Quantum Field Theory on Non-commutative Spacetimes

Von der Fakultät für Physik und Geowissenschaften

der Universität Leipzig

genehmigte

D I S S E R T A T I O N

zur Erlangung des akademischen Grades

Doctor rerum naturalium

Dr. rer. nat.

vorgelegt

von *Dipl.-Phys. Markus Borris*

geboren am *18. August 1981* **in** *Zwickau*

Gutachter: *Prof. Dr. Rainer Verch, Universität Leipzig*
Prof. Dr. Raimar Wulkenhaar, Universität Münster

Tag der Verleihung *18. April, 2011*

Bibliographic description:

Borris, Markus
Quantum Field Theory on Non-commutative Spacetimes
University of Leipzig, Thesis
86 pages, 36 bibliographical references, 1 figure

Abstract:

The time coordinate is a common obstacle in the theory of non-commutative (nc.) spacetimes. Despite that, this work shows how the interplay between quantum fields and an underlying nc. spacetime can still be analyzed, even for the case of nc. time. This is done for the example of a general Moyal-type external potential scattering of the Dirac field in Moyal-Minkowski spacetime. The spacetime is a rare example of a Lorentzian non-compact nc. geometry. Elements of the associated spectral function algebra are shown to be operationally involved at the level of quantum field operators by Bogoliubov's formula.

Furthermore, a similar task is attacked in the case of locally nc. spacetimes. An explicit star-product is constructed by a method of Kontsevich. It implements a decay of non-commutativity with increasing distance. This behavior should benefit the technical side – diverse interesting formal attempts are discussed.

It is striven for unification of several toy models of nc. spacetimes and a general strategy to define quantum field operators. Within the latter one has to implement the usual quantum behavior as well as a new kind of spacetime behavior. It is shown how this two-fold character causes key difficulties in understanding.

Studies and scientific history of the author:

10/2001-09/2003	Study of physics at the Technical University of Chemnitz with subsidiary subject Information Technology
09/2003	Intermediate diploma
10/2003-09/2005	Main studies at the University of Leipzig with focuses • general relativity • quantum field theory • mathematics (differential geometry and functional analysis)
10/2005-10/2006	Diploma thesis with topic “Dirac-Feld und nichtkommutative Raumzeit”, finished as diploma physicist (grade: 1.1)
Since 01/2007	PhD student in the field of “Quantum Field Theory on Non-commutative Spacetimes”
2006-2010	Participation in the 18th till 25th workshop “Foundations and Constructive Aspects of QFT”
11/2006	Seminar talk “Fermi-Fockraum und freies Dirac Feld”
01/2007-12/2008	Financial support within the scope of the DFG project “Nichtkommutative Geometrie und Elementarteilchen”
01/2007	Talk “Non-commutative potential scattering for Dirac fields on Moyal spacetime” at the abovementioned 19th workshop in Goettingen
02/2008	Participation in Spring School “Quantum Structures” in Leipzig
03/2008	Talk “Dirac Field on Moyal-Minkowski spacetime” at the DPG Spring Conference in Freiburg
07/2008	Participation in “CRCG Workshop on Quantum Fields and Non-commutative Geometry” in Goettingen
12/2008	Paper “Dirac field on Moyal-Minkowski spacetime and non-comm. potential scattering” in collaboration with Prof. R. Verch appears as preprint arXiv:0812.0786 [math-ph]
12/2008	Scientific guest stay to Jr. Prof. D. Bahns in Goettingen
01/2009-06/2010	Financial support by the IMPRS
06/2009	Seminar talk “Dirac field on Moyal-Minkowski spacetime”
01/2010	Abovementioned paper appears in Commun. Math. Phys. **293** (2010) 399, cf. [6]
02/2010	Talk “Dirac Field and Non-commutative Spacetimes” at IMPRS-Workshop in Leipzig

Contents

1 Introduction and summary

To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature ...

(Richard Feynman)

The cited famous physicist is even more right today. Mankind reached an enormous level of understanding of nature. Concerning the absolute fundamentals, real experiments barely keep up with theory. The issues discussed in this thesis are probably far apart from producing statements that could be checked e.g. in the recently popular Large Hadron Collider (LHC).
However, on the one hand this work can be seen as a source of creativity, making some new theoretical progress in a specific field. And on the other hand it contains attempts to arrange and tidy up things that are already *almost* explored. Especially the last point is often merely a matter of mathematical beauty and rigor. What makes this important and necessary? During the various attempts of distinct groups of physicists of pushing the frontiers further into the unknown, some of the even established theories suffer from a confusing variety of new alternative formulations of themselves. Probably the biggest child of sorrow in this regard is quantum field theory (QFT). Experimentally the predictions of the theory are absolutely sound, just like those from general relativity, the other half of the big two. Only exception is, when it comes to situations of extremely short distances and high energies where both of them become equally important.

From a mathematical point of view, general relativity is quite clean and almost easy. There is no need for many overly distinct approaches that are hard to compare. Whereas quantum field theory always seems to make problems and requests to be repaired in different ways. This makes it extra complicated to *extend* the known and working theory and to compare and align it with principles of the other.
In the big picture, the mathematical field of *non-commutative* (or better called: *spectral*) *geometry* is a very promising candidate to help us out here. The idea is to express differential geometry in a different "language" involving particularly algebra and functional analysis. Geometric spaces are described equivalently by so called *spectral data*, which are a set of objects, especially a certain *function*

algebra over the spacetime, and many relations. Physically relevant can be a specific option that opens up here. The option is to choose the function algebra either commutative and obtain classical geometric spaces, or to choose it non-commutative (nc.) and obtain something different. For physics the difference finally results in a quantization of spacetime, which is a concept suggested by various theoretical arguments nowadays. A nice motivation is given in [15].

The general and highly ambitious task is to understand the connection of spacetime quantization to the existing established theory of Einstein's geometry and the theory of quantum fields describing matter. In the thesis at hand this is broken down quite drastically, aiming at examples. There are various toy models (more or less physically reasonable) of nc. spaces around that will be conceptually reviewed in Section 5.1. Two of them are used and analysed in a much more extended manner, namely the special *Moyal-deformed Minkowski spacetime* and the more general concept of *locally non-commutative spacetimes.*

The main achievement of the thesis is covered in Chapter 2. It generalizes the results of the paper [6] to a nc. spacetime structure that *even* involves the time coordinate, which is quite nice. Time is an obstacle in a wider sense in nc. geometry. Some of the toy models are constructed in a Riemannian setting and even only for compact spaces. For those, Alain Connes (the founder of nc. geometry, [11]) gave a sound framework in principle. But the more important models, Moyal-Minkowski spacetime being one of them, need Lorentzian signature and non-compact spaces, for which such a general framework is still under construction. Lacking such a framework of so called *Lorentzian spectral triples* (LOST's), the Moyal-Minkowski spacetime at least comprises an example for which spectral data can indeed be constructed, according to [17].
The next question is how to construct a quantum field theory on such a spacetime. Again, no one knows the correct general strategy. Our choice is the most obvious and natural one. Since the spectral data contains a Hilbert space and a Dirac operator, this suggests usual CAR-quantization resulting in a Dirac field. In the commutative case, the procedure actually corresponds exactly to the usual approach of constructing the quantized Dirac field.
Then central emphasis is put on the construction of specific observables somehow connecting to elements of the nc. function algebra. The idea is to mathematically express an interaction between matter quantum fields and the underlying nc. spacetime. The easiest way to start with, is by external potential scattering. In the classical Minkowski case this amounts to a simple pointwise multiplication with a function $c \in \mathcal{A}$ as action of the potential operator V. The function algebra $\mathcal{A}$ is given by real-valued Schwartz functions $\mathscr{S}(\mathbb{R}^n, \mathbb{R})$ in $2 \leq n \in \mathbb{N}$ spacetime dimensions together with the pointwise product. For the Moyal-Minkowski spacetime the product structure is exchanged by a non-commutative $\star$-product and we must restrict ourselves to even dimensions $n \in 2\mathbb{N}$.

Finally, the scattering process is described by a scattering morphism $\beta_c := \beta_V$ (which exactly corresponds to the "unimplemented" well-known S-operator, just at the level of an abstract field algebra). A rather old idea by Bogoliubov roughly states: one obtains observable quantum fields by functional differentiation of the S-operator with respect to the interaction strength. We use this to arrive at a promising new concept of how to assign QFT observables to elements of the nc. function algebra $\mathcal{A}$. This concept could also be applied in a more general context.
Compared to the paper [6], here it seems even more difficult to prove implementability of the S-operator because of the nc. time. However, this is not necessary for many purposes. We can still prove *Bogoliubov's formula*

$$\frac{d}{d\lambda}\bigg|_{\lambda=0} \beta_{\lambda V}\Psi(f) = \Psi(VRf),$$

and show the existence and (essential) self-adjointness of an observable $\Phi(c) := \Phi(V)$ that generates the derivation $\delta_{\lambda V}$, defined by

$$\delta_{\lambda V}\Psi(f) := \frac{d}{d\lambda}\bigg|_{\lambda=0} \beta_{\lambda V}\Psi(f).$$

"Generates" means

$$[i\Phi(c), \Psi(f)] = \delta_{\lambda V}\Psi(f).$$

The occurring symbols are: the abstract Dirac field operators $\Psi(f)$ with $f \in \mathscr{S}_0$ (Schwartz, but compactly supported in time direction) and $R = R^+ - R^-$ the difference of the advanced/retarded fundamental solutions for the free Dirac operator. The potential V is chosen to be $c \star \cdot \star c$ with Moyal product. There is no indication that the choice $c \star \cdot + \cdot \star c$ could not be carried out as well – this is just a matter of taste.
This altogether gives some insight into the operational meaning of the elements c of the nc. spacetime algebra at the level of quantum field operators.
Besides, note that the mentioned results for time-non-commutativity do not contradict arguments stemming from [18]: the regularly cited lack of unitarity of the S-operator and thus lack of existence of a proper Bogoliubov morphism describing the scattering process refers to a case of self-interaction. However, here we deal with an issue of external potential scattering.
Concluding the chapter, also some attention is called to an interesting problem, occurring in the context of *Rieffel's deformed product*, [31], together with Bogoliubov's formula from above. In some way this joins our results with the "star-formalism" of those authors that usually start with a Langrangian. But it reveals, that it is still quite difficult to compare different approaches and resolve the two-fold nature of non-commutativity (QFT/spacetime).
The purpose here is just to try to establish some tidiness and understanding.

The next two Chapters 3 and 4 deal with the completely different concept of

locally nc. spacetimes (invented in [3]), which will be introduced as detailed as necessary at first. The idea is to make non-commutativity distance-dependent and define star-products between functions on $\mathbb{R}^n \times \mathbb{R}^n$ instead of $\mathbb{R}^n$. So *two* points are interrelated with each other, and nature should show its nc. effects only at very short distances, thinking of Planck scale, and not as a whole static background. This concept can in general be applied to different kinds of existing toy star-products. However, our construction of an explicit formula of a local product in flat Minkowski spacetime has some similarity with a deformed Moyal-product. The construction is based on a famous theorem of Maxim Kontsevich, [25], which is far from trivial.
Pushing the idea of distances further, one is inclined to consider two particles instead of one. Since naturally the next degree of difficulty would be reached when shifting from external potential scattering to two-particle/pair interaction. Then the aim are again similar investigations and results as presented above. In addition to the content of this chapter, especially also the later Section 5.3 supplements some interesting aspects. It reveals key difficulties during attempts to once more establish Bogoliubov's formula from above. For technical reasons we had to use cut-off functions as temporary auxiliaries in Chapter 2, that could be removed again at the end. This time, a cut-off is naturally built in and physically perfectly motivated. It implements the decay of non-commutativity for increasing distances. The hope is then that this does an equally nice job on the technical side again.
Unfortunately these ideas could not be brought to full conclusion and leave many open questions. But nevertheless they give some interesting impulses for further exploration for sure.

Chapter 5 reviews several toy models of nc. spacetimes and tries to put them on equal footing. In order to find connections with a broader range of literature, an explicit expression of the abstract field operator is proposed. The operator should contain and combine both types of (anti-)commutation behavior: the one of quantum field theoretic nature and the one caused by the new spacetime structure. And it should also be general enough to suit *various* spacetime models. Some connections are drawn to Weyl quantization and the approach of "DFR", [15].

A short conclusion plus outlook then ends the journey through selected topics and problems of quantum field theory together with non-commutative spacetimes. This field of research has lots of potential, offers room to improve and freedom of creativity and ideas. Although far from experiment, it still can help to conceptually and technically bring existing theories to perfection and lay the foundation for another breakthrough in the exploration of fundamentals.
Hopefully the one or other useful contribution could be made with this thesis.

2 Dirac field on Moyal-Minkowski space coupled to a potential with non-commutative time

2.1 Preliminaries

The **Moyal-deformed Minkowski spacetime** of even dimension $n = 1 + s \in 2\mathbb{N},\ s \geq 1$ will be described as $\mathbb{R}^n$ with the Minkowskian metric

$$\eta := (\eta_{\mu\nu})_{\mu,\nu=0}^{s} := \mathrm{diag}(1, -1, \ldots, -1)$$

equipped with a non-commutative algebra of functions thereon whose product is defined as follows. Let $n = 2l$ for $l \in \mathbb{N}$, and let $\theta > 0$. Then we define the $n \times n$-matrix

$$M := M_\theta := \frac{\theta}{2} \begin{bmatrix} 0_{l\times l} & \mathbb{1}_{l\times l} \\ -\mathbb{1}_{l\times l} & 0_{l\times l} \end{bmatrix} \tag{2.1}$$

With this notation, the **Moyal product**

$$f \star g(x) := \frac{1}{(2\pi)^n} \iint f(x - Mu) g(x + v) e^{-iuv} d^n u d^n v, \quad x \in \mathbb{R}^n, \tag{2.2}$$

is introduced for (complex-valued) Schwartz functions $f, g \in \mathscr{S}(\mathbb{R}^n)$. By $uv = u{\cdot}v$ we denote the standard Euclidean scalar product of vectors $u, v \in \mathbb{R}^n$. One can show, either directly or by adapting the arguments of [17], that $f \star g$ is again in $\mathscr{S}(\mathbb{R}^n)$ and that the product $f \star g$ is jointly continuous in f and g with respect to the usual test-function topology on $\mathscr{S}(\mathbb{R}^n)$.
For convenience, operators of left- and right-multiplication with a function $c \in \mathscr{S}(\mathbb{R}^n)$ are defined by

$$\begin{aligned} L_c f &:= c \star f \\ R_c f &:= f \star c. \end{aligned} \tag{2.3}$$

Obviously $M = M_\theta$ chosen as above is invertible, and one has

$$f \star g(x) = \frac{1}{(\pi\theta)^n} \iint f(x - u) g(x + v) e^{-iuM^{-1}v} d^n u d^n v, \tag{2.4}$$

which is the usual Moyal product investigated in several references (see [17],[19]). The product fulfills (Lemma 2.12 of [17])

$$\|f \star g\|_{L^2} \leq (2\pi\theta)^{-n/2} \|f\|_{L^2} \|g\|_{L^2}. \tag{2.5}$$

Furthermore, the operator of left multiplication L_c, mapping the space of $\mathscr{S}$-functions to itself, fits into the definition of a pseudo-differential operator, as we recall

Definition 2.1.1 *Let $h \in \mathscr{S}(\mathbb{R}^n)$. A linear operator A is called a **pseudo-differential operator** $A \in \Psi DO$ on $\mathbb{R}^n$, if it can be written as*

$$Ah(x) \quad = \quad (2\pi)^{-n} \iint \sigma[A](x,\xi) h(y) e^{i\xi(x-y)} d^n\xi d^n y.$$

*According to the **symbol** $\sigma[A]$, A falls into some class(es) $\Psi^d := \{A \in \Psi DO : \sigma[A] \in S^d\}$ of ΨDO's of **order** $d \in \mathbb{R}$, with*

$$S^d := \{\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) : \left|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)\right| \leq C_{K\alpha\beta}(1+|\xi|^2)^{(d-|\beta|)/2} \text{ for } x \in K\},$$

*where K is any compact subset of $\mathbb{R}^n$, $\alpha, \beta \in \mathbb{N}^n$ and $C_{K\alpha\beta}$ is some constant. $A \in \Psi DO$ is called **smoothing** (or **regularizing**), if $A \in \Psi^{-\infty} := \bigcap_{d\in\mathbb{R}} \Psi^d$*

Lemma 2.1.2 ([17]) *If $c \in \mathscr{S}(\mathbb{R}^n)$, then $L_c = c \star \cdot$ is a smoothing ΨDO.*

Proof Clearly $\sigma[c \star \cdot](x,\xi) = c(x - M\xi)$ works as symbol and the required estimates are fulfilled since $c \in \mathscr{S}$, as it was already outlined in [17]. □

Analogously this can be transferred to the operator of right multiplication as well.

For a subset G of n dimensional (Moyal-deformed) Minkowski spacetime we define, following usual convention, $J^{\pm}(G)$ as the **causal future**$(+)$/**past**$(-)$ set of G, defined as consisting of all points that can be reached from G by smooth future/past directed causal curves. And $J(G) := J^+(G) \cup J^-(G)$.
With an eye on Dirac fields some more structure is needed. For any given $n = 1 + s \in \mathbb{N}$, $s \geq 1$, we set

$$N := N(n) := \begin{cases} 2^{n/2} & : n \text{ even (the only relevant case for our purpose)} \\ 2^{(n-1)/2} & : n \text{ odd} \end{cases}. \tag{2.6}$$

Then we refer to a collection $(\gamma_0, \gamma_1, \ldots, \gamma_s)$ of $N \times N$-matrices as a set of **Dirac matrices** if the relations

$$\begin{aligned} \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}\mathbb{1} \quad (\mu,\nu = 0,1,\ldots,s) \\ \gamma_0^* = \gamma_0, \quad \gamma_k^* = -\gamma_k \quad (k = 1,\ldots,s) \end{aligned} \tag{2.7}$$

are fulfilled. A set of Dirac matrices thus corresponds to an irreducible Dirac representation of the complexified Clifford algebra $\mathbb{C}l_{1,s}$; it exists for all $n \geq 2$. Since we restricted ourselves to even dimensions n, it is possible to find a **charge conjugation operator** $C : \mathbb{C}^N \to \mathbb{C}^N$ for the Dirac matrices $(\gamma_0, \gamma_1, \dots, \gamma_s)$; this means that C is an antilinear involution ($C^2 = \mathbb{1}$) satisfying

$$C\gamma_\mu = -\gamma_\mu C. \tag{2.8}$$

For details we refer to [12] and [13].

As opposed to previous discussions (see [6]) of the Dirac field on $n = 1+s$ ($s \geq 1$) dimensional Moyal-deformed Minkowski spacetime we now want to drop some of the simplifications. This mainly concerns the special choices of "potential term" operator V involving Moyal products in our Dirac operator

$$D_V := D + V := (-i\partial\!\!\!/ + m) + V, \tag{2.9}$$

acting on $\mathscr{S}(\mathbb{R}^n, \mathbb{C}^N)$, with $m > 0$ constant, $\partial\!\!\!/ := \gamma_\mu \partial^\mu$. From now on the time dimension is no longer treated as being something special compared to the spatial dimensions along the action of V.
Concretely the potential operator $V : \mathscr{S}(\mathbb{R}^n, \mathbb{C}^N) \to \mathscr{S}(\mathbb{R}^n, \mathbb{C}^N)$ is of the form

$$V := M_\chi R_c L_c M_\chi, \tag{2.10}$$

where $c \in \mathscr{S}(\mathbb{R}^n, \mathbb{R})$ and $(M_\chi f)^A(x) := \chi(x) f^A(x)$ is the multiplication operator with a cut-off function

$$\chi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}) \text{ with maximum } \chi(0) = 1$$

(without loss of generality, the coordinate origin can be put into the center of interaction), which means

$$(Vf)^A(x) = \chi(c \star (\chi f^A) \star c).$$

R_c and L_c acting on vector-valued functions are of course defined component-wise.

Remark

(a) Since χ and c are real-valued and $R_c L_c = L_c R_c$ (associativity of $\star$), it is easy to see that the operator V is symmetric in $L^2(\mathbb{R}^n, \mathbb{C}^N)$ and commutes with complex conjugation.

(b) The function χ with compact support serves as "localization regulator". It it is questionable whether the following steps could be carried out without χ. At a later stage the limit $\chi \to 1$ will be investigated.

Proposition 2.1.3 *$R_c L_c$ is a smoothing ΨDO.*

Proof This follows easily from Lemma 2.1.2 and e.g. a Theorem in Hoermander's book [24, Theorem 18.1.8], that essentially says: Let P_1, P_1 be ΨDO's of orders m and n respectively, then $P_1 P_2$ is a ΨDO of order $m + n$. □

2.2 Review of the solution theory for the case $V = 0$

We aim at presenting a solution theory for the Dirac operator coupled to a potential of type (2.10). As a first step it is necessary to review the theory for the "free" Dirac operator

$$D = -i\partial\!\!\!/ + m,$$

which has been discussed comprehensively in [13] and [2] and just needs to be adapted to the spacetime being Moyal-deformed. In our previous paper [6] this has already been carried out even for a "non-free" Dirac operator and simplifies a lot for the free case. In fact the product structure of the function algebra defined on the underlying background manifold does not have any substantial influence on the elementary solution theory of the Dirac operator. So almost immediately we get the following analogue of Dimock's Theorem 2.1 (a) from [13].

Theorem 2.2.1 *There is a unique pair of linear maps*

$$R^{\pm} : C_c^{\infty}(\mathbb{R}^n, \mathbb{C}^N) \to C^{\infty}(\mathbb{R}^n, \mathbb{C}^N)$$

having the properties

$$\begin{gathered} DR^{\pm}f = f = R^{\pm}Df \ \textit{and} \\ \operatorname{supp} R^{\pm}f \subset J^{\pm}(\operatorname{supp} f) \quad (f \in C_c^{\infty}(\mathbb{R}^n, \mathbb{C}^N)). \end{gathered}$$

*$R^{\pm}$ are called **advanced(+)/retarded(-) fundamental solutions** of D. Additionally we set $R := R^{+} - R^{-}$.*

Proof The proof is given in [13] within the more general setting of an arbitrary globally hyperbolic manifold albeit for the non-Moyal-deformed case of course. If one insists on using the Moyal product, it appears only in the definition of the sesquilinear form

$$\begin{aligned} \langle f, h \rangle &:= \int_{\mathbb{R}^n} \gamma_{0AB}(\bar{f}^B \star h^A)(x) d^n x \\ &= \int_{\mathbb{R}^n} \gamma_{0AB} \bar{f}^B(x) h^A(x) d^n x \quad (f, h \in C_c^{\infty}(\mathbb{R}^n, \mathbb{C}^N)), \end{aligned}$$

and has no effect at all due to the tracial property of the Moyal product (Lemma 2.1 (v) in [17] resp. [19]). One of the crucial steps in the proof of Theorem 2.1 of Dimock's [13] is the property

$$\langle R^{\pm}h, f \rangle = \langle h, R^{\mp}f \rangle \tag{2.11}$$

which we just need to mention for later usage. □

2.3 Solution theory for the case $V \neq 0$

Our goal is an existence/uniqueness Theorem for fundamental solutions in the general case, i.e. opting for the potential (2.10) in

$$D_V = D + V = -i\partial\!\!\!/ + m + V.$$

Some intermediate steps are necessary.

Definition 2.3.1 *Obviously*

$$(f,g)_{L^2} := \sum_{A=1}^{N} (f^A, g^A)_{L^2} = \sum_{A=1}^{N} \int_{\mathbb{R}^n} \bar{f}^A(x) g^A(x) d^n x$$

defines a scalar product on $C_c^\infty(\mathbb{R}^n, \mathbb{C}^N)$ *or* $\mathscr{S}(\mathbb{R}^n, \mathbb{C}^N)$*, and*

$$\|f\|_{L^2} := \sqrt{(f,f)_{L^2}} = \left(\sum_{A=1}^{N} \int |f^A(x)|^2 d^n x \right)^{1/2}$$

is the associated norm.

In Appendix A.1 we prove the following theorem as an auxiliary result.

Theorem 2.3.2 *Let* $R^\pm : C_c^\infty(\mathbb{R}^n, \mathbb{C}^N) \to C^\infty(\mathbb{R}^n, \mathbb{C}^N)$*,* $n \geq 2$ *be the advanced and retarded fundamental solutions of* $D = -i\partial\!\!\!/ + m$ *on* n*-dim. Minkowski space-time:*

$$\begin{aligned} &DR^\pm f = f = R^\pm D f \text{ and} \\ &\operatorname{supp} R^\pm f \subset J^\pm(\operatorname{supp} f). \end{aligned}$$

Let $\triangle := \sum_{\mu=0}^{n-1} \partial_{x^\mu}^2$ *be the* n*-dimensional Laplace operator and*

$$Mf(y) := \sum_{\mu=0}^{n-1} y_\mu^2 f(y) \ (y \in \mathbb{R}^n),$$

and

$$\begin{aligned} \mathfrak{a} f &:= (1 - \triangle)^{-1} f, \\ \mathfrak{b} f &:= (1 + M)^{-1} f \end{aligned}$$

for $f \in L^2(\mathbb{R}^n)$*. Then there exist* $\alpha, \beta \in \mathbb{N}$*, such that* $\mathfrak{b}^\beta \mathfrak{a}^\alpha R^\pm \mathfrak{a}^\alpha \mathfrak{b}^\beta$ *can be extended to bounded operators on* $L^2(\mathbb{R}^n)$ *(taking values* $\subset L^2(\mathbb{R}^n)$*).*
As a byproduct it is shown that the domain of $R^\pm$ *can be extended to* $\mathscr{S}(\mathbb{R}^n, \mathbb{C}^N)$*. We will make use of this a lot and take*

$$R^\pm : \mathscr{S}(\mathbb{R}^n, \mathbb{C}^N) \to C^\infty(\mathbb{R}^n, \mathbb{C}^N).$$

Remark *Note that $R^\pm$ and $\mathfrak{a}^\alpha$ commute (easily shown with the help of explicit formulas from the proof in the appendix).*
This result can easily be applied in the Moyal-Minkowski setting as well. The product structure of the functions on the spacetime has no effect on the proven analytical properties.

For technical reasons to come we need the following definition. It introduces versions of $R^\pm$ which are restricted to time slices.

Definition 2.3.3 *Let*

$$\mathfrak{M}_\tau := \{(x^0, \underline{x}) \in \mathbb{R}^n : |x^0| < \tau\} \tag{2.12}$$

with arbitrarily fixed $0 < \tau \in \mathbb{R}$, chosen once and for all from now on. Set

$$\mathscr{S}_0(\mathfrak{M}_\tau) := C_c^\infty((-\tau, \tau)) \otimes \mathscr{S}(\mathbb{R}^{n-1})$$

as the space of Schwartz functions that are compactly supported in the time direction. Then define

$$R_\tau^\pm : \mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N) \to C^\infty(\mathfrak{M}_\tau, \mathbb{C}^N), \quad R_\tau^\pm f := \chi_{\mathfrak{M}_\tau} R^\pm f$$

for $f \in \mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N)$ with $\chi_{\mathfrak{M}_\tau}(x) := \begin{cases} 1 : x \in \mathfrak{M}_\tau \\ 0 : x \in \mathbb{R}^n \setminus \mathfrak{M}_\tau \end{cases}$.

Definition 2.3.4 *Further define the Dirac operator*

$$D_\tau := D = -i\partial\!\!\!/ + m$$

but restricted on $\mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N)$ and also

$$V_\tau := V = M_\chi R_c L_c M_\chi$$

being defined as in equation (2.10), but restricted to functions on $\mathfrak{M}_\tau$, along with the requirement $\operatorname{supp} \chi \subset \mathfrak{M}_\tau$.

Then obviously the solutions $R_\tau^\pm$ from Definition 2.3.3 fulfill the usual properties

- $R_\tau^\pm D_\tau f = f = D_\tau R_\tau^\pm f$ for the Dirac operator $D_\tau = -i\partial\!\!\!/ + m$ restricted on $f \in \mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N)$
- $\operatorname{supp} R_\tau^\pm f \subset J^\pm(\operatorname{supp} f) \cap \mathfrak{M}_\tau$ (a restricted version of the ordinary). Moreover there exists a compact $K_\tau \subset \mathbb{R}^n$ such that $\operatorname{supp} R_\tau^\pm f \subset K_\tau$.

Furthermore Theorem 2.3.2 can be transferred from $R^\pm$ to $R_\tau^\pm$ as well in a straightforward manner.

