Der Diracsee im äußeren Feld

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Kapitel 1

Einleitung

In dieser Einleitung soll ein kurzer Überblick über das Thema der vorliegenden Arbeit und über die erzielten Ergebnisse gegeben werden. Damit die Darstellung nicht zu technisch wird, ist die Beschreibung hauptsächlich in Worten gehalten. Dies kann natürlich nur ein qualitatives und stark vereinfachtes Bild vermitteln, doch sollte es hilfreich sein, um die mathematischen Konstruktionen von Kapitel 2 und 3 besser verstehen und einordnen zu können.

1.1 Physikalischer Hintergrund

In diesem Abschnitt soll anschaulich und sehr leicht verständlich in die physikalische Problemstellung eingeführt werden, zur Einfachheit beschränken wir uns dabei auf die elektromagnetische Wechselwirkung.

In der klassischen Mechanik werden Teilchen als punktförmig angesehen. Die Bewegung eines Teilchens wird durch eine Trajektorie (oder Teilchenbahn) beschrieben, also durch eine Funktion $\vec{x}(t)$, die den Ort $\vec{x}$ des Teilchens zu jeder Zeit t angibt. Falls keine Kräfte wirken, bewegt sich das Teilchen nach dem Newtonschen Trägheitsgesetz mit konstanter Geschwindigkeit $\dot{\vec{x}}$, die Trajektorie ist also eine Gerade. Im allgemeinen wird das Teilchen jedoch von Kräften auf seiner Bahn abgelenkt. Betrachtet man zum Beispiel ein Teilchen der Masse m und Ladung e in einem elektrischen Feld $\vec{E}$ und einem Magnetfeld $\vec{B}$, so wird die Bewegung durch die Newtonschen Gleichungen

$$m\,\ddot{\vec{x}} \;=\; e\vec{E} \;+\; e\,\dot{\vec{x}} \times \vec{B} \tag{1.1.1}$$

beschrieben ('$\times$' ist das Vektorprodukt). Die beiden Terme auf der rechten Seite von (1.1.1) sind die elektrische Kraft und die Lorentzkraft. Besonders einfach werden die Bewegungsgleichungen im Spezialfall, daß $\vec{E}$ und $\vec{B}$ statisch, also zeitunabhängig, sind. In diesem Fall ist nämlich die Energie

$$E \;=\; \frac{m}{2}\,|\dot{\vec{x}}|^2 \;+\; e\phi \tag{1.1.2}$$

zeitlich erhalten, dabei ist das elektrische Potential $\phi(\vec{x})$ über die Gleichung $\vec{E} = -\vec{\nabla}\phi$ bestimmt.

Mit den Newtonschen Gleichungen alleine wird noch keine realistische Dynamik klassischer Teilchen beschrieben. Damit mehrere Teilchen miteinander wechselwirken, müßte man nämlich berücksichtigen, daß jedes Teilchen ein elektromagnetisches Feld erzeugt, welches wiederum die Bewegung aller anderen Teilchen beeinflußt. Mathematisch ausgedrückt

müßte man also zu den Newtonschen Gleichungen die Maxwell-Gleichungen hinzufügen und das gekoppelte System der Newton-Maxwell-Gleichungen untersuchen. Dies ist ein schwieriges Problem, das hier nicht behandelt werden soll. Wir werden im folgenden nicht das gesamte wechselwirkende System betrachten, sondern das elektromagnetische Feld stets als vorgegebenes, *äußeres Feld* voraussetzen. Im äußeren Feld bewegt sich jedes Teilchen auf der durch die Newtonschen Gleichungen bestimmten Bahn; die Teilchen beeinflussen ihre Bewegung aber nicht gegenseitig. Es gibt physikalische Situationen, in denen diese Vereinfachung eine zulässige Näherung ist. Wird beispielsweise ein Elektronenstrahl im Magnetfeld einer Spule auf eine Kreisbahn abgelenkt, so ist die Bewegung der Elektronen im wesentlichen durch das vorgegebene Magnetfeld bestimmt, die elektrostatische Abstoßung der Elektronen des Strahls untereinander ist jedoch vernachlässigbar. Interessanter als solche unmittelbaren Anwendungen ist für uns, daß das Konzept des äußeren Feldes auch ein wichtiges Hilfsmittel ist, um das gesamte wechselwirkende System zu verstehen. Dazu stellt man sich vor, daß das äußere Feld von den Teilchen selbst erzeugt wird, als solches aber bereits bekannt ist. Mit dieser Vorstellung läßt sich das wechselwirkende System beispielsweise in einem iterativen Prozeß beschreiben: Man beginnt mit einer Anfangskonfiguration der Teilchen und löst die Newtonschen Gleichungen zunächst ohne elektromagnetisches Feld. Aus den erhaltenen Trajektorien kann man mit Hilfe der Maxwell-Gleichungen das von den Teilchen erzeugte elektromagnetische Feld bestimmen. Dieses Feld stimmt sicher nicht mit dem Feld im wechselwirkenden Fall überein, da es aus den Trajektorien gleichförmig bewegter Teilchen gewonnen wurde, doch kann man hoffen, daß es eine Näherung für das tatsächliche Feld ist. Um die Näherung zu verbessern, verwendet man das erhaltene elektromagnetische Feld als äußeres Feld und löst die Newtonschen Gleichungen zu den gegebenen Anfangsbedingungen erneut. Anschließend bestimmt man aus den Trajektorien ein neues elektromagnetisches Feld, u.s.w.. Falls diese induktive Konstruktion konvergiert, erhält man im Limes die exakten Teilchenbahnen und Felder des wechselwirkenden Systems. Dieses anschauliche Verfahren wird, in einer mathematisch präzisierten Form, bei der Lösung nichtlinearer Gleichungen mit Fixpunktsätzen ausgenutzt; die Lösung der Bewegungsgleichungen im äußeren Feld ist dabei ein wichtiger Zwischenschritt in der Konstruktion. Im folgenden spielt es keine Rolle, ob das äußere Feld wie im Beispiel des Elektronenstrahls in der Magnetspule tatsächlich „von außen“ angelegt ist oder wie bei der Fixpunktmethode als von den Teilchen selbst erzeugtes Feld angesehen wird; mit äußerem Feld ist lediglich gemeint, daß die Konfiguration der Felder vorgegeben ist.

Die Newtonsche Mechanik wurde in diesem Jahrhundert in mehreren Schritten grundlegend erweitert. Der erste Schritt in dieser Entwicklung war Einsteins spezielle Relativitätstheorie. Dabei werden Raum und Zeit nur noch relativ zu einem gleichförmig bewegten Bezugssystem eingeführt. Alle Bezugssysteme werden als gleichberechtigt angesehen; insbesondere nimmt man an, daß in allen Bezugssystemen die gleiche Lichtgeschwindigkeit c beobachtet wird. Aus diesen Annahmen folgt, daß beim Übergang von einem Bezugssystem zum anderen die vier Raum- und Zeitkoordinaten mit einer Lorentztransformation ineinander umgerechnet werden müssen. Teilchen werden weiterhin als punktförmig angesehen, ihre Bahn wird in jedem Bezugssystem durch eine Trajektorie $\vec{x}(t)$ beschrieben. Entlang einer Trajektorie kann man die Zeit einführen, die von einer längs der Bahn bewegten Uhr gemessen wird. Diese „Eigenzeit“ ist in infinitesimaler Form über die Gleichung

$$d\tau \;=\; \sqrt{1 - \frac{|\dot{\vec{x}}|^2}{c^2}}\, dt \tag{1.1.3}$$

gegeben. Um die Bewegung des Teilchens unabhängig von einem Bezugssystem zu beschreiben, faßt man Zeit und Ort zu einem Vierervektor $x = (t, \vec{x})$ zusammen und parametrisiert die Trajektorie nach der Eigenzeit. Die Teilchenbahn $x(\tau)$ ist dann eine Kurve in einer vierdimensionalen Raumzeit. Mathematisch wird die Raumzeit durch den Minkowski-Raum, einen Vektorraum mit einem Skalarprodukt der Signatur $(+---)$, beschrieben. Auch das elektromagnetische Feld läßt sich unabhängig von einem Bezugssystem im Minkowski-Raum formulieren; dazu faßt man das $\vec{E}$- und $\vec{B}$-Feld zu einer (4×4)-Matrix $F(x)$, dem elektromagnetischen Feldstärketensor, zusammen. Die relativistischen Bewegungsgleichungen lauten dann

$$m \frac{d^2}{d\tau^2} x^i \;=\; e F^i_j(x) \frac{d}{d\tau} x^j \quad , \tag{1.1.4}$$

