Large Deviations for Brownian Intersection Measures

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Introduction

Let us start with a number of independent Brownian motions from the origin in $\mathbb{R}^d$. Can one expect them to meet again in the future? We are interested in points which are hit by all the motions, possibly at different time points. Dvoretzky, Erdös, Kakutani and Taylor answered this question in a series of papers (see [DEK50], [DEK54] and [DEEKT57]). It turns out that arbitrarily many paths intersect in two dimensions, while in three dimensions, at most two paths intersect. The set of intersections is a rather peculiar random object and has been studied by many authors ([Ta64] , [Fr67],[DP00a],[DP00b],[DP00c],[DP00d]).
An upsurge in the research activities started with the construction of a natural measure which sits on the above mentioned intersection set, an object which counts how intense the paths mutually intersect. This object was called the *intersection local time*. Historically, the notion of this object goes back to physics. In an appendix to a paper by Symanzik ([Sy69]), Varadhan constructed a similar object for planar Brownian bridges. Later, Dynkin ([Dy81]) gave a general construction for the case of additive functionals of Markov processes. Geman, Horowitz and Rosen ([GHR84]) first carried out a rigorous construction of this measure. Heuristically, this can be written as, for every Borel set $A \subset \mathbb{R}^d$,

$$\ell_t(A) = \int_A dy \quad \prod_{i=1}^p \int_0^{t_i} ds \quad \delta_y(W_s^{(i)}) \qquad t = (t_1, \dots, t_p) \in (0, \infty)^p. \tag{0.0.1}$$

Although in $d \geq 2$, the above symbolical formula needs a rigorous justification, symbolically, the density of ℓ_t is the p-fold pointwise densities of each occupations measure, which is defined as

$$\ell_t^{(i)}(A) = \int_0^t \mathrm{d}s \, \mathbb{1}_A(W_s^{(i)}) \qquad i = 1, \dots, p.$$

This is one of the salient features of intersection local time measures: how far can these be understood as the product of the occupation measures? It is one of the goals of this thesis to answer this question in the large-t limit in terms of *large deviations*.

In the early 1970s, Donkser and Varadhan investigated the large-t behavior of the occupation time measures $\ell_t^{(i)}$. Although in $d \geq 2$, $\ell_t^{(i)}$ fails to have a density, it turns out that only in the limit $t \uparrow \infty$ the densities appear: Let $\mathcal{M}_1(B)$ be the space of probability measures on a bounded set B of $\mathbb{R}^d$ and τ_i be the exit time of the ith motion from B. Then, for $\mu \in \mathcal{M}_1(B)$, under the sub-probability density $\mathbb{P}_t(\cdot) = \mathbb{P}(\cdot \cap \{\tau_i > t\})$,

$$\lim_{t\uparrow\infty} \frac{1}{t} \log \mathbb{P}_t\bigg(\frac{1}{t}\ell_t^{(i)} \approx \mu^{(i)}\bigg) = -I(\mu^{(i)}) \qquad i = 1, \dots, p.$$

where

$$I(\mu^{(i)}) = \begin{cases} \frac{1}{2}\big\|\nabla\psi_i\big\|_2^2 & \text{if } \psi_i^2 := \frac{\mathrm{d}\mu^{(i)}}{\mathrm{d}x} \in H_0^1(B) \\ \infty & \text{else.} \end{cases}$$

Therefore, ψ_i^2 appears as *asymptotic density* of the ith occupation time distribution functional. The heuristic formula (0.0.1) suggests that the pointwise product $\prod_{i=1}^p \psi_i^2$ should describe the large-t density of ℓ_t. Theorem 3.1.2, the main result of this thesis makes precise this statement.

A second version of our principle is proved for the motions observed until the individual exit times $\tau_1, \dots, \tau_p$ from B, i.e., the time horizon $(t_1, \dots, t_p) \in (0, \infty)^p$ gets replaced by $(\tau_1, \dots, \tau_p)$ and subsequently ℓ_t transforms to $\ell = \ell_\tau$. König/Mörters ([KM02], [KM06]) studied the upper tails of ℓ in a compact set $U \subset B$. They showed that under the conditional measure $\mathbb{P}(\cdot\,|\ell(U) > a)$, the intersection measure $\ell/\ell(U)$ satisfies a law of large numbers, as the intersection mass $a \uparrow \infty$. In Theorem 3.2.1 we characterise the precise exponential rate of the convergence in terms of a large deviation principle.

Now we quickly describe how this thesis is organised. Chapter 1 deals with some classical facts about the Brownian intersection set and surveys on several constructions of the intersection local times. In the Chapter 2, we survey existing results for the total intersection local time on sets in $\mathbb{R}^d$. We particularly spell out results by X. Chen ([C09]) for large-t asymptotics of the total mass $\ell_t(\mathbb{R}^d)$ and by König and Mörters ([KM02], [KM06]) for large large-a asymptotics of the tails $\{\ell(U) > a\}$. Chapter 3 presents our large deviation results for intersection measures $\frac{\ell_t}{t^p}$ under $\mathbb{P}(\cdot \cap \{\tau_i > t\,\forall i\})$ as $t \uparrow \infty$, as well as for $\ell/\ell(U)$ under $\mathbb{P}(\cdot\,|\ell(U) > a)$ as $a \uparrow \infty$. We also summarise similar results for random walk intersections. though we are rather sketchy here as these are much simpler to deduce, (0.0.1) is meaningful for random walks. We also present the heuristic ideas which lead one to the main results. In Chapter 4 we present the proof of our main result, Theorem 3.1.2, modulo the proof of an super-exponential approximations of the intersection local times. Chapter 5 presents some moment formulas and combinatorial tricks used for the proof of the main estimate, Proposition 4.4.1, which we finish in Chapter 6. In Chapter 7, we pass from fixed time horizon t to random time horizon τ and derive the lower and upper bounds for Theorem 3.2.1. We conclude with some interesting open problems in this field of research.

Contents

Chapter 1

Brownian intersection local times.

1.1 Brownian Intersections.

Let us consider p independent Brownian motions $W^{(1)}, \dots W^{(p)}$ running in $\mathbb{R}^d$. We are interested in the random set of points in the space where these paths intersect:

$$S_b = \bigcap_{i=1}^{p} W^{(i)}[0, b_i) \qquad b = (b_1, \dots, b_p) \in (0, \infty)^p.$$

In other words, S_b is the intersection of the Brownian paths, the set of space points which are hit by all p motions before time $b_1 \wedge \cdots \wedge b_p$. S is a random set with high complexity and has kept many mathematicians interested. One of the classical results concerning this set is due to Dvoretzky, Erdös, Kakutani and Taylor (see [DEK50], [DEK54] and [DEEKT57]) which says that with probability one, S has points (other than the starting point, possibly) if and only if

$$p < \frac{d}{d-2}.$$

In other words, the intersection set is non-trivial if and only if

$$d = 2,\ p \geq 2 \quad \text{arbitrary} \qquad d = 3, \quad p = 2.$$

Throughout this work we'll be working with the above mentioned cases.

Subsequent research due to Taylor (see [Ta64]) and Fristedt (see [Fr67]) showed

- S is a Lebesgue-null set in $d \geq 2$ almost surely,
- $$\dim(S) = \begin{cases} 2 & \text{for } d = 2 \text{ and } p \in \mathbb{N} \\ 1 & \text{for } d = 3 \text{ and } p = 2 \\ 2 & \text{for } d \geq 2 \text{ and } p = 1. \end{cases} \tag{1.1.1}$$

where dim refers to the Hausdorff dimension.

Heuristically thinking, these results are not hard to believe. Indeed, it is known that a Brownian curve has Hausdorff dimension 2 and can be roughly thought of as a plane filling curve. If we have a number of intersecting planes in two dimensions, the intersection set happens to be a plane itself, which is of dimension two. Similarly, in dimension three, two planes intersect along a line, which is of dimension one. However, the geometry of S is quite non-trivial. For an example, it is an open problem to prove (or disprove) that S is totally disconnected in $\mathbb{R}^2$ (in $d = 3$, S has co-dimension 2 and hence can not have any connected components). There are many more finer results pertaining to S (see the last chapter of [MP10] for a comprehensive collection of open problems). But this would be outside the purview of our discussion as we now turn to our object of interest, a measure on S, the Brownian intersection local times.

1.2 Brownian intersection local times: definition and properties.

As mentioned in the introduction, we are interested in a measure on S_b which computes the intensity of path intersections in S_b. This measure ℓ_b can be formally defined as, for every Borel set $A \subset \mathbb{R}^d$,

$$\ell_b(A) = \int_A dy \quad \prod_{i=1}^p \int_0^{b_i} ds \quad \delta_y(W^{(i)}(s)) \qquad b = (b_1, \ldots, b_p) \in (0, \infty)^p. \tag{1.2.1}$$

Hence, informally ℓ_b is the pointwise product of the densities of the p occupation measures on the individual time horizons. This definition is rigorous in dimension $d = 1$, as the occupation measures of the motions have almost surely a density, which is jointly continuos in the space and the time variable. However, in $d \geq 2$, the occupation measures fail to have a density. Therefore, the above heuristic formula for ℓ_b needs an explanation, respectively a rigorous construction. We briefly review some constructions of this measure.

1. Confluent Brownian motions and Brownian intersection local times.
Geman, Horowitz and Rosen ([GHR84]) first carried out a rigorous constructio of ℓ_b as follows. They considered a random field, called the *confluent Brownin motion*, whose zero set, by definition, corresponds to time points when the intersections of p motions occur. It turns out that the occupation measure of the confluent Brownian motion carries a Lebesgue density. Keeping up with the notion of a local time, this (projected) density was called the Brownian intersection local time. To outline the complete picture, we spell out some details.

Let us fix two natural numbers N and D. Let $X : \mathbb{R}_+^N \to \mathbb{R}^D$ be a Borel function. Fix a Borel set A in $\mathbb{R}_+^N$. Then we can define the *occupation measure* of X relative to A by

$$\mu_A(B) = \lambda_N(A \cap X^{-1}(B)) \qquad \forall B \text{ Borel in } (\mathbb{R}^D).$$

If $\mu_A \ll \lambda_d$, we write

$$\alpha(y, A) = \frac{d\mu_A}{d\lambda_D}(y) \qquad \forall y \in \mathbb{R}^D$$

for the corresponding Radon-Nikodym derivative. This function $\alpha(y, A)$ is called the *occupation density* or the *local time* on A with respect to the Borel function X. If there is an occupation density for each A then we may choose $\alpha(y, A)$ to be a kernel (i.e., measurable in y and a finite measure in A).

Now the above general set up can be applied to a particular situation, namely, the *confluent Brownian motion* $W : \mathbb{R}_+^p \to \mathbb{R}^{d(p-1)}$ which is defined by

$$W(s_1, s_2, ..., s_p) = \big(W_1(s_1) - W_2(s_2), W_2(s_2) - W_3(s_3), ..., W_{p-1}(s_{p-1}) - W_p(s_p)\big). \tag{1.2.2}$$

It was shown ([GHR84]) that with probabiltity one, the occupation density $\alpha(y, A)$ for the confluent Brownian motion process W exists for every Borel set A in $\mathbb{R}_+^p$ and may be chosen so that $(y, t) \mapsto \alpha(y, Q_t)$ is jointly continuous, where $Q_t = \prod_{i=1}^p [0, t_i]$.

This implies that, with probability one, there is a family $\{\mu_y\colon y \in (\mathbb{R}^{d(p-1)})\}$ of finite measures on $\prod_{i=1}^p [0,T_i)$ such that

(i) The mapping $y \mapsto \mu_y$ is continuous with respect to the vague topology on the space $\mathbb{M}(\mathbb{R}^p)$ of locally finite measures on $\mathbb{R}^p$.

(ii) For all Borel functions $g\colon \mathbb{R}^{d(p-1)} \to [0,\infty]$ and $f\colon \Pi_{i=1}^p[0,T_i) \to [0,\infty]$

$$\int g(y)\langle f,\mu_y\rangle\,dy = \int_{\prod_{i=1}^p[0,T_i)} f\cdot g(W)\,ds_p...ds_1.$$

It follows from the above two properties that, for each y, the measure μ_y is supported by the level set

$$M_y = \{(s_1,s_2,...,s_p)\in\prod_{i=1}^p[0,T_i)\colon W(s_1,s_2,...,s_p)=y\}.$$

Note that M_0 is the set of time vectors at which the p motions coincide, which is the set we are interested in. Now we consider the mapping $T\colon \prod_{i=1}^p[0,T_i)\to\mathbb{R}^d$ defined by $T(t_1,t_2,...,t_p)=W_1(t_1)$. Then

$$T(M_0) \;=\; S.$$

Now for every Borel set B in S, define

$$\ell(B) = \mu_0(T^{-1}(B)),$$

i.e., ℓ is the image measure of μ_0 under T. The measure ℓ on S defined above is called the *Brownian intersection local time* of the p Brownian motions.

2. Wiener Sausages and Brownian intersection local time.

A much simpler construction of ℓ was carried out by Le Gall (see [LG86]) using Wiener sausages, collection of tubular neighborhoods around the Brownian path. The renormalised Lebesgue measure on the intersection of the sausages approach a measure as the sausage intersection shrinks to the path intersection. The limiting measure coincides with the Brownian intersection local time, previously constructed by Geman, Horowitz and Rosen. We formulate this discussion more precisely. For every $\varepsilon>0$, we define the *Wiener sausage* around each W_i by

$$S_\varepsilon^{(i)} = \{x\in\mathbb{R}^d\colon \text{there is } \ t\in[0,b_i) \ \text{ with } \ |x-W_i(t)|<\varepsilon\} \qquad i=1...,p$$

and take their intersection

$$S_\varepsilon = \bigcap_{i=1}^p S_\varepsilon^{(i)}.$$

We observe that $S=\bigcap_{\varepsilon>0}S_\varepsilon$. Now, for every $\varepsilon>0$, we consider the normalised Lebesgue measure ℓ_ε on $\mathbb{R}^d$ by

$$d\ell_\varepsilon(y) = s_d(\varepsilon)\cdot 1_{S_\varepsilon}(y)\,dy$$

where

$$s_d(\varepsilon) = \begin{cases} \pi^{-p}\log^p(\frac{1}{\varepsilon}) & \text{if } d=2\\ (2\pi\varepsilon)^{-2} & \text{if } d=3 \text{ and } p=2\\ \frac{2}{\omega_d(d-2)}\,\varepsilon^{2-d} & \text{if } d\geq 3 \text{ and } p=1.\end{cases}$$

Then it turns out that the limit $\varepsilon \downarrow 0$ yields the Brownian intersection local time.

$$\lim_{\varepsilon \to 0} \ell_\varepsilon(A) = \ell(A) \qquad \text{in } L^q(\mathbb{P}) \text{ for any } q \in [1, \infty), \tag{1.2.3}$$

for every $A \subset \mathbb{R}^d$ which is ℓ-continuous (i.e., $\ell(\partial A) = 0$) where ℓ is the (projected) intersection local time measure defined in the previous approach of Geman, Horowitz and Rosen.

To understand this construction better, we look at a simpler example. For a smooth curve $\gamma : [0,1] \to \mathbb{R}^2$ we have:

$$\lim_{\varepsilon \to 0} \frac{\lambda(\gamma_\varepsilon)}{2\varepsilon} = L(\gamma) \tag{1.2.4}$$

where

$$\gamma_\varepsilon = \{y \in \mathbb{R}^2 \ : \ |y - \gamma(t)| \leq \varepsilon \text{ for some } t \in [0,1]\}$$

is the ε-sausage around γ and and $L(\gamma)$ is the length of γ. For $d = 2$ and $p = 1$, according to (1.2.3)

$$\lim_{\epsilon \to 0} \frac{\lambda(S_\varepsilon \cap A)}{\frac{\pi}{\log \frac{1}{\varepsilon}}} = \ell(A). \tag{1.2.5}$$

A comparison between (1.2.4) and (1.2.5) reinforces the fact that the intersection local time is a measure of intersection intensities. The reasons for a simpler normalizing constant in the first case is attributed to the non-differentiablity of the Brownian path as well as the difference of the co-dimensions (for γ the co-dimension is $(2-1) = 1$ which appears as the exponent of ε in the denomenator, whereas the co-dimension of the Brownian curve is $(2-2) = 0$).

3. Hausdorff measures and Brownian intersection local times.

Le Gall ([LG87],[LG89]) showed that ℓ also coincides with (a multiple of) a Hausdorff measure induced by a suitably chosen gauge function. This approach was motivated by a problem concerning the size of the Brownian intersection set. Recall that the Hausdorff dimension of the intersection of any number of motions in $\mathbb{R}^2$ is 2. It is intuitively clear that the size of the intersection of p motions should be bigger than that of $p+1$ motions. This heuristic observation leads to the consideration of a Hausdorff measure on the intersection set induced by a suitable gauge function. More precisely, if $g_r(x) = x^2(\log \frac{1}{x})^r$ for any $r \in \mathbb{R}$, then in $d = 2$ we have

$$\mu_{g_r}(S) = \begin{cases} 0 & \text{if } r < p \\ \infty & \text{if } r > p \end{cases} \tag{1.2.6}$$

where μ_{g_r} is the g_r- Hausdorff measure. This result was conjectured by Taylor in [Ta73] and proved by Le Gall in [LG86]. The techniques used in [LG86] yields a similar result in $d = 3$. If $f_r(x) = x\left(\log \frac{1}{x}\right)^r$, then

$$\mu_{f_r}(S) = \begin{cases} 0 & \text{if } r \leq 0 \\ \infty & \text{if } r > 0. \end{cases} \tag{1.2.7}$$

It is worth making a comment about the case $p = 1$. The Hausdorff measure of the image of a single Brownian path was studied by many authors (see [Le53]). The correct gauge function was determined by Ciesielski and Taylor (see see [CT63]) for $d \geq 3$ and Taylor (see [Ta66]) for $d = 2$. The function is given by:

$$h(x) = \begin{cases} x^2\left(\log \frac{1}{x} \log\ \log\ \log \frac{1}{x}\right) & \text{if } d = 2 \\ x^2\left(\log\ \log \frac{1}{x}\right) & \text{if } d \geq 3. \end{cases}$$

However, we shall focus on the case of more than one motion in appropriate dimension. The results (1.2.6) and (1.2.7) were improved by the same author in [LG87] by computing a "correct" gauge function g such that the g-measure of S is positive and the measure is σ-finite. In fact, it turns out that for some positive constants C and C'

$$C\,\ell(A) \le \mu_{g_p}(A\cap S) \le C'\,\ell(A) \tag{1.2.8}$$

where

$$g_p(x) = \begin{cases} x^2\left(\log\,\log\,\log\,\log\frac{1}{x}\right)^p & \text{if } d=2\ ,\ p\in\mathbb{N} \\ x\left(\log\,\log\frac{1}{x}\right)^2 & \text{if } d=3,\ p=2 \end{cases}$$

and ℓ is the (projected) intersection local time of the Brownian paths.
The key argument leading to the above result is Le Gall's moment formula (detailed in section 5.1) and the existence of two postive constants M and M' (depending only on the d and p) such that for $d=2$,

$$M^k\ (\log R)^{pk}\,(k!)^p \le \mathbb{E}\left[\left(\ell(B(0;1))\right)^k\right] \le M'^k\ (\log R)^{pk}\,(k!)^p \qquad k\in\mathbb{N}$$

where it is assumed that the Brownian motions run until their first individual exit time from a fixed ball of radius R with $2\le R<\infty$. For $d=3$ and $p=2$,

$$M^k(k!)^2\,m \le \mathbb{E}\left[\left(\ell(B(0;1))\right)^k\right] \le M'^k(k!)^2 \qquad k\in\mathbb{N}.$$

The above result was farther sharpened by the same author in [LG89] by showing that with probability one, the intersection local time ℓ is exactly equal to a constant multiple of the g_p-Hausdorff measure on S:

$$\ell(A) = C_p\,\mu_{g_p}(A\cap S) \quad \text{for every } A\in\mathbb{B}(\mathbb{R}^2)$$

for some constant C_p.

From the above result it follows that the Hausdorff dimension of S is 2 for $d=2$ and $p\in\mathbb{N}$ since the exponent of x in the gauge function g_p is also 2 (of course, it contains some log terms too, but they do not influence the dimension). The same argument accounts for a similar result in the three dimensional case with two motions.

Furthermore, it is worth observing that ℓ is a random object which is equal to a Hausdorff measure induced by a suitable gauge function. Remarkably, the gauge function is non-random and depends on p and d in a rather simple manner.

4. Brownian intersection local time and convolutions of occupation measures.

As we shall see later on, we approach the intersection local times, by the product of convolved occupation measures of each path. In fact, let φ_ε be a smooth function that approximates the Dirac measure at 0 and $\ell^{(i)}_{\varepsilon,t}$ be i-th convolved occupation measure, i.e.,

$$\ell^{(i)}_{\varepsilon,t}(y) = \int \mathrm{d}s\,\varphi_\varepsilon(W^{(i)}_s - y).$$

Then we show that $\ell_{\varepsilon,t}(y) = \prod_{i=1}^p \ell^{(i)}_{\varepsilon,t}(y)$ converges ℓ_t in L^k for any $k\in\mathbb{N}$ (we in fact show that this convergence is exponential), see Section 4.2 and Proposition 4.4.1 for details.

Chapter 2

Asymptotic results for large total mass

2.1 Large-t asymptotics.

The large-t behaviour of the ISLT $\ell_t(\mathbb{R}^d)$ has been studied by X. Chen in a series of papers, see his monography [C09] for a comprehensive summary of these results and the concepts of the proofs and much more related material. The main result [C09, Theorem 3.3.2] is

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\ell_t(\mathbb{R}^d) > \gamma t^p) = -\gamma^{2/d(p-1)}\chi, \qquad \gamma > 0, \tag{2.1.1}$$

where

$$\chi = \inf \Big\{ \frac{p}{2} \|\nabla \psi\|_2^2 \colon \psi \in H^1(\mathbb{R}^d), \|\psi\|_{2p} = 1 = \|\psi\|_2 \Big\}. \tag{2.1.2}$$

As we will explain in more detail in Section 3.4, the term ψ^2 informally plays the role of the normalised occupation measure density of any of the p motions, and ψ^{2p} the one of the intersection measure $t^{-p}\ell_{t\mathbb{1}}$. This is one of the main features of intersection measures: How much rigorous meaning can be given to the intersection measure as a pointwise product of the occupation measures of the p motions? The above result indicates that some heuristic sense can be given in terms of a large-t limit in the interpretation of the characteristic variational formula. We shall come back to this question a bit later.

The number χ is also related to upper tail asymptotics of *self intersection local times* of discrete time random walks (see [K10, (1.12),(1.13)]).

2.2 Upper tail asymptotics.

Let us turn to the following question: How do the Brownian paths behave optimally when they are forced to produce a large intersection in certain region in the space? This question concerns extremely "thick" parts of the space i.e. random points having neighborhoods where ℓ piles up huge mass with high probability. ℓ being a measure intersection intensities, the above question boils down to studying probabilities of *upper tail events* of the form $\{\ell(U) > a\}$, for compact sets U. König and Mörters (see [KM02] and [KM06]) found out the logarithmic decay rate of these tail probabilities. Let $B \subset \mathbb{R}^d$ be an open bounded set and the motions run until their first individual exit times $\tau_1, \dots, \tau_p$ from B and are killed upon exiting B. We replace ℓ_t by $\ell = \ell_{\tau_1,\dots,\tau_p}$ (i.e., the time horizon $(t_1, \dots, t_p)$ gets replaced by $(\tau_1, \dots, \tau_p)$). We add that B could possibly be the whole space $\mathbb{R}^d$ for $d \geq 3$ while for $d = 2$ we assume that B is bounded (owing to the recurrence issues). We have the following Sobolev space:

$$\mathcal{D}(B) = \begin{cases} H_0^1(B) & \text{if } B \text{ is bounded} \\ D^1(\mathbb{R}^d) & \text{if } B = \mathbb{R}^d, \end{cases} \tag{2.2.1}$$

where $D^1(\mathbb{R}^d)$ is the Sobolev space of functions in $\mathbb{R}^d$ which vanish at infinity with their distributional gradient lying in $L^2(\mathbb{R}^d)$. Now we fix an open bounded set U in $\mathbb{R}^d$ such that U is compactly contained in B (i.e., $\overline{U} \subset B$). Then the main result in [KM02] and [KM06] can be summarised as

$$\lim_{a\to\infty} \frac{1}{a^{1/p}} \log \mathbb{P}\left[\ell(U) > a\right] = -\Theta(U) \tag{2.2.2}$$

where

$$\Theta(U) = \inf\{\frac{p}{2}\,||\nabla\psi||_2^2 \;:\; \psi \in \mathcal{D}(B), ||1_U\psi||_{2p}^2 = 1\}. \tag{2.2.3}$$

It turns out that the variational formula (2.2.3) admits minimiser(s) ψ which solve the Euler-Lagrange equation:

$$\triangle\psi(x) = -\frac{2}{p}\Theta(U)\psi^{2p-1}(x)\, 1\!\!1_U(x) \text{ for } x \in B\backslash\partial U. \tag{2.2.4}$$

We note that ψ is harmonically extended to B outside U. For $p > 1$ and general domains U, the question if the minimiser to (2.2.3) (equivalently, solution to (2.2.4)) is unique is still open. However, in $\mathbb{R}^3$, if U happens to be the unit ball $B(0;1)$ around the origin, thanks to the rotational symmetry of the Laplacian, ψ turns out to be a unique solution to an ordinary differential equation (see Theoem 1.3 in [KM02]).

