathbb{R} \cup \{\infty\} \longrightarrow \mathbb{R}^{2n}$ starting at 0 with $\|\tilde{p}_n\|_{C^0(\mathbb{R}\cup\{\infty\})} = 1$ for all $n \in \mathbb{N}$. Now each $\tilde{p}_n$ solves the equation

$$\begin{aligned} \partial_s \tilde{p}_n &= -\frac{1}{\lambda_n} \mathbb{P}^0(\nabla H(u_n + c_{x^-}) + \partial_s c_{x^-}), \text{for} \quad s \leq 0 \\ \partial_s \tilde{p}_n &= -\frac{1}{\lambda_n} \mathbb{P}^0(\nabla H(w_n + c_{x^+}) + \partial_s c_{x^+}), \text{for} \quad s > 0 \end{aligned} \tag{4.22}$$

in the corresponding sense. By a standard bootstrapping argument each $\tilde{p}_n$ is actually smooth and by (4.22) we obtain furthermore

$$\lim_{n\to\infty} \| \partial_s \tilde{p}_n \|_{C^0(\mathbb{R}\cup\{\infty\})} = 0 .$$

Due to the Arzelà-Ascoli Theorem the sequence $\tilde{p}_n$ is compact. Hence there is a subsequence $\tilde{p}_{n_k}$, which converges uniformly on $\mathbb{R} \cup \{\infty\}$ to a constant loop $\tilde{p}_\infty$ with $\| \tilde{p}_\infty \|_{C^0(\mathbb{R}\cup\{\infty\})} = 1$ and $\tilde{p}_\infty(\infty) = 0$, a contradiction. Thus the sequence λ_n is bounded, proving the claim. □

THEOREM 4.3.2. *Let $x^-, x^+ \in \mathcal{P}_0(H)$ and $K \in \mathcal{K}_{\theta,1}$. Then we have that $\mathcal{M}_{\text{hyb}}(x^-, x^+, H, J_0, \mathbb{X}_K)$, with $\mathbb{X}_K = -\nabla_{1/2} A_H + K$, is a $W^{1,2}_{\text{loc}}$-precompact set.*

PROOF. **Step 1 :** Let $w \in \Theta^{-1}_{x^-,c_{x^+}}(0)$. We show that for any bounded domain $\Omega_T = [0,T] \times \mathbb{S}^1 \subset [0,\infty) \times \mathbb{S}^1$ the restriction of the curve w to Ω_T is uniformly bounded. Set $u = w + c_{x^+}$ and recall that the energy of u is given by

$$E(u) = A_H(u(0,\cdot)) - A_H(x^+) \le A_H(x^-) - A_H(x^+) .$$

Therefore we compute

$$\begin{aligned} \| \partial_s w \|_{L^2(\Omega_T)} &\le \| \partial_s u \|_{L^2(\Omega_T)} + \| \partial_s c_{x^+} \|_{L^2(\Omega_T)} \\ &\le \sqrt{E(u)} + \| \partial_s c_{x^+} \|_{L^2(\Omega_T)} \\ &\le \sqrt{A_H(x^-) - A_H(x^+)} + \| \partial_s c_{x^+} \|_{L^2(\Omega_T)} \\ &=: c_0(x^-, x^+, c_{x^+}) . \end{aligned}$$

and

$$\begin{aligned} \| \partial_t w \|_{L^2(\Omega_T)} &\le \| \partial_t u - X_H(u) \|_{L^2(\Omega_T)} + \| X_H(u) \|_{L^2(\Omega_T)} + \| \partial_t c_{x^+} \|_{L^2(\Omega_T)} \\ &\le \sqrt{E(u)} + T \| H \|_{C^1(\mathbb{S}^1 \times \mathbb{T}^{2n})} + \| \partial_t c_{x^+} \|_{L^2(\Omega_T)} \\ &\le \sqrt{A_H(x^-) - A_H(x^+)} + T \| H \|_{C^1(\mathbb{S}^1 \times \mathbb{T}^{2n})} + \| \partial_t c_{x^+} \|_{L^2(\Omega_T)} \\ &=: c_1(x^-, x^+, T, H, c_{x^+}) . \end{aligned}$$