wobei die Vierer-Indizes i, j die Werte $0, \ldots, 3$ annehmen, und über j als doppelt auftretenden Index zu summieren ist. Auch die relativistischen Bewegungsgleichungen vereinfachen sich für statische Felder, also im Spezialfall, daß es ein Bezugssystem gibt, in welchem der Feldstärketensor zeitunabhängig ist. In diesem Fall ist nämlich die relativistische Energie

$$E \;=\; \frac{mc^2}{\sqrt{1 - \dfrac{|\dot{\vec{x}}|^2}{c^2}}} \;+\; e\phi \tag{1.1.5}$$

eine Erhaltungsgröße. Im Grenzfall, daß die Geschwindigkeit der Teilchen klein gegenüber der Lichtgeschwindigkeit ist, gehen die relativistischen Bewegungsgleichungen (1.1.4) in die Newtonschen Gleichungen (1.1.1) über. Die relativistische Energie (1.1.5) stimmt in diesem Grenzfall bis auf eine Größe mc^2, der Ruheenergie, mit der klassischen Energie (1.1.2) überein. Die Unterschiede zwischen der klassischen und relativistischen Dynamik werden deutlich, wenn die Geschwindigkeit der Teilchen von der Größenordnung der Lichtgeschwindigkeit ist. Insbesondere sieht man, daß die Wurzeln in (1.1.3) und (1.1.5) für $|\dot{\vec{x}}| = c$ verschwinden und für $|\dot{\vec{x}}| > c$ nicht mehr reell sind. Das bedeutet physikalisch, daß sich kein Teilchen schneller als mit Lichtgeschwindigkeit bewegen kann; die Lichtgeschwindigkeit ist absolute Grenzgeschwindigkeit. Verwendet man, daß Informationen von einem Punkt der Raumzeit zu einem anderen durch einen Lichtstrahl oder durch Teilchen übermittelt werden müssen, so folgt, daß in der Relativitätstheorie keine Information schneller als mit Lichtgeschwindigkeit übertragen werden kann. Für zwei Punkte der Raumzeit kann man unterscheiden, ob sie durch die Bahn eines Teilchens, das sich höchstens mit Lichtgeschwindigkeit bewegt, miteinander verbunden werden können oder nicht. Im ersten Fall nennt man die Punkte *zeitartig*, im zweiten Fall *raumartig* getrennt. Die Grenze zwischen diesen beiden Gebieten ist der *Lichtkegel*. Anders ausgedrückt liegt ein Punktepaar des Minkowski-Raums auf dem Lichtkegel, falls ein in einem der Punkte ausgesendeter Lichtblitz im anderen Punkt beobachtet werden kann. Die endliche Übertragungsgeschwindigkeit von Information ist die Grundlage für das *Kausalitätsprinzip*, welches besagt, daß sich Gebiete der Raumzeit physikalisch nicht beeinflussen können, falls sie raumartig getrennt sind.

Auf ganz andere Weise als durch die spezielle Relativitätstheorie wurde die Physik durch die Quantenmechanik weiterentwickelt. Dabei hält man an der Newtonschen Vorstellung von Raum und Zeit fest, es wird aber der Begriff des klassischen Punktteilchens aufgegeben. Ein Teilchen wird nicht mehr mit einer Trajektorie $\vec{x}(t)$, sondern mit einer komplexen Wellenfunktion $\Psi(t, \vec{x})$ beschrieben. Dies trägt der Beobachtung Rechnung, daß sich Teilchen auf sehr kleinen Längenskalen nicht wie klassische Punktteilchen, sondern wie

eine Welle verhalten (Welle-Teilchen-Dualismus). Die Längenskala, bei der solche Quanteneffekte wichtig werden, wird durch das Plancksche Wirkungsquantum $\hbar$ festgelegt. An die Stelle der Newtonschen Bewegungsgleichungen tritt die *Schrödinger-Gleichung*

$$i\hbar\,\partial_t\Psi \;=\; \frac{1}{2m}\,(-i\hbar\vec{\nabla} - e\vec{A})^2\;\Psi \;+\; e\phi\;\Psi \quad . \tag{1.1.6}$$

In dieser Gleichung wird das elektromagnetische Feld mit dem skalaren Potential ϕ und dem Vektorpotential $\vec{A}$ beschrieben. Dies ist äquivalent zum Arbeiten mit den elektrischen und magnetischen Feldern; diese Felder sind nun über die Gleichungen $\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\partial_t\vec{A}$ und $\vec{B} = \vec{\nabla}\times\vec{A}$ gegeben. Auch bei der Schrödinger-Gleichung hat man im Spezialfall eines statischen elektromagnetischen Feldes Energieerhaltung. Faßt man die Klammer in (1.1.6) als den Impulsoperator $\vec{p}$ auf,

$$\vec{p} \;=\; -i\hbar\vec{\nabla} \;-\; e\vec{A} \quad , \tag{1.1.7}$$

so ist die Energie der Erwartungswert des Hamiltonoperators

$$H \;=\; \frac{\vec{p}^{\,2}}{2m} \;+\; e\phi \quad . \tag{1.1.8}$$

Der genaue Zusammenhang zwischen der Schrödinger-Gleichung (1.1.6) und der Newtonschen Bewegungsgleichung (1.1.1) ist nicht ganz unproblematisch. Das liegt daran, daß die Bewegung eines quantenmechanischen Teilchens nicht wie bei einer klassischen Trajektorie deterministisch bestimmt ist. Nach der statistischen Interpretation der Quantenmechanik gibt das Absolutquadrat $|\Psi(t,\vec{x})|^2$ lediglich die Wahrscheinlichkeit an, daß sich das Teilchen zur Zeit t am Ort $\vec{x}$ befindet; ein Beobachter kann über den Aufenthalt des Teilchens aber keine genaue Aussage machen, ohne das physikalische System durch einen Meßprozeß zu verändern. Unter Berücksichtigung dieses prinzipiellen Unterschiedes zwischen der klassischen und quantenmechanischen Beschreibung kann man die Newtonschen Gleichungen aus der Schrödinger-Gleichung im Grenzfall $\hbar \to 0$ ableiten, indem man für Ψ Wellenpakete ansetzt, die im Raum mehr und mehr lokalisiert sind. In diesem Grenzprozeß geht der Impulsoperator (1.1.7) in den klassischen Impuls $\vec{p} = m\dot{\vec{x}}$ eines Punktteilchens über, so daß (1.1.8) mit der klassischen Energie (1.1.2) übereinstimmt.

In dieser Arbeit werden wir die Bewegung quantenmechanischer Teilchen im äußeren Feld untersuchen. Um dieses Problem richtig zu verstehen, sollte man beachten, daß die Schrödinger-Gleichung trotz des fehlenden Determinismus in der Quantenmechanik in dem Sinne deterministisch ist, daß die Wellenfunktion zu einem gegebenen Zeitpunkt eindeutig aus ihren Anfangswerten zu einem früheren Zeitpunkt bestimmt ist. Das bedeutet, daß das Problem des fehlenden Determinismus in der Quantenmechanik nicht auftritt, solange keine Meßprozesse stattfinden oder ein Beobachter das Geschehen beeinflußt. Für die Bewegung im äußeren Feld können wir uns darauf beschränken, die Lösungen der quantenmechanischen Wellengleichungen zu studieren; der Meßprozeß und Fragen der Interpretation der Quantenmechanik gehen in dieses Problem aber nicht ein.

Quantenmechanische Teilchen besitzen einen zusätzlichen inneren Freiheitsgrad, den *Spin*. Der Spin einer jeden Teilchensorte wird durch eine der Zahlen 0, $\frac{1}{2}$, 1, $\frac{3}{2}$,... beschrieben. Teilchen mit ganzzahligem Spin heißen *Bosonen*, Teilchen mit halbzahligem Spin *Fermionen*. Auf der Ebene der Elementarteilchen ist die Materie aus Fermionen (Leptonen und Quarks) aufgebaut, während die Bosonen (z.B. das Photon) die Wechselwirkung zwischen den Fermionen vermitteln. Deswegen macht es Sinn, wenn wir in dieser Arbeit nur die Bewegung von Fermionen untersuchen (das äußere Feld ist dagegen als ein klassischer Grenzfall der Bosonen anzusehen; deswegen wird es in Kapitel 2 und 3 manchmal

