Oscillatory Solutions to Hyperbolic Conservation Laws and Active Scalar Equations

(Oszillierende Lösungen von hyperbolischen Erhaltungsgleichungen und aktiven skalaren Gleichungen)

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Contents

1 Introduction

1.1 Partial Differential Equations in Matrix Space and Basic Questions

This thesis concerns itself with two different classes of nonlinear partial differential equations in matrix space. We first consider the following large class of active scalar equations, for which we will construct exact weak solutions. The form of these equations being

$$\begin{aligned} \partial_t \theta(t,x) + \operatorname{div}\left(u(t,x)\theta(t,x)\right) &= 0, \\ \operatorname{div} u(t,x) &= 0, \\ u(t,x) &= T[\theta(t,x)], \end{aligned} \tag{1.1}$$

where $\widehat{T[\theta]}(\xi) = m(\xi)\widehat{\theta}(\xi)$, and m is even and 0-homogeneous.

In the second part of this thesis we consider systems of hyperbolic conservation laws: we will present new compactness criteria and compactness results. In particular we consider conservation laws of the form

$$\partial_t u(t,x) + \partial_x f(u(t,x)) = 0. \tag{1.2}$$

In both cases, (1.1) and (1.2), u is a vector, whereas θ is a scalar.

Hyperbolic conservation laws such as (1.2) and more general balance laws (i.e. containing a forcing term) have been studied for a long time in continuum physics, occurring in various physical situations. Some examples include thermoelastic nonconductors of heat, the isentropic process of thermoelastic fluids, Maxwell's equation in nonlinear dielectrics, and many others. As a good reference for a broad overview, especially to the classical theory and for a more explicit presentation of the above mentioned examples, we recommend the books of Dafermos [Da05] and Bressan [Br07].

In physical theories which ignore mechanisms of dissipation such as viscous stress or heat conduction, the quantities u and f are functions of the state variables only but not of their derivatives.

In contrast to hyperbolic conservation laws, active scalar equations (1.1) are somewhat more specialized, however they still can describe various physical situations, such as the flow of an incompressible fluid through porous media, turbulence inside earth's core, and others. We refer to Chapter 3.1 for more detailed examples.

For a detailed description of these two classes of differential equations and for the precise assumptions on the nonlinearity f in (1.2) and the Fourier multiplier m that defines T from (1.1), we refer to the respective introductory Chapters 3.1 and 4.1.

The idea to study nonlinear partial differential equations in matrix space goes back to the work of Tartar and Murat [Ta79], which itself has its precursors in the work of Ball [Ba77]. The theory of compensated compactness plays a crucial role here, especially for the question of approximate solutions, cf. Chapters 3.4 and 2.2.
Recently many authors use this or a similar approach for deeper insights in questions of exact or approximate solutions, e.g. in the context of microstructure and rigidity, or to obtain non-uniqueness results. Theory and examples connected to this thesis can be found in [CFG11], [KMS03], [Mü98], [Sh11], [Sz11].

We now describe a framework for working with nonlinear partial differential equations in matrix space. As with many other nonlinear partial differential equations, (1.1) and (1.2) can be relaxed or be reformulated in matrix space. The resulting 'relaxed problem' for maps $z : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ then typically reads as a combination of a linear set of equations

$$\sum_{j=1}^{n} A_j \partial_j z = 0, \tag{1.3}$$

where the A_j are constant $s \times N$ matrices, and a pointwise algebraic constraint

$$z(y) \in K, \tag{1.4}$$

that takes care of the nonlinearity of the equation. Here K is a set lying in the ambient state space $\mathbb{R}^N$. It is called *constitutive set*.

In this general context it is natural to address the basic questions of exact and approximate solutions to the (relaxed) problem. For the case of genuine differential inclusions as in (1.8) below we refer the reader to [Mü98]. More precisely the two questions are:

Question 1: Approximate Solutions: Characterize all sequences z_j of maps that satisfy (1.3) and such that $\operatorname{dist}(z_j, K) \to 0$.

Question 2: Exact Solutions: Characterize all maps z that satisfy (1.3) and (1.4).

We will give answers to Question 1 for (1.2) and to Question 2 for (1.1).

An interesting related problem is the problem of a proper relaxation of the set K is interesting. The problem is to determine sets $\tilde{K}$ with $K \subset \tilde{K} \subset \mathbb{R}^N$ of maps such that Questions 1 or 2 have solutions that suffice the respective relations for $\tilde{K}$.

The question of which space the maps z, z_j should lie depends on the problem. In general one would consider some L^p space.

We first outline how we will attack Question 1 relating to approximate solutions:

If one now approximates (1.3) and (1.4) with a sequence say $\{z_j\}_{j\in\mathbb{N}}$ one is exposed to a lack of a priori estimates that could lead to compactness. The first approach to overcome this is to use the theory of compensated compactness. It can be quite successful, e.g. the Div-Curl-Lemma (which will also for us play an important role) is a well-known example. Still the more systematic approach is to use so-called measure valued solutions. If the sequence $\{z_j\}_{j\in\mathbb{N}}$ is bounded it converges to a family of probability measures $\{\nu_y\}_{y\in\Omega}$ (cf. Theorem 2.2.1). This forms the *Young measure valued solution* associated with the sequence $\{z_j\}_{j\in\mathbb{N}}$.
The problem of (1.3) and (1.4) then transforms to

$$\sum_{j=1}^{n} A_j \partial_j \langle \nu_y, id\rangle = 0 \qquad \text{in } \mathcal{D}', \tag{1.5}$$

$$\operatorname{supp} \nu_y \subset K \qquad \text{for a.e. } y, \tag{1.6}$$

$$\langle \nu_y, f\rangle \geq f(\langle \nu_y, id\rangle) \qquad \text{for a.e. } y \text{ and all } A\text{-quasiconvex functions } f. \tag{1.7}$$

Instead of going into detail here regarding A-quasiconvexity, we refer the reader to [FM98] and give the following instructive example:

Let the matrices A_j be such that (1.3) is $\operatorname{curl} z = 0$ for $z:\ \Omega \to \mathbb{R}^{m\times n} \cong \mathbb{R}^N$. Then $z = Dv$ and the combination of (1.3) and (1.4) is equivalent to

$$Dv \in K. \tag{1.8}$$

Reformulations into matrix space that take the form (1.8) are called *gradient differential inclusions*. These are of course a lot more specific than the combination of (1.3) and (1.4). Here $v : \mathbb{R}^n \to \mathbb{R}^m$ is typically a Lipschitz function, such that the gradient exists a.e. In fact, the reformulation for the system of conservation laws (1.2) will take this form. Kirchheim [Ki03] and Müller [Mü98] give a survey on differential inclusions of this type.

Young measure valued solutions that arise from (1.8) are called *gradient Young measures*, for a precise definition we refer to Definition 2.2.5. The theorem of Kinderlehrer and Pedregal (Theorem 2.2.6) characterizes gradient Young measures as quasiconvex measures (precise definitions in Chapter 2.1). Considering (1.7) we see that in this situation, A-quasiconvexity is now the same as the usual quasiconvexity we introduce in Chapter 2.1.
In other words, the Young measure valued solutions coming from (1.8) are exactly the quasiconvex measures. Similarly, one has for the general case of (1.3)-(1.4) that the measure valued solutions arising from an approximation are A-quasiconvex measures:

for details on this characterization which is not used in this thesis we refer to [FM98].

A special case of A-quasiconvex functions are the A-quasiaffine functions, that are the functions for which (1.7) holds with equality. They correspond to the commutativity relation (4.4) in Chapter 4: in the situation of (1.8) we have that f is quasiaffine if and only if f is a subdeterminant, cf. [Mü98], [Ba77], [Mo66], [Re67]. As subdeterminants define polyconvex measures we thus have also that if ν is a measure valued solution (and therefore a gradient Young measure) then ν is a polyconvex measure. More details regarding this can be found in Chapter 2.2.

Compactness follows now from polyconvex measures if the conditions above localize. That means to forget the y-dependency of z and therefore also the one of ν. Young measures that do not depend on y are called *homogeneous*. The equation (1.5) then plays no role and we are led to study

$$\begin{aligned} &\operatorname{supp}\nu \subset K, \\ &\langle \nu, M\rangle = M(\langle \nu, id\rangle), \end{aligned} \tag{1.9}$$

instead of (1.5)-(1.7) for compactness from polyconvex measures. Here M denotes the vector consisting of all minors. We will have compactness here if and only if ν turns out to be a Dirac measure, cf. Lemma 2.2.3. See also [DP83], [DP85], [KMS03] and [Sv93].

Beside the question of approximate solutions, there is a number of different possibilities to attack Question 2 relating to exact solutions. Basically for exact solutions there are two main paths.
The first one is to do an explicit construction. The advantage of this path is that it is quite robust. In many well known examples so-called degenerate T_k-configurations play a role. We present a typical T_4-configuration in Example 1, Chapter 2.1. More genreal information on T_k-configurations can be found in [KMS03].
The second path, which is the one we will use, is to construct the exact weak solutions of (1.3) and (1.4) abstractly. We will do so from subsolutions via convex integration. More precisely we will use the Baire category method, which will be presented in detail in Chapter 2.3. The Baire category method actually is one variant of convex integration that Gromov developed in [Gr86]. The latter approach is in general a lot more difficult, as one has to calculate the so-called Λ-convex hull of the constitutive set K. In some cases computing a big part of that set will be enough – one example for this situation is the m-unbounded case in this thesis, see Proposition 3.2.5. This can in many examples become quite complex and therefore be out of reach. In these cases it is best to try the first way with degenerate T_k-configurations.
If one is able to calculate K^Λ (for the definition see Chapter 2.1) and then construct weak solutions from subsolutions, one gets in return additional benefits. One then can identify compatible boundary and initial conditions for which the construction works. In the context of the Euler equations in [DS10] these conditions were called "wild initial data". In case of the incompressible porous media equation one can then gain existence results for the related Muskat problem, cf. [Sz11]. This is explained in more detail in

Chapter 1.2. There the benefits will also be discussed in connection with coarse-graining, with emphasis on (1.1).

The terminology 'convex integration' already indicates that convexity plays an important role in the analysis of the above questions. It turns out that there are different notions of convexity in matrix space (that all agree in $\mathbb{R}^n$): besides the usual convexity, there is polyconvexity, quasiconvexity, rank-one-convexity, and Λ-convexity, see Chapter 2.1 for the respective definitions. Among these notions of convexity the usual convexity is the strongest, that is a convex set or function is always also poly-, quasi-, rank-one-, and Λ-convex. In the context of the Baire category method as presented in Chapter 2.3 Λ-convexity is the most useful tool. More precisely the Λ-convex hull of the constitutive set K gives the proper relaxation for Question 2 from above and will help us in the construction of exact weak solutions.
In particular quasiconvexity plays a crucial role in vectorial variational problems. It was introduced first by Morrey in [Mo52] who proved that quasiconvex functions are exactly the lower semicontinuous functions for variational functionals of the form $\int f(Du)dx$, cf. Theorem 2.1.5. One can directly see here the connection to (1.8). This will also be explained in more detail in Chapter 2.1. But the analysis for quasiconvex functions is quite complicated, as the definition for quasiconvexity is a non-local one. Therefore we will use the other convexity notions to obtain necessary and sufficient conditions for compactness. All this will be presented in more detail throughout this thesis.

1.2 Main Results

For the precise formulations of our results we refer the reader to the statements in the respective chapters in which they are proven. Here we give just brief formulations which is enough to list the achievements of this thesis and compare them to known results. As already mentioned above we will give answers to both the question for exact, and approximate solutions.

In the context of the active scalar equations, as an answer to Question 2 above (exact weak solutions), we prove in Chapter 3 for even m (bounded or unbounded) the following result.

THEOREM 1.2.1. *There exist infinitely many periodic weak solutions to* (1.1) *with*

$$\theta \in L^\infty(\mathbb{T}^n \times \mathbb{R}), \qquad u \in L^\infty(\mathbb{T}^n \times \mathbb{R}),$$

such that

$$|\theta(x,t)| = \begin{cases} 1 & \textit{for a.e. } (x,t) \in \mathbb{T}^n \times (0,T) \\ 0 & \textit{for } t \notin (0,T) \end{cases}.$$

This forms the main result in that chapter (see Theorem 3.3.3 and 3.3.5 therein). So we have non-uniqueness in L^∞. This result was (with slightly more technical assumptions

on the Fourier multiplier m) already proved in [Sh11] using an explicit T_4-construction. For the special case of the incompressible porous media equation, this theorem was proved in [CFG11] (explicit construction) and [Sz11] (Baire category method).
This theorem generalizes the result in [Sh11] in two ways. Firstly, we do not assume the existence of 'regular points' of m. Secondly and much more importantly, we do not use the explicit T_4-construction, but prove the theorem with the Baire category method, which gives the benefits illustrated below. Additionally, it is a generalization of [Sz11] as we broaden the class of equations for which the theorem holds considerably, as the incompressible porous media equation makes one special choice of m that is bounded and smooth. For our proof, m does not have to be smooth bounded or even continuous.

As mentioned we give a proof of this theorem with help of the Baire category method, that is we construct our solutions from subsolutions. This can be illustrated with the following picture. Convex integration takes a subsolution and adds highly oscillating functions in the directions of the wave cone Λ (see (2.1) in Chapter 2.1) to it. Repeating this procedure gives then a weak exact solution that corresponds to that subsolution. In fact, as the process of adding the high frequency oscillations is highly non-unique, one gains not just one, but infinitely many weak solutions.
The notion subsolution is understood here in the way that z is a (strict) subsolution if it belongs to the (interior of the) Λ-convex hull of the constitutive set K from (1.4), shorthand int K^Λ.

So the picture to have in mind for convex integration is the following:

$$\text{subsolution from int } K^\Lambda \xrightarrow{\text{convex integration}} \text{weak solutions (lie in } K).$$

As we compute the set K^Λ to apply the Baire category method, one can then easily see for which initial data the theorem works.
The notion of initial conditions here is understood naturally as we have weak continuity in time for (1.1): we prove existence of weak solutions with $\theta \in L_t^\infty L_x^2$, we can then redefine our solution on a set of measure zero in t and get $\theta \in C((0,T), L_w^2(\mathbb{T}^n))$, where $L_w^2(\mathbb{T}^n)$ denotes the L^2 space equipped with the weak topology such that we have indeed $\theta(0,x) = 0$ for a.e. $x \in \mathbb{T}^n$. A proof of this can be found in Appendix A of [DS10].
So, in this sense Theorem 1.2.1 does not only hold for the initial conditions stated therein. For which initial condition the theorem holds can be seen from looking at the space of subsolutions X_0. The abstract properties X_0 has to satisfy are explained in Chapter 2.3, (A3). In our construction for weak solutions it takes the form

$$X_0 = \big\{ z \in C^\infty(\mathbb{T}^n \times (0,T)) : \operatorname{supp} z(x,\cdot) \subset (0,T),\ (1.3) \text{ holds and } z(x,t) \in \text{ int } K^\Lambda \\ \text{for all } (x,t) \in \mathbb{T}^n \times (0,T) \big\}.$$

Compare the spaces X_0 in the proofs of Theorem 3.3.3 and 3.3.5.
It is then important to show that X_0 is nonempty. For the initial conditions as stated in the theorem we need to show that $0 \in X_0$. If one wants to prove the same statement

for other initial conditions one can just replace the condition that $\operatorname{supp} z(x,\cdot) \subset (0,T)$ by $\operatorname{supp} z(x,\cdot) - \bar{z}(x,\cdot) \subset (0,T)$. Where $\bar{z}$ is a different subsolution (that is not zero as in our case). For every $\bar{z}$ for which then X_0 is not empty one obtains infinitely many exact weak solutions. Or to put it in another way, all $\bar{z}$ that lie in X_0 (defined as above) give all subsolutions for which we obtain weak exact solutions.
In this context we want to mention as nice example for an application of this idea the evolution of microstructure described in [Sz11]. For the case of the incompressible porous media equation modeling the flow of two immiscible fluids of different densities in a Hele-Shaw cell (cf. [ST58]) Székelyhidi exhibited examples of nontrivial admissible subsolutions.
Also it is possible to modify the bound by 1, this becomes clear from Remark 3.2.2.

In the above picture, convex integration is in some sense the reverse operation to coarse-graining. We will not go into depth regarding coarse-graining here, as it plays not a major role in this thesis. However in coarse-graining one averages (macroscopically) over weak solutions and gains the so called coarse-grained flow. Here it is not of particular importance if one considers long-time averages, ensemble-averages or local space-time-averages, see the discussion in [DS12].

The picture for coarse-graining looks as follows:

$$\text{“wild solutions”} \xrightarrow{\text{averaging}} \text{subsolution.}$$

So the natural question (stated in the context of (1.1)) comes up, if every subsolution representing some averaged θ corresponds to an initial data for which our construction works. In terms of the geometry of matrix space, introduced in Chapter 2.1, the subsolutions coming from the process of coarse-graining lie in the quasiconvex hull K^{qc} of the constitutive set K. So this question can be stated as: When is $K^{\Lambda} = K^{qc}$? For certain cases we were able to answer this question. In the proof of Proposition 3.2.1 we observe that m induces naturally an (asymmetric) norm. For the cases where this induced norm say M has the property that $M(y) = \langle y, Ay\rangle^{\frac{1}{2}}$ for A symmetric and positive definite, we call the norm quadratic and prove the following theorem:

Theorem 1.2.2. *Assuming the norm induced by m is quadratic, then we have that $K^{qc} = K^{\Lambda}$.*

This result includes the case of the incompressible porous media equation. It is a complete new result in the sense that it was not observed in [Sz11]. It is restated as Theorem 3.4.5. We first prove the above theorem in Proposition 3.4.4 for the special case of the incompressible porous media equation with help of the classical Div-Curl-Lemma, and afterwards for the more general case where $0 \in m(S^{n-1})$ and the Fourier multiplier induces a quadratic norm. Again, this means that we can identify all obtainable coarse-grained flows with all initial data that are compatible with the given boundary conditions.

In the context of genuinely nonlinear hyperbolic conservation laws we are concerned with Question 1 of approximate solutions. We approximate the equation and get a first solution in the sense of parametrized measures, also called Young measures. The notion of Young measure valued solutions is particularly weak such that they exist always for hyperbolic conservation laws cf. Theorem 2.2.1 and Chapter 4.1. If the measure valued solution associated to our limit process turns out to be a Dirac measure we have in fact convergence in L^1 and hence compactness. The measures that are associated to the limiting process are in this case the quasiconvex measures, cf. Theorem 2.2.6. As there is no useful characterization of quasiconvex functions (or even quasiconvex measures) available, we turn to the class of polyconvex measures in order to find sufficient conditions for compactness – if we prove compactness with help of polyconvex measures, we also have compactness for quasiconvex measure valued solutions and hence in L^1.
The problem of compactness from polyconvex measures, for which a fully exhaustive answer is still not available, was already addressed by Tartar in [Ta79]. We quote "I sincerely believe that this is the right way to attack nonlinear partial differential equation (from mechanics and physics)...". Tartar proved compactness for the case of the scalar conservation law. The next partial answer was derived by DiPerna in his marvelous paper [DP85] for the system of Lagrangian elasticity admitted by two entropy/entropy flux pairs. A more concise and systematic proof of his result can be found in Chapter 4.7.
After these authors [KMS03] formulated local problems for the situation that we are concerned with in the context of hyperbolic conservation laws. The properties to be investigated are taken from [KMS03]. Written down for the respective sets of measures they are:

(P1) Each constant matrix $A \in K$ has a neighborhood $U \subset \mathbb{R}^{2\times l}$ such that $\mathcal{P}^{rc}(K \cap \bar{U})$ is trivial.

(P2) Each constant matrix $A \in K$ has a neighborhood $U \subset \mathbb{R}^{2\times l}$ such that $\mathcal{P}^{pc}(K \cap \bar{U})$ is trivial.

If we show (P2), then we also will have compactness, see the exact definitions and explanations in Chapter 2.
The first achievement of this thesis on hyperbolic conservation laws is that we give a rather broad systematic approach of how to attack the question of compactness from polyconvex measures. We therefore derive a simple normal form for the curve $\gamma(u)$ that defines the constitutive set K in the matrix space formulation of (1.2). With help of this we can characterize the cases in which compactness coming from polyconvex measures holds.

Theorem 1.2.3. *Let $\gamma(u) \subset \mathbb{R}^{2\times l}$ have normal form and $M(\gamma(u)) \in \mathbb{R}^N$ denote the vector consisting of all 2×2 - minors of the matrix $\gamma(u)$. Then we have for $\varepsilon > 0$ the following dichotomy:*
either there exists a probability measure ν supported on $B_\varepsilon(0)$ that is not a Dirac measure

such that

$$\int M(\gamma(u))d\nu = M(A),$$

or there exists a vector $\alpha \in \mathbb{R}^N$ *such that for all probability measures* ν *supported on* $B_\varepsilon(0)$ *that are not a Dirac measure at* 0 *we have*

$$\alpha \cdot \int M(\gamma(u))d\nu < \alpha \cdot M(A).$$

This Theorem is restated and proved as Theorem 4.4.2, see also Theorem 4.4.1.
In the first case we have no compactness from polyconvex measures. On the one hand compactness could still be possible coming from quasiconvex measures (which would be rather difficult to prove). On the other hand there is the possibility of constructing a rank-one-convex measure as a counterexample to compactness.

In the second case we get a contradiction to the fact that equality must hold in the minor relation

$$\int M(\gamma(u))d\nu = M\left(\int \gamma(u)d\nu\right)$$

that is due to Tartar (he proved it in [Ta79]). Hence we have that ν must be δ_0, that is compactness. This minor relation (or commutativity relation) relies on the Div-Curl-Lemma (cf. [Ev90]). It is a simple application of the Div-Curl-Lemma onto the stream functions that are used to reformulate the conservation law (1.2) into matrix space, cf. Chapter 4.2. The arguments about whether we have compactness or not are explained in more detail in Chapter 4.4, see also Chapter 2.2.

Our general strategy is inspired by DiPerna. It is to insert the Taylor expansion of $\gamma(u)$ into the minor relation, take linear combinations of the individual minors therein, and then to conclude by standard analytical techniques. From this ansatz we derive some necessary (Lemmas 4.5.1 and 4.5.2) and a sufficient (Propositions 4.5.4) condition that will help to answer our compactness question. Unfortunately a necessary and sufficient condition is not available by now.

The assertions above hold for any number of hyperbolic conservation laws admitting an arbitrary number of entropy pairs. We then consider the narrower case of systems consisting of two equations (mostly admitting two entropy pairs). Here we give an equivalent alternative for taking linear combinations of minors. With help of this we prove that quadratic coordinate changes in u will not simplify compactness questions (Proposition 4.6.2). Also we show that rank-one directions of the constitutive set K are stable (Proposition 4.6.1) and see that the curvature of K is connected to the existence of convex entropies.

As further consequence of the necessary conditions coming from Theorem 1.2.3 we obtain the following result:

Theorem 1.2.4. *There is no compactness from polyconvex measures for systems of two hyperbolic conservation laws admitted by only one entropy.*
In other words: the corresponding constitutive set K does not have property (P2).

The question whether this is true or not was already raised in [KMS03]. There the system of Lagrangian elasticity was considered, admitting only one entropy pair. Compared to the case of two entropy pairs this broadens the space functions from which we approximate and hence makes establishing compactness results more challenging. Considering this situation, in [KMS03] the question was asked whether the set of polyconvex or rank-1-convex measures is trivial or not, that is: does the constitutive set K have properties (P1) or (P2) or not? The above theorem does not only apply for the special system from [KMS03] but for any hyperbolic system of two equations admitted by only one convex entropy pair. It answers the question regarding property (P2) negatively.

1.3 Organization of this Thesis

In Chapter 2 we present all geometrical and analytical tools that we need to prove the above mentioned results. We start therein (Chapter 2.1) with introducing basic notions for partial differential equations in matrix space. After that we turn to the various convexity notions in matrix space in the same section. Both of these topics are important throughout the whole thesis. As we use for the conservation laws Young measure theory we introduce the notion of measure valued solutions in Chapter 2.2 and exhibit how to relate them to the different convexity notions. The last part of Chapter 2 is then devoted to the Baire category method. Although it is not a development made in this thesis we prove it comprehensively for the specific way we use it for constructing exact solutions to active scalar equations.