Observe that

$$\| (\mathbb{P}^0)^\perp w \|^2_{L^2(\Omega_T)} = \int_0^T \sum_{k\neq 0} |w_k(s)|^2 ds \le \int_0^T \sum_{k\neq 0} |k|^2 |w_k(s)|^2 ds = \| \partial_t w \|^2_{L^2(\Omega_T)} \le c_1^2 ,$$

where $\mathbb{P}^0$ denotes the restriction to the 0-order coefficients. Showing that the t-derivative already bounds the non-constant part of w. Due to Lemma 4.3.1 we get the estimate $|w_0(0)| \le C$ with $C = C(x^-, x^+) > 0$. Thus, by writing

$$w_0(s) = \int_0^s \partial_\tau w_0(\tau) d\tau + w_0(0)$$

and by Hölder, we obtain that

$$\begin{aligned}
\| w_0 \|^2_{L^2(\Omega_T)} &= \int_0^T \int_0^1 \Big| \int_0^s \partial_\tau w_0(\tau) d\tau + w(0) \Big|^2 dt ds \\
&\leq 2 \int_0^T \left[|w(0)|^2 + \left(\int_0^s |\partial_\tau w_0(\tau)| d\tau \right)^2 \right] ds \\
&\leq 2TC^2 + 2T \int_0^T \int_0^s |\partial_\tau w_0(\tau)|^2 d\tau ds \\
&\leq 2TC^2 + 2T \int_0^T \| \partial_s w \|^2_{L^2(\Omega_T)} ds \\
&\leq 2TC^2 + 2T^2 c_0^2
\end{aligned}$$

Hence, there is a constant $c_2 = c_2(x^\pm, T, H, c_{x^+}, n)$ bounding $\| w \|_{L^2(\Omega_T)}$ and we are done with step 1.

Step 2 : Let $T > T' > 0$, $w_1, w_2 \in \Theta^{-1}_{x^-, c_{x^+}}(0)$ then we set $\delta w = w_1 - w_2$, $\Omega_T = [0, T] \times \mathbb{S}^1$ and claim that there is a constant $C = C(T', T, H)$ such that

$$\| \delta w \|_{W^{1,2}(\Omega_{T'}, \mathbb{R}^{2n})} \leq C\Big(\| \delta w \|_{L^2(\Omega_T, \mathbb{R}^{2n})} + \| \mathbb{P}^- \delta w(0, \cdot) \|_{H^{1/2}(\mathbb{S}^1)} \Big), \qquad \Omega_{T'} \subset \Omega_T . \tag{4.23}$$

Assume that (4.23) holds, then by step 1 we have that the restriction of any sequence $(w_n)_{n \in \mathbb{N}} \subset \Theta^{-1}_{x^-, c_{x^+}}(0)$ to Ω_T is uniformly bounded in $W^{1,2}(\Omega_T, \mathbb{R}^{2n})$. Since the embedding

$$W^{1,2}_{\mathcal{W}^u(x^-)}(\Omega_T, \mathbb{R}^{2n}) \hookrightarrow L^2(\Omega_T, \mathbb{R}^{2n})$$

is compact due to the Sobolev embedding Theorem, see [**10**], and due to Theorem 2.5.1, the set $\mathbb{P}^-\Big(\mathcal{W}^u(x^-) \cap \{ A_H \geq A_H(x^+) \} \Big)$ is precompact in $\mathbb{M}$. The estimate (4.23) implies that the restriction of $(w_n)_{n \in \mathbb{N}}$ to $\Omega_{T'}$ possesses a subsequence convergent in $W^{1,2}(\Omega_{T'}, \mathbb{R}^{2n})$, which proves the claimed result. It remains to prove (4.23). Therefore let β be a smooth function such that

$$\beta(s) = \begin{cases} 1 & , \quad s \leq T' \\ 0 & , \quad s \geq T' + (T - T')/2 \end{cases} .$$