However, it is worth looking at the case $p = 1$, where (2.2.4) is a linear eigenvalue problem and by the Rayleigh-Ritz principle, its unique (up to constant multiples) solution is the eigenvector corresponding to the principal eigenvalue of a compact symmetric L^2 operator. Moreover, for $p = 1$, the intersection local time is the just the occupation measure for a single Brownian path whose large t-asymptotics are well known. Indeed each occupation measure

$$\ell_t^{(i)}(A) = \int_0^t ds\, 1\!\!1_A(W_s^{(i)}) \text{ for } A \subset \mathbb{R}^d,$$

according to Donsker-Varadhan theory, satisfies a large deviation principle under $\mathbb{P}_t(\cdot) := \mathbb{P}(\cdot \cap t < \tau_i)$:

$$\mathbb{P}_t\left[\frac{\ell_t^{(i)}}{t} \approx \mu\right] = \exp\left[-t\frac{1}{2}\left\|\nabla\sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}x}}\right\|_2^2 + o(t)\right]. \tag{2.2.5}$$

The heuristic formula (1.2.1) suggests,

$$\ell_\tau(\mathrm{d}x) = \prod_{i=1}^{p} \frac{\mathrm{d}\ell_{\tau_i}^{(i)}(\mathrm{d}x)}{\mathrm{d}x}. \tag{2.2.6}$$

We can write:

$$\begin{aligned} \mathrm{e}^{-a\frac{p}{2}\|\nabla\psi\|_2^2} &= \mathrm{e}^{-p\; a\|\psi\|_2^2\;\frac{1}{2}\|\nabla\frac{\psi}{\|\psi\|_2}\|_2^2} \\ &\overset{(2.2.5)}{\approx} \mathbb{P}\left[\forall i = 1,\ldots,p,\; \frac{\ell^{(i)}_{a\|\psi\|_2^2}}{a\|\psi\|_2^2} \approx \frac{\psi^2(\cdot)}{\|\psi\|_2^2} \text{ in } B\right] \\ &\overset{(2.2.6)}{\approx} \mathbb{P}\left[\ell_{a\|\psi\|_2^2} \approx a^p\psi^{2p}(\cdot) \text{ in } B\right] \\ &\overset{\sup\psi:\int_U\psi^{2p}=1}{\approx} \mathbb{P}\left[\ell_{a\|\psi\|_2^2}(U) \approx a^p\right]. \end{aligned} \tag{2.2.7}$$

This is not too far from (2.2.2). The upshot of this calculation brings us to one of the basic features of intersection measures: If ψ^2 is the asymptotic density of each occupation time in U, then ψ^{2p} is the asymptotic density of the intersection local time in U. This was partially made precise (Theorem 1.3 [KM06]) in terms of a law of large numbers. Indeed, if d is a metric (in the weak sense) on the space $\mathcal{M}_U(B)$ of measures on B whose restriction to U is a probability measure and $\mathcal{M} = \{\mu \in \mathcal{M}_1(U) : \mu(\,dx) = \psi^{2p}(x)\mathrm{d}x\ \psi$ minimizes (2.2.3)$\}$ is the set of minimising functions ψ^{2p} and $\varepsilon > 0$, then under the conditional law $\mathbb{P}\{\cdot\ |\ell(U) > a\}$, $L := \ell/\ell(U)$ satisfies a law of large numbers:

$$\lim_{a\uparrow\infty} \mathbb{P}[d(L,\mathcal{M}) > \varepsilon|\ell(U) > a] = 0. \tag{2.2.8}$$

However, the logarithmic decay rate of these probabilities were not determined. In fact [KM06] failed to show that this convergence is exponential in $a^{1/p}$, and their proof was not a consequence of a large-deviation principle. It was the goal of [KM06] to get full control on the shape over $\ell/\ell(U)$ under $\mathbb{P}(\cdot \mid \ell(U) > a)$ in terms of asymptotics for test integrals against many test functions, but the technique used there (asymptotics for the k-th moments) turned out not to be able to give that; the technique precluded functions that assume negative values. This brings us to the start of our work. But before we formlate our mai results, we would like to finish this preamble with two interesting applications carried out in [KM02] and [KM06].

1) Exponential moments. From theorem (2.2.2), one easily defers a necessary and sufficient condition for the integrability of the random variable $\exp(\ell(U)^{\frac{1}{p}})$. More precisely,

$$\mathbb{E}\Big(\exp\big(\ell(U)^{1/p}\big)\Big)\begin{cases} < \infty & \text{if } \Theta(U) > 1\\ = \infty & \text{if } \Theta(U) < 1.\end{cases}$$

Indeed, for $g = \ell(U)^{\frac{1}{p}}$,

$$\mathbb{E}(\mathrm{e}^g) > \mathrm{e}^a\,\mathbb{P}\big(g > a\big) = \mathrm{e}^{a(1-\Theta(U)+o(1))}.$$

This implies $\mathbb{E}(\mathrm{e}^g) = \infty$ if $\Theta(U) < 1$.

Conversely, if $\Theta(U) > 1$,

$$\begin{aligned}\mathbb{E}(\mathrm{e}^g) &\le \int_0^\infty \mathrm{d}x\,\mathbb{P}(g > x)\mathrm{e}^x\\ &= \int_0^\infty \mathrm{d}x\,\mathrm{e}^{x(1-\Theta(U)+o(1))}\\ &< \infty.\end{aligned}$$

This question was first answered for the case $p = 1$ by Pinsky ([Pi86]) and was left open there for $p \ge 2$. This was generalized by König and Mörters (see Theorem 1.1 [KM06]): if $\phi_1, \dots, \phi_n$ are bounded non-negative Borel functions with compact support in B and if

$$\Theta(\phi_1,\dots,\phi_n) = \inf\Big\{\frac{p}{2}||\nabla\psi||_2^2\ :\ \psi \in \mathcal{D}(B),\ \sum_{i=1}^n ||\phi_i\psi||_{2p}^2 = 1\Big\},$$

then,

$$\mathbb{E}\Big[\exp\big(\sum_{i=1}^n \langle\phi_i^{2p},\ell\rangle^{\frac{1}{p}}\big)\Big]\begin{cases} < \infty & \text{if } \Theta(\phi) > 1\\ = \infty & \text{if } \Theta(\phi) < 1\end{cases} \tag{2.2.9}$$

Furthermore, (2.2.2) was extended to show that

$$\lim_{a\to\infty} \frac{1}{a} \log \mathbb{P}\Big[\sum_{i=1}^{n} \langle \phi_i^{2p}, \ell\rangle^{\frac{1}{p}} > a\Big] = -\Theta(\phi_1, \dots, \phi_n) \tag{2.2.10}$$

where for a function f and a measure μ, $\langle f, \mu\rangle$ denotes the integral $\int f\, d\mu$.
However, the main drawback was, as we mentioned earlier, to extend these results to full generality by considering all test functions with positive and negative values.

2) Hausdorff dimension spectrum for thick points. As mentioned at the beginning of this section, the aymptotic behavior of the tails events $\{\ell(U) > a\}$ determines the behavior of the Brownian sample paths in the space regions where they produce an untypically thick intersection. One could ask, *how many* such regions exist in the space? This was carried out by determining the size of the *thick points* of the space, i.e., points which have a neighbourhood around which the mass of ℓ is untypically large. The key argument leading to this result is is to find a gauge function ϕ such that the upper Hausdorff density of the intersecion local time is bounded. More precisely:

$$0 < \sup_{x\in s} \limsup_{r\downarrow 0} \frac{\ell(B(x;r))}{\phi(r)} < \infty.$$

Having found such a function ϕ, a point $x \in S$ is called *thick* , if

$$\limsup_{r\downarrow 0} \frac{\ell(B(x;r))}{\phi(r)} > 0.$$

Now the question concerning the size of the set of thick points in answered neatly by the *Hausdorff dimension spectrum* of the set which is defined as the function :

$$f(a) = \dim\{x \in S \,:\, \limsup_{r\downarrow 0} \frac{\ell(B(x;r))}{\phi(r)} = a\}$$

for each $a > 0$. It was shown by the authors (Theorem 1.4 [KM02]) that for $d = 3$ and $p = 2$,

(i)

$$\sup_{x\in\mathbb{R}^3} \limsup_{r\downarrow 0} \frac{\ell(B(x;r))}{r(\log\frac{1}{r})^2} = \frac{1}{\Theta(U)^2}$$

(ii)

$$\dim\{x \in S \,:\, \limsup_{r\downarrow 0} \frac{\ell(B(x;r))}{r(\log\frac{1}{r})^2} = a\} = 1 - \sqrt{a}\Theta(U).$$

Chapter 3

Main results: Large deviations and heuristic derivations

Recall from Chapter 1 that the Brownian intersection local time ℓ_t is a positive locally finite *measure* on $\mathbb{R}^d$. However, its *total mass* $\ell_t(A)$ of a given set $A \subset \mathbb{R}^d$ enjoys the same name. The difference between these two objects is significant in the context of our work. Henceforth we call the measure ℓ_t as *Brownian intersection measure* and its total mass $\ell_t(\mathbb{R}^d)$ as *Brownian intersection local time (ISLT)*. We state our main results.

3.1 Large deviations for intersection measures: Diverging time

Our first main result [KM11, Theorem 1.1] is a large-deviation principle for large time for the motions before exiting the set B. Assume that the p motions $W^{(1)}, \dots, W^{(p)}$ have some arbitrary starting distribution on B, possibly dependent on each other, which we suppress from the notation. Their occupation times measures are denoted by

$$\ell_t^{(i)} = \int_0^t \delta_{W_s^{(i)}} \,\mathrm{d}s, \qquad i = 1, \dots, p; t > 0. \tag{3.1.1}$$

We fix $b = (b_1, \dots, b_p) \in (0, \infty)^p$ and consider the time horizon $[0, tb_i]$ for the i-th motion. By

$$\mathbb{P}^{(tb)}(\cdot) = \mathbb{P}\Big(\cdot \cap \bigcap_{i=1}^p \{tb_i < \tau_i\}\Big)$$

we denote the sub-probability measure under which the i-th motion does not exit B before time tb_i. Then ℓ_{tb} is a random element of the set $\mathcal{M}(B)$ of positive measures on B. We equip it with the weak topology induced by test integrals with respect to continuous boundd functions $B \to \mathbb{R}$. By $\mathcal{M}_1(B)$ we denote the set of probability measures on B, and by $H_0^1(B)$ the usual Sobolev space with zero boundary condition in B.

Theorem 3.1.1 (LDP at diverging time)**.** *The tuple*

$$\Big(\frac{1}{t^p \prod_{i=1}^p b_i} \ell_{tb}; \frac{1}{tb_1} \ell_{tb_1}^{(1)}, \dots, \frac{1}{tb_p} \ell_{tb_p}^{(p)}\Big)$$

satisfies, as $t \to \infty$, a large deviation principle in the space $\mathcal{M}(B) \times \mathcal{M}_1(B)^p$ under $\mathbb{P}^{(tb)}$ with speed t and rate function

$$I\big(\mu; \mu_1, \dots, \mu_p\big) = \frac{1}{2}\sum_{i=1}^{p} b_i \|\nabla \psi_i\|_2^2, \tag{3.1.2}$$

if $\mu, \mu_1, \dots, \mu_p$ each have densities ψ^{2p} and $\psi_1^2, \dots, \psi_p^2$ such that $\psi, \psi_1, \dots, \psi_p \in H_0^1(B)$, $\|\psi_i\|_2 = 1$ $\forall i = 1, \dots, p$ and $\psi^{2p} = \prod_{i=1}^p \psi_i^2$; otherwise the rate function is ∞.

To be more explicit in the special case $b = \mathbb{1}$, Theorem 3.1.1 says that, for any continuous and bounded test functions $f, f_1, \dots, f_p \colon B \to \mathbb{R}$,

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{(t)} \Big[\exp\Big\{ t\Big(\langle t^{-p} \ell_{t\mathbb{1}}, f\rangle + \sum_{i=1}^{p} \langle \tfrac{1}{t} \ell_t^{(i)}, f_i\rangle \Big)\Big\}\Big] \\
&\quad = \sup\Big\{ \Big\langle \prod_{i=1}^p \psi_i^2, f \Big\rangle + \sum_{i=1}^p \langle \psi_i^2, f_i\rangle - \frac{1}{2}\sum_{i=1}^p \|\nabla\psi_i\|_2^2 \colon \psi_i \in H_0^1(B) \text{ and } \|\psi_i\|_2 = 1 \text{ for } i = 1, \dots, p \Big\}.
\end{aligned} \tag{3.1.3}$$

Theorem 3.1.1 is an extension of the well-known Donsker-Varadhan LDP for the occupation measures of a single Brownian motion in compacts [DV75-83], [G77] to the intersection measure. It gives a rigorous meaning to the heuristic formula in (1.2.1) in the limit $t \to \infty$. Since B is bounded, ℓ_t is a finite measure. However, there is no natural normalisation of ℓ_t that turned it into a probability measure. Our result shows that $t^{-p}\ell_t$ is asymptotically of finite order and admits a nice shape. A heuristic derivation of Theorem 3.1.1 in terms of the Donsker-Varadhan LDP is given in Section 3.4, the proof in Chapters 4 and 6.

Specialising to the first entry of the tuple, we get the following principle [KM11, Corollary 1.2] from the contraction principle, [DZ98, Theorem 4.2.1]:

Corollary 3.1.2. *Fix $b = (b_1, \dots, b_p) \in (0, \infty)^p$. Then the family of measures $((t^p \prod_{i=1}^p b_i)^{-1} \ell_{tb})_{t>0}$ satisfies, as $t \to \infty$, a large deviation principle in the space $\mathcal{M}(B)$ under $\mathbb{P}^{(tb)}$ with speed t and rate function*

$$I(\mu) = \inf\Big\{ \frac{1}{2}\sum_{i=1}^p b_i \|\nabla\psi_i\|_2^2 \colon \psi_i \in H_0^1(B), \|\psi_i\|_2 = 1 \,\forall i = 1, \dots, p, \text{ and } \prod_{i=1}^p \psi_i^2 = \frac{\mathrm{d}\mu}{\mathrm{d}x} \Big\}, \tag{3.1.4}$$

if μ has a density, and $I(\mu) = \infty$ otherwise.

To be more explicit, Corollary 3.1.2 says that, for any open set $G \subset \mathcal{M}(B)$ and every closed set $F \subset \mathcal{M}(B)$,

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big(t^{-p}\ell_t \in F, t < \tau_1 \wedge \dots \wedge \tau_p\big) &\le -\inf_{\mu\in F} I(\mu), \\
\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big(t^{-p}\ell_t \in G, t < \tau_1 \wedge \dots \wedge \tau_p\big) &\ge -\inf_{\mu\in G} I(\mu),
\end{aligned}$$

For some remarks on the rate function I, see Chapter 8.

3.2 Large deviations for intersection measures: Diverging mass

As a corollary of Theorem 3.1.1, we give now a related LDP for the normalized intersection local time for the motions stopped at their first exit from B under conditioning on $\{\ell(U) > a\}$ as $a \to \infty$, where we recall that $U \subset B$ is a compact set whose boundary is a Lebesgue null set. This solves a problem left open in [KM06]. That is, instead of diverging deterministic time, we now consider a random time horizon and diverging ISLT. The measure $\ell/\ell(U)$ is a positive measure on B, which is a probability measure on U. In the last chapter (see (2.2.8)) we mentioned that the normalised probability measure $\ell/\ell(U)$ satisfies a law of large masses under the conditional law $\mathbb{P}(\cdot \mid \ell(U) > a)$. Here we in particular identify the precise rate of the exponential convergence. By $\mathcal{M}_U(B)$ we denote the set of positive finite measures on B whose restriction to U is a probability measure. Our second main result [KM11, Theorem 1.3] is the following.

Theorem 3.2.1 (Large deviations at diverging mass)**.** *The normalized probability measures $\ell/\ell(U)$ under $\mathbb{P}(\cdot \,|\ell(U) > a)$ satisfy, as $a \to \infty$, a large deviation principle in the space $\mathcal{M}_U(B)$, with speed $a^{1/p}$ and rate function $J - \Theta_B(U)$, where*

$$J(\mu) = \inf\Big\{\frac{1}{2}\sum_{i=1}^p \|\nabla\phi_i\|_2^2 \colon \phi_1,\dots,\phi_p \in H_0^1(B), \prod_{i=1}^p \phi_i^2 = \frac{\mathrm{d}\mu}{\mathrm{d}x}\Big\}, \tag{3.2.1}$$

if μ has a density and $J(\mu) = \infty$ otherwise, where $\Theta_B(U)$ is the number appearing in (2.2.3).

The proof of Theorem 3.2.1 is in Chapter 7, a heuristic derivation from Theorem 3.1.1 is in Section 3.4. For some remarks concerning the rate function J see Chapter 8.

3.3 Large deviations for random walk intersection measures.

For Theorems 3.1.1 and 3.2.1 and Corollary 3.1.2, there are analogues for random walks on $\mathbb{Z}^d$ instead of Brownian motions on $\mathbb{R}^d$. These are much easier to formulate and to prove since the heuristic formula in (1.2.1) can be taken as a definition without problems. Let $X^{(1)},\dots,X^{(p)}$ be p independent continuous time random walks in $\mathbb{Z}^d$. Their *intersection local time* ℓ is just the product of their occupation measures:

$$\ell_t(z) = \prod_{i=1}^p \ell_t^{(i)}(z) \qquad z \in \mathbb{Z}^d,$$

where for each $i = 1,\dots,p$, their occupation measures $\ell^{(i)}$ is defined as

$$\ell_t^{(i)}(z) = \int_0^t ds\, \delta_z(X^{(i)}(s)). \tag{3.3.1}$$

Contrary to the Brownian case, for random walks, one can go beyond the *sub-critical regime* (i.e. $p < \frac{d}{d-2}$). In fact, a classical result by Dvroretzky and Erdös ([DE51]) says

$$\mathbb{P}(\ell_\infty < \infty) = \begin{cases} 1 & \text{if} \quad p > d/d-2, \\ 0 & \text{else.} \end{cases}$$

where ℓ_∞ referes to the infinite-time horizon intersection local time. We state results concerning both regimes.

Let us start with the *sub-critical* case: $p < \frac{d}{d-2}$. Here we necessarily restrict to random walks running until their first exit times $\tau^{(1)},\dots,\tau^{(p)}$ from a finite set $A \subset \mathbb{Z}^d$. Donsker-Varadhan theory says that each path measure $\frac{1}{t}\ell_t^{(i)}$ satisfies large deviation principle in the simplex $\mathcal{M}_1(\mathbb{Z}^d)$ of probability measures on $\mathbb{Z}^d$, under $\mathbb{P}(\cdot\,, t < \tau^{(i)})$. Loosely speaking,

$$\mathbb{P}\Big(\frac{1}{t}\ell_t^{(i)} \approx \psi_i^2(\cdot) \cap \,\{\tau^{(i)} > t\}\Big) \,= \exp\Big(-t\big[\frac{1}{2}\big\|\nabla\psi_i\big\|_2^2 + o(1)\big]\Big) \qquad t\uparrow\infty. \tag{3.3.2}$$

In $\mathbb{Z}^d$ the p-fold product in (3.3.1) is weakly continuous and hence by the contraction principle (see Theorem 4.2.1, [DZ98]), the normalised intersection local time measure $\frac{\ell_t}{\ell_t(A)}$ also satisfies a large deviation principle in the space $\mathcal{M}_1(A)$ of probability simplices on A, under the sub-probability density $\mathbb{P}_t = \mathbb{P}(\cdot \cap_{i=1}^p \{\tau^{(i)} > t\})$:

$$\mathbb{P}_t\Big(\frac{\ell_t(\cdot)}{\ell_t(A)} \approx\, \psi^{2p}(\cdot)\Big) =\, \exp\Big(-t\big[\,\Lambda(\psi) + o(1)\big]\Big) \qquad t\uparrow\infty, \tag{3.3.3}$$

for

$$\Lambda(\psi) = \inf\Big\{\frac{1}{2}\sum_{i=1}^p \|\nabla\psi_i\|_2^2 :\; \psi_i^2(A) = 1, \psi_i \in H_0^1(A)\;\forall i,\; \prod_{i=1}^p \frac{\psi_i^2(\cdot)}{\sum_{z\in A}\prod_{i=1}^p \psi_i^2(z)} = \psi^{2p}(\cdot)\Big\}. \tag{3.3.4}$$

This is an analogue to Theorem 3.1.2.
Like in the Brownian case, instead of fixed time horizon t we can work with random exit times and consider intersection measure $\ell = \ell_\tau$, i.e.,

$$\ell(z) = \prod_{i=1}^p \ell^{(i)}_{\tau^{(i)}}(z) \qquad z \in \mathbb{Z}^d.$$

Then using the previous result (3.3.2), and a time curtailing argument (see Section 3.4 for a heuristic sketch and Chapter 7 for a proof) we can show that the normalised intersection local time $\frac{\ell}{\ell(A)}$ satisfies a large deviation principle under $\mathbb{P}(\cdot \cap \{\ell(A) > a\})$:

$$\mathbb{P}\Big(\frac{\ell}{\ell(A)} \approx \psi^{2p}(\cdot),\; \ell(A) > a\Big) = \exp\Big(-a^{\frac{1}{p}}\big(\Gamma(\mu) + o(1)\big)\Big) \qquad a\uparrow\infty, \tag{3.3.5}$$

where

$$\Gamma(\mu) = \inf\Big\{\frac{1}{2}\sum_{i=1}^p \|\nabla\psi_i\|_2^2 :\; \psi_i \in H_0^1(A)\;\forall i,\; \prod_{i=1}^p \psi_i^2(\cdot) = \psi^{2p}(\cdot)\Big\}.$$

This is an analogue to Theorem 3.2.1.

Let us turn to the *super-critical* regime: $p > \frac{d}{d-2}$. We review the results from [CM09]. Now we need not restrict to finite boxes, as the intersection local time of $\mathbb{Z}^d$ is of finite order. Recall that $X^{(1)}, \ldots, X^{(p)}$ are p independent and identically distributed continuous time random walks in $\mathbb{Z}^d$ with the generator

$$Af(x) = \lim_{t\downarrow 0} \frac{\mathbb{E}_x(f(X^{(1)}(t)) - f(x)}{t}.$$

Furthermore, we assume that the walks are aperiodic and symmetric with finite variance so that A is a nonnegative definite, symmetric operator. Based on the Green's function

$$G(x) = \int_0^\infty \mathrm{d}t\,\mathbb{P}(X_t = x),$$

we define a bounded operator

$$\mathcal{B} :\; L^{\frac{2p}{2p-1}}(\mathbb{Z}^d) \to L^{2p}(\mathbb{Z}^d)$$

and $\mathcal{B}(f(x)) = \sum_{y\in\mathbb{Z}^d} G(x-)\, f(y)$.This is also called the *Green's operator* (note that the boundedness of $\mathcal{B}$ follows from the fact that $G \in L^{2p}(\mathbb{Z}^d)$ in $p > \frac{d}{d-2}$). Also note that $-\mathcal{B} \circ A = id$. Then, the main result (see Theorem 3 in [CM09]) characterises the upper tails of the infinite time total intersection local time $\ell = \ell_\infty(\mathbb{Z}^d)$:

$$\lim_{a\uparrow\infty} \frac{1}{a^{1/p}} \log \mathbb{P}\big(\ell > a\big) = -\frac{p}{\rho}, \tag{3.3.6}$$

where

$$\rho = \sup\big\{\langle f^{2p-1}, \mathcal{B} f^{2p-1}\rangle :\ \|f\|_{2p} = 1\big\}. \tag{3.3.7}$$

Now under some conditions,we can translate this variational formula into a more familiar representation as follows.

Given $\varepsilon > 0$, we choose f with $\|f\|_{2p} = 1$ such that $\langle f^{2p-1}, \mathcal{B} f^{2p-1}\rangle \geq \rho - \varepsilon$. By Hölder's inequality,

$$\big\|\mathcal{B} f^{2p-1}\big\|_{2p} \geq \langle f^{2p-1}, \mathcal{B} f^{2p-1}\rangle \big\|f\big\|_{2p}^{2p/2p-1} \geq \rho - \varepsilon.$$

Now

$$\begin{aligned} \rho &\geq \langle \mathcal{B} f^{2p-1}, f^{2p-1}\rangle \\ &= \langle \mathcal{B} f^{2p-1}, -A\mathcal{B} f^{2p-1}\rangle \\ &= (\rho-\varepsilon)^2 \Big\langle \frac{\mathcal{B} f^{2p-1}}{(\rho-\varepsilon)^2}, -A\frac{\mathcal{B} f^{2p-1}}{(\rho-\varepsilon)^2} \Big\rangle \\ &\geq (\rho-\varepsilon)^2 \inf\big\{\langle g, -Ag\rangle :\ \|g\|_{2p} = 1\big\}, \end{aligned}$$

where the last step follows from the continuity of the variational problem. We send $\varepsilon \downarrow 0$ o see that

$$\frac{1}{\rho} \geq \inf\big\{\langle g, -Ag\rangle :\ \|g\|_{2p} = 1\big\}.$$

To see the upper bound, we need to assume that there is a positive minimiser g for the above variational problem (this is, in fact, a non-trivial assumption). We take a bounded function $\varphi : \mathbb{Z}^d \to \mathbb{R}$ such that $\int \varphi = 0$. We write $f = g^{2p}$ so that $\|f\|_1 = 1$. For $\varepsilon > 0$ sufficiently small, $\|f + \varepsilon\varphi\|_1 = 1$. We perform a simple perturbative calculation:

$$\begin{aligned} 0 &= \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} \Big\langle (f+\varepsilon\varphi)^{\frac{1}{2p}}, A(f+\varepsilon\varphi)^{\frac{1}{2p}} \Big\rangle \\ &= \big\langle \varphi,\ f^{\frac{1}{2p}-1} A f^{\frac{1}{2p}} \big\rangle. \end{aligned}$$

We conclude that in the Hilbert space $L^2(\mathbb{Z}^d)$, the function $f^{\frac{1}{2p}-1} A f^{\frac{1}{2p}}$ is orthogonal to the orthogonal complement of the span of constant functions. Hence, there exists a constant λ such that $\lambda f^{1-\frac{1}{2p}} = A f^{\frac{1}{2p}}$. This means $\lambda g^{2p-1} = Ag$. Multiplying both sides by g, integrating over $\mathbb{Z}^d$, recalling $\|g\|_{2p} = 1$ and using the minimality of g, we conclude $-\rho^*\, Ag = g^{2p-1}$ where $1/\rho^*$ is the minimum. Since $A \circ -\mathcal{B} = id$, we infer $-\rho^* g = \mathcal{B} g^{2p-1}$ (we assumed A admits some maximum principle). Hence

$$1/\rho^* = \big\langle g, -Ag\big\rangle = \frac{1}{\rho^{*2}}\big\langle \mathcal{B} g^{2p-1}, g^{2p-1}\big\rangle \leq \frac{\rho}{\rho^{*2}}.$$

Hence, the right hand side of (3.3.6) equals

$$-p \inf\big\{\big\|\sqrt{-A} g\big\|_2^2 :\ \|g\|_{2p} = 1\big\}.$$

This reminds us of the classical form of our rate functions.

3.4 Heuristic derivation of main results

In this section we sketch the heuristics that leads one to Theorems 3.1.1 and 3.2.1. We strongly rely upon Donsker-Varadhan theory of large deviations. For simplicity, we drop compactness issues and formula the principle on $\mathbb{R}^d$ rather on some bounded domain B. We also put $b = 1\!\!1$ and write $\ell - t\mathrm{Q}$ instead of $\ell_{t1\!\!1}$.

Recall the occupation measure of the i-th Brownian motion defined in (3.1.1): That is, $\ell_t^{(i)}(A)$ is the amount of time that $W^{(i)}$ spends in $A \subset \mathbb{R}^d$. The famous Donsker-Varadhan LDP [G77], [DV75-83] states that

$$\mathbb{P}\big(\tfrac{1}{t}\ell_t^{(i)} \approx \mu\big) = \exp\Big[-t\frac{1}{2}\Big\|\nabla\sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}x}}\Big\|_2^2 + o(t)\Big], \qquad t \to \infty. \tag{3.4.1}$$

This is a simplified version of the statement that, under $\mathbb{P}(\cdot \mid W_{[0,t]}^{(i)} \subset B)$, the distributions of $\frac{1}{t}\ell_t^{(i)}$ satisfies an LDP with speed t and rate function $\mu \mapsto \frac{1}{2}\|\nabla\sqrt{\frac{\mathrm{d}\mu}{\mathrm{d}x}}\|_2^2$.

The heuristic formula in (1.2.1) states that

$$t^{-p}\ell_t(\mathrm{d}y) = \prod_{i=1}^{p} \frac{1}{t}\frac{\ell_t^{(i)}(\mathrm{d}y)}{\mathrm{d}y}. \tag{3.4.2}$$

Hence, $t^{-p}\ell_t$ is a function of the tuple $(\frac{1}{t}\ell_t^{(1)}, \dots, \frac{1}{t}\ell_t^{(p)})$. Let us ignore that this map is far from continuous. Now the LDP in Theorem 3.1.1 follows from a formal application of the contraction principle.