Definition 2.3.5 *Over $\Omega = \mathbb{R}^n$ or $\Omega = \mathfrak{M}_\tau$ define the* ***Sobolev space*** *(and its norm) of order $l \in \mathbb{N}$ by*

$$H^l := H^l(\Omega) := \{f \in L^2(\Omega) : \|f\|_{H^l} := \|\partial^l f\|_{L^2} + \|f\|_{L^2} < \infty\},$$

with ∂^l denoting the sum of multi-index derivatives to a combined order of l, $\partial^l f(x) = \sum_{l_1+\dots+l_n=l} \frac{\partial^l f(x)}{\partial x_1^{l_1} \cdots \partial x_n^{l_n}}$.

Proposition 2.3.6 *Let $V_\tau = M_\chi R_c L_c M_\chi$ be defined as in equation (2.10), but restricted to functions on $\mathfrak{M}_\tau$, along with the requirement $\operatorname{supp} \chi \subset \mathfrak{M}_\tau$. Then there is a constant $\mathscr{C} \in \mathbb{R}$ such that*

(a)

$$\|R_\tau^\pm V_\tau\|_{op} \leq \mathscr{C} \tag{2.13}$$

for $R_\tau^\pm V_\tau$ as mapping from

$$\left(\mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N), \|\cdot\|_{L^2}\right) \to \left(L^2(\mathfrak{M}_\tau, \mathbb{C}^N) \cap C^\infty, \|\cdot\|_{L^2}\right).$$

Beyond that, $R_\tau^\pm V_\tau$ maps to functions, that are spatially compactly supported in $\mathfrak{M}_\tau$.

(b)

$$\|V_\tau R_\tau^\pm\|_{op} \leq \mathscr{C} \tag{2.14}$$

for $V_\tau R_\tau^\pm$ as mapping from

$$\left(C_c^\infty(\mathfrak{M}_\tau, \mathbb{C}^N), \|\cdot\|_{L^2}\right) \to \left(C_c^\infty(\mathfrak{M}_\tau, \mathbb{C}^N), \|\cdot\|_{L^2}\right).$$

(c) Both cases (a) and (b) can be generalized from $\|\cdot\|_{L^2}$ to Sobolev norms $\|\cdot\|_{H^l}$ of arbitrary order $l \in \mathbb{N}_0$ (with a different constant, also called $\mathscr{C}$).

Proof

(a) Let's choose $\mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N)$ as domain and write

$$R_\tau^\pm V_\tau = \mathbb{1} R_\tau^\pm \mathbb{1} V_\tau = \mathfrak{b}^{-\beta} \mathfrak{b}^\beta R_\tau^\pm \mathfrak{a}^{2\alpha} \mathfrak{b}^\beta \mathfrak{b}^{-\beta} \mathfrak{a}^{-2\alpha} V_\tau,$$

with the maps $\mathfrak{a}, \mathfrak{b}$ from Theorem 2.3.2 and $\alpha, \beta \in \mathbb{N}$. Now V_τ, like V, is a smoothing pseudo-differential operator (smoothing ΨDO), since this is the case for $R_c L_c$ according to Proposition 2.1.3, and M_χ is just a ΨDO (in the set of ΨDO's the smoothing ones form an ideal). With $\mathfrak{b}^{-\beta}\mathfrak{a}^{-2\alpha}$ being a ΨDO, $\mathfrak{b}^{-\beta}\mathfrak{a}^{-2\alpha} V_\tau$ is also smoothing and therefore extends to a L^2-bounded operator with (in our case) values in $C_c^\infty(\mathfrak{M}_\tau)$.
Furthermore Theorem 2.3.2 along with its remark tells us, that $\mathfrak{b}^\beta R_\tau^\pm \mathfrak{a}^{2\alpha} \mathfrak{b}^\beta$ is L^2-bounded.

Due to the support property of $R^\pm_\tau$, that there is a compact $K_\tau \subset \mathbb{R}^n$ such that $\operatorname{supp} R^\pm_\tau f \subset J^\pm(\operatorname{supp} f) \cap \mathfrak{M}_\tau \subset K_\tau \subset \overline{\mathfrak{M}_\tau}$, the final mapping $\mathfrak{b}^{-\beta}$ ends up yielding L^2-integrable functions as range and being L^2-bounded. Beyond that, the resulting functions are smooth and spatially compactly supported.
Altogether this proves part (a).

(b) In this case it is not possible to choose $\mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N)$ as domain. It has to be $C^\infty_c(\mathfrak{M}_\tau, \mathbb{C}^N)$ instead, since $\mathfrak{b}^{-\beta}$ rightmost in

$$V_\tau R^\pm_\tau = V_\tau \mathbb{1} R^\pm_\tau \mathbb{1} = V_\tau \mathfrak{a}^{-2\alpha} \mathfrak{b}^{-\beta} \mathfrak{b}^{\beta} \mathfrak{a}^{2\alpha} R^\pm_\tau \mathfrak{b}^{\beta} \mathfrak{b}^{-\beta}$$

is only L^2-bounded from $C^\infty_c(\mathfrak{M}_\tau)$ to itself.
Then of course $\mathfrak{b}^{\beta} \mathfrak{a}^{2\alpha} R^\pm_\tau \mathfrak{b}^{\beta}$ and $V_\tau \mathfrak{a}^{-2\alpha} \mathfrak{b}^{-\beta}$ are L^2-bounded just like in part (a). And everything is finally mapped into $C^\infty_c(\operatorname{supp} \chi, \mathbb{C}^N)$, $\operatorname{supp} \chi \subset \mathfrak{M}_\tau$, which proves part (b).

(c) Let T be either the operator $R^\pm_\tau V_\tau$ or $V_\tau R^\pm_\tau$. From the proofs of (a) and (b) it is apparent that also $\partial^l T$ is L^2-bounded, since ∂^l acts just on C^∞_c-functions for both choices of T.

$$\begin{aligned} \|Tf\|_{H^l} &= \|\partial^l T f\|_{L^2} + \|Tf\|_{L^2} \leq C_1 \|f\|_{L^2} + C\|f\|_{L^2} = C_2 \|f\|_{L^2} \\ &\leq C_2 \|f\|_{L^2} + \|\partial^l f\|_{L^2} \leq C_3 \|f\|_{H^l} \end{aligned}$$

on $\mathfrak{M}_\tau$. □

Definition 2.3.7 *Let $\lambda \in \mathbb{R}$ and $D_\tau := D$ but restricted to $\mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N)$. Define*

$$D_{\tau,\lambda V} := D_\tau + \lambda V_\tau \tag{2.15}$$

on $\mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N)$ as the Dirac operator coupled to potential V_τ with interaction strength λ, restricted to the time-slice $\mathfrak{M}_\tau$.

Theorem 2.3.8 *Let λ be a sufficiently small real parameter, more precisely $|\lambda| \leq \mathscr{C}^{-1}$, where $\mathscr{C}$ is the constant from Proposition 2.3.6. Then*

$$\mathscr{L}^\pm_\tau := {R^\pm_{\tau,\lambda V}}^{\textit{(left)}} := (\mathbb{1} + R^\pm_\tau \lambda V_\tau)^{-1} R^\pm_\tau$$

mapping $\mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N) \to C^\infty(\mathfrak{M}_\tau, \mathbb{C}^N)$ are well-defined and fulfill

$$\mathscr{L}^\pm_\tau D_{\tau,\lambda V} f = f$$

for $f \in \mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N)$.
And also (see Proposition 2.3.6, second part)

$$\mathscr{R}^\pm_\tau := {R^\pm_{\tau,\lambda V}}^{\textit{(right)}} := R^\pm_\tau (\mathbb{1} + \lambda V_\tau R^\pm_\tau)^{-1}$$

mapping $C^\infty_c(\mathfrak{M}_\tau, \mathbb{C}^N) \to C^\infty(\mathfrak{M}_\tau, \mathbb{C}^N)$ are well-defined and fulfill

$$D_{\tau,\lambda V} \mathscr{R}^\pm_\tau f = f$$

for $f \in C^\infty_c(\mathfrak{M}_\tau, \mathbb{C}^N)$.

Proof From Proposition 2.3.6 we know that $\|R_\tau^\pm \lambda V_\tau\|_{\mathrm{op}} \leq 1$ for $|\lambda| \leq \mathscr{C}^{-1}$, and that in fact this holds between Sobolev spaces of arbitrary order. Hence $\mathbb{1} + R_\tau^\pm \lambda V_\tau$ is invertible in the sense of a Neumann series

$$(\mathbb{1} + R_\tau^\pm \lambda V_\tau)^{-1} = \sum_{j=0}^{\infty} (-R_\tau^\pm \lambda V_\tau)^j$$

converging uniformly for derivatives of arbitrary order, guaranteeing smoothness of the inverse. Then

$$\begin{aligned} \mathscr{L}_\tau^\pm D_{\tau,\lambda V} f &= (\mathbb{1} + R_\tau^\pm \lambda V_\tau)^{-1} R_\tau^\pm (D_\tau + \lambda V_\tau) f \\ &= (\mathbb{1} + R_\tau^\pm \lambda V_\tau)^{-1} (f + R_\tau^\pm \lambda V_\tau f) = f. \end{aligned}$$

The statement for $\mathscr{R}_\tau^\pm$ is proven analogously with the help of the remaining part of Proposition 2.3.6. □

Remark From now on, it is always implicitly assumed that $|\lambda|$ is chosen sufficiently small in accordance to a prespecified situation of fixed τ (setting up the thickness of the time-slice $\mathfrak{M}_\tau$) and χ (the cut-off function for the potential).
At a final stage we aim at $\tau \to \infty$ and $\chi \to 1$ of course.
Furthermore, $|\lambda|$ is assumed to be chosen sufficiently small in regard to the following Corollary 2.3.9 (For our practical purposes some fixed finite order l of differentiability will be sufficient there).

Corollary 2.3.9 *Actually on their smallest common domain the left and right fundamental solutions coincide for an arbitrary order $l \in \mathbb{N}$ of differentiability:*

$$R_{\tau,\lambda V}^\pm := \mathscr{R}_\tau^\pm = \mathscr{L}_\tau^\pm : C_c^l(\mathfrak{M}_\tau, \mathbb{C}^N) \to C^l(\mathfrak{M}_\tau, \mathbb{C}^N). \tag{2.16}$$

It holds

$$\mathscr{R}_\tau^\pm D_{\tau,\lambda V} f = f = D_{\tau,\lambda V} \mathscr{R}_\tau^\pm f$$

and

$$\operatorname{supp} \mathscr{R}_\tau^\pm f \subset J^\pm(\operatorname{supp} f) \cap \mathfrak{M}_\tau$$

for $f \in C_c^l(\mathfrak{M}_\tau, \mathbb{C}^N)$. Note that $\lambda = \lambda(l)$ has to be chosen smaller and smaller for increasing l.

Proof

$$\begin{aligned} \mathscr{L}_\tau^\pm &= (\mathbb{1} + R_\tau^\pm \lambda V_\tau)^{-1} R_\tau^\pm = \sum_{j=0}^{\infty} (-R_\tau^\pm \lambda V_\tau)^j R_\tau^\pm \\ &= R_\tau^\pm \sum_{j=0}^{\infty} (-\lambda V_\tau R_\tau^\pm)^j = \mathscr{R}_\tau^\pm, \end{aligned}$$

since both of the appearing series are shown to converge on $C_c^\infty(\mathfrak{M}_\tau, \mathbb{C}^N)$ w.r.t. Sobolev norms of arbitrary order l (see Prop. 2.3.6). □

2.4 Construction of the dynamics

On $\mathfrak{M}_\tau$ (see (2.12)) we have constructed fundamental solutions $R^\pm_{\tau,\lambda V} = \mathscr{R}^\pm_\tau$ with the properties of Corollary 2.3.9 for the Dirac operator $D_{\tau,\lambda V} = D_\tau + \lambda V_\tau$ coupled to λV_τ with a sufficiently small $\lambda \in \mathbb{R}$, cf. also Definition 2.3.4.

Definition 2.4.1

$$R_{\tau,\lambda V} := R^+_{\tau,\lambda V} - R^-_{\tau,\lambda V}$$

or shorthand notation

$$\mathscr{R}_\tau := R_{\tau,\lambda V}$$

for $\lambda V \neq 0$.

Remark on notation: In the following we will need the "free" versions (potential $\lambda V = 0$) of $R_{\tau,\lambda V}$ and $D_{\tau,\lambda V}$ again a lot, and also a few other objects to be defined with similar subscript notation.
The notational abbreviation then always goes like

$$R_\tau := R_{\tau,0}, \quad D_\tau := D_{\tau,0}.$$

We intend to get rid of the auxiliary τ, the restriction to the time-slice $\mathfrak{M}_\tau$, in a later subsection.

2.4.1 The CAR-algebra $\mathfrak{F}(\mathcal{K}_0^{\mathfrak{M}_\tau}, C)$ of the free Dirac field on $\mathfrak{M}_\tau$

At first, the free (meaning no potential, $\lambda V = 0$) one-particle Hilbert space, going to be called $\mathcal{K}_0^{\mathfrak{M}_\tau}$, needs to be constructed. The strategy is the same as in [6]. Although in this paper the construction from the beginning relied on the Dirac operator coupled to a potential (in the commutative case as well as in the non-commutative one), it is common and also simpler to carry this out for the free case.
Therefore we take the free advanced/retarded fundamental solutions $R^\pm_\tau = R^\pm_{\tau,0}$ and $R_\tau := R^+_\tau - R^-_\tau$ defined on $\mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N)$ (compactly supported in time direction, Schwartz in spatial directions, cf. Definition 2.3.3) for the free Dirac operator $D_\tau = D_{\tau,0} = -i\partial\!\!\!/ + m$ and set up the

Definition 2.4.2 *Let* $f, h \in \mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N)$.

$$\begin{aligned}
\langle f, h\rangle &:= \int_{\mathbb{R}^n} \gamma_{0AB} \bar{f}^B(x) h^A(x) d^n x = \gamma_{0AB}(f^B, h^A)_{L^2},\\
(f, h)_\tau &:= (f, h)_{\tau,0} := \langle f, iR_\tau h\rangle,\\
\mathcal{K}_0^{\mathfrak{M}_\tau} &:= \textit{completion of } \mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N)/\ker R_\tau\\
&\quad\ \textit{w.r.t. the following scalar product:}\\
([f]_\tau, [h]_\tau) &:= (f, h)_\tau,
\end{aligned}$$

denoting the elements (equivalence classes) of $\mathcal{K}_0^{\mathfrak{M}_\tau}$ *by* $[f]_\tau := [f]_{\tau,0}$.

Proposition 5 in [6] shows (actually in a more general case with a Moyal-type potential that slightly deforms the properties of the fundamental solutions) that everything is well-defined and $\left(\mathcal{K}_0^{\mathfrak{M}_\tau}, (\cdot,\cdot)\right)$ is indeed a Hilbert space.
Again, the Moyal product structure of our underlying spacetime doesn't affect the definition of $\langle\cdot,\cdot\rangle$ (probably a place, where one could argue to use the $\star$-product), because of its tracial property. Besides, it does not matter whether to integrate over $\mathbb{R}^n$ or $\mathfrak{M}_\tau$ there.
Of course it holds

$$[f]_\tau = [h]_\tau \Leftrightarrow R_\tau(f-h) = 0.$$

Remark One could easily choose the test-function space $C_c^\infty(\mathfrak{M}_\tau, \mathbb{C}^N)$ instead of $\mathscr{S}_0(\mathfrak{M}_\tau, \mathbb{C}^N)$, which is absolutely common. We just try to be a bit more general here. However, Schwartz w.r.t. all dimensions will not be possible, since we will need to prepare future and past scattering states distinct from the region where the potential is supported.

Definition 2.4.3 *Generators of the CAR-algebra* $\mathfrak{F}(\mathcal{K}_0^{\mathfrak{M}_\tau}, C)$, *$C$ being the charge conjugation from page 13, are the* $\mathbb{C}$*-linear*

$$B_\tau([f]_\tau) := B_{\tau,0}([f]_\tau), \quad [f]_\tau = [f]_{\tau,0} \in \mathcal{K}_0^{\mathfrak{M}_\tau}.$$

Writing $\Psi_\tau(f) := \Psi_{\tau,0}(f) := B_\tau([f]_\tau)$, *we demand the relations*

$$\begin{aligned} \Psi_\tau(f)^* &= \Psi_\tau(Cf) \\ \{\Psi_\tau(f)^*, \Psi_\tau(h)\} &= 2(f,h)_\tau \mathbb{1} \\ \Psi_\tau(D_\tau f) &= 0. \end{aligned}$$

2.4.2 The one-particle space dynamics

We rely on Lemma 4 (or 1) in [6], suitably adapted to the situation considered here. First of all, the geometrical situation is extended like in the picture:

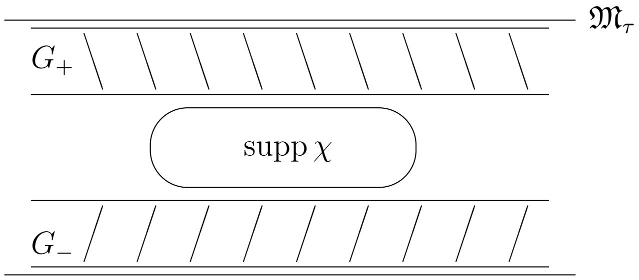

Figure 1: Definition of time-slices $G_\pm$

That is, in $\mathfrak{M}_\tau$ we define two open time-slices

$$G_+ = G_+(\tau,\chi), \quad G_- = G_-(\tau,\chi)$$

in the future $(+)$ and in the past $(-)$ of $\operatorname{supp}\chi$, the cut-off localization of the potential V_τ.
Analogously as in Subsection 2.4.1 we can construct the spaces $\mathcal{K}_0^{G_+}$ and $\mathcal{K}_0^{G_-}$ corresponding to the subspaces $G_+, G_- \subset \mathfrak{M}_\tau$ respectively.
$\mathcal{K}_0^{G_\pm}$ are a priori subspaces of $\mathcal{K}_0^{\mathfrak{M}_\tau}$, but in fact they coincide with $\mathcal{K}_0^{\mathfrak{M}_\tau}$ for functions $f \in \mathscr{S}_0(G_\pm, \mathbb{C}^N)$. We have unitary "embeddings" (in fact isomorphisms)

$$\begin{tikzcd} \mathcal{K}_0^{G_+} \arrow[dr, "u_{0,+}"] & \\ & \mathcal{K}_0^{\mathfrak{M}_\tau} \\ \mathcal{K}_0^{G_-} \arrow[ur, "u_{0,-}"'] & \end{tikzcd}$$

defined by

$$u_{0,\pm} : [f]_0^{G_\pm} \mapsto [f]_0^{\mathfrak{M}_\tau} := [f]_\tau$$

for $f \in \mathscr{S}_0(G_\pm, \mathbb{C}^N)$, where we have written $[f]_0^{G_\pm}$ for $f \mod \ker R_0^{G_\pm}$.
Now we define an operator

$$U_{\tau,\lambda V} : \mathcal{K}_0^{\mathfrak{M}_\tau} \to \mathcal{K}_0^{\mathfrak{M}_\tau}$$

in analogy to equation (13) in [6] by

$$U_{\tau,\lambda V} : [f]^{\mathfrak{M}_\tau} \overset{u_{0,+}^{-1}}{\longmapsto} [f^{G_+}]^{G_+} \overset{w}{\longmapsto} [f^{G_-}]^{G_-} \overset{u_{0,-}}{\longmapsto} [f]^{\mathfrak{M}_\tau}, \tag{2.17}$$

omitting the subscript 0 for the vanishing potential to simplify the notation. Actually, the role of $u_{0,\pm}$ here is quite trivial and the interesting thing is the action of w, where the potential λV comes into play. f^{G_+} is any element of $C_c^\infty(G_+, \mathbb{C}^N) \subset \mathscr{S}_0(G_+, \mathbb{C}^N)$ such that $R_\tau(f - f^{G_+}) = 0$. Then w is defined as follows: we take f^{G_-} in $\mathscr{S}_0(G_-, \mathbb{C}^N)$ such that

$$\mathscr{R}_\tau f^{G_+} \equiv R_{\tau,\lambda V} f^{G_+} = R_{\tau,0} f^{G_-} \equiv R_\tau f^{G_-} \text{ on } G_-. \tag{2.18}$$

We must show that this is well-defined, provided $|\lambda|$ is sufficiently small (applies to all statements made here; cf. Remark on p. 19). In particular it has to be shown that $\mathscr{R}_\tau f^{G_+}$ is independent of the choice of f^{G_+} in $[f^{G_+}]^{G_+}$. This is shown in the

Proof Let $[f^{G_+}]^{G_+} = [h^{G_+}]^{G_+}$ ($\Leftrightarrow R_\tau(f^{G_+} - h^{G_+}) = 0$). We have

$$\begin{aligned} \mathscr{R}_\tau &= \mathscr{R}_\tau^+ - \mathscr{R}_\tau^- \\ &= R_\tau + \sum_{j=1}^{\infty} (-)^j \left[R_\tau^+ (\lambda V_\tau R_\tau^+)^j - R_\tau^- (\lambda V_\tau R_\tau^-)^j \right]. \end{aligned} \tag{2.19}$$

The supports of f^{G_+} and h^{G_+} lie in the future of $\operatorname{supp}\chi$ and hence it follows that

$$V_\tau R_\tau^+ f^{G_+} = 0 = V_\tau R_\tau^+ h^{G_+}$$

and

$$V_\tau R_\tau^- f^{G_+} = -V_\tau R_\tau f^{G_+} = -V_\tau R_\tau h^{G_+} = V_\tau R_\tau^- h^{G_+}.$$

Inserting this in (2.19), one obtains $\mathscr{R}_\tau f^{G_+} = \mathscr{R}_\tau h^{G_+}$. □

The next step is to show that w is isometric (perhaps even unitary, but for our purpose isometry is in fact sufficient).

Proposition 2.4.4 *w is isometric.*

Proof We know that $\varphi := \mathscr{R}_\tau f^{G_+}$ is a solution of

$$D_{\tau,\lambda V}\varphi = (-i\partial\!\!\!/ + m + \lambda V_\tau)\varphi = 0 \text{ on } \mathfrak{M}_\tau.$$

To show that w is isometric, we have to check that

$$\left(f^{G_+}, h^{G_+}\right)^{G_+} = \left([f^{G_+}]^{G_+}, [h^{G_+}]^{G_+}\right)^{G_+} = \left(w[f^{G_+}]^{G_+}, w[h^{G_+}]^{G_+}\right)^{G_-}$$

for all f^{G_+}, h^{G_+} in $C_c^\infty(G_+, \mathbb{C}^N) \subset \mathscr{S}_0(G_+, \mathbb{C}^N)$. Since φ is at least C^1 (actually smooth) we can apply the Gaussian formula for $\varphi = \mathscr{R}_\tau f^{G_+}$ and $\psi = \mathscr{R}_\tau h^{G_+}$ (cf. Dimock's paper on Dirac fields [13]): Let $\Sigma_\pm$ be some Cauchy surfaces in $G_\pm$ respectively, and note that φ, ψ solve the free Dirac equation on G_+ and G_-.

$$\begin{aligned} &\left(f^{G_+}, h^{G_+}\right)^{G_+} - \left(w[f^{G_+}]^{G_+}, w[h^{G_+}]^{G_+}\right)^{G_-} \\ &= \int_{\Sigma_+} \bar{\varphi}^A(\underline{x})\delta_{AB}\psi^B(\underline{x})d^{n-1}\underline{x} - \int_{\Sigma_-} \bar{\varphi}^A(\underline{y})\delta_{AB}\psi^B(\underline{y})d^{n-1}\underline{y} \\ &= \int_{\partial\mathfrak{M}_{\Sigma_-,\Sigma_+}} \varphi^+\gamma^a\psi do_a, \end{aligned}$$

where do_a is the outer surface form of the boundary of $\mathfrak{M}_{\Sigma_-,\Sigma_+} = J^-(\Sigma_+) \cap J^+(\Sigma_-)$, $\partial\mathfrak{M}_{\Sigma_-,\Sigma_+} = \Sigma_+ \cup \Sigma_-$, and φ^+ denotes the Dirac adjoint spinor (a cospinor) $\varphi_C^+ := \bar{\varphi}^A\gamma^0_{AC}$. We used $\delta_{AB} = {\gamma^0}_{AC}{\gamma^0}^C{}_B$ and that the spatial directions are perpendicular to the surface normals. According to the Gaussian theorem the last line of the above calculation equals

$$\begin{aligned} &= \int_{\mathfrak{M}_{\Sigma_-,\Sigma_+}} \partial_a\left(\varphi^+\gamma^a\psi\right) d^n x = \int_{\mathfrak{M}_{\Sigma_-,\Sigma_+}} \left[\left(\partial_a\varphi^+\gamma^a\right)\psi + \varphi^+\gamma^a\partial_a\psi\right] d^n x \\ &= \int_{\mathfrak{M}_{\Sigma_-,\Sigma_+}} \left(\partial\!\!\!/\varphi^+\psi + \varphi^+\partial\!\!\!/\psi\right) d^n x, \end{aligned}$$

because the definitions of the differential operator $\not{\partial}$ on spinors and cospinors differ in the position of γ^a: $\not{\partial}\varphi^+ = \partial_a\varphi^+\gamma^a$, $\not{\partial}\psi = \gamma^a\partial_a\psi$. This can be found in Dimock [13] as well as the property $\not{\partial}\varphi^+ = (\not{\partial}\varphi)^+$, clear from $\not{\partial}\varphi^+ = \partial_a\bar{\varphi}\gamma^0\gamma^a = \partial_a\bar{\varphi}(\gamma^a)^*\gamma^0 = \partial_a\overline{(\varphi(\gamma^a)^T)}\gamma^0 = \partial_a\overline{(\gamma^a\varphi)}\gamma^0 = (\gamma^a\partial_a\varphi)^+$, which continues the indented calculation

$$
\begin{aligned}
&= \int_{\mathfrak{M}_{\Sigma_-,\Sigma_+}} \Big((\not{\partial}\varphi)^+\psi + \varphi^+\not{\partial}\psi \Big)\, d^n x \\
&= \int_{\mathfrak{M}_{\Sigma_-,\Sigma_+}} \Big(\left[-i\,(m+\lambda V_\tau)\,\varphi\right]^+\psi + \varphi^+\left[-i\,(m+\lambda V_\tau)\,\psi\right] \Big)\, d^n x \\
&= i\lambda \int_{\mathfrak{M}_{\Sigma_-,\Sigma_+}} \Big(V_\tau\varphi^+\psi - \varphi^+ V_\tau\psi \Big)\, d^n x = 0
\end{aligned}
$$

since V is L^2-symmetric as shown in Lemma 3 in [6], because c and χ appearing in $V = M_\chi R_c L_c M_\chi$ are real-valued and V acts as a scalar. □

Altogether this concludes the construction of the isometric (and probably even unitary) operator $U_{\tau,\lambda V} : \mathcal{K}_0^{\mathfrak{M}_\tau} \to \mathcal{K}_0^{\mathfrak{M}_\tau}$ which obviously commutes with the charge conjugation C.
Then by standard arguments, like used in Lemma 4 (or 1) in [6] (based on Araki [1], or [7], [8]), there is a C^*-algebraic endomorphism (probably automorphism, if $U_{\tau,\lambda V}$ is unitary)

$$\beta_{\tau,\lambda V} : \mathfrak{F}(\mathcal{K}_0^{\mathfrak{M}_\tau}, C) \to \mathfrak{F}(\mathcal{K}_0^{\mathfrak{M}_\tau}, C)$$

defined by

$$\beta_{\tau,\lambda V}\left(B_\tau([f]_\tau)\right) := B_\tau(U_{\tau,\lambda V}[f]_\tau), \tag{2.20}$$

which we call the Bogoliubov **scattering morphism**. Note that $\beta_{\tau,\lambda V}(B_\tau)|_{\lambda=0} = B_\tau$.