Now we apply the formula (3.17) of Lemma 3.3.1 to $\beta \delta w = \beta(w_1 - w_2)$, $w_1, w_2 \in \Theta^{-1}_{x^-, c_{x^+}}(0)$ and obtain

$$\begin{aligned}
\| \nabla(\beta \delta w) \|^2_{L^2(\Omega_T)} &= \| \overline{\partial}_{J_0}(\beta \delta w) \|^2_{L^2(\Omega_T)} + \| \mathbb{P}^- \delta w(0, \cdot) \|^2_{H^{1/2}(\mathbb{S}^1)} - \| \mathbb{P}^+ \delta w(0, \cdot) \|^2_{H^{1/2}(\mathbb{S}^1)} \\
&\leq 2(1 + \| \beta' \|^2_{C^0(\mathbb{R})}) \Big(\| \overline{\partial}_{J_0} \delta w \|^2_{L^2(\Omega_T)} + \| \delta w \|^2_{L^2(\Omega_T)} + \| \mathbb{P}^- \delta w(0, \cdot) \|^2_{H^{1/2}(\mathbb{S}^1)} \Big) .
\end{aligned}$$

Now, by the mean value theorem, we obtain

$$\| \overline{\partial}_{J_0} \delta w \|_{L^2(\Omega_T)} = \| \overline{\partial}_{J_0} u_1 - \overline{\partial}_{J_0} u_2 \|_{L^2(\Omega_T)} = \| X_H(u_1) - X_H(u_2) \|_{L^2(\Omega_T)} \leq \| H \|_{C^2(\mathbb{S}^1 \times \mathbb{T}^{2n})} \| \delta w \|_{L^2(\Omega_T)} .$$

Therefore we can find a constant $C' = C'(T', T, H)$ such that

$$\|\nabla(\beta\delta w)\|^2_{L^2(\Omega_T)} = \|\partial_s(\beta\delta w)\|^2_{L^2(\Omega_T)} + \|\partial_t(\beta\delta w)\|^2_{L^2(\Omega_T)} \leq C'\Big(\|\delta w\|^2_{L^2(\Omega_T)} + \|\mathbb{P}^-\delta w(0,\cdot)\|^2_{H^{1/2}(\mathbb{S}^1)}\Big),$$

which implies that there is $C = C(T', T, H)$ such that (4.23) holds. □

Combining the results of Propositions 2.5.3 and 3.3.6 we obtain the following statement.

PROPOSITION 4.3.3. *Let $H \in \mathcal{H}_{\text{reg}}$ be a nondegenerate Hamiltonian and $K \in \mathcal{K}_{\theta,1}$. Furthermore, let $x^-, x^+ \in \mathcal{P}_0(H)$ and let $(u_n)_{n\in\mathbb{N}} \in \mathcal{M}_{\text{hyb}}(x^-, x^+, H, J_0, \mathbb{X}_K)$ be a sequence of curves and $v_n = \varphi_{\mathbb{X}_K}(s, u_n(0.\cdot))$, $s < 0$, be the trajectories through $u_n(0,\cdot)) \in \mathbb{M}$. Then there exist $x_0, \ldots, x_{r_1}, y_0, \ldots, y_{r_2} \in \mathcal{P}_0(H)$ with $r_1, r_2 \in \mathbb{N}$ and*

$$A_H(x_0) > A_H(x_1) > \cdots > A_H(x_{r_1}) > A_H(y_0) > A_H(y_1) > \cdots > A_H(y_{r_2})$$

connecting trajectories

$$V_1 \subset \mathcal{W}^u(x_0) \cap \mathcal{W}^s(x_1), \ldots, V_{r_1} \subset \mathcal{W}^u(x_{r_1-1}) \cap \mathcal{W}^s(x_{r_1}),$$

and curves

$$U_1 \in \mathcal{M}_{\text{hyb}}(x_{r_1}, y_0, H, J_0, \mathbb{X}_K), U_2 \in \mathcal{M}_F(y_0, y_1, H, J_0), \ldots, U_{r_2} \in \mathcal{M}_F(y_{r_2-1}, y_{r_2}, H, J_0)$$

such that a subsequence (v_{n_k}, u_{n_k}) converges to $(V_1, \ldots, V_{r_1}, U_1, \ldots, U_{r_2})$ in the following sense. There exist reparametrization times $(\tau^j_k)_{k\in\mathbb{N}} \subset (-\infty, 0]$, $j \in \{1, \ldots, r_1\}$ and $(s^j_k)_{k\in\mathbb{N}} \subset [0, \infty)$, $j \in \{1, \ldots, r_2\}$ such that