Let us now give a heuristic derivation of the LDP in Theorem 3.2.1. The heuristic formula in (1.2.1) implies that

$$\frac{\ell(\mathrm{d}y)}{\ell(U)} = \frac{1}{\int_U \mathrm{d}x\, \prod_{i=1}^{p} \frac{\ell_{\tau_i}^{(i)}(\mathrm{d}x)}{\mathrm{d}x}} \prod_{i=1}^{p} \frac{\ell_{\tau_i}^{(i)}(\mathrm{d}y)}{\mathrm{d}y}. \tag{3.4.3}$$

Pick some $\mu \in \mathcal{M}_U(B)$ with density ϕ^{2p}. We make the ansatz that the event $\{\ell/\ell(U) \approx \mu, \ell(U) > a\}$ is realized by the event $\bigcap_{i=1}^{p} A(b_i, \psi_i)$, where

$$A(b_i, \psi_i) = \Big\{\tau_i > b_i a^{1/p}, \frac{1}{b_i a^{1/p}}\ell_{b_i a^{1/p}}^{(i)} \approx \psi_i^2(x)\,\mathrm{d}x \text{ on } B\Big\},$$

where $\psi_1, \dots, \psi_p \in H_0^1(B)$ are $L^2(B)$-normalized and $b_1, \dots, b_p \in (0, \infty)$. (Later we have to optimise over $\psi_1, \dots, \psi_p$ and $b_1, \dots, b_p$.) In other words, the i-th motion spends an amount of $\tau_i \approx b_i a^{1/p}$ time units in B until it leaves the set B, and its normalized occupation times measure resembles ψ_i^2 on B. We approximate $\ell(U) > a$ by $\ell(U) \approx a$ and have therefore the following condition for $b_1, \dots, b_p$:

$$1 \approx \frac{1}{a}\ell(U) = \prod_{i=1}^{p} b_i \int_U \mathrm{d}x \prod_{i=1}^{p} \psi_i^2(x). \tag{3.4.4}$$

Furthermore, from (3.4.3), we get the condition

$$\phi^{2p} = \frac{\ell}{\ell(U)} = \frac{\prod_{i=1}^{p} \psi_i^2}{\int_U \mathrm{d}x\, \prod_{i=1}^{p} \psi_i^2(x)} = \prod_{i=1}^{p} \big(b_i \psi_i^2\big). \tag{3.4.5}$$

Hence, we get, also using (3.4.1) with $t = b_i a^{1/p}$,

$$\begin{aligned}
&\lim_{a\to\infty} a^{-1/p} \log \mathbb{P}\Big(\frac{\ell}{\ell(U)} \approx \phi^{2p}, \ell(U) > a\Big) \\
&\quad = -\inf_{b_1,\dots,b_p,\psi_1,\dots,\psi_p} \lim_{a\to\infty} a^{-1/p} \log \mathbb{P}\Big(\bigcap_{i=1}^{p} A(b_i, \psi_i)\Big) \\
&\quad = -\inf_{b_1,\dots,b_p,\psi_1,\dots,\psi_p} \sum_{i=1}^{p} b_i \frac{1}{2} \|\nabla \psi_i\|_2^2,
\end{aligned} \tag{3.4.6}$$

where the infimum runs under the above mentioned conditions, in particular (3.4.4) and (3.4.5). Now substituting $\phi_i^2 = b_i \psi_i^2$ for $i = 1, \dots, p$, we see that the right-hand side of (3.4.6) is indeed equal to $-J(\mu)$. This ends the heuristic derivation of Theorem 3.2.1.

Chapter 4

Proof of Theorem 3.1.1: Large deviations for diverging time.

In this section, we prove our first main result, the LDP in Theorem 3.1.1. A summary of our proof is as follows. In Section 4.2 we introduce an approximation of the normalised intersection measure in terms of the pointwise product of smoothed versions of the normalized occupation times measures of the p motions and prove an LDP for the tuple built from them. This is quite easy, as this tuple is a continuous function of the normalised occupation times measures, for which we can apply the classical Donsker-Varadhan LDP. Furthermore, in Section 4.3 we show that the corresponding rate function converges to the rate function I of the LDP of Theorem 3.1.1 as the smoothing parameter vanishes. The convergence is in the sense of Γ-convergence, and its proof relies on standard analysis. In Section 4.4 we finish the proof of Theorem 3.1.1, subject to the fact that the smoothed versions of the intersection measure is indeed an exponentially good approximation of the (non-smoothed) intersection measure. This fact is formulated as a proposition, its proof is deferred to Chapter 5 and Chapter 6. In the following Section 4.1 we give some remarks on the relation to other proofs in this field in the literature.

4.1 Literature remarks on the proof.

In the last decades, with especially much success in this millennium, people have developed many techniques to derive the large-time or the large-mass asymptotics for the total mass of mutual intersections of several independent paths; we mentioned two important ones in Chapter 2. With the exception of the work in [KM06], these results concern only the total mass, but not integrals against test functions, as we consider in the present paper. Hence, the question arises which of the existing proof strategies are also amenable to the refined problem about test integrals. In our setting of large deviations in a bounded set B, we do not have the – technically very nasty – additional problem of compactifying the space, which cannot be overcome by the well-known periodisation technique, but was solved by Chen using an abstract compactness criterion by de Acosta. We are also not using the technique of comparing deterministic time t to random independent exponential time, as this works only in connection with the Brownian scaling property, which we cannot use for our refined problem.

The technique of finding the asymptotics of high polynomial moments and using them for the logarithmic asymptotics of probabilities was first carried out in [KM02] in the context of mutual Brownian intersection local times in a bounded set B, see Section 2.2 and a thorough presentation in [C09]. This has the advantage to avoid a smoothing approximation; these are always technically involved. In [KM06], this technique was extended to the analysis of test integrals against a large class of measurable and bounded test functions. However, this technique was not able to yield an LDP, since it could be applied only to nonnegative test functions. Hence, we believe that this technique will not be helpful for deriving LDPs.

Another possibility would be to use Le Gall's [LG86] approximation technique with the help of renormalised Lebesgue measure on the intersection of the Wiener sausages. The main task here would be to strengthen the L^p-convergence of test integrals of these measures to exponential convergence. However, we found no way to do this.

Chen developed a strategy of smoothing by convolution of the Dirac measure in the proof of [C09, Theorem 2.2.3] for finding the logarithmic asymptotics for the upper tails of the total mass of the intersection. However, the strategy of proving the exponentially good approximation was taylored there for the total mass and does not seem to be amenable to the study of test integrals against test functions that may take arbitrary, positive and negative, values.

On the other side, another technique developed in [C07] seems to be amenable to prove an exponentially good approximation of the intersection measure for $p = 2$ using Fourier inversion. However, for $p > 2$, the mollifier used in [C07] does not seem to admit an LDP, at least not without substantial work, and we did not see how.

Therefore, we chose to work with mollifying each occupation time and to approximate the intersection measure with their pointwise product, which itself is easily seen to satisfy an LDP. Our proof of the exponential approximation in Chapter 6 with this object requires combinatorial and analytical work.

4.2 Large deviations for smoothed intersection local times.

Recall from (3.1.1) the occupation measure $\ell_t^{(i)}$ of the i-th motion:

$$\ell_t^{(i)} = \int_0^t \mathrm{d}s\, \delta_{W_s^{(i)}}.$$

Let $\varphi = \varphi_1$ be a non-negative, $\mathcal{C}^\infty$-function on $\mathbb{R}^d$ with compact support, normalised such that $\int_{\mathbb{R}^d} \varphi_1(y)\,\mathrm{d}y = 1$. Now we define the approximation of the Dirac δ-function at zero by

$$\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi_1(x/\varepsilon).$$

Let us consider the convolution of the above occupation measures with φ_ε:

$$\ell_{\varepsilon,t}^{(i)}(y) = \varphi_\varepsilon \star \ell_t^{(i)}(y) = \int_0^t \mathrm{d}s\; \varphi_\varepsilon(W_s^{(i)} - y).$$

Then $\ell_{\varepsilon,t}^{(i)}$ is a bounded $\mathcal{C}^\infty$-function. As $\varepsilon \downarrow 0$, the measure with density $\ell_{\varepsilon,t}^{(i)}$ converges weakly towards the occupation measure $\ell_t^{(i)}$. Consider the point-wise product of the above densities:

$$\ell_{\varepsilon,t}(y) = \prod_{i=1}^p \ell_{\varepsilon,t}^{(i)}(y).$$

We will write $\ell_{\varepsilon,t}(y)\,\mathrm{d}y$ for the measure with density $\ell_{\varepsilon,t}$. It should come as no surprise that these measures are, for any fixed t, an approximation of the intersection local time ℓ_t as $\varepsilon \downarrow 0$, even though we could not find this statement in the literature. Actually, we will go much further and will show that they even are an exponentially good approximation of the intersection local time ℓ_t in the sense of [DZ98], see below.

First we state a large-deviation principle for the measures with density $\ell_{\varepsilon,t}$ as $t\to\infty$ for fixed $\varepsilon>0$. It is known by classical work by Donsker and Varadhan [DV75-83], [G77] that each $\frac{1}{t}\ell_t^{(i)}$ satisfies, as $t\to\infty$, a large-deviations principle. In the proof of Lemma 4.2.1 below we will see that $\ell_{\varepsilon,t}(y)\,\mathrm{d}y$ is a continuous functional of the tuple $(\ell_t^{(1)},\dots,\ell_t^{(p)})$. Hence, by the contraction principle, $\ell_{\varepsilon,t}(y)\,\mathrm{d}y$ itself satisfies an LDP with some (ε-dependent) rate function.

Recall that we equip $\mathcal{M}(\mathbb{R}^d)$, the space of finite measures on $\mathbb{R}^d$, with the weak topology induced by test integrals against continuous bounded functions. For a measure $\mu\in\mathcal{M}(\mathbb{R}^d)$ and a function $f\colon\mathbb{R}^d\to\mathbb{R}$, we denote by $\langle\mu,f\rangle$ the integral $\int f\,d\mu$.

Lemma 4.2.1 (LDP for smoothed measures)**.** *Fix $\varepsilon>0$ and $b=(b_1,\dots,b_p)\in(0,\infty)^p$. Then the tuple of random measures*

$$\Big(\frac{1}{t^p\prod_{i=1}^p b_i}\ell_{\varepsilon,tb};\tfrac{1}{tb_1}\ell^{(1)}_{\varepsilon,tb_1},\dots,\tfrac{1}{tb_p}\ell^{(p)}_{\varepsilon,tb_p}\Big)$$

satisfies, as $t\to\infty$, a large deviation principle in $\mathcal{M}(B)\times\mathcal{M}_1(B)^p$ under $\mathbb{P}^{(tb)}$ with speed t and rate function

$$\begin{aligned}I_\varepsilon\big(\mu;\mu_1,\dots,\mu_p\big)=\inf\Big\{\frac{1}{2}\sum_{i=1}^p b_i\|\nabla\psi_i\|_2^2\colon \psi_i\in H_0^1(B),\|\psi_i\|_2=1,\psi_i^2\star\varphi_\varepsilon=\frac{\mathrm{d}\mu_i}{\mathrm{d}x}\,\forall i=1,\dots,p,\\ \text{and}\ \prod_{i=1}^p\psi_i^2\star\varphi_\varepsilon=\frac{\mathrm{d}\mu}{\mathrm{d}x}\Big\},\end{aligned}\tag{4.2.1}$$

if μ has a density, and $I_\varepsilon(\mu)=\infty$ otherwise. The level sets of I_ε are compact.

Proof. First observe that the mapping

$$\big(\mathcal{M}_1(\mathbb{R}^d)\big)^p\longrightarrow\mathcal{M}(\mathbb{R}^d),\qquad(\mu_1,\dots,\mu_p)\mapsto\Big(\prod_{i=1}^p\mu_i\star\varphi_\varepsilon(x)\Big)\,\mathrm{d}x,\tag{4.2.2}$$

is weakly continuous. Indeed, first note that the map $(\mu_1,\dots,\mu_p)\mapsto\mu_1\otimes\cdots\otimes\mu_p$ is continuous from $\mathcal{M}_1(\mathbb{R}^d)^p$ to $\mathcal{M}_1((\mathbb{R}^d)^p)$ since $\mathcal{M}_1(\mathbb{R}^d)$ is a Polish space. Furthermore, for every continuous bounded test function $f\colon\mathbb{R}^d\to\mathbb{R}$ and any $\mu_1,\dots,\mu_p\in\mathcal{M}_1(\mathbb{R}^d)$, we have

$$\begin{aligned}\Big\langle f,\Big(\prod_{i=1}^p\mu_i\star\varphi_\varepsilon(x)\Big)\,\mathrm{d}x\Big\rangle&=\int_{\mathbb{R}^d}\mathrm{d}x f(x)\int_{(\mathbb{R}^d)^p}\mu_1(\,\mathrm{d}y_1)\dots\mu_p(\,\mathrm{d}y_p)\,\varphi_\varepsilon(x-y_1)\dots\varphi_\varepsilon(x-y_p)\\&=\Big\langle A_f,\mu_1\otimes\cdots\otimes\mu_p\Big\rangle,\end{aligned}$$

where

$$A_f(y_1,\dots,y_p)=\int_{\mathbb{R}^d}\mathrm{d}x\,f(x)\,\varphi_\varepsilon(x-y_1)\dots\varphi_\varepsilon(x-y_p).$$

As φ_ϵ is smooth and compactly supported in $\mathbb{R}^d$, the function A_f is continuous and bounded in $(\mathbb{R}^d)^p$. This shows the continuity of the map in (4.2.2). Now the claimed LDP follows from the contraction principle [DZ98, Theorem 4.2.1]. $\square$

4.3 Gamma-convergence of the rate function.

In this section, we pass to the limit $\varepsilon \downarrow 0$ in the variational formula (4.2.1). The sense of convergence is the Γ-convergence, as will be required in the proof of Theorem 3.1.1 in Section 4.4 below. The proof of this convergence is based on standard analytic tools. By $B_\delta(\mu) = \{\nu \in \mathcal{M}(B) \colon \mathrm{d}(\nu,\mu) < \delta\}$ we denote the open ball of radius δ around μ, where d is a metric which induces the weak topology in $\mathcal{M}(B)$. By d we also denote the product metric on $\mathcal{M}(B) \times \mathcal{M}_1(B)^p$ and by $B_\delta(\mu;\mu_1,\dots\mu_p)$ the open δ-ball around $(\mu,\mu_1,\dots,\mu_p)$ in this space.

Proposition 4.3.1. *For every $\mu \in \mathcal{M}(B)$, we have,*

$$\sup_{\delta>0} \liminf_{\varepsilon\downarrow 0} \inf_{B_\delta(\mu;\mu_1,\dots,\mu_p)} I_\varepsilon = I(\mu;\mu_1,\dots,\mu_p), \tag{4.3.1}$$

where I is the rate function defined in (3.1.2)*. Furthermore, the level sets of I are compact.*

Proof. We write $f(x)\,\mu(\mathrm{d}x)$ for the measure with density f with respect to μ. We denote the Lebesgue measure by $\mathrm{d}x$.

First we prove '$\leq$'. Let $\mu,\mu_1,\dots,\mu_p$ be given. Without loss of generality, we may assume that $\psi_i^2 = \frac{\mathrm{d}\mu_i}{\mathrm{d}x}$ exists, and $\frac{\mathrm{d}\mu}{\mathrm{d}x} = \prod_{i=1}^p \psi_i^2$. Fix $\delta > 0$ and take $\varepsilon > 0$ so small that $\psi_i^2 \star \varphi_\varepsilon(x)\,\mathrm{d}x \in B_{\delta/2p}(\mu_i)$ for $i = 1,\dots,p$ and $(\prod_{i=1}^p \psi_i^2 \star \varphi_\varepsilon(x))\,\mathrm{d}x \in B_{\delta/2p}(\mu)$. Hence, the tuple $((\prod_{i=1}^p \psi_i^2 \star \varphi_\varepsilon(x))\,\mathrm{d}x; \psi_1^2 \star \varphi_\varepsilon(x)\,\mathrm{d}x,\dots,\psi_p^2 \star \varphi_\varepsilon(x)\,\mathrm{d}x)$ lies in $B_\delta(\mu;\mu_1,\dots,\mu_p)$. Hence,

$$\inf_{B_\delta(\mu;\mu_1,\dots,\mu_p)} I_\varepsilon \leq I_\varepsilon\Big(\Big(\prod_{i=1}^p \psi_i^2 \star \varphi_\varepsilon(x)\Big)\,\mathrm{d}x; \psi_1^2 \star \varphi_\varepsilon(x)\,\mathrm{d}x,\dots,\psi_p^2 \star \varphi_\varepsilon(x)\,\mathrm{d}x\Big) \leq \frac{1}{2}\sum_{i=1}^p \|\nabla\psi_i\|_2^2,$$

where in the last step we used the definition of I_ε.

Now we prove '$\geq$'. Let $\mu,\mu_1,\dots,\mu_p$ be given and let $I(\mu;\mu_1,\dots,\mu_p)$ be finite. Without loss of generality, the left hand side of (4.3.1) is also finite. For $\delta,\varepsilon > 0$, we pick $(\mu^{(\delta,\varepsilon)},\mu_1^{(\delta,\varepsilon)},\dots,\mu_p^{(\delta,\varepsilon)})$ in $B_\delta(\mu;\mu_1,\dots,\mu_p)$ such that

$$\inf_{B_\delta(\mu;\mu_1,\dots,\mu_p)} I_\varepsilon \geq I_\varepsilon\big(\mu^{(\delta,\varepsilon)};\mu_1^{(\delta,\varepsilon)},\dots,\mu_p^{(\delta,\varepsilon)}\big) - \delta.$$

By definition of I_ε, there are L^2-normalized $\psi_i^{(\delta,\varepsilon)} \in H_0^1(B)$ for $i = 1,\dots,p$ such that $\mu_i^{(\delta,\varepsilon)}(\mathrm{d}x) = \psi_i^2 \star \varphi_\varepsilon(x)\,\mathrm{d}x$ and $\mu^{(\delta,\varepsilon)}(\mathrm{d}x) = (\prod_{i=1}^p \psi_i^2 \star \varphi_\varepsilon(x))\,\mathrm{d}x$ and

$$I_\varepsilon\big(\mu^{(\delta,\varepsilon)};\mu_1^{(\delta,\varepsilon)},\dots,\mu_p^{(\delta,\varepsilon)}\big) \geq \frac{1}{2}\sum_{i=1}^p \|\nabla\psi_i^{(\delta,\varepsilon)}\|_2^2 - \varepsilon.$$

Then, by well-known analysis [LL01, Chapter 8], along some subsequences, we may assume that $\psi_i^{(\delta,\varepsilon)} \to \psi_i^{(\delta)}$ as $\varepsilon \downarrow 0$, for some L^2-normalized $\psi_i^{(\delta)} \in H_0^1(B)$ for $i = 1,\dots,p$, such that $\|\nabla\psi_i^{(\delta)}\|_2^2 \leq \liminf_{\varepsilon\downarrow 0} \|\nabla\psi_i^{(\delta,\varepsilon)}\|_2^2$. This convergence is true strongly in L^q for any $q > 1$ in $d = 2$ and $1 < q < 6$ in $d = 3$, and we have

$$\liminf_{\varepsilon\downarrow 0} \inf_{B_\delta(\mu;\mu_1,\dots,\mu_p)} I_\varepsilon \geq \frac{1}{2}\sum_{i=1}^p \|\nabla\psi_i^{(\delta)}\|_2^2 - \delta. \tag{4.3.2}$$

In particular, we have $\mu_i^{(\delta,\varepsilon)} \Rightarrow \psi_i^{(\delta)}(x)^2\,\mathrm{d}x =: \mu_i^{(\delta)}(\mathrm{d}x)$ in the weak topology. It is elementary (using Hölder's inequality) to see that $(\psi_i^{(\delta,\varepsilon)})^2 \star \varphi_\varepsilon(x)\,\mathrm{d}x \Rightarrow \mu_i^{(\delta)}(\mathrm{d}x)$ in the weak topology. Hence, $\mu_i^{(\delta)} \in B_{\delta/2p}(\mu_i)$. Now we let $\delta \downarrow 0$ and take a subsequence of $\psi_i^{(\delta)}$ which converges to some ψ_i strongly in L^q for any $q > 1$ in $d = 2$ and $1 < q < 6$ in $d = 3$ and

$$\liminf_{\delta\downarrow 0} \sum_{i=1}^p \|\nabla\psi_i^{(\delta)}\|_2^2 \geq \sum_{i=1}^p \|\nabla\psi_i\|_2^2.$$

Since $\mu_i^{(\delta)} \in B_{\delta/2p}(\mu_i)$, ψ_i^2 must be a density of μ_i. Therefore, the right hand side of the last display is $2I(\mu;\mu_1,\dots,\mu_p)$. Sending $\delta \downarrow 0$ in (4.3.2), the proof is finished for the case when $I(\mu;\mu_1,\dots,\mu_p)$ is finite.

Now we consider the case $I(\mu;\mu_1,\dots\mu_p) = \infty$. First, we consider the case that all $\mu_1,\dots,\mu_p$ have densities $\psi_1^2,\dots,\psi_p^2$ such that $\psi_i \in H_0^1(B)$, but μ either fails to have a density or to be the pointwise product of the ψ_i^2. By way of contradiction, assume that the left hand side of (4.3.1) is finite. Now we follow the same line of arguments as above and define $\mu^{(\delta)} = (\prod_{i=1}^p (\psi_i^{(\delta)})^2(x))\,\mathrm{d}x$ and note that $\mu^{(\delta,\varepsilon)} \Rightarrow \mu^{(\delta)}$ as $\varepsilon \downarrow 0$. Indeed $\psi_i^{(\delta,\varepsilon)}$ converges as $\varepsilon \downarrow 0$ (strongly in L^q for $q > 1$ in $d = 2$ and $1 < q < 6$ in $d = 3$) to $\psi_i^{(\delta)}$, and taking the pointwise product of the densities is a weakly continuous operation. Hence $\mu^{(\delta)}$ lies in $B_{\delta/2p}(\mu)$. Now we send $\delta \downarrow 0$ and use the same argument to infer that $\mu^{(\delta)} \Rightarrow \mu = (\prod_{i=1}^p \psi_i^2(x))\,\mathrm{d}x$. This is a contradiction.

Furthermore, also in the case that one of the μ_i's does not have a density or its squareroot is not in $H_0^1(B)$, the same arguments above (by contradiction) shows

$$\liminf_{\delta\downarrow 0} \sum_{i=1}^p \|\nabla\psi_i^{(\delta)}\|_2^2 \geq +\infty = I(\mu;\mu_1,\dots,\mu_p).$$

□

4.4 Completion of the proof of Theorem 3.1.1.

The main step in the remaining part of the proof of Theorem 3.1.1 is to show that the intersection measure $t^{-p}\ell_{tb}$ is exponentially well approximated by $t^{-p}\ell_{\varepsilon,tb}$. This we formulate here as a result on its own interest.

Proposition 4.4.1 (Exponential approximation)**.** *Fix $b = (b_1,\dots,b_p) \in (0,\infty)^p$ and a measurable and bounded function $f\colon B \to \mathbb{R}$. Then, for any $\varepsilon > 0$, there is $C(\varepsilon) > 0$ such that*

$$\mathbb{E}^{(tb)}\Big[\Big|\big\langle \ell_{tb} - \ell_{\varepsilon,tb}, f\big\rangle\Big|^k\Big] \leq k!^p\, C(\varepsilon)^k, \qquad t \in (0,\infty), k \in \mathbb{N}. \tag{4.4.1}$$

and $\lim_{\varepsilon\downarrow 0} C(\varepsilon) = 0$.

Note that this result implicitly shows that ℓ_t is indeed approximated by $\ell_{\varepsilon,t}$ in L^k-topology for any k, as we announced before. The proof of Proposition 4.4.1 is given in Chapter 6. Now we finish the proof of our main result.

Proof of Theorem 3.1.1. Recall that we have a LDP for the ε-depending tuple in Lemma 4.2.1. We now use Proposition 4.4.1 to see that this tuple is an exponentially good approximation of the tuple in Theorem 3.1.1. Recall that d is a metric on $\mathcal{M}(B)$ that induces the weak topology. We also denote by d a metric on $\mathcal{M}(B) \times \mathcal{M}_1(B)^p$ that induces the product topology of this topology. Then we have to show that the probability that the d-distance of the two tuples in Lemma 4.2.1 and Theorem 3.1.1 being larger than any $\delta > 0$ has an exponential rate as $t \to \infty$ which tends to $-\infty$ as $\varepsilon \downarrow 0$. Since the topology on $\mathcal{M}(B)$ is induced by test integrals against continuous bounded functions, it is enough to show that, for any such test functions $f, f_1, \dots, f_p \colon B \to \mathbb{R}$,

$$\lim_{\varepsilon\downarrow 0}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}^{(tb)}\Big(\Big\{\Big|\Big\langle\frac{1}{t^p\prod_{i=1}^p b_i}(\ell_{tb}-\ell_{\varepsilon,tb}),f\Big\rangle\Big|>\delta\Big\}\cup\bigcup_{i=1}^p\Big\{\big|\langle\tfrac{1}{tb_i}(\ell^{(i)}_{tb_i}-\ell^{(i)}_{\varepsilon,tb_i}),f_i\rangle\big|>\delta\Big\}\Big)=-\infty.$$

This indeed follows from Proposition 4.4.1, together with a version of this for $p = 1$, which is indeed much simpler and also follows from [AC03, Lemma 3.1], e.g. Indeed, we have from Proposition 4.4.1 that

$$\lim_{\varepsilon\downarrow 0}\limsup_{t\uparrow\infty}\frac{1}{t}\log\mathbb{P}^{(tb)}\Big(\Big|\Big\langle\frac{1}{t^p\prod_{i=1}^p b_i}(\ell_{tb}-\ell_{\varepsilon,tb}),f\Big\rangle\Big|>\delta\Big)=-\infty, \tag{4.4.2}$$

which follows from the Markov inequality, applied to the function $x \mapsto x^k$ with $k = \lceil t\rceil$, as follows:

$$\begin{aligned}\mathbb{P}^{(tb)}\Big(\Big|\Big\langle\frac{1}{t^p\prod_{i=1}^p b_i}(\ell_{tb}-\ell_{\varepsilon,tb}),f\Big\rangle\Big|>\delta\Big)&\le\delta^{-k}t^{-pk}C^k\mathbb{E}^{(tb)}\Big[\Big|\Big\langle\ell_{tb}-\ell_{\varepsilon,tb},f\Big\rangle\Big|^k\Big]\\&\le\delta^{-k}t^{-pk}C^k\,k!^pC(\varepsilon)^k\le\widetilde{C}(\varepsilon)^t,\end{aligned}$$

for any $t > 0$, where C, $C(\varepsilon)$ and $\widetilde{C}(\varepsilon)$ depend on b, B, d, f and δ (but not on t) and satisfy $\lim_{\varepsilon\downarrow 0} C(\varepsilon) = 0 = \lim_{\varepsilon\downarrow 0}\widetilde{C}(\varepsilon)$, and $C(\varepsilon)$ is the constant from Proposition 4.4.1. Since $k = \lceil t\rceil$ and $\lim_{\varepsilon\downarrow 0}\widetilde{C}(\varepsilon) = 0$, (4.4.2) follows.