Chapter 3 is devoted to active scalar equations of the form (1.1). In the introductory section we give first a precise formulation of the equation and the conditions the individual entities have to satisfy. Then the main results for that chapter are stated precisely. Additionally we give some examples for physical systems that can be described by these equations and their respective Fourier multipliers (bounded and unbounded cases). In Chapter 3.2 we derive the relaxation of (1.1) in matrix space. We distinguish between the case where T is a bounded Operator (this holds as long as the set $\{m(S^{n-1})\}$ is bounded) for which we compute the Λ-convex hull of the constitutive set K, and the case where the set $\{m(S^{n-1})\}$ is unbounded (here we compute a large enough subset of K^Λ). After these computations we make a restriction to bounded subsets of K^Λ. This is required to obtain solutions that lie in L^∞ rather than just in L^2. In Chapter 3.3 we finally apply convex integration to establish existence of weak solutions and hence prove our main result, Theorem 1.2.1. In the Chapter 3.4 we investigate some properties of the induced norm and other functions and then prove Theorem 1.2.2, that is when $K^{qc} = K^\Lambda$.

Chapter 4 is devoted to hyperbolic conservation laws (1.2). In the introductory Chapter 4.1 we present basic concepts and examples where hyperbolic conservation laws play a role. We mention also how solutions can be approximated. We continue in Chapter 4.2 by reformulating (1.2) in matrix space. Conservation laws turn out to be a differential inclusion of type (1.8). In the subsequent Chapter 4.3 we develop our general strategy for gaining compactness from polyconvex measures. This contains the derivation of a simple normal form for the curve there called $\gamma(u)$, which is to lie in the constitutive set. In Chapter 4.4 we prove Theorem 1.2.3 that characterizes all cases in which it is possible to gain compactness from polyconvex measures. From this starting point we derive in Chapter 4.5 some necessary and also a sufficient condition. Then in Chapter 4.6 we prove some properties for systems of two equations admitted by two entropy pairs. The examples for compactness and noncompactness from polyconvex measures discussed in the subsequent Chapter 4.7 follow the path outlined in the foregoing sections. First we present the well-known case of a scalar conservation law in our setting. Next we show briefly the case of Lagrangian Elasticity, that was first proved by DiPerna. Third we prove Theorem 1.2.4.

The last Chapter 5 is split in two, where we discuss the respective results from Chapter 3 and 4 and point out some open problems.

2 Analysis and Geometric Considerations for Differential Inclusions

In this chapter we present all the tools needed to prove the results of this thesis. Most of the ideas presented here are known, and taken to some extend from [KMS03], [Mü98], and [Sz11]. In these references some statements are explained in more detail. For a good overview, we picked all ingredients needed in this thesis and glued them together, so that – except for some proofs we refer to – this thesis is all self-contained.

2.1 Basic Tools and Convexity in Matrix Space

The typical convex integration approach to investigating exact solutions (Question 2 from the introduction) is to consider the so called *wave cone* Λ. It describes plane wave solutions of the linear system (1.3) and is defined by

$$\Lambda := \left\{ \xi \in \mathbb{R}^n : \exists\, a \in \mathbb{R}^N \text{ s.t. } \sum_{i=1}^{n} \xi_i A_i a = 0 \right\}. \tag{2.1}$$

The cone Λ is related to one dimensional solutions $z(x) = h(x \cdot \xi)$ of (1.3) for Lipschitz maps $h : \mathbb{R} \to \mathbb{R}$, characterizing directions of one dimensional high frequency oscillations compatible with (1.3).

The question of whether such solutions also lie in the set K leads to the notion of lamination convexity. This notion was first introduced in a more general setting (of so called jet bundles) by Gromov [Gr86] (he called it P-convexity). For the definitions of other convexity notions in matrix space see below in this chapter. Gromov's method of convex integration is a significant generalization of the work of Nash [Na54] and Kuiper [Ku55] on C^1 isometric immersions.

A set M is called *lamination convex* with respect to a given cone Λ, if for any two points $A, B \in M$ are such that $A - B \in \Lambda$ then the whole segment $[A, B]$ belongs to M. The *lamination convex hull* of a set K then is the smallest lamination convex set containing K. The main point for convex integration, in the spirit of Gromov, is that (1.3) subject to (1.4) admits many solutions if the lamination convex hull of K, denoted as $K^{lc,\Lambda}$, is sufficiently large. This is because a large set $K^{lc,\Lambda}$ allows us to add high frequency oscillations in many directions to a subsolution with intent to construct a solution. A method of doing this is explained in detail in Chapter 2.3, where the general construction is presented. There we make no explicit use of the set $K^{lc,\Lambda}$. In fact, it turns out that instead of working in $K^{lc,\Lambda}$ one can also work in the Λ-*convex hull* K^{Λ}, which is defined by duality: a point does not belong to K^{Λ} if and only if there exists a Λ-convex function (i.e. a function that is convex in the directions given by the cone Λ) that separates it from K. A crucial fact is that K^{Λ} can be much larger than $K^{lc,\Lambda}$. The difference can already be observed in T_4-configurations that were already mentioned in the introduction. An example for such a configuration is given below.

Example 1. *We consider the space of diagonal* 2×2*-matrices and set the following four matrices* $A_1 = \mathrm{diag}(-1,-3) = -A_3$ *and* $A_2 = \mathrm{diag}(-3,1) = -A_4$*. Then we define* $K := \{A_1,\ A_2,\ A_3,\ A_4\}$.

One can see immediately that $\mathrm{rank}(A_i - A_j) \neq 1$*; still, they support a non-trivial minimizing sequence* $\{u_j\}$ *with* $\mathrm{dist}\,(Du_j, K) \to 0$ *in measure, but* Du_j *does not converge.*

For a complete proof of this we refer to Lemma 2.6 in [Mü98]. The idea is to use the rank-one-connections of the four matrices $J_1 = \mathrm{diag}(-1,1) = -J_3,\ J_2 = \mathrm{diag}(1,1) = -J_4$ *in a very clever manner. In a picture these matrices are located as follows:*

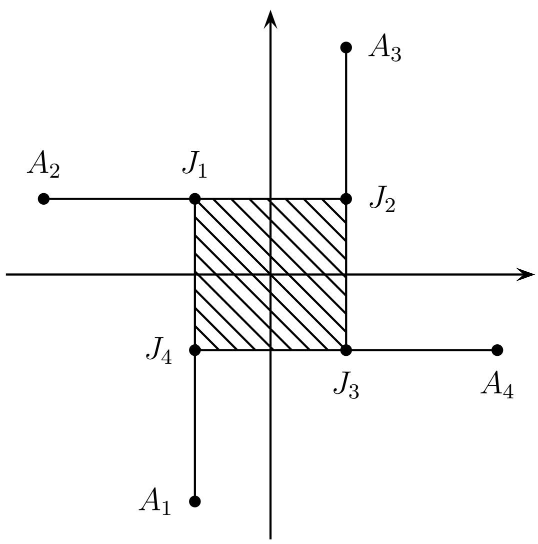

Figure 2.1: Typical T_4-configuration.

This example gives the proper idea of how to construct weak solutions explicitely. It is in fact the basic idea for the proofs in [CFG11] and [Sh11]. Therein one fixes 4 points of the K that lie in such a configuration. The counterpart to this configuration which is the Baire category method (Chapter 2.3) just sees a subset of K^{Λ} (for us mostly int K^{Λ}) and thus takes care of the whole set K but not only of four fixed points in K.

Now we turn to the definition of the further convexity notions that play a role in our matrix space analysis. These definitions are standard, cf. [Mü98].

DEFINITION 2.1.1. *For a matrix $A \in \mathbb{R}^{m\times n}$ let $M(A)$ denote the vector of all minors (i.e. subdeterminants) of A. Then a function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is called*

(i) convex *if for all $A, B \in \mathbb{R}^{m\times n}$ and all $\lambda \in (0,1)$*

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B); \tag{2.2}$$

(ii) polyconvex *if f is a convex function of its minors, i.e. there exists a convex function g such that $f(A) = g(M(A))$;*

(iii) quasiconvex *if for every open and bounded set Ω with $|\partial\Omega| = 0$ and any $\phi \in W^{1,\infty}(\Omega, \mathbb{R}^m)$ one has*

$$\int_\Omega f(A + D\phi)dx \geq \int_\Omega f(A)dx = |\Omega| f(A);$$

(iv) rank-1 convex *if f is convex along rank-1 lines, i.e.* (2.2) *for all $A, B \in \mathbb{R}^{m\times n}$ with* rank $(B - A) = 1$ *and all $\lambda \in (0,1)$.*

REMARK 2.1.2. *(i) If $m = 1$ or $n = 1$, then all notions of convexity agree.*

(ii) The definition of quasiconvexity is indeed independent of Ω. If the quasiconvexity condition holds for one Ω one can extend ϕ by 0 outside Ω and then translate and scale Ω.
Still the definition of quasiconvexity is not local, a fact that makes it often difficult to handle in concrete calculations.

(iii) For $f \in C^2$ rank-1 convexity is equivalent to the Legendre-Hadamard condition.

Regarding these convexity notions we have the following well-known lemma:

LEMMA 2.1.3. *Convexity $\Rightarrow$ polyconvexity $\Rightarrow$ quasiconvexity $\Rightarrow$ rank-1 convexity.*

PROOF. A proof of this can be found in [Mü98] (Lemma 4.3 therein) or [Pe93]. □

REMARK 2.1.4. *The opposite implications are in general false for $n \geq 2$, $m \geq 2$. The only pending question here is whether rank-1 convexity implies quasiconvexity. Šverák answered this question negatively for the case where $m \geq 3$ by giving a counterexample in [Sv92]. For $m = 2$, $n \geq 2$ this question remains open.*

For a closed set K one then can define the **-hull*, where the '*' stands for quasiconvex, polyconvex or rank-1 convex, as the set of points that cannot be separated by the corresponding class of functions:

$$K^* := \left\{ M \in \mathbb{R}^{m\times n} : \, f(M) \leq \sup_K f, \text{ where } f \text{ is } * \right\}. \tag{2.3}$$

In view of Lemma 2.1.3 we have then

$$K^{rc} \subset K^{qc} \subset K^{pc} \subset K^{c}. \tag{2.4}$$

Furthermore $K^{lc,\Lambda} \subset K^{rc}$, which is a direct corollary of Lemma 2.1.3. One can also define the * hull for open or arbitrary sets, cf. [KMS03]. As we have no application for these sets in this thesis we omit the definitions in this place.

The natural convexity notion for vector valued variational problems is the one of quasiconvexity. It was first introduced by Morrey in [Mo52]. He proved the following theorem:

Theorem 2.1.5 (Morrey). *Suppose $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is continuous. And let $I(u) = \int_\Omega f(Du)dx$ for a bounded Lipschitz domain Ω.*

(i) $I(\cdot)$ is weakly--sequentially lower semicontinuous (w*slsc) if and only if f is quasiconvex.*

(ii) If in addition for some $p \geq 1$ and $C > 0$ the inequalities $0 \leq f(A) \leq C(|A|^p + 1)$ hold, then $I(\cdot)$ is wslsc on $W^{1,p}(\Omega; \mathbb{R}^m)$.

Proof. A proof of (i) can be found in [Mü98]. This Theorem is Theorem 4.4 in [Mü98] and is proved in Chapter 4.8 therein. For statement (ii) we refer to Morrey's original paper. In fact, (ii) does not play a role in this thesis. ☐

As mentioned in the introduction, the main issue of characterizing approximate and exact solutions (Questions 1 and 2 in the introduction) is that it is technically very difficult to show whether a function is quasiconvex or not. This relies on the fact that the definition is non-local. Therefore with polyconvexity and rank-1 convexity one has tools to obtain sufficient or necessary conditions for lower semicontinuity as these notions are a lot easier to handle. For the space of measure valued solutions introduced in the following section one has a similar inclusion as (2.4) giving again necessary and sufficient criteria for compactness coming from the respective class of measure valued solutions.

2.2 Young Measures and Convexity

In this part we introduce the notion of measure valued solutions, which is very weak. Then we show how it relates to usual weak solutions and how the different convexity notions from above come into play in the class of measure valued solutions.

Theorem 2.2.1 (Fundamental Theorem for Young Measures). *Given bounded $\Omega \subset \mathbb{R}^n$ and a sequence $\{u_k\}_{k=1}^\infty$ bounded in $L^\infty(\Omega, \mathbb{R}^m)$, then there exists a subsequence $\{u_{k_j}\}$ and for a.e. $x \in \Omega$ a probability measure ν_x on $\mathbb{R}^m$ such that for all $F \in C(\mathbb{R}^m)$*

$$F(u_{k_j}) \overset{*}{\rightharpoonup} \overline{F} \textit{ in } L^\infty(\Omega), \textit{ where } \overline{F} \textit{ is given by } \overline{F}(x) := \langle \nu_x, F\rangle = \int_{\mathbb{R}^m} F(x)d\nu_x,$$

that is for all $\phi \in L^1(\Omega)$ and $F \in C(\mathbb{R}^m)$

$$\int\limits_{\Omega} \phi(x)F(u_{k_j}(x))dx \to \int\limits_{\Omega} \phi(x) \int\limits_{\mathbb{R}^m} F(\xi)d\nu_x(\xi)dx.$$

PROOF. A proof for this theorem can be found in [Ev90]. □

DEFINITION 2.2.2. *We call the family $\{\nu_x\}_{x\in\Omega}$ defined in Theorem 2.2.1 a* family of Young measures *associated with the subsequence $\{u_{k_j}\}$. Also we say that the Young measure ν is generated by the sequence $\{u_{k_j}\}$. Furthermore the center of mass of the Young measure also called* barycenter *is denoted by $\overline{\nu}_x := \langle \nu_x, id\rangle = \int\limits_{\mathbb{R}^m} \xi d\nu_x(\xi)$.*

The following lemma is well-known (for instance Corollary 2.1 in [DP85]), it plays a crucial role in the proofs in Chapter 4:

LEMMA 2.2.3. *If the sequence sequence $\{u_k\}$ lies in $L^\infty(\Omega, \mathbb{R}^m)$, then the sequence $\{u_{k_j}\}$ converges strongly if and only if $\nu_x = \delta_{u(x)}$ for a.e. x.*

Before delve deeper into the theory of Young Measures, we want to give two simple examples for Young measures that exhibit distinguished features. As side remark, we want to mention that the notion of measure valued solutions was even further generalized to capture not only oscillation but also concentration effects. We will not go into details regarding that as our situation is not effected by such phenomena.

1. As already seen in the above Lemma 2.2.3, we have for $u_n \to u$ strongly in L^1 that the generated Young measure is simply $\nu_x = \delta_{u_x}$. This can be interpreted as that in strongly converging sequences no oscillation effect will take place.

2. For $u : \mathbb{R} \to \mathbb{R}$ consider the functional $I(u) = \int_0^1 (1-u_x^2)^2 + u^2 dx$. Minimize $I(u)$ subject to $u(0) = u(1) = 0$.
   Let v be the periodic extension of the sawtooth function
   $$s(x) = \begin{cases} x & \text{on } [0, \frac{1}{2}) \\ 1-x & \text{on } [\frac{1}{2}, 1) \end{cases}.$$
   Then $u_k(x) := \frac{1}{k}v(kx)$ gives an faster and faster oscillating sequence such that $I(u_k)$ converges to its infimum 0. Also $u_k \rightharpoonup u = 0$ but the infimum is not attained as no function will satisfy $u' = \pm 1$ and $u \equiv 0$ at the same time. The generated Young measure then would be $\nu_x = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$. Note that this measure is in fact independent of x. Hence it is called a homogeneous Young measure. In some sense one can interpret this in the following way: at a point x this very weak kind of solution has value 0 (as $u_k \rightharpoonup 0$) and has slope $+1$ or -1 with a probability of 50% each.

Let now $\mathcal{P}(K)$ denote the set of probability measures supported on K. For $\nu \in \mathcal{P}(K)$ we denote as above by $\bar{\nu} = \int A d\nu(A)$ its barycenter or center of mass. We consider the following subsets of $\mathcal{P}(K)$ which satisfy a Jensen inequality with respect to the convexity notions from Chapter 2.1:

$$\mathcal{P}^{rc}(K) := \left\{ \nu \in \mathcal{P}(K) : \int f(A) d\nu(A) \geq f(\bar{\nu}) \text{ for all rank-1 convex } f \right\},$$
$$\mathcal{P}^{qc}(K) := \left\{ \nu \in \mathcal{P}(K) : \int f(A) d\nu(A) \geq f(\bar{\nu}) \text{ for all quasiconvex } f \right\},$$
$$\mathcal{P}^{pc}(K) := \left\{ \nu \in \mathcal{P}(K) : \int f(A) d\nu(A) \geq f(\bar{\nu}) \text{ for all polyconvex } f \right\}.$$

In view of Definition 2.1.1 we have for the polyconvex measures the following characterization:

Corollary 2.2.4.

$$\mathcal{P}^{pc}(K) = \left\{ \nu \in \mathcal{P}(K) : \int M(A) d\nu(A) = M(\bar{\nu}) \textit{ for all minors } M \right\}.$$

Looking back to (2.3) for closed sets K we then have

$$K^* = \{\bar{\nu} : \nu \in \mathcal{P}^*(K)\}, \qquad \text{where } * \in \{rc, qc, pc\}$$

and derive the following inclusions in the space of probability measures:

$$\mathcal{P}^{rc} \subset \mathcal{P}^{qc} \subset \mathcal{P}^{pc}. \tag{2.5}$$

Gradient Young Measures

Besides the general problem (1.3)-(1.4) we are in view of the analysis on hyperbolic conservation laws also interested in so called Gradient Young measures. In the matrix space formulation to be derived in Chapter 4.2 z will be a gradient. This means we want to solve/approximate the differential inclusion

$$Dv \in K,$$

for some set K. As we will use Young measure theory to answer it, this question motivates to the following definition:

Definition 2.2.5. *A weakly-*-measurable map $\nu : \Omega \to \mathcal{P}(\mathbb{R}^{m\times n})$ is called a $W^{1,\infty}$-*gradient Young measure *if there exists a sequence of maps $v_j : \Omega \to \mathbb{R}^n$ such that*

$$\begin{aligned} v_j &\overset{*}{\rightharpoonup} v && \textit{in } W^{1,\infty} \quad \textit{and} \\ \delta_{Dv_j} &\overset{*}{\rightharpoonup} \nu && \textit{in } L^\infty_w(\Omega, \mathcal{P}(\mathbb{R}^{m\times n})). \end{aligned}$$

The problem of approximate solutions (Question 1 from the introduction) is now the same as asking for all $W^{1,\infty}$-gradient Young measures that do arise from gradients. Kinderlehrer and Pedregal gave in [KP91] a complete characterization of such maps:

THEOREM 2.2.6 (Kinderlehrer-Pedregal). *A weakly-*-measurable map $\nu : \Omega \to \mathcal{P}(\mathbb{R}^{m\times n})$ is a $W^{1,\infty}$-gradient Young measure, if and only if $\nu_x \geq 0$ a.e. and there exist $K \subset \mathbb{R}^{m\times n}$ compact and $u \in W^{1,\infty}(\Omega;\mathbb{R}^m)$, such that*

(i) $\operatorname{supp}\nu_x \subseteq K$ *for a.e.* $x \in \Omega$*;*

(ii) $\overline{\nu}_x = Du(x)$ *for a.e.* $x \in \Omega$*;*

(iii) $\int f d\nu_x \geq f(\overline{\nu}_x)$ *for a.e.* $x \in \Omega$ *and all quasiconvex* $f : \mathbb{R}^{m\times n} \to \mathbb{R}$*.*

The important point in this theorem is (*iii*). It tells us that the quasiconvex Young measures are the ones that arise from approximating the differential inclusion. As already pointed out it is quite difficult to work with quasiconvexity. In view of (2.5) and Lemma 2.2.3 we obtain the following criteria for compactness questions on differential inclusions:

1. Compactness for approximate solutions $\Leftrightarrow$ $\mathcal{P}^{qc}$ trivial, i.e. $\mathcal{P}^{qc}$ consists only of Dirac measures.

2. $\mathcal{P}^{pc}$ trivial $\Rightarrow$ compactness for approximate solutions.

3. $\mathcal{P}^{rc}$ not trivial $\Rightarrow$ no compactness for approximate solutions.

The second point is the one we will mainly use for our compactness criteria in Chapters 4.4 to 4.7.

2.3 Baire Category Method

As we presented some general analytic tools for solving differential inclusions above, and also some more specific facts that will be used for systems of hyperbolic conservation laws. We want to present now the Baire category method in the form we will apply it to the active scalar equations (1.1) in Chapter 3.3.

This part is mainly taken from [Sz11]. We just adjusted a few things in the proofs and added some more interpretations.

We recall our set-up for solving systems partial differential equations in matrix space:

$$\sum_{i=1}^{n} A_i \partial_i z = 0 \qquad \text{in } \Omega,$$
$$z(y) \in K \qquad \text{for a.e. } y \in \Omega,$$

where $z : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ is the unknown state variable, $A_i \in \mathbb{R}^{n\times N}$, and $K \subset \mathbb{R}^N$ is a closed set.

To apply the Baire category theorem we make the following three assumptions:

(A1) Assumption on the wave cone:

There exist a closed cone $\Lambda \subset \mathbb{R}^N$ and a constant $C > 0$ such that we have for every $\bar{z} \in \Lambda$ a sequence $\{w_j\}_{j=1}^{\infty} \subset C_0^{\infty}(B_1(0), \mathbb{R}^N)$ that satisfies

(i) $\sum_{i=1}^{n} A_i \partial_i w_j = 0$ for all j,

(ii) $\operatorname{dist}(w_j, [-\bar{z}, \bar{z}]) \to 0$ uniformly,

(iii) $w_j \rightharpoonup 0$ in L^2,

(iv) $\int |w_j|^2 dy > C|\bar{z}|^2$.

The wave cone Λ was already introduced in (2.1). So the crucial point here is the fact that it admits for any point in Λ the oscillation property expressed above. Although the oscillation converge weakly to zero, they are detected by the strong L^2-norm. Condition (ii) will be responsible, for our technical assumption in Theorem 3.3.5 that m be bounded in a neighborhood of the ξ_i. For more detail cf. Chapter 3.3.

(A2) Existence of some open set $U \subset \mathbb{R}^N$ with the following properties:

(i) $U \cap K = \emptyset$,

(ii) for all $z \in U$ with $\operatorname{dist}(z, K) \geq \alpha > 0$ there exists $\bar{z} \in \Lambda \cap S^{N-1}$ such that

$$z + t\bar{z} \in U \text{ for } |t| < \beta.$$

Here β is a positive constant depending only on α.
A natural choice for this set U is the interior of the Λ-convex hull of K. See also Chapter 2.1, where we introduced the different convexity notions. In cases where it is too difficult to compute K^{Λ} other choices for U are of course possible. In fact we will use the set $\operatorname{int} K^{\Lambda}$ for all directions of Λ only for bounded m. Still we will restrict to a bounded subset thereof to obtain L^{∞} instead of just L^2 solutions.

(A3) Space of subsolutions:

There exists a nonempty, bounded set $X_0 \subset L^2(\Omega)$ and an open subdomain $\mathscr{U} \subset \Omega$ with $|\mathscr{U}| < \infty$ such that the following holds:

(i) $z(y) \in U$ for all $y \in \mathscr{U}$,

(ii) for $w \in C_0^{\infty}(\mathscr{U})$ solving the linear equations (1.3) and $z \in X_0$ with $(z+w)|_{\mathscr{U}} \in U$ we have that $z + w \in X_0$.

So in this sense one can say that the elements of X_0 are perturbable on the subdomain $\mathscr{U}$.

Having these three assumptions satisfied, we define X to be the closure of X_0 with respect to the weak L^2-topology. As X_0 is bounded, the topology of weak L^2 convergence is metrizable on X. Hence X becomes a complete metric space with some metric say $d_X(\cdot,\cdot)$.