2.4.3 Bogoliubov's formula

Carefully checking the arguments of Chapter 8 in [6] one realizes that they apply here as well, thanks to the localization of V by χ. Moreover, the situation at hand here is actually better behaved, because of the ordinary propagation property of the fundamental solutions in Corollary 2.3.9.
We can hence define the derivation

$$\delta_{\tau,\lambda V}\left(B_\tau([f]_\tau)\right) := \left.\frac{d}{d\lambda}\right|_{\lambda=0} B_\tau(U_{\tau,\lambda V}[f]_\tau) \tag{2.21}$$

and get indeed the expected result

$$\delta_{\tau,\lambda V}\left(B_\tau([f]_\tau)\right) = B_\tau([V_\tau R_\tau f]_\tau). \tag{2.22}$$

2.4.4 Getting rid of the dependence on τ

We want to extend the results on $\mathfrak{M}_\tau$ to the whole $\mathbb{R}^n$. The potential cut-off χ still remains fixed. For $\tau' > \tau$ one has the canonical isomorphism

$$\alpha_{\tau',\tau} : \mathfrak{F}(\mathcal{K}_0^{\mathfrak{M}_\tau}, C) \to \mathfrak{F}(\mathcal{K}_0^{\mathfrak{M}_{\tau'}}, C)$$

given by

$$\alpha_{\tau',\tau} : B_\tau([f]_\tau) \mapsto B_{\tau'}([f]_{\tau'}).$$

With

$$\delta_{\tau,\lambda V}\left(B_\tau([f]_\tau)\right) = B_\tau([V_\tau R_\tau f]_\tau)$$

one has

$$\delta_{\tau',\lambda V}\alpha_{\tau',\tau}B_\tau([f]_\tau) = \alpha_{\tau',\tau}\delta_{\tau,\lambda V}B_\tau([f]_\tau).$$

So in fact, $\delta_{\tau,\lambda V}$ defines already all $\delta_{\tau',\lambda V}$. Hence, we have a derivation

$$\delta_{\lambda V}B([f]) = \alpha_{\infty,\tau}\delta_{\tau,\lambda V}\alpha^{-1}_{\infty,\tau}B([f]) = B([VRf]), \tag{2.23}$$

where

$$\alpha_{\infty,\tau} : \mathfrak{F}(\mathcal{K}_0^{\mathfrak{M}_\tau}, C) \to \mathfrak{F}(\mathcal{K}_0, C),$$

with $\mathcal{K}_0 = \mathcal{K}_0^{\mathfrak{M}_\infty}$ for all of Moyal-Minkowski spacetime. And it doesn't matter which τ one takes as long as $\operatorname{supp}\chi$ is compactly contained in $\mathfrak{M}_\tau$.

2.4.5 Final step: Removing the cut-off χ for the potential

Due to the special choice of χ, see the lines below equation (2.10), we can take the limit $\chi \to 1$ e.g. in the form $a \to \infty$ for $\chi_a(x) := \chi\left(\frac{1}{a}x\right)$, replacing χ by χ_a everywhere.
The problem which arises is that one must check that $\lim_{a\to\infty}[V_a Rf]$ exists as an element in $\mathcal{K}_0$. Put differently, one must show that the limit

$$\lim_{a\to\infty} i\langle V_a Rf, RV_a Rf\rangle$$

exists, or whether $i\langle c \star Rf \star c, R(c \star Rf \star c)\rangle$ makes sense formally.
The problem here is that Rf is not contained in $\mathscr{S}$; probably Rf is bounded. Finally, when we manage to check that the integral can be formed or the limit exists, we would have

$$\delta_{\lambda V}B([f]) = \lim_{a\to\infty}\delta_{\lambda V_a}B([f]),$$

and this should be a derivation since it is the limit of derivations.
Actually, all of this can indeed be carried out, since it is possible to prove the following

Proposition 2.4.5 *Let $f \in \mathscr{S}_0(\mathbb{R}^n, \mathbb{C}^N)$ and $c \in \mathscr{S}(\mathbb{R}^n, \mathbb{R})$. Then it holds*

$$\left(c \star (Rf)^B \star c, (R(c \star Rf \star c))^A\right)_{L^2} < \infty.$$

Proof The paper [17] (Lemma 2.12 and text below) tells us that $\star$ can be extended to $L^2 \times L^2$, or $\mathscr{S} \times L^1$, or $\mathscr{S} \times \mathcal{F}L^1$ and even, which is the key to our result, $\mathscr{S} \times L^\infty$; and thereby always mapping to L^2 (at least). Together with Lemma A.2.3, showing that Rf is indeed bounded, this implies $c \star (Rf)^B \star c \in L^2$. It is easy to see that this is also still smooth and Lemma A.2.4 can be applied, ensuring the existence of the integral $(g, Rg)_{L^2}$, with $g := c \star Rf \star c \in L^2 \cap C^\infty$.□

So finally Bogoliubov's formula holds true in the following form.

Theorem 2.4.6

$$\delta_{\lambda V} \Psi(f) = \Psi(VRf),$$

with $VRf = c \star Rf \star c$, $c \in \mathscr{S}(\mathbb{R}^n, \mathbb{R})$, $\star$ denoting the ordinary Moyal product (implying non-commutativity also in time direction), $R = R^+ - R^-$ the difference of adv./ret. fundamental solutions for the free Dirac operator $D = -i\not{\partial} + m$, and $\Psi(f)$, $f \in \mathscr{S}_0(\mathbb{R}^n, \mathbb{C}^N)$, the CAR-quantized abstract free Dirac field operators $\mathfrak{F}(\mathcal{K}_0, C)$, and finally $\delta_{\lambda V}$ a derivation defined by

$$\delta_{\lambda V} \Psi(f) := \left.\frac{d}{d\lambda}\right|_{\lambda=0} \beta_{\lambda V} \Psi(f) = \left.\frac{d}{d\lambda}\right|_{\lambda=0} \Psi(U_{\lambda V} f).$$

Proof Result of the stepwise construction in this chapter. □

2.4.6 Existence of an operator generating the derivation

To complete the picture, we want to establish the existence of an operator $\Phi(c)$, that generates the derivation $\delta_{\lambda V}$. Confer Theorem 2.4.6 for a summary of notation and context. **Generating** in this sense means that

$$[i\Phi(c), \Psi(f)] = \delta_{\lambda V} \Psi(f) \tag{2.24}$$

should hold, where $\Phi(c)$ exists as an essentially self-adjoint operator in the algebra of the CAR-quantized abstract free Dirac field operators $\mathfrak{F}(\mathcal{K}_0, C)$. In other words $\Phi(c)$ then describes an observable quantum field, and Bogoliubov's formula takes the form

Theorem 2.4.7 *It exists an essentially self-adjoint operator $\Phi(c)$ in $\mathfrak{F}(\mathcal{K}_0, C)$, such that*

$$[i\Phi(c), \Psi(f)] = \Psi(VRf) \tag{2.25}$$

holds. The assumptions are the same as in Theorem 2.4.6.

Proof The proof completely relies on the strategy of [6], specifically Prop. 7, its Proof and the explanations in between. The essential criterion for the existence of $\Phi(c)$ is $[dT_{\text{sc}}^{(V)}, p_+]$ being Hilbert-Schmidt (cf. Sec. 10 in [34]). Here $dT_{\text{sc}}^{(V)} g := -i \frac{d}{d\lambda}\Big|_{\lambda=0} T_{\text{sc}}^{(\lambda V)} g$, and $T_{\text{sc}}^{(\lambda V)}$ denotes the scattering transformation defined on functions $g \in \mathscr{S}(\Sigma, \mathbb{C}^N)$ on a Cauchy surface $\Sigma \cong \mathbb{R}^s$, with $s = n-1$. $T_{\text{sc}}^{(\lambda V)}$ is the analogue of $U_{\lambda V}$ introduced in subsection 2.4.2 and finally connected to $\beta_{\lambda V}$ there. Let $H_0 : \mathscr{S}(\mathbb{R}^s, \mathbb{C}^N) \to \mathscr{S}(\mathbb{R}^s, \mathbb{C}^N)$ be the Hamiltonian of the free Dirac equation (when put in Hamiltonian form), on $n = s+1$-dimensional (Moyal-)Minkowski spacetime. Then finally p_+ is the spectral projection of H_0 corresponding to the spectral interval $[m, \infty)$ (the positive spectral subspace), $p_- = \mathbb{1} - p_+$, $m > 0$.

However, in our case with the time-coordinate being involved, it will be necessary to understand the functions $f \in \mathscr{S}_0(\mathbb{R}^n, \mathbb{C}^N)$ as functions $\chi \in \mathscr{S}(\mathbb{R}^s, \mathbb{C}^N)$ for each fixed "time" t, with $\chi(\underline{x}) := \chi_t(\underline{x}) := f(t, \underline{x}) = f(x)$, where $\underline{x} = (x^1, \ldots, x^s)$ and $t = x^0$.

Paper [6] deals with the following explicit expression for $p_+ dT_{\text{sc}}^{(V)} p_-$, an operator acting on $\mathscr{S}(\mathbb{R}^s, \mathbb{C}^N) \subset L^2(\mathbb{R}^s, \mathbb{C}^N)$, for which it is sufficient to show the Hilbert-Schmidt property:

$$p_+ dT_{\text{sc}}^{(V)} p_- = -\int_{\mathbb{R}} e^{iH_0 t} p_+ \gamma_0 a(t, x^1)^2 L_b R_b p_- e^{-iH_0 t} dt.$$

Back then the Moyal multiplication $L_c R_c$ was split due to $c(x) = a(t) b(\underline{x})$ – a simplification that we want to drop now while adapting the expression to the actual situation at hand.

Let, for $\chi \in \mathscr{S}(\mathbb{R}^s, \mathbb{C}^N)$,

$$\phi_{p_-\chi}(t, \underline{x}) := \left(e^{iH_0 t} p_- \chi\right)(\underline{x}), \quad (t \in \mathbb{R}, \underline{x} \in \mathbb{R}^s), \tag{2.26}$$

i.e. $\phi_{p_-\chi}$ is the (weak) solution of the free Dirac equation (in Hamiltonian form) with initial datum $\phi_{p_-\chi}(0, \underline{x}) = p_-\chi(\underline{x})$ at $t = 0$.

Let $c \in \mathscr{S}(\mathbb{R}^n, \mathbb{R})$, then $L_c R_c \phi_{p_-\chi} = c \star \phi_{p_-\chi} \star c$ denotes the Moyal product of c from left and right with $\phi_{p_-\chi}$.

One can then form the operator G_c (which is just $-p_+ dT_{\text{sc}}^{(V)} p_-$) defined by

$$(G_c \chi)(\underline{x}) := \int_{\mathbb{R}} p_+ e^{-iH_0 t} \gamma_0 (c \star \phi_{p_-\chi} \star c)(t, \underline{x}) dt. \tag{2.27}$$

Here, $p_+ e^{-iH_0 t}$ acts on $f_t(\cdot) = (c \star \phi_{p_-\chi} \star c)(t, \cdot)$ which (as needs to be shown) is in $L^2(\mathbb{R}^s, \mathbb{C}^N)$ for all t.

We wish to prove

- G_c is a well-defined operator $G_c : L^2(\mathbb{R}^s, \mathbb{C}^N) \to L^2(\mathbb{R}^s, \mathbb{C}^N)$
- G_c is Hilbert-Schmidt (w.r.t. the Hilbert space $L^2(\mathbb{R}^s, \mathbb{C}^N)$).

This rather technical part is shifted to the Appendix A.3. $\square$

2.5 Connection to Rieffel's deformed product

In [6] we investigated the Dirac field on Moyal-deformed Minkowski spacetime coupled to a potential, which acts by Moyal-multiplication with respect to spatial coordinates (commutative time). This Chapter updated the essentials to the general case of non-commutative time. Already due to the former work we got some enlightening insights concerning the operational meaning of elements of the non-commutative spacetime algebra at the level of quantum field operators.

One step was to obtain observables

$$\Phi(c) := -id/d\lambda|_{\lambda=0}\, S_{\lambda c}$$

labeled by elements c of the non-commutative function algebra. This idea stems from a principle of Bogoliubov, obtaining observables by functional differentiating the resulting scattering operator $S_{\lambda c}$ with respect to the interaction strength.

The paper then proves the proper existence of the objects $\Phi(c)$ as (essentially) self-adjoint operators and beyond that derives the relation

$$[i\Phi(c), \psi(f)] = \psi(VRf),$$

showing the derivative action of these objects on the generating elements $\psi(f)$ of the field algebra (in Fock-vacuum representation, and to some extent also at the abstract level). Here $R = R^+ - R^-$ is the advanced minus retarded Green's operator of the free Dirac field. And V is the external potential, an operator chosen to be $L_c{+}R_c$ or L_cR_c alternatively. The left and right Moyal multiplication is defined almost like in eq. (2.3), but with commutative action w.r.t. the time instead and also an accordingly simplified space for c.

Finishing this very sketchy overview of past results, we end up at the relation

$$[:\psi^\dagger\psi:(c), \psi(f)] = -i\psi(cRf) \tag{2.28}$$

with the Wick-ordered absolute squared field strength (see appendix of [6] for precise definition and proof). Actually it turned out that this is the same object as $\Phi(c) =: \psi^\dagger\psi : (c)$, but at first only in the case of classical Minkowski spacetime with a classical potential, acting as multiplication operator $Vf(x) = c(x)f(x)$.

Now we want to extend this equation (2.28) to the non-commutative case, an interesting possibility overlooked beforehand. First note that

$$f \star g := \frac{1}{(2\pi)^n} \iint (\tau_{Mu} f)(\tau_v g) e^{iuv} d^n u d^n v, \tag{2.29}$$

with translation map $(\tau_v f)(x) := f(x-v),\ v \in \mathbb{R}^n$ and $f, g \in \mathscr{S}(\mathbb{R}^n)$, coincides with our common definition of the Moyal product (2.2)

$$f \star g(x) = \frac{1}{(2\pi)^n} \iint f(x - Mu) g(x+v) e^{-iuv} d^n u d^n v.$$

Starting from $\psi(VRf) = \psi(c \star (Rf) + (Rf) \star c)$ a formal computation yields for the first term

$$\begin{aligned}
\psi(c \star (Rf)) &= \psi\left((2\pi)^{-n} \iint (\tau_{Mu}c)(\tau_v Rf) e^{iuv} d^n u d^n v\right) \\
&= (2\pi)^{-n} \iint \psi\left((\tau_{Mu}c)(\tau_v Rf)\right) e^{iuv} d^n u d^n v \\
&= (2\pi)^{-n} \iint \psi\left((\tau_{Mu}c)R(\tau_v f)\right) e^{iuv} d^n u d^n v \\
&\overset{(2.28)}{=} (2\pi)^{-n} \iint i\left[: \psi^\dagger\psi : (\tau_{Mu}c), \psi(\tau_v f)\right] e^{iuv} d^n u d^n v \\
&= (2\pi)^{-n} \iint i\left[\alpha_{Mu} : \psi^\dagger\psi : (c), \alpha_v \psi(f)\right] e^{iuv} d^n u d^n v \\
&=: i\left(: \psi^\dagger\psi : (c) \star_R \psi(f) - \psi(f)_R\star : \psi^\dagger\psi : (c)\right) \\
&=: i\left[: \psi^\dagger\psi : (c), \psi(f)\right]',
\end{aligned}$$

with **Rieffel deformation products** $\star_R$ and ${}_R\star$ at quantum field operator algebra level, corresponding to the Moyal-$\star$ at nc. spacetime function algebra level. Be careful, that the definition $[\cdot,\cdot]'$ is not a commutator in the well-known sense: The meaning of $\star_R$ (and also of ${}_R\star$) is two-fold. It multiplies the field operators not only in the sense of the nc. spacetime but also in the sense of the quantum operators. With respect to the product $\star_R$ in the sense of *spacetime*, the first factor is always $: \psi^\dagger\psi : (c)$, that is why the notation ${}_R\star$ is used to denote the product "from right to left" in the second last line. $[\cdot,\cdot]$ is a commutator only in the quantum operator sense.

To summarize it once again, we actually have non-commutativity at two levels going on here at the same time, firstly at the usual quantum operator level, and secondly at the spacetime level.

The calculation for the second term results in

$$\psi((Rf) \star c) = -i\left[\psi(f), : \psi^\dagger\psi : (c)\right]' =: i\left[: \psi^\dagger\psi : (c), \psi(f)\right]'',$$

defining $[\cdot,\cdot]''$, which does not equal $[\cdot,\cdot]'$. And altogether

$$\left[: \psi^\dagger\psi : (c), \psi(f)\right]' + \left[: \psi^\dagger\psi : (c), \psi(f)\right]'' = -i\psi(c \star (Rf) + (Rf) \star c). \tag{2.30}$$

It is possible to pair the four terms on the left hand side in another way:

$$\left[: \psi^\dagger\psi : (c), \psi(f)\right]_{\star_R} + \left[: \psi^\dagger\psi : (c), \psi(f)\right]_{{}_R\star} = -i\psi(c \star (Rf) + (Rf) \star c), \tag{2.31}$$

with the *properly* defined commutators

$$\begin{aligned}
[\psi(f), \psi(g)]_{\star_R} &:= \psi(f) \star_R \psi(g) - \psi(g) \star_R \psi(f) \qquad (2.32)\\
[\psi(f), \psi(g)]_{{}_R\star} &:= \psi(f)_R \star \psi(g) - \psi(g)_R \star \psi(f).
\end{aligned}$$

These are more natural, since when the composition order of the operators is changed, the $\star_R$ product type stays the same and hence changes order of action w.r.t. *spacetime* as well. Note, that the terms of the sums in eq. (2.31) on the left hand side do not individually correspond to the ones on the right. This is only the case in eq. (2.30), with the definitions (not commutators)

$$\begin{aligned} [\psi(f), \psi(g)]' &:= \psi(f) \star_R \psi(g) - \psi(g)_R \star \psi(f) \\ [\psi(f), \psi(g)]'' &:= \psi(f)_R \star \psi(g) - \psi(g) \star_R \psi(f). \end{aligned} \tag{2.33}$$

Of course we restricted ourselves to a purely formal derivation. The mathematical rigorous justification is provided by Rieffel's work [31]. The three critical points are

(a) Commuting ψ and integration.

(b) Existence of α_v corresponding to τ_v.

(c) Commuting R and τ_v.

The last one is easily proven in the following Lemma.

Lemma 2.5.1 $(\forall v \in \mathbb{R}^n)(R^\pm \tau_v = \tau_v R^\pm)$.

Proof Using the explicit forms of $R^\pm$ from [6] (Proof of Prop. 5.(b)),

$$\begin{aligned} (R^\pm \tau_v f)(t, \underline{x}) &= \pm i\gamma^0 \int \theta(\pm(t-t')) f_{t'-v^0}(\underline{x} - \underline{v}) dt' \\ &= \pm i\gamma^0 \int \theta(\pm(t - v^0 - t'')) f_{t''}(\underline{x} - \underline{v}) dt'' \\ &= (\tau_v R^\pm f)(t, \underline{x}). \end{aligned}$$

□

So far about how one can handle the case $\psi(VRf) = \psi(c \star (Rf) + (Rf) \star c)$. Unfortunately for $\psi(VRf) = \psi(c \star (Rf) \star c)$ it is not that easy. There is a rough idea, but we will not take any risks to make a conjecture regarding the shape of a possible formula.
However, this section shows the danger in mistakenly mixing up the two different natures, the field operators have to fulfill in the setting of nc. spacetimes.
Except here, everywhere in this work we avoided defining a star-product at the field operator level. For those who are more familiar with such approaches, this section can perhaps serve as some interface.
Recently appeared literature worthy to point out in this regard is [23] and [10].

3 Locally non-commutative spacetimes

It has been shown in Chapter 2 that the results of [6] can also be obtained without the restriction of time-space-commutativity (i.e. for a Moyal matrix of full rank). Although this is quite an achievement, it is too hard to see even the least serious physical drive for the usage of Moyal-type spacetime non-commutativity. That is one reason why we do not want to follow traditional Moyal spacetimes here anymore. The other reason is that a really nice new approach to nc. spacetimes by D. Bahns and S. Waldmann [3] has attracted attention. Their reasonable argument is to expect effects of spacetime non-commutativity only at very small distances, where one commonly is tempted to think of Planck scale. So the idea is to define a star-product between functions on $\mathbb{R}^n \times \mathbb{R}^n$ instead of $\mathbb{R}^n$. Of course in [3] this is done for general manifolds. However, the main ingredient then is to introduce a suitable decay of non-commutativity with respect to the increasing distance between two points.
We are still only interested in the simple case of flat spacetime. There we can neglect the dependence on the "center of mass"-coordinate of the two points, because the non-commutativity should not depend on absolute positions in a flat universe. Of course one could (probably should, as a second step) also think of generalizing this according to variations of curvature.
But here we restrict ourselves to $\mathbb{R}^n$ where the framework of locally nc. spacetimes can be described most vividly and is introduced technically as follows.

3.1 Basics

The setting to start with is just Minkowski spacetime $(\mathbb{R}^n, \eta)$. The tangent bundle is $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$, and the exponential map for $p \in \mathbb{R}^n$ has the simple form $\exp_p : T_p\mathbb{R}^n \to \mathbb{R}^n$, $\exp_p(v_p) = p + v_p$. What will be essential in order to implement *locality* (of the nc. spacetimes) later, is the introduction of an open neighbourhood $\mathcal{U}$ of the zero section of $T\mathbb{R}^n$, that is $T\mathbb{R}^n \supseteq \mathcal{U} \supseteq \mathbb{R}^n \times \{0\}$, where 0 is an element of the typical fiber $\mathbb{R}^n$ (think of "short" vectors being attached to spacetime points). Furthermore let $\mathcal{V} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ be a suitable open neighbourhood of the diagonal $\Delta_{\mathbb{R}^n} := \{(p,p) | p \in \mathbb{R}^n\}$, i.e. $\mathbb{R}^n \times \mathbb{R}^n \supseteq \mathcal{V} \supseteq \Delta_{\mathbb{R}^n}$, such that the following map Φ acts as a diffeomorphism between $\mathcal{U}$ and $\mathcal{V}$:

$$\Phi : \mathcal{U} \ni v_p \mapsto \Phi(v_p) := (\exp_p(-v_p), \exp_p(v_p)) = (p - v_p, p + v_p) \in \mathcal{V}.$$

Of course for our flat case scenario Φ is a global diffeomorphism, and there is no need to restrict to regions $\mathcal{U}$ and $\mathcal{V}$ from a mathematical point of view. But it is done anyway in order to localize non-commutativity later, when the supports of Poisson structures, used to define star-products, will be chosen to be compact within these open neighbourhoods.

Now set $(x_1, x_2) := (p - v_p, p + v_p)$ and view it as an arbitrary pair of points in $\mathcal{V} \in \mathbb{R}^n \times \mathbb{R}^n$. For such a point-pair one is interested in the distance perpendicular to the diagonal $\Delta_{\mathbb{R}^n}$, which is obviously related to the extension of $\mathcal{V}$. To this end it is helpful to consider the mirrored point (x_2, x_1). Using just the ordinary vector space structure of $\mathbb{R}^n \times \mathbb{R}^n$ one realizes that the midpoint $\frac{1}{2}\left((x_1, x_2) + (x_2, x_1)\right)$ coincides with (p, p). And so does $\frac{1}{2}\left((x_1, x_2) - (x_2, x_1)\right)$ with $(-v_p, v_p)$. Besides, putting together only the second components of the last two equations this also gives us the inverse of Φ.

However, we want to slightly differ from this convention and use the coordinate transformation $\varkappa$ instead of Φ:

$$\varkappa^{-1} : \mathcal{V} \ni (x_1, x_2) \mapsto (x_1 + x_2, x_1 - x_2) =: (h, v) \in \mathcal{U}, \tag{3.1}$$

$$\varkappa : \mathcal{U} \ni v_h := (h, v) \mapsto \left(\frac{h+v}{2}, \frac{h-v}{2}\right) = (x_1, x_2) \in \mathcal{V}.$$

So v denotes the so called vertical or relative coordinate between the points x_1, x_2, which will play the most crucial role in the process of defining so called *vertical* and *local* star-products.

3.2 Star-products

Star-products in general got an axiomatic foundation by [4]. For a review confer [33], written by D. Sternheimer. At first this was proposed for symplectic manifolds, whereas more recent results allow a more general setting of Poisson manifolds. It is the latter that will finally be of main interest for us, but let's briefly sketch the original axioms for $(\mathbb{R}^d, \omega)$ symplectic: A $\mathbb{C}[[\lambda]]$-bilinear[1] map

$$\star : C^\infty(\mathbb{R}^d)[[\lambda]] \times C^\infty(\mathbb{R}^d)[[\lambda]] \to C^\infty(\mathbb{R}^d)[[\lambda]]$$

that can be written as formal power series

$$f \star g := \sum_{k=0}^{\infty} \lambda^k B_k(f, g)$$

for $f, g \in C^\infty(\mathbb{R}^d)$ and bidifferential operators $B_k : C^\infty(\mathbb{R}^d) \times C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$, is called **star-product**, if and only if for all $f, g \in C^\infty(\mathbb{R}^d)[[\lambda]]$ the following properties are fulfilled:

[1] $X[[\lambda]]$ denotes polynomials in λ with coefficients in space X.

(a) $\star$ is associative

(b) $B_0(f,g) = fg$ ($\star$ is a deformation of the usual pointwise product)

(c) $B_1(f,g) - B_1(g,f) = \{f,g\}$ (deformation of the Poisson bracket)

(d) $f \star 1 = f = 1 \star f$

(e) $\operatorname{supp}(f \star g) \subseteq \operatorname{supp} f \cap \operatorname{supp} g$ in every power of λ.

In our setting, we will finally need a star-product $\tilde{\star}$ defined on $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$-functions (from now on we omit the $[[\lambda]]$-dependence), which is locally non-commutative in a sense still to be made explicitly clear. To this end it is more convenient to start with using the function space $C^\infty(T\mathbb{R}^n)$ instead (because the coordinate in vertical direction v is more directly accessible). Of course actually $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$, and the diffeomorphism $\varkappa$ simply transforms the variables $(h,v) \mapsto (x_1,x_2)$ into each other.

Let $f,g \in C^\infty(T\mathbb{R}^n)$, then on the way towards "locality" the intermediate notion of a **vertical** star-product $\star$ is simply defined by $(f \star g)(h,v)$ not containing any $\partial/\partial h$ derivative within the bidifferential operators B_k at all, i.e. they are required to differentiate exclusively w.r.t. the vertical direction v. This amounts to the corresponding differentiation

$$\frac{\partial}{\partial v} f(h,v) = \frac{\partial}{\partial v}(\tilde{f} \circ \varkappa)(h,v) = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)\tilde{f}(x_1,x_2)$$

for functions $\tilde{f} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

Seeking a generalization of the Moyal-Minkowski case, the star-product is defined as follows: With the Minkowski spacetime dimension $n = 2l \in 2\mathbb{N}$, where coordinates are indexed from 0 to $n-1$, set

$$\Omega := \begin{bmatrix} 0_{l\times l} & \mathbb{1}_{l\times l} \\ & \\ -\mathbb{1}_{l\times l} & 0_{l\times l} \end{bmatrix} \tag{3.2}$$

and $\lambda := \frac{\theta}{2}$, $\theta > 0$ (i.e. $\lambda\Omega = M = M_\theta$ along the definition in Section 2.1). Then define

$$\begin{aligned} & f \star g(h,v) \\ := \; & \sum_{k=0}^{\infty} \frac{(i\lambda)^k}{k!} \sum_{I,J \in \{0,\dots,n-1\}^k} \Omega^{IJ} \frac{\partial^k}{\partial v^I} f(h,v) \frac{\partial^k}{\partial v^J} g(h,v) \\ = \; & \sum_{k=0}^{\infty} \frac{(i\lambda)^k}{k!} \sum_{i_1,\dots,j_k=0}^{n-1} \Omega^{i_1 j_1} \cdots \Omega^{i_k j_k} \frac{\partial^k}{\partial v^{i_1} \cdots v^{i_k}} f(h,v) \frac{\partial^k}{\partial v^{j_1} \cdots v^{j_k}} g(h,v) \end{aligned} \tag{3.3}$$

for $f, g \in C^\infty(T\mathbb{R}^n)$, $(h, v) = v_h \in \mathbb{R}^{2n}$, where h plays a passive role compared to v. One can also obtain an equation of nicer form:

$$f \star g = \mu \circ \exp\left(i\lambda \sum_{i,j=0}^{n-1} \Omega^{ij} \partial_{v^i} \otimes \partial_{v^j} \right) (f \otimes g), \tag{3.4}$$

with $\mu(f \otimes g) := fg$. This can be translated easily into

$$\tilde{f} \tilde{\star} \tilde{g} = \mu \circ \exp\left(\frac{i\lambda}{4} \sum_{i,j} \Omega^{ij} (\partial_{x_1^i} - \partial_{x_2^i}) \otimes (\partial_{x_1^j} - \partial_{x_2^j}) \right) (\tilde{f} \otimes \tilde{g}),$$

for $\tilde{f}, \tilde{g} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Note that in the following the $\tilde{\cdot}$-signs above the function symbols are dropped, since the associated spaces do not differ significantly. Using Fourier transformation and its inverse, one can also arrive at an integral form of the last equation for $f, g \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$:

$$\begin{aligned} & f \tilde{\star} g(x_1, x_2) \\ = \ & (2\pi)^{-2n} \iiiint e^{ix_1(k_1+u_1)+ix_2(k_2+u_2)} e^{-\frac{i\lambda}{4}(k_1-k_2)\Omega(u_1-u_2)} \cdot \\ & \hat{f}(k_1, k_2) \hat{g}(u_1, u_2) d^n k_1 d^n k_2 d^n u_1 d^n u_2 \\ = \ & (2\pi)^{-2n} \iiiint f\left(x_1 - \frac{\lambda}{4}\Omega(u_1 - u_2), x_2 + \frac{\lambda}{4}\Omega(u_1 - u_2) \right) \cdot \\ & g(x_1 + v_1, x_2 + v_2) e^{-iu_1 v_1} e^{-iu_2 v_2} d^n u_1 d^n u_2 d^n v_1 d^n v_2. \end{aligned}$$

This looks quite similar compared to the more familiar version of the Moyal product on $f, g \in C^\infty(\mathbb{R}^n)$:

$$f \star g(x) = (2\pi)^{-n} \iint f(x - \lambda\Omega u) g(x + v) e^{-iuv} d^n u d^n v.$$

For the sake of completeness here comes the analogue of (3.4) ($f, g \in C^\infty(T\mathbb{R}^n)$):

$$f \star g(h, v) = (2\pi)^{-n} \iint f(h, v - \lambda\Omega u) g(h, v + w) e^{-iuw} d^n u d^n w,$$

which for $f, g \in C^\infty(T_h\mathbb{R}^n)$ could also be written as

$$f \star g(v) = (2\pi)^{-n} \iint f(v - \lambda\Omega u) g(v + w) e^{-iuw} d^n u d^n w.$$

Now what's still missing is the implementation of "locality". The most straightforward idea, which we will stick to, is to start with the Moyal special case and make a transition from the constant symplectic matrix Ω to a $v = x_1 - x_2$-dependent

$$\Omega_\chi(v) := \chi(v)\Omega, \tag{3.5}$$

with a suitable cut-off $C_c^\infty(B_\epsilon, [0,1])$-function χ, where $B_\epsilon := \{x \in \mathbb{R}^n | \|x\| < \epsilon\}$ for some fixed $\epsilon > 0$. But of course it is not that easy. You cannot just artificially plug in such a χ and expect all the properties of a star-product still to hold. The most crucial problem is associativity, as there is absolutely no obvious way to repair or prove this. But a glimpse at the theorem appearing at the end of the next section reveals that in fact a solution for this problem exists. To understand this, one at first has to learn something about a beautiful and famous piece of theory constructed by M. Kontsevich.