auch „bosonisches Feld“ genannt). Da alle fermionischen Elementarteilchen Spin $\frac{1}{2}$ haben, ist es außerdem sinnvoll, wenn wir uns im folgenden auf Spin-$\frac{1}{2}$-Teilchen beschränken. Der Spin ist ein reiner Quanteneffekt. Er hat keine klassische Entsprechung und kann deswegen nur schwer in Worten beschrieben werden. Manchmal wird er als eine „Eigenrotation“ des quantenmechnischen Teilchens um seine eigene Achse veranschaulicht, doch ist diese Vorstellung eigentlich nicht richtig. Am deutlichsten zeigt sich der Spin im Stern-Gerlach-Versuch: ein aus gleich erscheinenden Elektronen bestehender Teilchenstrahl wird beim Durchfliegen eines starken, inhomogenen Magnetfeldes in zwei Teilstrahlen aufgespalten. Man sagt, daß die Teilchen des einen Teilstrahls ihren Spin in Richtung des Magnetfeldes ausgerichtet haben, während bei den Teilchen des anderen Teilstrahls der Spin dem Magnetfeld entgegengesetzt ist. Zur mathematischen Beschreibung dieses Experiments verwendet man eine zweikomponentige Wellenfunktion, wobei die Komponenten den beiden Einstellungen des Spins entsprechen. Die zweikomponentige Wellenfunktion Ψ erfüllt die *Pauli-Gleichung*

$$i\hbar\,\partial_t\Psi \;=\; \frac{1}{2m}\,(-i\hbar\vec{\nabla} - e\vec{A})^2\;\Psi \;+\; e\phi\;\Psi \;-\; \frac{\hbar e}{2m}\,\vec{B}\vec{\sigma}\;\Psi \quad , \tag{1.1.9}$$

dabei besteht der Vektor $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ aus den drei Pauli-Matrizen

$$\sigma^1 \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 \;=\; \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(das Skalarprodukt $\vec{B}\vec{\sigma}$ ist also eine (2×2)-Matrix, die aus den Komponenten des Magnetfeldes aufgebaut ist). Die Pauli-Gleichung unterscheidet sich von der Schrödinger-Gleichung (1.1.6) lediglich um den letzten Summanden auf der rechten Seite von (1.1.9). Dieser Term beschreibt eine unterschiedliche Kopplung der beiden Spinorkomponenten an das Magnetfeld und erklärt so den Stern-Gerlach-Versuch. Die Pauli-Gleichung beschreibt auch andere physikalische Effekte, beispielsweise die Aufspaltung von Spektrallinien in Magnetfeldern (anomaler Zeeman-Effekt). In vielen Anwendungen, beispielsweise in schwachen Magnetfeldern, spielt der Spin keine wichtige Rolle; deshalb ist es oft ausreichend, mit der einfacheren Schrödinger-Gleichung zu arbeiten.

Bereits kurze Zeit nach Formulierung der Quantenmechanik wurde versucht, die Vorstellung quantisierter Teilchen mit der speziellen Relativitätstheorie zu verbinden. Dies gelang mit der *Dirac-Gleichung*, die ein relativistisches, quantenmechanisches Teilchen mit Spin $\frac{1}{2}$ beschreibt. Die entscheidende Idee bei der Aufstellung dieser Gleichung war es, die Anzahl der Spinorkomponenten gegenüber der Pauli-Gleichung von zwei auf vier zu verdoppeln, dafür aber mit einer Differentialgleichung erster Ordnung zu arbeiten. Um das elektromagnetische Feld unabhängig von einem Bezugssystem zu beschreiben, faßt man das skalare Potential und das Vektorpotential zu einem Vierervektor $A_j = (-\phi, \vec{A})$, dem elektromagnetischen Potential, zusammen; den elektromagnetischen Feldstärketensor erhält man daraus über die Gleichung $F_{jk} = \partial_j A_k - \partial_k A_j$ (∂_j steht für die vier partiellen Ableitungen $\partial_t = \partial_0$, $\vec{\nabla} = (\partial_1, \partial_2, \partial_3)$). Außerdem wählen wir zur Einfachheit die Einheiten für die Zeit so, daß die Lichtgeschwindigkeit $c = 1$ ist. Die Dirac-Gleichung lautet dann

$$\left(i\hbar\,\gamma^j\frac{\partial}{\partial x^j} \;+\; e\gamma^j A_j \;-\; m\right)\Psi \;=\; 0 \quad , \tag{1.1.10}$$

wobei die Dirac-Matrizen γ^j (4×4)-Matrizen sind, die aus den Pauli-Matrizen aufgebaut

sind, genauer

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad \text{und} \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad . \tag{1.1.11}$$

Im Spezialfall statischer Potentiale hat man auch bei der Dirac-Gleichung Energieerhaltung; die Energie ist der Erwartungswert des zugehörigen Hamiltonoperators

$$H = \begin{pmatrix} m & \vec{p}\vec{\sigma} \\ \vec{p}\vec{\sigma} & -m \end{pmatrix} + e\phi \quad , \tag{1.1.12}$$

wobei $\vec{p}$ wieder der Impulsoperator (1.1.7) ist. Die statistische Deutung der Quantenmechanik läßt sich unmittelbar auf die Dirac-Gleichung übertragen, wenn man das Absolutquadrat $|\Psi|^2$ der Dirac-Wellenfunktion als die Wahrscheinlichkeitsdichte des Teilchens interpretiert. Trotz ihrer statistischen Interpretation ist die Dirac-Gleichung, genau wie die Schrödinger-Gleichung, in dem Sinne deterministisch, daß die Wellenfunktion durch ihre Anfangswerte zu einem gegebenen Zeitpunkt eindeutig für alle Zeiten festgelegt ist. Außerdem ist die Dirac-Gleichung kausal; die Wellenfunktion $\Psi(x)$ hängt also nur vom elektromagnetischen Feld und den Anfangswerten von Ψ an denjenigen Raumzeitpunkten ab, die von x zeitartig getrennt sind. Im nichtrelativistischen Grenzfall (also der Näherung, daß die kinetische Energie des Teilchens klein gegenüber seiner Ruheenergie ist) geht die Dirac-Gleichung in die Pauli-Gleichung über. Außerdem liefert die Dirac-Gleichung relativistische Korrekturen zur Pauli-Gleichung, die experimentell bestätigt wurden. Man kann auch einen direkten Zusammenhang zwischen der Dirac-Gleichung und der relativistischen Bewegungsgleichung (1.1.4) herstellen, indem man, ähnlich wie beim klassischen Grenzfall der Schrödinger-Gleichung, die Dynamik von Wellenpaketen im Limes $\hbar \to 0$ untersucht. In diesem Grenzfall geht der Impulsoperator in den relativistischen Impuls eines Punktteilchens

$$\vec{p} = \frac{m\dot{\vec{x}}}{\sqrt{1 - |\dot{\vec{x}}|^2}} \tag{1.1.13}$$

über. Durch diese Identifikation wird der Hamiltonoperator (1.1.12) nicht, wie man aus der klassischen Physik erwarten würde, zu einer skalaren Energiefunktion, sondern bleibt eine (4×4)-Matrix. Das bedeutet, daß die Energie eines Dirac-Teilchens selbst im klassischen Grenzfall von der Orientierung des 4-Spinors abhängt. Um die Energiezustände zu bestimmen, muß diese (4×4)-Matrix diagonalisiert werden. Die Eigenwerte der Matrix sind die zugehörigen Energien E, man erhält

$$E = \pm\sqrt{\vec{p}^2 + m^2} + e\phi \quad . \tag{1.1.14}$$

Wählt man in dieser Gleichung das Pluszeichen, so erhält man nach Einsetzen von (1.1.13) genau den Ausdruck (1.1.5) für die relativistische Energie eines Punktteilchens.

Trotz ihres unmittelbaren Erfolges gab es mit der Dirac-Gleichung zunächst ein schwerwiegendes Problem: ihre Lösungen negativer Energie. Daß die Dirac-Gleichung Lösungen negativer Energie besitzt, kann man schon an ihrem klassischen Grenzfall sehen. Wählt man nämlich in (1.1.14) das Minuszeichen, so wird die Energie für hinreichend großes $\vec{p}^2$ negativ und kann beliebig klein gemacht werden. Es gibt also nicht nur einzelne Lösungen negativer Energie, die Energie der Lösungen der Dirac-Gleichung ist sogar nach unten unbeschränkt. Dies ist physikalisch nicht sinnvoll. Abgesehen davon, daß noch nie Teilchen negativer Energie beobachtet wurden, wäre ein physikalisches System mit nach unten