THEOREM 2.3.1. *Let (A1)-(A3) be satisfied. Then the set*

$$\{z \in X : z(y) \in K \quad \textit{for a.e. } y \in \mathscr{U}\}$$

is residual in X.

DEFINITION 2.3.2. *In a topological space a set is called* nowhere dense *if its closure has nonempty interior.*
A set is called meagre *is it can be expressed as the union of countably many nowhere dense sets.*
A set is called comeagre *or* residual set *if it is the complement of a meagre set.*

REMARK 2.3.3. *Therefore a set to be residual in X implies that it is dense in X. So this argument gives us infinitely many solutions.*

Before we are ready to prove this more or less well-known theorem, we state the following lemma which plays a decisive role in the proof of the theorem above.

LEMMA 2.3.4. *In the situation described above let (A1)-(A3) be satisfied. Let furthermore $z \in X_0$ with $\int_{\mathscr{U}} \min(1, \operatorname{dist}(z(y), K))dy \geq \varepsilon > 0$. Then for all $\eta > 0$ there exists $\tilde{z} \in X_0$ with $d_X(z, \tilde{z}) < \eta$ and $\int_{\mathscr{U}} |z - \tilde{z}|^2 dy \geq \delta$, with $\delta = \delta(\varepsilon) > 0$.*

PROOF OF THEOREM 2.3.1. We note first that the functional $I(z) = \int_{\mathscr{U}} |z(y)|^2 dy$ is a Baire-one function, i.e. the the pointwise limit of a sequence of continuous functions.

To see this choose $\eta_j \in C_c^\infty(B_{\frac{1}{j}}(0))$ as the standard mollifier sequence. Then observe that

$$I_j(z) := \int\limits_{y \in \mathscr{U} :\, \operatorname{dist}(y, \partial\mathscr{U}) > \frac{1}{j}} |z * \eta_j(y)|^2 dy$$

is continuous as a map from X to $\mathbb{R}$, and that $I_j(z) \to I(z)$ as $j \to \infty$.

Now the Baire category theorem asserts that the set

$$Y := \{z \in X : I \text{ is continuous at } z\}$$

is residual in X. Next we show that $z \in Y$ implies $\int_{\mathscr{U}} \min(1, \operatorname{dist}(z(y), K))dy = 0$. Then the proof of the theorem is finished.
Let us assume the opposite, $\int_{\mathscr{U}} \min(1, \operatorname{dist}(z(y), K))dy =: \varepsilon > 0$ for some $z \in Y$. Furthermore $\{z_j\}_{j\in\mathbb{N}} \subset X_0$ be a sequence such that $d_X(z_j, z) \to 0$. Since I is continuous at z (as $z \in Y$), we have that $z_j \to z$ strongly in $L^2(\mathscr{U})$. For j large enough we may assume that $\int_{\mathscr{U}} \min(1, \operatorname{dist}(z_j(y), K))dy > \frac{\varepsilon}{2}$.
By applying Lemma 2.3.4 to each element of the sequence $\{z_j\}_{j\in\mathbb{N}}$ we get a new sequence $\{\tilde{z}_j\}_{j\in\mathbb{N}} \subset X_0$, such that $d_X(\tilde{z}_j, z) \to 0$ in X, and hence strongly in L^2. But the Lemma also says that $\int_{\mathscr{U}} |z - \tilde{z}|^2 dy \geq \delta(\varepsilon) > 0$, which contradicts the strong convergence of both sequences $\{z_j\}_{j\in\mathbb{N}}$ and $\{\tilde{z}_j\}_{j\in\mathbb{N}}$ to z. $\square$

We conclude this Chapter by proving Lemma 2.3.4. For notational convenience we define $F(z) := \min\{1, \mathrm{dist}\,(z, K)\}$.

Proof of Lemma 2.3.4. First, since $z \in X_0$ is continuous and $z(y_0) \notin K$ for any $y_0 \in \mathscr{U}$, there exists $r_0 > 0$ depending on y_0 such that

$$\frac{1}{2}F(z(y_0)) \leq F(z(y)) \leq 2F(z(y_0)) \quad \text{for all } y \in B_{r_0}(y_0) \subset \mathscr{U}.$$

Then we can cover at least half of $\mathscr{U}$ with say N balls $B_k = B_{r_k}(y_k)$, for $1 \leq k \leq N$, i.e. $| \cup_{k=1}^{N} B_k| \geq \frac{1}{2}|\mathscr{U}|$. Additionally we can do it in such a manner that

$$\sum_{k=1}^{N} \int\limits_{B_k} F(z(y))dy \geq \frac{1}{2} \int\limits_{\mathscr{U}} F(z(y))dy \tag{2.6}$$

Second, we see that (A2) implies that there exists some $\phi : [0,1] \to [0,1]$ continuous with $\phi(0) = 0$ and $\phi(t) > 0$ for $t \neq 0$. Additionally for any $z \in U$ there exists $\bar{z} \in \Lambda \cap S^{N-1}$ such that

$$z + t\bar{z} \in U \text{ for } |t| < \phi(F(z(y))). \tag{2.7}$$

Without loss of generality we may also assume that ϕ is convex and monotone increasing (if not consider the convexification of ϕ instead).

In view of a rescaled version of (A1), which gives us sequences of smooth functions on each ball B_k for $1 \leq k \leq N$. We can, given η, pick for each of those sequences a late member denoted by $\tilde{z}_k$ (the k denotes not the member of the sequence but the ball in which we apply the rescaled (A1)) such that

(i) $z(y) + \tilde{z}_k \in U$ for all $y \in \mathscr{U}$,

(ii) $d_X(\tilde{z}_k, 0) < \frac{\eta}{N}$,

(iii) $\int_{B_k} |\tilde{z}_k|^2 dy > C\phi(F(z(y_k)))|B_k|$.

For the last property we used also (2.7). Now we define $w := \sum_{k=1}^{N} \tilde{z}_k \in C_c(\mathscr{U})$ and see that $z(y) + w(y) \in U$ for $y \in \mathscr{U}$). (A3) thus gives that $\tilde{z} := z + w$ lies in X_0. Clearly,

$$d_X(\tilde{z}, z) \leq \sum d_X(\tilde{z}_k, 0) < \eta.$$

For the L^2 distance of z and $\tilde{z}$ we calculate thanks to the convexity and monotonicity of ϕ, and in view of (2.6)

$$\int\limits_{\mathscr{U}} |z - \tilde{z}|^2 dy \overset{\text{(iii)}}{\geq} C \sum_{k=1}^{N} \phi(F(z(y_k)))|B_k|$$

$$
\begin{aligned}
&\geq C\phi\left(\sum_{k=1}^{N}(F(z(y_k)))|B_k|\right)\\
&\geq C\phi\left(\frac{1}{4}\int_{\mathscr{U}}(F(z(y)))dy\right).
\end{aligned}
$$

This ends the proof.

$\square$

3 Active Scalar Equations

3.1 Introduction to Active Scalar Equations

In this chapter we consider the following class of active scalar equations:

$$\partial_t \theta + \operatorname{div} u\theta = 0, \tag{3.1}$$
$$\operatorname{div} u = 0, \tag{3.2}$$
$$u = T[\theta] \tag{3.3}$$

on the torus $\mathbb{T}^n$ with zero mean condition:

$$\int\limits_{\mathbb{T}^n} u(t,x)dx = \int\limits_{\mathbb{T}^n} \theta(t,x)dx = 0. \tag{3.4}$$

Here T is a Fourier multiplier in the space variables only:

$$\widehat{T[\theta]}(\xi) = m(\xi)\widehat{\theta}(\xi) \quad \xi \in \mathbb{Z}^n. \tag{3.5}$$

We assume in addition that $m : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ is even, 0-homogeneous, and $m(\xi)\cdot\xi = 0$. The latter condition implies incompressibility of u. We do not assume m to be smooth or even bounded, in particular the operator T is not necessarily of Calderón-Zygmund type. If m turns out to be unbounded we need an additional technical assumption, cf. Theorem 3.1.3.

Note that since u is divergence free (3.1) can be written as $\partial_t\theta + u\nabla\theta = 0$. For such an equation one uses the term *passive scalar equation* for a problem where the velocity u is given, and the term *active scalar equation* whenever u is determined from θ, e.g. as in (3.3). As this equation has no apparent smoothing mechanism we are led to consider weak solutions.

Definition 3.1.1. *A pair* $(\theta, u) \in L^2_{loc}(\mathbb{T}^n \times \mathbb{R})$ *is called a* weak solution of the system (3.1)-(3.4)*, if for every* $\phi \in C_0^\infty(\mathbb{T}^n \times \mathbb{R})$ *the following holds:*

(i) $\int_{\mathbb{T}^n\times\mathbb{R}} \theta(\partial_t\phi + u\cdot\nabla\phi)dxdt = 0,$

(ii) $T[\theta]$ *defines a distribution for a.e.* $t \in \mathbb{R}$*,*

(iii) (3.2) *and* (3.3) *hold in the distributional sense,*

(iv) (3.4) *holds in the usual sense.*

The main result in this chapter are the following two theorems:

THEOREM 3.1.2. *Suppose m is even, 0-homogeneous, and bounded and that the convex hull of the set $\{m(\xi) : \xi \in S^{n-1}\}$ has nonempty interior. Then there exist infinitely many periodic weak solutions (θ, u) with $\theta \in L^\infty(\mathbb{T}^n \times \mathbb{R})$ and $u \in L^\infty(\mathbb{T}^n \times \mathbb{R})$ to* (3.1)-(3.4) *with*

$$|\theta(x,t)| = \begin{cases} 1 & \text{for a.e. } (x,t) \in \mathbb{T}^n \times (0,T) \\ 0 & \text{for } t \notin [0,T] \end{cases}.$$

This theorem is restated and proved in Theorem 3.3.5. We furthermore can establish the same statement for an unbounded Fourier multiplier m, at least if its image under S^{n-1} contains a bounded subset of points that spans a set with nonempty interior, and such that this image is still bounded in some arbitrary small neighborhood of the respective points:

THEOREM 3.1.3. *Suppose m is even and 0-homogeneous and that there exists a subset $S_0 := \{\xi_i\} \subset S^{n-1}$ for $i \in I$ and some index set I such that m is bounded in a small neighborhood of ξ_i and* conv $\{m(\xi_i) : \xi_i \in S^{n-1}\}$ *has nonempty interior. Then there exist infinitely many periodic weak solutions (θ, u) with $\theta \in L^\infty(\mathbb{T}^n \times \mathbb{R})$ and $u \in L^\infty(\mathbb{T}^n \times \mathbb{R})$ to* (3.1)-(3.4) *with*

$$|\theta(x,t)| = \begin{cases} 1 & \text{for a.e. } (x,t) \in \mathbb{T}^n \times (0,T) \\ 0 & \text{for } t \notin [0,T] \end{cases}.$$

REMARK 3.1.4. *In fact Theorem 3.1.2 is just a special case of 3.1.3*

REMARK 3.1.5. *Note that the divergence-free condition $\xi \cdot m(\xi) = 0$ does not assure that int conv$\{m(S^{n-1})\} \neq \emptyset$ as the three dimensional example*

$$m(\xi) = \frac{1}{|\xi|^2}\left(-\xi_1\xi_2 - \xi_1\xi_3, \xi_1^2, \xi_1^2\right)$$

tells us. Its image lies in the two dimensional plane, where $\xi_2 = \xi_3$, and hence has clearly no three dimensional interior.

These theorems generalize the main result of [Sh11] and [Sz11]. We want to emphasize that we do not assume the existence of regular points for m. Furthermore the proof given below in Chapter 3.3 does not use the explicit T_4-construction as does the proof of Shvydkoy: in Chapter 3.2 we explicitely calculate the Λ-convex hull. That makes it possible to apply the abstract Baire category method presented in Chapter 2.3. Thus we treat this equation similar as the incompressible porous media equation was treated in [Sz11]. As we will see below (cf. Definition 3.4.1) m induces naturally an (asymmetric) norm. For the case in which this norm is a quadratic norm (the incompressible porous media equation is a special case thereof) we show in Chapter 3.4 that then $K^\Lambda = K^{qc}$. In the general case (2.5) this may not be true. However $K^\Lambda = K^{qc}$ means that we determined the proper relaxation in the sense that we have the nice duality to coarse-graining pointed out in the introduction of this thesis. In fact, this relaxation was for the incompressible porous media equation already derived in [Sz11], but the fact that $K^\Lambda = K^{qc}$ was neither stated nor proved.

In the following we present some applications of the equations (3.1)-(3.3). They indicate that for both bounded and unbounded m there are relevant physical examples covered by our general situation.

Example 2. *The 2D porous media equation for an incompressible fluid u is*

$$\rho_t + \operatorname{div}(\rho u) = 0.$$

u and the density ρ are related by Darcy's law

$$u + \nabla p = -(0, \rho).$$

The pressure p can be eliminated and we get that the set $\{m(\xi) : \xi \in S^1\}$ is the sphere centered at $(0, -\frac{1}{2})$ with radius $\frac{1}{2}$. More precisely, the symbol may be written explicitely as

$$m(\xi) = \frac{1}{|\xi|^2}(\xi_1\xi_2, -\xi_1^2).$$

So m clearly satisfies the assumptions from Theorem 3.1.2. This is case is exactly considered in [Sz11] and before in [CFG11]. See also the discussion in the beginning of Chapter 3.4.

For the 3 dimensional incompressible porous media equation the symbol is

$$m(\xi) = \frac{1}{|\xi|^2}(\xi_1\xi_3, \xi_2\xi_3, -\xi_1^2 - \xi_2^2).$$

In particular $\{m(\xi) : \xi \in S^2\}$ is the sphere of radius $\frac{1}{2}$ centered at $(0, 0, -\frac{1}{2})$.

Example 3. *This example is from magnetostrophic turbulence inside the Earth core in the way proposed by Moffat [Mo08]. Clearly the active scalar θ depends on three spacial dimensions and it represents the buoyancy coefficient. The symbol m first was derived by Friedlander and Vicol in [FV11] and is given by*

$$m(\xi) = \left(\frac{\xi_2\xi_3|\xi|^2 + \xi_1\xi_2^2\xi_3}{\xi_3^2|\xi|^2 + \xi_2^4}, \frac{-\xi_1\xi_3|\xi|^2 + \xi_2^3\xi_3}{\xi_3^2|\xi|^2 + \xi_2^4}, \frac{-\xi_2^2(\xi_1^2 + \xi_2^2)}{\xi_3^2|\xi|^2 + \xi_2^4} \right).$$

This symbol is not bounded on S^2, hence the operator T is not bounded on L^2. Still m satisfies the assumptions of Theorem 3.1.3.
The graph of m looks is shown in Figure 3.1.

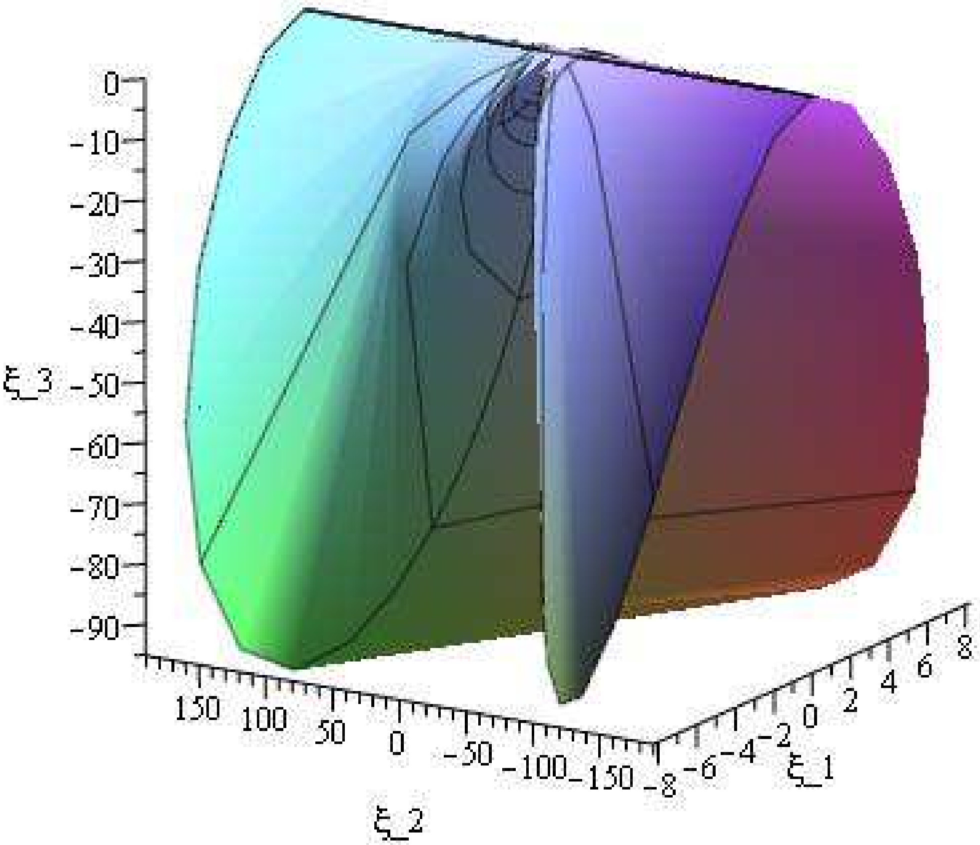

Figure 3.1: The graph of the symbol for magnetostrophic turbulence inside Earth's core

Example 4. *Another example for an active scalar equation is the surface quasi-geostrophic equation. Its symbol is*

$$m(\xi) = \frac{i}{|\xi|}(-\xi_2, \xi_1).$$

Note that the symbol is odd, so that Theorem 3.1.2 does not apply. Still there exists a weak solution $\theta \in L_t^\infty L_x^2$ as proved in [Re95] basically with help of commutator estimates and partial integration.

This part of the thesis is organized as follows. In Chapter 3.2 we reformulate the equations as a differential inclusion, similarly as in [Sh11]. Then we calculate the Λ-convex hull of the constitutive set. Therefore we need to distinguish several cases. After that we show in which cases K^Λ agrees with K^{qc}. In Chapter 3.3 we use the explicit form of the Λ-convex hull to construct weak solutions with help of the Baire category method from Chapter 2.3.

3.2 Relaxation

We rewrite the equations as differential inclusion. For a state variable $z = (\theta, u, q) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ we consider the linear equations

$$\begin{aligned} \partial_t \theta + \operatorname{div} q &= 0, \\ u &= T[\theta] \end{aligned} \tag{3.6}$$

subject to

$$\begin{aligned} |\theta| &= 1, \\ q &= u\theta, \end{aligned}$$

which defines the constitutive set K. Note that the incompressibility $\operatorname{div} u = 0$ is covered by our assumption $m(\xi) \cdot \xi = 0$ a.e.

Hence the constitutive set is

$$K = \{(\theta, u, q) : |\theta| = 1, \quad q = u\theta\}. \tag{3.7}$$

The associated wave cone (cf. (2.1)), which characterizes periodic solutions of (3.6) is then

$$\Lambda = \left\{(\theta, u, q) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n : \exists\, \xi \in S^{n-1} \text{ such that } u = m(\xi)\theta\right\}.$$

3.2.1 First Case: $\{m(\xi) : \xi \in S^{n-1}\} \subset \mathbb{R}^n$ is a Bounded Set

Before we are ready to calculate the Λ-convex hull of the constitutive set K from (3.7) in this section we introduce the following notation.

We define the set $S_0 := \{m(\xi) : \xi \in S^{n-1}\} \subset \mathbb{R}^n$ and denote its convex hull by S. From our assumption in the theorem we know that $\operatorname{int} S \neq \emptyset$. Note that we do not have in general that $\partial S = S_0$. In the case of the incompressible porous media equation S_0 is the boundary of a convex set, cf. Example 2 in the previous chapter where $S_0 = \partial B_{\frac{1}{2}}((0, -\frac{1}{2}))$.

PROPOSITION 3.2.1.

$$K^{\Lambda} = \left\{(\theta, u, q) : |\theta| \leq 1, \quad q - \theta u \in (1-\theta^2)S\right\}.$$

PROOF. First we show "$\supset$":

Let $|\theta_0| < 1$. And let $u_0 \in \mathbb{R}^n$. Then define for $\xi \in S^{n-1}$

$$\begin{aligned} z_1 &= (1, u_0 + (1-\theta_0)m(\xi), q_1) \qquad \text{and} \\ z_2 &= (-1, u_0 - (1+\theta_0)m(\xi), q_2), \end{aligned} \tag{3.8}$$

where $q_i = \theta_i u_i$, for $i = 1, 2$ (this ensures that $z_i \in K$). Then clearly $z_1 - z_2 = (2, 2m(\xi), 2u_0 - 2\theta_0 m(\xi)) \in \Lambda$. Furthermore with $z_0 := \frac{1+\theta_0}{2} z_1 + \frac{1-\theta_0}{2} z_2 = (\theta_0, u_0, \theta_0 u_0 + (1-\theta_0^2)m(\xi))$ we have that $q_0 = \theta_0 u_0 + (1-\theta_0^2)m(\xi)$, so any z with

$$|\theta| \leq 1 \qquad \text{and} \qquad q - \theta u \in (1-\theta^2)S_0$$

is contained in K^{Λ}. Since for any $\bar{q}$ the point $(0, 0, \bar{q}) \in \Lambda$ we get the same for any z with

$$|\theta| \leq 1 \qquad \text{and} \qquad q - \theta u \in (1-\theta^2)S.$$

Hence $K^{\Lambda} \supset \{(\theta, u, q) : |\theta| \leq 1, \quad q - \theta u \in (1-\theta^2)S\}$.

Now we show "$\subset$":

Let $y_0 \in \operatorname{int} S$ so that $0 \in S - y_0$. Define

$$M(y) := \min\{\lambda : \; y \in \lambda(S - y_0)\}. \tag{3.9}$$

Note that as S is convex we have the triangle inequality for M, i.e. $M(y_1 + y_2) \leq M(y_1) + M(y_2)$ and that $M(\mu y) = \mu M(y)$ for $\mu > 0$. In general M is not a full norm but just an asymmetric norm as $M(-y) \neq M(y)$ is possible.

Claim: The function $g(\theta, u, q) := M(q - \theta u - (1-\theta^2)y_0) + \theta^2 - 1$ is Λ-convex.

Then observing that $K \subset \{z : \; g(z) \leq 0\}$ it follows directly that $K^{\Lambda} \subset \{z : \; g(z) \leq 0\}$ and we are done.

What remains to show is the above claim: g is Λ-convex. Let therefore z be arbitrary, $\overline{z} \in \Lambda$, and $t \in \mathbb{R}$.

The following calculation uses just the triangle inequality.

$$\begin{aligned}
g(z+t\overline{z}) &= M\left(q-\theta u-(1-\theta^2)y_0+t(\overline{q}-\overline{\theta}u-\theta\overline{u}+2\theta\overline{\theta}y_0)-t^2(\overline{\theta}\overline{u}-\overline{\theta}^2y_0)\right)\\
&\quad +\theta^2+2t\theta\overline{\theta}+t^2\overline{\theta}^2-1\\
&\geq g(z)+ct-t^2\left(M(\overline{\theta}\overline{u}-\overline{\theta}^2y_0)-\overline{\theta}^2\right)\\
&= g(z)+ct-t^2\overline{\theta}^2\left(M\left(\frac{\overline{u}}{\overline{\theta}}-y_0\right)-1\right)\\
&\geq g(z)+ct,
\end{aligned}$$

where

$$c=2\theta\overline{\theta}-M(\overline{q}-\overline{\theta}u-\theta\overline{u}+2\theta\overline{\theta}y_0).$$

The last inequality holds as for $z\in\Lambda$ we have for some ξ that $\overline{u}=m(\xi)\overline{\theta}$ and therefore

$$\begin{aligned}
M\left(\frac{\overline{u}}{\overline{\theta}}-y_0\right) &= \min\{\lambda:\ \frac{\overline{u}}{\overline{\theta}}-y_0\in\lambda(S-y_0)\}\\
&= \min\{\lambda:\ m(\xi)-y_0\in\lambda(S-y_0)\}\\
&\leq 1.
\end{aligned}$$

Again here we have $\min\{\lambda:\ m(\xi)-y_0\in\lambda(S-y_0)\}=1$, if $\partial S=S_0$.