So even though it does not yet seem to be mathematically well-defined at the moment (after the next two sections it actually will be), we understand the notion of **locality** of the nc. spacetime in our concrete example precisely as the carried out transition from Ω to Ω_χ.

3.3 Kontsevich star-products

The question is, whether there still exists an associative star-product, if the constant symplectic structure is replaced by a Poisson structure and beyond that depends on spacetime points (in our case only on relative distances). M. Kontsevich gave a solution in [25] for any finite dimensional Poisson manifold. One of the more comprehensible short explanations can be found e.g. in [14], which we will use to get started.

Let $(\mathbb{R}^d, \pi)$ be a Poisson manifold, i.e. $\pi(f,g) = \{f,g\}$ is the Poisson bracket for $f, g \in C^\infty(\mathbb{R}^d)$. A global coordinate chart at hand, we can write

$$\{f,g\} = \sum_{i,j=1}^{d} \pi^{ij} \partial_i f \partial_j g,$$

where $\pi^{ij} \in C^\infty(\mathbb{R}^d)\ \forall i,j$ (components of the Poisson tensor) and ∂_i is the derivative in x^i-direction. Ultimately Kontsevich's formula for his star-product $f \star_K g$ will depend on nothing else except derivatives of f, g and π, combined in a complicated combinatorial way. Now let's define all the ingredients one by one.

Let $k \in \mathbb{N}_0$. To each k there is assigned a family of graphs G_k. And to each graph $\Gamma \in G_k$ one associates a bidifferential operator $\mathcal{B}_\Gamma$ and a weight $w(\Gamma) \in \mathbb{R}$. Then

$$f \star_K g := \sum_{k=0}^{\infty} \lambda^k \sum_{\Gamma \in G_k} w(\Gamma) \mathcal{B}_\Gamma(f,g), \quad f, g \in C^\infty(\mathbb{R}^d),$$

will finally give the formula for the star-product.

An oriented graph Γ belongs to G_k, iff

(a) Γ consists of $k+2$ vertices labeled $\{1, 2, \ldots, k, L, R\}$ and $2k$ oriented edges labeled $\{i_1, j_1, i_2, j_2, \ldots, i_k, j_k\}$, L, R stand for "Left", "Right"

(b) The ordered pair of edges (i_m, j_m), $1 \leq m \leq k$, starts at vertex m

(c) Γ has no loop (edge starting and ending at the same vertex) and no parallel multiple edges (edges starting at the same and also ending at the same vertex).

To deepen the understanding of what these graphs look like, let us count the number of possible graphs in one family G_k: Obviously G_0 contains only one element, the graph having just two vertices $\{L, R\}$. For $k \geq 1$ the first edge starting at some arbitrary vertex different from $\{L, R\}$ (since from there never starts any edge) has $k+1$ possible ending vertices (all except the starting vertex itself), while the second edge then has only k possible ends left (no parallels). Since there are k possible vertices having an ordered pair of edges starting from, this results in $((k+1)k)^k$ different graphs in G_k.

Given a graph Γ one associates a bidifferential operator B_Γ acting on the functions f, g by the following algorithm:

1. View a vertex $m \in \{1, \ldots, k\}$ as symbolically standing for $\pi^{i_m j_m}$ (component function of the Poisson tensor), and view vertices L and R as the functions f and g respectively.

2. Put derivatives in front of $\pi^{i_m j_m}$ and f and g with respect to coordinate-indices named like the edge-labels of edges (if any) ending in the associated vertex.

3. Multiply the resulting $k+2$ terms and sum over $1 \leq i_1, j_1, \ldots, i_k, j_k \leq d$.

An example helps best: Let $\Gamma \in G_2$ consist of the vertices $\{1, 2, L, R\}$ and the edges i_1, j_1, i_2, j_2 ending in $2, L, R, L$ respectively:

$$\begin{tikzcd} 1 \arrow[r, "i_1"] \arrow[d, "j_1"'] & 2 \arrow[dl, "j_2"'] \arrow[d, "i_2"] \\ L & R \end{tikzcd}$$

Then $\Gamma \mapsto \mathcal{B}_\Gamma$ along the algorithm results in

$$\mathcal{B}_\Gamma(f, g) = \sum_{i_1, j_1, i_2, j_2 = 1}^{d} \pi^{i_1 j_1} \partial_{i_1} \pi^{i_2 j_2} \partial_{j_1 j_2} f \partial_{i_2} g. \tag{3.6}$$

What remains, is the definition of the weight $w(\Gamma)$. Let $\mathcal{H} := \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$, $\mathcal{H}_k := \{z_1, \ldots, z_k \in \mathcal{H} | z_i \neq z_j \text{ for } i \neq j\}$ and $\phi : \mathcal{H}_2 \to \mathbb{R}/2\pi\mathbb{Z}$ be the function defined by

$$\begin{aligned} \phi(z_1, z_2) &:= \frac{1}{2i} \mathrm{Log} \frac{(z_2 - z_1)(\bar{z}_2 - z_1)}{(z_2 - \bar{z}_1)(\bar{z}_2 - \bar{z}_1)} \\ &= \arg((z_1 - z_2)(z_1 - \bar{z}_2)) \end{aligned}$$

and extended to the real line $z_1, z_2 \in \mathbb{R}$, $z_1 \neq z_2$, by continuity.

The symbolic associations used this time for $\Gamma \mapsto w(\Gamma)$ are that each vertex $m \in \{1, \ldots, k\}$ stands for the variable $z_m \in \mathcal{H}$, and the vertices L, R are translated into the numbers $0 \in \mathbb{R}$, $1 \in \mathbb{R}$ respectively. Define

$$w(\Gamma) := \frac{1}{k!(2\pi)^{2k}} \int_{\mathcal{H}_k} \bigwedge_{1 \leq m \leq k} \left(d\phi(z_m, I_m) \wedge d\phi(z_m, J_m) \right), \tag{3.7}$$

where I_m, J_m denote the variables or numbers $\{0, 1\}$ associated with the ending vertex of the edges i_m, j_m respectively. Within our recent example of $\Gamma \in G_2$ from above one has to integrate the 4-form $d\phi(z_1, z_2) \wedge d\phi(z_1, 0) \wedge d\phi(z_2, 1) \wedge d\phi(z_2, 0)$ over $\mathcal{H}_2$.

Remark It seems very sophisticated to do the calculations for as many graphs as $((k+1)k)^k$. It is, but there are some benefits, too: Many graphs are similar in the way that permutations of edges or vertices yield the same term $w(\Gamma)\mathcal{B}_\Gamma(f, g)$, since these permutations just amount to sign-flips in both factors w and $\mathcal{B}$ at once. Also there are "bad" graphs that have not a single edge ending in one of the vertices L or R; for those the weight w vanishes. Actually e.g. in G_2 there are only 3 graphs out of overall 36 that need to be calculated and that just contribute multiple times.

Note also that the weights $w(\Gamma)$ are universal in the sense that they do not depend on π and not even on the dimension d.

This ends all the definitions necessary to understand (without trying to prove) the statement of the following

Theorem 3.3.1 (Kontsevich) *For any Poisson structure π on $\mathbb{R}^d$*

$$f \star_K g := \sum_{k=0}^{\infty} \lambda^k \sum_{\Gamma \in G_k} w(\Gamma) \mathcal{B}_\Gamma(f, g), \quad f, g \in C^\infty(\mathbb{R}^d), \tag{3.8}$$

*defines an associative product, which we call **Kontsevich star-product**.*

3.4 Finally: a vertical, local star-product $\tilde{\star}_K$ on $\mathbb{R}^n \times \mathbb{R}^n$

Recall the problem that we had with non-constant Poisson structures at the end of Section 3.2 to properly define a vertical and local star-product on twice the Minkowski spacetime $\mathbb{R}^n \times \mathbb{R}^n$. Now Theorem 3.3.1 suddenly opens up the opportunity to use such Poisson structures that depend smoothly on spacetime coordinates rather than being constant like the default symplectic matrix Ω.

Thus we are going to take

$$\pi^{ij} := \Omega_\chi^{ij} = \chi \Omega^{ij} \tag{3.9}$$

as Poisson tensor (components), with Ω and χ (the cut-off function) defined like in Section 3.2. Also like before λ remains to be set to $\theta/2$ and in analogy to the Moyal case we use $i\lambda$ instead of λ in the series expansion. Note that the term $1/k!$ appearing in the series of equation (3.3) is now implicitly contained within the weights $w(\Gamma)$.

Since we would like to check all the premises carefully, recall that a **Poisson structure** on $\mathbb{R}^d$ is defined by a bilinear mapping (also called the **Poisson bracket**)

$$\{\cdot,\cdot\} : C^\infty(\mathbb{R}^d) \times C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d),$$

such that

(a) $\{f,g\} = -\{g,f\}$ (antisymmetry).

(b) $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0$ (Jacobi identity).

(c) $\{fg,h\} = f\{g,h\} + g\{f,h\}$ (derivation).

In local (here also global) coordinates

$$\{f,g\}(x) = \sum_{i,j=1}^{d} \pi^{ij}(x)\partial_i f(x)\partial_j g(x)$$

holds, where $\pi^{ij}(x)$ are the components of the so called **Poisson tensor**, that have to fulfill $\pi^{ij}(x) = -\pi^{ji}(x)\ \forall x, i, j$ and a differential equation imposed by the Jacobi identity. The derivation property is built in trivially.

Proposition 3.4.1 *Let $d \in 2\mathbb{N}$, $\chi \in C_c^\infty(\mathbb{R}^d)$ and Ω be the default symplectic $d \times d$-matrix of* (3.2)*, then $\pi^{ij}(x) := \chi(x)\Omega^{ij}$, $x \in \mathbb{R}^d$, $1 \leq i,j \leq d$, properly defines a Poisson tensor.*

Proof Obviously $\pi^{ij}(x) = -\pi^{ji}(x)\ \forall x, i, j$. Hence what needs to be shown, is the Jacobi identity. Consider

$$\{f,g\}(x) = \sum_{i,j=1}^{d} \chi(x)\Omega^{ij}\partial_i f(x)\partial_j g(x) = \chi(x)\{f,g\}'(x), \tag{3.10}$$

where $\{\cdot,\cdot\}'$ denotes the well-known usual Poisson bracket. Calculate

$$\begin{aligned}
&\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} \\
&= \chi\left(\{f,\chi\{g,h\}'\}' + \{h,\chi\{f,g\}'\}' + \{g,\chi\{h,f\}'\}'\right) \\
&= \chi\left(\chi\{f,\{g,h\}'\}' + \{f,\chi\}'\{g,h\}' + \text{w.r.t. } f,g,h \text{ cyclic exchanged terms}\right) \\
&= \chi\left(\chi\cdot 0 + \{f,\chi\}'\{g,h\}' + \{h,\chi\}'\{f,g\}' + \{g,\chi\}'\{h,f\}'\right),
\end{aligned}$$

where (3.10) and then the derivation- and Jacobi-properties of $\{\cdot,\cdot\}'$ have been used. Now it suffices to show that $\{f,\chi\}'\{g,h\}' + \{h,\chi\}'\{f,g\}' + \{g,\chi\}'\{h,f\}'$

vanishes, which will be done by induction on $l := d/2 \in \mathbb{N}$. For $l = 1$ this expression equals explicitly

$$\begin{aligned} &\underbrace{\partial_1 f \partial_2 \chi \partial_1 g \partial_2 h}_{1} - \underbrace{\partial_2 f \partial_1 \chi \partial_1 g \partial_2 h}_{2} - \underbrace{\partial_1 f \partial_2 \chi \partial_2 g \partial_1 h}_{3} + \underbrace{\partial_2 f \partial_1 \chi \partial_2 g \partial_1 h}_{4} \\ + \; &\underbrace{\partial_1 h \partial_2 \chi \partial_1 f \partial_2 g}_{3} - \underbrace{\partial_2 h \partial_1 \chi \partial_1 f \partial_2 g}_{5} - \underbrace{\partial_1 h \partial_2 \chi \partial_2 f \partial_1 g}_{6} + \underbrace{\partial_2 h \partial_1 \chi \partial_2 f \partial_1 g}_{2} \\ + \; &\underbrace{\partial_1 g \partial_2 \chi \partial_1 h \partial_2 f}_{6} - \underbrace{\partial_2 g \partial_1 \chi \partial_1 h \partial_2 f}_{4} - \underbrace{\partial_1 g \partial_2 \chi \partial_2 h \partial_1 f}_{1} + \underbrace{\partial_2 g \partial_1 \chi \partial_2 h \partial_1 f}_{5}, \end{aligned}$$

containing 6 pairs of terms adding up to zero each. Since being quite cumbersome, we abandon writing down an explicit expression again. We just remark that for "$l = l'+1$" the expression differs from the one for "$l = l'$" (which vanishes because of the induction premise) only additively by an expression looking exactly like the one above for $l = 1$ with only 1 replaced by $2l' - 1$ and 2 replaced by $2l'$, which obviously forces it to vanish anyway again. □

Now we have everything at hand, necessary to properly define a star-product for functions $f, g \in C^\infty(X)$, with $X \in \{\mathbb{R}^n \times \mathbb{R}^n, T\mathbb{R}^n, T_h\mathbb{R}^n\}$, which is not only vertical but also local in the sense defined within Section 3.2. Let's recall the notations of our main setting: Proceeding from $n = 2l \in 2\mathbb{N}$ dimensional Minkowski spacetime equipped with coordinates $\{x^0, \ldots, x^{n-1}\}$ in a Lorentz frame, we introduced coordinates (h, v) on $T\mathbb{R}^n \cong \mathbb{R}^{2n}$, respectively v on $T_h\mathbb{R}^n \cong \mathbb{R}^n$, and (x_1, x_2) on twice the spacetime $\mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$. The spaces/coordinates were brought into relation with each other due to the diffeomorphism $\varkappa$, cf. (3.1).

Definition 3.4.2 (& Corollary) *Let $\chi \in C_c^\infty(\mathbb{R}^n)$, $\lambda = \theta/2 > 0$ and Ω be the default symplectic $n \times n$-matrix like (3.2). Setting $\pi^{ij}(v) := \chi(v)\Omega^{ij}$ for all $v \in \mathbb{R}^n$, $0 \leq i, j \leq n-1$, as Poisson tensor, then*

$$f_h \star_K g_h := \sum_{k=0}^{\infty} (i\lambda)^k \sum_{\Gamma \in G_k} w(\Gamma) \mathcal{B}_\Gamma(f_h, g_h), \quad f_h, g_h \in C^\infty(T_h\mathbb{R}^n),$$

defines a vertical and local star-product for every $h \in \mathbb{R}^n$ (we write $\star_{K_h} = \star_K$, since there is actually no dependence). One can also formulate the same product for functions $f, g \in C^\infty(T\mathbb{R}^n)$ by setting $f(h, v) := f_h(v)$, $g(h, v) := g_h(v)$, $\forall h, v$.

Example for one of the graphs $\Gamma_{ex} \in G_2$:

$$\mathcal{B}_{\Gamma_{ex}}(f, g)(h, v) = \sum_{i_1, j_1, i_2, j_2 = 0}^{n-1} \pi^{i_1 j_1}(v) \partial_{i_1} \pi^{i_2 j_2}(v) \partial_{j_1 j_2} f(h, v) \partial_{i_2} g(h, v),$$

where $\partial_i = \partial_{v^i} = \frac{\partial}{\partial v^i}$ denotes the derivative into the direction of the m'th component of the relative/vertical coordinate v.

Last but not least the final form of the product for $f, g \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$:

$$f \tilde{\star}_K g := \sum_{k=0}^{\infty} (i\lambda)^k \sum_{\Gamma \in G_k} w(\Gamma) \tilde{\mathcal{B}}_\Gamma(f, g), \tag{3.11}$$

where $\tilde{\mathcal{B}}$ defers from $\mathcal{B}$ by a change of variables (through $\varkappa$) within its constituents $\pi^{ij}(v) = \pi^{ij}(x_1 - x_2)$ and the derivatives "$\frac{\partial}{\partial v^i} = \frac{1}{2}\left(\frac{\partial}{\partial x_1^i} - \frac{\partial}{\partial x_2^i}\right)$".

Since (3.11) is an extremely complicated series expansion, for selected practical calculations one would be contented with the formula up to 2nd order:

$$\begin{aligned} f\tilde{\star}_K g(x_1,x_2) \;=\;& fg(x_1,x_2) + i\lambda\Omega^{ij}\chi(x_1-x_2)\tilde{\partial}_i f\tilde{\partial}_j g(x_1,x_2) \\ &+(i\lambda)^2\Omega^{i_1j_1}\Omega^{i_2j_2}\Big[\frac{1}{2}\chi(x_1-x_2)^2\tilde{\partial}_{i_1i_2}f\tilde{\partial}_{j_1j_2}g(x_1,x_2) \\ &+\frac{1}{3}\chi(x_1-x_2)\tilde{\partial}_{i_1}\chi(x_1-x_2)\left(\tilde{\partial}_{j_1j_2}f\tilde{\partial}_{i_2}g+\tilde{\partial}_{i_2}f\tilde{\partial}_{j_1j_2}g\right)(x_1,x_2) \\ &-\frac{1}{6}\tilde{\partial}_{j_2}\chi(x_1-x_2)\tilde{\partial}_{j_1}\chi(x_1-x_2)\tilde{\partial}_{i_1}f\tilde{\partial}_{i_2}g(x_1,x_2)\Big] + O\left((i\lambda)^3\right) \end{aligned} \tag{3.12}$$

with summation convention, and "$\tilde{\partial}_i := \frac{1}{2}\left(\frac{\partial}{\partial x_1^i} - \frac{\partial}{\partial x_2^i}\right)$".

Note that the desired distance behavior of the star-product at its heart is regulated by the function χ. The idea is that, due to this physically reasonable cut-off, the product is analytically sufficiently nice behaved, such that one can reach similar results like in Chapter 2.

4 Pair-interaction for Dirac fields

In the previous chapter a star-product $\tilde{\star}_K$ was constructed that implements the principle of locality in a nc. spacetime. This means that non-commutativity is equipped with some fall-off behavior for growing distances. Speaking of distances it seems natural to consider the interaction between *two* particles/points, instead of scattering only one particle in an external potential. That is why we now want to deal a bit with many-particle quantum mechanics and pair-interaction first.

Let $n = 1+s \in \mathbb{N}_{\geq 2}$ be the (Minkowski) spacetime dimension and $N(n)$ an appropriately chosen spinor dimension, like (2.6). We take $\mathcal{H} := \Big(L^2(\mathbb{R}^n, \mathbb{C}^N), \langle\cdot|\cdot\rangle\Big)$ as one-particle Hilbert space, with

$$\langle\varphi|\psi\rangle := \sum_{A,B=1}^{N} \int_{\mathbb{R}^n} \delta_{AB}\bar{\varphi}^A(x)\psi^B(x)d^n x,$$

for spinors $\varphi, \psi \in \mathcal{H}$. Consider the following many-particle Dirac-Hamiltonian

$$H^{\mathcal{N}} = H_0^{\mathcal{N}} + V^{\mathcal{N}},$$

acting on $\phi \in \mathcal{D}^{\mathcal{N}} = \mathscr{S}\Big((\mathbb{R}^n)^{\mathcal{N}}, (\mathbb{C}^N)^{\mathcal{N}}\Big) \subseteq \mathcal{H}^{\mathcal{N}}$ and mapping into $\mathcal{H}^{\mathcal{N}}$, where $\mathcal{N} \in \mathbb{N}_0$ is the particle number (becoming a variable later due to usage of the occupation number representation):

$$\begin{aligned}
&\Big(H^{\mathcal{N}}\phi\Big)^{A_1\cdots A_{\mathcal{N}}}(x_1,\ldots,x_{\mathcal{N}}) \\
&:= \sum_{j=1}^{\mathcal{N}} \left(i{\gamma_0}^{A_j}{}_{B_j}{\gamma_k}^{B_j}{}_{C_j}\frac{\partial}{\partial x_j^k} + {\gamma_0}^{A_j}{}_{C_j} m\right)\phi^{C_1\cdots C_{\mathcal{N}}}(x_1,\ldots,x_{\mathcal{N}}) \\
&\quad + \sum_{1\leq j<k\leq\mathcal{N}} \Big(V^{(jk)}\phi\Big)^{A_1\cdots A_{\mathcal{N}}}(x_1,\ldots,x_{\mathcal{N}}) \quad \text{(and implicit summations)},
\end{aligned}$$

with two-particle operator $V^{(jk)}$ defined by

$$\begin{aligned}
\Big(V_{(0)}^{(jk)}\phi\Big)^{A_1\cdots A_{\mathcal{N}}}(x_1,\ldots,x_{\mathcal{N}}) &:= c(x_j,x_k)\cdot\phi^{A_1\cdots A_{\mathcal{N}}}(x_1,\ldots,x_{\mathcal{N}}) \\
\Big(V_{(\mathrm{i})}^{(jk)}\phi\Big)^{A_1\cdots A_{\mathcal{N}}}(x_1,\ldots,x_{\mathcal{N}}) &:= \left(c\tilde{\star}_K\phi^{(jk)A_1\cdots A_{\mathcal{N}}} + \phi^{(jk)A_1\cdots A_{\mathcal{N}}}\tilde{\star}_K c\right)(x_j,x_k) \\
\Big(V_{(\mathrm{ii})}^{(jk)}\phi\Big)^{A_1\cdots A_{\mathcal{N}}}(x_1,\ldots,x_{\mathcal{N}}) &:= \left(c\tilde{\star}_K\phi^{(jk)A_1\cdots A_{\mathcal{N}}}\tilde{\star}_K c\right)(x_j,x_k)
\end{aligned}$$

either-way, with $c \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$. Here $\phi^{(jk)} : \mathbb{R}^n \times \mathbb{R}^n \to (\mathbb{C}^N)^{\mathcal{N}}$ is declared to be equal to ϕ but treating all the coordinates as parameters, except x_j and x_k. The latter two coordinates are the only ones acted upon by $c\tilde{\star}_K\cdot$ and $\cdot\tilde{\star}_K c$.

Since $\mathcal{H}$ is separable one can choose a complete, orthonormal basis $\{\varphi_k\}_{k\in\mathbb{N}} \subseteq \mathcal{H}$. Thus $\langle\varphi_j|\varphi_k\rangle = \delta_{jk}$, $\sum_j |\varphi_j\rangle\langle\varphi_j| = \mathbb{1} = \mathrm{id}_{\mathcal{H}}$, when using common physics notation $|\varphi_k\rangle := \varphi_k$ and $\langle\varphi_k|$ for the functional in $L(\mathcal{H}, \mathbb{C})$ "canonically" induced by φ_k. Of course every so-called **state vector** $\varphi \in \mathcal{H}$ can then be expanded in a series $|\varphi\rangle = \sum_j \alpha_j|\varphi_j\rangle = \sum_j\langle\varphi_j|\varphi\rangle|\varphi_j\rangle$. And furthermore a basis transformation onto $\{\psi_k\}_{k\in\mathbb{N}} \subseteq \mathcal{H}$ is given by $|\psi_k\rangle = \sum_j\langle\varphi_j|\psi_k\rangle|\varphi_j\rangle$. Just remember, that for our concrete space $\mathcal{H} = L^2(\mathbb{R}^n, \mathbb{C}^N)$, a representation by functions defined on spacetime, also the notation $|x, A\rangle = \sum_j |\varphi_j\rangle\langle\varphi_j|x, A\rangle = \sum_j |\varphi_j\rangle\bar{\varphi}_j^A(x)$, $A = 1, \dots, N$, enjoys popularity, as well as short forms like $|k\rangle := |\varphi_k\rangle$.

The next step is to introduce a Fermionic Fock space $\mathfrak{H} := \mathcal{F}_{\mathrm{F}}(\mathcal{H})$ with its scalar product denoted also by $\langle\cdot|\cdot\rangle$, and vacuum vector $|0\rangle := \Omega \in \mathfrak{H}$. This shall correspond exactly to the so called **vacuum representation** space, if we had started with the algebraic approach to quantum field theory. For every one-particle basis state φ_k, a creation operator $a_k^\dagger := a_{\varphi_k}^\dagger$ and annihilation operator $a_k := a_{\varphi_k}$ are defined on $\mathfrak{H}$ in the usual manner ($\psi_j \in \mathcal{H}, j = 1, \dots, n$):

$$\begin{aligned} a_{\varphi_k}^\dagger(\psi_1 \wedge \dots \wedge \psi_n) &:= \varphi_k \wedge \psi_1 \wedge \dots \wedge \psi_n \\ a_{\varphi_k}(\psi_1 \wedge \dots \wedge \psi_n) &:= \sum_{j=1}^n (-)^{j+1}\langle\varphi_k|\psi_j\rangle\psi_1 \wedge \dots \wedge \hat{\psi}_j \wedge \dots \wedge \psi_n, \end{aligned}$$

obeying $\{a_j, a_k^\dagger\} = \delta_{jk}$, $\{a_j^\dagger, a_k^\dagger\} = 0 = \{a_j, a_k\}$. A vector $\psi \in \mathfrak{H}$ in Fock space is denoted in various ways: $\psi = |\psi\rangle = |n_1 n_2 \cdots\rangle$ with n_k particles in state φ_k, such that e.g. $a_2^\dagger|0\rangle = |010\cdots\rangle$. Or even shorter one writes e.g. $a_j^\dagger a_k^\dagger|0\rangle = a_j^\dagger|k\rangle = |jk\rangle$ for $|0\cdots01_{j\text{'th}}0\cdots01_{k\text{'th}}0\cdots\rangle$, resp. $\varphi_j \wedge \varphi_k$, which would be 0 for $j = k$, modeling the Pauli principle. Now point-dependent creators and annihilators are introduced according to the transformation law between bases:

$$\begin{aligned} \Psi^{*A}(x) &:= \sum_j\langle\varphi_j|x, A\rangle a_j^\dagger = \sum_j \bar{\varphi}_j^A(x)a_j^\dagger, \quad {a^\dagger}^A(x)|0\rangle = |x, A\rangle, \\ \Psi^A(x) &:= \sum_j \varphi_j^A(x)a_j, \end{aligned}$$

which are also called **field operators**.