unbeschränkter Energie nämlich zwangsläufig instabil, da man beliebig viel Energie freisetzen könnte, indem man die Teilchen in Zustände genügend negativer Energie bringt. Das Problem der Lösungen negativer Energie wurde durch die Einführung des *Diracsees* beseitigt, was nun anschaulich beschrieben werden soll. Gemäß der ursprünglichen Vorstellung von Dirac sind im Vakuum alle Zustände negativer Energie durch Teilchen besetzt. Man hat sich das Vakuum also nicht einfach als einen „leeren Raum“ vorzustellen, sondern es besteht aus einem „See“ unendlich vieler Teilchen negativer Energie, die sich in der Raumzeit bewegen. Dieser See ist im Vakuum homogen und isotrop aufgebaut. Bewegt sich ein Teilchen des Sees mit Impuls $\vec{p}$, so bewegt sich gleichzeitig ein anderes Teilchen des Sees mit Impuls $-\vec{p}$ in umgekehrter Richtung. Auf diese Weise kompensieren sich die Effekte aller Teilchen gegenseitig[1], so daß der Diracsee im Vakuum nicht beobachtbar ist. Um ein physikalisches System mit Teilchen zu beschreiben, fügt man zu dem See des Vakuums weitere Teilchen hinzu. Da alle Zustände negativer Energie bereits besetzt sind, muß die Energie aller hinzugefügten Teilchen positiv sein. Auf diese Weise verschwindet das Problem der Zustände negativer Energie, und das physikalische System wird stabil. Anstatt Teilchen zu dem See hinzuzufügen, kann man auch Systeme betrachten, bei denen einzelne Teilchen aus dem See entfernt wurden. In diesem Fall kompensieren sich die Beiträge aller Teilchen nicht vollständig; die „Löcher“ im Diracsee werden als scheinbare Teilchen beobachtet. Genauer erscheint ein nicht besetzter Zustand negativer Energie und z.B. negativer Ladung als ein Teilchen positiver Energie und positiver Ladung. Auf diese Weise führt die Vorstellung des Diracsees auf das Postulat, daß es zu jedem Dirac-Teilchen (Elektron, Quark, Neutrino,...) ein zugehöriges *Antiteilchen* gibt, welches die gleiche Masse, aber umgekehrte Ladung besitzt. Außerdem erklärt der Diracsee einen physikalischen Prozeß, bei dem Antiteilchen enstehen, die *Paarerzeugung*. Wird beispielsweise mit einem Teilchenbeschleuniger an einem Ort eine sehr hohe Energiedichte erzeugt, so kann dort ein Teilchen des Diracsees in einen Zustand positiver Energie übergehen. Auf diese Weise enstehen gleichzeitig ein Teilchen positiver Energie und ein „Loch“ im Diracsee, also ein Teilchen/Antiteilchen-Paar. Umgekehrt können sich ein Teilchen und ein Antiteilchen auch gegenseitig vernichten, wobei Energie beispielsweise in Form elektromagnetischer Strahlung frei wird. Alle diese physikalischen Effekte können experimentell beobachtet werden und geben eine Bestätigung sowohl für die Dirac-Gleichung als auch für das Konzept des Diracsees.

Ausgehend von der relativistischen Quantenmechanik hat sich die moderne Quantenfeldtheorie entwickelt. Im Formalismus der kanonischen Quantisierung baut man dabei aus allen Lösungen der Dirac-Gleichung und den möglichen Konfigurationen des elektromagnetischen Feldes durch Bildung von Tensorprodukten den sog. Fockraum auf. Die physikalischen Gleichungen werden dann als Operatorgleichungen auf dem Fockraum formuliert. Durch die Quantisierung des elektromagnetischen Feldes wird der Teilchencharakter elektromagnetischer Wellen, wie er sich z.B. im Photoeffekt zeigt, berücksichtigt. Außerdem werden damit spezielle Quanteneffekte wie die Planck-Strahlung und der Casimir-Effekt

[1] Man sollte jedoch beachten, daß sich die Energien der Teilchen nicht gegenseitig kompensieren. Naiv betrachtet besitzt der Diracsee also eine unendlich große negative Energie. Zur Einfachheit wird dieses Phänomen in der Einleitung nicht angesprochen, es wird aber in Kapitel 2 und 3 selbstverständlich berücksichtigt. Die unendliche Energie des Diracsees führt in dieser Arbeit auf keine Probleme; sie zeigt sich in unserem Formalismus darin, daß der Diracsee des Vakuums durch eine Distribution beschrieben wird, die Singularitäten auf dem Lichtkegel hat. Problematisch wird die unendliche Energie des Diracsees erst dann, wenn die Kopplung der Dirac-Teilchen an die Gravitation berücksichtigt wird. In unserem Formalismus ließe sich das Problem der unendlichen Energiedichte beseitigen, indem man vom Energie-Impuls-Tensor des Diracsees den singulären Beitrag auf dem Lichtkegel abzieht.

erklärt. Die „zweite“ Quantisierung des Diracfeldes liefert dagegen einen Formalismus, mit dem Antiteilchen und die Paarerzeugung mathematisch befriedigend beschrieben werden können. Die Konstruktion des Diracsees spielt auch in der Quantenfeldtheorie eine wichtige Rolle. Sie entspricht dort der Umbenennung und formalen Vertauschung von Erzeugungs- und Vernichtungsoperatoren für die Einteilchen-Zustände negativer Energie. Trotzdem tritt der Diracsee in der Quantenfeldtheorie etwas in den Hintergrund. Er ist heute in den Augen vieler Physiker nicht mehr wie bei Dirac ein realer See miteinander wechselwirkender Teilchen, sondern wird oft nur noch als eine formale Konstruktion der Quantenfeldtheorie angesehen. Andererseits ist die Frage, ob der Diracsee physikalische Wirklichkeit oder nur eine mathematische Konstruktion ist, rein philosophischer Art; sie kann in der vorliegenden Arbeit nicht behandelt werden und ist im Rahmen dieser Arbeit auch nicht relevant. So betrachtet sind sich der Diracsee in der relativistischen Quantenmechanik und der Quantenfeldtheorie sehr ähnlich; der einzige Unterschied besteht darin, daß die Dirac-Teilchen in der relativistischen Quantenmechanik mit einem klassischen elektromagnetischen Feld, in der Quantenfeldtheorie dagegen mit einem quantisierten Photonfeld wechselwirken.

Nach diesem allgemeinen Überblick kann nun die eigentliche Problemstellung der vorliegenden Arbeit beschrieben werden. In Kapitel 2 wird die Frage untersucht, wie der Diracsee im äußeren klassischen Feld formal definiert werden kann. Obwohl dies ein recht abstraktes mathematisches Problem ist, kann die zugrundeliegende Schwierigkeit auch physikalisch erklärt werden. Wir beginnen dazu mit dem einfachsten Spezialfall, dem Diracsee im Vakuum. Im Vakuum sind alle Lösungen der Dirac-Gleichung, die Ebenen-Wellen-Lösungen, explizit bekannt. Die Energie jeder Lösung ist, ganz ähnlich wie im „klassischen Grenzfall“ (1.1.14), durch die Dispersionsrelation $E = \pm\sqrt{\vec{p}^2 + m^2}$ gegeben, wobei $\vec{p}$ der Impuls des Teilchens ist. Diese Dispersionsrelation gibt eine natürliche Aufspaltung des Lösungsraums der Dirac-Gleichung in Lösungen positiver und negativer Energie. Man kann den Diracsee beispielsweise als den Projektor auf den Raum der Ebenen-Wellen-Lösungen negativer Energie einführen.

Nicht mehr ganz so einfach ist die Konstruktion des Diracsees im statischen äußeren Feld. In diesem Fall kann man nämlich die Dirac-Gleichung i.a. nicht mehr explizit lösen und folglich für den Diracsee keinen geschlossenen Ausdruck mehr hinschreiben. Formal gesehen bereitet die Konstruktion des Diracsees aber immer noch keine Schwierigkeiten: Da die Potentiale in der Dirac-Gleichung (1.1.10) zeitunabhängig sind, kann man die Zeit abseparieren, also für die Wellenfunktion

$$\Psi(t, \vec{x}) \;=\; e^{-\frac{i}{\hbar}Et}\, \Phi(\vec{x})$$

mit einem komplexen 4-Spinor Φ und einem reellen Parameter E ansetzen. Die Dirac-Gleichung vereinfacht sich dann zu einer Differentialgleichung in $\vec{x}$, nämlich zur Gleichung $H\Phi = E\Phi$ mit dem Hamiltonoperator (1.1.12). Die Energie des Zustandes stimmt gerade mit dem Parameter E überein. Etwas allgemeiner ausgedrückt ist im statischen äußeren Feld die Energie jedes Zustandes erhalten, und man kann die Lösungen der Dirac-Gleichung als Eigenzustände der Energie wählen. Das Vorzeichen der Energie dieser Eigenzustände gibt eine Aufspaltung des Lösungsraums der Dirac-Gleichung in zwei Teilräume, und man kann den Diracsee ganz ähnlich wie im Vakuum als den Projektor auf den Raum aller Eigenzustände negativer Energie definieren.