So in fact g is Λ-convex. This completes the proof. □

REMARK 3.2.2. *For the above computations it is not essential that we have. $\theta=\pm1$. It is very easy to generalize to cases where $\theta\in -a,b$ for $a,b>0$. Further generalizations will become more involved but seem possible as well.*

REMARK 3.2.3. *For the construction of weak solutions in the following section we will define U from Chapter 2.3 as the interior of K^Λ.*

REMARK 3.2.4. *The functions M and g from the above proof are discussed in more detail in Chapter 3.4.*

3.2.2 Second Case: $\{m(\xi):\xi\in S^{n-1}\}\subset\mathbb{R}^n$ is Unbounded

Here we restrict twice:

First we do not consider the whole set $\{m(\xi)\ :\ \xi\in S^{n-1}\}$. Instead of considering S we choose a restricted set S_r, being the convex hull of points $m(\xi_i)\in\mathbb{R}^n$, such that $\xi_i\in S^{n-1}$ are chosen in a way that we still have $\operatorname{int}(\operatorname{conv} m(\xi_i))\neq\emptyset$. Additionally m should be bounded in at least a small neighborhood of each $m(\xi_i)$. This additional requirement makes our restricted cone Λ_r defined below compatible with the oscillation Lemma 3.3.1. In particular the set S_r is bounded. Here $i\in I$ for some index set I, just as in the assumption of Theorem 3.1.3.

Second we restrict the wave cone Λ as follows to the bounded directions of m from above:

$$\begin{aligned}\Lambda_r &= \left\{z = (\theta, u, q) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n : \exists\, \xi \in S^{n-1} \,\text{s.t.}\, m(\xi) \in S_r \,\text{and}\, u = m(\xi)\theta\right\} \\ &= \{z \in \Lambda \text{ such that } m(\xi) \in S_r\}\,.\end{aligned}$$

As in the m-bounded case we have the following proposition for the restricted wave cone:

Proposition 3.2.5.

$$K^{\Lambda_r} = \left\{(\theta, u, q) : |\theta| \leq 1, \quad q - \theta u \in (1 - \theta^2) S_r\right\}.$$

Proof. The proof is exactly the same as the proof for Proposition 3.2.1, just replace S by S_r, Λ by Λ_r, and ξ by ξ_i. □

Although K^{Λ_r} is not the whole Λ-convex hull for the equation but just a part of it, K^{Λ_r} is indeed large enough to apply the Baire category method.

The transition from the m-bounded to the m-unbounded case we performed above is surprisingly simple. For almost every case we consider below it works in the same way. We always have to replace the set S by the restricted set S_r, which implies to select points ξ_i instead of all $\xi \in S^{n-1}$ with the above mentioned properties. That again gives us the restricted the wave cone Λ_r instead of Λ. In the process of this transition one should still always have in mind that just a part of the wave cone is under consideration.

We will be point out m-unbounded cases where one has to be a bit more careful. Otherwise one can follow the m-bounded proofs line by line.

3.2.3 Restriction to Bounded Subsets of K^Λ

Working in K^Λ or merely in K^{Λ_r} suffices to construct weak solutions to (3.1)-(3.3). But as consequence of the fact that K, K^Λ, and K^{Λ_r} are not bounded in the variables u and q, the weak solutions u will be in L^2 only but not in L^∞. The reason for this can be found in the proof of Theorem 3.3.3 in the following section. To overcome this we consider now bounded sets U_b and K_b in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ of $U = \text{int}\, K^\Lambda$ and K respectively.

Our way to restrict to bounded subsets of K^Λ is completely different to the one taken in [Sz11]. In [Sz11] the method was to restrict K to the bounded set $K \cap \{z : |u| \leq \gamma\}$, and then to compute the Λ-convex hull thereof. While the restriction was very easy, the computation of the Λ-convex hull became quite complicated. It had to distinguish various cases and used in particular Pythagorean theorem at some point which is not possible for us, as it does not hold for our function $M(y)$ from the proof of Proposition 3.2.1.

So instead we construct a bounded subset of the Λ-convex hull ad hoc that will not be Λ-convex itself, but has all necessary properties for the Baire category method. Our construction is somewhat more complicated compared to [Sz11], but in return the calculations [Sz11] had to go through are not needed. All in all this makes life much simpler. We benefit again from this approach in the proof of Proposition 3.3.4 in Chapter 3.3.

We construct now the sets U_b and K_b in several steps, in such a way that we only add Λ-directions to our initial set U_0. We define the set U_0 for fixed $\gamma > \sup_{\xi \in S^{n-1}} |m(\xi)|$ as

$$U_0 := \{(0, u, m(\xi)) : |u| \leq \gamma, \, \xi \in S^{n-1}\}.$$

Now for $z = (0, u, m(\xi)) \in U_0$ similar as in the proof of Proposition 3.2.1 we set

$$\begin{aligned} z_1(u, \xi) &:= (1, u + m(\xi), u + m(\xi)), \\ z_2(u, \xi) &:= (-1, u - m(\xi), -u + m(\xi)). \end{aligned} \tag{3.10}$$

Again we have for any $z \in U_0$ that $z_1 - z_2 \in \Lambda$. Furthermore a short calculation gives for fixed $z = (0, u, m(\xi)) \in U_0$ that

$$[z_1, z_2] = \left\{ \tfrac{1+\theta}{2} z_1 + \tfrac{1-\theta}{2} z_2 : \, \theta \in [-1, 1] \right\} = \{(\theta, u + \theta m(\xi), \theta u + m(\xi)) : \, \theta \in [-1, 1]\}.$$

This motivates the definition of the functions

$$\begin{aligned} f(\theta, u, \xi) &:= u + \theta m(\xi), \\ g(\theta, u, \xi) &:= \theta u + m(\xi), \end{aligned} \tag{3.11}$$

and the next set

$$\begin{aligned} U_1 &:= \{(\theta, f(\theta, u, \xi), g(\theta, u, \xi)) : \, (0, u, m(\xi)) \in U_0, \, \theta \in [-1, 1]\} \\ &= \left\{ (\theta, u + \theta m(\xi), \theta u + m(\xi)) : \, |\theta| \leq 1, \, |u| \leq \gamma, \, \xi \in S^{n-1} \right\}. \end{aligned} \tag{3.12}$$

By construction U_1 is a compact set with the property that every point in U_1 lies on a Λ-line segment connecting the levels $\theta = 1$ and $\theta = -1$ according to (3.10). Note that $U_1|_{\theta=0} = U_0$ and that U_1 is a $2n$-dimensional set.
A short calculation gives that a point $z = (\theta, v, q) \in U_1$ can be written as

$$z = (\theta, v, q) = \left(\theta, v, \theta v + (1 - \theta^2) m(\xi)\right), \tag{3.13}$$

where v solves

$$v = u + \theta m(\xi) \qquad \text{for } |u| \leq \gamma \text{ and } \xi \in S^{n-1}. \tag{3.14}$$

Written in this form one can already see the similarity to the set K^Λ from Proposition 3.2.1. The Λ-line segment on which this point $z = (\theta, v, q)$ lies was spanned by the two points $z_1 = (\theta, v, q)|_{\theta=1}$ and $z_2 = (\theta, v, q)|_{\theta=-1}$.
The set $V \subset \mathbb{R}^n$ which gives all possible points v that satisfy (3.14) can also be written as

$$V = \{v \in \mathbb{R}^n : \, \exists \xi \in S^{n-1} \text{ with } |v - m(\xi)| \leq \gamma\}.$$

To use as in the proof of Proposition 3.2.1 the fact that $(0,0,\bar{q})$ lies in Λ we define the set U_2 by taking the convex hull in the third variable for all points in U_1 more precisely:

$$\begin{aligned} U_2 := \Big\{ & (\theta, f(\theta,u,\xi), q) : \, |\theta| \leq 1, \, |u| \leq \gamma, \, \xi \in S^{n-1} \text{and} \\ & q \in \operatorname{conv} \big\{ g(\theta,v,\zeta) : \text{ with } v \in \mathbb{R}^n \text{ and } \zeta \in S^{n-1} \\ & \text{s.t.}\, f(\theta,u,\xi) = f(\theta,v,\zeta) \big\} \Big\} \\ = \Big\{ & (\theta, u + \theta m(\xi), q) : \, |\theta| \leq 1, \, |u| \leq \gamma, \, \xi \in S^{n-1} \text{and} \\ & q \in \operatorname{conv} \big\{ \theta v + m(\zeta) : \text{ with } v \in \mathbb{R}^n \text{ and } \zeta \in S^{n-1} \\ & \text{s.t.}\, u + \theta m(\xi) = v + \theta m(\zeta) \big\} \Big\}. \end{aligned} \tag{3.15}$$

Note that the equation $f(\theta,u,\xi) = f(\theta,v,\zeta)$ has for every θ many solutions as long as γ is not too small, more precisely $\gamma > \sup_{\xi \in S^{n-1}} |m(\xi)|$ will be large enough. Furthermore U_2 is bounded, even compact and still constructed just by adding Λ-directions to the initial set U_0. Furthermore U_2 is a $2n+1$-dimensional set.

As in (3.13) a point in U_2 can be written as $z = (\theta, v, q)$, with v again solving (3.14). And q such that

$$q - \theta v \in (1-\theta^2)\tilde{S}, \tag{3.16}$$

where $\tilde{S}$ being the convex hull of all points $m(\xi)$ coming from (3.14).

Now we are ready to define

$$\begin{aligned} U_b := \big\{ & (\theta, u + \theta m(\xi), q) \in U_2 : \, |\theta| < 1, |u| < \gamma, \xi \in S^{n-1}, \\ & \text{and}\, q \in \operatorname{int}\operatorname{conv}\{g(\theta,v,\zeta) : \text{ as in (3.15)}\} \big\} \end{aligned} \tag{3.17}$$

and

$$K_b := K \cap U_2.$$

We want to remark here that neither U_1 nor U_2 nor U_b is a Λ-convex set. Still U_b has thanks to our construction by adding Λ-directions and obtaining U_2 as a compact set with full measure in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ the properties needed to apply the Baire category method as proposed in Chapter 2.3. The Property (A2) therein will be proved for U_b below in Proposition 3.3.4.
We recall that U_2 depends on γ and hence write $U_2 = U_2(\gamma)$. Similarly we write $K_b = K_b(\gamma)$. Then we have for $\gamma < \gamma'$

$$U_2(\gamma) \subset U_2(\gamma').$$

And, as for $\gamma \to \infty$ the condition on v in (3.14) vanishes, we have that

$$\bigcup_{\gamma>0} U_2(\gamma) = K^\Lambda$$
$$\text{and} \bigcup_{\gamma>0} K_b(\gamma) = K.$$

LEMMA 3.2.6. *The set U_b is not empty.*

PROOF. This is clear since $(0,0,y_0) \in U_b$: we have

$$U_b|_{(0,0,q)} = \{(0,0,\operatorname{int}\operatorname{conv}\{m(\xi) : \xi \in S^{n-1}\})\}$$

and that y_0 lies in int S. □

PROPOSITION 3.2.7. *U_b is open.*

PROOF. We denote $(\theta, u_i, \xi_i) = z_i$.

Now let $z^0 \in U_b$. Hence there exist z_1^0, ..., z_{n+1}^0 with $f(z_i^0) = f(z_j^0)$ and $\lambda_i^0 > 0$ with $\sum_{i=1}^{n+1} \lambda_i^0 = 1$, such that

$$z^0 = (\theta^0, u^0, q^0) = \left(\theta^0, u_1^0 + \theta^0 m(\xi_1^0), \sum_{i=1}^{n+1} \lambda_i^0 g(z_i^0)\right). \tag{3.18}$$

Now let $z = (\theta, u, q)$ be a point nearby, that is for given $0 < \varepsilon \ll 1$ we pick z such that $|z - z^0| < \varepsilon$. We will show that z still is of the form (3.18), and hence in U_b.
As $|z - z^0| < \varepsilon$ we have that

$$\begin{aligned} |\theta - \theta^0| &< \varepsilon, \\ |u - u^0| = |u - (u_i^0 + \theta^0 m(\xi_i^0))| &< \varepsilon, \\ |q - q^0| = |q - \sum_{i=1}^{n+1} \lambda_i^0 g(z_i^0)| &< \varepsilon, \end{aligned}$$

where g is defined as in (3.11). Now we choose $u_i := u - \theta m(\xi_i^0)$ such that we have $u = u_i + \theta m(\xi_i^0)$ as desired and furthermore that $|u_i - u_i^0| < c\varepsilon$.

What remains to show is that there exist λ_i such that $q = \sum \lambda_i g(\theta, u_i, \xi_i^0)$. But as we have that q is close to q^0, second $q^0 \in \operatorname{conv}\{g(\theta^0, u_i^0, \xi_i^0)\}$ and third that the $g(\theta^0, u_i^0, \xi_i^0)$ are close to $g(\theta, u_i, \xi_i^0)$ for each i:

$$\begin{aligned} |g(\theta^0, u_i^0, \xi_i^0) - g(\theta, u_i, \xi_i^0)| &= |\theta^0 u_i^0 - \theta u_i| = |\theta^0 u_i^0 - \theta^0 u_i + \theta^0 u_i - \theta u_i| \\ &\le |\theta^0||u_i^0 - u_i| + |\theta^0 - \theta||u_i| < c\varepsilon, \end{aligned}$$

we can conclude that for z close enough to z^0 also $q \in \operatorname{conv}\{g(\theta, u_i, \xi_i^0)\}$, which proves the existence of such λ_i. □

3.3 Construction of Weak Solutions

We construct here the weak solutions with help of the Baire category method as presented in Chapter 2.3. We will closely follow the proof therein and hence verify the three assumptions (A1)-(A3) made there.

First we note that we have the existence of plane wave solutions as the following lemma tells us:

LEMMA 3.3.1. *Fix $\Omega \subset \Omega_T = (0,T) \times \mathbb{T}^n$ open, $\varepsilon > 0$, and $\mu \in (0,1)$ and denote $(t,x) = y$. Then for every $\bar{z} \in \Lambda$ there exists a sequence $\{z_k\} \subset C_0^\infty(\Omega_T)$, satisfying (3.6), and such that*

(i) $z_k \overset{*}{\rightharpoonup} 0$,

(ii) $\sup_{y \notin \Omega} |z_k(y)| < \varepsilon$ *for all* $k \in \mathbb{N}$,

(iii) $\sup_{y \in \Omega} \operatorname{dist}(z_k, [-(1-\mu)\bar{z}, \mu\bar{z}]) < \varepsilon$,

(iv) $|\{y \in \Omega : z_k(y) \in B_\varepsilon(\mu\bar{z})\}| > (1-\mu)(1-\varepsilon)|\Omega|$,

(v) $|\{y \in \Omega : z_k(y) \in B_\varepsilon(-(1-\mu)\bar{z})\}| > \mu(1-\varepsilon)|\Omega|$.

A proof of this lemma is given in [Sh11]. For the proof it is an essential fact that the Fourier multiplier m is even. In the proof the oscillations are constructed around a point $\bar{z}_0 = (\theta_0, \theta_0 m(\xi_0), q_0)$, where $\theta_0 \neq 0, \xi_0 \in S^{n-1}, q_0 \in \mathbb{R}^n$. It is also noticed in [Sh11] that one can additionally ensure the localization of the Fourier support of the sequence $\{z_k\}$ in the following sense:

(vi) For any given open neighborhood $W \subset S^{n-1}$ of ξ one can choose z_k in such a way that the inclusion $\operatorname{supp} \widehat{z_k(t,\cdot)} \subset W \cup (-W)$ holds, for all $t \in (0,T)$ and $k \in \mathbb{N}$.

The neighborhood W of ξ can be chosen arbitrarily small but cannot be shrinked to a point.

This lemma verifies (A1). For the case where m is bounded the statements $(i) - (v)$ will be enough. (vi) then is needed for the m-unbounded case and also the reason for our assumption on m in Theorem 3.1.3 that m stay bounded on a small neighborhood of a set that spans $\mathbb{R}^n$.

We now define a set U that will satisfy (A2). For the m-bounded case let

$$U := \operatorname{int} K^\Lambda = \left\{(\theta, u, q) : |\theta| < 1, \quad q - \theta u \in (1-\theta^2) \operatorname{int} S\right\}. \tag{3.19}$$

Analogously we choose in the m-unbounded case $U := \operatorname{int} K^{\Lambda_r}$. All following calculations hold in either case. For m unbounded it is always sufficient to consider directions in Λ_r instead of Λ-directions.

PROPOSITION 3.3.2. *There exists a constant $c > 0$ such that for any $z \in U$ there exists a $\bar{z} \in \Lambda \cap S^{n-1}$ such that*

$$z + t\bar{z} \in U \qquad \text{for } |t| < c(1-\theta^2).$$

PROOF. We proof this Proposition for bounded m. If m is unbounded one can follow the proof line by line with the usual changes (S becomes S_r, ξ becomes ξ_i, Λ becomes Λ_r).

So, first we consider the case where $\partial S = S_0$.
Let now $|\theta| < 1$. We consider the following three subordinate cases (still $\partial S = S_0$):

(i) $q - \theta u \in (1-\theta^2)\partial S$,

(ii) $q - \theta u \in \left((1-\theta^2)S \setminus \frac{1-\theta^2}{2}S\right)$,

(iii) $q - \theta u \in \frac{1-\theta^2}{2}S$.

ad (i) We know from the formulae in the proof of Proposition 3.2.1 choosing $\bar{z}$ in the direction $z_1 - z_2$ from (3.8) that

$$z + t\bar{z} \in K^\Lambda \quad \text{for} \quad |t| < \min\left\{\tfrac{1-\theta}{2}, \tfrac{1+\theta}{2}\right\} < c_0(1-\theta^2).$$

ad (ii) Here we can take $\bar{z}$ close to the direction from (i) conclude by continuity for $0 < c < c_0$:
$q - \theta u \in \left((1-\theta^2)S \setminus \frac{1-\theta^2}{2}S\right)$ means that $q = \theta u + \mu(1-\theta^2)m(\xi)$ for some $\mu \in [\frac{1}{2}, 1)$ (for $\mu = 1$ we are again in case (i)). Also the point z is now in $U = \operatorname{int} K^\Lambda$
We set:

$$\begin{aligned} z_1' &= (1, u + (1-\theta)m(\xi), q_1'), \\ z_2' &= (-1, u - (1+\theta)m(\xi), q_2'), \end{aligned}$$

with

$$\begin{aligned} q_1' &= q_1 - (1-\mu)(1-\theta)m(\xi) = u + (1-\theta)m(\xi) - (1-\mu)(1-\theta)m(\xi) \\ \text{and} \quad q_2' &= q_2 - (1-\mu)(1+\theta)m(\xi) = -u + (1+\theta)m(\xi) - (1-\mu)(1+\theta)m(\xi). \end{aligned}$$

For $\mu = 1$ we had that $z_i' = z_i$. Note that changing the third variable does not harm $z_1' - z_2' =: \bar{z}'$ being a Λ-direction. As in case (i) we have that $z = \frac{1+\theta}{2}z_1' + \frac{1-\theta}{2}z_2'$. Furthermore we have that

$$z + t\bar{z}' = (\theta + 2t, u + 2tm(\xi), u(\theta + 2t) + \mu m(\xi)(1 - \theta^2 - 2t\theta))$$

As $z \in U$ we want now to calculate the value of t when $z + t\bar{z}' \in \partial U$. As $U = \{z = (\theta, u, q) : |\theta| < 1, q - \theta u \in (1-\theta^2)S\}$ we have that

$$\partial U = \{z = (\rho, w, p) : |\rho| = 1, p - \rho w \in (1-\rho^2)\partial S\}$$

$$= \{z = (\rho, w, p) : |\rho| = 1,\, p - \rho w = 0\}.$$

$z + t\bar{z}'$ will in the first two variables not meet ∂U as long as $t < c_0(1-\theta^2)$ just as in case (i) as we did not change the z_i in these variables. Now we look at the condition for the third variable. Direct calculation gives:

$$\begin{aligned} p - \rho w &= 0, \\ \Leftrightarrow u(\theta + 2t) + \mu m(\xi)(1-\theta^2 - 2t\theta) - (\theta + 2t)(u + 2tm(\xi)) &= 0, \\ \Leftrightarrow 4t^2 + 2\theta(1+\mu)t - \mu(1-\theta^2) &= 0. \end{aligned} \tag{3.20}$$

As $|\theta| < 1$ was given above and $\mu \in [\frac{1}{2}, 1]$ we can hence deduce that $z + t\bar{z}' \in U$ for $t < c(1-\theta^2)$.
From (3.20) it becomes clear that this argument does not work for $\mu \to 0$.

ad (iii) In this case one can simply take $\bar{z} = (0, 0, \bar{q})$ with $\bar{q}$ parallel to $q - \theta u$.

As these arguments only depend on the direction of $\bar{z}$ we can normalize it such that $\bar{z} \in S^{n-1}$. This concludes the proof for the case $\partial S = S_0$.

Now for the case where $\partial S \neq S_0$ we consider $z = (\theta, u, q)$ with $|\theta| < 1$ and $q - \theta u \in (1-\theta^2)\partial S \setminus S_0$. Then there exist two points $z' = (\theta, u, q')$ and $z'' = (\theta, u, q'')$ with $q' - \theta u \in (1-\theta^2)S_0$ and $q'' - \theta u \in (1-\theta^2)S_0$, such that

$$z = \lambda z' + (1-\lambda)z'' = (\theta, u, \lambda q' - (1-\lambda)q'')$$

for some $\lambda \in (0,1)$. For λ close to 0 and 1 we can pick the direction from the explicit formulae (3.8) for z' or z'' respectively again and argue by continuity as in (ii). If we are away from q' and q'' we can just as in (iii) choose $\bar{z}$ parallel to $q' - q''$. Now we have the statement again on all of ∂S and proceed as in the first case where $\partial S = S_0$. This finishes the proof.

□

Theorem 3.3.3. *There exist infinitely many periodic weak solutions to (3.1)-(3.3) with*

$$\theta \in L^\infty(\mathbb{T}^n \times \mathbb{R}), \qquad u \in L^\infty(\mathbb{R}; L^2(\mathbb{T}^n)),$$

such that

$$|\theta(x,t)| = \begin{cases} 1 & \text{for a.e. } (x,t) \in \mathbb{T}^n \times (0,T) \\ 0 & \text{for } t \notin [0,T] \end{cases}.$$

Proof. The proof is similar the proof of Theorem 3.2. in [Sz11]. We construct solutions (θ, u, q) of (3.6) that lie inside K for almost every (x,t). By the definition (3.7) of K we have then that $q = \theta u$ and $|\theta| = 1$ a.e. Therefore they satisfy (3.1).

First we consider the case where m is bounded.

We now define the space of subsolutions with the notation from Chapter 2.3. Let $\Omega = \mathbb{T}^n \times \mathbb{R}$ and $\mathscr{U} = \mathbb{T}^n \times (0,T)$. Recall the definition of U from (3.19) denote as usual $z = (\theta, u, q)$ and let

$$X_0 = \{z \in C^\infty(\mathscr{U}) : \operatorname{supp} z(x,\cdot) \subset (0,T),\ (3.6) \text{ holds and } z(x,t) \in U \quad \forall (x,t) \in \mathscr{U}\}.$$

Note that we can assume $0 \in X_0$ as we may without loss of generality assume $0 \in U$ (eventually we need a constant shift by some y_0 in the q-variable). Hence X_0 is nonempty. For any $z \in X_0$ we have that $|\theta| \leq 1$ and therefore

$$||\theta||_{L_t^\infty L_x^2(\mathscr{D})} \leq 1.$$

Then as m is bounded we get for $u = T[\theta]$ that

$$||u||_{L_t^\infty L_x^2(\mathscr{D})} \leq C \tag{3.21}$$

for some constant C. This implies that X_0 is a bounded subset of L^2: the set X_0 satisfies the condition of (A3) from Chapter 2.3. Lemma 3.3.1 and Proposition 3.3.2 give conditions (A1) and (A2). Applying Theorem 2.3.1 concludes the proof for the case where m is bounded.