$$\varphi_{\mathbb{X}_K}(\tau^1_k, v_{n_k}) \longrightarrow V_1, \ldots, \varphi_{\mathbb{X}_K}(\tau^{r_1}_k, v_{n_k}) \longrightarrow V_{r_1} \quad \textit{in the Hausdorff-distance}$$

and

$$u_{n_k} \longrightarrow U_1, u_{n_k}(\cdot + s^1_k, \cdot) \longrightarrow U_2, \ldots, u_{n_k}(\cdot + s^{r_2}_k, \cdot) \longrightarrow U_{r_2} \quad \textit{in} \quad W^{1,2}_{\text{loc}}.$$

PROOF. By Theorem 4.3.2 we have that $\mathcal{M}_{hyb}(x^-, x^+, H, J_0, \mathbb{X}_K) = c_{x^+} + \Theta^{-1}_{x^-,c_{x^+}}(0)$ is $W^{1,2}_{\text{loc}}$-precompact. Thus, each sequence $(u_n)_{n\in\mathbb{N}} \subset c_{x^+} + \Theta^{-1}_{x^-,c_{x^+}}(0)$ possesses subsequence $(u_{n_k})_{k\in\mathbb{N}}$ convergent in $W^{1,2}_{\text{loc}}$ with limit curve U, which due to a finite energy estimate as in Proposition 3.3.6 satisfies $U(+\infty, \cdot) \in \mathcal{P}_0(H)$. For Exactly the same reason also (v_{n_k}) converges in the Hausdorff-distance to some trajectory V with $V(-\infty) \in \mathcal{P}_0(H)$. Now either $U(+\infty, \cdot) = x^+$ and $V(-\infty) = x^-$ and we are done, or new critical points $V(-\infty) = x, U(+\infty, \cdot) = y \in \mathcal{P}_0(H)$ appear with $A_H(x^-) < A_H(x) < A_H(y) < A_H(x^+)$. In this case we can proceed as in Proposition 3.3.6 using the fact that both gradients satisfy the (PS) property to get the remaining reparametrization times and connecting trajectories and curves. □

We want to remark that, indeed, there is no braking at the boundary, i.e., if $u_{n_k}(0, \cdot) \in \mathbb{M}$ converges to some critical point, then this would result in a curve that runs into a critical point in finite time, which is therefore constant, see [**21**]. Of course, also the curve v_n would be constant in this case so the whole curve would converge to some critical point, which cannot happen if $A_H(u_{n_k}(0 + s^i_k, \cdot))$ is assumed to be a regular value as we do.

4.4. The isomorphism in $\mathbb{Z}_2$- coefficients

By Corollary 3.2.4 we can choose a generic Hamiltonian H such that (H, J_0) becomes a regular pair. Furthermore, due to Theorem 4.2.7 we can choose $K \in \widehat{\mathcal{K}}_{\mathrm{reg}}$ such that $\mathbb{X}_K$ satisfies the Morse-Smale property up to order 2, i.e., the unstable and stable manifolds meet transversally and the moduli spaces $\mathcal{M}_F(x^-, x^+, H, J_0)$ and $\mathcal{M}_{\mathrm{hyb}}(x^-, x^+, H, J_0, \mathbb{X}_K)$ are manifolds for all $x^\pm \in \mathcal{P}_0(H)$ with $\mu(x^-) - \mu(x^+) \le 2$. In this section we show that, in this situation, there is a chain isomorphism

$$\Phi_* : (C_*(H), \partial_*^M) \longrightarrow (C_*(H), \partial_*^F), \quad i.e., \Phi_{*-1} \circ \partial_*^M = \partial_*^F \circ \Phi_* .$$

For technical reasons we restrict ourselves to the case of $\mathbb{Z}_2$-coefficients, i.e.,

$$C_k(H) = \bigoplus_{\substack{x \in \mathcal{P}_0(H), \\ \mu(x)=k}} \mathbb{Z}_2 x .$$

The first part is to prove that Φ_* is a chain homomorphism. This fact is an immediate consequence of the following result.