Das eigentliche Problem bei der Konstruktion des Diracsees tritt auf, wenn das äußere Feld zeitabhängig ist. In diesem Fall ist die Energie der einzelnen Dirac-Teilchen nämlich i.a. keine Erhaltungsgröße mehr. Es kann insbesondere passieren, daß ein Teilchen, das

zunächst positive Energie hat, zu einem späteren Zeitpunkt negative Energie besitzt oder umgekehrt. Das Vorzeichen der Energie ist also keine wohldefinierte Eigenschaft der Dirac-Teilchen mehr. Damit geht die natürliche Aufspaltung des Lösungsraums der Dirac-Gleichung in zwei Teilräume verloren; es ist nicht mehr klar, aus welchen Zuständen der Diracsee aufgebaut werden soll. Besonders deutlich sieht man dieses Problem in einem Gedankenexperiment, das als *Kleinsches Paradox* bezeichnet wird. Dabei betrachtet man Dirac-Teilchen in einem zeitabhängigen, homogenen elektrischen Potential $\phi = \phi(t)$, das die Form einer „Potentialschwelle“ hat. Beispielsweise kann man annehmen, daß $\phi(t)$ für negative Zeiten verschwindet und für $t \geq 1$ einen konstanten Wert $\phi_0 > 0$ annimmt; im Bereich $0 < t < 1$ soll ϕ glatt zwischen den Randwerten 0 und ϕ_0 interpolieren. Dieses Potential hat eine so einfache Form, daß die Dirac-Gleichung explizit gelöst werden kann. Die Lösungen sind im Ort ebene Wellen mit Impuls $\vec{p}$ und haben die Energie $E(t) = \pm\sqrt{\vec{p}^2 + m^2} + \phi(t)$. Die Energie ist also tatsächlich zeitabhängig. Falls ϕ_0 größer als zweimal die Ruheenergie ist, $\phi_0 > 2m$, gibt es Zustände, deren Energie ihr Vorzeichen ändert. Dadurch bricht die übliche Definition des Diracsees als „alle Zustände negativer Energie“ der Dirac-Gleichung zusammen. Manchmal wird das Kleinsche Paradox auch etwas physikalischer als eine „Erzeugung von Teilchen“ an der Potentialschwelle interpretiert. Konstruiert man nämlich den Diracsee zu einem Zeitpunkt $t < 0$ genau wie im Vakuum aus allen Ebenen-Wellen-Lösungen negativer Energie, so sind zu einer Zeit $t > 1$ zusätzlich zu allen Zuständen negativer Energie alle diejenigen Zustände besetzt, deren Energie durch die Energieverschiebung an der Potentialschwelle positiv wurde. Gemäß der Interpretation positiver Energiezustände als „Teilchen“ wurden also an der Potentialschwelle Teilchen erzeugt. Dieser Prozeß ist aber physikalisch nicht sinnvoll, weil die Teilchen nicht, wie durch experimentell gefundene Erhaltungssätze (die Lepton- und Baryonerhaltung) gefordert, in Teilchen/Antiteilchen-Paaren generiert werden. Noch problematischer als beim Kleinschen Paradox ist die Situation in einem allgemeinen orts- und zeitabhängigem elektromagnetischen Potential $A(t, \vec{x})$. In diesem Fall geht die natürliche Zerlegung des Lösungsraumes in zwei Teilräume nämlich schon für beliebig kleine Potentiale $|A(t, \vec{x})| < \varepsilon$ verloren. Wegen dieser prinzipiellen Schwierigkeiten war es vor unserer Definition von Kapitel 2 noch nie gelungen, den Diracsee für zeitabhängige äußere Felder sinnvoll einzuführen. Alle Konstruktionen waren a-priori auf äußere Felder beschränkt, die entweder statisch sind oder eine nur asymptotisch kleine, „adiabatische“ Zeitabhängigkeit besitzen.

In Kapitel 3 wird der Diracsee, ausgehend von unserer formalen Definition, im Ortsraum untersucht. Ziel ist es dabei, die Abhängigkeit des Diracsees vom äußeren Feld möglichst genau und explizit zu beschreiben. Dieses Problem läßt sich physikalisch leicht verstehen, wenn man sich den Diracsee wie oben beschrieben als eine Vielzahl von Teilchen vorstellt, die sich im äußeren Feld bewegen: Im Vakuum ist der Diracsee homogen und isotrop aufgebaut; die Beiträge aller Teilchen heben sich exakt gegeneinander weg. Im äußeren Feld ist die Situation jedoch nicht so einfach. Nehmen wir beispielsweise an, daß es zu einem Teilchen des Sees mit Impuls $\vec{p}$ ein anderes Teilchen des Sees gibt, das sich zur gleichen Zeit mit Impuls $-\vec{p}$ in umgekehrter Richtung bewegt. Liegt ein elektrisches Feld an, so werden beide Teilchen durch die elektrische Kraft beschleunigt. Da die Teilchen gleiche Ladung und Masse haben, zeigt die Beschleunigung beider Teilchen in die gleiche Richtung, so daß sich die Beschleunigungseffekte der beiden Teilchen nicht gegenseitig kompensieren. Betrachtet man auf der anderen Seite ein Magnetfeld, so werden die Teilchen senkrecht zu ihrer Bewegungsrichtung abgelenkt, wodurch ebenfalls die Gegensätzlichkeit in der Bewegung beider Teilchen verloren geht. Allgemeiner gesagt kann man erwarten, daß sich im äußeren Feld die Effekte aller Teilchen nicht mehr exakt gegen-

einander wegheben; es sollte einen kollektiven Beitrag der Teilchen zum Diracsee geben. Da der Diracsee das Vakuum der Dirac-Theorie ist, kann man diesen Effekt auch als eine „Störung“ oder „Polarisierung“ des Vakuums durch das äußere Feld ansehen. Man sollte ihn aber nicht mit der Vakuumpolarisation der Quantenfeldtheorie verwechseln. Bei der Vakuumpolarisation wird nämlich ein virtuelles Teilchen/Antiteilchen-Paar betrachtet, während wir hier die kollektive Bewegung aller Teilchen des Sees untersuchen. Der hier beschriebene Effekt wird in der Quantenfeldtheorie nicht berücksichtigt. Das liegt daran, daß der Diracsee dort nur für das Vakuum (oder manchmal im statischen äußeren Feld) konstruiert und das wechselwirkende, zeitabhängige System anschließend über eine naive Störungsentwicklung gewonnen wird. Unser Effekt ist deswegen physikalisch interessant, weil es als Folge dieses Effektes möglich sein sollte, den Diracsee direkt zu beobachten. Bevor man sinnvoll über die physikalischen Konsequenzen nachdenken kann, ist es jedoch notwendig, den Effekt auch quantitativ zu verstehen. Dies ist genau die Problemstellung von Kapitel 3.

Abschließend geben wir einige allgemeine Literaturhinweise, ohne Anspruch auf Vollständigkeit. Die Grundlagen der relativistischen Mechanik und Feldtheorie können in [1] nachgeschlagen werden. Für die relativistische Quantenmechanik ist [2] eines der Standardwerke. Die Dirac-Gleichung und der Diracsee in der Quantenfeldtheorie ist in [3] gut beschrieben. Eine recht umfassende Darstellung der Dirac-Gleichung und des Diracsees im statischen äußeren Feld mit vielen weiteren Referenzen findet sich in [4], dort ist auch das Kleinsche Paradox erklärt. Der Leser, der sich für die Dirac-Gleichung auch im Gravitationsfeld interessiert, findet mit [5] oder [6] einen geeigneten Ausgangspunkt.

1.2 Verwendete mathematische Methoden

Der Diracsee besteht aus unendlich vielen Teilchen. Um dies mathematisch zu beschreiben, summieren/integrieren wir über alle Lösungen der Dirac-Gleichung negativer Energie. Genauer führen wir für den Diracsee des Vakuums die Funktion

$$P(x,y) \;=\; \sum_{a=1,2} \int_{\mathbb{R}^3} d\vec{p}\; \Psi_{\vec{p}a}(x)\, \overline{\Psi_{\vec{p}a}(y)} \tag{1.2.15}$$

ein, wobei $\Psi_{\vec{p}a}$ die Ebenen-Wellen-Lösungen mit Impuls $\vec{p}$ und Energie $E = -\sqrt{\vec{p}^2 + m^2}$ sind (der Index a steht für die beiden Einstellungen des Spins). Das Integral in (1.2.15) existiert nicht punktweise (also für festes x und y), $P(x,y)$ ist aber eine wohldefinierte Distribution auf $\mathbb{R}^4 \times \mathbb{R}^4$. Wir fassen $P(x,y)$ auch als den Integralkern eines Operators auf, setzen also

$$(P\,\Psi)(x) \;=\; \int P(x,y)\, \Psi(y)\, d^4y \quad . \tag{1.2.16}$$

Das Bild des Operators P wird von den Ebenen-Wellen-Lösungen negativer Energie aufgespannt.