If now m is unbounded, looking at (3.3) we see that T becomes an unbounded operator and (3.21) might no longer hold. As consequence X_0 would not be a bounded subset of L^2 and (A3) fails. But this lack of boundedness can be fixed by two observations.

First, we construct the weak solutions only at some points where m is bounded, which is the set S_r defined in Chapter 3.2.2.

Second, as we have on the one hand property *(vi)* in Lemma 3.3.1 and on the other hand from our assumption in the theorem m is bounded even in a neighborhood of $\{\xi_i\}$ – the operator T acts here as a bounded operator. Hence we can re-establish (3.21) for a redefined T and finish the proof. □

As already remarked earlier the crucial point that we get $u \in L_t^\infty L_x^2$ only, but not in $L_{t,x}^\infty$ is the unboundedness of the sets U, K. In the following we prove that we can establish (A2) and (A3) also for the bounded sets U_b and K_b to get a strengthening of the above theorem to L^∞.

Proposition 3.3.4. *There exists a constant $c > 0$ such that for any $z \in U_b$ there exists a $\bar{z} \in \Lambda \cap S^{n-1}$ such that*

$$z + t\bar{z} \in U_b \qquad \textit{for } |t| < c(1-\theta^2).$$

Proof. The proof is similar to the one of Proposition 3.3.4. Again we deal with different cases. We give the proof for the case where m is bounded and $\partial S = S_0$. Exactly as in the proof of Proposition 3.3.2 one then can generalize to both, the m-unbounded case and to the case where $\partial S \supsetneq S_0$.
Similar as in the unbounded case we consider several cases. Recall the definitions of U_1 and U_2 from (3.12) and (3.15) respectively. As described there a point in U_1 can be

written as in (3.13), and for a point in U_2 we recall the definition of the set $\tilde{S}$, that was the convex subset of S subject to the restriction in the definition of U_2 in (3.15).

As in the proof of Proposition 3.3.2 let now $|\theta| < 1$ be given. And $z = (\theta, v, q)$ with the coordinates as in (3.16), that is v solves $v = u + \theta m(\xi)$ for $|u| \leq \gamma$ and $\xi \in S^{n-1}$, and $q - \theta v \in (1-\theta^2)\tilde{S}$.

First we consider $z \in U_1$. This corresponds to case (i) in the proof of Proposition 3.3.2. We choose $\bar{z}$ corresponding to (3.10) and have $z + t\bar{z} \in U_1$ for $|t| < c_0(1-\theta^2)$.

If $z \notin U_1$. Then $z \in U_2 \setminus U_1$. The condition therefore now reads as

$$q - \theta v \in (1-\theta^2)\tilde{S}'$$

where still v solves $v = u + \theta m(\xi)$ for $|u| \leq \gamma$ and $\xi \in S^{n-1}$. And $\tilde{S}' := \tilde{S} \setminus S_0$.

We assume first that $\tilde{S} = S$, then we also have $\tilde{S}' = \operatorname{int} S$ (as we deal with the case $\partial S = S_0$).

As in the proof of Proposition 3.3.2 this falls again in two parts (corresponding to case (ii) and (iii) therein):

- $q - \theta v \in (1-\theta^2) \operatorname{int} S \setminus \frac{1-\theta^2}{2} S$ and
- $q - \theta v \in \frac{1-\theta^2}{2} S$.

In the first case of which we argue again by continuity:
A point from this case has the form $z = (\theta, v, \theta v + \mu(1-\theta^2)m(\xi))$ for some $\mu \in [\frac{1}{2}, 1)$. We change the points z_i from (3.10) similarly as in the proof of the above proposition and set:

$$\begin{aligned} z_1' &:= (1, u + m(\xi), u + m(\xi) - (1-\mu)(1-\theta)m(\xi)), \\ z_2' &:= (-1, u - m(\xi), -u + m(\xi) - (1-\mu)(1+\theta)m(\xi)), \end{aligned}$$

such that again $z = \frac{1+\theta}{2} z_1' + \frac{1-\theta}{2} z_2'$ and still $z_1' - z_2' =: \bar{z}' \in \Lambda$. As in the proof for unbounded U we still have that $z + t\bar{z}' \in U_b$ for $t < c_0(1-\theta^2)$ if we look at the first two variables. And with the same calculation as above for the third variable we again establish that $z + t\bar{z} \in U_b$ for $t < c(1-\theta^2)$.

Finally, if $q - \theta v \in \frac{1-\theta^2}{2} S$ we naturally choose $\bar{z} = (0, 0, \bar{q})$ with $\bar{q}$ parallel to $q - \theta v$. This concludes the proof for $\partial S = S_0$ and $\tilde{S} = S$.

If now $\tilde{S} \neq S$ we argue for $\partial \tilde{S} \setminus S_0$ as in the proof of Proposition 3.3.2, where we generalized to $\partial S \neq S_0$. This again closes the proof for m bounded and $\partial S = S_0$. The generalizations to the cases $\partial S \neq S_0$ and to unbounded m work exactly as in Proposition 3.3.2. $\square$

Now we define the space X_0 of subsolutions as

$$X_0 = \{z \in C^\infty(\mathscr{U}) : \operatorname{supp} z(x,\cdot) \subset (0,T),\ (3.6) \text{ holds and } z(x,t) \in U_b \quad \forall (x,t) \in \mathscr{U}\}.$$

That gives us the following strengthening of Theorem 3.3.3, the main result in this chapter:

Theorem 3.3.5. *There exist infinitely many periodic weak solutions to (3.1)-(3.3) with*

$$\theta \in L^\infty(\mathbb{T}^n \times \mathbb{R}), \qquad u \in L^\infty(\mathbb{T}^n \times \mathbb{R}),$$

such that

$$|\theta(x,t)| = \begin{cases} 1 & \text{for a.e. } (x,t) \in \mathbb{T}^n \times (0,T) \\ 0 & \text{for } t \notin [0,T] \end{cases}.$$

3.4 When is $K^\Lambda = K^{qc}$?

Before we prove for certain cases that $K^\Lambda = K^{qc}$, we exploit some properties of the functions in the proof of Proposition 3.2.1.

First we come to the function M from (3.9). Recall that it was defined as

$$M(y) = \min\{\lambda : y \in \lambda(S - y_0)\}.$$

In the general case M is an asymmetric norm. That means that M has all properties of a norm except for the condition that $M(x) = M(-x)$. If however the set S is point symmetric and one chooses y_0 to be the reflection point, then M becomes a full norm.

The 2D incompressible porous media equation serves as an example where M is a full norm (this example generalizes easily to the n-dimensional incompressible porous media equation). Here we have $S = B_{\frac{1}{2}}((0, -\frac{1}{2}))$. Hence choosing $y_0 = (0, -\frac{1}{2})$ we get that $M(y) = \frac{1}{2}|y|$, a rescaled Euclidean norm. This fact allowed [Sz11] to restrict to compact subsets of K^Λ in such a way which is not possible in our situation, as Pythagorean theorem does not hold for our general M but for the euclidean case. We had to take a new approach in Chapter 3.2.3 that in fact simplified the corresponding proof in [Sz11].

This motivates the following definition:

Definition 3.4.1. *We say that $M(y) := \min\{\lambda : y \in \lambda(S - y_0)\}$ is the* (asymmetric) norm induced *by m. If M is of the form $M(y) = \langle Ay, y\rangle$ for $A \in \mathbb{R}^{n\times n}$ positive definite and symmetric, we say that m induces a* quadratic norm *and also write shorthand $\langle Ax, y\rangle = \langle x, y\rangle_A$ for the corresponding scalar product.*

Remark 3.4.2. *Of course a priori M depends not only on m but also on $y_0 \in \operatorname{int} \operatorname{conv}\{m(S^{n-1}\}$. But as y_0 is kept fixed through all our operations, we can omit the latter dependency.*

Next we turn to the Λ-convex function g. It can be decomposed into a convex part, say f, and a remainder. Recall that g was defined as

$$g(\theta, u, q) := M(q - \theta u - (1 - \theta^2)y_0) + \theta^2 - 1.$$

The next lemma gives its convex part f.

LEMMA 3.4.3. *The the function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ with*

$$f(\theta, u, q) = M(q - \theta u - (1 - \theta^2)y_0) + \frac{1}{2}\theta^2 + \frac{1}{2}M(u - \theta y_0)^2$$

is convex.

PROOF. The proof works similar to the proof of Λ-convexity of g after (3.9). Just take arbitrary z and arbitrary $\bar{z}$ and compute with help of triangle inequality that

$$\begin{aligned} f(z + t\bar{z}) &\geq f(z) + ct + t^2\left(\frac{1}{2}\bar{\theta}^2 + \frac{1}{2}M(\bar{u} - \bar{\theta}y_0)^2 - M(\bar{\theta}\bar{u} - \bar{\theta}^2 y_0)\right) \\ &\geq f(z) + ct + \frac{1}{2}t^2(\bar{\theta} - M(\bar{u} - \bar{\theta}y_0))^2 \\ &\geq f(z) + ct. \end{aligned}$$

□

If M is a quadratic norm, i.e. $M(x) = \langle Ax, x\rangle$, then the remainder can be written as

$$g - f = \frac{1}{2}\langle Au, 2\theta y_0 - u\rangle - 1. \tag{3.22}$$

If we show for an approximating sequence $\{z_k\}_{k\in\mathbb{N}}$ that the remainder converges for example in the sense of distributions, then the function g is proven to be semicontinuous. Therefore as K^Λ is a level set of this function, we get that K^Λ equals K^{qc}, cf. the definitions in Chapter 2.1.

PROPOSITION 3.4.4. *In case of the incompressible porous media equation we have $K^\Lambda = K^{qc}$.*

PROOF. As mentioned above it is enough to show the convergence of the remainder (3.22) in $\mathcal{D}'_{x,t}$. In case of the IPM we have that in (3.22) $A = Id$ and therefore $M(y) = \frac{1}{2}|y|$.

Consider a sequence $\{z_k\}_{k\in\mathbb{N}}$ such that:

(i) $z_k \in C^\infty(\mathbb{T}^n \times [0, T])$

(ii) z_k satisfies the linear equations (3.6)

(iii) $z_k \in K$.

Then we have (for a not relabeled subsequence) that

$$\begin{aligned} \theta_k &\overset{*}{\rightharpoonup} \theta && \text{in } L^\infty((0,T)\times\mathbb{T}^n),\\ u_k &\overset{*}{\rightharpoonup} u && \text{in } L^\infty(0,T;L^2(\mathbb{T}^n)),\\ q_k &\overset{*}{\rightharpoonup} q && \text{in } L^\infty(0,T;L^2(\mathbb{T}^n)). \end{aligned}$$

As consequence of $\partial_t\theta_k + \operatorname{div} q_k = 0$ we also have

$$\partial_t\theta_k \rightharpoonup \partial_t\theta \qquad \text{in } L^\infty(0,T;H^{-1}(\mathbb{T}^n)). \tag{3.23}$$

Here we recall that the space $H^{-1}(\mathbb{T}^n)$ is the dual space of $H^1(\mathbb{T}^n)$. So we have for the sequence θ_k that

$$\begin{aligned} ||\theta_k(t,\cdot)-\theta_k(s,\cdot)||_{H^{-1}} = ||\int_s^t \partial_\tau\theta_k(\tau,\cdot)d\tau||_{H^{-1}} &\le \int_s^t ||\partial_\tau\theta_k(\tau,\cdot)||_{H^{-1}}d\tau\\ &\le C\cdot|t-s|. \end{aligned}$$

As a result of (3.23) $\{\partial_t\theta_k\}_{k\in\mathbb{N}}$ lies for a.e. $t\in[0,T]$ in a bounded subset of $H^{-1}(\mathbb{T}^n)$ and the constant C is independent of k. So we have equicontinuity and get by Arzelà-Ascoli theorem that $\{\theta_k\}$ converges uniformly in t to some $\theta\in C(0,T;H^{-1}(\mathbb{T}^n))$.

Now define $u := T[\theta]$ (in the IPM case T is a bounded operator in H^{-1} as it is bounded in L^2) which lies again in $(C(0,T;H^{-1}(\mathbb{T}^n)))^n$ and furthermore $\tilde{u}_k := 2u_k + (0,\theta_k)$. Then we have for

$$\begin{aligned} v_k &:= \tilde{u}_k - (0,\theta_k),\\ w_k &:= \tilde{u}_k + (0,\theta_k) \end{aligned}$$

that $\{\operatorname{div} v_k\}$ and $\{\operatorname{curl} w_k\}$ lie for a.e. t in compact subsets of H^{-1} and $(H^{-1})^{n\times n}$ respectively. See also [Sz11] for the reformulation of the incompressible porous media equation into this div-curl structure. Also it follows from the above and $u = T[\theta]$ that

$$\begin{aligned} v_k &\rightharpoonup v && \text{in } L^2_x \text{ for a.e. } t,\\ w_k &\rightharpoonup w && \text{in } L^2_x \text{ for a.e. } t. \end{aligned}$$

Hence we have by the Div-Curl-Lemma (Theorem 4 in [Ev90]) in space that

$$v_k w_k \xrightarrow{\mathcal{D}'_x} vw \qquad \text{for a.e. fixed } t, \tag{3.24}$$

Now we show that the convergence (3.24) also holds in space-time. For this purpose define for arbitrary $\varphi\in C_0^\infty([0,T]\times\mathbb{T}^n)$

$$\psi_k(t) := \int_{\mathbb{T}^n} \left(v_k(t,x)w_k(t,x) - v(t,x)w(t,x)\right)\varphi(t,x)dx.$$

In view of (3.24) we have that for a.e. t that $\psi_k(t) \to 0$. Also ψ_k lies for every k in L^∞, since v_k, w_k, v, w, and φ all are in L^∞.

Then applying the dominated convergence theorem allows us to interchange the integral with the limit to obtain:

$$\lim_{k\to\infty} \int_{\mathbb{T}^n} \psi_k(t)dt = \int_{\mathbb{T}^n} \lim_{k\to\infty} \psi_k(t)dt = 0,$$

which implies convergence in the sense of distributions in space-time of $v_k w_k$. So indeed,

$$\begin{aligned} v_k w_k &\xrightarrow{\mathcal{D}'_{x,t}} vw, \\ \text{i.e.} \quad |\tilde{u}_k|^2 - \theta_k^2 &\xrightarrow{\mathcal{D}'_{x,t}} |\tilde{u}|^2 - \theta^2, \\ \text{or} \quad u_k \cdot (u_k - 2\theta_k y_0) &\xrightarrow{\mathcal{D}'_{x,t}} u \cdot (u - 2\theta y_0), \end{aligned}$$

as $y_0 = (0, -\frac{1}{2})$, which is exactly the Div-Curl-Lemma in space-time. Hence we have the convergence in the sense of distributions for the remainder $\frac{1}{2}\langle Au, 2\theta y_0 - u\rangle - 1$. □

Theorem 3.4.5. *If m induces a quadratic norm and 0 lies on $m(S^{n-1})$ then we have that $K^\Lambda = K^{qc}$.*

Proof. We note that the in the proof of the previous Proposition 3.4.4 the only point, on which the special structure from IPM entered was to use the Div-Curl-Lemma to obtain (3.24). So this is the point where we have to adjust the proof, we need only to show convergence of the remainder (3.22) in the sense of distributions with respect to space.

Therefore we first note that $m(\xi)$ and $(2y_0 - m(\xi))$ are orthogonal with respect to the scalar product induced by m:

$$\langle Am(\xi), 2y_0 - m(\xi)\rangle = \langle Ay_0, y_0\rangle - \langle A(y_0 - m(\xi)), y_0 - m(\xi)\rangle = 1 - 1 = 0.$$

The second last equality is on the one hand true since for all $\xi \in S^{n-1}$ we have that $M(m(\xi) - y_0) = \min\{\lambda : m(\xi) - y_0 \in \lambda(S - y_0)\} = \min\{\lambda : m(\xi) - y_0 \in \lambda(m(\xi) - y_0)\} = 1$. On the other hand as 0 lies on $m(S^{n-1})$ we the existence of some $\xi_0 \in S^{n-1}$ with $m(\xi_0) = 0$, and hence have also that $\langle Ay_0, y_0\rangle = M(y_0) = 1$. (Note also that M being a quadratic norm assures that $\partial S = S_0$.)
As this orthogonality is in Fourier space, we want to prove the required convergence therein. We define

$$v_k(t,x) := A(2\theta_k(t,x)y_0 - u_k(t,x)).$$

Applying Plancherel (on the torus) we have convergence of the remainder $\frac{1}{2}\langle u, A(2\theta y_0 - u)\rangle - 1$ in $\mathcal{D}'_x$ if and only if $\sum_{\xi\in\mathbb{Z}^n} (\widehat{u_k} * \widehat{\varphi})(\xi)\widehat{v}_k(\xi)$ converges for all $\varphi \in C_0^\infty$. As already mentioned we just need to show this convergence for fixed t to then apply the dominated convergence theorem.

We calculate

$$
\begin{aligned}
&\sum_{\xi\in\mathbb{Z}^n}\left\langle \sum_{\eta\in\mathbb{Z}^n}\widehat{\varphi}(\eta)\widehat{u}_k(\xi-\eta),\widehat{v}_k(\xi)\right\rangle\\
=&\sum_{\xi}\left\langle \sum_{\eta}\widehat{\varphi}(\eta)m(\xi-\eta)\widehat{\theta}_k(\xi-\eta),A(2y_0-m(\xi))\widehat{\theta}_k(\xi)\right\rangle\\
=&\sum_{\xi}\left\langle \sum_{\eta}\widehat{\varphi}(\eta)\left(m(\xi-\eta)-m(\xi)\right)\widehat{\theta}_k(\xi-\eta),A(2y_0-m(\xi))\widehat{\theta}_k(\xi)\right\rangle.
\end{aligned}
$$

The last inequality holds because of the above observed orthogonality.

We claim that there exists a constant C independent of k, such that

$$
\sum_{\xi}(1+|\xi|)^2\left|\sum_{\eta}\widehat{\varphi}(\eta)\left(m(\xi-\eta)-m(\xi)\right)\widehat{\theta}_k(\xi-\eta)\right|^2 < C. \tag{3.25}
$$

Having this claim we can apply Plancherel backwards and have that $\varphi\cdot u_k$ lies in a bounded subset of H^1. Then we apply Rellich-Kondrashov and obtain strong convergence (for a subsequence) of $\varphi\cdot u_k$ in L^2. Together with the weak convergence of v_k we thus have the desired convergence of the remainder.

Before we show (3.25) we have a closer look at the difference $m(\xi-\eta)-m(\xi)$. Our assumption that M is a quadratic norm means that $m(\zeta)-y_0$ forms an ellipse with the origin as point of symmetry. Thus we calculate:

$$
\begin{aligned}
|m(\xi-\eta)-m(\xi)| &= |m(\xi-\eta)-y_0-(m(\xi)-y_0)|\\
&=\left|\frac{\langle\xi-\eta,A(\xi-\eta)\rangle}{|\xi-\eta|^2}-\frac{\langle\xi,A\xi\rangle}{|\xi|^2}\right|\\
&=\frac{|2\langle\xi,A\xi\rangle\langle\xi,\eta\rangle-2\langle\xi,A\eta\rangle\langle\xi,\xi\rangle+\langle\eta,A\eta\rangle\langle\xi,\xi\rangle-\langle\xi,A\xi\rangle\langle\eta,\eta\rangle|}{|\xi|^2|\xi-\eta|^2}\\
&\le C\frac{|\xi|^3|\eta|+|\xi|^2|\eta|^2}{|\xi|^2|\xi-\eta|^2}=C\frac{|\xi||\eta|+|\eta|^2}{|\xi-\eta|^2}.
\end{aligned}
$$

We will use the last inequality only for $\xi\not\sim\eta$, as for this case the boundedness of m is of course more useful.

Now we split the sum over ξ in (3.25) into two parts. The first part being when $\{|\xi|<R\}$ for some fixed R. This part is always finite. This comes from the fact that the η-sum converges and the sum over ξ is only a finite sum for fixed R.

So lets now consider the part of (3.25) where $|\xi|>R$. Applying Jensen's inequality it is enough to estimate the sum

$$
\sum_{\{|\xi|>R\}}(1+|\xi|)^2\sum_{\eta\in\mathbb{Z}^n}|\widehat{\varphi}|^2(\eta)\left|m(\xi-\eta)-m(\xi)\right|^2|\widehat{\theta}_k|^2(\xi-\eta) \tag{3.26}
$$

$$
\begin{aligned}
&= \sum_{\{|\xi|>R\}} (1+|\xi|)^2 \sum_{|\eta|\lesssim|\xi|} |\widehat{\varphi}|^2(\eta)\, |m(\xi-\eta)-m(\xi)|^2\, |\widehat{\theta}_k|^2(\xi-\eta) \\
&\quad + \sum_{\{|\xi|>R\}} (1+|\xi|)^2 \sum_{|\eta|\sim|\xi|} |\widehat{\varphi}|^2(\eta)\, |m(\xi-\eta)-m(\xi)|^2\, |\widehat{\theta}_k|^2(\xi-\eta) \\
&\quad + \sum_{\{|\xi|>R\}} (1+|\xi|)^2 \sum_{|\eta|\gtrsim|\xi|} |\widehat{\varphi}|^2(\eta)\, |m(\xi-\eta)-m(\xi)|^2\, |\widehat{\theta}_k|^2(\xi-\eta) \\
&= I + II + III.
\end{aligned}
$$

We estimate these three sums independently.

ad I: This is the most complicated case. We note that here $|m(\xi-\eta)-m(\xi)| \lesssim \frac{|\eta|}{|\xi|}$, and hence

$$
\sum_{|\eta|\lesssim|\xi|} |\widehat{\varphi}|^2(\eta)\, |m(\xi-\eta)-m(\xi)|^2\, |\widehat{\theta}_k|^2(\xi-\eta) \lesssim \frac{1}{\xi^2} \sum_{|\eta|\lesssim|\xi|} (|\eta||\widehat{\varphi}(\eta)|)^2 |\widehat{\theta}_k|^2(\xi-\eta).
$$

We define $f(\eta) := |\eta\widehat{\varphi}(\eta)|^2$ and $g_k(\zeta) := |\widehat{\theta}_k|^2(\zeta)$ and use Young's inequality for convolution (cf. [Bo07] Theorem 3.9.4) to compute:

$$
\begin{aligned}
\sum_{\{|\xi|>R\}} \frac{(1+|\xi|)^2}{|\xi|^2} f * g_k(\xi) &\le C \sum_{\xi\in\mathbb{Z}^n} f * g_k(\xi) \le C||f * g_k||_{L^1} \\
&\le C||f||_{L^1}||g_k||_{L^1} < C,
\end{aligned}
$$

independently of k as the θ_k converge weakly in any L^p.

ad II: As every summand of the η-sum is bounded and continuous we can estimate pointwise:

$$
\begin{aligned}
&\sum_{\{|\xi|>R\}} (1+|\xi|)^2 \sum_{|\eta|\sim|\xi|} |\widehat{\varphi}|^2(\eta)\, |m(\xi-\eta)-m(\xi)|^2\, |\widehat{\theta}_k|^2(\xi-\eta) \\
&\le C \sum_{\{|\xi|>R\}} (1+|\xi|)^2 |\xi|^n \widehat{\varphi}^2(\xi).
\end{aligned}
$$

As $\varphi \in C_0^\infty$ we may assume that $|\widehat{\varphi}|^2(\xi) \lesssim C|\xi|^{-N}$ for N as large as we want, hence the sum converges.

ad III: Here $|\eta|$ is large, we may again assume therefore again that $\widehat{\varphi}^2(\eta) \lesssim |\eta|^{-N} \lesssim |\xi|^{-N}$. This makes the sum in (3.26) converge.

Hence (3.25) is true and we conclude the proof of the theorem.

□

Before we close this section on the question when $K^\Lambda = K^{qc}$ we want to give the following lemma concerning the condition of 0 to lie in $m(S^{n-1})$ from Theorem 3.4.5.