Proposition 4.4.1. *Assume we have chosen a regular pair (H, J_0) and $K \in \widehat{\mathcal{K}}_{\mathrm{reg}}$. Let $x^-, x^+ \in \mathcal{P}_0(H)$ with $\mu(x^-) - \mu(x^+) = 1$. Then $\overline{\mathcal{M}_{\mathrm{hyb}}(x^-, x^+, H, J_0, \mathbb{X}_K)}$ is a $W^{1,2}_{\mathrm{loc}}$- compact manifold with boundary of dimension 1. Its boundary consists of all broken hybrid type trajectories from x^- to x^+.*

Proof. By Corollary 4.2.9 $\mathcal{M}_{\mathrm{hyb}}(x^-, x^+, H, J_0, \mathbb{X}_K)$ is a manifold of dimension 1, which is $W^{1,2}_{\mathrm{loc}}$- precompact, according to Theorem 4.3.2. We claim that if $\partial\mathcal{M}_{\mathrm{hyb}}(x^-, x^+, H, J_0, \mathbb{X}_K)$ is non-empty, then

$$W \in \overline{\mathcal{M}_{\mathrm{hyb}}(x^-, x^+, H, J_0, \mathbb{X}_K)} \setminus \mathcal{M}_{\mathrm{hyb}}(x^-, x^+, H, J_0, \mathbb{X}_K)$$

is a broken hybrid type trajectory that is of the following form . Either there is $y \in \mathcal{P}_0(H)$ with $\mu(y) = \mu(x^+)$ and $W = (V, U)$, where V is a connecting trajectory from x^- to y and $U \in \mathcal{M}_{\mathrm{hyb}}(y, x^+, H, J_0, \mathbb{X}_K)$, or $\mu(y) = \mu(x^-)$ and $W = (V', U')$, where $V' \in \mathcal{M}_{\mathrm{hyb}}(x^-, y, H, J_0, \mathbb{X}_K)$ and $U' \in \mathcal{M}_F(y, x^+, H, J_0)$. Thus the ends of $\overline{\mathcal{M}_{\mathrm{hyb}}(x^-, x^+, H, J_0, \mathbb{X}_K)}$ are half open intervals whose boundary points correspond to the broken hybrid type trajectories. To prove this result one uses the gluing methods established in [**3**] and [**34**], both of which localize either at the Morse part or the Floer part and can therefore be applied. Further discussions are omitted here and will be subject to future work. □

Now we construct the isomorphism as follows :

Let $H \in \mathcal{H}_{\mathrm{reg}}$ be given, then due to Corollary 3.2.4 we can assume that, up to a small perturbation which leaves the periodic orbits fixed, H is such that (H, J_0) is a regular pair. We choose $K \in \widehat{\mathcal{K}}_{\mathrm{reg}}$

and set for $\mathcal{V} = \mathbb{M} \times (\mathbb{R}^n \times \mathbb{H}^+)$

$$\upsilon(x, y) = \#\mathcal{M}_{\text{hyb}}(x, y, H, J_0, \mathbb{X}_K) \mod 2\,, \quad m(x, \mathcal{V}) = \mu(y)$$

to be the number of connected components of the corresponding moduli spaces. Due to Theorem 4.3.2, Theorem 4.2.7 and Lemma 4.2.8 the above identity is well defined. Observe furthermore that, since the solutions $(v, u) \in \mathcal{M}_{\text{hyb}}(x, y, H, J_0, \mathbb{X}_K)$ are half Morse trajectories and half pseudo-holomorphic curves, due to the Carleman similarity principle in Theorem 4.2.6, each solution is isolated and is therefore a connected component. Consider the abelian groups

$$C_k(H) = \bigoplus_{\substack{x \in \mathcal{P}_0(H),\\ \mu(x)=k}} \mathbb{Z}_2 x\,.$$

Then we define the following map $\Phi_k : (C_k(H), \partial_k^M) \longrightarrow (C_k(H), \partial_k^F)$ between the Morse- and Floer-complex generator-wise as

$$\Phi_k(x) = \sum_{\substack{y \in \mathcal{P}_0(H),\\ \mu(x)=\mu(y)}} \upsilon(x, y) y \tag{4.24}$$

and state the following Theorem.