Wir bemerken, daß man den Diracsee alternativ auch beschreiben könnte, indem man aus den Einteilchen-Wellenfunktionen den zugehörigen fermionischen Fockraum aufbaut. Dies wäre aber technisch aufwendiger. Um sich davon zu überzeugen, daß auch unsere Beschreibung des Diracsees physikalisch sinnvoll ist, kann man unmittelbar einen Zusammenhang zwischen einem Operator der Form (1.2.16), (1.2.15) und einem Zustand des Fockraums herstellen. Dazu wählt man im Bild von P eine Basis und faßt die Basisvektoren als die Einteilchen-Zustände auf. Aus diesen Einteilchen-Zuständen erhält man durch

Bildung des Tensorproduktes einen Zustand im Fockraum. Die Abhängigkeit von der Wahl der Basis fällt wegen der Antisymmetrie des Tensorproduktes heraus.

Der fermionische Projektor im äußeren Feld wird auf ähnliche Weise wie (1.2.15) aus den Lösungen Ψ der Dirac-Gleichung im äußeren Feld aufgebaut. Wir arbeiten im folgenden in „natürlichen“ Einheiten mit $\hbar = 1$ und schreiben die Dirac-Gleichung in der Form

$$(i\partial\!\!\!/ + \mathcal{B} - m)\,\Psi \;=\; 0 \quad , \tag{1.2.17}$$

wobei der Querstrich eine Kurznotation für die Kontraktion mit den Dirac-Matrizen ist, also $\partial\!\!\!/ \equiv \gamma^j \partial_j$. Im elektromagnetischen Feld (1.1.10) ist $\mathcal{B} = eA\!\!\!/$; $\mathcal{B}$ kann aber nun ein allgemeinerer Operator sein. Die Dirac-Gleichung ist eine lineare Gleichung für die Wellenfunktion. Die Abhängigkeit der Lösungen vom äußeren Feld ist aber nichtlinear. Für die Definition des Diracsees in Kapitel 2 arbeiten wir mit einer formalen Störungsentwicklung im äußeren Potential. Dazu entwickelt man die Wellenfunktion nach Potenzen von $\mathcal{B}$ und löst die Dirac-Gleichung zu jeder Ordnung mit Hilfe von Greenschen Funktionen auf. Dies führt auf eine Darstellung von Ψ mittels einer Reihe von Operatorprodukten

$$\Psi \;=\; \sum_{n=0}^{\infty} (-1)^n\, s^{(n)}\, \mathcal{B}\, s^{(n-1)} \,\cdots\, s^{(1)}\, \mathcal{B}\, s^{(0)}\, \Phi \quad , \tag{1.2.18}$$

wobei die Faktoren $s^{(.)}$ Greensche Funktionen des Dirac-Operators sind, und Φ eine Lösung der freien Dirac-Gleichung $(i\partial\!\!\!/ - m)\Phi = 0$ ist. Die einzelnen Summanden in (1.2.18) werden in der Physik *Feynman-Diagramme* genannt.

Das Kleinsche Paradox zeigt sich in diesem Formalismus darin, daß die naive Störungsentwicklung nicht eindeutig ist. Diese Uneindeutigkeit sieht man beispielsweise schon daran, daß man für jeden Faktor $s^{(.)}$ in (1.2.18) willkürlich die avancierte oder retardierte Greensfunktion oder eine Linearkombination der beiden wählen kann. Um die Uneindeutigkeit zu beseitigen, nutzen wir aus, daß die Störungsentwicklung für die avancierte und retardierte Greensche Funktion als Folge der Kausalität eindeutig ist. Indem wir die Störungsentwicklung des Diracsees auf diejenige für die Greenschen Funktionen zurückführen, erhalten wir auch eine eindeutige Störungsentwicklung für den Diracsee. Der Zusammenhang zwischen den Greenschen Funktionen und dem Diracsee wird mit einem speziellen *Operatorkalkül* hergestellt, der Ähnlichkeit mit Methoden der Funktionalanalysis hat. Damit lassen sich mathematische Operationen wie Multiplikation oder Wurzelziehen auf Feynman-Diagramme anwenden. Dies führt auf die sog. *kausale Störungsentwicklung* für den Diracsee.

In Kapitel 3 werden zunächst die einzelnen Feynman-Diagramme analytisch untersucht. Sie sind Distributionslösungen einer inhomogenen Dirac-Gleichung. Da die Dirac-Gleichung hyperbolisch ist, kann man das Verhalten der Lösungen in der Nähe des Lichtkegels mit der Methode der *Integration längs Charakteristiken* untersuchen (siehe z.B. [7]). Um diese Methode am einfachsten Beispiel zu erklären, betrachten wir eine Lösung f der zweidimensionalen, inhomogenen Wellengleichung

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) f(t,x) \;=\; g(t,x) \quad .$$

Der Lichtkegel um den Ursprung besteht hier aus den beiden Geraden $t = \pm x$. In „Lichtkegelkoordinaten“ $u = \frac{1}{2}\,(t+r)$, $v = \frac{1}{2}\,(t-r)$ nimmt der Wellenoperator die Form

$$\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \;=\; \frac{\partial}{\partial u} \frac{\partial}{\partial v}$$

an. Da der Wellenoperator nur ersten Ableitungen in u und v enthält, kann man eine der beiden Ableitungen durch Integration der Wellengleichung eliminieren, z.B.

$$\frac{\partial}{\partial v} f(u,v) \;-\; \frac{\partial}{\partial v} f(0,v) \;=\; \int_0^u g(\tau, v)\, d\tau \quad . \tag{1.2.19}$$

Für $v = 0$ erlaubt diese Gleichung, die transversale Ableitung von f auf der Geraden $t = x$ mit Hilfe eines Linienintegrals längs des Lichtkegels auszudrücken. Durch Iteration lassen sich auch explizite Formeln für die höheren partiellen Ableitungen von f bestimmen. Leider kann die Methode der Integration längs Charakteristiken in dieser einfachen Form nicht auf unser Problem angewendet werden. Grob gesagt besteht die Schwierigkeit darin, daß die Feynman-Diagramme nicht, wie bei Lösungen hyperbolischer Gleichungen üblich, über ein Anfangswertproblem gegeben, sondern wie in (1.2.18) als ein Operatorprodukt definiert sind. Bei der einfachen Integration längs Charakteristiken kann man aber die spezielle Form der Operatorprodukte nicht ausnutzen. Um diese Schwierigkeit zu umgehen, wird in dieser Arbeit eine Technik bereitgestellt, die *Lichtkegelentwicklung* genannt wird. Damit können Operatorprodukte der Form wie in (1.2.18) in einer Umgebung des Lichtkegels entwickelt werden. Ähnlich wie in (1.2.19) treten dabei Linienintegrale längs des Lichtkegels auf. Bei den Feynman-Diagrammen höherer Ordnung erhält man aber nicht nur ein einfaches Integral, sondern mehrere ineinander geschachtelte Linienintegrale. Außerdem liefert die Lichtkegelentwicklung in einem Schritt die Formeln für die partiellen Ableitungen beliebiger Ordnung auf dem Lichtkegel. Dazu wird das Feyman-Diagramm mit einer unendlichen Reihe von Linienintegralen ausgedrückt.

Mit der Lichtkegelentwicklung können alle Feynman-Diagramme explizit im Ortsraum untersucht werden. Die verbleibende Aufgabe besteht darin, die unendliche Summe über alle Feynman-Graphen zu kontrollieren. Dies führt auf eine Reihe von kombinatorischen und analytischen Problemen, die hier in Worten und ohne ihren genauen technischen Zusammenhang nicht beschrieben werden können. Durch die Untersuchung der Konvergenz der Störungsreihe wird die Lichtkegelentwicklung zu einem Verfahren, um Lösungen der Dirac-Gleichung mit Wechselwirkung (1.2.17) zu analysieren. Insbesondere können damit sowohl die Greenschen Funktionen als auch der Diracsee nicht-perturbativ untersucht werden.

1.3 Die Ergebnisse in Worten

Die kausale Störungsentwicklung liefert eine eindeutige Definition des Diracsees im äußeren Feld. Das äußere Feld kann eine beliebige Orts- und Zeitabhängigkeit haben; wir müssen lediglich voraussetzen, daß das äußere Potential genügend regulär ist und im Unendlichen hinreichend schnell abfällt. Um den Fall mehrerer Teilchensorten zu berücksichtigen, wird die Konstruktion für ein System von Diracseen durchgeführt, das *fermionischer Projektor* genannt wird. Außerdem ist die Störung des Dirac-Operators so allgemein gehalten, daß sie neben einem elektromagnetischen Feld auch beliebige Yang-Mills-Felder und das Gravitationsfeld umfaßt. Für den Fall, daß der fermionische Projektor chirale Teilchen (also Teilchen einer ausgezeichneten Händigkeit wie beispielsweise die Neutrinos im Standardmodell) enthält, erhält man eine zusätzliche Bedingung an das äußere Feld, es muß *kausalitätskompatibel* sein. Diese Bedingung gibt eine Einschränkung für die Wechselwirkung der chiralen Teilchen mit den übrigen Dirac-Teilchen. Sie scheint physikalisch sinnvoll zu sein, denn sie ist für die Wechselwirkungen des Standardmodells erfüllt.