Hence, according to [DZ98, Theorem 4.2.16], the LDP of Theorem 3.1.1 is true with the rate function on the left-hand side of (4.3.1). But Proposition 4.3.1 identifies this as I given in (3.1.4).

Note that by (4.3.1) and [DZ98, Theorem 4.2.16], I is a lower semicontinuous functional. Hence, its level sets are closed in $\mathcal{M}(B)\times\mathcal{M}_1(B)^p$. Since the infimum in (3.1.2) extends only over functions in $H_0^1(B)$ (i.e., with zero boundary conditions), I can be seen also as a lower semicontinuous functional on $\mathcal{M}(\overline{B})\times\mathcal{M}_1(\overline{B})^p$, which is weakly compact by Prohorov's theorem. Hence, the levels sets of I are also compact. That is, the proof of Theorem 3.1.1 is finished. □

Chapter 5

Preparations for Proof of Proposition 4.4.1: Moment formulas.

This chapter provides the guiding philosophy of the proof of Proposition 4.4.1 and collects some material to be used later in the next chapter. Recall that we have to estimate the object $\mathbb{E}^{(t)}(|\langle \ell_t, f\rangle - \langle \ell_{\varepsilon,t}, f\rangle|^k)$. The basic idea is to split the k-th moment of difference of ℓ_t and $\ell_{\varepsilon,t}$ using the binomial theorem and write the mixed moments (consisting of ε-dependent and ε-free objects) in terms of a product of k-step transition densities. These densities possess an eigenvalue expansion consisting of several eigenfunctions of $-\frac{1}{2}\Delta$ with 0 boundary condition. The ε convolution latently existing in our formula also manifests here and stick to some eigenfunctions (while the rest of the eigenfunctions remain ε-free). Using some fine combinatorial trick we trace back our path and again using the binomial theorem we recover an estimate (paying little cost) consisting of the kth power of difference of objects which are close when ε is small. We start with some preparatory steps.

5.1 Moment formula.

We begin with a moment formula for the left-hand side of (4.4.1), which is an adaptation of Le Gall's formula for the moments of $\ell(U)$ for compact subsets U of B [LG86, LG87, LG89].

We write $\mathbb{P}^{(t)}_{x,y}$ and $\mathbb{E}^{(t)}_{x,y}$ for the Brownian bridge sub-probability measure $\otimes_{l=1}^{p}\mathbb{P}_{x^{(l)}}(\cdot\,, t<\tau; W_t \in \mathrm{d}y^{(l)})/\mathrm{d}y^{(l)}$ (where $x=(x^{(1)},\dots,x^{(p)}), y=(y^{(1)},\dots,y^{(p)})\in B^p$) and the corresponding expectation. In other words, under $\mathbb{P}^{(t)}_{x,y}$, we consider p independent Brownian bridges in B with time interval $[0,t]$ from $x^{(l)}$ to $y^{(l)}$, for $l=1,\dots,p$. Later we integrate over $x,y\in B^p$ with respect to $\nu(\mathrm{d}x)\mathrm{d}y$, where ν is the joint starting distribution of the p motions and hence $\mathbb{P}^{(t)}=\int_{B^p}\nu(\mathrm{d}x)\int_{B^p}\mathrm{d}y\,\mathbb{P}^{(t)}_{x,y}$.

Furthermore, we denote by $p^{(B)}_s(x,y)=\mathbb{P}_x(W_s\in \mathrm{d}y;\tau>s)/\mathrm{d}y$ the density of the distribution of a single Brownian motion at time s before the exit time τ from B when started at $x\in B$. By $\mathfrak{S}_k$ we denote the set of permutations of $1,\dots,k$.

Lemma 5.1.1 (Moment formula)**.** *For any continuous function $f\colon B\to\mathbb{R}$ and any $k\in\mathbb{N}$ and any $t>0$,* *and any* $x_0 = (x_0^{(1)}, \dots, x_0^{(p)})$ *and* $x_{k+1} = (x_{k+1}^{(1)}, \dots, x_{k+1}^{(p)}) \in B^p$,

$$\mathbb{E}^{(t)}_{x_0,x_{k+1}}\Big[\big(\langle f, \ell_t\rangle - \langle f, \ell_{\varepsilon,t}\rangle\big)^k\Big] = \sum_{m=0}^{k}(-1)^m\binom{k}{m}\int_{B^k}\prod_{i=1}^{k}\big(f(y_i)\,\mathrm{d}y_i\big)$$
$$\prod_{i=1}^{p}\Big[\sum_{\sigma\in\mathfrak{S}_k}\int_{[0,t]^k}\mathrm{d}r_k\dots\mathrm{d}r_1\,\mathbb{1}\{\textstyle\sum_{i=1}^k r_i \le t\}\displaystyle\int_{B^{k-m}}\prod_{j=m+1}^{k}\Big(\varphi_\varepsilon(y_j - z_j)\,\mathrm{d}z_j\Big)\prod_{j=1}^{k+1}p^{(B)}_{r_j}(x^{(i)}_{j-1}, x^{(i)}_j)\Big], \tag{5.1.1}$$

where we abbreviate $r_{k+1} = t - \sum_{i=1}^k r_i$ *and, for* $j = 1, \dots, k$,

$$x_j = x_j^{(i)} = \begin{cases} y_{\sigma^{-1}(j)} & \text{if } \sigma^{-1}(j) \le m, \\ z_{\sigma^{-1}(j)} & \text{if } \sigma^{-1}(j) > m. \end{cases} \tag{5.1.2}$$

Proof. We use the binomial theorem to split the k-th moment as follows.

$$\mathbb{E}^{(t)}_{x_0,x_{k+1}}\Big[\big(\langle f, \ell_t\rangle - \langle f, \ell_{\varepsilon,t}\rangle\big)^k\Big] = \sum_{m=0}^{k}(-1)^m\binom{k}{m}\mathbb{E}^{(t)}_{x_0,x_{k+1}}\Big[\langle f, \ell_t\rangle^m\langle f, \ell_{\varepsilon,t}\rangle^{k-m}\Big]. \tag{5.1.3}$$

Now we handle the mixed moments above. We formulate the proof in a somewhat lose way, a mathematically correct way to turn the following way is described in [LG86]. For any $m \in \{0, \dots, k\}$,

$$\mathbb{E}^{(t)}_{x_0,x_{k+1}}\Big[\langle f, \ell_t\rangle^m\langle f, \ell_{\varepsilon,t}\rangle^{k-m}\Big] = \int_{B^k}\prod_{l=1}^{k}f(y_i)\,\mathbb{E}^{(t)}_{x_0,x_{k+1}}\Bigg[\bigotimes_{j=1}^{m}\ell_t(\mathrm{d}y_j)\bigotimes_{j=m+1}^{k}\ell_{\varepsilon,t}(y_j)\,\mathrm{d}y_j\Bigg], \tag{5.1.4}$$

where we recall that ℓ_t does not have a density, but $\ell_{\varepsilon,t}$ is a smooth function. By definition of $\ell_{\varepsilon,t}$ and independence of paths, the expectation on the right-hand side of (5.1.4) can be written as

$$\prod_{i=1}^{p}\Bigg[\int_{[0,t]^k}\mathrm{d}s_k\dots\mathrm{d}s_1\int_{B^{k-m}}\prod_{j=m+1}^{k}\Big(\varphi_\varepsilon(y_j - z_j)\,\mathrm{d}y_j\Big)\mathbb{P}^{(t)}_{x_0^{(i)},x_{k+1}^{(i)}}\left(\begin{cases} W_{s_j} \in \mathrm{d}y_j & \text{if } j \le m, \\ W_{s_j} \in \mathrm{d}z_j & \text{if } j > m. \end{cases}\right)\Bigg],$$

where we remark that the integral over B^{k-m} refers to $\mathrm{d}z_{m+1}\dots\mathrm{d}z_k$. Now we time-order the k-dimensional cube $[0,t]^k$ and write the last expression as

$$\prod_{i=1}^{p}\Bigg[\sum_{\sigma\in\mathfrak{S}_k}\int_{0\le s_1\le\dots\le s_k\le t}\mathrm{d}s_k\dots\mathrm{d}s_1\int_{B^{k-m}}\prod_{j=m+1}^{k}\varphi_\varepsilon(y_j - z_j)\mathbb{P}^{(t)}_{x_0^{(i)},x_{k+1}^{(i)}}\left(\begin{cases} W_{s_{\sigma(j)}} \in \mathrm{d}y_j & \text{if } j \le m \\ W_{s_{\sigma(j)}} \in \mathrm{d}z_j & \text{if } j > m. \end{cases}\right)\Bigg] \tag{5.1.5}$$

The time-ordering allows us to invoke the Markov property at the consecutive times $s_1 < s_2 < \dots < s_k$ and to split the path into k pieces. Each of the pieces is a Brownian motion before leaving B. Therefore the joint probability distribution above also splits into the corresponding k-step transition probability densities.

$$\begin{aligned}\mathbb{P}^{(t)}_{x_0^{(i)},x_{k+1}^{(i)}}\left(\begin{cases} W_{s_{\sigma(j)}} \in \mathrm{d}y_j & \text{if } j \le m, \\ W_{s_{\sigma(j)}} \in \mathrm{d}z_j & \text{if } j > m. \end{cases}\right) &= \mathbb{P}^{(t)}_{x_0^{(i)},x_{k+1}^{(i)}}\left(\begin{cases} W_{s_j} \in \mathrm{d}y_{\sigma^{-1}(j)} & \text{if } \sigma^{-1}(j) \le m, \\ W_{s_j} \in \mathrm{d}z_{\sigma^{-1}(j)} & \text{if } \sigma^{-1}(j) > m. \end{cases}\right) \\ &= \mathbb{P}^{(t)}_{x_0^{(i)},x_{k+1}^{(i)}}\Big(W_{s_j} \in \mathrm{d}x_j^{(i)}, j = 1, \dots, k\Big) \\ &= \Big(\prod_{j=1}^{k+1}p^{(B)}_{s_j - s_{j-1}}(x^{(i)}_{j-1}, x^{(i)}_j)\Big)\,\mathrm{d}y_1\dots\mathrm{d}y_m\mathrm{d}z_{m+1}\dots\mathrm{d}z_k.\end{aligned} \tag{5.1.6}$$

Substituting $r_j = s_j - s_{j-1}$ and putting all the material together proves the lemma. □

Remark: A heuristic proof for $k \ll t$.
In order to give some guidance to the reader, let us briefly describe heuristically in which way we will succeed to estimate the bulky expression on the right of (5.1.1) in terms of $k!^p C(\varepsilon)^k$ with a small $C(\varepsilon)$. We do this only for the regime $k \ll t$, which we actually do not consider in Proposition 4.4.1, but this only meant as a demonstration of the philosophy of our proof. Apart from the formulation of Lemma 5.1.2 below, the material of this section will not be used later in the proof of Proposition 4.4.1.

The problem is to extract an extinction coming from a difference of two close (for small ε) terms with a power of order k by use of the binomial theorem. Since this works only if certain powers of these close terms appear, one has to expand the probability terms on the right of (5.1.1) into sums of powers.

Our second main ingredient is a standard eigenvalue expansion with respect to the spectrum of the Laplace operator in B with zero boundary condition, which follows from the well-known spectral theorem for compact, self-adjoint operators [B95, Theorem 4.13]:

Lemma 5.1.2 (Eigenvalue expansion). *There exist a system of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and an $L^2(B)$-orthonormal basis of corresponding eigenfunctions $\psi_1, \psi_2, \dots$ in B of $-\frac{1}{2}\Delta$ with zero boundary condition in B, that is, $-\frac{1}{2}\Delta\psi_n = \lambda_n\psi_n$ for any $n \in \mathbb{N}$. Furthermore,*

$$p_s^{(B)}(x,y) = \sum_{n=1}^{\infty} \mathrm{e}^{-s\lambda_n}\, \psi_n(x)\psi_n(y), \qquad s > 0, \tag{5.1.7}$$

and the convergence is absolute and uniform in $x, y \in B$.

In the regime $k \ll t$, we use that r_j is large for any j and use the approximation

$$p_r^{(B)}(x,y) = \mathrm{e}^{-r\lambda_1}(\psi_1(x)\psi_1(y) + o(1)), \qquad r \to \infty. \tag{5.1.8}$$

That is, instead of plugging in the full eigenvalue expansion (6.1.6) we just pick the leading term of the expansion (5.1.8) in the last line of (5.1.1). This gives, for any $i = 1, \dots, p$,

$$\begin{aligned}\prod_{j=1}^{k+1} p_{r_j}^{(B)}(x_{j-1}^{(i)}, x_j^{(i)}) &\approx \prod_{j=1}^{k+1} \Big(\mathrm{e}^{-r_j\lambda_1}\psi_1(x_{j-1}^{(i)})\psi_1(x_j^{(i)})\Big)\\ &= \mathrm{e}^{-t\lambda_1}\psi_1(x_0^{(i)})\psi_1(x_{k+1}^{(i)})\prod_{j=1}^{k}\psi_1^2(x_j)\\ &= \mathrm{e}^{-t\lambda_1}\psi_1(x_0^{(i)})\psi_1(x_{k+1}^{(i)})\Big(\prod_{j=1}^{m}\psi_1^2(y_j)\Big)\Big(\prod_{j=m+1}^{k}\psi_1^2(z_j)\Big).\end{aligned} \tag{5.1.9}$$

Note that the last term does not depend on $\sigma \in \mathfrak{S}_k$ or any $r_1, \dots, r_k \in [0,t]$. Also note that $|\mathfrak{S}_k| = k!$ and $\int_{[0,t]^k} \mathrm{d}r_k \dots \mathrm{d}r_1 1\!\!1\{\sum_{i=1}^k r_k \leq t\} = t^k/k!$. Substituting the last term of (5.1.9) in (5.1.1), we can integrate out the convolution integrals over $z_{m+1}, \dots, z_k$ and afterwards the integrals over $y_1, \dots, y_k$ and see that

$$\begin{aligned}
&\mathbb{E}^{(t)}_{x_0,x_{k+1}}\Big[(\langle f,\ell_t\rangle - \langle f,\ell_{\varepsilon,t}\rangle)^k\Big] \\
&\quad\approx \mathrm{e}^{-tp\lambda_1} t^{kp}\,\Big(\prod_{i=1}^p \psi_1(x_0^{(i)})\psi_1(x_{k+1}^{(i)})\Big)\sum_{m=0}^k (-1)^m \binom{k}{m} \\
&\qquad\times \int_{B^k} \mathrm{d}y_1\dots\mathrm{d}y_k\,\Big(\prod_{j=1}^k f(y_j)\Big)\Big(\prod_{j=1}^m \psi_1^{2p}(y_j)\Big)\Big(\prod_{j=m+1}^k \left(\varphi_\varepsilon\star\psi_1^2\right)^p(y_j)\Big) \\
&\quad= \mathrm{e}^{-tp\lambda_1} t^{kp}\,\Big(\prod_{i=1}^p \psi_1(x_0^{(i)})\psi_1(x_{k+1}^{(i)})\Big)\sum_{m=0}^k (-1)^m \binom{k}{m}\langle f,\psi_1^{2p}\rangle^m \langle f,(\varphi_\varepsilon\star\psi_1^2)^p\rangle^{k-m} \\
&\quad= \mathrm{e}^{-tp\lambda_1} t^{kp}\,\Big(\prod_{i=1}^p \psi_1(x_0^{(i)})\psi_1(x_{k+1}^{(i)})\Big)\Big(\langle f,\psi_1^{2p}\rangle - \langle f,(\varphi_\varepsilon\star\psi_1^2)^p\rangle\Big)^k,
\end{aligned} \tag{5.1.10}$$

according to the binomial theorem. Since φ_ε is an approximation of the Dirac delta measure at zero, it is clear that $\langle f,\psi_1^{2p}\rangle - \langle f,(\varphi_\varepsilon\star\psi_1^2)^p\rangle$ tends to zero as $\varepsilon \downarrow 0$. Hence, we have derived an upper bound as claimed in (4.4.1).

Now we insert the eigenvalue expansion of Lemma 5.1.2 in the right-hand side of (5.1.1). This will be combinatorially very cumbersome, and this comes a bit milder if we integrate the measure $\mathbb{P}^{(t)}_{x_0,x_{k+1}}$ over $x_0 = x_{k+1} \in B^p$, i.e., if we replace it by the measure $\mathbb{P}^{(t)} = \int_{B^p} \mathrm{d}x\,\mathbb{P}^{(t)}_{x,x}$. By $\mathbb{E}^{(t)}$ we denote the corresponding expectation.

Recall that we fixed a continuous bounded function $f\colon B \to \mathbb{R}$. For $n_1, \dots, n_k \in \mathbb{N}$, denote

$$F_t(n_1,\dots,n_k) = \int_{[0,t]^k} \mathrm{d}r_k \dots \mathrm{d}r_1 \mathbb{1}\{\textstyle\sum_{i=1}^k r_k \le t\} \displaystyle\prod_{j=1}^{k+1} \mathrm{e}^{-r_j\lambda_{n_j}}, \qquad n_{k+1} = n_1. \tag{5.1.11}$$

We abbreviate $\mathcal{N} = (n_j^{(i)})_{j=1,\dots,k;i=1,\dots,p} \in \mathbb{N}^{p\times k}$. For fixed $j \in \{1,\dots,k\}$, denote the j-th column $(n_j^{(1)},\dots,n_j^{(p)})$ of $\mathcal{N}$ by $\mathcal{N}_j$ and for $i \in 1,\dots,p$ the i-th row $(n_1^{(i)},\dots,n_k^{(i)})$ by $\mathcal{N}^{(i)}$. Furthermore, we abbreviate

$$\begin{aligned}
a(\mathcal{N}_j,\mathcal{N}_{j+1}) &= \Big\langle f, \prod_{i=1}^p \psi_{n_j^{(i)}}\psi_{n_{j+1}^{(i)}}\Big\rangle, \\
a_\varepsilon(\mathcal{N}_j,\mathcal{N}_{j+1}) &= \Big\langle f, \prod_{i=1}^p \varphi_\epsilon\star\big(\psi_{n_j^{(i)}}\psi_{n_{j+1}^{(i)}}\big)\Big\rangle.
\end{aligned} \tag{5.1.12}$$

Lemma 5.1.3. *For any continuous function $f\colon B \to \mathbb{R}$ and any $k \in \mathbb{N}$ and any $t > 0$,*

$$\begin{aligned}
\mathbb{E}^{(t)}\Big[(\langle f,\ell_t\rangle - \langle f,\ell_{\varepsilon,t}\rangle)^k\Big] = &\sum_{\mathcal{N}\in\mathbb{N}^{p\times k}} \prod_{i=1}^p F_t\left(n_1^{(i)},\dots,n_k^{(i)}\right) \sum_{\sigma_1,\dots,\sigma_p\in\mathfrak{S}_k} \\
&\sum_{m=0}^k (-1)^m \binom{k}{m} \prod_{j=1}^m a\big(\mathcal{N}_{\sigma(j)},\mathcal{N}_{\sigma(j)+1}\big) \prod_{j=m+1}^k a_\varepsilon\big(\mathcal{N}_{\sigma(j)},\mathcal{N}_{\sigma(j)+1}\big),
\end{aligned} \tag{5.1.13}$$

where we abbreviated $\mathcal{N}_{\sigma(j)} = (n^{(i)}_{\sigma_i(j)})_{i=1,\dots,p}$ and $\mathcal{N}_{\sigma(j)+1} = (n^{(i)}_{\sigma_i(j)+1})_{i=1,\dots,p}$.

Proof. We pick up from the last line in (5.1.1) and write $x_0 = x_{k+1}$ and $r_0 = 0$ and $r_{k+1} = t - \sum_{i=1}^k r_i$.

$$\begin{aligned}
&\int_B \mathrm{d}x_{k+1} \prod_{j=1}^{k+1} p_{r_j}^{(B)}(x_{j-1}, x_j) \\
&\quad = \int_B \mathrm{d}x_{k+1} \prod_{j=1}^{k+1} \left(\sum_{n=1}^{\infty} \mathrm{e}^{-r_j \lambda_n} \psi_n(x_{j-1}) \psi_n(x_j) \right) \\
&\quad = \int_B \mathrm{d}x_{k+1} \sum_{n_1,\dots,n_{k+1}=1}^{\infty} \left(\prod_{j=1}^{k+1} \mathrm{e}^{-r_j \lambda_{n_j}} \psi_{n_j}(x_{j-1}) \psi_{n_j}(x_j) \right) \\
&\quad = \sum_{n_1,\dots,n_{k+1}=1}^{\infty} \prod_{j=1}^{k+1} \left(\mathrm{e}^{-r_j \lambda_{n_j}} \right) \langle \psi_{n_1}, \psi_{n_{k+1}} \rangle \left(\prod_{j=1}^{k} \psi_{n_j}(x_j) \psi_{n_{j+1}}(x_j) \right) \\
&\quad = \sum_{n_{k+1}=n_1,\dots,n_k=1}^{\infty} \left(\prod_{j=1}^{k+1} \mathrm{e}^{-r_j \lambda_{n_j}} \right) \left(\prod_{j=1}^{k} \psi_{n_j}(x_j) \psi_{n_{j+1}}(x_j) \right).
\end{aligned} \tag{5.1.14}$$

The last step follows from the orthonormality of the eigenfunctions and thus the sum on n_1 and on n_{k+1} may be restricted to the sum on $n_1 = n_{k+1}$. Looking back at (5.1.2) we see that the last line of (5.1.14) is equal to

$$\sum_{n_1,\dots,n_k=1}^{\infty} \left(\prod_{j=1}^{k+1} \mathrm{e}^{-r_j \lambda_{n_j}} \right) \prod_{j=1}^{m} \left(\psi_{n_{\sigma(j)}} \psi_{n_{\sigma(j)+1}} \right)(y_j) \times \prod_{j=m+1}^{k} \left(\psi_{n_{\sigma(j)}} \psi_{n_{\sigma(j)+1}} \right)(z_j) \tag{5.1.15}$$

Using this in (5.1.14), substituting this in (5.1.1) and using the notation in (5.1.11), we see that

$$\begin{aligned}
&\mathbb{E}^{(t)} \Big[(\langle f, \ell_t \rangle - \langle f, \ell_{\varepsilon,t} \rangle)^k \Big] = \int_{B^p} \mathrm{d}x_0 \, \mathbb{E}_{x_0,x_0}^{(t)} \Big[(\langle f, \ell_t \rangle - \langle f, \ell_{\varepsilon,t} \rangle)^k \Big] \\
&\quad = \sum_{m=0}^{k} (-1)^m \binom{k}{m} \int_{B^k} \prod_{i=1}^{k} (f(y_i) \, \mathrm{d}y_i) \\
&\quad \times \Big[\sum_{n_{k+1}=n_1,\dots,n_k=1}^{\infty} F_t(n_1, \dots, n_k) \sum_{\sigma \in \mathfrak{S}_k} \prod_{j=1}^{m} \left(\psi_{n_{\sigma(j)}} \psi_{n_{\sigma(j)+1}} \right)(y_j) \\
&\quad \times \prod_{j=m+1}^{k} \varphi_\varepsilon \star \Big(\psi_{n_{\sigma(j)}} \psi_{n_{\sigma(j)+1}} \Big)(y_j) \Big]^p.
\end{aligned} \tag{5.1.16}$$

In order to carry out the k integrals over $y_1, \dots, k$, we write the p-th power explicitly as a p-fold sum over multiple indices. Recall that we abbreviate $\mathcal{N} = (n_j^{(i)})_{j=1,\dots,k; i=1,\dots,p} \in \mathbb{N}^{p \times k}$. Therefore we obtain from the last display that

$$\begin{aligned}
&\mathbb{E}^{(t)} \Big[(\langle f, \ell_t \rangle - \langle f, \ell_{\varepsilon,t} \rangle)^k \Big] \\
&= \sum_{\mathcal{N} \in \mathbb{N}^{p \times k}} \prod_{i=1}^{p} F_t \left(n_1^{(i)}, \dots, n_k^{(i)} \right) \sum_{\sigma_1,\dots,\sigma_p \in S_k} \prod_{j=1}^{m} \left(\int_U \mathrm{d}y f(y) \prod_{i=1}^{p} \left(\psi_{n^{(i)}_{\sigma_i(j)}} \psi_{n^{(i)}_{\sigma_i(j)+1}} \right)(y) \right) \\
&\times \prod_{j=m+1}^{k} \left(\int_U \mathrm{d}y f(y) \prod_{i=1}^{p} \varphi_\epsilon \star \left(\psi_{n^{(i)}_{\sigma_i(j)}} \psi_{n^{(i)}_{\sigma_i(j)+1}} \right)(y) \right).
\end{aligned} \tag{5.1.17}$$

Using the notation in (5.1.12), we see that this is the assertion that we wanted to prove. □

5.2 Combinatorial counting.

We want to simplify the right-hand side of (5.1.13). The starting point is that many of the multi-indices $\mathcal{N} \in \mathbb{N}^{p\times k}$ on the right-hand side give precisely the same contribution. Our task here is to identify what classes of such $\mathcal{N}$ do this and to evaluate their cardinality. First, we set up some notations and summarise the right hand side of (5.1.13).

First we note that the two products in the last line do not depend on each value of $(n_{\sigma(j)}, n_{\sigma(j)+1})$ for $j = 1, \dots, k$, but only on their occupation numbers, i.e., on the number $A(l)$ of occurrences of a given vector $l \in (\mathbb{N}^p)^2$ in the vector $(n_{\sigma(j)}, n_{\sigma(j)+1})_{j=1,\dots,k}$. Hence, $A\colon (\mathbb{N}^p)^2 \to \{0, \dots, k\}$ is a map satisfying $\sum_{l\in(\mathbb{N}^p)^2} A(l) = k$, and we will be summing on all such maps. However, in order to describe the right-hand side of (5.1.13), we also have to sum on all occupation numbers $r(l)$ of the vectors $(n_{\sigma(1)}, n_{\sigma(j)+1})$ in the first product and the occupation numbers (which are necessarily $A(l) - r(l)$) in the second product. This leads to a further sum on all maps $r\colon (\mathbb{N}^p)^2 \to \{0, \dots, k\}$ satisfying $\sum_{l\in(\mathbb{N}^p)^2} r(l) = m$ and $0 \le r(l) \le A(l)$ for any $l \in (\mathbb{N}^p)^2$. We denote by $M_{k,m}$ the set of all pairs (A, r) of such maps and by M_k the set of all maps A as above. Our strategy is to write the right-hand side of (5.1.13) as a sum on $m = 0, \dots, k$ and a sum on $(A, r) \in M_{k,m}$, express both the product over the a-terms and the product over the F_t-terms as a function of A and r, and finally to count all the tuples $(\mathcal{N}, \sigma_1, \dots, \sigma_p)$ such that (A, r) is the pair of occupation number vectors of the vectors $(n_{\sigma(j)}, n_{\sigma(j)+1})$ for $j = 1, \dots, k$. By the last we mean that $A(l)$ is equal to the number of $j \in \{1, \dots, k\}$ such that l is equal to $(n_{\sigma(j)}, n_{\sigma(j)+1})$.