Theorem 1. *The Morse-complex of the Hamiltonian action A_H is chain-isomorphic to the Floer-complex of (H, J_0).*

Proof. First, we show that Φ is an homomorphism. In this regard consider the following identity

$$\sum_{\substack{z \in \mathcal{P}_0(H),\\ \mu(y)=\mu(z)}} \sum_{\substack{y \in \mathcal{P}_0(H),\\ \mu(x)-\mu(y)=1}} \upsilon(y, z)\rho(x, y)z + \sum_{\substack{z \in \mathcal{P}_0(H),\\ \mu(y)-\mu(z)=1}} \sum_{\substack{y \in \mathcal{P}_0(H),\\ \mu(x)=\mu(y)}} \nu(y, z)\upsilon(x, y)z = 0 \mod 2 \tag{4.25}$$

where $\rho(x, y)$ is the number of connected components of the intersection $\mathcal{W}^u(x) \cap \mathcal{W}^s(y)$ with respect to $\mathbb{X}_K$ and $\nu(y, z)$ is the number of connected components of $\widehat{\mathcal{M}_F}(x, y, H, J_0)$. Indeed, the coefficient in (4.25) is the number of broken hybrid type trajectories from x to z, which by Proposition 4.4.1 equals the number of boundary points of $\mathcal{M}_{\text{hyb}}(x, z, H, J_0, \mathbb{X}_K)$. Since the boundary of a compact one-dimensional manifold always consists of an even number of points, (4.25) holds. In particular

$$\sum_{\substack{z \in \mathcal{P}_0(H),\\ \mu(y)=\mu(z)}} \sum_{\substack{y \in \mathcal{P}_0(H),\\ \mu(x)-\mu(y)=1}} \upsilon(y, z)\rho(x, y)z = \sum_{\substack{z \in \mathcal{P}_0(H),\\ \mu(y)-\mu(z)=1}} \sum_{\substack{y \in \mathcal{P}_0(H),\\ \mu(x)=\mu(y)}} \nu(y, z)\upsilon(x, y)z \mod 2\,.$$

Hence

$$\Phi_{*-1} \circ \partial_*^M = \partial_*^F \circ \Phi_*$$

and therefore Φ is a homomorphism as claimed. Now we order the critical points $x_1, \ldots, x_n \in \mathcal{P}_0(H)$, $n \in \mathbb{N}$ with $\mu(x_i) = k$, $i = 1, \ldots, n$ by increasing action, choosing any order for subsets of solutions with identical action, i.e., $A_H(x_1) \leq A_H(x_2) \leq \cdots \leq A_H(x_{n-1}) \leq A_H(x_n)$. Then, by Lemma 4.2.8, Φ_k is of the form

$$\Phi_k = \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & 0 & 1 \end{pmatrix},$$

where $* = 1$ or $* = 0$. Hence Φ is an isomorphism. $\square$

Bibliography

[1] Alberto Abbondandolo. *Morse theory for Hamiltonian systems*, volume 425 of *Chapman & Hall/CRC Research Notes in Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2001.

[2] Alberto Abbondandolo and Pietro Majer. A Morse complex for infinite dimensional manifolds. II. *to appear*.

[3] Alberto Abbondandolo and Pietro Majer. Morse homology on Hilbert spaces. *Comm. Pure Appl. Math.*, 54(6):689–760, 2001.

[4] Alberto Abbondandolo and Pietro Majer. Ordinary differential operators in Hilbert spaces and Fredholm pairs. *Math. Z.*, 243(3):525–562, 2003.

[5] Alberto Abbondandolo and Pietro Majer. A Morse complex for infinite dimensional manifolds. I. *Adv. Math.*, 197(2):321–410, 2005.

[6] Alberto Abbondandolo and Pietro Majer. Lectures on the Morse complex for infinite-dimensional manifolds. In *Morse theoretic methods in nonlinear analysis and in symplectic topology*, volume 217 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 1–74. Springer, Dordrecht, 2006.

[7] Alberto Abbondandolo and Pietro Majer. Infinite dimensional Grassmannians. *J. Operator Theory*, 61(1):19–62, 2009.

[8] Alberto Abbondandolo and Matthias Schwarz. On the Floer homology of cotangent bundles. *Comm. Pure Appl. Math.*, 59(2):254–316, 2006.