Mit unserer Definition des Diracsees wird das Kleinsche Paradox zumindest auf formaler Ebene gelöst. Wir beschreiben nun, wie dies anschaulich zu verstehen ist. Wie bereits erwähnt, wird der Diracsee im äußeren Feld mit einer Operatorkalkülkonstruktion aus den Greenschen Funktionen gewonnen. Genauer wird dabei der Absolutbetrag eines Operators, der auf den Wellenfunktionen der Raumzeit wirkt, gebildet. Das Bilden des Absolutbetrages ist eine *nicht-kausale* Operation, d.h. es werden dabei auch Gebiete der Raumzeit miteinander verknüpft, die raumartig getrennt sind. Dies hat zur Folge, daß der Diracsee an einem gegebenen Punkt der Raumzeit auch vom äußeren Feld an Gebieten der Raumzeit abhängt, die in der Zukunft dieses Punktes liegen oder davon raumartig getrennt sind. Betrachten wir zur Erläuterung die Situation des Kleinschen Paradoxes etwas genauer. Streng genommen ist die kausale Störungsentwicklung nicht auf die Potentialschwelle $\phi(t)$ anwendbar, weil das Potential im Unendlichen nicht abfällt; dies ist aber kein grundlegendes Problem, da man das Potential im Unendlichen abändern kann, ohne dadurch die physikalische Situation wesentlich zu verändern. Die kausale Störungsentwicklung liefert dann auf eindeutige Weise einen Diracsee. Dieser Diracsee stimmt weder für $t < 0$ noch für $t > 1$ mit dem „Raum aller Zustände negativer Energie“ überein, in beiden asymptotischen Gebieten sind sowohl Zustände positiver als auch negativer Energie besetzt. Im Grenzfall $\phi_0 \to 0$ des Vakuums geht der Diracsee aber in den Raum aller Ebenen-Wellen-Lösungen negativer Energie über. Man sieht also, daß der Diracsee zur Zeit $t < 0$ davon abhängt, wie sich das elektrische Potential zu einem späteren Zeitpunkt verhält. Daran zeigt sich die Nicht-Kausalität der Konstruktion. Auf diese Weise verschwindet aber gleichzeitig das Kleinsche Paradox, denn der Diracsee bleibt für ein beliebiges Potential wohldefiniert. Etwas allgemeiner kann man sagen, daß mit der kausalen Störungsentwicklung die Vorstellung des Diracsees als „alle Zustände negativer Energie“, die in der zeitabhängigen Situation keinen Sinn mehr macht, ganz aufgegeben wird. Zwar ist der Diracsee im statischen äußeren Feld immer noch genau aus allen Eigenlösungen negativer Energie aufgebaut, doch wird der Diracsee nicht mehr durch diese Eigenschaft charakterisiert. Anstatt dessen geht die Kausalität in die allgemeine Definition des Diracsees ein.

Es mag auf den ersten Blick merkwürdig erscheinen, daß zwar die Kausalität für die Konstruktion des Diracsees verwendet wird, der so erhaltene Diracsee aber trotzdem ein nicht-kausales Objekt ist. In der Tat ist die genaue Rolle der Kausalität an der formalen Definition des Diracsees nicht unmittelbar erkennbar. Da die Kausalität eng mit der Geometrie des Lichtkegels zusammenhängt, bietet die Lichtkegelentwicklung von Kapitel 3 die geeignete Methode, um die kausale Struktur des Diracsees aufzuklären. Es wird gezeigt, daß der Diracsee aus der Summe eines *kausalen* und eines *nicht-kausalen Anteils* aufgebaut ist. Der kausale Anteil besitzt *Singularitäten auf dem Lichtkegel*. Mit Hilfe der Lichtkegelentwicklung können diese Singularitäten vollständig mit geschachtelten Linienintegralen über das äußere Potential und dessen partielle Ableitungen beschrieben werden. Die „Kausalität“ der Störungsrechnung für den Diracsee bedeutet, daß diese Linienintegrale als beschränkte, konvexe Integrale das Kausalitätsprinzip erfüllen. Bei einer naiven Störungsrechnung für den Diracsee können die Singularitäten dagegen i.a. nur mit Linienintegralen beschrieben werden, die bis ins Unendliche reichen und so die Kausalität verletzen. Der nicht-kausale Anteil ist in jeder Ordnung Störungstheorie eine glatte Funktion. Man kann ihn in den sog. *Hochenergie-* und *Niederenergieanteil* zerlegen, die einen unterschiedlichen physikalischen Ursprung haben. Der Hochenergieanteil kommt dadurch zustande, daß durch Multiplikation mit dem Potential Zustände der freien Dirac-Gleichung positiver und negativer Energie miteinander gemischt werden. Er ist ein Effekt höherer Ordnung Störungstheorie und spielt nur dann eine wichtige Rolle, wenn die Ener-

gie (d.h. Frequenz) des äußeren Potentials von der Größenordnung der Ruheenergie der Dirac-Teilchen ist. Der Niederenergieanteil ist dagegen unabhängig von der Energie des äußeren Feldes von Bedeutung. Sein Auftreten hängt letztlich damit zusammen, daß der fermionische Projektor $P(x,y)$ des Vakuums (1.2.15) (im Gegensatz zur retardierten und avancierten Greenschen Funktion) auch dann einen Beitrag hat, wenn die Punkte x und y raumartig getrennt sind.

Wenn man unsere Konstruktion ernst nimmt, ist das Kausalitätsprinzip der speziellen Relativitätstheorie in seiner strengen Form verletzt. Sieht man nämlich den Diracsee als ein physikalisches Objekt an, so beeinflussen sich auch Gebiete der Raumzeit gegenseitig, die raumartig getrennt sind. Allerdings kann man diesen Effekt nicht dazu ausnutzen, um Information schneller als mit Lichtgeschwindigkeit zu übertragen. Um dies am Beispiel zu erklären, betrachten wir zwei Punkte x und y der Raumzeit, die raumartig getrennt sind. Um Information von x nach y zu übertragen, müßte man im Punkt x ein Signal aussenden, das in y empfangen und entschlüsselt werden kann. Mit Dirac-Teilchen läßt sich das Signal auf zwei Arten senden. Die erste Möglichkeit besteht darin, die Wellenfunktionen einzelner Teilchen zu verändern, also beispielsweise in x einen Teilchenstrahl zu emittieren. Da die Wellenfunktionen Lösungen der Dirac-Gleichung sind und die Dirac-Gleichung eine kausale Gleichung ist, breitet sich bei einem solchen Prozeß das Signal höchstens mit Lichtgeschwindigkeit aus und kann folglich im Punkt y nicht empfangen werden. Als zweite Möglichkeit kann man in x das äußere Feld modulieren, also beispielsweise das elektrische Potential sprunghaft verändern. Durch die nicht-kausalen Effekte wird dadurch der Diracsee im Punkt y tatsächlich beeinflußt. Allerdings kann das Signal im Punkt y nicht entschlüsselt werden. Denn selbst unter der Annahme, daß ein Beobachter den Diracsee in einer Umgebung des Punktes y genau kennt, kann er nicht zurückschließen, wie das ursprünglich gesendete Signal aussah. Das sieht man mathematisch daran, daß der nicht-kausale Anteil Integrale über das äußere Feld enthält, die nicht nach dem äußeren Potential aufgelöst werden können. Wir kommen zu dem Schluß, daß die Kausalität in einer leicht abgeschwächten Form als „Kausalität bei der Übertragung von Information“ weiterhin gültig ist.

Um Mißverständnisse zu vermeiden, weisen wir darauf hin, daß die strenge Kausalität schon in der Störungstheorie erster Ordnung für den Diracsee verletzt ist, ganz gleich, mit welcher Methode die Störungsrechnung durchgeführt wird. So gesehen ist die Nicht-Kausalität des Diracsees also kein neuer Effekt. Allerdings wurde der Effekt noch nie systematisch untersucht und seine Bedeutung beispielsweise für das Kleinsche Paradox nicht erkannt. Es gibt zwei Gründe dafür, daß das Studium der nicht-kausalen Effekte des Diracsees bisher so sehr vernachlässigt wurde: Zum einen werden Feynman-Diagramme in der Quantenfeldtheorie meist nur im Impulsraum betrachtet, wo die kausale Struktur schwer erkennbar ist. Außerdem fehlte mit der kausalen Störungsentwicklung der geeignete Rahmen, um den Diracsee im zeitabhängigen Feld sinnvoll zu beschreiben.