LEMMA 3.4.6. *If m is such that for all ξ we have $m(\xi) \cdot \xi = 0$, then 0 cannot lie outside $m(S^{n-1})$.*

PROOF. This is just a simple consequence of separation theorem. If 0 would lie outside $m(S^{n-1})$ then there would exist some ζ such that $m(\xi) \cdot \eta \geq c > 0$. This contradicts our assumption. □

So in Theorem 3.4.5 we just exclude the case where $0 \in \operatorname{int} m(S^{n-1})$.

4 Systems of Hyperbolic Conservation Laws

4.1 Introduction to Hyperbolic Conservation Laws

A general system of conservation laws for $u : \Omega_T \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ has the following form

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0, \tag{4.1}$$

where the so called *flux* $f : \mathbb{R}^m \to \mathbb{R}^m$ is given.

This thesis will only be concerned with hyperbolic nondegenerate systems in one space dimension ($n = 1$).

The system being *hyperbolic* means that the Jacobian of f has m real eigenvalues. *Strict hyperbolicity* demands additionally that all m eigenvalues of Df are distinct. We want to point out here that the existence of m distinct eigenvalues does not imply that the system as a whole is diagonalizable. For each fixed $\bar{u}$ the system is certainly diagonalizable, however for a diagonalization independent of $\bar{u}$ one would need the existence of a so-called Riemann invariant coordinate system, cf. Chapter 4.3 or [Da05].

The nondegeneracy condition essentially imposes that f is not allowed to be linear. The standard hypothesis here is the condition of genuine nonlinearity due to Lax [La57], that is strict monotonicity of the wave speeds as function of the wave amplitude.

In the theory of weak solutions to (4.1) one already has for Burger's equation ($m = 1$ and $f = \frac{u^2}{2}$: a standard example for a hyperbolic conservation law) non-uniqueness, cf. [Ev98]. To pick among these non-unique solutions the physical relevant ones and achieve uniqueness in a proper space of functions two criteria play an important role:

First, the Rankine-Hugoniot condition, a natural condition for handling physical shocks properly. As this condition will not play a role in our proofs, we will not go into further detail regarding this condition and refer the reader to [Da05] or [Ev98].

Second, the so-called entropy condition, which is due to Lax [La71]. The entropy condition rules out nonphysical shocks, thus it selects a unique solution in a 'good' class of functions. For more detail cf. [Ev98]. A special role here is played by convex entropies. One example for a convex entropy is the first entropy in the system for Lagrangian elasticity: it is given below in (4.22).

As entropies play a decisive role also on the compactness proofs in Chapter 4.7 we give in the following the definition and some properties that will be useful for our purposes.

A pair of functions η, $q : \mathbb{R}^m \to \mathbb{R}$, is called *entropy/entropy flux pair*, or shorthand *entropy pair*, if they satisfy the additional conservation law

$$\partial_t \eta(u) + \partial_x q(u) = 0, \tag{4.2}$$

and furthermore they have to satisfy the following compatibility condition which we also will refer to as entropy condition:

$$\nabla q = (Df)^t \nabla \eta. \tag{4.3}$$

The equality in (4.2) is usually not exactly an equality but rather "$\leq$". As we look at approximate solutions in the Young measure space, we can omit the sign that comes in from the approximation. This comes from Murat's lemma [Mu81] and is also explained in [Ev90].
The condition in (4.3) is made in a way that smooth solutions of the conservation law automatically satisfy it. Of course, if we take linear combinations of the conservation laws (4.1) by multiplying several u_i and $f_i(u)$ from (4.1) by a constant we do get an entropy and a corresponding flux. However these kind of entropy pairs will not help us for our compactness argument, a fact that will become clear in the subsequent sections.

The question whether systems (4.1) are admitted by convex or other nonlinear entropy pairs and by how many of them highly depends on the individual system itself. For more information we suggest [Da05].

However entropies play a somehow mysterious role: although their existence lies already in the system itself, they will explicitely be needed to obtain compactness results, see the sections below.

Since in general we do not have exact solutions to systems of conservation laws, we approximate (4.1). There are a bunch of possibilities and ways to do so. The way one does, will again have no deep impact for the compactness arguments we are aiming at. We essentially need the boundedness of the sequence for the existence of Young measures as in Theorem 2.2.1. One good option would be to choose a viscosity approximation as suggested in [Br07] or [DP83].
We want to note that one has the weak-strong-uniqueness property for hyperbolic conservation laws as was proved in [BLS09]. This means that measure valued solutions agree with classical solutions if the latter exist.

4.2 Conservation Laws in Matrix Space

We now will reformulate (4.1)-(4.2) as a differential inclusion into the form (1.8), i.e. $Dv \in K$. As already pointed out in the introduction the task then is the following:

Let $\Omega \subset \mathbb{R}^n$ be open and $K \subset \mathbb{R}^{m\times n}$ be compact. Then find $v : \Omega \longrightarrow \mathbb{R}^m$ which is Lipschitz continuous such that $Dv(x) \in K$ a.e. Furthermore the boundary condition $v|_{\partial\Omega} = v_0$ should hold. The set K is as in the previous chapters called constitutive set. For more general information we refer to [Ki03] or [Mü98].

We will reformulate not only the system of m conservation laws (4.1) but at the same time the augmented system with the eventual companion equations for entropies (4.2). We assume to have k entropy pairs (η_j, q_j) for $1 \leq j \leq k$. How to perform the reformulation now is already known, e.g. [KMS03] raised their question on hyperbolic conservation laws (cf. Chapter 1.2) in this notation. The reader might also find some insights in [Ev90].

In order to reformulate this problem one considers potentials ψ_i for the original conservation laws and $\varphi_j : \mathbb{R}^m \to \mathbb{R}$ for the additional conservation laws of the entropies. Here $1 \leq i \leq m$ and $1 \leq j \leq k$. The potentials or stream functions (whose existence is guaranteed by Poincaré's lemma) now should satisfy:

$$\begin{array}{rclcrcl} \partial_x \psi_i &=& u_i & \qquad & \partial_t \psi_i &=& -f_i(u) \\ \partial_x \varphi_j &=& \eta_j(u) & & \partial_t \varphi_j &=& -q_j(u) \end{array}.$$

For a function $v : \mathbb{R}^{m+k} \to \mathbb{R}^{m+k}$ given by $v = (v_1, ..., \varphi_{m+k}) := (\psi_1, ...\psi_m, \varphi_1, ..., \varphi_k)$ this can be rewritten as

$$Dv = \begin{pmatrix} \partial_x v_1 & \partial_t v_1 \\ \vdots & \vdots \\ \partial_x v_{m+k} & \partial_t v_{m+k} \end{pmatrix} \in K := \{\gamma(u) : \, u \in \mathbb{R}^m\},$$

where $\gamma(u)$ is the block matrix given by

$$\gamma(u) := \begin{pmatrix} u & -f(u) \\ \eta(u) & -q(u) \end{pmatrix} \in \mathbb{R}^{(m+k)\times 2}.$$

In the case of two conservation laws admitted by two entropy pairs (a situation that will be concerned in Chapter 4.6) the system becomes $Dv \in K := \{\gamma(u) : \, u \in \mathbb{R}^2\}$, where

$$\gamma(u_1, u_2) := \begin{pmatrix} u_1 & -f^1 \\ u_2 & -f^2 \\ \eta_1 & -q_1 \\ \eta_2 & -q_2 \end{pmatrix}.$$

We see in particular that the reformulation for conservation laws (4.1) take the form of a genuine differential inclusion, that is (1.8). Hence to study compactness we consider (1.9) as explained in the introduction. Recall that (1.9) was

$$\begin{aligned} &\operatorname{supp} \nu \subset K, \\ &\langle \nu, M \rangle = M(\langle \nu, id \rangle). \end{aligned}$$

Furthermore we note that in our case the measure ν is in a certain sense reduced. It is already determined on all of $\mathbb{R}^{(m+k)\times 2}$ by the m entries u_1, ..., u_m such that we consider ν as a probability measure on $\mathbb{R}^m$ instead of being a measure on $\mathbb{R}^{(m+k)\times 2}$.

4.3 Strategy for Compactness from $\mathcal{P}^{pc}$

As explained in Chapter 2.2, Corollary 2.2.4, (and also mentioned in the introduction, see (1.9)) polyconvex measures are characterized by the following relation for minors:

Let $M_{ij}(\gamma(u))$ be the minor taking the i-th and j-th row of $\gamma(u)$. Then

$$\int M_{ij}(\gamma(u))d\nu = M_{ij}\left(\int \gamma(u)d\nu\right). \tag{4.4}$$

This relation was first introduced by Tartar in [Ta79]. It is sometimes referred to as *commutativity relation* as the integral with respect to the Young measure ν commutes with the process of taking minors.

We want to recall from Chapter 2.2 that we have compactness for approximate solutions if and only if quasiconvex measures are Dirac. This is result of Theorem 2.2.6 which characterizes Gradient Young measures, together with Lemma 2.2.3, which says that we have compactness if and only if the measure valued solution is Dirac.

As we are for good reasons mentioned earlier not dealing with quasiconvex measures we have in view of (2.5) a sufficient condition for compactness if we show that the set of polyconvex measures is trivial. With Theorem 4.4.1 and 4.4.2 below we develop a dichotomic alternative that gives a characterization on whether this question (compactness from $\mathcal{P}^{pc}$) can be answered affirmatively or not.

For each minor we have one equation from (4.4). The idea is now just to look at these equations and linear combinations thereof and then force the Young measure being a Dirac measure by standard analytic techniques.

Our approach is to expand γ into a Taylor series. It is inspired by DiPerna [DP85], who treated the equations for Lagrangian elasticity similarly. In fact, he did the expansion more explicitly and just for some parts of certain minors and did not have the broad approach we present below. Still, it is natural to do an expansion, as it directly filters out the linear part of the equations. In the formulation of Theorem 4.4.2 the number of equations (4.4) gives exactly determines the number of degrees of freedom for our Taylor expanded version of (4.4) from which we want to conclude compactness.

Counting these gives all in all $\frac{(m+k)(m+k+1)}{2}$ degrees of freedom. It turns out that it is important, whether or not these will involve second, third, or fourth order terms as lowest order terms (other cases will be excluded). Considering the normalized form of γ that will be derived below in this chapter, we count:

- $\frac{m(m+1)}{2}$ terms that have only second and higher order terms (coming from two original conservation law rows of γ),
- km terms that have only third and higher order terms (coming from one conservation law and one entropy row of γ),
- $\frac{k(k+1)}{2}$ terms that have only fourth and higher order terms involved (coming from two entropy rows of γ).

We expand γ now into a Taylor series at the point $\bar{u}$. We want to establish (4.4) then for all terms up to order 4 (in the variables u_i) plus some remainder. To compute all minors up to fourth order properly we have to expand γ up to third order. After the expansion is done in the general case, we will see that a lot of normalizations are possible to make further calculations a lot easier.

In the notation we are going to use, the Taylor expansion for γ has the following form:

$$\begin{aligned}\gamma(u) =& \gamma(\bar{u}) + D\gamma(\bar{u})[u-\bar{u}] + \frac{1}{2}D^2\gamma(\bar{u})[u-\bar{u}, u-\bar{u}] \\ &+ \frac{1}{6}D^3\gamma(\bar{u})[u-\bar{u}, u-\bar{u}, u-\bar{u}] + \mathcal{O}(|u-\bar{u}|^4).\end{aligned} \tag{4.5}$$

The following observations lead to our simple normalized form announced in the introduction (Chapter 1.2).

Lemma 4.3.1. *Condition (4.4) holds after translation.*

Proof. Let A be a constant matrix and B_{ij} the bilinear form associated with M_{ij}.

$$\begin{aligned}&\int M_{ij}(\gamma(u)-A)d\nu = \int M_{ij}(\gamma(u)) - B_{ij}(\gamma(u), A) + M_{ij}(A)d\nu \\ =& M_{ij}\left(\int \gamma(u)d\nu\right) - B_{ij}\left(\int \gamma(u)d\nu, A\right) + M_{ij}(A) = M_{ij}\left(\int \gamma(u) - Ad\nu\right).\end{aligned}$$

□

Hence we can assume choosing $A = \gamma(\bar{u})$ without loss of generality that $\gamma(\bar{u}) = 0$, and in particular $\bar{u} = 0$.

Lemma 4.3.2. *Condition (4.4) holds after multiplication of γ from either side by constant matrices.*

Proof. This follows directly from the linearity of the integral. □

Applying this now to a curve γ with $\gamma(0) = 0$, we can choose the constant matrix say A as in the following calculation and get:

$$A \cdot \gamma(u) := \begin{pmatrix} I_m & 0 \\ -(\partial_i \eta_j(0))_{ij} & I_k \end{pmatrix} \cdot \gamma(u) = \begin{pmatrix} u & f(u) \\ Q\eta_1(u) & Q^* q_1(u) \\ \vdots & \vdots \\ Q\eta_k(u) & Q^* q_k(u) \end{pmatrix},$$

where

$$\begin{aligned} Q\eta_j(u) &= \eta_j(u) - \nabla\eta_j(0) \cdot u \qquad \text{and} \\ Q^* q_j(u) &= q_j(u) - \nabla\eta_j(0) \cdot f(u). \end{aligned}$$

Note that $(Q\eta(u), Q^*q(u))$ is an entropy/entropy flux pair again and both $Q\eta(u)$ and $Q^*q(u)$ are of quadratic order (i.e. their Taylor expansion involves only terms with u^l for $l \geq 2$) by using (4.3).

If we did not have $\gamma(0) = 0$, the transition to quadratic entropies is of course also possible. Just take

$$\begin{aligned} Q\eta(u) &:= \eta(u) - \eta(\bar{u}) - \nabla\eta(\bar{u}) \cdot (u - \bar{u})\,, \\ Q^*q(u) &:= q(u) - q(\bar{u}) - \nabla\eta(\bar{u}) \cdot (f(u) - f(\bar{u}))\,, \\ \text{and} \quad A &= \begin{pmatrix} I_m & 0 \\ -(\partial_i \eta_j(\bar{u}))_{ij} & I_k \end{pmatrix}, \end{aligned}$$

for any point $\bar{u}$. This fact was already was observed and used by different authors, e.g. [DP85], [Da05].

Of course one could instead or additionally choose another linear coordinate change Φ in the matrix space as the commutativity relation is still preserved:

$$\begin{aligned} &M\left(\int \Phi \circ \gamma\right) = M\left(\Phi \circ \int \gamma\right) = M(\Phi) \cdot M\left(\int \gamma\right) = M(\Phi) \int M(\gamma) \\ &= \int M(\Phi)\, M(\gamma) = \int M(\Phi \circ \gamma)\,. \end{aligned}$$

However, we found that at least in the general case the matrix A from above gives the maximal simplification.

The third simplification we suggest is a linear diagonalization of $Df(0)$:

As the system is hyperbolic there exists a coordinate system of $\mathbb{R}^m$ such that $Df(0)$ becomes diagonal. Note that in general $Df(u)$ might not be diagonal away from 0, even in the coordinate system where $Df(0)$ is diagonal. Getting Df diagonal also away from 0 is possible only if one has a coordinate system of so called Riemann invariants. For systems of size $m \geq 3$ such a coordinate system does not exist in general, as one has to solve here an overdetermined system of equations. For further information on Riemann invariants we refer to [Da05], as they do not play a decisive role in this thesis. Still we will do the linear diagonalization of Df at 0.

So let the coordinate system of eigenvectors of $Df(0)$ be v_1, ..., v_m. Then take as new coordinates $v = Tu$ where T is the constant matrix with v_1, ..., v_m in the columns. Furthermore we multiply γ from the left by the block matrix $B := \begin{pmatrix} T^{-1} & 0 \\ 0 & Id \end{pmatrix}$, which is not an issue according to Lemma 4.3.2.

Then we get:

$$B \cdot \gamma(Tu) = \begin{pmatrix} T^{-1} & 0 \\ 0 & Id \end{pmatrix} \cdot \begin{pmatrix} Tu & f(Tu) \\ \eta(Tu) & q(Tu) \end{pmatrix} = \begin{pmatrix} u & T^{-1}f(Tu) \\ \eta(Tu) & q(Tu) \end{pmatrix}.$$

This defines a new $\tilde{\gamma}(u) = \begin{pmatrix} u & \tilde{f} \\ \tilde{\eta} & \tilde{q} \end{pmatrix}$ with $D\tilde{f}(0)$ diagonal. The entropy condition still holds as we have:

$$\begin{aligned} D\tilde{f}(Tu) &= T^{-1}Df(Tu)T, \\ \nabla\tilde{\eta}(Tu) &= T^t\nabla\eta(Tu), \\ \nabla\tilde{q}(Tu) &= T^t\nabla q(Tu). \end{aligned}$$

Putting this together gives

$$\begin{aligned} (D\tilde{f}(Tu))^t \cdot \nabla\tilde{\eta}(Tu) &= (T^{-1}Df(Tu)T)^t T^t\nabla\eta(Tu) = (TT^{-1}Df(Tu)T)^t\nabla\eta(Tu) \\ &= T^t(Df(Tu))^t\nabla\eta(Tu) = T^t\nabla q(Tu) \\ &= \nabla\tilde{q}(Tu). \end{aligned}$$

Remark 4.3.3. *Of course one has the same amount of simplifications in subsequent calculations if one does not diagonalize $Df(0)$, but instead transforms $Df(0)$ into an off-diagonal matrix.*

So after all the above simplifications, i.e. a shift in matrix space, a linear coordinate change in matrix space, and a linear coordinate change in the variables, we get a curve, which we will not relabel and still call $\gamma(u)$ that looks like

$$\gamma(u) := \begin{pmatrix} u & -f \\ \eta & -q \end{pmatrix} \tag{4.6}$$

with $\gamma(0) = 0$, $Df(0)$ diagonal and quadratic entropy/entropy flux pairs at $\bar{u} = 0$.

With all the aforementioned simplifications we may rewrite the above expressions as:

$$\gamma(u) = B \cdot A \cdot (\gamma_{\text{old}}(Tu) - \gamma_{\text{old}}(0)).$$

Looking at the Taylor expansion (4.5) we have in block matrix notation as prefactors of the 0-th and first order terms:

$$\gamma(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{(m+k)\times 2}, \tag{4.7}$$

$$D\gamma(0) = \begin{pmatrix} Id & \operatorname{diag}(\lambda_1, \ldots, \lambda_m) \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{(m+k)\times 2m}. \tag{4.8}$$

For the lower right entry in this block matrix we used the entropy condition (4.3).

Remark 4.3.4. *We want to point out here that an eventual quadratic transformation in the coordinates u will at least in the case $m = k = 2$ not help in our proofs in Chapter 4.7. The point is that we will not gain additional degrees of freedom in the linear combination $\alpha \cdot M(\gamma)$, which is the base for compactness arguments (see the following sections). This might also hold for the general case, where $m, k \geq 2$.*
This is because after taking the minors the quadratic parts of the transform will not show up in the Taylor expansion below fifth order. This will be proved for the case $m = k = 2$, where we have two conservation laws and two entropy pairs below in Chapter 4.6, Proposition 4.6.2.

4.4 A Characterizing Criterion for Compactness from Polyconvex Measures

In this section we will give a dichotomic alternative that characterizes the cases where one can gain compactness from $\mathcal{P}^{pc}$ and where not. This alternative is presented in the next two theorems. After having this alternative, we will plug in the Taylor expansion for γ from Chapter 4.3 to obtain some necessary and also a sufficient condition.

Before we state the main theorem in this section we introduce the following sets:

$$C_A := \{\nu \text{ probability measure} : \bar{\nu} = 0,\ A = \int \gamma(u) d\nu,\ \operatorname{supp} \nu \subset B_1\}, \tag{4.9}$$

$$C_A^\varepsilon := \{\nu \in C_A : \operatorname{supp} \nu \subset B_\varepsilon\}, \tag{4.10}$$

where $B_1 = B_1(0) \subset \mathbb{R}^m$ and $B_\varepsilon = B_\varepsilon(0) \subset \mathbb{R}^m$. We observe that if a measure ν' belonging to one of these sets is a Dirac measure at say v, then the condition that $\bar{\nu}' = 0$ forces v to be 0 as

$$0 = \bar{\nu}' = \int \operatorname{id} d\nu' = v.$$

THEOREM 4.4.1. *Let $\gamma(u)$ be as in (4.6) and $M(\gamma(u)) \in \mathbb{R}^N$ denote the vector consisting of all 2×2-minors of the matrix $\gamma(u)$. Furthermore let C_A be defined as in (4.9). Then we have the following dichotomy:*
either there exists a measure $\nu \in C_A$ that is not a Dirac measure such that

$$\begin{aligned} &\int M(\gamma(u)) d\nu = M(A) \\ \Leftrightarrow \forall_{\alpha \in \mathbb{R}^N} \qquad &\alpha \cdot \int M(\gamma(u)) d\nu = \alpha M(A) \end{aligned} \tag{4.11}$$

or there exists some $\alpha \in \mathbb{R}^N$ such that for every non-Dirac measure $\nu \in C_A$ we have that

$$\alpha \cdot \int M(\gamma(u)) d\nu < \alpha \cdot M(A). \tag{4.12}$$

PROOF. Let $\mathcal{M}_r$ denote the space of probability measures and

$$\begin{aligned} T: \quad \mathcal{M}_r &\longrightarrow \mathbb{R}^N \\ \nu &\longmapsto \int M(\gamma(u)) d\nu. \end{aligned}$$

Note that C_A is convex: i.e. for $\lambda_1,\ \lambda_2 \in C_A$ and $t \in (0, 1)$ we have

$$\overline{t\lambda_1 + (1-t)\lambda_2} = t\bar{\lambda}_1 + (1-t)\bar{\lambda}_2 = 0 \qquad \text{and}$$

$$\int \gamma(u)d(t\lambda_1 + (1-t)\lambda_2) = t\int \gamma(u)d\lambda_1 + (1-t)\int \gamma(u)d\lambda_2 = A.$$

Furthermore C_A is closed. To see this, let $\{\lambda_k\} \subset C_A$ and $\lambda_k \overset{k\to\infty}{\longrightarrow} \lambda$, i.e. $\int f d\lambda_k \overset{k\to\infty}{\longrightarrow} \int f d\lambda$ for every continuous bounded f. Choosing $f = id$ and $f = \gamma(u)$ shows that the limit λ again lies in C_A.
One directly gets that $T(C_A) \subset \mathbb{R}^N$ again is closed and convex. Precisely, take $a,\, b \in T(C_A)$ and $t \in (0,\, 1)$, then there exist μ and ν such that $T\mu = a$ and $T\nu = b$. Now for every t we have:

$$ta + (1-t)b = tT\mu + (1-t)T\nu = T(t\mu + (1-t)\nu).$$

For closedness of $T(C_A)$ argue similarly.

Now we have two alternatives:

- either $M(A) \in T(C_A)$: then we are in case of (4.11);
- or $M(A) \notin T(C_A)$: then we have case (4.12) as separation theorem tells us, (here the $\alpha \in \mathbb{R}^N$ is the linear functional as $(\mathbb{R}^N)' \cong \mathbb{R}^N$).

This proves the theorem if C_A contains no Dirac measure, but what happens if C_A contains a Dirac measure say ν?

As seen directly after the definition of C_A in (4.9) we then have that $\nu = \delta_0$.
Recalling that in our normalized form (4.6) we have that $f(0) = \eta(0) = q(0) = 0$ and thus the matrix A must be the zero matrix. Hence we are left with C_0. If now C_0 contains only δ_0 then $C_0 \setminus \{\delta_0\} = \emptyset$ is trivially again closed and convex. Hence the above proof still holds.

So let us assume the contrary, i.e. there exists some $\mu \in C_0$ with $\mu \neq \delta_0$ say with $\mu(\{0\}) = 1 - \delta$.
But

$$0 \overset{A=0}{=} \int \eta(u)d\mu$$

gives a contradiction, hence $C_0 = \{\delta_0\}$ and we are done.

$\square$

A slightly different version of the above theorem is the following statement which follows immediately.