Let us consider the term $\prod_{i=1}^p F_t\left(n_1^{(i)}, \dots, n_k^{(i)}\right)$ in the first line of (5.1.13). It turns out that it does not depend on the full information contained in $\mathcal{N}$, but only on the occupation numbers $A(l)$ introduced above. To formulate this precisely, we have to introduce certain marginals of A. For every $i = 1, \dots, p$, we define the i-th marginal of $A \in M_k$ by

$$A_i(l^{(i)}) = \sum_{(l^{(j)})_{j\neq i}\in(\mathbb{N}^2)^{p-1}} A(l^{(1)}, \dots, l^{(p)}), \qquad l^{(i)} \in \mathbb{N}^2. \tag{5.2.1}$$

Then F_t is only a function of the occupation times vector A_i, i.e., there is a function $\overline{F}_t$ such that

$$F_t\left(n_1^{(i)}, \dots, n_k^{(i)}\right) = \overline{F}_t(A_i), \tag{5.2.2}$$

if $A \in M_k$ is the occupation times vector of $(n_{\sigma(j)}, n_{\sigma(j)+1})$ for $j = 1, \dots, k$.

Furthermore, let $\Phi(A)$ be the multi-indices $\mathcal{N}$ that produce the occupation times vectors A_i:

$$\Phi(A) = \left\{\mathcal{N} \in \mathbb{N}^{p\times k}\colon\ \forall\, i = 1, \dots, p,\ \forall\, l \in \mathbb{N}^2, \#\{j \le k\colon\ (n_j^{(i)}, n_{j+1}^{(i)}) = l\} = A_i(l)\right\}. \tag{5.2.3}$$

Lemma 5.2.1. *For any continuous function $f\colon B \to \mathbb{R}$, for any $t > 1$ and $k \in \mathbb{N}$,*

$$\begin{aligned}\mathbb{E}^{(t)}\left[(\langle f, \ell_t\rangle - \langle f, \ell_{\varepsilon,t}\rangle)^k\right] = k! \sum_{A\in M_k} \#\Phi(A)\Big(&\frac{\prod_{i=1}^p \prod_{l^{(i)}\in\mathbb{N}^2} A_i(l^{(i)})!}{\prod_{l\in(\mathbb{N}^2)^p} A(l)!}\Big) \\ &\prod_{l\in(\mathbb{N}^2)^p} (a(l) - a_\varepsilon(l))^{A(l)} \prod_{i=1}^p \overline{F}_t(\overline{A}_i).\end{aligned} \tag{5.2.4}$$

Proof. Rewriting the right-hand side of (5.1.13) in terms of the above summary, we obtain that

$$\mathbb{E}^{(t)}\Big[(\langle f,\ell_t\rangle-\langle f,\ell_{\varepsilon,t}\rangle)^k\Big]=\sum_{m=0}^{k}\binom{k}{m}\sum_{(A,r)\in M_{k,m}}\overline{F}_t(\overline{A}_i)\prod_{l\in(\mathbb{N}^2)^p}\Big[(-a(l))^{r(l)}a_\varepsilon(l)^{A(l)-r(l)}\Big]\#\Psi(A,r),\quad(5.2.5)$$

where the set Ψ is given by

$$\begin{aligned}\Psi(A,r)=\big\{(\mathcal{N},\sigma_1,\dots,\sigma_p)\in\mathbb{N}^{p\times k}\times\mathfrak{S}_k^p\colon\forall l=(l^{(1)},\dots,l^{(p)})\in(\mathbb{N}^2)^p,\\ r(l)=\#\{j\le m\colon(n^{(i)}_{\sigma_i(j)},n^{(i)}_{\sigma_i(j)+1})=l^{(i)}\ \forall i\},\\ A(l)-r(l)=\#\{j>m\colon(n^{(i)}_{\sigma_i(j)},n^{(i)}_{\sigma_i(j)+1})=l^{(i)}\ \forall i\}\big\}.\end{aligned}\quad(5.2.6)$$

In other words, $\Psi(A,r)$ is the set of those multi-indices $\mathcal{N}$ and permutations $\sigma_1,\dots,\sigma_p$ such that r is the occupation times vector of the vector $(n_{\sigma(j)},n_{\sigma(j)+1})_{j\le m}$ and $A-r$ is the one of $(n_{\sigma(j)},n_{\sigma(j)+1})_{j>m}$.

Now we evaluate this counting term. We will decompose this in the two steps of counting first the multi-indices and afterwards the permutation. First we consider the multi-indices $\mathcal{N}$ that produce the occupation times vectors A_i:

$$\Phi(A)=\big\{\mathcal{N}\in\mathbb{N}^{p\times k}\colon\ \forall\,i=1,\dots,p,\ \forall\,l\in\mathbb{N}^2,\#\{j\le k\colon\ (n^{(i)}_j,n^{(i)}_{j+1})=l\}=A_i(l)\big\}.\quad(5.2.7)$$

Given $\mathcal{N}\in\Phi(A)$, we denote

$$\Psi\,(A,r,\mathcal{N})=\big\{(\sigma_1,\dots,\sigma_p)\in\mathfrak{S}_k^p\colon(\mathcal{N},\sigma_1,\dots,\sigma_p)\in\Psi(A,r)\big\}.\quad(5.2.8)$$

Then it is clear that $\#\Psi(A,r)=\sum_{\mathcal{N}\in\Phi(A)}\#\Psi(A,r,\mathcal{N})$. The cardinalities of $\Psi(A,r,\mathcal{N})$ is counted as follows:

$$\#\Psi(A,r,\mathcal{N})=m!(k-m)!\,\Big(\frac{\prod_{i=1}^p\prod_{l^{(i)}\in\mathbb{N}^2}A_i(l^{(i)})!}{\prod_{l\in(\mathbb{N}^2)^p}A(l)!}\Big)\prod_{l\in(\mathbb{N}^2)^p}\binom{A(l)}{r(l)}\qquad\text{for}\quad A\in\Phi(A).\quad(5.2.9)$$

Let us defer the proof of (5.2.9) to the end of the proof of this lemma and use it in (5.2.5). Replacing m by $\sum_l r(l)$, the only condition on r in the set $\bigcup_{m=0}^k M_{k,m}$ that is left is that $r(l)\in\{0,\dots,A(l)\}$ for any l. Therefore, we deduce from (5.2.5) that

$$\begin{aligned}\mathbb{E}^{(t)}\Big[(\langle f,\ell_t\rangle-\langle f,\ell_{\varepsilon,t}\rangle)^k\Big]&=\sum_{A\in M_k}\sum_{\mathcal{N}\in\Phi(A)}\prod_{i=1}^p\overline{F}_t(\overline{A}_i)\\ &\quad\sum_{r(l)\in\{0,\dots,A(l)\}\,\forall l\in(\mathbb{N}^2)^p}\binom{k}{\sum_l r(l)}\prod_{l\in(\mathbb{N}^2)^p}\Big[(-a(l))^{r(l)}a_\varepsilon(l)^{A(l)-r(l)}\Big]\#\Psi(A,r,\mathcal{N})\\ &=k!\sum_{A\in M_k}\#\Phi(A)\prod_{i=1}^p\overline{F}_t(\overline{A}_i)m!(k-m)!\,\Big(\frac{\prod_{i=1}^p\prod_{l^{(i)}\in\mathbb{N}^2}A_i(l^{(i)})!}{\prod_{l\in(\mathbb{N}^2)^p}A(l)!}\Big)\\ &\quad\times\prod_{l\in(\mathbb{N}^2)^p}\Big[\sum_{r(l)=0}^{A(l)}\Big[(-a(l))^{r(l)}a_\varepsilon(l)^{A(l)-r(l)}\Big]\binom{A(l)}{r(l)}\Big].\end{aligned}\quad(5.2.10)$$

By the binomial theorem, the last line is equal to $\prod_{l\in(\mathbb{N}^2)^p}(a(l)-a_\varepsilon(l))^{A(l)}$.

Now we turn to the proof of (5.2.9) . We proceed by induction on $p \in \mathbb{N}$. For $p = 1$, we want to find out the the number of permutations σ of the numbers $1, \dots, k$ such that any $l \in \mathbb{N}^2$ appears $r(l)$ times as a pair $(n_{\sigma(j)}, n_{\sigma(j)+1})$ for $j \le m$ and $A(l) - r(l)$ times as a pair $(n_{\sigma(j)}, n_{\sigma(j)+1})$ for $j > m$. We will now describe a two-step procedure that constructs all such σ. For each $l \in \mathbb{N}^2$, choose $r(l)$ out of $A(l)$ indices $j \in \{1, \dots, k\}$ such that $(n_j, n_{j+1}) = l$. Let D be the set of those j. Then D has precisely m elements and there are $\prod_{l\in\mathbb{N}^2} \binom{A(l)}{r(l)}$ choices. Now any $\sigma \in \mathfrak{S}_k$ that maps $\{1, \dots, m\}$ onto D has the above property. Obviously, for a given D, there are $m!(k-m)!$ such σs. This shows that there are at least as many as $m!(k-m)! \prod_{l\in\mathbb{N}^2} \binom{A(l)}{r(l)}$ such σs. In other words,

$$\#\Psi(A, r, \mathcal{N}) \ge \prod_{l\in\mathbb{N}^2} \binom{A(l)}{r(l)} m!(k-m)!. \tag{5.2.11}$$

To see that also the upper bound $\le$ holds, pick a $\sigma \in \Psi$ and put $D = \{\sigma(1), \dots, \sigma(m)\}$. Then, by definition of Ψ, D contains, for any l, precisely $r(l)$ out of $A(l)$ indices j satisfying $(n_j, n_{j+1}) = l$. This means that the above construction produces also the chosen σ. This shows that equality holds in (5.2.11).

For $p = 2$, we can go ahead similarly. Without loss of generality, we may assume that $\mathcal{N} \in \Phi(A)$. First we argue that

$$\{\sigma_1 \in \mathfrak{S}_k \colon \exists \sigma_2 \in \mathfrak{S}_k \colon (\sigma_1, \sigma_2) \in \Psi(A, r, \mathcal{N})\} = \Psi_1\big(A_1, r_1, \mathcal{N}^{(1)}\big) \tag{5.2.12}$$

where $\Psi_1(A_1, r_1, \mathcal{N}^{(1)})$ is defined in (5.2.6) for $p = 1$ and A and r replaced by their first marginals A_1 an r_1 respectively. Indeed, let $\sigma_1, \sigma_2 \in \mathfrak{S}_k$ be such that $r(\cdot)$ and $A(\cdot) - r(\cdot)$ are the occupation times vectors of $\big(n^{(i)}_{\sigma_i(j)}, n^{(i)}_{\sigma_i(j)+1}\big)_{i=1,2}$ for $j = 1, \dots, m$ and of $\big(n^{(i)}_{\sigma_i(j)}, n^{(i)}_{\sigma_i(j)+1}\big)_{i=1,2}$ for $j = m+1, \dots, k$, respectively. By projecting on the first row, we see that r_1 and $A_1 - r_1$ are the occupation numbers of $\big(n^{(1)}_{\sigma_1(j)}, n^{(1)}_{\sigma_1(j)+1}\big)$ for $j = 1, \dots, m$ and $\big(n^{(1)}_{\sigma_1(j)}, n^{(1)}_{\sigma_1(j)+1}\big)$ for $j = m+1, \dots, k$. This shows that $\sigma_1 \in \Psi_1(A_1, r_1, \mathcal{N}^{(1)})$.
Let us show that also $\supset$ holds in (5.2.12). Pick $\sigma_1 \in \Psi_1(A_1, r_1, \mathcal{N}^{(1)})$. Since $\mathcal{N} \in \Phi(A)$, for each $l^{(2)} \in \mathbb{N}^2$, there are precisely $A_2(l^{(2)})$ indices j such that $\big(n^{(2)}_j, n^{(2)}_{j+1}\big) = l^{(2)}$. Therefore, there is an order (i.e., a permutation σ_2 of the second row) such that, for any $l^{(1)}$ and any $r(l^{(1)}, l^{(2)})$, the set $\{j \le m \colon \big(n^{(1)}_{\sigma_1(j)}, n^{(1)}_{\sigma_1(j)+1}\big) = l^{(1)}\}$ contains precisely as many as $r(l^{(1)}, l^{(2)})$ indices j satisfying $\big(n^{(2)}_{\sigma_2(j)}, n^{(2)}_{\sigma_2(j)+1}\big) = l^{(2)}$, for any $l^{(2)} \in \mathbb{N}^2$ and the set $\{j > m \colon \big(n^{(1)}_{\sigma_1(j)}, n^{(1)}_{\sigma_1(j)+1}\big) = l^{(1)}\}$ contains precisely as many as $A(l^{(1)}, l^{(2)}) - r(l^{(1)}, l^{(2)})$ indices j satisfying $\big(n^{(2)}_{\sigma_2(j)}, n^{(2)}_{\sigma_2(j)+1}\big) = l^{(2)}$, for any $l^{(2)} \in \mathbb{N}^2$. Therefore, $(\sigma_1, \sigma_2) \in \Psi(A, r, \mathcal{N})$. This proves (5.2.12).
Hence we have

$$\#\Psi_2(A, r, \mathcal{N}) = \sum_{\sigma_1 \in \Psi_1(A, r, \mathcal{N}^{(1)})} \#\big\{\sigma_2 \in \mathfrak{S}_k \colon (\sigma_1, \sigma_2) \in \Psi_2(A, r, \mathcal{N})\big\}. \tag{5.2.13}$$

Fix $\sigma_1 \in \Psi_1(A_1, r_1, \mathcal{N}^{(1)})$. We now give a two-step construction of all $\sigma_2 \in \mathfrak{S}_k$ satisfying $(\sigma_1, \sigma_2) \in \Psi_2(A, r, \mathcal{N})$. For each $l^{(1)}, l^{(2)} \in \mathbb{N}^2$, we decompose the set $\{j \le m \colon \big(n^{(1)}_{\sigma_1(j)}, n^{(1)}_{\sigma_1(j)+1}\big) = l^{(1)}\}$ into disjoint sets $D_{l^{(1)}, l^{(2)}}$ of cardinality $r(l^{(1)}, l^{(2)})$ and the set $\{j > m \colon \big(n^{(1)}_{\sigma_1(j)}, n^{(1)}_{\sigma_1(j)+1}\big) = l^{(1)}\}$ into sets $\overline{D}_{l^{(1)}, l^{(2)}}$ of cardinality $A(l^{(1)}, l^{(2)}) - r(l^{(1)}, l^{(2)})$. For doing this, we have

$$\prod_{l^{(1)}\in\mathbb{N}^2} \frac{r_1(l^{(1)})!(A_1 - r_1)(l^{(1)})!}{\prod_{l^{(2)}\in\mathbb{N}^2} \big(r(l^{(1)}, l^{(2)})!\big)\big((A-r)(l^{(1)}, l^{(2)})!\big)}$$

choices. Having fixed these sets, every permutation σ_2 satisfying $\sigma_2\big(\{j \le k \colon \big(n^{(2)}_j, n^{(2)}_{j+1}\big) = l^{(1)}\}\big) = \bigcup_{l^{(1)}\in\mathbb{N}^2} \big(D_{l^{(1)}, l^{(2)}} \cup \overline{D}_{l^{(1)}, l^{(2)}}\big)$, $\forall l^{(2)} \in \mathbb{N}^2$, has the property that each pair $(l^{(1)}, l^{(2)})$ appears precisely $r(l^{(1)}, l^{(2)}$

times in $\big(n^{(i)}_{\sigma_i(j)}, n^{(i)}_{\sigma_i(j)+1}\big)_{i=1,2}$ for $j = 1,\dots,m$ and precisely $(A-r)(l^{(1)}, l^{(2)})$ times $\big(n^{(i)}_{\sigma_i(j)}, n^{(i)}_{\sigma_i(j)+1}\big)_{i=1,2}$ for $j = m+1,\dots,k$. That is, $(\sigma_1,\sigma_2) \in \Psi_2(A, r, \mathcal{N})$. Obviously, there are $\prod_{l^{(2)}} A_2(l^{(2)})!$ such permutations σ_2. Different choices of D and $\overline{D}$ produces different choices of permutations σ_1, σ_2. A little reflection shows that every σ_2 satisfying $(\sigma_1,\sigma_2) \in \Psi_2$ can be constructed in this way (put $D_{(l^{(1)},l^{(2)})} = \big\{j \le m \colon \big(n^{(i)}_{\sigma_i(j)}, n^{(i)}_{\sigma_i(j)+1}\big)_{i=1,2} = (l^{(1)}, l^{(2)})\big\}$ and $\overline{D}_{(l^{(1)},l^{(2)})} = \big\{j > m \colon \big(n^{(i)}_{\sigma_i(j)}, n^{(i)}_{\sigma_i(j)+1}\big)_{i=1,2} = (l^{(1)}, l^{(2)})\big\}$).

Therefore, we have

$$
\begin{aligned}
\#\Psi_2(A, r, \mathcal{N}) &= \#\Psi_1(A_1, r_1, \mathcal{N}^{(1)}) \times \prod_{l^{(2)}\in\mathbb{N}^2} A_2(l^{(2)})! \prod_{l^{(1)}\in\mathbb{N}^2} \frac{r_1(l^{(1)})!\,(A_1 - r_1)(l^{(1)})!}{\prod_{l^{(2)}\in\mathbb{N}^2} r(l^{(1)}, l^{(2)})!\,\big(A - r\big)(l^{(1)}, l^{(2)})!} \\
&= m!(k-m)!!\frac{\prod_{l^{(1)}} A_1(l^{(1)})! \prod_{l^{(2)}} A_2(l^{(2)})!}{\prod_{l^{(1)},l^{(2)}} r(l^{(1)}, l^{(2)})!\,(A-r)(l^{(1)}, l^{(2)})!} \\
&= m!(k-m)!\,\frac{\prod_{i=1}^2 \prod_{l^{(i)}\in\mathbb{N}^2} A_i(l^{(i)})!}{\prod_{l\in(\mathbb{N}^2)^2} A(l)!} \prod_{l\in(\mathbb{N}^2)^2} \binom{A(l)}{r(l)}.
\end{aligned}
\tag{5.2.14}
$$

We leave the proof for the higher $p > 2$ to the reader, as it is similar and can be carried out in a recursive manner. This proves (5.2.9) and hence Lemma 5.2.1. □

Chapter 6

Proof of Proposition 4.4.1: Super-exponential estimate.

We turn to the proof of Proposition 4.4.1. We shall do this only for $b = \mathbb{1}$. Fix a continuous bounded function f on B. Then our task is to prove that, for any $t > 1$ and $\varepsilon > 0$,

$$\Big|\mathbb{E}^{(t)}\Big[\big(\langle \ell_t, f\rangle - \langle \ell_{\varepsilon,t}, f\rangle\big)^k\Big]\Big| \leq \; k!^p \, C(\varepsilon)^k \qquad k \in \mathbb{N} \tag{6.0.1}$$

and $\lim_{\varepsilon\downarrow 0} C(\varepsilon) = 0$.

Note that we have now the absolute value signs outside the expectation, in contrast to (4.4.1). This is sufficient for proving (4.4.1), since, for k even, we can drop the absolute value signs anyway, and for k odd, we use Jensen's inequality to go from the power k to $k+1$ and use that $((k+1)!^p)^{k/(k+1)} \leq k!^p C^k$ for some $C \in (0, \infty)$ and all $k \in \mathbb{N}$.

In proving (6.0.1), the main task is to get hold of k-th power of a small constant. Our proof follows similar methods developed in the last chapter, see (5.2.4). However, while estimating the right hand side of (5.2.4), we see that ls in the product $\prod_{l\in(\mathbb{N}^2)^p}(a_l - a_l^\varepsilon)^{A_l}$ are not bounded and therefore, the difference $a_l - a_l^\varepsilon$ is not uniformly small. The obstacle stems from two singularities: (1) the time parameters r_j getting small and (2) the indices n_j attached to the corresponding eigenfunction ψ_{n_j} getting large. These two singularities hinder us from integrating $\int_{[0,t]} \mathrm{d}r_j$ along with the infinite sum $\sum_{n_j\in\mathbb{N}}$. Hence, we expand only those transition densities $p_{r_j}^{(B)}(x, y)$ for which $r_j > \delta$, $\delta > 0$ being an auxiliary parameter. For this part, large n_j indices can easily be summed out, thanks to the presence of the exponentially small factors $\exp\{-\lambda_{n_j} r_j\}$. The rest of the transition densities (for which $r_j \leq \delta$) stay over and are finally integrated out in terms of the Green's function. We employ our binomial trick to those $j \in \{1, \dots, k\}$ for which both $r_j >$ and n_j stay away from the respective singularities. We turn to details.

6.1 Eigenvalue expansion and elimination of small contributions.

Recall that we have to show (6.0.1). We pick up from our moment formula in (5.1.1):

$$\mathbb{E}^{(t)}_{x_0,x_{k+1}}\Big[\big(\langle f,\ell_t\rangle-\langle f,\ell_{\varepsilon,t}\rangle\big)^k\Big]=\sum_{m=0}^k(-1)^m\binom{k}{m}\int_{B^k}\prod_{i=1}^k\big(f(y_i)\,\mathrm{d}y_i\big)$$

$$\prod_{i=1}^p\Big[\sum_{\sigma\in\mathfrak{S}_k}\int_{[0,t]^k}\mathrm{d}r_k\dots\mathrm{d}r_1\,\mathbb{1}\{\textstyle\sum_{i=1}^k r_i\le t\}\displaystyle\int_{B^{k-m}}\prod_{j=m+1}^k\Big(\varphi_\varepsilon(y_j-z_j)\,\mathrm{d}z_j\Big)\prod_{j=1}^{k+1}p^{(B)}_{r_j}(x^{(i)}_{j-1},x^{(i)}_j)\Big],$$

where we abbreviate $r_{k+1}=t-\sum_{i=1}^k r_i$ and, for $j=1,\dots,k$,

$$x_j=x_j^{(i)}=\begin{cases}y_{\sigma^{-1}(j)} & \text{if }\sigma^{-1}(j)\le m,\\ z_{\sigma^{-1}(j)} & \text{if }\sigma^{-1}(j)>m.\end{cases}$$

For brevity, we set forth the following notations. We abbreviate, with a slight abuse of notation,

$$\begin{aligned}\int\mathrm{d}y\prod f &= \int_B\mathrm{d}y_1\dots\int_B\mathrm{d}y_k\prod_{j=1}^k f(y_j),\\ \int_<\mathrm{d}r &= \int_{[0,t]^k}\mathrm{d}r_k\dots\mathrm{d}r_1\mathbb{1}\{\textstyle\sum_{i=1}^k r_i\le t\}\displaystyle\qquad\Big(r_{k+1}=t-\sum_{i=1}^k r_i\Big),\\ \int\mathrm{d}z\,\varphi_\varepsilon &= \int_B\mathrm{d}z_{m+1}\dots\int_B\mathrm{d}z_k\prod_{j=m+1}^k\varphi_\varepsilon(y_j-z_j).\end{aligned}$$

Our next main step is to expand the transition density terms $p^{(B)}_{r_i}(x_{i-1},x_i)$ in a standard Fourier series with respect to all the eigenvalues and eigenfunctions of $-\frac12\Delta$ in B with zero boundary condition, see Lemma 5.1.2. However, as mentioned in the preceding discussion, this series has only then good convergence properties if the time parameter r_i is bounded away from zero. Therefore, we introduce a new small parameter $\delta\in(0,\infty)$ and distinguish, for each integration variable r_i, if $r_i\le\delta$ or $r_i>\delta$. Introducing another small parameter $\eta\in(0,\infty)$, we isolate the contribution from those multi-indices $(r_1,\dots,r_k)$ such that less than ηk of the indices i satisfy $r_i\le\delta$. In other words, we write

$$\int_<\mathrm{d}r=\sum_{D\subset\{1,\dots,k+1\}}\int_<\mathrm{d}r\prod_{j\in D}\mathbb{1}_{r_j\le\delta}\prod_{j\notin D}\mathbb{1}_{r_j>\delta}$$

and see from (5.1.1) that

$$\mathbb{E}^{(t)}_{x_0,x_{k+1}}\Big[\big(\langle f,\ell_t\rangle-\langle f,\ell_{\varepsilon,t}\rangle\big)^k\Big]=(I)_{t,k}(\eta,\delta,\varepsilon)+(II)_{t,k}(\eta,\delta,\varepsilon),\tag{6.1.1}$$

where

$$\begin{aligned}(I)_{t,k}(\eta,\delta,\varepsilon)=\sum_{m=0}^k(-1)^m\binom{k}{m}\int\mathrm{d}y\prod f\sum_{\substack{\forall i=1,\dots,p:\ D_i\subset\{1,\dots,k+1\}\\ \#D_i\le\eta k}}\\ \prod_{i=1}^p\Big[\sum_{\sigma\in\mathfrak{S}_k}\int_<\mathrm{d}r\prod_{j\in D_i}\mathbb{1}_{r_j\le\delta}\prod_{j\in D_i^{\mathrm{c}}}\mathbb{1}_{r_j>\delta}\int\mathrm{d}z\,\varphi_\varepsilon\prod_{j=1}^{k+1}p^{(B)}_{r_j}(x_{j-1},x_j)\Big],\end{aligned}\tag{6.1.2}$$

and $(II)_{t,k}(\eta,\delta,\varepsilon)$ is defined accordingly, that is, with the sum on the D_i replaced by the sum on $D_1,\dots,D_p\subset\{1,\dots,k+1\}$ satisfying $\#D_i>\eta k$ for at least one $i\in\{1,\dots,p\}$. This last term has a small exponential rate for fixed η if δ is small, since there are at least ηk integrations $r_i\in[0,\delta]$:

Lemma 6.1.1 (Riddance of small δ). *For every $\eta,\delta>0$, there is $C(\eta,\delta)>0$ such that, for any $\varepsilon\in(0,1]$,*

$$\Big|(II)_{t,k}(\eta,\delta,\varepsilon)\Big|\le k!^pC(\eta,\delta)^k,\qquad t\in(0,\infty),k\in\mathbb{N},\tag{6.1.3}$$

where $C(\eta,\delta)\downarrow 0$ as $\delta\downarrow 0$.