So what was done is essentially a transition from $\mathcal{H}^{\mathcal{N}}$, the many-particle product space, to $\mathfrak{H}$, the Fock space with arbitrary particle number. Both are possible choices of state spaces to work with – the major drawback in case of $\mathcal{H}^{\mathcal{N}}$ being that one would have to take care of anti-symmetrization by hand permanently. Finally this transition must be made also for operators, especially the Hamiltonian $H^{\mathcal{N}}$, resulting in the following operator

$$\hat{H} = \hat{H}_0 + \hat{V} \tag{4.1}$$

on Fock space $\mathfrak{H}$:

$$\hat{H} = \int_{\mathbb{R}^n} \Psi_A^*(x)(H_0\Psi)^A(x)d^nx + \frac{1}{2}\iint_{\mathbb{R}^{2n}} \Psi_A^*(x)\Psi_B^*(x')(V\Psi^2)^{AB}(x,x')d^nx'd^nx, \tag{4.2}$$

with

$$(H_0\Psi)^A(x) := \left({\gamma_0}^A{}_B{\gamma_k}^B{}_C\frac{\partial}{\partial x^k} + {\gamma^0}^A{}_C m\right)\Psi^C(x),$$

$$\Psi^{2AB}(x,x') := \Psi^A(x)\Psi^B(x')$$

and V is one choice out of

$$\begin{aligned}\left(V_{(0)}\Psi^2\right)^{AB}(x,x') &:= \left(c\cdot\Psi^{2AB}\right)(x,x')\\ \left(V_{(\mathrm{i})}\Psi^2\right)^{AB}(x,x') &:= \left(c\tilde{\star}_K\Psi^{2AB} + \Psi^{2AB}\tilde{\star}_K c\right)(x,x')\\ \left(V_{(\mathrm{ii})}\Psi^2\right)^{AB}(x,x') &:= \left(c\tilde{\star}_K\Psi^{2AB}\tilde{\star}_K c\right)(x,x'),\end{aligned}$$

$c \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$. In case of the commutative potential operator $V_{(0)}$ the notation is not necessarily that ugly:

$$(V_{(0)}\Psi^2)^{AB}(x,x') = c(x,x')\Psi^A(x)\Psi^B(x') =: V(x,x')\Psi^A(x)\Psi^B(x').$$

Note that the field operator Ψ conceptually still originates from the free field, despite constructing the two-particle interaction operator $\hat{V}$ with its help. One more remark on notation: the hat ˆ above one-particle operators A is sometimes used to denote their second quantization just like in more modern manner $d\Gamma(A) = \hat{A}$.

The origin of equation (4.2) and all the fundamentals of this chapter can be comprehended at lecture level e.g. in [9].

Now let's try to connect these things, stemming more likely from physicists' literature, to the language used in more rigorous mathematical approaches – despite still leaving out diverse technical difficulties. Consider the definition of the field operator $\Psi^A(x) = \sum_{j=1}^{\infty} \varphi_j^A(x)a_j,\ A,\ldots,N$ from above. In physics [21] it is common to use the plane-wave solutions

$$\{\varphi^A_{(\underline{p},r)}(x)\}_{\underline{p}\in\mathbb{R}^3, r\in\{1,\ldots,4\}}$$

of the Dirac equation as a complete orthonormal basis, labeled by $(\underline{p}, r)$ instead of just $j \in \mathbb{N}$, i.e. continuously by momentum and discretely by r. Where $r = 1, 2$ means "positive energy" and spin $+1/2$, $-1/2$ resp., and $r = 3, 4$ "negative energy" and spin $-1/2$, $+1/2$ resp. Then the field operator reads

$$\Psi^A(x) = \sum_{r=1}^{4}\int_{\mathbb{R}^3} \varphi^A_{(\underline{p},r)}(x)a_{(\underline{p},r)}d^3\underline{p}.$$

Defining $a^A(x,r) := \int_{\mathbb{R}^3} \varphi^A_{(\underline{p},r)}(x) a_{(\underline{p},r)} d^3\underline{p}$, this becomes

$$\Psi^A(x) = \sum_{r=1}^{4} a^A(x,r).$$

How does this relate to a version "smeared" with $f \in C_c^\infty(\mathbb{R}^n, \mathbb{C}^N) \subset L^2 = \mathcal{H}$, as in [34]: $\Psi(f) = b(e_+ f) + d^\dagger(e_- f)$, where $e_\pm$ are resp. the projectors onto positive and negative energy solutions and b is a particle annihilation and $d^\dagger$ an antiparticle creation operator? Actually

$$\begin{aligned}
\Psi(f) &= \sum_{A=1}^{4} \int_{\mathbb{R}^4} \Psi_A(x) f^A(x) d^4x \\
&= \sum_A \int_{\mathbb{R}^4} \sum_{r=1}^{4} a_A(x,r)((e_+ + e_-)f)^A(x) d^4x \\
&= \sum_A \int_{\mathbb{R}^4} \Big((a_A(x,1) + a_A(x,2)) (e_+ f)^A(x) \\
&\quad + (a_A(x,3) + a_A(x,4)) (e_- f)^A(x) \Big) d^4x,
\end{aligned}$$

since $e_+ \varphi^A_{(\underline{p},r')} = 0$ for $r' \in \{3,4\}$, $e_- \varphi^A_{(\underline{p},r'')} = 0$ for $r'' \in \{1,2\}$, which equals

$$\begin{aligned}
&= \sum_A \int_{\mathbb{R}^4} \Big(b^\dagger_A(x)(e_+ f)^A(x) + d_A(x)(e_- f)^A(x) \Big) d^4x \\
&= b^\dagger(e_+ f) + d(e_- f),
\end{aligned}$$

with obvious definitions (implying also $\Psi^A(x) = {b^\dagger}^A(x) + d^A(x)$). This deviates from the convention in Thaller's book [34], where $f \mapsto \Psi(f)$ is chosen to be antilinear instead of linear. The latter shall be our choice (so the definitions of $\Psi(f), \Psi^*(f)$ are interchanged compared to [34]!). Please note that $b, b^\dagger$ are to be viewed to make sense only for arguments $e_+ \cdot$ (particles) and $d, d^\dagger$ only for $e_- \cdot$ (antiparticles), i.e. otherwise they are defined to vanish.

Since very fitting in this context, we would also like to opt for Araki's so-called "self-dual" form of quantization (originating in the quantization at algebraic level), like in [6], which means (here already at the level of Fock space operators)

$$\begin{aligned}
\Psi(f)^* &= \Psi(Cf), \\
\{\Psi(f)^*, \Psi(h)\} &= 2(f,h)\mathbb{1}. \qquad (4.3)
\end{aligned}$$

Here $Cf := \bar{f}$ denotes complex conjugation, and $(\cdot,\cdot) := (\cdot,\cdot)_{(R)}$ is some appropriate scalar product on solution space ($R := R_V$ denoting the difference of advanced and retarded solutions as usual) defined with the help of $\langle \cdot | \cdot \rangle$. And it holds

$$\Psi(D_0 f) = 0,$$

with the free Dirac operator D_0, in connection with the free field operator. To fully reconstruct our conventions in [6], new creation and annihilation operators $A^\dagger, A$ are defined by

$$A(e_+f) := b(e_+f) \text{ and } A^\dagger(Ce_-f) := d^\dagger(e_-f)$$

for all $f \in C_c^\infty$, and likewise for their adjoint versions, which results in

$$\Psi(f) = A(e_+Cf) + A^\dagger(e_+f). \tag{4.4}$$

These are also consistent with the earlier definitions of $a^\dagger, a$ ($\psi_j \in C_c^\infty \subset \mathcal{H}, j = 1, \dots, n$):

$$\begin{aligned} A^\dagger(\varphi)(\psi_1 \wedge \cdots \wedge \psi_n) &:= \varphi \wedge \psi_1 \wedge \cdots \wedge \psi_n \\ A(\varphi)(\psi_1 \wedge \cdots \wedge \psi_n) &:= \sum_{j=1}^n (-)^{j+1} (\varphi, \psi_j)\, \psi_1 \wedge \cdots \wedge \hat{\psi}_j \wedge \cdots \wedge \psi_n, \end{aligned}$$

with A being antilinear and $A^\dagger$ linear. And for consistency it has to hold

$$\{A^\dagger(e_+f), A(e_+h)\} = (f,h)\, \mathbb{1} \text{ as well as "}\{A, A\} = 0 = \{A^\dagger, A^\dagger\}\text{"}.$$

For comparison, we list the non-vanishing anti-commutators in alternative notation:

$$\begin{aligned} \{b^\dagger(e_+f), b(e_+h)\} &= (f,h)\, \mathbb{1} \\ \{d^\dagger(e_-f), d(e_-h)\} &= (Cf, Ch)\, \mathbb{1} \\ \{b^\dagger(e_+f), d(e_-h)\} &= (f, Ch)\, \mathbb{1} \\ \{d^\dagger(e_-f), b(e_+h)\} &= (Cf, h)\, \mathbb{1}. \end{aligned}$$

To illustrate the interpretation of the field operator in eq. (4.4) we depict by

- - -|- - -

the "Dirac sea" of negative energy solutions on the left hand side of "|" and the space of positive energy solutions on the right, both spaces empty so far (resulting in total energy set to zero by convention, just as a first choice, see the remarks concluding this illustration further below). Then

- - -|- ● -

corresponds to a particle (or state, just using these notions interchangeably for simplicity) of positive energy,

- ○ -|- - -

to an antiparticle or hole of positive energy,

- ● -|- - -

to an unphysical "particle" of negative energy (a potential slot for an antiparticle), and finally

$$\text{- - -|- } \circ \text{ -}$$

corresponding to an unphysical "particle" of effectively negative energy.

$A^\dagger(e_+f)$ does the following:

$$\text{- - -|- - -} \quad \longrightarrow \quad \text{- - -|- } \bullet \text{ -} = e_+f,$$

i.e. it creates a particle called $\bullet_+ = e_+f$ ($\bullet_\pm$ means a $\bullet$ on the left ($-$) or on the right ($+$) of "|", same then for $\circ$). But since $e_+f = Ce_-Cf$, this can also mean to create an antiparticle

$$\text{- } \circ \text{ -|- - -} = Ce_-Cf$$

called $\circ_- = Ce_-Cf$, which illustrates a *missing* "particle" $\bullet_-$ of negative energy on the left and hence actually a *positive* energy contribution to the world, just the existence of a so-called antiparticle or hole $\circ_-$. This naturally incorporates the idea of pair creation

$$\text{- - -|- - -} \quad \longrightarrow \quad \text{- } \circ \text{ -|- } \bullet \text{ -.}$$

Or perhaps with - - -|- - - replaced by $\bullet\bullet\bullet$|- - - (not the same, but a shift to infinitely more negative energy) the process

$$\bullet\bullet\bullet\text{|- - -} \quad \longrightarrow \quad \bullet\circ\bullet\text{|- } \bullet \text{ -.}$$

provides a nicer view of that. To complete this first part of the illustration consider next to $A^\dagger(e_+f) = A^\dagger(Ce_-Cf)$ also for a moment $A(e_-Cf)$ (which is not the same as the right hand side of the last equation) describes

$$e_-Cf = \text{- } \bullet \text{ -|- - -} \quad \longrightarrow \quad \text{- - -|- - -}$$

(annihilation of negative energy "particles") or also

$$Ce_+f\text{- - -|- } \circ \text{ -} \quad \longrightarrow \quad \text{- - -|- - -}$$

(annihilation of unphysical negative energy "particles" as well) since $e_-Cf = Ce_+f$. The latter two illustrate the physically not so vivid side of the mirror.

The other term in eq. (4.4), $A(e_+Cf)$ deals with a possibly different particle e_+g, with $g := Cf$, which obviously gets annihilated:

$$e_+g = \text{- - -|- } \bullet \text{ -} \quad \longrightarrow \quad \text{- - -|- - -.}$$

The remaining alternate meaning according to

$$A(e_+g) = A(Ce_-Cg),$$

and for completeness also the two effects of $A^\dagger(e_- Cg) = A^\dagger(Ce_+ g)$, are illustrated by

$$\begin{array}{rcl} Ce_- Cg = \text{- } \circ \text{ -|- - -} & \longrightarrow & \text{- - -|- - -} \\ \text{- - -|- - -} & \longrightarrow & \text{- } \bullet \text{ -|- - -} = e_- Cg \\ \text{- - -|- - -} & \longrightarrow & \text{- - -|- } \circ \text{ -} = Ce_+ g, \end{array}$$

from top to bottom: annihilating antiparticle or hole $\circ_- = Ce_- Cg$ (completing the picture of pair annihilation); creating unphysical negative energy "particle" $\bullet_- = e_- Cg = e_- f$; creating unphysical "hole" (in effect negative energy) $\circ_+ = Ce_+ g$ in the world of positive energies.
We could also write $\Psi(f) = A^\dagger(e_- f) + A^\dagger(e_+ f)$, which means possibly creating $\bullet_-$'s, i.e. physically annihilating antiparticles $\circ_-$, or creating $\bullet_+$'s, i.e. physically creating particles.

An interesting question is from which kind of ground state or vacuum state one should start. It seems a bit difficult to imagine and describe a process of spontaneously creating twice some positive energy out of zero. Unless in nature this is of course somehow realized as experimentally approved. Therefore the idea came up by Dirac to provide an infinitely exhaustible reservoir of already occupied negative energy states. Because these can explain pair creation just by excitation (quantum jump) of at first unphysical states. The second and even more important reason for that concept is of course the desire of a lower bound of energy. That is, we need some $\bullet_-$'s to start with, which are able to spontaneously "climb" twice the energy distance (towards zero) to their corresponding physical particles of inverse energy, thereby leaving one half of the freed energy for a particle and the other for its antiparticle. To stress our picture further one last time:

$$\bullet\bullet\bullet \,|\, \text{- - -} \quad \longrightarrow \quad \text{“}\bullet \text{ - } \bullet \text{ } \!\!\!\!\mid\!\!\!\bullet \text{- - -”} \quad \longrightarrow \quad \bullet\circ\bullet \,|\, \text{- } \bullet \text{ -}.$$

Of course a ground state like that (leftmost above) assumes an always equal number of particles and antiparticles in the world, which is somehow debatable.

Since we do not want to get even further involved into things that are pretty much standard, let us get back to the second quantized version $\hat{H}_0$ of the Hamilton operator H_0. Actually (4.2) does not represent the physically correct choice yet. Beyond that it is also not very useful even from the mathematical point of view because it only admits Hamiltonians H, that are trace-class, see [34, Sec. 10.2.4]. To be brief, one wants the vacuum expectation value of $\hat{H}_0$ to vanish (same e.g. for charge and number operators), implement the lower boundedness of energy according to our foregoing illustrating remarks, and one also wants to resolve ambiguities concerning operator ordering – a fundamental quantum mechanical issue. The solution for that is **normal ordering** of an operator $\hat{A}$ on Fock space, defined by

$$:\hat{A}: := \hat{A} - \langle 0|\hat{A}|0\rangle,$$

or equivalently (can be shown using the anti-commutation relations) in words: to obtain $:\hat{A}:$ from $\hat{A}$, anti-commute all creation operators to the left of all annihilation operators (get a minus sign for each commutation), i.e. just omit the additional terms that normally arise from non-vanishing anti-commutators.

So let us keep that in mind and choose the normal ordered version $:\hat{H}:$, and with it also $:\hat{V}:$, whenever appropriate.

4.1 Remarks on time-evolution

Please mind that this section serves as a rough sketch only, its content is still vague.
For the free field governed by the free Hamiltonian H_0, resp. by its second quantized version $d\Gamma(H_0) = \hat{H}_0$, Heisenberg's equation of motion

$$-i\frac{\partial}{\partial t}\Psi(f) = [d\Gamma(H_0), \Psi(f)] \tag{4.5}$$

should hold. What is also true, is the axiomatic relation $\Psi(D_0 f) = 0$, for the free Dirac operator

$$D_0 f = (-i\gamma_\mu\partial^\mu + m)f = \left(-i\frac{\partial}{\partial t} + H_0\right)f.$$

When we add $-\Psi(D_0 f)$ (i.e. nothing) to the left hand side of (4.5), this results in

$$[d\Gamma(H_0), \Psi(f)] = -\Psi(H_0 f).$$

This equation is consistent with one that could have also been obtained from the basic principles/definitions of second quantization of one-particle operators (see [34, Sec. 10.2.4]).

In case of a wide range of types of external potentials, realized by simple multiplication operators $(vf)(x) := v(x)f(x)$, $f \in \operatorname{dom}(D_0)$, one expects still the same

$$[d\Gamma(H_0 + v), \Psi_v(f)] = -\Psi_v((H_0 + v)f),$$

among $\Psi_v((D_0 + v)f) = 0$.
Just as an observation at this point, consider

$$[: d\Gamma(v) :, \Psi_v(f)] = -\Psi_v(vf)$$

and compare it with a central relation

$$[: \Psi_v^*\Psi_v : (v), \Psi_v(f)] = -\Psi_v(viR_0 f)$$

of [6], where R_0 was the difference of advanced/retarded solutions of D_0. And $: \Psi_v^*\Psi_v : (v)$ was defined as some coincidence limit, described in detail in the

appendix of [6].
The dynamical connection between these two equations does not get clear in this context.

The most difficult task is of course the issue of a two-particle interaction. We are in quest of operators $\Psi_V(f)$ obeying

$$\Psi_V(D_0 f) + \left[\hat{V}, \Psi_V(f)\right] = 0,$$

but $\Psi_V(D_0 f)$ not vanishing in general. Due to the two-particle interaction, $\hat{V}$ is in general not the second quantization of a one-particle operator. From the scattering theoretical point of view one would like to start from a picture like

$$\begin{tikzcd}
\Psi_V(\infty, \underline{x})\chi \arrow[r, equal] & \Psi(\infty, \underline{x})\chi_{\mathrm{out}} \\
\Psi_V(t, \underline{x})\chi \arrow[u, "t\to\infty"] \arrow[d, "t\to-\infty"'] & \\
\Psi_V(-\infty, \underline{x})\chi \arrow[r, equal] & \Psi(-\infty, \underline{x})\chi_{\mathrm{in}} \arrow[uu, "S_V"']
\end{tikzcd}$$

with S_V being the S-operator, $\chi, \chi_{\#} \in \mathfrak{H}$ vectors in Fock space, where χ_{in} is assumed to be of even particle number with non overlapping velocity supports.

On suitable spaces of asymptotic states perhaps $\frac{d}{d\lambda}\Big|_{\lambda=0} S_{\lambda V}\chi_{\mathrm{as}} = \hat{V}\chi_{\mathrm{as}}$ can hold, $\lambda \in \mathbb{R}$ parameter.
However, these are just some sketchy ideas. It seems to be a very difficult task to find some way to proceed. The general hope is that the cut-off χ in the construction of the Kontsevich star-product $\tilde{\star}_K$ of Chapter 3, which is used in the potential $\hat{V}$, does an equally well job like the diverse cut-offs that were introduced for the Moyal-Minkowski case in Chaper 2. In the latter these were just temporary tools for technical purposes. Admittedly, this time the cut-off would always be present. But in return it is physically perfectly motivated.
Section 5.3 makes some further suggestions that could perhaps help.

5 An attempt to define quantum field operators on non-commutative spacetimes

5.1 Overview of non-commutative toy spacetimes

We review some of the most common toy models for nc. spacetimes by just listing their basic structures for a comparison on equal footing.
The initial point for these models is usually the motivation and then postulation of some special kind of commutation behavior for the quantum mechanical position operators $X^\mu, \mu = 0, \ldots, s$ (here including the time as well). As you know, this is also equivalent to imposing specific uncertainty relations between them. The general setting is classical $n = 1 + s \in \mathbb{N}_{\geq 2}$ dimensional Minkowski spacetime. In quantum mechanics, these X^μ are unbounded operators in some Hilbert space, e.g. $L^2(\mathbb{R}^n, \mathbb{C}^N)$. They are defined on a dense domain and act by just multiplying real numbers x^μ (coordinate functions) corresponding to X^μ. Hence they all easily commute with each other. Things are getting difficult, when these operators X^μ are also elements of some nc. spacetime algebra, call it $\mathcal{A}_{\text{st}}$. Such a scenario is conveniently realized by changing the product structure for functions on the spacetime manifold to a star product, hence also affecting the coordinate chart functions x^μ. That is, the classical algebra $\mathcal{A} = (C^\infty(\mathbb{R}^n, \mathbb{C}), \cdot)$ of functions on the spacetime is replaced for example by the algebra $\mathcal{A}_\star = (C_c^\infty(\mathbb{R}^n, \mathbb{C}), \star)$. Actually this kind of algebra (whatever choice) is understood as exactly the one that is associated with the notion of a spectral triple. Be careful not to mix up $\mathcal{A}_\star$ with $\mathcal{A}_{\text{st}}$, their correlation will be made clear later. We use upper case X^μ for elements in $\mathcal{A}_{\text{st}}$, and lower case $x^\mu \in \mathcal{A}_\star$.

Some of the common models are:

- *Classical commutative spacetime*
  $[X^\mu, X^\nu] = 0$.
  As a first step into the direction of defining field theories uniformly in all the following cases, it became convenient to consider the so called **Weyl form** (see [16] and references therein; also [15] for precise technicalities, that we want to omit here as far as possible)

$$W(k) := e^{-iX^\mu k_\mu}, \quad k \in \mathbb{R}^n, \tag{5.1}$$

Obviously it holds

$$W(k)W(k') = W(k+k'), \quad W(k)^* = W(-k) = W(k)^{-1}.$$

- *Moyal-Minkowski spacetime* (first introduced in [22] and [30])
  $[X^\mu, X^\nu] = iM^{\mu\nu}$,
  with $M = \frac{\theta}{2}\Omega$, $\theta > 0$ and
  $$\Omega := \begin{bmatrix} 0_{l\times l} & \mathbb{1}_{l\times l} \\ -\mathbb{1}_{l\times l} & 0_{l\times l} \end{bmatrix},$$
  $n = 2l \in 2\mathbb{N}$ the even spacetime dimension. Again define $W(k) := e^{-iX^\mu k_\mu}$ like above. Here we get the relations
  $$W(k)W(k') = W(k+k')e^{-\frac{i}{2}kMk'}, \quad W(k)^* = W(-k) = W(k)^{-1}.$$

- *Lie algebra structure*
  $[X^\mu, X^\nu] = iC^{\mu\nu}{}_\rho X^\rho$,
  $C^{\mu\nu}{}_\rho \in \mathbb{C}$. Define $W(k) := e^{-iX^\mu k_\mu}$. Then
  $$W(k)W(k') = W\left(k + k' + \frac{1}{2}g(k,k')\right), \quad W(k)^* = W(-k) = W(k)^{-1},$$
  with g just abbreviating the terms coming from the Baker-Campbell-Hausdorff formula.

- *κ-Minkowski spacetime* (see [26], [28])
  $[X^0, X^j] = -\frac{i}{\kappa}X^j$,
  $\kappa > 0$, $j = 1, \dots, s$. Define $W(k) := e^{-iX^\mu k_\mu}$. As κ-Minkowski is only a special case of a Lie algebra structure, the relations are the same.

- *Quantum space structure*, here especially the *Manin-plane*, $n = 2$ (see [29])
  $X^0X^1 = cX^1X^0$,
  $c \in \mathbb{C}$. Here one defines
  $$W(f(X^0, X^1)) := \,:f(X^0, X^1):,$$
  with some normal ordering of functions f of the coordinates. For details cf. [27].

- *"DFR"* (Doplicher, Fredenhagen, Roberts, see [15])
  $X^{\mu *} = X^\mu$, $[X^\rho, Q^{\mu\nu}] = 0$, $Q^{\mu\nu}Q_{\mu\nu} = 0$, $\frac{1}{4}[X^0, \dots, X^s]^2 = \mathbb{1}$,
  where $iQ^{\mu\nu} := [X^\mu, X^\nu]$ and $[X^0, \dots, X^s] := -\frac{1}{2}Q^{\mu\nu}(*Q)_{\mu\nu}$. As usual the Weyl form $W(k) := e^{-iX^\mu k_\mu}$ is introduced and fulfills
  $$W(k)W(k') = W(k+k')e^{-\frac{i}{2}kQk'}, \quad W(k)^* = W(-k) = W(k)^{-1}.$$

As one can see, the classical commutative spacetime is a special case of all the others.

5.2 Quantum field operator candidate

We try to define a reasonable expression for a quantum field operator on non-commutative backgrounds. At least this should reproduce the suggestion, that was already made in the DFR-paper [15] for the special model there. For the Klein-Gordon field they propose, $f \in C_c^\infty(\mathbb{R}^n, \mathbb{C})$,

$$\Phi_{\mathrm{KG}}(f) = \int_{\mathbb{R}^n} \Phi_{\mathrm{KG}}(X + a\mathbb{1}) f(a) d^n a, \tag{5.2}$$

where $X \mapsto \Phi_{\mathrm{KG}}(X)$ is formally just the usual plane-wave expansion of the Klein-Gordon field, except for the tensor product signs $\otimes$ to appear, and with possibly non-commuting variables X^μ:

$$\Phi_{\mathrm{KG}}(X) = (2\pi)^{-s/2} \int_{\mathbb{R}^s} \left(e^{-iX^\mu p_\mu} \otimes a(\underline{p}) + e^{iX^\mu p_\mu} \otimes a^\dagger(\underline{p}) \right)\Big|_{p_0 = \omega_{\underline{p}}} \frac{d^s \underline{p}}{2\omega_{\underline{p}}},$$

where $\omega_{\underline{p}} = \sqrt{\underline{p}^2 + m^2}$, and $a, a^\dagger$ are the bosonic annihilation, creation operators. So we have $\Phi_{\mathrm{KG}}(f) \in \mathcal{A}_{\mathrm{st}} \otimes L(\mathcal{F})$. Since the idea is, that $\Phi_{\mathrm{KG}}(f)$ is not just a linear operator in some Fock space $\mathcal{F}$. It should also contain another part, which is an element of the nc. algebra of spacetime elements $X \in \mathcal{A}_{\mathrm{st}}^n$.

We want to stick with the Klein-Gordon field for a while, just for the sake of notational simplification. But the transition to Dirac fields will actually be easy and straightforward.

In the foregoing section some emphasis has been put on the Weyl form $W(k) = e^{-iX^\mu k_\mu}$, $k \in \mathbb{R}^n$, $X \in \mathcal{A}_{\mathrm{st}}^n$. This will be the key element for the definition of the field operator. It will allow us to treat all the nc. toy spacetimes in an equal manner. As an intermediate step let's define

$$(Q_W f)(a) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ik^\mu a_\mu} W(k) \hat{f}(k) d^n k, \tag{5.3}$$

for $f \in C_c^\infty(\mathbb{R}^n, \mathbb{C})$, where $\hat{f} = \mathcal{F} f$ here denotes Fourier transformation with Minkowski product.

$$\hat{f}(k) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ik^\mu x_\mu} f(x) d^n x,$$

and $\check{f} = \mathcal{F}^{-1} f$ the inverse one. Hence formally "$(Q_W f)(a) = f(a\mathbb{1} - X)$" holds, which helps to see the coincidence between the DFR-proposal (5.2) and the following one:

$$\Phi_{\mathrm{KG}}^{\otimes}(Q_W f) := \int_{\mathbb{R}^n} (Q_W f)(a) \otimes \Phi_{\mathrm{KG}}(a) d^n a \in \mathcal{A}_{\mathrm{st}} \otimes L(\mathcal{F}). \tag{5.4}$$

Here the tensor product splitting is made clearer, and Φ_{KG} is exactly just the usual field operator (the pointwise version). The Dirac field analogue is

$$\Psi^{\otimes}(Q_W f) := \sum_{A=1}^{N} \int_{\mathbb{R}^n} (Q_W f^A)(a) \otimes \Psi_A(a) d^n a, \quad f \in C_c^\infty(\mathbb{R}^n, \mathbb{C}^N). \tag{5.5}$$

Shortly we will mention some nice facts related to the objects $Q_W f$, which then motivates their introduction in retrospect. The field operator itself could have been defined nicely without them, too:

$$\Phi^{\otimes}_{\mathrm{KG,W}}(f) := \int_{\mathbb{R}^n} W(k) \otimes \hat{f}(k)\check{\Phi}_{\mathrm{KG}}(k)d^n k = \Phi^{\otimes}_{\mathrm{KG}}(Q_W f), \tag{5.6}$$

as one easily checks.

The definition of $Q_W f$ is very similar to Weyl's quantization, associating an operator with a function of classical (commuting) variables (see [35], [36]):

$$\mathfrak{W}(f) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{iX^\mu k_\mu} \hat{f}(k) d^n k = (2\pi)^{-n/2} \int_{\mathbb{R}^n} W(-k)\hat{f}(k)d^n k, \tag{5.7}$$

which is again in $\mathcal{A}_{\mathrm{st}}$ like $Q_W f$ and $W(k)$; it is quite common to suppress the actual dependence on X in the notation. One could write $\mathfrak{W}(f) = f(X)$, known from functional calculus, which provides the precise mathematical background. The analogue is $(Q_W f)(a) = f(a\mathbb{1} - X)$ for all $a \in \mathbb{R}^n$.