THEOREM 4.4.2. *Let $0 < \varepsilon \leq 1$ and all other expressions as in Theorem 4.4.1. Then we have the following dichotomy:*

either there exists a measure $\nu \in C_A^\varepsilon$ that is not a Dirac measure such that

$$\begin{aligned} &\int M(\gamma(u))d\nu = M(A) \\ \Leftrightarrow \forall_{\alpha\in\mathbb{R}^N} \qquad &\alpha \cdot \int M(\gamma(u))d\nu = \alpha \cdot M(A) \end{aligned} \tag{4.13}$$

or there exists some $\alpha \in \mathbb{R}^N$ such that for every non-Dirac measure $\nu \in C_A^\varepsilon$ we have that

$$\alpha \cdot \int M(\gamma(u))d\nu < \alpha \cdot M(A). \tag{4.14}$$

Remark 4.4.3. *(i) The direction of the inequality in (4.12) and (4.14) is of course arbitrary. One can obtain the reverse inequality by choosing $-\alpha$ instead of α. The essential fact is that it holds strictly and for all measures in $C_A \setminus \{\delta_v\}$.*

(ii) If we have in the above theorems the second case (4.12) and (4.14) we have a contradiction to the minor relation (4.4), which tells that equality must hold for all ν coming from approximation with polyconvex functions. Hence ν must be a Dirac mass and we have compactness in L^1.

(iii) If we have (4.11) or (4.13), we have a genuine example for a polyconvex measure that is not L^1. This means that we cannot gain compactness from polyconvex measures. In this situation one could look for a rank one convex measure that serves as a real counterexample to compactness, also on the relevant class $\mathcal{P}^{qc}$ of quasiconvex Young measures.

So the question of gaining compactness from polyconvex measures is now reformulated as either (4.13) which means we cannot gain compactness or (4.14), i.e. we have compactness.

4.5 Necessary and Sufficient Conditions

Starting from the dichotomy in Theorem 4.4.2 we will in the following state necessary and also sufficient conditions for compactness in terms of this formulation. We therefore plug in the Taylor expansion of the normalized γ from Chapter 4.3 into the commutativity relation (4.4).

Recall the commutativity relation for the vector M

$$\int M(\gamma(u))d\nu = M\left(\int \gamma(u)d\nu\right) \tag{4.15}$$

and the Taylor expansion of γ

$$\gamma(u) = \gamma(0) + D\gamma(0)[u] + \frac{1}{2}D^2\gamma(0)[u,u] + \frac{1}{6}D^3\gamma(0)[u,u,u] + \mathcal{O}(|u|^4),$$

with the prefactors as in (4.7) and (4.17). As earlier, let B be the bilinear form associated with M, i.e. B is such that

$$M(X+Y) = M(X) + B(X,Y) + M(Y).$$

This gives (as $\gamma(0)=0$) for the integrand of the left hand side of (4.15) up to fourth order:

$$\begin{aligned} M(\gamma(u)) =& M\left(D\gamma(0)[u] + \frac{1}{2}D^2\gamma(0)[u,u] + \frac{1}{6}D^3\gamma(0)[u,u,u] + \mathcal{O}(|u|^4)\right) \\ =& \underbrace{M\left(D\gamma(0)[u]\right)}_{\text{2nd order}} + \underbrace{\frac{1}{2}B\left(D\gamma(0)[u], D^2\gamma(0)[u,u]\right)}_{\text{3rd order}} \\ &+ \underbrace{\frac{1}{4}M\left(D^2\gamma(0)[u,u]\right) + \frac{1}{6}B\left(D\gamma(0)[u], D^3\gamma(0)[u,u,u]\right)}_{\text{4th order}} + \mathcal{O}(|u|^5). \end{aligned}$$

For the right hand side of (4.15) this expression is a bit simpler as we have that $\int u d\nu = \bar{u} = 0$. The first term showing up is already fourth order:

$$\begin{aligned} M\left(\int \gamma(u) d\nu\right) &= M\left(\int D\gamma(0)[u] + \frac{1}{2}D^2\gamma(0)[u,u] + \frac{1}{6}D^3\gamma(0)[u,u,u] + \mathcal{O}(|u|^4) d\nu\right) \\ &= \frac{1}{4}M\left(\int D^2\gamma(0)[u,u] d\nu\right) + \int u^2 d\nu \int \mathcal{O}(|u|^3) d\nu. \end{aligned}$$

In the following two lemmas we derive as necessary conditions, that the linear combination of the second and the third order terms have to be zero if we want to prove compactness from polyconvex measures, i.e. (4.14).

Lemma 4.5.1. *(4.14)* $\Longrightarrow \alpha M(D\gamma(0)[u]) = 0$.

Proof. It is equivalent to show that $\alpha M(D\gamma(0)[u]) \neq 0 \Longrightarrow$ (4.13).

Observe that

$$M_{ij}(D\gamma(0)[u]) = \begin{cases} (\lambda_j - \lambda_i)u_i u_j & \text{if } i \leq m \text{ and } j \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

First let $m = 2$.

If now $M(\gamma(u))$ is purely quadratic, then the measure

$$\nu_0 := \frac{1}{2}\delta_{(\frac{\varepsilon}{2},\frac{\varepsilon}{2})} + \frac{1}{2}\delta_{(\frac{\varepsilon}{2},-\frac{\varepsilon}{2})}$$

is a measure that satisfies (4.13) and we are done.

If the Taylor expansion of $M(\gamma(u))$ involves higher order terms then we make the following claim.

Claim: For ε small enough there exist $\eta \in (0,1)$ such that for all α and for $\nu := \eta\delta_{(\frac{\varepsilon}{2},\frac{\varepsilon}{2})} + (1-\eta)\delta_{(\frac{\varepsilon}{2},-\frac{\varepsilon}{2})}$ we have $\alpha \int M(\gamma(u))d\nu = 0$.
Clearly ν is a probability measure, hence we have an example for (4.13).

Proof of claim:
The only term $\neq 0$ is M_{12} (we are proving the case $m = 2!$). So

$$\begin{aligned}
\alpha \int M(\gamma(u))d\nu &= \int \alpha M(D\gamma(0)[u]) + o(|u|^3)d\nu = \int \alpha_{12}(\lambda_2 - \lambda_1)u_1u_2 + o(|u|^3)d\nu \\
&= \alpha_{12}(\lambda_2-\lambda_1)\frac{\varepsilon^2}{4}(\eta-(1-\eta)) + \eta g\left(\tfrac{\varepsilon}{2},\tfrac{\varepsilon}{2}\right) + (1-\eta)g\left(\tfrac{\varepsilon}{2},-\tfrac{\varepsilon}{2}\right) \\
&= \eta\left(\alpha_{12}(\lambda_2-\lambda_1)\frac{\varepsilon^2}{2} + g\left(\tfrac{\varepsilon}{2},\tfrac{\varepsilon}{2}\right) - g\left(\tfrac{\varepsilon}{2},-\tfrac{\varepsilon}{2}\right)\right) - \left(\alpha_{12}(\lambda_2-\lambda_1)\frac{\varepsilon^2}{4} - g\left(\tfrac{\varepsilon}{2},-\tfrac{\varepsilon}{2}\right)\right),
\end{aligned}$$

where $g\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right) \overset{\varepsilon\to 0}{\sim} \varepsilon^3 \overset{\varepsilon\to 0}{\sim} g\left(\frac{\varepsilon}{2},-\frac{\varepsilon}{2}\right)$.

Now

$$\eta = \frac{\alpha_{12}(\lambda_2-\lambda_1)\frac{\varepsilon^2}{4} - g\left(\frac{\varepsilon}{2},-\frac{\varepsilon}{2}\right)}{\alpha_{12}(\lambda_2-\lambda_1)\frac{\varepsilon^2}{2} + g\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right) - g\left(\frac{\varepsilon}{2},-\frac{\varepsilon}{2}\right)}$$

does the job for some $\varepsilon > 0$ which is so small that $\eta \in (0,1)$. Such an ε exists clearly as $\eta(\varepsilon) \overset{\varepsilon\to 0}{\longrightarrow} \frac{1}{2}$.

If now $m \geq 3$ the procedure is similar. The measure ν then can be chosen as the weighted sum of 2^m Dirac measures of the type $\delta_{(..,\frac{\varepsilon}{2},..,-\frac{\varepsilon}{2},..)}$. Alternatively one can take similar to the case where $m = 2$ two Dirac measures $\delta_{(\frac{\varepsilon}{2},-\frac{\varepsilon}{2},0,..,0)}$ and $\delta_{(\frac{\varepsilon}{2},\frac{\varepsilon}{2},0,..,0)}$. □

Lemma 4.5.2. *(4.14)* $\Longrightarrow \alpha B(D\gamma(0)[u], D^2\gamma(0)[u,u]) = 0$.

Proof. The essential fact in the proof of the foregoing lemma was that the leading (=second) order terms were indefinite. We could simply weight a measure consisting of 2^m Dirac measures in a way that compensated the higher order terms independent of their sign exactly, yielding $\int M(\gamma(u))d\nu = 0$. As all third order polynomials are indefinite around 0, a modified proof still holds. □

Lemma 4.5.3. *(4.14) $\Longrightarrow$ 4th order terms are (positive) semidefinite.*

Proof. Again, if they were indefinite one could find in the above manner a non-Dirac measure satisfying (4.13). □

After choosing a linear combination that suffices the above necessary conditions we are left with only fourth and higher order terms in (4.14). In the following Proposition we give a sufficient condition on the fourth order terms to gain compactness. We have to admit that this condition is for sure not necessary as examples discussed below will show. Still it features nicely how the principle structure for a compactness proof runs.

PROPOSITION 4.5.4. *If ε is small enough and additionally*

(i) $\alpha M(D\gamma(0)[u]) = 0$ *(second order terms),*

(ii) $\alpha B(D\gamma(0)[u], D^2\gamma(0)[u,u]) = 0$ *(third order terms),*

(iii) $\alpha B(D\gamma(0)[u], D^3\gamma(0)[u,u,u])$ *(fourth order) is positive definite,*

(iv) $\alpha M(D^2\gamma(0)[u,u])$ *(fourth order) is positive definite,*

then we have (4.14), i.e. compactness.

PROOF. We want to show that (4.14) holds. A Taylor expansion gives that (4.14) is equivalent to:

$$\alpha \int \Big(M(D\gamma(0)[u]) + \frac{1}{2} B(D\gamma(0)[u], D^2\gamma(0)[u,u]) + \frac{1}{4} M(D^2\gamma(0)[u,u]) + \frac{1}{6} B(D\gamma(0)[u], D^3\gamma(0)[u,u,u]) + \mathcal{O}(|u|^5) \Big) d\nu < \alpha M(A).$$

Having (i) and (ii), it follows that

$$\alpha \int \Big(\frac{1}{4} M(D^2\gamma(0)[u,u]) + \frac{1}{6} B(D\gamma(0)[u], D^3\gamma(0)[u,u,u]) + \mathcal{O}(|u|^5) \Big) d\nu < \alpha M(A).$$

The right hand side of which is

$$\begin{aligned} \alpha M(A) &= \alpha M \left(\int \frac{1}{2} D^2\gamma(0)[u,u] d\nu + \int \mathcal{O}(u^3) d\nu \right) \\ &= \alpha \frac{1}{4} M \left(\int \frac{1}{2} D^2\gamma(0)[u,u] d\nu \right) + \int |u|^2 d\nu \int \mathcal{O}(|u|^3) d\nu. \end{aligned}$$

Now we use (iii) and choose ε so small that

$$\alpha \int \frac{1}{6} B(D\gamma(0)[u], D^3\gamma(0)[u,u,u]) d\nu + \int \mathcal{O}(|u|^5) d\nu - \int |u|^2 d\nu \int \mathcal{O}(|u|^3) d\nu > 0,$$

which again implies that

$$\alpha \int \frac{1}{4} M(D^2\gamma(0)[u,u]) d\nu < \alpha \frac{1}{4} M \left(\int \frac{1}{2} D^2\gamma(0)[u,u] d\nu \right).$$

The last inequality is (for supp ν small) true by (iv), i.e. (iv) implies convexity around $\bar{\nu} = 0$ it is just Jensen's inequality. □

REMARK 4.5.5. *In view of Remark 4.4.3 it would also be sufficient if conditions (iii) and (iv) from the above proposition would both hold with negative definiteness replacing positive definiteness.*

4.6 Systems of Two Equations

4.6.1 Geometry of the Constitutive Set

In this section we discuss what the above simplifications mean geometrically. To simplify calculations we restrict here to the case where $m = k = 2$, such that $\gamma(u) \in \mathbb{R}^{4\times 2}$. Some of the statements should also hold for larger systems.

First, the two simplifications that $\gamma(0) = 0$ and consequently also $\bar{u} = 0$ mean that we always can shift the curve γ such that it intersects the origin, moreover the center of mass is located there.

Tangent space

The tangent space of the constitutive set K is associated with $D\gamma(u)$. As we take measure valued solutions with small support around the origin we are only interested in the area around 0. There we observe that rank-one-directions stay stable.

Proposition 4.6.1. *Let $D\gamma(0)[u] \in \mathbb{R}^{4\times 2}$ be as above. Then rank-one directions stay stable in a neighborhood of* 0.

Proof. The rank-1-directions of $D\gamma(0)[u]$ are the coordinate axes: the minor $M_{12}(D\gamma(0)[u]) = (\lambda_2 - \lambda_1)u_1u_2$ equals zero only if $u_1 = 0$ or $u_2 = 0$ as a consequence of strict hyperbolicity (that is $\lambda_2 - \lambda_1 \neq 0$). So we have that

$$J(w_1, w_2, u_1, u_2) := M_{12}\left(D\gamma(w)[u]\right) = 0 \qquad \text{at the point } p = (0, 0, 1, 0),$$

as

$$\begin{aligned}
&\frac{d}{du_2} M_{12}(D\gamma(w)[u])\Big|_p \\
=&\frac{d}{du_2}\left(f_1^2(w)u_1^2 + (f_2^2(w) - f_1^1(w))u_1u_2 - f_2^1(w)u_2^2\right)\Big|_p \\
=&\left((f_2^2(w) - f_1^1(w))u_1 - 2f_2^1(w)u_2\right)\Big|_p = \lambda_2 - \lambda_1 \\
\neq&0
\end{aligned}$$

The implicit function theorem (cf. [Ko04]) gives now existence of a function $g \in C^1(\mathbb{R}^3)$ such that $M_{12}\left(D\gamma(w)\left[\begin{pmatrix} u_1 \\ g(w_1, w_2, u_1)\end{pmatrix}\right]\right) = 0$ in some neighborhoods around the point p, i.e.

$$f_1^2(w)u_1^2 + (f_2^2(w) - f_1^1(w))u_1 g(w, u_1) - f_2^1(w)g(w, u_1)^2 = 0. \tag{4.16}$$

Fortunately all other minors M_{ij} stay 0. Thus we have

$$D\gamma(w)[u] = \begin{pmatrix} u_1 & \nabla f^1(w)\cdot u \\ u_2 & \nabla f^2(w)\cdot u \\ \nabla\eta^1(w)\cdot u & \nabla q^1(w)\cdot u \\ \nabla\eta^2(w)\cdot u & \nabla q^2(w)\cdot u \end{pmatrix},$$

and hence:

$$
\begin{aligned}
&M_{13}\left(D\gamma(w)\left[\begin{pmatrix} u_1 \\ g(w_1,w_2,u_1)\end{pmatrix}\right]\right)\\
&= u_1\left(u_1(f_1^1(w)\eta_1^1(w)+f_1^2(w)\eta_2^1(w))+g(w,u_1)(f_2^1(w)\eta_1^1(w)+f_2^2(w)\eta_2^1(w))\right)\\
&\quad - u_1(u_1 f_1^1(w)\eta_1^1(w)+g(w,u_1)(f_2^1(w)\eta_1^1(w)+f_1^1(w)\eta_2^1(w))) - g(w,u_1)f_2^1\eta_2^1\\
&= \eta_2^1\left(f_1^2(w)u_1^2+(f_2^2(w)-f_1^1(w))u_1 g(w,u_1)-f_2^1(w)g(w,u_1)^2\right)\\
&\overset{(4.16)}{=} 0,\\
&M_{23}\left(D\gamma(w)\left[\begin{pmatrix} u_1 \\ g(w_1,w_2,u_1)\end{pmatrix}\right]\right)\\
&= g(w,u_1)\left(u_1(f_1^1(w)\eta_1^1(w)+f_1^2(w)\eta_2^1(w))+g(w,u_1)(f_2^1(w)\eta_1^1(w)+f_2^2(w)\eta_2^1(w))\right)\\
&\quad - (u_1 f_1^2(w)+g(w,u_1)f_2^2)(u_1\eta_1^1(w)+g(w,u_1)\eta_2^1)\\
&= \eta_1^1\left(-f_1^2(w)u_1^2-(f_2^2(w)-f_1^1(w))u_1 g(w,u_1)+f_2^1(w)g(w,u_1)^2\right)\\
&\overset{(4.16)}{=} 0,\\
&M_{34}\left(D\gamma(w)\left[\begin{pmatrix} u_1 \\ g(w_1,w_2,u_1)\end{pmatrix}\right]\right)\\
&= \det\left(\begin{pmatrix} \eta_1^1(w) & \eta_2^1(w) \\ \eta_1^2(w) & \eta_2^2(w)\end{pmatrix}\cdot\begin{pmatrix} u_1 & \nabla f^1(w)\cdot\begin{pmatrix} u_1 \\ g(w,u_1)\end{pmatrix} \\ g(w,u_1) & \nabla f^2(w)\cdot\begin{pmatrix} u_1 \\ g(w,u_1)\end{pmatrix}\end{pmatrix}\right)\\
&= \det\begin{pmatrix} \eta_1^1(w) & \eta_2^1(w) \\ \eta_1^2(w) & \eta_2^2(w)\end{pmatrix}\cdot M_{12}\left(\gamma(w)\left[\begin{pmatrix} u_1 \\ g(w,u_1)\end{pmatrix}\right]\right) \overset{(4.16)}{=} 0.
\end{aligned}
$$

The minors M_{23} and M_{24} are calculated analogously. □

Curvature

Similarly the curvature of K around zero is connected with $D^2\gamma(0)$. This has the form

$$
D^2\gamma(0)[u,u] = \begin{pmatrix} 0 & D^2 f(0)[u,u] \\ D^2\eta(0)[u,u] & D^2\eta(0)[u,\operatorname{diag}(\lambda_1,\, ...,\, \lambda_m)\cdot u]\end{pmatrix} \in \mathbb{R}^{(m+k)\times 2}. \quad (4.17)
$$

Written in this form one can see that a positive curvature is connected to the existence of convex entropies.

4.6.2 Alternative to Taking Linear Combinations of Minors

Instead of taking linear combinations of the minors. One could try to derive a formula of the following form

$$
\alpha M(\gamma) = \det(P\gamma), \quad (4.18)
$$

where P is a constant 2×4-matrix, such that the determinant in (4.18) is really just a determinant of a 2×2 matrix.

Direct calculation gives:

$$\det(P \cdot \gamma) = \sum_{(i,j)=(1,2)}^{(3,4)} (p_{1i}p_{2j} - p_{1j}p_{2i})M_{ij}(\gamma),$$

such that (4.18) is satisfied if $\alpha_{ij} = p_{1i}p_{2j} - p_{1j}p_{2i}$. For given α this system is clearly overdetermined. However, it turns out that if we restrict to the α, that respect the necessary conditions from Lemmas 4.5.1 and 4.5.2, then the following matrix P does the job:

$$\begin{pmatrix} 1 & \frac{\alpha_{24}}{\alpha_{14}} & \frac{1}{\alpha_{14}} & 0 \\ 0 & 0 & \beta\alpha_{14} & \alpha_{14} \end{pmatrix}.$$

Here $\beta := -\frac{\eta^1_{u_2u_2}(0)}{\eta^2_{u_2u_2}(0)}$. All calculations needed to get this P are straight-forward, so we do not put them in here line by line.

Nevertheless this formula is useful to prove the already announced following proposition.

Proposition 4.6.2. *An additional quadratic coordinate change in u has no effect on the alternatives in Theorem 4.4.2. Hence it does help to obtain compactness from polyconvex measures.*

Proof. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the coordinate transform. We suppose that it consists of a linear part and a purely quadratic part, such that $\Phi(u) = Tu + R[u, u] =: v + r$. We observe that $\Phi(0) = 0$ and insert Φ into the Taylor expansion of γ to obtain:

$$\begin{aligned}\gamma(\Phi(u)) =& D\gamma(0)[\Phi(u)] + \frac{1}{2}D^2\gamma(0)[\Phi(u), \Phi(u)] + \frac{1}{6}\gamma(0)[\Phi(u), \Phi(u), \Phi(u)] + \mathcal{O}(|u|^4)\\ =& D\gamma(0)[v]\\ &+ D\gamma(0)[r] + \frac{1}{2}D^2\gamma(0)[v, v]\\ &+ \frac{1}{2}D^2\gamma(0)[v, r] + \frac{1}{6}\gamma(0)[v, v, v]\\ &+ \mathcal{O}(|u|^4).\end{aligned}$$

Next we look at both sides of the commutativity relation (4.15) to see which terms show up additionally.

First we look at the integrand on the left hand side. We multiply the transformed γ by the matrix P and take the determinant. Furthermore we use the fact that P lets the second and third order terms we already had before the quadratic transform vanish to compute:

$$\det(P\gamma(\Phi(u))) = \det(PD\gamma(0)[v]) + B_2(PD\gamma(0)[v], PD\gamma(0)[r])$$

$$
\begin{aligned}
&+\frac{1}{2}B_2(PD\gamma(0)[v], PD^2\gamma(0)[v,v] + \det(PD\gamma(0)[r])\\
&+\frac{1}{4}\det(PD^2\gamma(0)[v,v]) + B_2(PD\gamma(0)[v], PD^2\gamma(0)[v,r])\\
&+\frac{1}{6}B_2(PD\gamma(0)[v], PD^3\gamma(0)[v,v,v]) + \mathcal{O}(|u|^5)\\
=&B_2(PD\gamma(0)[v], PD\gamma(0)[r]) + \frac{1}{4}\det(PD^2\gamma(0)[v,v])\\
&+B_2(PD\gamma(0)[v], PD^2\gamma(0)[v,r]) + \frac{1}{6}B_2(PD\gamma(0)[v], PD^3\gamma(0)[v,v,v])\\
&+\mathcal{O}(|u|^5),
\end{aligned}
$$

where B_2 indicates as usual the bilinear form associated with the determinant. Note that $\det(PD\gamma(0)[r]) = 0$ also because of the structure of P.

So the new terms of the left hand sides integrand are the first and the third in the last equality. Both turn out to be zero. For the first, $B_2(PD\gamma(0)[v], PD\gamma(0)[r])$, this is just a short calculation. The calculation for $B_2(PD\gamma(0)[v], PD^2\gamma(0)[v,r])$ is a bit more involved, one has to use the derivative of the entropy condition (4.3). Still it is straight forward.

So on the left hand side of (4.15) no additional terms show up. Hence we now turn to the right hand side. After inserting the quadratic transform as above it is:

$$
\begin{aligned}
\int \gamma(\Phi(u))d\nu =& \int D\gamma(0)[v]d\nu + \int D\gamma(0)[r]d\nu + \frac{1}{2}\int D^2\gamma(0)[v,v]d\nu\\
&+\int D^2\gamma(0)[v,r]d\nu + \int D^3\gamma(0)[v,v,v]d\nu + \int \mathcal{O}(|u|^4)d\nu.
\end{aligned}
$$

As $\bar{\nu} = 0$, the first term is zero. Multiplying by P and taking the determinant gives

$$
\begin{aligned}
\det\left(P\cdot\int\gamma(\Phi(u))d\nu\right) =& \det\left(\int PD\gamma(0)[r]d\nu\right)\\
&+B_2\left(\int PD\gamma(0)[r]d\nu, \int PD^2\gamma(0)[v,v]d\nu\right)\\
&+\frac{1}{4}\det\left(\int PD\gamma(0)[v,v]d\nu\right) + \int\mathcal{O}(|u|^2)d\nu\int\mathcal{O}(|u|^3)d\nu.
\end{aligned}
$$

But as $P\cdot D\gamma(0)[\cdot] = 0$ we have also on this side no additional terms, which closes the proof. □

4.6.3 General Form for Minors

We call a minor a *second (third, fourth) order minor*, if it involves second (third, fourth) order terms as lowest order term.