Proof. Note that the only i-dependence of the factors in the last line of (6.1.2) sits in the starting and ending points, $x_0^{(i)}$ and $x_{k+1}^{(i)}$. We neglect the changing signs $(-1)^m$ and estimate $\binom{k}{m}\le 2^k$ and estimate against the supremum over all $x_0^{(i)}\in B$ and all $x_{k+1}^{(i)}$ for each $i=1,\dots,p$. Hence, the sum on $D_1,\dots,D_p$ satisfying $\#D_i>\eta k$ for at least one i is equal to p times the sum on those $D_1,\dots,D_p$ satisfying $\#D_1>\eta k$. Estimating also $|f|\le C$ and dropping the indicator on $\{\sum_{j=1}^k r_j\le t\}$ and carrying out the integration on r_j, we obtain,

$$\begin{aligned}|(II)|\le p(2C)^k\sup_{x_0,x_{k+1}\in B^k}\sum_{m=0}^k\int_{B^k}\mathrm{d}y_1\dots\mathrm{d}y_k\prod_{i=2}^p\Big[\sum_{\sigma_i\in\mathfrak{S}_k}\int\varphi_\varepsilon\prod_{j=1}^{k+1}G(x_{j-1},x_j)\Big]\\ \times\sum_{D_1\colon\#D_1>\eta k}\sum_{\sigma_1\in\mathfrak{S}_k}\int\varphi_\varepsilon\prod_{j\in D_1}G_\delta(x_{j-1},x_j)\prod_{j\in D_1^{\mathrm{c}}}G(x_{j-1},x_j),\end{aligned}$$

where G is the Green's function in B and $G_\delta(v,w)=\int_0^\delta\mathrm{d}s\,p_s^{(B)}(v,w)$ is the truncated Green's function. Now we carry out the convolution integrals over $\mathrm{d}z_{m+1}\dots\mathrm{d}z_k$, which turns some of the (truncated) Green's functions into convolved (truncated) Green's functions, each of which can be estimated against $G^{(\star\varepsilon)}$ and $G_\delta^{(\star\varepsilon)}$, respectively, where

$$G^{(\star\varepsilon)}(x,y)=\max\Big\{G(x,y),(G(x,\cdot)\star\varphi_\varepsilon)(y)\Big\},\tag{6.1.4}$$

and an analogous notation for G replaced by G_δ.

Now we interchange the integration over $y_1,\dots,y_k$ and the sum on σ_1, such that, after some elementary substitutions involving all the permutations, this sum on σ_1 is turned into $k!$ times the term with σ_1 equal to the identical permutation. This gives

$$\begin{aligned}|(II)|\le k!\,p(2C)^k\sup_{x_0,x_{k+1}\in B^k}\sum_{m=0}^k\int_{B^k}\mathrm{d}y_1\dots\mathrm{d}y_k\prod_{i=2}^p\Big[\sum_{\sigma_i\in\mathfrak{S}_k}\prod_{j=1}^{k+1}G^{(\star\varepsilon)}(x_{j-1},x_j)\Big]\\ \times\sum_{D_1\colon\#D_1>\eta k}\prod_{j\in D_1}G_\delta^{(\star\varepsilon)}(y_{j-1},y_j)\prod_{j\in D_1^{\mathrm{c}}}G^{(\star\varepsilon)}(y_{j-1},y_j).\end{aligned}$$

Note that, for any $\tilde\delta>0$,

$$\limsup_{\delta\downarrow0}\sup_{\varepsilon\in(0,1]}\sup_{\substack{v,w\in B\colon\\|v-w|\ge\tilde\delta}}G_\delta^{(\star\varepsilon)}(v,w)=0,\qquad\text{and}\qquad\limsup_{\tilde\delta\downarrow0}\sup_{\varepsilon\in(0,1]}\sup_{x\in B}\int_{|x-y|\le\tilde\delta}G^{(\star\varepsilon)}(x,y)^p\,\mathrm{d}y=0.\tag{6.1.5}$$

In order to employ these two facts, we separate the product over $i=2,\dots,p$ from the last line with the help of Hölder's inequality and distinguish in the latter term those integrals over $\mathrm{d}y_1\dots\mathrm{d}y_k$ that satisfy

$\#\{j \in D_1\colon |y_{j-1}-y_j| \le \tilde\delta\} > \tilde\eta k$ and the remainder, where $\tilde\delta > 0$ and $\tilde\eta > 0$ are new small auxiliary parameters. The first contribution gives at least $\tilde\eta k$ integrals over $G_\delta^{(\star\varepsilon)}(y_{j-1},y_j)^p\,\mathrm{d}y_j$ with $|y_{j-1}-y_j| \le \tilde\delta$ (and therefore a small number) and in the second, we have at least $\tilde\eta k$ indices j with $|y_{j-1}-y_j| > \tilde\delta$, which makes it possible to estimate $G_\delta^{(\star\varepsilon)}(y_{j-1},y_j)$ against a small number. Hence, the contribution from the last line is bounded by $k!\widetilde C(\delta,\eta)^k$ for some suitable $\widetilde C(\delta,\eta) \in (0,\infty)$ satisfying $\lim_{\delta\downarrow 0}\widetilde C(\delta,\eta) = 0$. The other terms (that is, those that stem from the product over $i = 2,\dots,p$) can be bounded against $k!^{p-1}C^k$ for some constant C that does not depend on k. Summarizing, we obtain the estimate in (6.1.3) with some suitable $C(\delta,\eta)$. The details are pretty standard and we refer the reader to the proof of [KM02, Lemma 3.3]. □

Now we go on with the term (I) defined in (6.1.2) and recall the eigenvalue expansion of Lemma 5.1.2:

Lemma 6.1.2. *There exist a system of eigenvalues $0 < \lambda_1 \le \lambda_2 \le \dots$ and an $L^2(B)$-orthonormal basis of corresponding eigenfunctions $\psi_1, \psi_2, \dots$ in B of $-\frac12\Delta$ with zero boundary condition in B, that is, $-\frac12\Delta\psi_n = \lambda_n\psi_n$ for any $n \in \mathbb{N}$. Furthermore,*

$$p_s^{(B)}(x,y) = \sum_{n=1}^\infty \mathrm{e}^{-s\lambda_n}\,\psi_n(x)\psi_n(y), \qquad s > 0, \tag{6.1.6}$$

and the convergence is absolute and uniform in $x, y \in B$.

For every $i = 1,\dots,p$ and each $j \in D_i^{\mathrm c}$, i.e., for any time duration $r_j \ge \delta$, we expand $p_{r_j}^{(B)}(x_{j-1},x_j)$ into a eigenvalue series as in the above Lemma, introducing a sum on $\mathcal N^{(i)} = (n_j^{(i)})_{j\in D_i^{\mathrm c}} \in \mathbb{N}^{D_i^{\mathrm c}}$. Because $r_j \ge \delta$ and the appearance of the factor $\exp\{-r_j\lambda_{n_j^{(i)}}\}$, the sum on $n_j^{(i)}$ converges exponentially fast.

The eigenfunctions $\psi_{n_j^{(i)}}$ will later be used for an application of the binomial theorem, but this will turn out to be helpful only if all indices $n_j^{(i)}$ appearing are taken from some bounded set. Therefore, we truncate this infinite sum at a large cut off level $R \in \mathbb{N}$. We write $\mathcal R = \{1,\dots,R\}$ and split each sum on $n_j^{(i)}$ into the two sums on $n_j^{(i)} \in \mathcal R$ and $n_j^{(i)} \in \mathcal R^{\mathrm c}$. This gives, for every i, sums of the form

$$\prod_{j\in D_i^{\mathrm c}}\Big(\sum_{n_j^{(i)}\in\mathcal R} + \sum_{n_j^{(i)}\in\mathcal R^{\mathrm c}}\Big) = \sum_{E_i\subset D_i^{\mathrm c}}\ \sum_{\mathcal N^{(i)}\in\mathcal R^{E_i}}\ \sum_{\mathcal N^{(i)}\in(\mathcal R^{\mathrm c})^{D_i^{\mathrm c}\setminus E_i}},$$

with the understanding that $\mathcal N^{(i)} \in \mathcal R^{E_i}$ and $\mathcal N^{(i)} \in (\mathcal R^{\mathrm c})^{D_i^{\mathrm c}\setminus E_i}$ may be concatenated to some map $\mathcal N^{(i)}\colon D_i^{\mathrm c} \to \mathbb{N}$.

We now introduce another small parameter $\gamma \in (0,\infty)$ and distinguish the contribution coming from those multi-sums with sets E_i satisfying $\#(D_i^{\mathrm c}\setminus E_i) \le \gamma k$ for all i and the remainder. This implies the decomposition $(I)_{t,k}(\eta,\delta,\varepsilon) = (Ia)_{t,k}(\eta,\gamma,\delta,\varepsilon,R) + (Ib)_{t,k}(\eta,\gamma,\delta,\varepsilon,R)$, where $(Ia) = (Ia)_{t,k}(\eta,\gamma,\delta,\varepsilon,R)$ is defined as

$$\begin{aligned}(Ia) = &\sum_{\substack{\forall i\colon D_i\subset\{1,\dots,k+1\}\\ \#D_i\le\eta k}}\ \sum_{\substack{\forall i\colon E_i\subset D_i^{\mathrm c}\\ \#(D_i^{\mathrm c}\setminus E_i)\le\gamma k}}\ \sum_{\forall i\colon \mathcal N^{(i)}\in\mathcal R^{E_i}}\ \sum_{\forall i\colon \mathcal N^{(i)}\in(\mathcal R^{\mathrm c})^{D_i^{\mathrm c}\setminus E_i}}\ \sum_{m=0}^k(-1)^m\binom{k}{m}\\ &\int\mathrm{d}y\prod f\prod_{i=1}^p\Big[\sum_{\sigma\in\mathfrak S_k}\int_<\mathrm{d}r\,H_r(\mathcal N^{(i)}|_{D_i^{\mathrm c}};D_i)\int\mathrm{d}z\,\varphi_\varepsilon\prod_{j\in D_i}p_{r_j}^{(B)}(x_{j-1},x_j)\prod_{j\in D_i^{\mathrm c}}\psi_{n_j^{(i)}}(x_{j-1})\psi_{n_j^{(i)}}(x_j)\Big]\end{aligned} \tag{6.1.7}$$

where

$$H_r(\mathcal{N}^{(i)}; D_i) = \Big(\prod_{j\in D_i} 1\!\!1_{r_j\le\delta}\Big) \prod_{j\in D_i^{\rm c}} \Big(1\!\!1_{r_j>\delta}\, \exp\Big\{-r_j\lambda_{n_j^{(i)}}\Big\}\Big). \tag{6.1.8}$$

The definition of (Ib) is according, i.e., for at least one $i\in\{1,\dots,p\}$, the set E_i satisfies $\#(D_i^{\rm c}\setminus E_i)>\gamma k$. That is, for at least one i, the sum on $n_j^{(i)}$ runs over the remainder set $\mathcal{R}^{\rm c}$ for at least γk different js and gives therefore, for large R, a small factor with power at least γk. Let us first show that therefore $(Ib)_{t,k}(\eta,\gamma,\delta,\varepsilon,R)$ is a small error term if R is large for fixed γ:

Lemma 6.1.3 (Riddance of large $\mathcal{N}$). *For every $\eta,\gamma,\delta\in(0,1)$ and $R\in\mathbb{N}$, there is $C^{(b)}(\eta,\gamma,\delta,R)>0$ such that, for any $\varepsilon\in(0,1)$,*

$$(Ib)_{t,k}(\eta,\gamma,\delta,\varepsilon,R)\le k!^p C^{(b)}(\eta,\gamma,\delta,R)^k, \qquad t\in(0,\infty), k\in\mathbb{N}, \tag{6.1.9}$$

and $C^{(b)}(\eta,\gamma,\delta,\varepsilon,R)\downarrow 0$ as $R\uparrow\infty$.

Proof. We use a generic contant C that does not depend on the parameters involved, but only on B, f or d. In (6.1.7) (with the neccessary changes for (Ib)), we estimate $\sum_{m=0}^k(-1)^m\binom{k}{m}\le 2^k$ and $\|f\|_\infty\le C$ and $\int_< {\rm d}r\le\int_{[0,\infty)^k}{\rm d}r_1\dots{\rm d}r_k$ and

$$H_r(\mathcal{N}^{(i)}; D_i)\le\Big(\prod_{j\in D_i^{\rm c}\setminus E_i} 1\!\!1_{r_j>\delta}\,\exp\Big\{-r_j\lambda_{n_j^{(i)}}\Big\}\Big)\prod_{j\in E_i}\exp\Big\{-r_j\lambda_1\Big\}.$$

Next, in (Ib) we estimate all the terms against their absolute value and then apply the uniform eigenfunction estimate [Gr02]

$$\|\psi_n\|_\infty\le C\lambda_n^{\frac{d-1}{4}}, \qquad n\in\mathbb{N}, \tag{6.1.10}$$

to the eigenfunction product $\prod_{j\in D_i^{\rm c}}\psi_{n_j^{(i)}}(x_{j-1})\psi_{n_j^{(i)}}(x_j)$ to see that (recall the notation in (6.1.4))

$$\begin{aligned}(Ib)\le C^k &\sum_{\substack{\forall i\colon D_i\subset\{1,\dots,k+1\}\\ \#D_i\le\eta k}}\ \sum_{\substack{\forall i\colon E_i\subset D_i^{\rm c}\\ \exists j\colon \#(D_j^{\rm c}\setminus E_j)>\gamma k}} \int{\rm d}y\prod_{i=1}^p\Big[\Big(\sum_{\sigma\in\mathfrak{S}_k}\prod_{j\in D_i}G^{(\star\varepsilon)}(x_{j-1},x_j)\Big)\Big(\prod_{j\in E_i}\sum_{n_j^{(i)}\in\mathcal{R}}\lambda_{n_j^{(i)}}^{\frac{d-1}{2}}\Big)\\ &\times\Big(\prod_{j\in D_i^{\rm c}\setminus E_i}\sum_{n_j^{(i)}\in\mathcal{R}^{\rm c}}\int_\delta^\infty{\rm d}r\,{\rm e}^{-r\lambda_{n_j^{(i)}}}\lambda_{n_j^{(i)}}^{\frac{d-1}{2}}\Big)\Big(\int_{[0,\infty)^{E_i}}{\rm d}r\prod_{j\in E_i}{\rm e}^{-r_j\lambda_1}\Big)\Big]\\ \le C^kC_\delta(R)^{\gamma k}C(R)^{pk}&\sum_{\substack{\forall i\colon D_i\subset\{1,\dots,k+1\}\\ \#D_i\le\eta k}}\ \sum_{\substack{\forall i\colon E_i\subset D_i^{\rm c}\\ \exists j\colon \#(D_j^{\rm c}\setminus E_j)>\gamma k}}\int{\rm d}y\prod_{i=1}^p\Big(\sum_{\sigma\in\mathfrak{S}_k}\prod_{j\in D_i}G^{(\star\varepsilon)}(x_{j-1},x_j)\Big),\end{aligned} \tag{6.1.11}$$

where $C_\delta(R)=\sum_{n\in\mathcal{R}^{\rm c}}\int_\delta^\infty{\rm d}r\,{\rm e}^{-r\lambda_n}\lambda_n^{(d-1)/2}$ and $C(R)=\sum_{n\in\mathcal{R}}\lambda_n^{(d-1)/2}$, and we have estimated $\int_0^\infty{\rm d}r\,{\rm e}^{-r\lambda_1}\le C$ for some $C>1$. We assumed that R is so large that $C_\delta(R)<1$ and $C(R)\ge 1$. Use that $\sup_{\varepsilon\in(0,1]}\sup_{x\in B}\int_B{\rm d}y\,G^{(\star\varepsilon)}(x,y)^p\le C$ (see the second statement in (6.1.5)) to see that the sum on $\sigma\in\mathfrak{S}_k$ is not larger than $k!^pC^k$. The two sums on the sets D_i and E_i have no more than C^k terms.

By the well-known Weyl lemma, λ_n tends to ∞ like $n^{2/d}$. Hence, $C_\delta(R)$ decays stretched-exponentially fast to zero as $R\uparrow\infty$ (the rate depends on δ only), and C_R tends to ∞ only polynomially, hence we may estimate $C^kC_\delta(R)^{\gamma k}C(R)^{pk}\le C^{(b)}(\eta,\gamma,\delta,R)^k$ with some constant satisfying $C^{(b)}(\eta,\gamma,\delta,\varepsilon,R)\downarrow 0$ as $R\uparrow\infty$. This finishes the proof. □

6.2 Estimating the main term

After the preparations in Lemma 6.1.1 and 6.1.3, we now estimate the main term (Ia) defined in (6.1.7), which is the heart of the proof. The proof of (6.0.1), and therefore the proof of Proposition 4.4.1, is finished by the two lemmas, together with the following proposition, see (6.1.1) and recall the decomposition $(I) = (Ia) + (Ib)$.

Proposition 6.2.1 (The main estimate)**.** *For every $\eta, \gamma, \delta, \varepsilon \in (0, 1)$ such that $\eta + \gamma < 1/2p$ and for every $R \in \mathbb{N}$, there is a constant $C^{(a)}(\eta, \gamma, \delta, \varepsilon, R) > 0$ such that,*

$$\Big|(Ia)_{t,k}(\eta, \gamma, \delta, \varepsilon, R)\Big| \leq k!^p C^{(a)}(\eta, \gamma, \delta, \varepsilon, R)^k, \qquad t \in (0, \infty), k \in \mathbb{N}, \tag{6.2.1}$$

and $C^{(a)}(\eta, \gamma, \delta, \varepsilon, R) \downarrow 0$ as $\varepsilon \downarrow 0$.

Proof. Step 1: Rewrite of eigenfunction terms. First we unravel the last term involving the eigenfunctions appearing in the right hand side of (6.1.7). Observe that $z_j = z_j^{(i)}$ and $x_j = x_j^{(i)}$ in the i-th factor both depend on i, and we write σ_i instead of σ. Recall from Lemma 5.1.1 that

$$x_j^{(i)} = \begin{cases} y_{\sigma_i^{-1}(j)} & \text{if } \sigma_i^{-1}(j) \leq m, \\ z^{(i)}_{\sigma_i^{-1}(j)} & \text{if } \sigma_i^{-1}(j) > m. \end{cases} \tag{6.2.2}$$

Therefore, the last term in the second line of (6.1.7) reads as follows.

$$\begin{aligned} \prod_{j \in D_i^c} \big(\psi_{n_j^{(i)}}(x_{j-1}^{(i)}) \psi_{n_j^{(i)}}(x_j^{(i)})\big) = \Big(\prod_{\substack{j \in \sigma_i^{-1}(D_i^c) \\ j \leq m}} \psi_{n^{(i)}_{\sigma_i(j)}}(y_j) \Big) \Big(\prod_{\substack{j \in \sigma_i^{-1}(D_i^c - 1) \\ j \leq m}} \psi_{n^{(i)}_{\sigma_i(j)+1}}(y_j) \Big) \\ \times \Big(\prod_{\substack{j \in \sigma_i^{-1}(D_i^c) \\ j > m}} \psi_{n^{(i)}_{\sigma_i(j)}}(z_j^{(i)}) \Big) \Big(\prod_{\substack{j \in \sigma_i^{-1}(D_i^c - 1) \\ j > m}} \psi_{n^{(i)}_{\sigma_i(j)+1}}(z_j^{(i)}) \Big). \end{aligned}$$

We now carry out the φ_ε-convolution integration over all $z_j^{(i)}$ and the integration over all those y_j that satisfy the following: (1) they exclusively appear in the above product twice for every $i \in \{1, \dots, p\}$ (but not in the product over the $p^{(B)}_{r_j}$-terms with $j \in D_i$ for any i), i.e., $\sigma_i(j)$ and $\sigma_i(j) + 1$ both lie in D_i^c, and (2) the index $n^{(i)}_{\sigma_i(j)}$ respectively $n^{(i)}_{\sigma_i(j)+1}$ at the corresponding ψ lies in $\mathcal{R}$, i.e., both indices $\sigma_i(j)$ and $\sigma_i(j) + 1$ lie in E_i. Since $E_i \subset D_i^c$, these are precisely those j that satisfy $j \in S(\sigma)$, where we set, for each $\sigma = (\sigma_1, \dots, \sigma_p) \in \mathfrak{S}_k^p$,

$$S(\sigma) = \bigcap_{i=1}^p \sigma_i^{-1}(F_i), \qquad \text{where} \qquad F_i = E_i \cap (E_i - 1).$$

Certainly, we have to obey that, for $j \leq m$, the integration is over y_j and for $j > m$ it is the convolution with φ_ε. To express this, we write, for every subset $S \subset \{1, \dots, k\}$,

$$S_\leq = S \cap \{1, \dots, m\} \qquad \text{and} \qquad S_> = S \cap \{m + 1, \dots, k\}.$$

Each $j \in S(\sigma)$ appears only in the product over $\psi_{(\dots)}$ or $\varphi_\varepsilon \star \psi_{(\dots)}$, whereas for $j \in S(\sigma)^c = \{1, \dots, k\} \setminus S(\sigma)$, the eigenfunction products stay over and remain unconvolved. We write $\mathcal{N} = (\mathcal{N}^{(1)}, \dots, \mathcal{N}^{(p)})$ and $\mathcal{N}_j =$

$(n_j^{(1)}, \dots, n_j^{(p)})$ and recall that, for $j \in S(\sigma)$,

$$a(\mathcal{N}_j, \mathcal{N}_{j+1}) = \Big\langle f, \prod_{i=1}^p \psi_{n_j^{(i)}} \psi_{n_{j+1}^{(i)}} \Big\rangle, \tag{6.2.3}$$

$$a_\varepsilon(\mathcal{N}_j, \mathcal{N}_{j+1}) = \Big\langle f, \prod_{i=1}^p \varphi_\varepsilon \star \big(\psi_{n_j^{(i)}} \psi_{n_{j+1}^{(i)}}\big) \Big\rangle. \tag{6.2.4}$$

Substituting this in (6.1.7), we conclude

$$\begin{aligned}(Ia) = &\sum_{\substack{\forall i\colon D_i \subset \{1,\dots,k+1\} \\ \# D_i \le \eta k}} \sum_{\substack{\forall i\colon E_i \subset D_i^{\mathrm{c}} \\ \#(D_i^{\mathrm{c}} \setminus E_i) \le \gamma k}} \sum_{\forall i\colon \mathcal{N}^{(i)} \in \mathcal{R}^{E_i}} \sum_{\forall i\colon \mathcal{N}^{(i)} \in (\mathcal{R}^{\mathrm{c}})^{D_i^{\mathrm{c}} \setminus E_i}} \sum_{m=0}^k (-1)^m \binom{k}{m} \\ &\times \sum_{\sigma = (\sigma_1, \dots, \sigma_p) \in \mathfrak{S}_k^p} \Big[\prod_{j \in S(\sigma)_\le} a\big(\mathcal{N}_{\sigma(j)}, \mathcal{N}_{\sigma(j)+1}\big)\Big] \Big[\prod_{j \in S(\sigma)_>} a_\varepsilon\big(\mathcal{N}_{\sigma(j)}, \mathcal{N}_{\sigma(j)+1}\big)\Big] G_t\big(m, D, E, \sigma, \mathcal{N}\big),\end{aligned} \tag{6.2.5}$$

where we wrote $\mathcal{N}_{\sigma(j)} = (n_{\sigma_i(j)}^{(i)})_{i=1,\dots,p}$ and $D = (D_1, \dots, D_p)$ and $E = (E_1, \dots, E_p)$, and the remainder term is given as

$$\begin{aligned} G_t\big(m, D, E, \sigma, \mathcal{N}\big) = &\int_{B^{S(\sigma)^{\mathrm{c}}}} \mathrm{d}y \prod_{j \in S(\sigma)^{\mathrm{c}}} f(y_j) \\ \prod_{i=1}^p \Big[\int_< \mathrm{d}r\, H_r(\mathcal{N}^{(i)}; D_i) &\int \prod_{j \in W_i\colon j > m} \big(\mathrm{d}z_j^{(i)} \varphi_\varepsilon(y_j - z_j^{(i)})\big) \prod_{j \in D_i} p_{r_j}^{(B)}(x_{j-1}^{(i)}, x_j^{(i)}) \\ &\times \Big(\prod_{j \in \sigma_i^{-1}(D_i^{\mathrm{c}} \setminus F_i)\colon j \le m} \psi_{n_{\sigma_i(j)}^{(i)}}(y_j)\Big)\Big(\prod_{j \in \sigma_i^{-1}((D_i^{\mathrm{c}} - 1) \setminus F_i)\colon j \le m} \psi_{n_{\sigma_i(j)+1}^{(i)}}(y_j)\Big) \\ &\times \Big(\prod_{j \in \sigma_i^{-1}(D_i^{\mathrm{c}} \setminus F_i)\colon j > m} \psi_{n_{\sigma_i(j)}^{(i)}}(z_j^{(i)})\Big)\Big(\prod_{j \in \sigma_i^{-1}((D_i^{\mathrm{c}} - 1) \setminus F_i)\colon j > m} \psi_{n_{\sigma_i(j)+1}^{(i)}}(z_j^{(i)})\Big)\Big],\end{aligned} \tag{6.2.6}$$

where we recall that $F_i = E_i \cap (E_i - 1)$. Note that G_t depends on $\mathcal{N}^{(i)}$ only via its restriction to D_i^{c} and on σ_i only via its restriction to

$$W_i^{\mathrm{c}} = \sigma_i^{-1}\big((D_i^{\mathrm{c}} \setminus F_i) \cup ((D_i^{\mathrm{c}} - 1) \setminus F_i) \cup D_i \cup (D_i - 1)\big) = \sigma_i^{-1}(F_i^{\mathrm{c}}), \tag{6.2.7}$$

where $^{\mathrm{c}}$ denotes the complement in $\{1, \dots, k\}$.

Step 2: Cutting and permutation symmetry.

We write $m = m_1 + m_2$ and $k - m = m_3 + m_4$, where $m_1 = \# S(\sigma)_\le$ and $m_3 = \# S(\sigma)_>$. With $\sum_{m=0}^k (-1)^m \binom{k}{m}$ in front, the second line of (6.2.5) reads

$$\begin{aligned} &\sum_{\substack{m_1, m_2, m_3, m_4 \in \mathbb{N}_0 \\ \sum_{l=1}^4 m_l = k}} (-1)^{m_2} \binom{k}{m_1 + m_2} \sum_{\substack{S_\le \subset \{1, \dots, m_1 + m_2\} \\ \# S_\le = m_1}} \sum_{\substack{S_> \subset \{m_1 + m_2 + 1, \dots, k\} \\ \# S_> = m_3}} \\ &\times \sum_{\sigma = (\sigma_1, \dots, \sigma_p) \in \mathfrak{S}_k^p} \mathbb{1}_{\left\{\substack{S_\le = S(\sigma)_\le \\ S_> = S(\sigma)_>}\right\}} \Big[\prod_{j \in S_\le} \big(-a(\mathcal{N}_{\sigma(j)}, \mathcal{N}_{\sigma(j)+1})\big)\Big] \Big[\prod_{j \in S_>} a_\varepsilon(\mathcal{N}_{\sigma(j)}, \mathcal{N}_{\sigma(j)+1})\Big] \\ &\times G_t\big(m_1 + m_2, D, E, \sigma, \mathcal{N}\big).\end{aligned}$$

We claim that the term in the last two lines above is constant on the *sets* $S_\le$ and $S_>$ and depends only on the cardinalities m_1 of $S_\le$ and m_3 of $S_>$. More precisely, for $m = m_1 + m_2$, and any permutation $\tau \in \mathfrak{S}_k$ such that $\tau(\{1,\dots,m\}) = \{1,\dots,m\}$, we claim (putting $\sigma\circ\tau = (\sigma_1\circ\tau,\dots,\sigma_p\circ\tau)$)

(i)

$$\tau^{-1}(S(\sigma)_\le) = S(\sigma\circ\tau)_\le \qquad \text{and} \qquad \tau^{-1}(S(\sigma)_>) = S(\sigma\circ\tau)_>,$$

(ii)

$$\begin{aligned}&\prod_{j\in S(\sigma)_\le} a\big(\mathcal{N}_{\sigma(j)},\mathcal{N}_{\sigma(j)+1}\big)\prod_{j\in S(\sigma)_>} a_\varepsilon\big(\mathcal{N}_{\sigma(j)},\mathcal{N}_{\sigma(j)+1}\big)\\ &= \prod_{j\in S(\sigma\circ\tau)_\le} a\big(\mathcal{N}_{(\sigma\circ\tau)(j)},\mathcal{N}_{(\sigma\circ\tau)(j)+1}\big)\prod_{j\in S(\sigma\circ\tau)_>} a_\varepsilon\big(\mathcal{N}_{(\sigma\circ\tau)(j)},\mathcal{N}_{(\sigma\circ)\tau)(j)+1}\big),\end{aligned}$$

(iii)

$$G_t\big(m_1+m_2,D,E,\sigma,\mathcal{N}\big) = G_t\big(m_1+m_2,D,E,\sigma\circ\tau,\mathcal{N}\big).$$

Proofs of these facts are rather easy and involve straightforward computations. Indeed, (i) is seen as follows.

$$\begin{aligned}\tau^{-1}\big(S(\sigma)_\le\big) &= \tau^{-1}\Big(\bigcap_{i=1}^p S_i(\sigma_i)\Big)\cap\{1,\dots,m\} = \bigcap_{i=1}^p \tau^{-1}\big(\sigma_i^{-1}(F_i)\big)\cap\{1,\dots,m\}\\ &= \bigcap_{i=1}^p(\sigma_i\circ\tau)^{-1}(F_i)\cap\{1,\dots,m\} = S(\sigma\circ\tau)_\le.\end{aligned}$$

This proves (i) and similarly one can prove (ii). For the third part, we substitute $\widetilde{y}_j = y_{\tau(j)}$ and can perform a similar computation. Therefore, the sums on $S_\le$ and $S_>$ may be replaced by the number of summands, which is $\binom{m_1+m_2}{m_1}\times\binom{k-m_1-m_2}{m_3}$ and the definite choices

$$S^*_\le = \{1,\dots,m_1\} \qquad \text{and} \qquad S^*_> = \{m_1+m_2+1,\dots,m_1+m_2+m_3\}.$$

Multiplied with the factor $\binom{k}{m_1+m_2}$, the number gives $\frac{k!}{m_1!m_2!m_3!m_4!}$.