Now, like already indicated in the foregoing section, it is convenient to associate a star product with each of the nc. model structures. In [27] this is done for Moyal, the Lie algebra structure and the Manin-plane, where the strategy always builds upon implementing the relation

$$\mathfrak{W}(f)\mathfrak{W}(g) = \mathfrak{W}(f \star g). \tag{5.8}$$

It is easy to show, that then also

$$(Q_W f)(Q_W g)(a) = (Q_W(f \star g))(a) \quad \forall a \in \mathbb{R}^n \tag{5.9}$$

holds. If one quickly wants to check this e.g. just for Moyal (note: here just for once defined with Minkowski product in the exponential),

$$\begin{aligned} (f \star g)(x) &= (2\pi)^{-n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{-iu^\mu v_\mu} f\,(x - Mu)\, g(x + v) d^n u d^n v \\ &= (2\pi)^{-n} \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{-i(k+p)^\mu x_\mu} e^{-ik^\nu (Mp)_\nu} \hat{f}(k)\hat{g}(p) d^n k d^n p \end{aligned}$$

will be useful.

Let $f = f(x)$, $f \in C^\infty_c(\mathbb{R}^n) = \mathcal{A}_\star$ be a function of spacetime variables x, respectively coordinate functions x, equipped with either the classical or some non-commutative product. And let $F = F(X)$, $F : \mathcal{A}^n_{\mathrm{st}} \to \mathcal{A}_{\mathrm{st}}$ be the corresponding map to f for the abstractly postulated (non-)commuting spacetime variables X (upper case). Then, when we take a look at the definition (5.7) of Weyl quantization, we see that the transition from a function f with $f = f(x)$ to F with $F = F(X) = \mathfrak{W}(f)$ is just realized by inverse Fourier transformation followed by an ordinary one and besides just "renaming" x into X. In the preceding paragraphs the same symbol was used for f and F, which is reasonable, since the distinction was made sufficiently by the use of upper and lower case x's.

But to avoid confusion, let's use the case-sensitive notation for the mappings (f, F) one last time, namely for the purpose of explaining the relation between $[X^\mu, X^\nu] = X^\mu X^\nu - X^\nu X^\mu$ and $[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu$. Let $F(X) := X^\mu = \mathrm{pr}_\mu$, $G(X) := X^\nu = \mathrm{pr}_\nu$ be the projections onto components $X \overset{F}{\mapsto} X^\mu$, $X \overset{G}{\mapsto} X^\nu$, and f, g the corresponding ones for the lower case $x \in \mathbb{R}^n$. Now $\mathfrak{W}(X^\mu)\mathfrak{W}(X^\nu) = \mathfrak{W}(x^\mu \star x^\nu)$ holds, which can be expressed as $X^\mu(X)X^\nu(X) = (x^\mu \star x^\nu)(X)$, too. Furthermore it is quite common to write X^μ instead of $X^\mu(X)$, i.e. $X^\mu X^\nu = (x^\mu \star x^\nu)(X)$. And $(x^\mu \star x^\nu)(X)$ arises from $(x^\mu \star x^\nu)(x) = x^\mu \star x^\nu$ just by the renaming-process $x \mapsto X$ due to $\mathfrak{W}$, like explained within the foregoing paragraph.

Remark The article [15] makes some interesting further suggestions. They define states on the algebra of operators $\Phi_{\mathrm{KG}}(X)$, which, when localized properly, are interpretable as analogues of classical spacetime points.
We will not exploit this further. But it is worth mentioning and could still be a valuable ingredient for alternative strategies.

5.3 Field operators for tensor product function spaces

A general question, that was already thoroughly discussed in Chapters 3 and 4, is how to apply principles of locally non-commutative spacetimes to quantum field theory. This section suggests another probably useful idea of a technical interface to combine these two things. With an eye on the sketchy ideas of the pair interaction topic from Chapter 4, we want to make in fact *two* slightly deviating proposals of how to define field operators for this special purpose.

5.3.1 Option 1

The idea is to take the usual test function space (smooth compactly supported spinors in case of the Dirac field) normally used to label field operators $\Psi(f)$,

$$f \in C_c^\infty(\mathbb{R}^n, \mathbb{C}^N) =: \mathscr{D}^{n,N},$$

$n, N \in \mathbb{N}$ appropriately chosen (cf. also Section 2.1), and tensor-multiply it with an auxiliary space, doubling the count of spacetime coordinates n. Define

$$\mathscr{S}^{n,N} := \mathscr{S}(\mathbb{R}^n, \mathbb{C}^N),$$

and let this auxiliary space be $\mathscr{S}^{n,1}$. Obviously

$$\mathscr{D}^{n,N} \otimes \mathscr{S}^{n,1} \subseteq \mathscr{S}^{n,N} \otimes \mathscr{S}^{n,1} \cong \mathscr{S}^{2n,N}$$

holds, and with $h \in \mathscr{S}^{n,1}$ we have $f \otimes h \in \mathscr{D}^{n,N} \otimes \mathscr{S}^{n,1} \subseteq \mathscr{S}^{2n,N}$. $\mathscr{S}^{2n,N}$ then arises as the "label-space" for a new kind of field operator (already represented on vacuum-Fock-space $\mathcal{H}^{\mathrm{vac}}$), defined by

$$\Psi_\varepsilon^{(2)}(f \otimes h) := \sum_{A=1}^{N} \int_{\mathbb{R}^n} \Psi_A(x)(f \diamond_\varepsilon h)^A(x) d^n x = \Psi(f \diamond_\varepsilon h), \tag{5.10}$$

with the last equality showing its connection to the ordinary well-known field operator. Where $\diamond_\varepsilon : \mathscr{D}^{n,N} \otimes \mathscr{S}^{n,1} \to \mathscr{D}^{n,N}$ is defined as follows:

$$(f \diamond_\varepsilon h)^A(x) := f^A(x) \int_{\mathbb{R}^n} \delta_\varepsilon(x-y) h(y) d^n y, \tag{5.11}$$

with $\delta_\varepsilon \in \mathscr{S}(\mathbb{R}^n, \mathbb{R})$, $0 < \varepsilon \in \mathbb{R}$, being Gaussian functions $\delta_\varepsilon(x) := \frac{1}{\varepsilon\sqrt{\pi}} e^{-\frac{x^2}{\varepsilon^2}}$, fulfilling $\int_{\mathbb{R}^n} \delta_\varepsilon(x) d^n x = 1$ and approaching the Dirac delta peak measure

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} g(y) \delta_\varepsilon(x-y) d^n y = g(x)$$

for some $g \in \mathscr{S}$. This amounts to

$$\Psi_\varepsilon^{(2)}(f \otimes h) = \Psi(f \diamond_\varepsilon h) \xrightarrow[\varepsilon \to 0]{} \Psi(fh)$$

with pointwise product fh.

In the next step we need to adjust the definition of the potential operator to the new setting. That is $V := V^{(2)} : \mathscr{S}^{2n,N} \to \mathscr{S}^{2n,N}$ is chosen to be one out of

$$\begin{aligned} \text{(i) } (V(f \otimes h))^A &:= (V_{(\mathrm{i})}(f \otimes h))^A := c \star (f^A \otimes h) + (f^A \otimes h) \star c \\ \text{(ii) } (V(f \otimes h))^A &:= (V_{(\mathrm{ii})}(f \otimes h))^A := c \star (f^A \otimes h) \star c, \end{aligned}$$

where $c := c^{(2)} \in \mathscr{S}(\mathbb{R}^{2n}, \mathbb{R})$ is an element of the algebra $\mathcal{A}_\star^{(2)} = (\mathscr{S}(\mathbb{R}^{2n}, \mathbb{C}), \star)$ in the sense of spectral triples, and $\star \equiv \tilde{\star}_K$ is the vertical, local Kontsevich star-product from Section 3.4. Alternatively for a given $h \in \mathscr{S}^{n,1}$ one could consider $V_h : \mathscr{S}^{n,N} \to \mathscr{S}^{2n,N}$,

$$(V_h f)^A := (V(f \otimes h))^A,$$

understood as composition of maps $f \mapsto f \otimes h \mapsto V(f \otimes h)$.

One central object of interest is an expression like $\Psi(V^{(1)} R_0 g)$, $g \in \mathscr{D}^{n,N}$, with $V^{(1)}$ being some one-particle external potential and R_0 the difference of free advanced/retarded solutions of the free Dirac operator. This object is involved in promising operational relations, namely Bogoliubov's formula (to be found as a result of Chapter 2 and the paper [6]). It reveals some hint at possible methods to analyse effects of spacetime non-commutativity at the level of quantum field operators.

When we try to establish similar things for field operators $\Psi_\varepsilon^{(2)}(f \otimes h)$, a quite obvious starting point could be the consideration of the correctly defined expression $\Psi(V^{(1)} R_0 (f \diamond_\varepsilon h))$. But our potential $V = V^{(2)}$ is defined on the tensor

product of function spaces, respectively on $\mathscr{S}^{2n,N}$, shifting the focus to an expression like $\Psi_\varepsilon^{(2)}(V(R_0 f \otimes h))$.
Where however, it is not clear whether $V(R_0 f \otimes h)$ or $VR_0^{(2)}(f \otimes h)$ is needed, with some unknown $R_0^{(2)}$.
$\Psi_\varepsilon^{(2)}(V(R_0 f \otimes h))$ could possibly correspond to an expression $\Psi(P_1(V(R_0 f \otimes h)) \diamond_\varepsilon P_2(V(R_0 f \otimes h)))$, with P_1, P_2 being projectors on the two factors of the tensor product, respectively on the first or second half of the $2n$ coordinate dependencies. This is probably hard to "compare" with $\Psi(V^{(1)} R_0(f \diamond_\varepsilon h))$.

5.3.2 Option 2

Define

$$\Psi^{(2)}(\bar{f} \otimes g) := \Psi^*(f)\Psi(g) \tag{5.12}$$

at the level of vacuum-Fock-representation space $\mathcal{H}^{\mathrm{vac}}$, for $f, g \in \mathscr{D}^{n,N}$ (defined in Sec. 5.3.1). $V := V^{(2)} : \mathscr{S}^{2n,N^2} \to \mathscr{S}^{2n,N^2}$ is chosen to be one out of

$$\begin{aligned}
\text{(i) } (V(f \otimes g))^{AB} &:= (V_{\text{(i)}}(f \otimes g))^{AB} := c \star (f^A \otimes g^B) + (f^A \otimes g^B) \star c \\
\text{(ii) } (V(f \otimes g))^{AB} &:= (V_{\text{(ii)}}(f \otimes g))^{AB} := c \star (f^A \otimes g^B) \star c,
\end{aligned}$$

where $c := c^{(2)} \in \mathscr{S}(\mathbb{R}^{2n}, \mathbb{R})$ is an element of the algebra $\mathcal{A}_\star^{(2)} = (\mathscr{S}(\mathbb{R}^{2n}, \mathbb{C}), \star)$ in the sense of spectral triples, and $\star \equiv \tilde{\star}_K$ is the vertical, local Kontsevich star-product from Section 3.4.

Let $\mathfrak{F}^{\mathrm{vac}}_{\Psi^{(2)}} \subseteq L(\mathcal{H}^{\mathrm{vac}})$ be the algebra generated by the elements $\Psi^{(2)}(\bar{f} \otimes g)$ and $\mathbb{1}$. And define $\delta_V : \mathfrak{F}^{\mathrm{vac}}_{\Psi^{(2)}} \to \mathfrak{F}^{\mathrm{vac}}_{\Psi^{(2)}}$ by derivation-property on products and by

$$\delta_V\left(\Psi^{(2)}(\bar{f} \otimes g)\right) := \Psi^{(2)}\left(V \circ (R_0 \otimes R_0)(\bar{f} \otimes g)\right) \tag{5.13}$$

on single elements. The question arises whether there exists an operator $G(V) \in L(\mathcal{H}^{\mathrm{vac}})$, such that

$$\left[G(V), \Psi^{(2)}(\bar{f} \otimes g)\right] = \delta_V\left(\Psi^{(2)}(\bar{f} \otimes g)\right).$$

This equation would then represent an analogon to

$$\left[i\Phi(V^{(1)}), \Psi(f)\right] = \Psi(V^{(1)} R_0 f)$$

formerly derived for the case of Moyal-Minkowski spacetime, with $\Phi(V^{(1)}) := \Phi(c) = -id/d\lambda|_{\lambda=0} S_{\lambda c}$, the S-operator differentiated with respect to the interaction strength. In case of implementability; otherwise $\Phi(c)$ is just seen as the generator of the derivation and the property of its (essential) self-adjointness is desirable to be shown.
However, the tough task at this stage is to show the existence of $G(V)$. But this will not be investigated further here.

6 Conclusion and outlook

Apart from inventing something completely new, like e.g. string theorists do, we take well-known concepts of quantum field theory completely serious. The new ingredient of non-commutative spacetimes is physically reasonable motivated and mathematically anyway without a doubt.
Using tools and ideas of solid foundation, it was indeed possible to construct an operational connection between quantum fields and the underlying nc. spacetime structure. This was even achieved for the general case of even the time coordinate being non-trivially involved. Although carried out for the special example of Moyal-Minkowski spacetime, the strategy could well be a pattern for more general situations.
Locally nc. spacetimes are probably better candidates. For those some groundwork could be established.
However, the level of difficulty in this field of research leads to many distinct tentative approaches. These are often quite far apart from their conceptual point of view, or let it just be a notational one. So it is important to invest some energy in translation. This is unfortunately by no means easy, and we made only modest progress in this regard.
Further, one has to ask, how far examples can be a guidance towards general principles and methods. They can surely help to dig one's way through the confusing variety of non-commutative geometry. But in a way this field seems like such a powerful toolbox that some day it should inspire a new idea from a more "philosophical" point of view that simplifies things again. Probably one cannot expect spectacular experimental predictions and explanations so soon. So the main drive is almost one of mere mathematical beauty.

For a detailed summary of the discussed topics and the content I refer back to the Introduction, Chapter 1.

Appendix A

Technical details

A.1 Continuity of the fundamental solutions with respect to Schwartz norms

A.1.1 Klein-Gordon case

For the sake of less notational effort, the following theorem is proven for the Klein-Gordon case first.

Theorem A.1.1 *Let $E^{\pm} : C_c^{\infty}(\mathbb{R}^4) \to C^{\infty}(\mathbb{R}^4)$ be the advanced and retarded fundamental solutions of $\Box + m^2$ on 4-dim. Minkowski spacetime:*

$$\begin{aligned} &(\Box + m^2)E^{\pm} f = f = E^{\pm}(\Box + m^2) f \textit{ and} \\ &\operatorname{supp} E^{\pm} f \subset J^{\pm}(\operatorname{supp} f). \end{aligned}$$

Let $\triangle := \sum_{\mu=0}^{3} \partial_{x^\mu}^2$ be the 4-dimensional Laplace operator and

$$Mf(y) := \sum_{\mu=0}^{3} y_\mu^2 f(y) \ (y \in \mathbb{R}^4),$$

and

$$\begin{aligned} \mathfrak{a} f &:= (1 - \triangle)^{-1} f, \\ \mathfrak{b} f &:= (1 + M)^{-1} f \end{aligned}$$

for $f \in L^2(\mathbb{R}^4)$. Then there exist $\alpha, \beta \in \mathbb{N}$, such that $\mathfrak{b}^\beta \mathfrak{a}^\alpha E^{\pm} \mathfrak{a}^\alpha \mathfrak{b}^\beta$ can be extended to bounded operators on $L^2(\mathbb{R}^4)$ (taking values $\subset L^2(\mathbb{R}^4)$).
As a byproduct it is shown, that the domain of $E^{\pm}$ can be extended to $\mathscr{S}(\mathbb{R}^4)$.

Remark *The statement can be proven for arbitrary dimensions $\mathbb{R}^n$, $n \geq 2$, as well.*
Note also, that $E^{\pm}$ and $\mathfrak{a}^\alpha$ commute (easily shown with the help of explicit formulas from the following proof).

Proof

1. Notation. For $p, x \in \mathbb{R}^4$ define:
$$\begin{aligned}\text{Minkowski product } p[x] &:= p_\mu x^\mu = \eta_{\mu\nu} p^\mu x^\nu = p^0 x^0 - \underline{p} \cdot \underline{x} \\ \text{Euklidean norm } |p|^2 &:= p_0^2 + p_1^2 + p_2^2 + p_3^2,\end{aligned}$$
for $f \in \mathscr{S}(\mathbb{R}^4)$:
$$\hat{f}(p) := \frac{1}{(2\pi)^2} \int f(x) e^{-ip[x]} d^4x$$
the Fourier transform with Minkowski product.

2. Fourier space representation of $E^\pm$. According to [32]
$$E^\pm(x,y) = \lim_{\varepsilon \to 0+} \frac{1}{(2\pi)^4} \int \frac{e^{-ip[x-y]}}{m^2 - p[p] \mp ip_0\varepsilon} d^4p$$
holds, i.e. for $f, g \in \mathscr{S}(\mathbb{R}^4)$:
$$(f, E^\pm g)_{L^2(\mathbb{R}^4)} = \lim_{\varepsilon \to 0+} \int \frac{\bar{\hat{f}}(p)\hat{g}(-p)}{m^2 - p^2 \mp ip_0\varepsilon} d^4p.$$
Remark According to the definition of $E^\pm$ in the assumptions of the theorem the limit exists for the time being for $f, g \in C_c^\infty(\mathbb{R}^4)$. If it exists also for all $f, g \in \mathscr{S}(\mathbb{R}^4)$, then $f, g \mapsto (f, E^\pm g)_{L^2(\mathbb{R}^4)}$ is a sesquilinear form on $\mathscr{S}(\mathbb{R}^4) \subset L^2(\mathbb{R}^4)$.

3. Let $F \in \mathscr{S}(\mathbb{R}^4)$. For the spatial coordinates $\underline{p} = (p_1, p_2, p_3)$ we introduce spherical coordinates:
$$F(p) = F(p_0, \underbrace{r, \Omega}_{\underline{p}}), \text{ with } \Omega = (\vartheta, \varphi) \in S^2.$$
Set
$$\tau^+(F) := \lim_{\varepsilon \to 0+} \int \frac{F(p)}{m^2 - p^2 - ip_0\varepsilon} d^4p.$$
In spherical coordinates:
$$\tau^+(F) = \lim_{\varepsilon \to 0+} \int \frac{F(p_0, r, \Omega)}{m^2 + r^2 - p_0^2 - ip_0\varepsilon} r^2 dr d\Omega dp_0$$
with $d\Omega = \sin\vartheta d\vartheta d\varphi$, integration over $r \in (0, \infty),\ \Omega \in S^2,\ p_0 \in \mathbb{R}$. Partial integration w.r.t. r:
$$\tau^+(F) = \frac{1}{2} \lim_{\varepsilon \to 0+} \int \partial_r \left(rF(p_0, r, \Omega) \right) \ln\left(m^2 + r^2 - p_0^2 - ip_0\varepsilon\right) dr d\Omega dp_0,$$
with complex logarithm $\ln z = \ln|z| + i \arg z$. Here the limit can be taken:
$$\lim_{\varepsilon \to 0+} \ln(m^2 + r^2 - p_0^2 - ip_0\varepsilon) = \ln|m^2 + r^2 - p_0^2| + \frac{i\pi}{2} \left(1 - \operatorname{sgn}\,(m^2 + r^2 - p_0^2)\right),$$
taking into account the complex logarithm $\ln z = \ln|z| + i \arg z$.

4. It holds $|-i\pi\mathrm{sgn}\ p_0| = \pi$, and $r, p_0 \mapsto \ln|m^2 + r^2 - p_0^2|$ is locally L^1 on $\mathbb{R}_+ \times \mathbb{R}$ (follows from $\varrho^\varepsilon \ln\rho \to_{\varrho\to 0+} 0\ \forall\varepsilon > 0$). Further

$$\int_{|r-r^*|\le 1}\int_{|p_0-p_0^*|\le 1}\left|\ln|m^2+r^2-p_0^2|\right| drdp_0 \le C_{(0)}(1+|r^*|^2)^m(1+|p_0^*|^2)^n$$

holds true for arbitrary $m, n > 0$. This implies: It exists $\nu \in \mathbb{N}$, such that

$$\xi : r, p_0 \mapsto \left(\frac{1}{1+r^2+p_0^2}\right)^{2\nu}\left(\left|\ln|m^2+r^2-p_0^2|\right| + C_\xi\right),$$

$C_\xi \in \{0, \pi\}$, is in $L^1\left((0,\infty)\times\mathbb{R}, drdp_0\right)$. Hence we get the estimation

$$\left|\tau^+(F)\right| \le C_{(1)} \sup_{p_0,r,\Omega}\left|\partial_r\left(rF(p_0,r,\Omega)\right)(1+r^2+p_0^2)^{2\nu}\right|$$

with $C_{(1)} = \int |\xi(r,p_0)|drdp_0 d\Omega$.

5. Now consider the case $F(p) = \hat{\hat{f}}(p)\hat{g}(-p)$. Then

$$\begin{aligned}\partial_r(rF)(p) &= r\partial_r F(p) + F(p)\\ &= r\left[\left(\partial_r\hat{\hat{f}}(p)\right)\hat{g}(-p) + \hat{\hat{f}}(p)\partial_r\hat{g}(-p)\right] + \hat{\hat{f}}(p)\hat{g}(-p)\end{aligned}$$

and

$$|\partial_r(rF)(p)| \le (1+r^2)^2\left[\left|\hat{f}(p)\right| + \left|\partial_r\hat{f}(p)\right|\right]\left[|\hat{g}(-p)| + |\partial_r\hat{g}(-p)|\right],$$

which implies

$$\sup_{p_0,r,\Omega}|\partial_r(rF)(p_0,r,\Omega)|(1+r^2+p_0^2)^{2\nu} \le N(\hat{f})N(\hat{g})$$

with $N(\hat{f}) = \sup_{p\in\mathbb{R}^4}(1+|p|^2)(1+|p|^2)^\nu\left(\left|\hat{f}(p)\right| + \left|\partial_r\hat{f}(p)\right|\right)$.

6. It holds

$$\begin{aligned}\left|\partial_r(r\hat{f})(p)\right| &\le \left|\hat{f}(p)\right| + r\left|\partial_r\hat{f}(p)\right|\\ &\le \left|\hat{f}(p)\right| + (1+|p|^2)\sum_{k=1}^{3}\left|\partial_{p_k}\hat{f}(p)\right|\end{aligned}$$

and

$$N(\hat{f}) \le \sup_{p\in\mathbb{R}^4}(1+|p|^2)^{\nu+1}\left(\left|\hat{f}(p)\right| + \sum_{k=1}^{3}\left|\partial_{p_k}\hat{f}(p)\right|\right).$$

Moreover we have

$$
\begin{aligned}
&(1+|p|^2)^{\nu+1}\sum_{k=1}^{3}\left|\partial_{p_k}\hat{f}(p)\right| \\
\leq\;& \sum_{k=1}^{3}\left|\partial_{p_k}\left((1+|p|^2)^{\nu+1}\hat{f}(p)\right)\right| + \sum_{k=1}^{3}\left|\left(\partial_{p_k}(1+|p|^2)^{\nu+1}\right)\hat{f}(p)\right| \\
\leq\;& C_{(2)}\left[(1+|p|^2)^{\nu+1}\left|\hat{f}(p)\right| + \sum_{k=1}^{3}\left|\partial_{p_k}\left((1+|p|^2)^{\nu+1}\hat{f}(p)\right)\right|\right].
\end{aligned}
$$

This implies

$$
N(\hat{f}) \leq \sup_{p\in\mathbb{R}^4} C_{(3)}\left[(1+|p|^2)^{\nu+1}\left|\hat{f}(p)\right| + \sum_{k=1}^{3}\left|\partial_{p_k}\left((1+|p|^2)^{\nu+1}\hat{f}(p)\right)\right|\right].
$$

With the help of

$$
\begin{aligned}
(1+|p|^2)^{\mu}\hat{f}(p) &= \left((1-\triangle)^{\mu} f\right)^{\wedge}(p) \text{ and} \\
\sup_p\left|\hat{f}(p)\right| &\leq \frac{1}{(2\pi)^2}\int_{\mathbb{R}^4}|f(x)|\,dx
\end{aligned}
$$

one obtains

$$
\begin{aligned}
N(\hat{f}) &\leq \frac{C_{(3)}}{(2\pi)^2}\int_{\mathbb{R}^4}(1+|x_1|+|x_2|+|x_3|)\left|(1-\triangle)^{\nu+1}f(x)\right| d^4x \\
&\leq C_{(4)}\int_{\mathbb{R}^4}(1+|x|^2)\left|(1-\triangle)^{\nu+1}f(x)\right| d^4x \\
&\leq C_{(5)}\left(\int_{\mathbb{R}^4}(1+|x|^2)^{2\mu}\left|(1-\triangle)^{\nu+1}f(x)\right|^2 d^4x\right)^{1/2},
\end{aligned}
$$

by a standard argument. So altogether it has been shown:

$$
\left|(f, E^{+}g)_{L^2}\right| \leq C_{(6)}\left\|(1+M)^{\mu}(1-\triangle)^{\nu+1}f\right\|_{L^2}\left\|(1+M)^{\mu}(1-\triangle)^{\nu+1}g\right\|_{L^2}
$$

for all $f, g \in \mathscr{S}(\mathbb{R}^4)$. The argument for E^- works along the same lines and results in the same estimate. With $\alpha := \nu+1$, $\beta := \mu$, $u := \mathfrak{b}^{-\beta}\mathfrak{a}^{-\alpha}f$, $w := \mathfrak{b}^{-\beta}\mathfrak{a}^{-\alpha}g$, this implies

$$
\left|\left(\mathfrak{a}^{\alpha}\mathfrak{b}^{\beta}u, E^{\pm}\mathfrak{a}^{\alpha}\mathfrak{b}^{\beta}w\right)_{L^2}\right| \leq C_{(6)}\|u\|_{L^2}\|w\|_{L^2}
$$

for all $u, w \in \mathscr{S}(\mathbb{R}^4)$,which is basically the statement of the theorem. □

A.1.2 Dirac case

Theorem A.1.2 *Let $R^{\pm} : C_c^{\infty}(\mathbb{R}^4, \mathbb{C}^4) \to C^{\infty}(\mathbb{R}^4, \mathbb{C}^4)$ be the advanced and retarded fundamental solutions of $D = -i\partial\!\!\!/ + m$ on 4-dim. Minkowski spacetime:*

$$
\begin{aligned}
&DR^{\pm}f = f = R^{\pm}Df \textit{ and} \\
&\operatorname{supp} R^{\pm}f \subset J^{\pm}(\operatorname{supp} f).
\end{aligned}
$$

Let $\triangle := \sum_{\mu=0}^{3} \partial_{x^\mu}^2$ *be the 4-dimensional Laplace operator and*

$$Mf(y) := \sum_{\mu=0}^{3} y_\mu^2 f(y) \ (y \in \mathbb{R}^4),$$

and

$$\begin{aligned} \mathfrak{a}f &:= (1-\triangle)^{-1} f, \\ \mathfrak{b}f &:= (1+M)^{-1} f \end{aligned}$$

for $f \in L^2(\mathbb{R}^4)$*. Then there exist* $\alpha, \beta \in \mathbb{N}$*, such that* $\mathfrak{b}^\beta \mathfrak{a}^\alpha R^\pm \mathfrak{a}^\alpha \mathfrak{b}^\beta$ *can be extended to bounded operators on* $L^2(\mathbb{R}^4)$ *(taking values* $\subset L^2(\mathbb{R}^4)$*).*
As a byproduct it is shown, that the domain of $R^\pm$ *can be extended to* $\mathscr{S}(\mathbb{R}^4, \mathbb{C}^4)$*.*

Remark *The statement can be proven for arbitrary dimensions* $\mathbb{R}^n$*,* $n \geq 2$*, as well.*
Note also, that $R^\pm$ *and* $\mathfrak{a}^\alpha$ *commute (easily shown with the help of explicit formulas from the following proof).*