Second order minors

In the case $m = 2$ we have only one minor involving second order terms which is M_{12}. Lemma 4.5.1 tells us that we always have to omit it.

Third order minors

Then, for each entropy pair (η_k, q_k) we have two minors M_{ik}, for $i = 1, 2$ that involve third order terms. These third order terms have a commutator-structure of the following form:

$$\left(\begin{array}{c} M_{1k}(\gamma) \\ M_{2k}(\gamma) \end{array} \right) \Bigg|_{\text{3rd order}} = D^2\eta_k(u, Dfu)u - D^2\eta_k(u,u)Dfu.$$

Fourth order minors

Similarly, the fourth order terms of the minors M_{ik} from above calculate on the left hand side of the minor relation as integral over

$$\begin{aligned} \left(\begin{array}{c} M_{1k}(\gamma) \\ M_{2k}(\gamma) \end{array} \right) \Bigg|_{\text{4th order}} =& \frac{1}{3} D^2\eta(u, D^2f(u,u))u - \frac{1}{4} D^2\eta(u,u)D^2f(u,u) \\ &+ \frac{1}{6} \left(D^3\eta(u,u,Df \cdot u)u - D^3\eta(u,u,u)Df \cdot u \right). \end{aligned}$$

Whereas on the right hand side we just have:

$$\left(\begin{array}{c} M_{1k}(\int \gamma d\nu) \\ M_{2k}(\int \gamma d\nu) \end{array} \right) \Bigg|_{\text{4th order}} = -\frac{1}{4} \int D^2f(u,u)d\nu \int D^2\eta(u,u)d\nu.$$

4.7 Examples for Compactness and Noncompactness for Conservation laws

In this part of the thesis we present three results on compactness in our new setting. The positive results (scalar conservation law and system of Lagrangian elasticity) that already were known ([Ev90], [DP85]) first. Then the negative example for a system of two equations admitted by only one entropy, which is a new result, Theorem 1.2.4.

The Scalar Conservation Law

We now look at the case of a genuinely nonlinear scalar conservation law admitted by one convex entropy, i.e. $m = k = 1$. That is looking for solutions u of

$$\partial_t u + \partial_x f(u) = 0, \tag{4.19}$$

$$\partial_t \eta(u) + \partial_x q(u) = 0, \tag{4.20}$$

where η is strictly convex entropy and q the corresponding entropy flux. The entropy condition (4.3) here just reads as $q' = f'\eta'$.

We assumed:

(a) genuine nonlinearity, i.e. $f''(0) \neq 0$ and

(b) strict convexity of the entropy, i.e. $\eta''(0) > 0$.

And considering our normal form for γ from Chapter 4.3 we may assume that:

$$\begin{aligned} \gamma(0) &= 0, \\ D\gamma(0)[u] &= \begin{pmatrix} u & f'(0)u \\ 0 & 0 \end{pmatrix}, \\ D^2\gamma(0)[u,u] &= \begin{pmatrix} 0 & f''(0)u^2 \\ \eta''(0)u^2 & q''(0)u^2 \end{pmatrix}, \\ D^3\gamma(0)[u,u,u] &= \begin{pmatrix} 0 & f'''(0)u^3 \\ \eta'''(0)u^3 & q'''(0)u^3 \end{pmatrix}. \end{aligned}$$

So for these 2×2 matrices we only have to consider the usual determinant as the only minor.

The necessary conditions are after the normalization of γ automatically fulfilled. It is $\det D\gamma(0) = 0$, which means that second order terms vanish. Also the third order terms do:

$$\begin{aligned} B(D\gamma(0)[u], D^2\gamma(0)[u,u]) &= \det \begin{pmatrix} u & f'(0)u \\ \eta''(0)u^2 & q''(0)u^2 \end{pmatrix} + \det \begin{pmatrix} 0 & f''(0)u^2 \\ 0 & 0 \end{pmatrix} \\ &= u^3(q''(0) - f'(0)\eta''(0)) = 0. \end{aligned}$$

In the last equation we used the identity $q'' = f''\eta' + f'\eta''$, which comes directly from the entropy condition, and the fact that $\eta'(0) = 0$.

For the fourth order terms we calculate similarly

$$\begin{aligned} \det D^2\gamma(0)[u,u] &= -u^4 f''(0)\eta''(0), \\ B(D\gamma(0)[u], D^3\gamma(0)[u,u,u]) &= 2u^4 f''(0)\eta''(0). \end{aligned}$$

We see that the necessary conditions from Proposition 4.5.4 are not satisfied. Still we can gain compactness: we look at the full minor relation (4.4) and plug in the Taylor expansion. As we only have one minor we can choose α conveniently such that $\alpha \cdot f''(0) \cdot \eta''(0) = 1$ and get

$$\frac{1}{12}\int u^4 d\nu + \int \mathcal{O}(u^5) d\nu = \frac{1}{4}\left(\int u^2 d\nu\right)^2 + \int u^2 d\nu \int \mathcal{O}(u^3) d\nu$$

as $u \to 0$. We assume $\nu \neq \delta_0$ and absorb the higher order terms for supp ν small by a part of the right hand side to obtain

$$\frac{1}{12}\int u^4 d\nu > \frac{1}{8}\left(\int u^2 d\nu\right)^2.$$

Now Jensen's inequality implies

$$\frac{1}{12}\left(\int u^2 d\nu\right)^2 > \frac{1}{8}\left(\int u^2 d\nu\right)^2,$$

which of course false. Hence ν must be a Dirac measure at 0.

System of Lagrangian Elasticity

Let

$$f^1(u_1,u_2) = -a(u_2) \qquad f^2(u_1,u_2) = -u_1. \tag{4.21}$$

With $a'(u_2) > 0$ the system is hyperbolic and $a''(u_2) \neq 0$ implies genuine nonlinearity. This system is admitted by the two entropy pairs:

$$\eta_1(u) = \frac{1}{2}u_1^2 + A(u_2) \qquad q_1(u) = -u_1 a(u_2) \text{ and} \tag{4.22}$$

$$\eta_2(u) = u_1 u_2 \qquad q_2(u) = -\frac{1}{2}u_1^2 - \tau(u_2), \tag{4.23}$$

where τ is the Legendre-transformation of $A(u_2) = \int^{u_2} a(w)dw$.

Their quadratic parts are:

$$Q\eta_1(u) = \frac{1}{2}u_1^2 + QA(u_2) \qquad Q^*q_1(u) = -u_1 a(u_2) \text{ and} \tag{4.24}$$

$$Q\eta_2(u) = u_1 u_2 \qquad Q^*q_2(u) = -\frac{1}{2}u_1^2 - \bar{Q}A(u_2). \tag{4.25}$$

This first order system comes from the quasilinear wave equation

$$w_{tt} - a(w_x)_x = 0$$

by introducing velocity $u_1 = w_t$ and strain $u_2 = w_x$. The quasilinear wave equation again is the Euler-Lagrange-equation of the variational functional

$$\int\int \frac{1}{2}w_t^2 - A(w_x)dxdt.$$

As the integrand of this functional is invariant under spatial and temporal translations, all smooth solutions of the first order system satisfy the two additional conservation laws: $\partial_t\eta_i + \partial_x q_i = 0$, for $i = 1, 2$. This comes from Noether's theorem, which states that any differentiable symmetry of the action of a physical system has a corresponding conservation law.

THEOREM 4.7.1. *If we have (4.4) for the system consisting of (4.21), (4.22) and (4.23) and* $\operatorname{supp}\nu$ *sufficiently small then* ν *is Dirac.*

As already mentioned earlier, this directly implies compactness.

Note that the proof given below does not use all the simplifications from Chapter 4.3. We just assume $\gamma(0) = 0$ and take the quadratic parts of the entropies. As Df is already off-diagonal a linear diagonalization will not simplify the calculations, so we skip that. In view of Remark 4.3.3 one can use the following formula to transform the linear combination for the diagonal case into the linear combination of the off-diagonal case:

$$\lambda \cdot M(\gamma) = \mu \cdot M(B \cdot \gamma)$$

$$\Leftrightarrow \mu = \begin{pmatrix} \frac{1}{ad-bc}\lambda_{12} \\ \frac{c}{bc-ad}\lambda_{23} - \frac{d}{bc-ad}\lambda_{13} \\ \frac{c}{bc-ad}\lambda_{24} - \frac{d}{bc-ad}\lambda_{14} \\ \frac{b}{bc-ad}\lambda_{13} - \frac{a}{bc-ad}\lambda_{23} \\ \frac{b}{bc-ad}\lambda_{14} - \frac{a}{bc-ad}\lambda_{24} \\ \lambda_{34} \end{pmatrix}$$

with the matrix

$$B = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2\alpha} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2\alpha} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

transforming the γ with off-diagonal Df into a diagonal at 0 (and $\alpha := \sqrt{a'(0)}$). So if the linear combination for the off-diagonal case is represented by the vector λ, the vector μ gives the same linear combination for the transformed, diagonalized case.

PROOF. The assumption that $\gamma(0) = 0$ means that $\bar{u}_1 = \bar{u}_2 = a(\bar{u}_2) = 0$. Let furthermore for notational convenience $\alpha := \sqrt{a'(0)}$ and $\beta := a''(0)$.

Straight forward calculation gives for the prefactors of the Taylor expansion of $\gamma(u)$:

$$D\gamma(0)[u] = \begin{pmatrix} u_1 & \alpha^2 u_2 \\ u_2 & u_1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$D^2\gamma(0)[u,u] = \begin{pmatrix} 0 & \beta u_2^2 \\ 0 & 0 \\ u_1^2 + \alpha^2 u_2^2 & 2\alpha^2 u_1 u_2 \\ 2u_1 u_2 & u_1^2 + \alpha^2 u_2^2 \end{pmatrix},$$

$$D^3\gamma(0)[u,u,u] = \begin{pmatrix} 0 & \delta u_2^3 \\ 0 & 0 \\ \beta u_2^3 & 3\beta u_1 u_2^2 \\ 0 & 2\beta u_2^3 \end{pmatrix}.$$

Let M denote the vector consisting of all minors but M_{12} and B the associated bilinear form. We skip M_{12} (and of course the companion B_{12}) as M_{12} is the only minor containing second order terms and as Lemma 4.5.1 forces the second order terms to be zero. So here $M = (M_{13}, M_{14}, M_{23}, M_{24}, M_{34})$ and B correspondingly. Computing now the relevant terms from the Taylor expansion gives the following third and fourth order summands:

$$B(D\gamma(0)[u], D^2\gamma(0)[u,u]) = \begin{pmatrix} \alpha^2 u_1^2 u_2 - \alpha^4 u_2^3 \\ u_1^3 - \alpha^2 u_1 u_2^2 \\ -u_1^3 + \alpha^2 u_1 u_2^2 \\ -u_1^2 u_2 + \alpha^2 u_2^3 \\ 0 \end{pmatrix},$$

$$M(D^2\gamma(0)[u,u]) = \begin{pmatrix} -\beta u_1^2 u_2^2 - \alpha^2 \beta u_2^4 \\ -2\beta u_1 u_2^3 \\ 0 \\ 0 \\ (u_1^2 - \alpha^2 u_2^2)^2 \end{pmatrix},$$

$$B(D\gamma(0)[u], D^3\gamma(0)[u,u,u]) = \begin{pmatrix} 3\beta u_1^2 u_2^2 - \alpha^2 \beta u_2^4 \\ 2\beta u_1 u_2^3 \\ 2\beta u_1 u_2^3 \\ 2\beta u_2^4 \\ 0 \end{pmatrix}.$$

The properly weighted fourth order terms in the expansion on the left hand side of (4.4) inside the integral are:

$$\frac{1}{4}M(D^2\gamma(0)[u,u]) + \frac{1}{6}B(D\gamma(0)[u], D^3\gamma(0)[u,u,u]) = \begin{pmatrix} \frac{1}{4}\beta u_1^2 u_2^2 - \frac{5}{12}\alpha^2 \beta u_2^4 \\ -\frac{1}{6}\beta u_1 u_2^3 \\ \frac{1}{3}\beta u_1 u_2^3 \\ \frac{1}{3}\beta u_2^4 \\ \frac{1}{4}(u_1^2 - \alpha^2 u_2^2)^2 \end{pmatrix}. \tag{4.26}$$

Our next task is to respect the necessary condition from Lemma 4.5.2. This means to find a vector $\lambda = (\lambda_{13}, \lambda_{14}, \lambda_{23}, \lambda_{24}, \lambda_{34})$, such that

$$\lambda \cdot B(D\gamma(0)[u], D^2\gamma(0)[u,u]) = 0.$$

As linear combinations that respect the necessary conditions we get

$$\lambda = (\alpha^2 \lambda_{24}, \lambda_{14}, \lambda_{14}, \lambda_{24}, 1).$$

Here we set without loss of generality $\lambda_{34} = 1$, as third order terms are not involved for the minor M_{34}.

Now we are only left with the fourth order terms in the minor relation. As all necessary conditions from Chapter 4.4 are satisfied at this point, we have to choose the remaining degrees of freedom of λ_{14} and λ_{24} in a clever way. In view of Proposition 4.5.4 we want to avoid indefiniteness. Looking at (4.26) we hence choose $\lambda_{14} = 0$. To weight out the β, we take $\lambda_{24} = \frac{2}{\beta}$.

Then we calculate the linear combination of the entries of (4.26) to obtain the surprisingly simple form:

$$\begin{aligned}&(2\frac{\alpha^2}{\beta}, 0, 0, 2\frac{\alpha^4}{\beta}, 1) \cdot (\frac{1}{4}M(D^2\gamma(0)[u,u]) + \frac{1}{6}B(D\gamma(0)[u], D^3\gamma(0)[u,u,u])) \\ =&\frac{1}{4}u_1^4 + \frac{\alpha^4}{12}u_2^4.\end{aligned}$$

Calculating both sides of (4.15) for the above linear combination of minors yields:

$$\int \left(\frac{1}{4}u_1^4 + \frac{\alpha^4}{12}u_2^4\right) d\nu + H = \frac{1}{4}\left(\int u_1^2 d\nu\right)^2 - \alpha^2 \left(\int u_1 u_2 d\nu\right)^2 - \frac{\alpha^4}{4}\left(\int u_2^2 d\nu\right)^2, \tag{4.27}$$

where H denotes all higher order terms. They turn out to have the following structure:

$$\begin{aligned} H = &\int u_1 u_2 d\nu \int \mathcal{O}(u_1 u_2^2) d\nu + \int u_1^2 d\nu \int \mathcal{O}(u_2^3) d\nu \\ &+ \int u_2^2 d\nu \int \mathcal{O}(u_2^3) d\nu + \int \mathcal{O}(u_1^2 u_2^3) d\nu + \int \mathcal{O}(u_2^5) d\nu. \end{aligned}$$

This H can be estimated in the following way:

$$|H| < \tilde{C} \cdot \int u_2^4 d\nu \cdot \sup\{u_2 : \operatorname{supp} \nu\}. \tag{4.28}$$

Hence all higher order terms from both sides can be absorbed by $C \int u_2^4 d\nu$, where the constant $C > 0$ depends only on the support of ν. The main ingredients to obtain (4.28) are Hölder's inequality, and the equation $\int u_1^2 d\nu \leq \mathit{const} \int u_2^2 d\nu$, which comes from the commutativity relation for M_{12}. Although the latter equation is a specific feature for the system of Lagrangian elasticity, from the proof of Proposition 4.5.4 for the sufficient condition we see that higher order terms can also be absorbed differently.

So the complete minor relation (4.15) including the 'absorbed' remainder and dropping terms that are negative on the right hand side can be written as:

$$\int \left(u_1^4 + C u_2^4\right) d\nu \leq \left(\int u_1^2 d\nu\right)^2,$$

with $C > 0$. The fact that Jensen's inequality is strict for strictly convex functions gives now that ν must be a Dirac measure at zero in the u_1 variable. Then

$$\int C u_2^4 d\nu \leq 0$$

gives that ν must also be Dirac at zero in the u_2 variable. This finishes the proof. $\square$

Systems of Two Equations with One Entropy Pair

Here

$$\gamma(u) = \begin{pmatrix} u_1 & f^1(u) \\ u_2 & f^2(u) \\ \eta(u) & q(u) \end{pmatrix}.$$

Omitting the only minor that involves second order terms M_{12} (Lemma 4.5.1 forces us to do so), we have that $M = \begin{pmatrix} M_{13} \\ M_{23} \end{pmatrix} =: \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$, where both entries are minors that involve third and higher order terms.

To satisfy the necessary condition from Lemma 4.5.2, we now look at these minors. As consequence of the entropy condition (4.3), which can be written as

$$\partial_l q = \partial_j \eta \partial_l f^j,$$

we have for the second derivative of q that

$$\partial_{kl} q = \partial_j \eta \partial_{kl} f^j + \partial_{jk} \eta \partial_l f^j.$$

Therefore the minors M_i may be calculated to be:

$$\begin{aligned} &\frac{1}{2} u^i \left(\partial_{kl} q u^l u^k - \partial_j \eta \partial_{kl} f^j u^l u^k \right) - \frac{1}{2} \partial_{jk} \eta \partial_l f^i u^l u^j u^k \\ =&\frac{1}{2} \partial_{jk} \eta \partial_l f^j u^i u^l u^k - \frac{1}{2} \partial_{jk} \eta \partial_l f^i u^j u^l u^k \\ =&\frac{1}{2} \partial_{jk} \eta u^k \partial_l f^j u^l u^i - \frac{1}{2} \partial_{jk} \eta u^j u^k \partial_l f^i u^l. \end{aligned}$$

Here we employ the Einstein summation convention, summing over j, k, l.

If we now take the two minors for $i = 1, 2$ we get the following vector valued expression for the third order terms:

$$D^2\eta(u, Dfu)u - D^2\eta(u, u)Dfu$$

– compare to Chapter 4.6. This has a commutator-like structure. As the system is assumed to be strictly hyperbolic, Df is linearly diagonalizable at zero with two distinct real eigenvalues, say have the form $B = \operatorname{diag}(\lambda, \mu)$. Furthermore $D^2\eta(0)[\cdot, \cdot]$ is a quadratic form, say $A(\cdot,\cdot)$. This comes from the fact that η is assumed to be a convex entropy.

With this notation the third order terms become:

$$\begin{aligned} A(u,u)Bu - A(u,Bu)u =& [A(u,u)Bu - A(u,Bu)u] \cdot e_1 \\ &+ [A(u,u)Bu - A(u,Bu)u] \cdot e_2 \\ =& (\lambda - \mu) \begin{pmatrix} A(u,u^2e^2)u^1 \\ A(u,-u^1e^1)u^2 \end{pmatrix}. \end{aligned}$$

As proposed in Lemma 4.5.2 we now take linear combinations of the form αM, where M is the vector in $\mathbb{R}^2$ consisting of the two minors M_i for $i = 1,2$ and $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$, with $\alpha_i \in \mathbb{R}$. Furthermore let $A = (a_{ik})_{i,k=1,2}$ correspond to the Hessian of the entropy η. Then the condition that 3rd order terms vanish reads as

$$\begin{pmatrix} a_{12} & -a_{11} \\ a_{22} & -a_{12} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0. \tag{4.29}$$

As for a convex entropy $D^2\eta(0)$ is not degenerate, i.e. $\det D^2\eta(0) \neq 0$, this is not solvable for $\alpha \neq 0$.

Thus we directly run into the case (4.13) from Chapter 4.4 on necessary conditions for compactness, and obtain the following theorem.

Theorem 4.7.2. *For a system of two strictly hyperbolic conservation laws with only one convex entropy/entropy flux pair one cannot gain compactness from looking at polyconvex measures.*

5 Open Problems

5.1 Active Scalar Equations

The chapter on active scalar equations is pretty well closed. Possible future directions include:

- First one should get rid of the technical assumption in Chapter 3.4 that $0 \in m(S^{n-1})$ and prove the following conjecture: if m is even, 0-homogeneous and $m(\xi) \perp \xi$ and such that $m(S^{n-1})$ has nonempty interior, then $0 \in m(S^{n-1})$. We already prove in Lemma 3.4.6 that 0 cannot lie outside $m(S^{n-1})$. Furthermore we were not able to find an example where $0 \in \text{int conv } m(S^{n-1})$.

- The cases for which we prove $K^\Lambda = K^{qc}$ (where m induces a quadratic norm) do have no fundamental differences in the calculation of K^Λ. Hence it is reasonable to assume that $K^\Lambda = K^{qc}$ holds for many more cases. Considering the way we handle cases for unbounded m, one can 'embed' in a first step graphs $\{m(\xi_i) : i \in I\}$ (notation as in Chapter 3.2.2) that induce quadratic norms again. Beyond this one cannot follow the proof we gave, which has two reasons. First, without a m that induces a quadratic norm one can not switch by Plancherel to the Fourier side as we did. Second, it is totally unclear if one stays in (x, t) coordinates (we did that in the incompressible porous media case) how the Div-Curl-Lemma generalizes. Therefore this is a difficult task.

- For odd multipliers only the SQG case is known to us (cf. Example 4 in Chapter 3.1). This is not possible to prove by convex integration as the plane wave solutions do not localize (one does not get an oscillation lemma like Lemma 3.3.1). Still it is an interesting question whether one can establish the same result for odd multipliers of the same generality or if the impact of being an even multiplier is greater than i9mposing that the plane waves are localized. (Looking at the proof in [Re95] this seems really hard if T becomes unbounded.)

5.2 Hyperbolic Conservation Laws

In the chapter on hyperbolic conservation laws, there are different points where the results we obtained are not fully satisfactory and should be improved. Here we list the most important points:

- First we regret that we do not have a necessary and sufficient condition for compactness from $\mathcal{P}^{pc}$. The sufficient condition we gave in Proposition 4.5.4 is too restrictive as already the example of the scalar conservation law tells us. But all of the three positive examples for compactness from $\mathcal{P}^{pc}$ – the scalar conservation law, the system of Lagrangian elasticity (both in Chapter 4.7), and the sufficient condition (Proposition 4.5.4) – have in common that the sum of the fourth order terms on the left hand side of (4.15), that is $\alpha \cdot \left(\frac{1}{4}D^2\gamma(0)[u,u] + \frac{1}{6}B(D\gamma(0)[u], D^3\gamma(0)[u,u,u])\right)$ is definite. So this (together with the necessary conditions from Lemmas 4.5.1 and 4.5.2) would be a natural candidate for another necessary condition or even a necessary and sufficient condition.

- Another object connected to any sufficient condition (let it be Proposition 4.5.4 or an improved version of it) would be to perform a dimension counting argument. In Chapter 4.3 we counted already the number of second, third and fourth order minors that come up in the minor relation (4.15). One then has to spend some degrees of freedom to make second an third order terms zero. This results in a number of degrees of freedom that is left to satisfy the sufficient condition, which always will be a condition on definiteness of fourth order terms. In [GL64] one has an algorithm to check positive definiteness of quartics. We want to remark here that given a certain number of entropies it is always possible to satisfy the conditions from Proposition 4.5.4.

- Of course generalizing the assertions from Chapter 4.6 on systems of two equations to arbitrary systems is a step that might be useful for further calculations.

- Another important task is to investigate the case of noncompactness from $\mathcal{P}^{pc}$ in more detail. In which cases is it here possible to construct counterexamples to compactness in the class of laminates $\mathcal{P}^{rc}$? Here one can distinguish two cases. First, if the necessary conditions from Lemmas 4.5.1 and 4.5.2 cannot be satisfied. In the proofs of these lemmas we have a method of finding a measure that contradicts the compactness from $\mathcal{P}^{pc}$. Now one can ask if this measure belongs also to $\mathcal{P}^{rc}$. Second, would be noncompactness from $\mathcal{P}^{pc}$ in the case in which our necessary conditions are satisfied. Here it is by now open how to proceed.
  In this context, it would also be interesting to determine cases in which one has $\mathcal{P}^{pc} = \mathcal{P}^{qc}$ or $\mathcal{P}^{rc} = \mathcal{P}^{qc}$ (similar to considering the cases where $K^{\Lambda} = K^{qc}$ in the chapter on active scalar equations – for hyperbolic conservation laws we have that $K^{\Lambda} = K^{rc}$).

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