Recall that G_t depends on any permutation σ_i only via its restriction to $W_i^{\rm c} = \sigma_i^{-1}(F_i^{\rm c})$, see (6.2.7). Therefore, we split each permutation $\sigma_i \in \mathfrak{S}_k$ into two bijections $\sigma_i\colon W_i\to F_i$ and $\tau_i\colon W_i^{\rm c}\to F_i^{\rm c}$ and we write

$$\sum_{\sigma\in\mathfrak{S}_k^p} = \sum_{\substack{\forall i\colon W_i\subset\{1,\dots,k\}\\ \#W_i=\#F_i}}\ \sum_{\forall i\colon \sigma_i\colon W_i\to F_i}\ \sum_{\forall i\colon \tau_i\colon W_i^{\rm c}\to F_i^{\rm c}},$$

where the two latter sums go over bijections σ_i and τ_i. Furthermore, from (6.2.3) we see that the a and a_ε terms depend on $\mathcal{N}^{(i)}$ via its restriction to $F_i = E_i\cap(E_i-1)$. With this in mind, we decompose the sum on $\mathcal{N}$ as

$$\sum_{\forall i\colon \mathcal{N}^{(i)}\in\mathcal{R}^{E_i}} = \sum_{\forall i\colon \mathcal{N}^{(i)}\in\mathcal{R}^{F_i}}\ \sum_{\forall i\colon \mathcal{N}^{(i)}\in\mathcal{R}^{E_i\setminus F_i}}.$$

Putting all the material together, we conclude

$$
\begin{aligned}
(Ia) = & \sum_{\substack{\forall i\colon D_i\subset\{1,\dots,k\}\\ \#D_i\leq\eta k}} \sum_{\substack{\forall i\colon E_i\subset D_i^{\rm c}\\ \#(D_i^{\rm c}\setminus E_i)\leq\gamma k}} \sum_{\substack{\forall i\colon W_i\subset\{1,\dots,k\}\\ \#W_i=\#F_i}} \sum_{\substack{m_1,m_2,m_3,m_4\in\mathbb{N}_0\\ \sum_{l=1}^4 m_l=k}} (-1)^{m_2}\frac{k!}{m_1!m_2!m_3!m_4!}\\
& \times \sum_{\forall i\colon \mathcal{N}^{(i)}\in(\mathcal{R}^{\rm c})^{D_i^{\rm c}\setminus E_i}} \sum_{\forall i\colon \mathcal{N}^{(i)}\in\mathcal{R}^{E_i\setminus F_i}} \sum_{\forall i\colon \tau_i\colon W_i^{\rm c}\to F_i^{\rm c}} \sum_{\forall i\colon \mathcal{N}^{(i)}\in\mathcal{R}^{F_i}} G_t\big(m_1+m_2,D,E,\tau,\mathcal{N}\big)\\
& \times \sum_{\forall i\colon \sigma_i\colon W_i\to F_i} \Big[\prod_{j\in S^*_{\leq}}\big(-a(\mathcal{N}_{\sigma(j)},\mathcal{N}_{\sigma(j)+1})\big)\Big]\Big[\prod_{j\in S^*_{>}} a_\varepsilon(\mathcal{N}_{\sigma(j)},\mathcal{N}_{\sigma(j)+1})\Big].
\end{aligned}
\tag{6.2.8}
$$

Step 3: Counting permutations and multi-indices.

Our next goal is to simplify the terms starting from the sum on $\mathcal{N}^{(i)}\in\mathcal{R}^{F_i}$ on the right hand side of (6.2.8) and to show that these terms contain the k-th power of a small number if ε is small, which lays the basis of an upper bound like in (6.2.1) with a small number to the power k. This is pretty similar to the combinatorial methods develpoed in Section 5.2. However, we spell out the details for our current set up for the sake of completeness. We will count the number of $\mathcal{N}^{(1)},\dots,\mathcal{N}^{(p)}$ and of $\sigma_1,\dots,\sigma_p$ that give precisely the same contribution and to apply the binomial theorem (incorporating the sum on m_1 and m_3) for a large power of terms of the form $a_\varepsilon(l)-a(l)$, which is uniformly small if ε is small. This is the point after which we are finally allowed to use more stable estimates like the triangle inequality for absolute signs.

The starting point is that many of the multi-indices $\mathcal{N}^{(i)}\in\mathcal{R}^{F_i}$ and of the permutations $\sigma_1,\dots,\sigma_p$, $i=1,\dots,p$, give precisely the same contribution. Our task here is to identify what classes of such $\mathcal{N}$ and σ do this and to evaluate their cardinality.

First we note that the two products in the third line do not depend on each value of $(\mathcal{N}_j,\mathcal{N}_{j+1})$ for $j\in S^*$, but only on their occupation numbers, i.e., on the number $A(l)$ of occurrences of a given vector $l\in(\mathcal{R}^2)^p$ in the vector $((\mathcal{N}_j,\mathcal{N}_{j+1}))_{j\in S^*}$. Hence, $A\colon(\mathcal{R}^2)^p\to\mathbb{N}_0$ is a map satisfying $\sum_{l\in(\mathcal{R}^2)^p}A(l)=m_1+m_3$, and we will be summing on all such maps. Note that the dependence of the term G_t defined in (6.2.6) on $\mathcal{N}^{(i)}|_{F_i}$ is only via the occupation numbers $A(l)$, since these indices enter only as a product over all $j\in F_i$. Since also m_2+m_4 can be constructed from $m=m_1+m_2$ and A, we therefore may write

$$
G_t\big(m_1+m_2,D,E,\tau,\mathcal{N}\big)=\widetilde{G}_t\big(m_2+m_4,D,E,\tau,A,(\mathcal{N}^{(i)}|_{D_i^{\rm c}\setminus F_i})_{i=1,\dots,p}\big)
$$

for some suitable function $\widetilde{G}_t$ which we do not make explicit here.

However, in order to describe the last line on the right-hand side of (6.2.8), we also have to sum on all occupation numbers $r(l)$ of the vectors $(\mathcal{N}_j,\mathcal{N}_{j+1})$ in the first product and the occupation numbers (which are necessarily $A(l)-r(l)$) in the second product. This leads to a further sum on all maps $r\colon(\mathcal{R}^2)^p\to\mathbb{N}_0$ satisfying $\sum_{l\in(\mathcal{R}^2)^p}r(l)=m_1$ and $0\leq r(l)\leq A(l)$ for any $l\in(\mathcal{R}^2)^p$. We denote by M_{m_1,m_3} the set of all pairs (A,r) of such maps and by $M_{m_1+m_3}$ the set of all maps A as above. Our strategy is to write the right-hand side of (6.2.8) as a sum on $A\in M_{m_1+m_3}$ and a sum on $(A,r)\in M_{k,m}$, express both the product over the a-terms as functions of A and r, and finally to count all the tuples $(\mathcal{N}^{(i)}|_{F_i},\sigma_i)$, $i=1,\dots,p$, such that (A,r) is the pair of occupation number vectors of the vectors $(\mathcal{N}_{\sigma(j)},\mathcal{N}_{\sigma(j)+1})$ for $j\in S^*$. By the last we mean that $A(l)$ is equal to the number of $j\in S^*$ such that $l=(\mathcal{N}_{\sigma(j)},\mathcal{N}_{\sigma(j)+1})$.

In view of this discussion, the terms starting from the sum on $\mathcal{N}^{(i)}\in\mathcal{R}^{F_i}$ on the right hand side of (6.2.8) read as

$$
\sum_{(A,r)\in M_{m_1,m_3}}\widetilde{G}_t\big(m_2+m_4,D,E,\tau,A,\mathcal{N}\big)\prod_{l\in(\mathcal{R}^2)^p}\Big[(-a(l))^{r(l)}a_\varepsilon(l)^{A(l)-r(l)}\Big]\#\Psi(A,r),
\tag{6.2.9}
$$

where the set Ψ is given by

$$\Psi(A,r)=\Big\{\big(\mathcal{N}^{(i)}|_{F_i},\sigma_i\big)_{i=1,\dots,p}\colon\forall l\in(\mathcal{R}^2)^p, r(l)=\#\{j\in S^*_{\le}\colon(\mathcal{N}_{\sigma(j)},\mathcal{N}_{\sigma(j)+1})=l\},\\ A(l)-r(l)=\#\{j\in S^*_{>}\colon(\mathcal{N}_{\sigma(j)},\mathcal{N}_{\sigma(j)+1})=l\}\Big\},\tag{6.2.10}$$

where the domains of the $\mathcal{N}^{(i)}|_{F_i}$ and the σ_i are as in (6.2.8).

Now we evaluate this counting term. We will decompose this in the two steps of counting first the multi-indices and afterwards the permutation. For every $i=1,\dots,p$, we define the i-th marginal of $A\in M_{m_1+m_3}$ by

$$A_i(l^{(i)})=\sum_{(l^{(j)})_{j\neq i}\in(\mathcal{R}^2)^{p-1}}A(l^{(1)},\dots,l^{(p)}),\qquad l^{(i)}\in\mathcal{R}^2.\tag{6.2.11}$$

Now we consider the multi-indices $\mathcal{N}$ that produce the occupation times vectors A_i:

$$\Phi(A_1,\dots,A_p)=\big\{(\mathcal{N}^{(i)}|_{F_i})_{i=1,\dots,p}\colon\\ \forall\, i=1,\dots,p,\ \forall l^{(i)}\in\mathcal{R}^2,\#\{j\in S^*\colon\ (\mathcal{N}^{(i)}_j,\mathcal{N}^{(i)}_{j+1})=l^{(i)}\}=A_i(l^{(i)})\big\}.\tag{6.2.12}$$

Given $\mathcal{N}\in\Phi(A)$, we denote

$$\Psi\,(A,r,\mathcal{N})=\big\{(\sigma_i)_{i=1,\dots,p}\in\otimes_{i=1}^p\mathcal{B}(W_i,F_i)\colon(\mathcal{N},\sigma_1,\dots,\sigma_p)\in\Psi(A,r)\big\},\tag{6.2.13}$$

where we denote by $\mathcal{B}(W,F)$ the set of bijections $W\to F$. Then it is clear that $\#\Psi(A,r)=\sum_{\mathcal{N}\in\Phi(A)}\#\Psi(A,r,\mathcal{N})$. The cardinality of $\Psi(A,r,\mathcal{N})$ is given in the next lemma.

Lemma 6.2.2 (Cardinality of $\Psi(A,r,\mathcal{N})$)**.** *For any $m_1,m_3\in\mathbb{N}_0$ and any $(A,r)\in M_{m_1,m_3}$ and any $\mathcal{N}\in\Phi(A)$,*

$$\#\Psi(A,r,\mathcal{N})=m_1!m_3!\,\frac{\prod_{i=1}^p\prod_{l^{(i)}\in\mathcal{R}^2}A_i(l^{(i)})!}{\prod_{l\in(\mathcal{R}^2)^p}A(l)!}\prod_{l\in(\mathcal{R}^2)^p}\binom{A(l)}{r(l)}.\tag{6.2.14}$$

Proof. We count the number of p independent bijections $\sigma_i\colon W_i\to F_i$ for $i=1,\dots,p$ with the prescribed properties. Since $\#(\cap_{i=1}^pW_i)=\#(\cap_{i=1}^pF_i)=\#S^*$, clearly this task boils down to counting all permutations σ_i of $S^*=S^*_{\le}\cup S^*_{>}$. Therefore, we shall be counting permutations σ_i of S^*. But this is known from (5.2.9). $\square$

Now we use (6.2.14) in (6.2.9) and this in (6.2.8). Replacing m_1 on the right-hand side of (6.2.8) by $\sum_l r(l)$, the only condition on r in the set $\bigcup_{m=0}^{m_1+m_3}M_{m_1,m_3}$ that is left is that $r(l)\in\{0,\dots,A(l)\}$ for any l. Therefore, we infer from (6.2.9) and (6.2.8) that

$$(Ia)=\sum_{\substack{\forall i\colon D_i\subset\{1,\dots,k\}\\ \#D_i\le\eta k}}\ \sum_{\substack{\forall i\colon E_i\subset D_i^c\\ \#(D_i^c\setminus E_i)\le\gamma k}}\ \sum_{\substack{\forall i\colon W_i\subset\{1,\dots,k\}\\ \#W_i=\#F_i}}\ \sum_{m_2+m_4\le k}(-1)^{m_2}\frac{k!}{m_2!m_4!}\\ \sum_{\forall i\colon\mathcal{N}^{(i)}\in(\mathcal{R}^c)^{D_i^c\setminus E_i}}\ \sum_{\forall i\colon\mathcal{N}^{(i)}\in\mathcal{R}^{E_i\setminus F_i}}\ \sum_{\forall i\colon\tau_i\colon W_i^c\to F_i^c}\ \sum_{A\in M_{k-m_2-m_4}}\widetilde{G}_t\big(m_2+m_4,D,E,\tau,A,\mathcal{N}\big)\\ \times\#\Phi(A)\frac{\prod_{i=1}^p\prod_{l^{(i)}\in\mathcal{R}^2}A_i(l^{(i)})!}{\prod_{l\in(\mathcal{R}^2)^p}A(l)!}\prod_{l\in(\mathcal{R}^2)^p}\Big[\sum_{r(l)=0}^{A(l)}\big[(-a(l)^{r(l)}a_\varepsilon(l)^{A(l)-r(l)}\big]\binom{A(l)}{r(l)}\Big].\tag{6.2.15}$$

By the binomial theorem, the last term in the brackets is equal to $(a(l) - a_\varepsilon(l))^{A(l)}$.

Step 4: Finishing: some estimates.

In this step we shall prove (6.2.1) and finish the proof of Proposition 6.2.1. From now on, we will use that $|a(l) - a_\varepsilon(l)|$ is, for fixed R, small uniformly in $l \in \mathcal{R}^{2p}$ if $\varepsilon > 0$ is small, and we are allowed to use the triangle inequality to estimate all the other terms appearing in (6.2.15) in absolute value. We will use C to denote a generic positive constant that depends on f, B or d only and may change its value from appearance to appearance.

The main task now is to estimate the second line of (6.2.15) as follows. We claim that there is some $C_\delta \in (0,\infty)$ such that, for any $k, m_2, m_4 \in \mathbb{N}$ satisfying $m_2 + m_4 \le k$ and for any $A \in M_{k-m_2-m_4}$ and for any $t \in (0,\infty)$,

$$\sum_{\forall i:\, \mathcal{N}^{(i)} \in (\mathcal{R}^{\mathrm{c}})^{D_i^{\mathrm{c}} \setminus E_i}} \sum_{\forall i:\, \mathcal{N}^{(i)} \in \mathcal{R}^{E_i \setminus F_i}} \sum_{\forall i:\, \tau_i:\, W_i^{\mathrm{c}} \to F_i^{\mathrm{c}}} \big|\widetilde{G}_t\big(m_2 + m_4, D, E, \tau, A, \mathcal{N}\big)\big| \le C_\delta^k \prod_{i=1}^p \#(F_i^{\mathrm{c}})! \tag{6.2.16}$$

We defer the proof of (6.2.16) to the end of this step.

Next, it is a standard fact from combinatorics [dH00, II.2] that, for $A \in M_{k-m_2-m_4}$,

$$\#\Phi(A) \le k^p \prod_{i=1}^p \frac{\prod_{l_1^{(i)} \in \mathcal{R}} \overline{A}_i(l_1^{(i)})!}{\prod_{l^{(i)} \in \mathcal{R}^2} A_i(l^{(i)})!} \tag{6.2.17}$$

where $\overline{A_i}$ is the marginal of A_i on the first component, i.e., $\overline{A_i}(l_1) = \sum_{l_2 \in \mathcal{R}} A_i(l_1, l_2)$ for every $l_1 \in \mathcal{R}$. We estimate the sum over W_i against $\binom{k}{\#F_i}$ and the sum over D_i and E_i against C^k. Combining everything, we conclude

$$\begin{aligned}
(Ia) &\le k^p C^k C_\delta^k \sum_{m_2+m_4 \le k} \frac{k!}{m_2! m_4!} \prod_{i=1}^p \Big[\binom{k}{\#F_i} \#F_i^{\mathrm{c}}!\Big] \\
&\quad \times \sum_{A \in M_{k-m_2-m_4}} \frac{\prod_{i=1}^p \prod_{l_1^{(i)} \in \mathcal{R}} \overline{A}_i(l_1^{(i)})!}{\prod_{l \in (\mathcal{R}^2)^p} A(l)!} \prod_{l \in (\mathcal{R}^2)^p} |a(l) - a_\varepsilon(l)|^{A(l)} \\
&\le k^p C^k C_\delta^k k!^p \sum_{m_2+m_4 \le k} \frac{k!}{m_2! m_4! (k - m_2 - m_4)!} \\
&\quad \times \sum_{A \in M_{k-m_2-m_4}} \frac{(k - m_2 - m_4)!}{\prod_{l \in (\mathcal{R}^2)^p} A(l)!} \prod_{l \in (\mathcal{R}^2)^p} |a(l) - a_\varepsilon(l)|^{A(l)},
\end{aligned} \tag{6.2.18}$$

where we estimated $\#F_i! \ge (k - m_2 - m_4)!$, which is true for any i since $S^* \subset \sigma_i^{-1}(F_i)$, and $\prod_{i=1}^p \prod_{l_1^{(i)} \in \mathcal{R}} \overline{A}_i(l_1^{(i)})! \le (k - m_2 - m_4)!$, which is true since the numbers $\overline{A}_i(l_1^{(i)})$ sum up to $k - m_2 - m_4$.

Now we use the multinomial theorem to see that the last sum is equal to $C_{\varepsilon,R}^{k-m_2-m_4}$, where $C_{\varepsilon,R} = \sum_{l \in (\mathcal{R}^2)^p} |a(l) - a_\varepsilon(l)|$. Take ε so small that $C_{\varepsilon,R} < 1$, then we can estimate $C_{\varepsilon,R}^{k-m_2-m_4} \le C_{\varepsilon,R}^{k(1-2p(\eta+\gamma))}$, since

$$k - m_2 - m_4 = \#S^* = \#\bigcap_{i=1}^p W_i = \#\bigcap_{i=1}^p \big(E_i \cap (E_i - 1)\big) \ge k(1 - 2p(\eta + \gamma)),$$

since $\#D_i^{\mathrm{c}} \ge k(1-\eta)$ and $\#(D_i^{\mathrm{c}} \setminus E_i) \le \gamma k$ (and also $\#(D_i^{\mathrm{c}} \setminus (E_i - 1)) \le \gamma k$) and therefore $\#(E_i \cap (E_i - 1)) \ge k(1 - 2(\eta + \gamma))$.

The sum over $m_2+m_4\le k$ on the right-hand side of (6.2.18) equal to 3^k, which we absorb in the C^k. Hence, we derive the estimate

$$(Ia)\le k!^p k^p C^k C_\delta^k C_{\varepsilon,R}^{k(1-2p(\eta+\gamma))}.$$

Since $\lim_{\varepsilon\downarrow 0} C_{\varepsilon,R}=0$ and $\eta+\gamma<1/2p$, this estimate proves (6.2.1) and therefore finishes the proof of Proposition 6.2.1.

Now we owe the reader only the proof of (6.2.16). We estimate, recalling from (6.1.8),

$$\begin{aligned}
H_r(\mathcal{N}^{(i)};D_i) &= \Big(\prod_{j\in D_i} 1\!\!1_{r_j\le\delta}\Big)\prod_{j\in D_i^c}\Big(1\!\!1_{r_j>\delta}\ \exp\Big\{-r_j\lambda_{n_j^{(i)}}\Big\}\Big)\\
&\le \prod_{j\in D_i^c}\Big(1\!\!1_{r_j>\delta}\ \exp\Big\{-\frac{r_j}{2}\lambda_{n_j^{(i)}}\Big\}\Big)\times\prod_{j\in(D_i^c-1)}\Big(1\!\!1_{r_{j+1}>\delta}\ \exp\Big\{-\frac{r_{j+1}}{2}\lambda_{n_{j+1}^{(i)}}\Big\}\Big)\\
&\le \prod_{j\in D_i^c\setminus F_i}\Big(1\!\!1_{r_j>\delta}\ \exp\Big\{-\frac{r_j}{2}\lambda_{n_j^{(i)}}\Big\}\Big)\times\prod_{j\in(D_i^c-1)\setminus F_i}\Big(1\!\!1_{r_{j+1}>\delta}\ \exp\Big\{-\frac{r_{j+1}}{2}\lambda_{n_{j+1}^{(i)}}\Big\}\Big)\\
&\quad\times\prod_{j\in F_i}\exp\Big\{-r_j\lambda_1\Big\}.
\end{aligned}$$

Furthermore, we drop the indicator on $\{\sum_{j=1}^{k+1} r_j\le t\}$, such that all integrations on r_j can be executed freely (over $[\delta,\infty)$ for $j\notin F_i$ and over $[0,\infty)$ for $j\in F_i$) as an upper bound. In (6.2.6), we estimate the absolute value of G_t by using the triangle inequality and invoke the uniform eigenfunction estimate (recall (6.1.10)):

$$\|\psi_n\|_\infty\le C\lambda_n^{\frac{d-1}{4}}$$

to the eigenfunction products appearing in the last two lines of (6.2.6). Furthermore, in the left hand side of (6.2.16) we also summarize and estimate the sums over $\mathcal{N}^{(i)}|_{D_i^c\setminus E_i}$ and $\mathcal{N}^{(i)}|_{E_i\setminus F_i}$ as a sum over $\mathcal{N}^{(i)}|_{D_i^c\setminus F_i}\in\mathbb{N}^{D_i^c\setminus F_i}$, for $i=1,\dots,p$. Hence, we obtain, also using the notation of (6.1.5),

$$\begin{aligned}
\text{l.h.s. of (6.2.16)} &\le C^k\int_{B^{(S^*)^c}}\mathrm{d}y\prod_{j\in(S^*)^c}\prod_{i=1}^p\Bigg[\Big(\sum_{\tau_i\colon W_i^c\to F_i^c}\prod_{j\in D_i}G^{(\star\varepsilon)}(y_{\tau_i^{-1}(j-1)},y_{\tau_i^{-1}(j)})\Big)\\
&\quad\times\Big(\prod_{j\in D_i^c\setminus F_i}\sum_{n_j^{(i)}\in\mathcal{R}^c}\int_\delta^\infty\mathrm{d}r\,\mathrm{e}^{-r\lambda_{n_j^{(i)}}}\lambda_{n_j^{(i)}}^{\frac{d-1}{2}}\Big)\Big(\int_{[0,\infty)^{F_i}}\mathrm{d}r\prod_{j\in F_i}\mathrm{e}^{-r_j\lambda_1}\Big)\Bigg] \qquad (6.2.19)\\
&\le C^kC_\delta^k\Big(\prod_{i=1}^p\#F_i^c!\Big)\int_{B^{(S^*)^c}}\mathrm{d}y\prod_{i=1}^p\prod_{j\in D_i}G^{(\star\varepsilon)}(y_{j-1},y_j)
\end{aligned}$$

where $C_\delta=\sum_{n\in\mathbb{N}}\int_\delta^\infty\mathrm{d}r\,\mathrm{e}^{-r\lambda_n}\lambda_n^{(d-1)/2}\vee 1$, and we absorbed the $\#F_i$-fold power of $\int_0^\infty\mathrm{d}r\,\mathrm{e}^{-r\lambda_1}=1/\lambda_1$ in the term C^k, and we used the Jensen's inequality to the sum over $\tau_1,\dots,\tau_p$ to get hold of the term $\prod_{i=1}^p(\#F_i^c)!$. The integrals over the y_j are now bounded by C^k, thanks to the classical fact $\sup_{x\in B}\int_B\mathrm{d}y\,G^p(x,y)\le C$ for $p<d/(d-2)$. Altering the value of C_δ suitably, we finish the proof of (6.2.16). $\square$

Chapter 7

From large time to large mass: Proof of Theorem 3.2.1.

In this section we prove Theorem 3.2.1. To do this, we carry over our LDP for ℓ_{tb} as the time t diverges (Theorem 3.1.1) to an LDP for $\ell = \ell_{(\tau_1,\cdots,\tau_p)}$ with random time horizon $[0,\tau_1)\times\cdots\times[0,\tau_p)$ as the mass $\ell(U)$ diverges. Recall that U is a compact subset of B whose boundary is a Lebesgue null set. We want large deviations for the probability measures $\ell/\ell(U)$ conditional on $\mathbb{P}(\cdot \mid \ell(U) > a)$, as $a \uparrow \infty$ with rate function J defined in (3.2.1). The basic idea is to replace ℓ with ℓ_{tb} where $t = a^{1/p}$ and to optimise over $b = (b_1, \ldots, b_p)$. In other words, we cut each i-th Brownian path at some time tb_i smaller than τ_i, for some $b_i > 0$ and control the cut-off part. Theorem 3.1.1 gives the large-deviations rate for ℓ_{tb} as $t \to \infty$. Optimising over $b_1, \cdots, b_p$ gives us the desired asymptotics. Lemmas 7.1.1 and 7.2.1 below give the lower resp. upper bound in the LDP.