Proof We want to restrict ourselves to just pointing out the few differences compared to the proof of Theorem A.1.1. Apart from those, the Dirac case works exactly along the same lines.
The explicit formula for the fundamental solutions is replaced by

$$R^\pm_{AB}(x,y) = (i\partial\!\!\!/_x + m)_{AB} E^\pm(x,y)$$

(see [5]; which is the same as derivating with $\partial\!\!\!/_{x-y}$ instead) resulting in

$$(f, R^\pm g)_{L^2(\mathbb{R}^4)} = \lim_{\varepsilon \to 0+} \int \frac{\hat{\hat{f}}^A(p)(p\!\!\!/ + m)_{AB}\hat{g}^B(-p)}{m^2 - p^2 \mp ip_0\varepsilon} d^4p.$$

Paragraphs 3. and 4. of the Klein-Gordon proof can be transferred completely unchanged.
Then consider the Schwartz function $F(p) = \hat{\hat{f}}^A(p)(p\!\!\!/ + m)_{AB}\hat{g}^B(-p)$. Calculate

$$\begin{aligned} &|\partial_r(rF)(p)| \\ &= \Big| r \Big[\Big(\partial_r \hat{\hat{f}}^A(p) \Big) (p\!\!\!/ + m)_{AB}\hat{g}^B(-p) \\ &\quad + \hat{\hat{f}}^A(p)(\gamma_1 \sin\vartheta\cos\varphi + \gamma_2 \sin\vartheta\sin\varphi + \gamma_3\cos\vartheta)_{AB}\hat{g}^B(-p) \\ &\quad + \hat{\hat{f}}^A(p)(p\!\!\!/ + m)_{AB}\partial_r\hat{g}^B(-p) \Big] + \hat{\hat{f}}^A(p)(p\!\!\!/ + m)_{AB}\hat{g}^B(-p) \Big| \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{A,B=1}^{N} r\left[\left|\partial_r \hat{\hat{f}}^A(p)\right| C_1(|p_0|+r+1)\left|\hat{g}^B(-p)\right| + \left|\hat{\hat{f}}^A(p)\right| C_2 \left|\hat{g}^B(-p)\right|\right.\\
&\quad \left.\left|\hat{\hat{f}}^A(p)\right| C_1(|p_0|+r+1)\left|\partial_r \hat{g}^B(-p)\right|\right] + \left|\hat{\hat{f}}^A(p)\right| C_1(|p_0|+r+1)\left|\hat{g}^B(-p)\right|\\
&\leq \sum_{A,B}^{N}(1+r^2)C_3(|p_0|+r+1)\left[\left|\hat{f}^A(p)\right| + \left|\partial_r \hat{f}^A(p)\right|\right]\left[\left|\hat{g}^B(-p)\right| + \left|\partial_r \hat{g}^B(-p)\right|\right]\\
&\leq C_4^2 \sum_{A,B}^{N}(1+|p|^2)^2\left[\left|\hat{f}^A(p)\right| + \left|\partial_r \hat{f}^A(p)\right|\right]\left[\left|\hat{g}^B(-p)\right| + \left|\partial_r \hat{g}^B(-p)\right|\right],
\end{aligned}
$$

due to $|(\not{p}+m)_{AB}| \leq C_1(|p_0|+r+1)$ and other straightforward estimates, $|p|^2 = p_0^2 + r^2,\ r = |\underline{p}|$. This leads to

$$
\sup_{p_0,r,\Omega} |\partial_r(rF)(p_0,r,\Omega)|\,(1+|p|^2)^{2\nu} \leq N(\hat{f})N(\hat{g})
$$

with $N(\hat{f}) = C_4 \sum_{A=1}^{N} \sup_{p\in\mathbb{R}^4}(1+|p|^2)^{\nu+1}\left(\left|\hat{f}^A(p)\right| + \left|\partial_r \hat{f}^A(p)\right|\right)$.
Along the lines of paragraph 6. of proof A.1.1 this slight deviation carries over till

$$
\begin{aligned}
N(\hat{f}) &\leq C_{(5)}C_4\left(\sum_{A=1}^{N}\int_{\mathbb{R}^4}(1+|x|^2)^{2\mu}\left|(1-\triangle)^{\nu+1}f^A(x)\right|^2 d^4x\right)^{1/2}\\
&= C_{(5)}C_4\left\|(1+M)^{\mu}(1-\triangle)^{\nu+1}f\right\|_{L^2},
\end{aligned}
$$

which then proves the Dirac case in the same manner. □

A.2 Further analytical properties of the fundamental solutions

A.2.1 Klein-Gordon case

Lemma A.2.1 *Let $E^{\pm} : \mathscr{S}(\mathbb{R}^n) \to C^{\infty}(\mathbb{R}^n)$ be the advanced and retarded fundamental solutions of $\Box + m^2$ on n-dim. Minkowski spacetime like in Thm. A.1.1, defined on $\mathscr{S}$-functions.*
Then it holds

$$
\sup_{x\in\mathbb{R}^n}\left|E^{\pm}f(x)\right| < \infty.
$$

Proof We take the explicit formula for $E^{\pm}$ from the proof of Thm. A.1.1, $f \in \mathscr{S}(\mathbb{R}^n)$, and calculate:

$$
\begin{aligned}
E^{\pm}f(x) &= \int_{\mathbb{R}^n} E^{\pm}(x,y)f(y)d^ny = \int_{\mathbb{R}^n}\lim_{\varepsilon\to 0+}(2\pi)^{-n}\int_{\mathbb{R}^n}\frac{e^{-ip[x-y]}f(y)}{m^2-p^2\mp ip_0\varepsilon}d^npd^ny\\
&= \lim_{\varepsilon\to 0+}(2\pi)^{-n/2}\int_{\mathbb{R}^n}\frac{e^{-ip[x]}\check{f}(p)}{m^2-p^2\mp ip_0\varepsilon}d^np,
\end{aligned}
$$

$\check{f}$ denoting the inverse Fourier transform with Minkowski product. To simplify notation for the introduction of spherical coordinates we restrict ourselves to the case $n = 4$ from now on (without loss of generality). Then

$$\begin{aligned} & E^{\pm} f(x) \\ = \; & \lim_{\varepsilon \to 0+} (2\pi)^{-2} \int_{\mathbb{R}\times\mathbb{R}_+\times S^2} \frac{e^{-ip_0x_0+ir(x_1\cos\vartheta\cos\varphi+x_2\cos\vartheta\sin\varphi+x_3\sin\vartheta)}\check{f}(p_0,r,\vartheta,\varphi)}{m^2-p_0^2+r^2\mp ip_0\varepsilon} \cdot \\ & r^2\sin\vartheta d\varphi d\vartheta dr dp_0. \end{aligned}$$

Similar as in the proof of Thm. A.1.1 this equals

$$\int_{\mathbb{R}\times\mathbb{R}_+\times S^2} \partial_r \left(re^{-ip_0x_0+i\underline{p}(r,\Omega)\underline{x}}\check{f}(p_0,r,\Omega)\right) \left(\ln \left| m^2+r^2-p_0^2 \right| + \begin{cases} i\pi \\ 0 \end{cases} \right) d\Omega dr dp_0,$$

omitting the analytical meaningless prefactor $\frac{1}{2}(2\pi)^{-2}$.

$$\begin{aligned} & \partial_r \left(re^{-ip_0x_0+i\underline{p}(r,\Omega)\underline{x}}\check{f}(p_0,r,\Omega)\right) \\ = \; & e^{-ip_0x_0+i\underline{p}(r,\Omega)\underline{x}}\check{f}(p_0,r,\Omega) \\ & +re^{-ip_0x_0+i\underline{p}(r,\Omega)\underline{x}} i\left(x_1\cos\vartheta\cos\varphi+x_2\cos\vartheta\sin\varphi+x_3\sin\vartheta\right)\check{f}(p_0,r,\Omega) \\ & +re^{-ip_0x_0+i\underline{p}(r,\Omega)\underline{x}}\partial_r\check{f}(p_0,r,\Omega). \end{aligned}$$

With these three terms $E^{\pm}f(x)$ splits up into six integrals $\sum_{j=1}^{6} I_j$ that are going to be investigated separately now.

(a)

$$\begin{aligned} I_1 &= \int e^{-ip_0x_0+i\underline{p}(r,\Omega)\underline{x}}\check{f}(p_0,r,\Omega) i\pi d\Omega dr dp_0 \\ |I_1| &\leq \int \left|\check{f}(p_0,r,\Omega)\right| \pi d\varphi d\vartheta dr dp_0 < \infty. \end{aligned}$$

(b)

$$\begin{aligned} I_2 = & \int re^{-ip_0x_0+i\underline{p}(r,\Omega)\underline{x}} i\left(x_1\cos\vartheta\cos\varphi+x_2\cos\vartheta\sin\varphi+x_3\sin\vartheta\right) \cdot \\ & \check{f}(p_0,r,\Omega) i\pi d\Omega dr dp_0. \end{aligned}$$

Define $\check{g} := r\check{f}$, which is still $\in \mathscr{S}$. Then the Fourier transforms, i.e. the integrals w.r.t. p_0 and r can be carried out:

$$\begin{aligned} I_2 = & \int \left(-x_1\cos\vartheta\cos\varphi-x_2\cos\vartheta\sin\varphi-x_3\sin\vartheta\right) \cdot \\ & \tilde{g}(x_0, \underbrace{-x_1\cos\vartheta\cos\varphi-x_2\cos\vartheta\sin\varphi-x_3\sin\vartheta}_{=:z}, \Omega)\pi d\Omega, \end{aligned}$$

where $\tilde{g}$ is still Fourier transformed with respect to Ω. We can define $\tilde{h}(x_0, z, \Omega) := z\tilde{g}(x_0, z, \Omega)$, which is still Schwartz. Then of course

$$|I_2| \leq \int \left|\tilde{h}(x_0, z(\Omega, x_1, x_2, x_3), \Omega)\right| \pi d\Omega = \int H(\Omega)\pi d\Omega < \infty,$$

with $H(\Omega) := \sup_{x\in\mathbb{R}^4} |h(x_0, z(\Omega, x_1, x_2, x_3), \Omega)| < \infty \forall \Omega$.

(c)

$$\begin{aligned} I_3 &= \int r e^{-ip_0x_0 + i\underline{p}(r,\Omega)\underline{x}} \partial_r \check{f}(p_0, r, \Omega) i\pi d\Omega dr dp_0 \\ |I_3| &\leq \int r \left|\partial_r \check{f}(p_0, r, \Omega)\right| \pi d\varphi d\vartheta dr dp_0 < \infty. \end{aligned}$$

(d)

$$\begin{aligned} I_4 &= \int e^{-ip_0x_0 + i\underline{p}(r,\Omega)\underline{x}} \check{f}(p_0, r, \Omega) \ln\left|m^2 + r^2 - p_0^2\right| d\Omega dr dp_0 \\ |I_4| &\leq \int \left|\check{f}(p_0, r, \Omega)\right| (1 + r^2 + p_0^2)^{2\nu} \cdot \\ &\qquad (1 + r^2 + p_0^2)^{-2\nu} \left|\ln\left|m^2 + r^2 - p_0^2\right|\right| d\varphi d\vartheta dr dp_0 < \infty, \end{aligned}$$

since there exists a $\nu \in \mathbb{N}$ such that

$$\int (1 + r^2 + p_0^2)^{-2\nu} \left|\ln\left|m^2 + r^2 - p_0^2\right|\right| d\varphi d\vartheta dr dp_0 < \infty;$$

and $\sup_{p_0,r,\Omega} \left|\check{f}(p_0, r, \Omega)\right| (1 + r^2 + p_0^2)^{2\nu} < \infty$ due to $f \in \mathscr{S}$ (cf. proof of Thm. A.1.1).

(e)

$$\begin{aligned} I_5 &= \int r e^{-ip_0x_0 + i\underline{p}(r,\Omega)\underline{x}} i \left(x_1 \cos\vartheta \cos\varphi + x_2 \cos\vartheta \sin\varphi + x_3 \sin\vartheta\right) \cdot \\ &\qquad \check{f}(p_0, r, \Omega) \ln\left|m^2 + r^2 - p_0^2\right| d\Omega dr dp_0 \\ |I_5| &\leq \int r \sum_{j=1}^{3} |x_j| \left|\check{f}(p_0, r, \Omega)\right| \left|\ln\left|m^2 + r^2 - p_0^2\right|\right| d\varphi d\vartheta dr dp_0, \end{aligned}$$

where then again $(1 + r^2 + p_0^2)^{2\nu}(1 + r^2 + p_0^2)^{-2\nu}$ is inserted, yielding

$$|I_5| \leq C_1 \sum_{j=1}^{3} |x_j|,$$

with some constant $C_1 \in \mathbb{R}^+$.

(f)

$$\begin{aligned} I_6 &= \int re^{-ip_0x_0+i\underline{p}(r,\Omega)\underline{x}}\partial_r\check{f}(p_0,r,\Omega)\ln\left|m^2+r^2-p_0^2\right|d\Omega drdp_0 \\ |I_6| &\leq \int r\left|\partial_r\check{f}(p_0,r,\Omega)\right|\left|\ln\left|m^2+r^2-p_0^2\right|\right|d\varphi d\vartheta drdp_0 < \infty, \end{aligned}$$

again by the same argument.

Altogether we have

$$|E^{\pm}f(x)| \leq C + C_1\sum_{j=1}^{3}|x_j|.$$

At least this shows $\sup_{x_0\in\mathbb{R}}|E^{\pm}f(x)|$ is finite for every $\underline{x}=(x_1,x_2,x_3)$. For the spatial coordinates our estimate was actually to rough. Because [2] tells us that $E^{\pm}f$ is indeed spatially compact, i.e.

- $\exists K$ compact such that $\operatorname{supp} E^{\pm}f \subseteq J^{\pm}(K)$ and
- the global hyperbolicity of the spacetime

imply for every Cauchy surface $S\subset\mathbb{R}^4$ that $\operatorname{supp} E^{\pm}f|_S \subset S\cap J^{\pm}(K)$, which is compact according to Corollary A.5.4 in the book [2]. This implies

$$\sup_{x\in\mathbb{R}^4}|E^{\pm}f(x)| < \infty,$$

and generalizes easily to arbitrary dimensions $2\leq n\in\mathbb{N}$. □

Lemma A.2.2 *Again let $E^{\pm}:\mathscr{S}(\mathbb{R}^n)\to C^{\infty}(\mathbb{R}^n)$ be the advanced and retarded fundamental solutions of $\Box+m^2$, $m>0$, on n-dim. Minkowski spacetime like in Lemma A.2.1.*
Then it holds

$$\left(f,E^{\pm}g\right)_{L^2} < \infty,$$

even extended to functions $f,g\in L^2(\mathbb{R}^n)\cap C^{\infty}(\mathbb{R}^n)$.

Proof (only for $n=4$) Let $f,g\in L^2(\mathbb{R}^n)\cap C^{\infty}(\mathbb{R}^n)$. Explicitly

$$(f,E^{\pm}g)_{L^2} = \lim_{\varepsilon\to 0+}\int_{\mathbb{R}^n}\frac{\hat{\bar{f}}(p)\check{g}(p)}{m^2+\underline{p}^2-p_0^2\mp ip_0\varepsilon}d^np.$$

We make a case differentiation

(a) $|m^2+\underline{p}^2-p_0^2|\leq m^2$

(b) $|m^2+\underline{p}^2-p_0^2|> m^2$

and split the integral

$$\left|(f, E^{\pm} g)_{L^2}\right| \leq \lim_{\varepsilon \to 0+} \left(\int_{(a)} + \int_{(b)} \right) \frac{\left|\hat{\bar{f}}(p) \check{g}(p)\right|}{\left|m^2 + \underline{p}^2 - p_0^2 \mp i p_0 \varepsilon\right|} d^n p =: I_{(a)} + I_{(b)}.$$

Case (b) is the easier one. The Limit can be carried out immediately:

$$\begin{aligned} I_{(b)} &= \int_{(b)} \frac{\left|\hat{\bar{f}}(p) \check{g}(p)\right|}{\left|m^2 + \underline{p}^2 - p_0^2\right|} d^n p < m^{-2} \int_{(b)} \left|\hat{\bar{f}}(p) \check{g}(p)\right| d^n p \\ &\leq m^{-2} \int_{\mathbb{R}^n} \left|\hat{\bar{f}}(p) \check{g}(p)\right| d^n p \leq m^{-2} \|f\|_{L^2} \|g\|_{L^2} < \infty. \end{aligned}$$

The calculation for part (a) starts with some coordinate transformations/substitutions:

$$\begin{aligned} I_{(a)} &= \lim_{\varepsilon \to 0+} \int_{(a)} \frac{\left|\hat{\bar{f}}(p) \check{g}(p)\right|}{\sqrt{(m^2 + \underline{p}^2 - p_0^2)^2 + p_0^2 \varepsilon^2}} d^n p \\ &= \lim_{\varepsilon \to 0+} \int_{(a)} \frac{\left|\hat{\bar{f}}(p_0, r, \Omega) \check{g}(p_0, r, \Omega)\right| r^2}{\sqrt{(m^2 + r^2 - p_0^2)^2 + p_0^2 \varepsilon^2}} dr d\Omega dp_0 \\ &= \frac{1}{2} \lim_{\varepsilon \to 0+} \int_{(a)} \left|\hat{\bar{f}}(p_0, z, \Omega) \check{g}(p_0, z, \Omega)\right| \sqrt{\frac{z}{(m^2 + z - p_0^2)^2 + p_0^2 \varepsilon^2}} dz d\Omega dp_0 \\ &= \frac{1}{4} \lim_{\varepsilon \to 0+} \int_{(a)} \left|\hat{\bar{f}}(u, z, \Omega) \check{g}(u, z, \Omega)\right| \sqrt{\frac{z}{\left[(m^2 + z - u)^2 + u \varepsilon^2\right] u}} dz d\Omega du, \end{aligned}$$

with $\Omega \in [0, 4\pi]$, $u \in \mathbb{R}^+$, $z \in \mathbb{R}^+$, combined with the condition (a): $|m^2 + z - u| \leq m^2$. We disentangle (a) into two subcases:

(1) $m^2 + z - u \geq 0 \Leftrightarrow u \leq m^2 + z$. (a)$\Leftrightarrow$ $u \geq z$. Finally this implies $0 \leq z \leq u \leq m^2 + z$.

(2) $m^2 + z - u < 0 \Leftrightarrow u > m^2 + z$. (a)$\Leftrightarrow$ $u \leq 2m^2 + z$. Finally this implies $0 \leq m^2 + z < u \leq 2m^2 + z$.

This suggests to split the u-integration as follows:

$$\begin{aligned} I_{(a)} &= \frac{1}{4} \lim_{\varepsilon \to 0+} \int_0^{4\pi} \int_0^{\infty} \left(\int_z^{z+m^2} + \int_{z+m^2}^{z+2m^2} \right) \left|\hat{\bar{f}}(u, z, \Omega) \check{g}(u, z, \Omega)\right| \\ &\quad \sqrt{\frac{z}{\left[(m^2 + z - u)^2 + u \varepsilon^2\right] u}} du dz d\Omega. \end{aligned}$$

However, one has to take care of the singularity at $u \to m^2 + z$ before sending ε to 0. Therefore it is necessary to introduce a $\delta > 0$ sufficiently small, in fact $\delta < m^2/2$, to proceed with

$$\begin{aligned} I_{(\mathrm{a})} &= \frac{1}{4}\int_0^{4\pi}\int_0^{\infty}\left(\int_z^{z+m^2-\delta} + \int_{z+m^2+\delta}^{z+2m^2} + \fint_{z+m^2-\delta}^{z+m^2+\delta}\right)\left|\hat{\bar{f}}(u,z,\Omega)\check{g}(u,z,\Omega)\right| \\ &\quad \frac{1}{|m^2+z-u|}\sqrt{\frac{z}{u}}dudzd\Omega =: \frac{1}{4}\int_0^{4\pi}\int_0^{\infty}(I_{(\mathrm{a},1)} + I_{(\mathrm{a},2)} + I_{(\mathrm{a},3)})dzd\Omega, \end{aligned}$$

and define

$$T(z,u) := \frac{1}{|m^2+z-u|}\sqrt{\frac{z}{u}}.$$

The most critical integral is $I_{(\mathrm{a},3)}$. We will see, that its Cauchy principal value exists and hence justifies the limit $\varepsilon \to 0+$. But let's investigate the integrals one after another.

- $I_{(\mathrm{a},1)}$:

$$\begin{aligned} I_{(\mathrm{a},1)} &= \int_z^{z+m^2-\delta}\left|\hat{\bar{f}}(u,z,\Omega)\check{g}(u,z,\Omega)\right| T(z,u)du \\ &= \int_z^{z+m^2-\delta}\left|\hat{\bar{f}}(u,z,\Omega)\check{g}(u,z,\Omega)\right| duT(z,\xi_1), \end{aligned}$$

  applying the mean value theorem for integration with $\xi_1 \in [z, z+m^2-\delta]$. Then even for all $z \in [0,\infty)$ it holds

$$T(z,\xi_1) = \frac{1}{|m^2+z-\xi_1|}\sqrt{\frac{z}{\xi_1}} \leq C_1.$$

  And of course the u-integration in $I_{(\mathrm{a},1)}$ converges, leaving some L^1-function in z and Ω.

- $I_{(\mathrm{a},2)}$:

$$\begin{aligned} I_{(\mathrm{a},2)} &= \int_{z+m^2+\delta}^{z+2m^2}\left|\hat{\bar{f}}(u,z,\Omega)\check{g}(u,z,\Omega)\right| T(z,u)du \\ &= \int_{z+m^2+\delta}^{z+2m^2}\left|\hat{\bar{f}}(u,z,\Omega)\check{g}(u,z,\Omega)\right| duT(z,\xi_2), \end{aligned}$$

  with $\xi_2 \in [z+m^2+\delta, z+2m^2]$. Then even for all $z \in [0,\infty)$ it holds

$$T(z,\xi_1) = \frac{1}{|m^2+z-\xi_2|}\sqrt{\frac{z}{\xi_2}} \leq C_2.$$

  And again the u-integration in $I_{(\mathrm{a},2)}$ converges, leaving some L^1-function in z and Ω.

- $I_{(\mathrm{a},3)}$:
  Here we have to start the calculation from scratch and postpone the application of the absolute value to a later step. I.e. the starting point is

$$\tilde{I}_{(\mathrm{a})} = \lim_{\varepsilon \to 0+} \int_{(\mathrm{a})} \frac{\hat{\hat{f}}(p)\check{g}(p)}{m^2 + \underline{p}^2 - p_0^2 \mp ip_0\varepsilon} d^n p.$$

Then all the steps are the same, till we arrive at

$$\begin{aligned}
\tilde{I}_{(\mathrm{a},3)} &= \lim_{\varepsilon \to 0+} \int_{z+m^2-\delta}^{z+m^2+\delta} \frac{\hat{\hat{f}}(u,z,\Omega)\check{g}(u,z,\Omega)}{m^2 + z - u \mp i\sqrt{u}\varepsilon} \sqrt{\frac{z}{u}} du \\
&= ⨍_{z+m^2-\delta}^{z+m^2+\delta} \frac{\hat{\hat{f}}(u,z,\Omega)\check{g}(u,z,\Omega)}{m^2 + z - u} \sqrt{\frac{z}{u}} du \\
&= ⨍_{z+m^2-\delta}^{z+m^2+\delta} \frac{1}{m^2 + z - u} \sqrt{\frac{z}{u}} du \hat{\hat{f}}(\xi_3, z, \Omega)\check{g}(\xi_3, z, \Omega),
\end{aligned}$$

again applying the mean value theorem (just for a different factor in the integral), with $\xi_3 \in [z + m^2 - \delta, z + m^2 + \delta]$. Now the Cauchy principal value is computed:

$$\begin{aligned}
&⨍_{z+m^2-\delta}^{z+m^2+\delta} \frac{1}{m^2 + z - u} \sqrt{\frac{z}{u}} du \\
&= \lim_{\eta \to 0} \left(\int_{z+m^2-\delta}^{z+m^2-\eta} + \int_{z+m^2+\eta}^{z+m^2+\delta} \right) \frac{1}{m^2 + z - u} \sqrt{\frac{z}{u}} du \\
&= \lim_{\eta \to 0} \left(\left(-\operatorname{arcoth}\sqrt{1 - \frac{\eta}{m^2 + z}} - \operatorname{artanh}\sqrt{\frac{m^2 + z}{z + m^2 - \delta}} \right) \right. \\
&\quad \left. + \left(-\operatorname{artanh}\sqrt{\frac{m^2 + z}{z + m^2 + \delta}} + \operatorname{artanh}\sqrt{\frac{m^2 + z}{\eta + m^2 + z}} \right) \right) \frac{2z}{\sqrt{(m^2 + z)z}}.
\end{aligned}$$

This simplifies to

$$\begin{aligned}
= \frac{z}{\sqrt{(m^2 + z)z}} &\left(2\operatorname{artanh}\sqrt{\frac{m^2 + z}{z + m^2 - \delta}} - 2\operatorname{artanh}\sqrt{\frac{m^2 + z}{z + m^2 + \delta}} \right. \\
&\left. + \ln\left(-\frac{1}{m^2 + z} \right) - \ln\left(\frac{1}{m^2 + z} \right) \right).
\end{aligned}$$

Investigating the terms for $z \in [0, \infty)$ one by one:

(i) $\frac{z}{\sqrt{(m^2+z)z}} = \frac{1}{\sqrt{m^2/z+1}} \to 1$ for $z \to \infty$. For $z \to 0$ l'Hospital helps: $\frac{1}{\frac{m^2+2z}{2\sqrt{(m^2+z)z}}} = \frac{2\sqrt{(m^2+z)z}}{m^2+2z} \to 0$ for $z \to 0$. Continuity and positivity then imply

$$0 \leq \frac{z}{\sqrt{(m^2 + z)z}} \leq C_3.$$

(ii) $a_1 := \frac{m^2+z}{z+m^2-\delta} = \frac{1}{1-\frac{\delta}{z+m^2}} \in (1,2)$, since $\frac{\delta}{z+m^2} \in (0,\frac{1}{2})$, since $z \in [0,\infty)$ and $0<\delta<m^2/2$. And $a_2 := \frac{m^2+z}{z+m^2+\delta} = \frac{1}{1+\frac{\delta}{z+m^2}} \in (\frac{2}{3},1)$. Hence

$$0 < \tilde{C}_4 \leq 2\mathrm{artanh}\sqrt{a_1} - 2\mathrm{artanh}\sqrt{a_2} \leq C_4.$$

(iii) Set $l := \frac{1}{m^2+z}$. $\ln(-l) - \ln l = \ln|-l| + i\arg(-l) - \ln|l| - i\arg(l)$, which equals just $i\pi$.

Altogether

$$I_{(\mathrm{a},3)} = \left|\tilde{I}_{(\mathrm{a},3)}\right| \leq \left|\hat{\tilde{f}}(\xi_3,z,\Omega)\check{g}(\xi_3,z,\Omega)\right| C_3(C_4+\pi).$$

Putting it all together

$$\begin{aligned} I_{(\mathrm{a})} &= \frac{1}{4}\int_0^{4\pi}\int_0^\infty (I_{(\mathrm{a},1)} + I_{(\mathrm{a},2)} + I_{(\mathrm{a},3)}) dz d\Omega \\ &\leq \frac{1}{4}\int_0^{4\pi}\int_0^\infty (C_1+C_2+C_3(C_4+\pi)) \int_0^\infty \left|\hat{\tilde{f}}(u,z,\Omega)\check{g}(u,z,\Omega)\right| du dz d\Omega \\ &< \infty. \end{aligned}$$

□

A.2.2 Dirac case

The results of the last subsection can be transferred to the Dirac case as well.