[9] Ralph Abraham and Joel Robbin. *Transversal mappings and flows*. An appendix by Al Kelley. W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[10] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics*. Elsevier, Academic Press, New York, 2nd edition, 2003.

[11] C. C. Conley and E. Zehnder. The Birkhoff-Lewis fixed point theorem and a conjecture of V. I. Arnold. *Invent. Math.*, 73(1):33–49, 1983.

[12] J. Eells and K. D. Elworthy. Open embeddings of certain Banach manifolds. *Ann. of Math. (2)*, 91:465–485, 1970.

[13] A. Floer and H. Hofer. Coherent orientations for periodic orbit problems in symplectic geometry. *Math. Z.*, 212(1):13–38, 1993.

[14] Andreas Floer. Morse theory for Lagrangian intersections. *J. Differential Geom.*, 28(3):513–547, 1988.

[15] Andreas Floer. Symplectic fixed points and holomorphic spheres. *Comm. Math. Phys.*, 120(4):575–611, 1989.

[16] Andreas Floer, Helmut Hofer, and Dietmar Salamon. Transversality in elliptic Morse theory for the symplectic action. *Duke Math. J.*, 80(1):251–292, 1995.

[17] G. H. Hardy. On the mean value of the modulus of an analytic function. *Proceedings of the London mathematical society*, series 2(14):269â277, 1915.

[18] Dan Henry. *Perturbation of the boundary in boundary-value problems of partial differential equations*, volume 318 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005. With editorial assistance from Jack Hale and Antônio Luiz Pereira.

[19] Amann Herbert. *Lectures on Some Fixed Point Theorems*. IMPA, Rio de Janeiro 1974.

[20] H. Hofer and D. A. Salamon. Floer homology and Novikov rings. In *The Floer memorial volume*, volume 133 of *Progr. Math.*, pages 483–524. Birkhäuser, Basel, 1995.

[21] H. Hofer and E. Zehnder. *Symplectic invariants and Hamiltonian dynamics*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 1994.

[22] Tosio Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[23] Serge Lang. *Fundamentals of differential geometry*, volume 191 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[24] Robert B. Lockhart and Robert C. McOwen. Elliptic differential operators on noncompact manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(3):409–447, 1985.

[25] D. McDuff and D. Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

[26] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[27] Paul H. Rabinowitz. Periodic solutions of Hamiltonian systems. *Comm. Pure Appl. Math.*, 31(2):157–184, 1978.

[28] Paul H. Rabinowitz. Periodic solutions of a Hamiltonian system on a prescribed energy surface. *J. Differential Equations*, 33(3):336–352, 1979.

[29] Joel Robbin and Dietmar Salamon. The Maslov index for paths. *Topology*, 32(4):827–844, 1993.

[30] Dietmar Salamon. Lectures on Floer homology. In *Symplectic geometry and topology (Park City, UT, 1997)*, volume 7 of *IAS/Park City Math. Ser.*, pages 143–229. Amer. Math. Soc., Providence, RI, 1999.

[31] Dietmar Salamon and Eduard Zehnder. Morse theory for periodic solutions of Hamiltonian systems and the Maslov index. *Comm. Pure Appl. Math.*, 45(10):1303–1360, 1992.

[32] Cornelia Schneider. Trace operators in Besov and Triebel-Lizorkin Spaces. *Preprint available at www.math.uni-leipzig.de/preprint/2008/p3-2008.ps*, 2008.

[33] M. Schwarz. *Morse homology*, volume 111 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1993.

[34] M. Schwarz. *Cohomology Operations from S^1-Corbodisms in Floer Homology*. PhD thesis. http://www.math.uni-leipzig.de/~schwarz/, ETH Zürich, 1995.

[35] Michael Shub. *Global stability of dynamical systems*. Springer-Verlag, New York, 1987. With the collaboration of Albert Fathi and Rémi Langevin, Translated from the French by Joseph Christy.

[36] S. Smale. An infinite dimensional version of Sard's theorem. *Amer. J. Math.*, 87:861–866, 1965.

[37] Michael E. Taylor. *Partial differential equations. I*, volume 115 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1996. Basic theory.