We pick a metric d on $\mathcal{M}(B)$ which induce the weak topology. Recall that $\mathcal{M}_U(B)$ is the subspace of positive measures on B whose restriction to U is a probability measure and for $\mu \in \mathcal{M}_U(B)$,

$$J(\mu) = \inf\Big\{\frac{1}{2}\sum_{i=1}^{p}\|\phi_i\|_2^2\colon\ \phi_1,\ldots,\phi_p \in H_0^1(B);\ \prod_{i=1}^{p}\phi_i^2 = \frac{\mathrm{d}\mu}{\mathrm{d}x}\Big\}$$

if μ has a density, else $J(\mu) = \infty$.

7.1 The lower bound.

Lemma 7.1.1 (Lower bound)**.** *For every open set $G \subset \mathcal{M}_U(B)$, we have*

$$\liminf_{a\uparrow\infty}\frac{1}{a^{1/p}}\log\mathbb{P}\left(\frac{\ell}{\ell(U)} \in G, \ell(U) > a\right) \geq -\inf_{\mu\in G} J(\mu). \tag{7.1.1}$$

Proof. Set $t = a^{1/p}$ and fix $b = (b_1, \ldots, b_p) \in (0,\infty)^p$. We use that, for any $\delta_1, \delta_2 > 0$,

$$\begin{aligned}\{\ell(U) > a\} &\supset \big\{a < \ell(U) < a(1+\delta_1)\big\} \cap \bigcap_{i=1}^{p}\{tb_i < \tau_i < t(b_i+\delta_2)\}\\ &\supset \big\{a < \ell_{tb}(U) < a(1+\delta_1) - \big(\ell_{t(b+\delta_2\mathbb{1})}(U) - \ell_{tb}(U)\big)\big\} \cap \bigcap_{i=1}^{p}\{tb_i < \tau_i < t(b_i+\delta_2)\}.\end{aligned}$$

On the set on the right-hand side, we want to replace $\ell/\ell(U)$ by $\frac{1}{t^p}\ell_{tb} = \frac{1}{a}\ell_{tb}$. The difference is estimated as

$$\Big|\frac{\ell}{\ell(U)} - \frac{\ell_{tb}}{a}\Big| = \Big|\frac{\ell-\ell_{tb}}{\ell(U)} + \frac{1}{t^p}\ell_{tb}\Big(\frac{a}{\ell(U)}-1\Big)\Big| \le \frac{\ell_{t(b+\delta_2 1\!\!1)}-\ell_{tb}}{t^p} + \frac{1}{t^p}\ell_{tb}\frac{\delta_1}{1+\delta_1}. \tag{7.1.2}$$

Pick some open set $\widetilde{G}\subset\mathcal{M}(B)$ such that $G=\widetilde{G}\cap\mathcal{M}(B)$. Fix $\varepsilon>0$. Denote by $\widetilde{G}_\varepsilon=\{\mu\in\widetilde{G}\colon\ \mathrm{d}(\mu,\widetilde{G}^{\mathrm{c}})>\varepsilon\}$ the inner ε-neighbourhood of $\widetilde{G}$. Hence, for any $M>0$, on the event $\{\mathrm{d}(\frac{1}{t^p}\ell_{tb},0)<M\}\cap A$, where

$$A=\Big\{\mathrm{d}\Big(\frac{\ell_{t(b+\delta_2 1\!\!1)}-\ell_{tb}}{t^p},0\Big)<\frac{\varepsilon}{2},\ell_{t(b+\delta_2 1\!\!1)}(U)-\ell_{tb}(U)\le a\frac{\delta_1}{2}\Big\}, \tag{7.1.3}$$

we have, for sufficiently small $\delta_1,\delta_2>0$, that the event $\{\ell/\ell(U)\in G\}$ contains the event $\{\frac{1}{t^p}\ell_{tb}\in\widetilde{G}_\varepsilon\}$. Thus, we have the following lower bound.

$$\begin{aligned}
&\mathbb{P}\Big(\frac{\ell}{\ell(U)}\in G,\ell(U)>a\Big)\\
&\ge\mathbb{P}\Big(\tfrac{1}{t^p}\ell_{tb}\in\widetilde{G}_\varepsilon, a<\ell_{tb}(U)<a(1+\tfrac{\delta_1}{2}),\mathrm{d}(\tfrac{1}{t^p}\ell_{tb},0)<M,A,\forall i\colon tb_i<\tau_i<t(b_i+\delta_2)\Big)\\
&=\mathbb{E}\Big(1\!\!1\{\tfrac{1}{t^p}\ell_{tb}\in\widetilde{G}_\varepsilon,1<\tfrac{1}{t^p}\ell_{tb}(U)<1+\tfrac{\delta_1}{2},\mathrm{d}(\tfrac{1}{t^p}\ell_{tb},0)<M,\forall i\colon tb_i<\tau_i\}F\big(W^{(1)}_{tb_1},\dots,W^{(p)}_{tb_p}\big)\Big),
\end{aligned} \tag{7.1.4}$$

where we used the Markov property at times $tb_1,\dots,tb_p$ and introduced

$$F(x)=\mathbb{P}_x\Big(\mathrm{d}(\tfrac{1}{t^p}\ell_{t\delta_2 1\!\!1},0)<\frac{\varepsilon}{2},\ell_{t\delta_2 1\!\!1}(U)\le t^p\frac{\delta_1}{2},\forall i\colon\tau_i<tb_i\delta_2\Big);$$

we recall that $\mathbb{P}_x$ denotes expectation with respect to the p motions starting in the sites $x_1,\dots,x_p$, respectively. It is easy to see, by chosing some appropriate joint strategy of the p motions, that $\liminf_{t\to\infty}\frac{1}{t}\log\inf_{x\in B^p}F(x)\ge 0$. To the remaining term on the right-hand side of (7.1.4), we can apply the lower bound in the LDP for $(t^p\prod_{i=1}^p b_i)^{-1}\ell_{tb}$ from Corollary 3.1.2 and obtain

$$\begin{aligned}
&\liminf_{a\to\infty}\frac{1}{a^{1/p}}\log\mathbb{P}\Big(\frac{\ell}{\ell(U)}\in G,\ell(U)>a\Big)\\
&\ge-\inf\Big\{\frac{1}{2}\sum_{i=1}^p b_i\|\nabla\psi_i\|_2^2\colon\psi_i\in H_0^1(B),\|\psi_i\|_2=1\,\forall i,\\
&\qquad\prod_{i=1}^p(b_i\psi_i^2)\in\widetilde{G}_\varepsilon,1<\int_U\prod_{i=1}^p(b_i\psi_i^2)<1+\tfrac{\delta_1}{2},\mathrm{d}\Big(\prod_{i=1}^p(b_i\psi_i^2),0\Big)<M\Big\},
\end{aligned}$$

where we conceive the function $\prod_{i=1}^p(b_i\psi_i^2)$ as a measure on B. Now let $M\to\infty$ to see that the last condition is immaterial, let $\delta_1\downarrow 0$, substitute $\phi_i^2=b_i\psi_i^2$ and take the supremum over $b_1,\dots,b_p$ on the right-hand side (i.e., drop the condition $\|\phi_i\|_2^2=b_i$), to see that

$$\begin{aligned}
&\liminf_{a\to\infty}\frac{1}{a^{1/p}}\log\mathbb{P}\Big(\frac{\ell}{\ell(U)}\in G,\ell(U)>a\Big)\\
&\ge-\inf\Big\{\frac{1}{2}\sum_{i=1}^p\|\nabla\phi_i\|_2^2\colon\phi_i\in H_0^1(B)\,\forall i,\prod_{i=1}^p\phi_i^2\in\widetilde{G}_\varepsilon,1=\int_U\prod_{i=1}^p\phi_i^2\Big\}\\
&=-\inf_{\widetilde{G}_\varepsilon}\widetilde{J},
\end{aligned}$$

where $\widetilde{J}$ is the extension of J defined in (3.2.1) from $\mathcal{M}_U(B)$ to $\mathcal{M}(B)$ with $J(\mu)=\infty$ for $\mu\in\mathcal{M}(B)\setminus\mathcal{M}_U(B)$. Now let $\varepsilon\downarrow 0$ and use the lower semicontinuity of J to see that (7.1.1) holds. This concludes the proof of Lemma 7.1.1. □

7.2 The upper bound.

Now we handle the upper bound part.

Lemma 7.2.1 (Upper bound)**.** *For every closed set $F \subset \mathcal{M}_U(B)$,*

$$\limsup_{a\uparrow\infty} \frac{1}{a^{1/p}} \log \mathbb{P}\left(\frac{\ell}{\ell(U)} \in F, \ell(U) > a\right) \leq -\inf_{\mu\in F} J(\mu). \tag{7.2.1}$$

Proof. For any $R \in (0,\infty)$ and $\delta_1 \in (0,\infty)$, we have the following upper bound estimate:

$$\begin{aligned} \mathbb{P}\Big(\frac{\ell}{\ell(U)} \in F, \ell(U) > a\Big) \leq \sum_{j\in\mathbb{N}\cap[0,R/\delta_1]} & \mathbb{P}\Big(\frac{\ell}{\ell(U)} \in F, a(1+(j-1)\delta_1) < \ell(U) \leq a(1+j\delta_1)\Big) \\ & + \mathbb{P}\big(\ell(U) > aR\big). \end{aligned} \tag{7.2.2}$$

The exponential rate of the second probability is known from [KM02], see (2.2.2):

$$\mathbb{P}\big(\ell(U) > aR\big) = \exp\big(-a^{1/p}R^{1/p}\big(\Theta_B(U) + o(1)\big)\big), \tag{7.2.3}$$

where $\Theta_B(U) \in (0,\infty)$ is the variational formula appearing in (2.2.3).

With this in mind, let us now focus on one of the summands of the first term on the right-hand side of (7.2.2). By monotonicity in j, is sufficient to consider the event for $j = 1$, as this gives the dominant term. Then, for any $\widetilde{R} \in \mathbb{N}$ and $\delta_2 \in (0,\infty)$,

$$\begin{aligned} &\mathbb{P}\Big(\frac{\ell}{\ell(U)} \in F, a < \ell(U) \leq a(1+\delta_1)\Big) \\ &\leq \sum_{b_1,\dots,b_p\in\delta_2\mathbb{N}\cap[0,\widetilde{R}]} \mathbb{P}\Big(\frac{\ell}{\ell(U)} \in F, a < \ell(U) \leq a(1+\delta_1), \forall i\colon a^{1/p}b_i < \tau_i \leq a^{1/p}(b_i+\delta_2)\Big) \\ &\qquad + \sum_{i=1}^p \mathbb{P}\big(\tau_i > a^{1/p}\widetilde{R}\big) + \sum_{i=1}^p \mathbb{P}\big(\ell(U) > a, \tau_i \leq a^{1/p}\delta_2\big). \end{aligned} \tag{7.2.4}$$

The first probability on the last line has a strongly negative exponential rate for large $\widetilde{R}$:

$$\mathbb{P}\big(\tau_i > a^{1/p}\widetilde{R}\big) = \exp\big(-\widetilde{R}a^{1/p}\lambda_1 + o(a^{1/p})\big), \qquad a \uparrow \infty, \tag{7.2.5}$$

$\lambda_1 \in (0,\infty)$ being the principal eigenvalue of $-\frac{1}{2}\Delta$ in B with zero boundary condition. Furthermore, the last probability on the last line has a strongly negative exponential rate for small δ_2, since

$$\lim_{\delta_2\downarrow 0} \limsup_{a\uparrow\infty} \frac{1}{a^{1/p}} \log \mathbb{P}\big(\ell(U) > a, \tau_i \leq a^{1/p}\delta_2\big) = -\infty, \qquad i \in \{1,\dots,p\}. \tag{7.2.6}$$

This is shown as follows. For any $K \in (0,\infty)$, estimate

$$\mathbb{P}\big(\ell(U) > a, \tau_i \leq a^{1/p}\delta_2\big) \leq \mathbb{P}\big(\ell(U) > a, \tau_i \leq a^{1/p}\delta_2, \forall j \neq i\colon \tau_j \leq a^{1/p}K\big) + \sum_{j\neq i} \mathbb{P}(\tau_j > a^{1/p}K).$$

The last term has a very negative exponential rate for large K (see (7.2.5)), and for fixed K, we estimate the first term on the right against $\mathbb{P}(\ell_{a^{1/p}v}(U) > a)$, where v is the vector in $(0,\infty)^p$ with δ_2 in the i-th component and K in all the other $p-1$ components (we use the notation introduced in (1.2.1)). Now use the Markov inequality to estimate, for any $m \in \mathbb{N}$,

$$\begin{aligned}\mathbb{P}(\ell_{a^{1/p}v}(U) > a) &\leq a^{-m}\mathbb{E}\big[\ell_{a^{1/p}v}(U)^m\big] \leq a^{-m}\mathbb{E}_0\big[\ell_{a^{1/p}v}(\mathbb{R}^d)^m\big] \\ &\leq a^{-m}\mathbb{E}_0\big[\ell_{a^{1/p}\delta_2\mathbb{1}}(\mathbb{R}^d)^m\big]^{1/p}\mathbb{E}_0\big[\ell_{a^{1/p}K\mathbb{1}}(\mathbb{R}^d)^m\big]^{(p-1)/p},\end{aligned}$$

where we used the fact that the total mass of the intersection local time is stochastically larger if all the p motions start from the origin (see [C09, (2.2.24)]) and used Hölder's inequality in the last step (see [C09, (2.2.12)]); recall the notation $\mathbb{1} = (1, \dots, 1) \in \{1\}^p$. Now use the Brownian scaling property and the bound

$$\mathbb{E}_0\big[\ell_{a^{1/p}\delta_2\mathbb{1}}(\mathbb{R}^d)^m\big] = \big(a^{1/p}\delta_2\big)^{\frac{2p-d(p-1)}{2}m}\mathbb{E}_0\big[\ell_{\mathbb{1}}(\mathbb{R}^d)^m\big] \leq m!^{\frac{d(p-1)}{2}}\big(a^{1/p}C_{\delta_2}\big)^{\frac{2p-d(p-1)}{2}m}$$

with some C_{δ_2} satisfying $\lim_{\delta_2\downarrow 0} C_{\delta_2} = 0$ and an analogous bound for $\mathbb{E}_0[\ell_{a^{1/p}K\mathbb{1}}(\mathbb{R}^d)^m]$ (see [C09, (2.2.22)] and the last display in the proof of [C09, Theorem 2.2.9]), and pick $m \approx a^{1/p}$ and summarize to see that (7.2.6) holds.

Hence, we focus on one of the summands of the first sum on the right-hand side of (7.2.4), for fixed $\delta_2, \widetilde{R} \in (0,\infty)$. Set $t = a^{1/p}$ and $b = (b_1, \dots, b_p)$. We want to replace $\ell/\ell(U)$ by $\frac{1}{t^p}\ell_{tb}$. The difference is estimated as in (7.1.2) on the event $\{a < \ell(U) < a(1+\delta_1)\} \cap \bigcap_{i=1}^p \{tb_i < \tau_i \leq t(b_i+\delta_2)\}$; this difference is small on the event $\{\mathrm{d}(\frac{1}{t^p}\ell_{tb}, 0) \leq M\} \cap A$, with A as in (7.1.3), for any M and small δ_1. Furthermore, note that, on the event $\bigcap_{i=1}^p \{tb_i < \tau_i \leq t(b_i+\delta_2)\}$,

$$\big\{a < \ell(U) < a(1+\delta_1)\big\} \subset \big\{a - \big(\ell_{t(b+\delta_2\mathbb{1})}(U) - \ell_{tb}(U)\big) < \ell_{tb}(U) < a(1+\delta_1)\big\}. \tag{7.2.7}$$

Fix $\varepsilon > 0$. Note that F is also closed in $\mathcal{M}(B)$. Denote by $F_\varepsilon = \{\mu \in \mathcal{M}(B) \colon \mathrm{d}(\mu, F) \leq \varepsilon\}$ the outer closed ε-neighborhood of F. Hence, for any $M > 0$, on the event $\{\mathrm{d}(\frac{1}{t^p}\ell_{tb}, 0) \leq M\} \cap A$, we have, for sufficiently small $\delta_1 > 0$, that the event $\{\ell/\ell(U) \in F\}$ is contained in the event $\{\frac{1}{t^p}\ell_{tb} \in F_\varepsilon\}$, and furthermore we may estimate $\ell_{t(b+\delta_2\mathbb{1})}(U) - \ell_{tb}(U) \leq a\delta_1/2$ and use this on the right-hand side of (7.2.7). Thus,

$$\begin{aligned}&\mathbb{P}\Big(\frac{\ell}{\ell(U)} \in F, a < \ell(U) \leq a(1+\delta_1), \forall i \colon a^{1/p}b_i < \tau_i \leq a^{1/p}(b_i+\delta_2)\Big) \\ &\leq \mathbb{P}\Big(\frac{1}{t^p}\ell_{tb} \in F_\varepsilon, 1 - \frac{\delta_1}{2} < \frac{1}{t^p}\ell_{tb}(U) < 1+\delta_1, \mathrm{d}\Big(\frac{1}{t^p}\ell_{tb}, 0\Big) \leq M, A, \forall i \colon \tau_i > tb_i\Big) \\ &\quad + \mathbb{P}\Big(\mathrm{d}\Big(\tfrac{1}{t^p}\ell_{tb}, 0\Big) > M\, \forall i \colon \tau_i > tb_i\Big) + \mathbb{P}(A^{\mathrm{c}}) \\ &\leq \mathbb{P}\Big(\frac{1}{t^p}\ell_{tb} \in F_\varepsilon, 1 - \frac{\delta_1}{2} < \frac{1}{t^p}\ell_{tb}(U) < 1+\delta_1,\ \forall i \colon \tau_i > tb_i\Big) \\ &\quad + \mathbb{P}\Big(\mathrm{d}\Big(\tfrac{1}{t^p}\ell_{tb}, 0\Big) > M, \forall i \colon \tau_i > tb_i\Big) \\ &\qquad + \mathbb{P}\Big(\mathrm{d}\Big(\tfrac{1}{t^p}\big(\ell_{t(b+\delta_2\mathbb{1})} - \ell_{tb}\big), 0\Big) > \frac{\varepsilon}{2}\Big) + \mathbb{P}\Big(\tfrac{1}{t^p}\big(\ell_{t(b+\delta_2\mathbb{1})}(U) - \ell_{tb}(U)\big) > \frac{\delta_1}{2}\Big).\end{aligned} \tag{7.2.8}$$

Note that the exponential rates of the last three terms are strongly negative for large M, respectively for small δ_2. For the first of these this follows from an application of the LDP for $\frac{1}{\beta t^p}\ell_{tb}$ (with $\beta = \prod_{i=1}^p b_i$) from Corollary 3.1.2 noting that large values of $\mathrm{d}(\mu, 0)$ imply large values of $\mu(B)$. For the two latter terms, this follows from our proof of (7.2.6) (use the Markov property at times $tb_1, \dots, tb_p$, respectively).

For the first term on the right-hand side of (7.2.8), we put $\beta = \prod_{i=1}^p b_i$, use the upper bound for the LDP of $\frac{1}{\beta t^p}\ell_{tb}$ from Corollary 3.1.2 and the continuity of the map $\mu \mapsto \mu(U)$ (recall that U is a Lebesgue-continuity set), to see that

$$
\begin{aligned}
\limsup_{a\to\infty} \frac{1}{a^{1/p}} \log \mathbb{P}\Big(\frac{1}{\beta t^p}\ell_{tb} \in \frac{F_\varepsilon}{\beta}, \frac{1-\frac{\delta_1}{2}}{\beta} < \frac{1}{\beta t^p}\ell_{tb}(U) < \frac{1+\delta_1}{\beta}, \forall i\colon \tau_i > tb_i\Big) \\
\le -\inf\Big\{\frac{1}{2}\sum_{i=1}^p b_i\|\nabla\psi_i\|_2^2 \colon \psi_i \in H_0^1(B),\ \|\psi_i\|_2 = 1\forall i, \\
\prod_{i=1}^p \psi_i^2 \in \frac{F_\varepsilon}{\beta}, \frac{1-\frac{\delta_1}{2}}{\beta} \le \int_U \prod_{i=1}^p \psi_i^2 \le \frac{1+\delta_1}{\beta}\Big\} \\
\le -\inf\Big\{\frac{1}{2}\sum_{i=1}^p \|\nabla\phi_i\|_2^2 \colon \phi_1,\dots,\phi_p \in H_0^1(B), \prod_{i=1}^p \phi_i^2 \in F_\varepsilon,\ 1-\frac{\delta_1}{2} \le \int_U \prod_{i=1}^p \phi_i^2 \le 1+\delta_1\Big\},
\end{aligned}
$$

where we substituted $\phi_i^2 = b_i\psi_i^2$ and dropped the condition $\|\psi_i\|_2 = 1$. Now let $\delta_1 \downarrow 0$ and note that the right-hand side converges to $-\inf_{F_\varepsilon} \widetilde{J}$, where $\widetilde{J}$ is the extension of J defined in (3.2.1) from $\mathcal{M}_U(B)$ to $\mathcal{M}(B)$ with $J(\mu) = \infty$ for $\mu \in \mathcal{M}(B) \setminus \mathcal{M}_U(B)$. By lower semicontinuity, this in turn tends to the right-hand side of (7.2.1). Collecting all preceding steps, this concludes the proof of Lemma 7.2.1. □

Chapter 8

Outlook and open questions.

We conclude with some interesting open questions which might open up directions for future research.

- **Variational characterization.** Recall the rate function I from (3.1.4). It is tempting to conjecture that, for $(\psi_1, \dots, \psi_p)$ a minimising tuple in (3.1.4), all the ψ_i should be identical. This would simplify the formula to $I(\mu) = \frac{p}{2}\|\nabla\psi\|_2^2$ if ψ^{2p} is a density of μ with $\psi \in H_0^1(B)$. However, we found no evidence for that and indeed conjecture that this is not true for general μ. But note that the result by Chen in (2.1.1)–(2.1.2), after replacing $\ell_t(\mathbb{R}^d)$ by $\ell_t(B)$ and $H^1(\mathbb{R}^d)$ by $H_0^1(B)$, for $a = 1$ suggests that, at least for the miniser μ of $I(\mu)$, all the ψ_i should be identical, since the minimiser in (2.1.2) is just some ψ^{2p}. Like for the rate function I, we do not know whether or not the minimising $\phi_1, \dots, \phi_p$ are identical in the variational formula for J in (3.2.1). However, when minimising also over $\mu \in \mathcal{M}_U(B)$, we see that $\min_{\mu\in\mathcal{M}_U(B)} J(\mu) = \Theta_B(U)$, and an inspection of (2.2.3) shows that a minimising tuple is given by picking all ϕ_i equal to ϕ, where ϕ^{2p} is the minimiser in (2.2.3). It is an open problem to give a sufficient condition on μ for having a minimising tuple of p identical functions $\phi_1, \dots, \phi_p$.

- **Uniqueness of solutions to a non-linear PDE.** We recall the PDE from (2.2.4):

$$\triangle\psi(x) = -\frac{2}{p}\Theta(U)\psi^{2p-1}(x)\, \mathbb{1}_U(x) \quad \text{for } x \in B\backslash\partial U,$$

  where B is a domain in $\mathbb{R}^d$ and U is a compactly contained subset of B. For $p = 1$, this is a linear eigenvalue problem and the solution is unique (the eigenvector for the principle eigenvalue of $-\Delta$ in B). However, as we mentioned before, for $p > 1$, the uniqueness of solutions to this PDE is an open problem, unless U happens to be the unit ball in $\mathbb{R}^3$. We indeed conjecture that it is possible to find examples of U for which more than one solution can be constructed.

- **LDP on the whole set B.** When the domain of motions B is bounded, we can also focus on the *entire set* B and try to understand the distribution of $\frac{\ell}{\ell(B)}$ in terms of a large deviation principle.

- **Unbounded domains in $\mathbb{R}^3$.** As our techniques borrow from Donsker-Varadhan LDP theory on compacts as well as the spectral theory of the Laplacian, so far we have been able to prove all the results when the domain B is bounded. This is in fact a necessity in $d = 2$, as in this regime, recurrence of paths makes the intersection local time mass of an unbounded domain in $\mathbb{R}^2$ infinity. However, this can be relaxed in $\mathbb{R}^3$ and the next goal will be to have similar results for $p = 2$ and $B = \mathbb{R}^3$.

- **Intersections of Wiener Sausages.** Powerful results of Van den Berg, Bolthausen and den Hollander ([BBH01]) concern deviations of the volume of a wiener sausage *on the scale of mean*: Let $S_\epsilon(t)$ be the ϵ-neighborhood of a Brownian path starting from the origin and observed until time t in $\mathbb{R}^d$. Then, their main result is the following: For $d \geq 3$ and $c > 0$,

$$\lim_{t\uparrow\infty} t^{-\frac{d-2}{d}} \log \mathbb{P}(|S_\varepsilon(t)| \leq ct) = -I^{\kappa_\varepsilon}(c) \tag{8.0.1}$$

$$I^{\kappa_\varepsilon}(c) = \inf_{\psi\in\Psi^{\kappa_\varepsilon}(c)} \left[\|\nabla\psi\|_2^2\right] \tag{8.0.2}$$

and

$$\Psi^{\kappa_\varepsilon}(c) = \{\psi \in H^1(\mathbb{R}^d) \colon \ \|\psi\|_2 = 1, \int_{\mathbb{R}^d} \big(1 - \exp(-\kappa_\varepsilon \psi^2)\big) \leq c\}$$

where κ_ε is the Newtonian capacity of a ball of radius ε around the origin. The optimal strategy for the above large deviations is different from the Donsker-Varadhan setting ([DV75-83]): Instead of filling out a ball of volume $o(t)$ completely and nothing outside, the Wiener sausage fills only a part of the space and leaves random holes whose sizes are of oder 1 and whose density varies on sclae $t^{\frac{1}{d}}$ according to some optimal profile. However, showing that the law of the Brownian path conditional on the large deviation event $\{S_\varepsilon(t) \leq ct\}$ follows the above mentioned optimal strategy is an open problem, at least to the best of our knowledge and is worth an investigation.
Furthermore, in a subsequent paper ([BBH04]), the same authors considered the volume of intersection of two sausages and proved a similar large deviation result like in (8.0.1) with $|S_\varepsilon(t)|$ replaced by $|S_\varepsilon^{(1)}(t) \cap S_\varepsilon^{(2)}(t)|$. It is intriguing to pass to the $\lim_{c\to\infty}$ in the corresponding asymptotics like (8.0.1) to get a rate function for infinite time intersection volume. However, as the intersection volume can take over the value t on a scale *larger than* t and therefore, the order of the limits $\lim_{t\to\infty}$ and $\lim_{c\to\infty}$ can not, in general, be switched.

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Selbstständigkeitserklärung

Hiermit erkläre ich, die vorliegende Dissertation selbständig und ohne unzulässige fremde Hilfe angefertigt zu haben. Ich habe keine anderen als die angeführten Quellen und Hilfsmittel benutzt und sämtliche Textstellen, die wörtlich oder sinngemäss aus veröffentlichen Schriften entnommen wurden, und alle Angaben, die auf mündlichen Auskünften beruhen, als solche kenntlich gemacht. Ebenfalls sind alle von anderen bereitgestellen Materiallen oder erbrachten Dienstleistungen als solche gekennzeichnet.

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