Lemma A.2.3 *Let $R^\pm : \mathscr{S}(\mathbb{R}^n,\mathbb{C}^N) \to C^\infty(\mathbb{R}^n,\mathbb{C}^N)$ be the advanced and retarded fundamental solutions of $-i\not\partial + m$ on n-dim. Minkowski spacetime like in Thm. A.1.2, defined on $\mathscr{S}$-functions.*
Then it holds

$$\sup_{x\in\mathbb{R}^n}\left|(R^\pm f)^A(x)\right| < \infty.$$

Proof (only for $n=4$) The strategy is exactly the same as for the Klein-Gordon case in Lemma A.2.1, hence we will only give a rough sketch. Starting point is

$$\begin{aligned} (R^\pm f)^A(x) &= \int_{\mathbb{R}^n} R^{\pm A}{}_B(x,y) f^B(y) d^n y \\ &= \lim_{\varepsilon\to 0+}(2\pi)^{-2}\int_{\mathbb{R}^n}\frac{e^{-ip[x]}(\not p + m)^A{}_B \check{f}^B(p)}{m^2-p^2\mp ip_0\varepsilon} d^n p. \end{aligned}$$

The only differences to Lemma A.2.1 are an additional term

$$re^{-ip_0x_0+i\underline{p}(r,\Omega)\underline{x}}\left(\gamma_1\cos\vartheta\cos\varphi + \gamma_2\cos\vartheta\sin\varphi + \gamma_3\sin\vartheta\right)\check{f}(p_0,r,\Omega)$$

in the derivative $\partial_r \left(re^{-ip_0x_0+i\underline{p}(r,\Omega)\underline{x}} (\not{p}+m)\check{f}(p_0,r,\Omega)\right)$ and the factor $(\not{p}+m)$, which can always be estimated by $C_2(|p_0|+r+1)$. Finally we get

$$|(R^{\pm}f)^A(x)| \leq C + C_1 \sum_{j=1}^{3} |x_j|,$$

which then again together with the argument from the book [2] proves the Lemma. $\square$

Lemma A.2.4 *Again let $R^{\pm} : \mathscr{S}(\mathbb{R}^n, \mathbb{C}^N) \to C^{\infty}(\mathbb{R}^n, \mathbb{C}^N)$ be the advanced and retarded fundamental solutions of $-i\not{\partial}+m$, $m>0$, on n-dim. Minkowski spacetime like in Lemma A.2.3.*
Then it holds

$$\left(f, R^{\pm}g\right)_{L^2} < \infty$$

even extended to functions $f, g \in L^2(\mathbb{R}^n) \cap C^{\infty}(\mathbb{R}^n)$.

Proof (only for $n=4$) Let $f, g \in L^2(\mathbb{R}^n) \cap C^{\infty}(\mathbb{R}^n)$. Explicitly

$$(f, R^{\pm}g)_{L^2} = \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^n} \frac{\hat{\bar{f}}^A(p)(\not{p}+m)_{AB}\check{g}^B(p)}{m^2-p^2 \mp ip_0\varepsilon} d^np.$$

In other words, the connection to the Klein-Gordon case is given by $(f, R^{\pm}g)_{L^2} = (f, (i\not{\partial}+m)E^{\pm}g)_{L^2} = ((i\gamma_0\partial^0 - i\gamma_k\partial^k + m)f, E^{\pm}g)_{L^2}$, note $i\not{\partial} = i\gamma_0\partial^0 + i\gamma_k\partial^k$. Since $(i\gamma_0\partial^0 - i\gamma_k\partial^k + m)f$ is again in $L^2(\mathbb{R}^n) \cap C^{\infty}(\mathbb{R}^n)$ Lemma A.2.2 can be applied (componentwise). $\square$

A.3 Proof of the technicalities of Theorem 2.4.7

The context is given within the abovementioned theorem and its proof.

Proposition A.3.1 *For the operator G_c defined in* (2.27) *it holds*

- *G_c is a well-defined operator $G_c : L^2(\mathbb{R}^s, \mathbb{C}^N) \to L^2(\mathbb{R}^s, \mathbb{C}^N)$*
- *G_c is Hilbert-Schmidt (w.r.t. the Hilbert space $L^2(\mathbb{R}^s, \mathbb{C}^N)$).*

Proof

(a) *Preliminaries.* The proof works along the same lines as the one of Prop. 7 in [6]. With the help of Fourier transforms $\mathcal{F}^{-1}\mathcal{F}$ and smooth unitary matrices $\mathrm{U}(\underline{k})^{-1}\mathrm{U}(\underline{k})$ the operator expression (2.27)

$$(G_c\chi)(\underline{x}) := \int_{\mathbb{R}} p_+ e^{-iH_0t}\gamma_0(c \star \phi_{p_-\chi} \star c)(t, \underline{x})dt$$

is brought into a more conveniently calculable shape. Here the Fourier transform $\mathcal{F} := \mathcal{F}_s$ for $\xi \in L^2(\mathbb{R}^s, \mathbb{C}^N)$ is given by

$$\hat{\xi}(\underline{k}) := (\mathcal{F}\xi)(\underline{k}) := (\mathcal{F}_s\xi)(\underline{k}) := (2\pi)^{-s/2} \int_{\mathbb{R}^s} e^{-i\underline{k}\underline{x}}\xi(\underline{x})d^s\underline{x}. \tag{A.1}$$

$\mathrm{U}(\underline{k})$ is a family of smooth unitary matrices that exists due to properties of H_0, that diagonalizes its Fourier transform $\hat{H}_0(\underline{k}) = \mathcal{F}H_0\mathcal{F}^{-1}(\underline{k})$. As you know $\hat{H}_0(\underline{k})^*\hat{H}_0(\underline{k}) = |\hat{H}_0(\underline{k})|^2 \mathbb{1}_{N\times N}$ holds with $|\hat{H}_0(\underline{k})| = \sqrt{\underline{k}^2 + m^2}$. The diagonalization reads

$$\mathrm{U}(\underline{k})\hat{H}_0(\underline{k})\mathrm{U}(\underline{k})^{-1} = \begin{pmatrix} -|\hat{H}_0(\underline{k})| & 0 \\ 0 & |\hat{H}_0(\underline{k})| \end{pmatrix}.$$

For the Fourier transforms of $p_\pm$, i.e. $\mathcal{F}p_\pm\mathcal{F}^{-1}(\underline{k}) = \hat{p}_\pm(\underline{k})$, we have the properties

$$\mathrm{U}(\underline{k})\hat{p}_+(\underline{k})\mathrm{U}(\underline{k})^{-1} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} =: P_+, \mathrm{U}(\underline{k})\hat{p}_-(\underline{k})\mathrm{U}(\underline{k})^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} =: P_-,$$

with $\mathbb{1} = \mathbb{1}_{N/2\times N/2}$. Then

$$\begin{aligned} \mathrm{U}(\underline{k})e^{i\hat{H}_0(\underline{k})t}\hat{p}_+(\underline{k})\mathrm{U}(\underline{k})^{-1} &= e^{-i|\hat{H}_0(\underline{k})|t}P_+, \\ \mathrm{U}(\underline{k})\hat{p}_-(\underline{k})e^{-i\hat{H}_0(\underline{k})t}\mathrm{U}(\underline{k})^{-1} &= e^{-i|\hat{H}_0(\underline{k})|t}P_-. \end{aligned}$$

During later calculations within the expression of G_c we will implicitly insert some $\mathrm{U}^{-1}\mathrm{U}$ to make use of the above relations, without explicitly mentioning them again.

Dealing with the Moyal product on fully n-dimensional spacetime we will also need the following notation of the Fourier transform $\mathcal{F}_n$ for functions $f \in L^2(\mathbb{R}^n, \mathbb{C}^N)$:

$$\tilde{f}(k) := (\mathcal{F}_n f)(k) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ikx} f(x)d^n x. \tag{A.2}$$

Note that the scalar product in the phase factor is always the Euclidean scalar product. Note also that underscored letters like $\underline{k}$ will be used throughout to denote elements in $\mathbb{R}^s$, while those lacking an underscore denote elements in $\mathbb{R}^n$. We will also write $(k_0, \underline{k}) = k$ as is customary, understanding that $k_0 \in \mathbb{R}$ and $\underline{k} \in \mathbb{R}^s$. A further convention will be used to avoid having to write down factors of $(2\pi)^{-\sharp/2}$ in the integrals: we will denote arbitrary factors in that sense by the same symbol C_π.

(b) *Fourier integral expression for $c \star g \star c$.* Let $g \in \mathscr{S}(\mathbb{R}^n, \mathbb{C}^N)$. Then we have that $\mathcal{F}_n(c \star g \star c) = \mathcal{F}_n L_c R_c \mathcal{F}_n^{-1}\tilde{g}$, where L_c/R_c denotes left/right multiplication by c. On iterating the Fourier integral expression for Moyal multiplication (cf. equations (75), (76) in [6]), one obtains

$$\mathcal{F}_n L_c R_c \mathcal{F}_n^{-1} \tilde{g}(y) = C_\pi \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{ikM(y-w)} \tilde{c}(y-k)\tilde{c}(k-w)\tilde{g}(w) d^n w d^n k.$$

Carrying out the k-integration yields for this expression, up to a constant factor, the convolution

$$\left[\left(\mathcal{F}_n^{-1} (\sigma \tilde{c})_y \right) * \left(\mathcal{F}_n^{-1} \tilde{c}_w \right) \right] (M(y-w)),$$

where we have used

$$(\sigma h)(q) := h(-q), \quad h_y(q) := h(q-y).$$

Using the antisymmetry of M, the last expression becomes equal to

$$\begin{aligned} & C_\pi \iint_{\mathbb{R}^n \times \mathbb{R}^n} e^{iyMw} c(u + M(y-w)) c(u) e^{-iu(y-w)} \tilde{g}(w) d^n u d^n w \\ &= C_\pi \int_{\mathbb{R}^n} e^{iyMw} \psi_c(y-w) \tilde{g}(w) d^n w, \end{aligned}$$

where

$$\psi_c(v) := \int_{\mathbb{R}^n} c(u + Mv) c(u) e^{-iuv} d^n u$$

is in $\mathscr{S}(\mathbb{R}^n, \mathbb{C})$.

In summary, we have found: There is a function $\psi_c \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$ so that

$$(\mathcal{F}_n (c \star g \star c))(y) = C_\pi \int_{\mathbb{R}^n} e^{iyMw} \psi_c(y-w) \tilde{g}(w) d^n w \tag{A.3}$$

for all $g \in \mathscr{S}(\mathbb{R}^n, \mathbb{C}^N)$. Since $\psi_c \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$, one can clearly extend this to a greater set of g, even certain distributions, and we will use this in the next steps.

(c) *Inserting $\phi_{p_-\chi}$ for g.* It is not difficult to see that $\phi_{p_-\chi}$, defined in (2.26), is C^∞ jointly in t and $\underline{x}$. This follows from the following observations and usage of the Sobolev lemma:

- H_0 is elliptic
- $p_-\chi$ is in the C^∞-domain of H_0
- $\phi_{p_-\chi}$ is in the C^∞-domain of $\partial_t^2 + H_0^* H_0$ and hence in the C^∞-domain of $\triangle_{\mathbb{R}^n}$.

We note that $\mathcal{F}_n \phi_{p_-\chi} = \mathcal{F}_0 \mathcal{F}_s \phi_{p_-\chi}$, where $\mathcal{F}_0$ is the Fourier transform with respect to t. One has

$$\begin{aligned} \left(\mathcal{F}_s \phi_{p_-\chi} \right) (t, \underline{k}) &= \mathcal{F}_s \left(e^{iH_0 t} p_- \chi \right) (\underline{k}) = \left(\mathcal{F}_s e^{-i|H_0|t} \mathcal{F}_s^{-1} \widehat{p_-\chi} \right) (\underline{k}) \\ &= \left(e^{-i|\hat{H}_0(\underline{k})|t} \widehat{p_-\chi} \right) (\underline{k}), \end{aligned}$$

using that $H_0 p_- = -|H_0| p_-$. Performing now also the Fourier transform w.r.t. t, one obtains

$$\begin{aligned}\left(\mathcal{F}_n \phi_{p_-\chi}\right)(k_0, \underline{k}) &= \left(\mathcal{F}_0 \mathcal{F}_s \phi_{p_-\chi}\right)(k_0, \underline{k}) = C_\pi \int_{\mathbb{R}} e^{-ik_0 t} e^{-i|\hat{H}_0(\underline{k})|t} \widehat{p_-\chi}(\underline{k}) dt \\ &= C_\pi \delta(-(k_0 + |\hat{H}_0(\underline{k})|)) \widehat{p_-\chi}(\underline{k}),\end{aligned}$$

which is a distribution in $\mathscr{S}'(\mathbb{R}^n, \mathbb{C}^N)$.

Now we can insert $\mathcal{F}_n \phi_{p_-\chi}$ for $\tilde{g}$ in the expression obtained in (A.3) for the Fourier transform of $c \star g \star c$. This yields that

$$\begin{aligned}&\left(\mathcal{F}_n(c \star \phi_{p_-\chi} \star c)\right)(k_0, \underline{k}) \\ &= C_\pi \int_{\mathbb{R}^n} e^{ikMw} \psi_c(k_0 - w_0, \underline{k} - \underline{w}) \delta(-(w_0 + |\hat{H}_0(\underline{w})|)) \widehat{p_-\chi}(\underline{w}) d^n w \\ &= C_\pi \int_{\mathbb{R}^s} e^{i(k_0, \underline{k})M\left(-|\hat{H}_0(\underline{w})|, \underline{w}\right)} \psi_c(k_0 + |\hat{H}_0(\underline{w})|, \underline{k} - \underline{w}) \widehat{p_-\chi}(\underline{w}) d^s \underline{w}.\end{aligned}$$

It is easy to see that the function is in $C^\infty(\mathbb{R}^n, \mathbb{C}^N)$ and also bounded. Moreover, for each fixed k_0, $\underline{k} \mapsto \left(\mathcal{F}_n(c \star \phi_{p_-\chi} \star c)\right)(k_0, \underline{k})$ is in $L^2(\mathbb{R}^s, \mathbb{C}^N)$.

Now we can form $\left(\mathcal{F}_s(c \star \phi_{p_-\chi} \star c)\right)(t, \underline{k})$ by forming

$$\left(\mathcal{F}_0^{-1} \mathcal{F}_n(c \star \phi_{p_-\chi} \star c)\right)(t, \underline{k}) = C_\pi \int_{\mathbb{R}} e^{ik_0 t} \left(\mathcal{F}_n(c \star \phi_{p_-\chi} \star c)\right)(k_0, \underline{k}) dk_0.$$

The result is of the form

$$C_\pi \int_{\mathbb{R}^s} e^{iQ(\underline{k}, \underline{w})} (\mathcal{F}_0 \psi_c)(t + R(\underline{w}), \underline{k} - \underline{w}) e^{i|\hat{H}_0(\underline{w})|t} \widehat{p_-\chi}(\underline{w}) d^s \underline{w},$$

where $Q(\underline{k}, \underline{w})$ is C^∞ and $R(\underline{w})$ is C^∞ and the latter is of order $|\underline{w}|$ for large $|\underline{w}|$. This shows that for fixed t, $\underline{k} \mapsto \left(\mathcal{F}_s(c \star \phi_{p_-\chi} \star c)\right)(t, \underline{k})$ is in $L^2(\mathbb{R}^s, \mathbb{C}^N)$ and therefore, $\underline{x} \mapsto (c \star \phi_{p_-\chi} \star c)(t, \underline{x})$ is in $L^2(\mathbb{R}^s, \mathbb{C}^N)$. Thus, $p_+ e^{-iH_0 t}$ can be applied on $(c \star \phi_{p_-\chi} \star c)(t, \cdot)$.

(d) *Final analysis of G_c.* The next step is to form (2.27)

$$(G_c \chi)(\underline{x}) = \int_{\mathbb{R}} p_+ e^{-iH_0 t} \gamma_0 (c \star \phi_{p_-\chi} \star c)(t, \underline{x}) dt.$$

We will instead try to inspect

$$(\mathcal{F}_s G_c \chi)(\underline{k}) = C_\pi \int_{\mathbb{R}} \mathcal{F}_s p_+ e^{-iH_0 t} \mathcal{F}_s^{-1} \gamma_0 \mathcal{F}_s (c \star \phi_{p_-\chi} \star c)(t, \underline{k}) dt.$$

One has

$$\left(\mathcal{F}_s p_+ e^{-iH_0 t} g\right)(\underline{k}) = \left(\mathcal{F}_s p_+ e^{-i|H_0|t} \mathcal{F}_s^{-1} \hat{g}\right)(\underline{k}) = \hat{p}_+(\underline{k}) e^{-i|\hat{H}_0(\underline{k})|t} \hat{g}(\underline{k}),$$

for all $g \in L^2(\mathbb{R}^s, \mathbb{C}^N)$ with some bounded, continuous, matrix-valued function $\hat{p}_+(\underline{k})$. Hence we obtain[1]

$$\begin{aligned}(\mathcal{F}_s G_c \chi)(\underline{k}) &= C_\pi \int_{\mathbb{R}} \hat{p}_+(\underline{k}) e^{-i|\hat{H}_0(\underline{k})|t} \gamma_0 \mathcal{F}_s(c \star \phi_{p_-\chi} \star c)(t, \underline{k}) dt \\ &= C_\pi \hat{p}_+(\underline{k}) \gamma_0 \mathcal{F}_n(c \star \phi_{p_-\chi} \star c)(|\hat{H}_0(\underline{k})|, \underline{k}) \\ &= \hat{p}_+(\underline{k}) \gamma_0 C_\pi \int_{\mathbb{R}^s} e^{i(|\hat{H}_0(\underline{k})|, \underline{k}) M\left(-|\hat{H}_0(\underline{w})|, \underline{w}\right)} \psi_c \left(|\hat{H}_0(\underline{k})| + |\hat{H}_0(\underline{w})|, \underline{k} - \underline{w}\right) \\ &\quad \cdot \hat{p}_-(\underline{w}) \hat{\chi}(\underline{w}) d^s \underline{w},\end{aligned}$$

where similarly as above, we have used that $\widehat{p_-\chi}(\underline{w}) = \hat{p}_-(\underline{w})\hat{\chi}(\underline{w})$ with a bounded, continuous, matrix-valued function $\hat{p}_-(\underline{w})$.

Now the important observation is: since

- $\psi_c \in \mathscr{S}(\mathbb{R}^n, \mathbb{C})$
- $|\hat{H}_0(\underline{k})| > 0$ and $|\hat{H}_0(\underline{k})|$ being of order $|\underline{k}|$ for large $|\underline{k}|$,

the function $(\underline{k}, \underline{w}) \mapsto \psi_c \left(|\hat{H}_0(\underline{k})| + |\hat{H}_0(\underline{w})|, \underline{k} - \underline{w}\right)$ is in $L^2(\mathbb{R}^s \times \mathbb{R}^s, \mathbb{C})$. And consequently,

$$\mathcal{F}_s G_c \mathcal{F}_s^{-1} \hat{\chi}(\underline{k}) = \int_{\mathbb{R}^s} K(\underline{k}, \underline{w}) \hat{\chi}(\underline{w}) d^s \underline{w},$$

with $K \in L^2(\mathbb{R}^s \times \mathbb{R}^s, \mathbb{C}^{N\times N})$. However, this shows that $\mathcal{F}_s G_c \mathcal{F}_s^{-1}$ is Hilbert-Schmidt on $L^2(\mathbb{R}^s, \mathbb{C}^N)$, which in turn shows (even though somewhat indirectly) that G_c is well-defined as an operator on $L^2(\mathbb{R}^s, \mathbb{C}^N)$, and also that G_c is Hilbert-Schmidt. $\square$

[1] **Remark** Actually, the Fourier integral of the following computation exists only weakly in t – one would actually have to insert a sequence of $\mathscr{S}(\mathbb{R})$-testfunctions $j_n(t)$ with $j_n \nearrow 1$ as $n \to \infty$ and control the limit, showing that the last integral of the computation is obtained in the limit.

Appendix B

Notation

B.1 Abbreviations

QFT	Quantum field theory
nc.	non-commutative

B.2 On index-placement

				from alphabet-
space-time coordinates $0 \dots 3$	greek	lower-case	letters	-middle
space-time coordinates $1 \dots 3$	latin	lower-case	letters	-middle
abstract tensor indices	latin	lower-case	letters	-beginning
spinor indices $1 \dots 4$	latin	upper-case	letters	-beginning

Summation convention is used throughout (pairwise occurring indices, one in upper and one in lower position, are implicitly summed over).

B.3 Symbols

$L(X,Y)$	Set of linear maps (=:operators) between vector spaces X and Y, $L(X) := L(X,X)$
$\mathcal{B}(X,Y)$	Set of bounded operators between normed spaces X, Y
$\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$	Self-adjoint operators subset $\mathcal{B}(\mathscr{H})$, $\mathscr{H}$ Hilbert space
$C(A,B)$	Set of continuous maps between topological spaces A and B
$C_b(A,B)$	Set of continuous and bounded maps between metric spaces A, B
$C^\infty(A,B)$	Set of infinitely-times continuously differentiable maps between normed spaces A, B
$C_b^\infty(A,B)$	$C^\infty(A,B)$ and bounded
$C_c^\infty(A,B)$	$C^\infty(A,B)$ with compact support (“test-functions”)

$L^p(A,B)$	Equivalence class of measurable maps f, for which $\lvert f\rvert^p$ $(1 \le p < \infty)$ is integrable w.r.t. the Lebesgue-measure (A, B measure spaces)
$L^\infty(A,B)$	Equivalence class of essentially (almost everywhere) bounded maps
$\mathscr{S}(\mathbb{R}^n, \mathbb{C}^N)$	Set of Schwartz-functions from $\mathbb{R}^n$ to $\mathbb{C}^N$
$\lVert \cdot \rVert_{\mathrm{op}}$	Some operator norm, specifically defined by the surrounding context

As a rule, above sets are considered as normed spaces (equipped with usual norms). Other symbols:

$\mathring{K}$	Interior of a subset K of a topological space
$\overline{K}$	Closure of K
∂K	Boundary of K
$\bar{f}$	Complex conjugation of a $\mathbb{C}^N$-valued function
$\operatorname{supp} f$	Support, i.e. closure of the set of points at which f (e.g. as map between topological vector spaces) doesn't vanish
$\mathbb{K}^{m\times n}$	Set of $m \times n$-matrices with elements out of field $\mathbb{K}$

Remark

(a) For every sesquilinear form, linearity is demanded for the 2nd component.

(b) Integrals without explicit specification of the domain of integration are considered as stretched over the whole domain of the integrand.

(c) If marked by anything at all, spacial $\mathbb{R}^{n-1}$-vectors get an underscore: $\underline{x}$.

(d) For the Dirac **delta-distribution** $\delta_y \in (C_c^\infty)'$, defined by $\delta_y f := f(y)$, more often than not we use the usual physicist-notation

$$\int f(x)\delta(x-y)dx := f(y),$$

"$\delta(y-x) = \delta(x-y)$". The discrete Kronecker-delta is declared by $(m, n \in \mathbb{Z})$

$$\delta_{mn} := \begin{cases} 0 & : m \neq n \\ 1 & : m = n \end{cases}.$$

Bibliography

[1] Araki, H., *Bogoliubov automorphisms and Fock representations of canonical anticommutation relations*, Am. Math. Soc. **62** (1987) 23

[2] Bär, C., Ginoux, N., Pfäffle, F., *Wave Equations on Lorentzian Manifolds and Quantization*, ESI Lectures in Mathematics and Physics, European Mathematical Society, Zürich, 2007

[3] Bahns, D., Waldmann, S., *Locally noncommutative spacetimes*, Rev. Math. Phys. **19** (2007) 273

[4] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D., *Deformation theory and quantization*, Ann. Phys. **111** (1978) 61–151

[5] Bogoliubov, N.N., Logunov, A.A., Todorov, I.T., *Introduction to Axiomatic Quantum Field Theory*, Benjamin, 1975

[6] Borris, M., Verch, R., *Dirac field on Moyal-Minkowski spacetime and non-commutative potential scattering*, Commun. Math. Phys. **293** (2010) 399

[7] Bratteli, O., Robinson, D.W., *Operator Algebras and Quantum Statistical Mechanics vol. 1*, Springer, 2002

[8] Bratteli, O., Robinson, D.W., *Operator Algebras and Quantum Statistical Mechanics vol. 2*, Springer, 2002

[9] Bruus, H., Flensberg, K., *Many-Body Quantum Theory in Condensed Matter Physics: An Introduction*, Oxford University Press, USA, 2004

[10] Buchholz, D., Lechner, G., Summers, S.J., *Warped convolutions, Rieffel deformations and the construction of quantum field theories*, arXiv:1005.2656 [math-ph] (2010)

[11] Connes, A., *Noncommutative Geometry*, Academic Press, 1994

[12] Coquereaux, R., *Spinors, reflections and Clifford algebras: A Review*, in: Spinors in Physics and Geometry (Trieste, 1986), 135-190, World Scientific, Singapore, 1988

[13] Dimock, J., *Dirac quantum fields on a manifold*, Trans. Am. Math. Soc. **269** (1982) 133

[14] Dito, G., *Kontsevich star product on the dual of a Lie algebra*, Lett. Math. Phys. **48, 4** (1999) 307–322

[15] Doplicher, S., Fredenhagen, K., Roberts, J.E., *The Quantum structure of space-time at the Planck scale and quantum fields*, Commun. Math. Phys. **172** (1995) 187

[16] Filk, T., *Divergencies in a field theory on quantum space*, Phys. Lett. B **376** (1996) 53

[17] Gayral, V., Gracia-Bondía, J., Iochum, B., Schücker, T., Várilly, J.C., *Moyal planes are spectral triples*, Commun. Math. Phys. **246** (2004) 569

[18] Gomis, J., Mehen, T., *Space-time noncommutative field theories and unitarity*, Nucl. Phys. B **591** (2000) 265

[19] Gracia-Bondía, J.M., Várilly, J.C., *Algebras of distributions suitable for phase-space quantum mechanics 1*, J. Math. Phys. **29** (1988) 869–879

[20] Gracia-Bondía, J.M., Várilly, J.C., *On the ultraviolet behavior of quantum fields over noncommutative manifolds*, Int. J. Mod. Phys. **A14** (1999) 1305

[21] Greiner, W., Reinhardt, J., *Field Quantization*, Springer, 1996

[22] Groenewold, H.J., *On the Principles of elementary quantum mechanics*, Physica **12** (1946) 405–460

[23] Grosse, H., Lechner, G., *Noncommutative deformations of Wightman quantum field theories*, JHEP 0809 (2008) 131

[24] Hoermander, L., *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*, Springer-Verlag, Berlin, 1994

[25] Kontsevich, M., *Deformation quantization of Poisson manifolds, I*, Lett. Math. Phys. **66** (2003) 157–216

[26] Lukierski, J., Ruegg, H., Zakrzewski, W.J., *Classical quantum mechanics of free kappa relativistic systems*, Annals Phys. **243** (1995) 90

[27] Madore, J., Schraml, S., Schupp, P., Wess, J., *Gauge theory on noncommutative spaces*, Eur. Phys. J. **C 16** (2000) 161

[28] Majid, S., Ruegg, H., *Bicrossproduct structure of kappa Poincare group and non-commutative geometry*, Phys. Lett. B **334** (1994) 348

[29] Manin, Yu.I., *Quantum groups and non-commutative geometry*, Commun. Math. Phys. **123** (1989) 163

[30] Moyal, J.E., *Quantum mechanics as a statistical theory*, Proc. of the Cambridge Phil. Soc. **45** (1949) 99–124

[31] Rieffel, M., *Deformation Quantization for Actions of* $\mathbb{R}^d$, Memoirs AMS **106**.506 (1993)

[32] Scharf, G., *Finite Quantum Electrodynamics*, Springer-Verlag, 1989

[33] Sternheimer, D., *Deformation quantization: Twenty years after*, In: Rembieliński, J. (ed.) Particles, Fields and Gravitation, Proc. of the Lodz meeting 1998. AIP Press. NY (1998) 107–145

[34] Thaller, B., *The Dirac Equation*, Springer-Verlag, Berlin, 1992

[35] Weyl, H., *Quantenmechanik und Gruppentheorie*, Z. Physik **46** (1927) 1; *The Theory of Groups and Quantum Mechanics*, Dover, New-York, 1931, translated from *Gruppentheorie und Quantenmechanik*, Hirzel Verlag, Leipzig, 1928

[36] Wigner, E.P., *Quantum corrections for thermodynamic equilibrium*, Phys. Rev. **40** (1932) 749

Acknowledgements

In the first place I would like to thank my supervisor Professor Rainer Verch for his patience, creativity and support in all kinds of scientific difficulties. He selected a very ambitious but also very interesting topic of research for us. Beyond that, he helped simplifying the process of requesting financial support by contacting the relevant persons. Which brings me to thank Professor Klaus Sibold, who fortunately had overlapping scientific interests. Hence, he was able to integrate me into a research project sponsored by the German Research Foundation (DFG) for the duration of two years. Many thanks go to the foundation itself. Furthermore, I am very grateful towards the International Max Planck Research Schools, that were supporting me financially for additional one and a half years. I received further very noticeable scientific support from Jr. Professor Dorothea Bahns. She shied absolutely no effort to help me out on specific questions and took plenty of her valuable time.
Last but not least, I would like to thank my family for supporting me along this very extended period of education and research.

Declaration

Herewith I affirm that I wrote this thesis independently. I did not use any other than the cited sources. Passages that are verbally or analogously taken over from these sources are marked accordingly. This work has not been used in any examination before and has not been published in its entirety. Also I have never unsuccessfully attempted any doctorate before.
Furthermore, nobody else except the acknowledged persons from the previous page contributed to this work. And of course there have not been pursued any financial interests in that regard.
Finally I accept the “Promotionsordnung der Fakultät vom 23. März 2010”.

Place	Date	Signature