Außer der Untersuchung der Kausalität kann mit der Lichtkegelentwicklung auch die formale Definition des Diracsees auf eine solidere mathematische Grundlage gestellt und die Abhängigkeit vom äußeren Feld expliziter beschrieben werden. Dies soll nun kurz dargestellt werden. Alle Feynman-Diagramme sind als Distributionen wohldefiniert. Wie bereits erwähnt, führen wir die Lichtkegelentwicklung zunächst für alle Feynman-Diagramme durch. Dabei nehmen wir an, daß das äußere Feld aus chiralen und skalaren/pseudoskalaren Potentialen, also beispielsweise aus beliebigen links- oder rechtshändigen Yang-Mills-Feldern, aufgebaut ist. Diese Annahme ist eine technische Vereinfachung, unsere Methoden sind aber auch auf allgemeinere Potentiale anwendbar. Die Lichtke-

gelentwicklung liefert für jedes Feynman-Diagramm eine unendliche Reihe geschachtelter Linienintegrale über das äußere Potential und dessen partielle Ableitungen. Diese Entwicklung ist zunächst nur formal. Nach Abspaltung des nicht-kausalen Anteils, von dem gezeigt wird, daß er eine glatte Funktion ist, ist die Lichtkegelentwicklung des verbleibenden kausalen Anteils aber mathematisch wohldefiniert. Ein wichtiges Ergebnis dieser Arbeit besteht darin zu zeigen, daß für die Lichtkegelentwicklung des kausalen Anteils die unendliche Summe über alle Feynman-Diagramme explizit ausgeführt werden kann. Dadurch wird der kausale Anteil des Diracsees zu einem unabhängig von der Störungstheorie definierten mathematischen Objekt. Außerdem erhält man auf diese Weise nicht-perturbative Formeln für die Lichtkegelentwicklung des kausalen Anteils. Mit der sog. *Reduktion auf den phasenfreien Anteil* wird eine nützliche Methode bereitgestellt, mit der die Koeffizienten dieser Lichtkegelentwicklung berechnet werden können. Die Lichtkegelentwicklung kann in eine *eichinvariante Form* gebracht werden, an der beispielsweise die Abhängigkeit des kausalen Anteils von den Yang-Mills-Feldern und -Strömen direkt abgelesen werden kann.

Als ein kurzer Ausblick soll abschließend auf einen Punkt hingewiesen werden, den der Autor besonders interessant findet. Wie wir gesehen haben, geht in der relativistischen Quantenmechanik der Determinismus an zwei ganz unterschiedlichen Stellen verloren: bei der statistischen Interpretation und bei den nicht-kausalen Effekten des Diracsees. Es scheint naheliegend, daß diese beiden Phänomene grundlegend miteinander zusammenhängen. Mit den derzeitigen Gleichungen der Physik kann ein solcher Zusammenhang aber nicht hergestellt werden. Dazu müßte man eine Formulierung der Physik finden, die den quantenmechanischen Meßprozeß intrinsisch aus den physikalischen Gleichungen erklärt, und in die der Diracsee unmittelbar als ein physikalisches Objekt eingeht. Der Autor hofft, daß die vorliegende Arbeit dazu beitragen kann, physikalische Gleichungen dieser Art zu finden.

Kapitel 2

Definition of the Dirac Sea in the Presence of External Fields

It is shown that the Dirac sea can be uniquely defined for the Dirac equation with general interaction, if we impose a causality condition on the Dirac sea. We derive an explicit formula for the Dirac sea in terms of a power series in the bosonic potentials.

The construction is extended to systems of Dirac seas. If the system contains chiral fermions, the causality condition yields a restriction for the bosonic potentials.

2.1 Introduction

The Dirac equation has solutions of negative energy, which have no meaningful physical interpretation. This popular problem of relativistic quantum mechanics was originally solved by Dirac's concept that all negative-energy states are occupied in the vacuum forming the so-called Dirac sea. Fermions and anti-fermions are then described by positive-energy states and "holes" in the Dirac sea, respectively. Although this vivid picture of a sea of interacting particles is nowadays often considered not to be taken too literally, the construction of the Dirac sea also plays a crucial role in quantum field theory. There it corresponds to the formal exchanging of creation and annihilation operators for the negative-energy states of the free field theory.

Usually, the Dirac sea is only constructed in the vacuum. This is often considered to be sufficient, because the interacting system can be described by a perturbation of the vacuum. Unfortunately, the situation is more difficult: In relativistic quantum mechanics with interaction, the fermionic wave functions are solutions of the Dirac equation

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{\Psi} \;=\; 0 \quad , \tag{2.1.1}$$

where the operator $\mathcal{B}$ is composed of the bosonic potentials (for example, we can describe the electromagnetic interaction by choosing $\mathcal{B} = eA\!\!\!/$ with the electromagnetic potential A). In contrast to the free Dirac equation $(i\partial\!\!\!/ - m)\, \Psi = 0$, it is not obvious how to characterize the negative-energy solutions of (2.1.1). Qualitatively, the problem is that the perturbation $\mathcal{B}$ leads to a mixing of the free solutions and destroys the natural splitting into solutions of positive and negative energy. As a consequence, it is not clear how the Dirac sea of the system (2.1.1) can be constructed. We point out that this problem is not solved by a simple perturbation expansion in $\mathcal{B}$; it is then hidden in the non-uniqueness of this expansion (see section 2.2 for details). In quantum field theory, the problem of defining

the Dirac sea is even more complicated, because the virtual pair creation/annihilation must be taken into account. We will not deal these problems here and restrict to the limit of "classical" potentials and wave functions. Nevertheless, our considerations are also relevant for quantum field theory, because it is in many situations (e.g. for a quantum system in a classical background field) preferable to use the Dirac equation (2.1.1) as the starting point for the fermionic field quantization. In this sense, the construction of the Dirac sea of (2.1.1) is preliminary for the description of interacting quantum fields.

We conclude that the definition of the Dirac sea is basic for a reasonable physical interpretation of the Dirac equation (2.1.1). In the present paper, we will discuss the difficulty in constructing the Dirac sea and finally solve the problem in terms of a formal perturbation expansion in $\mathcal{B}$. Before starting the analysis, we describe the problem in more mathematical terms: Every solution of the free Dirac equation $(i\partial\!\!\!/ - m)\,\Psi = 0$ is a linear combination of plane wave solutions of the form

$$\Psi(t,\vec{x}) \;=\; e^{-i(\omega t - \vec{k}\vec{x})}\, \chi_{\omega,\vec{k}} \quad , \qquad\qquad \omega \;=\; \pm\sqrt{\vec{k}^2 + m^2}$$

with a 4-spinor $\chi_{\omega,\vec{k}}$ which is independent of t and $\vec{x}$. The sign of ω gives a natural splitting of the solutions into solutions of positive and negative frequency. Identifying frequency and energy via Planck's formula, these solutions are commonly called the positive and negative energy solutions of the free Dirac equation. Since the simple identification of frequency and energy might lead to confusion (sometimes the "energy" of a negative-frequency state denotes the positive energy of the corresponding anti-particle state), we prefer the notion of positive and negative "frequency" in the following. We denote the negative-frequency solutions by $\Psi_{\vec{k}a}$, where $\vec{k}$ is the momentum and $a = 1, 2$ are the two spin states (for an explicit formula for $\Psi_{\vec{k}a}$ see e.g. [2]). If the states $\Psi_{\vec{k}a}$ were normalized with respect to the usual scalar product

$$(\Psi \mid \Phi) \;=\; \int_{\mathbb{R}^3} (\overline{\Psi}\, \gamma^0\, \Phi)(t,\vec{x})\; d\vec{x} \quad , \qquad\qquad \overline{\Psi} = \Psi^* \gamma^0 \quad , \tag{2.1.2}$$

we could form the projector $P_{<\Psi_{\vec{k}a}>}$ on the one-dimensional subspace $<\Psi_{\vec{k}a}>$ by

$$\left(P_{<\Psi_{\vec{k}a}>}\, \Psi\right)(t,\vec{x}) \;=\; \int_{\mathbb{R}^3} \left(\Psi_{\vec{k}a}(t,\vec{x})\; \overline{\Psi_{\vec{k}a}(t,\vec{y})}\right)\; \gamma^0\; \Psi(t,\vec{y})\; d\vec{y} \quad .$$

In this sense, the product $\Psi_{\vec{k}a}(x)\, \overline{\Psi_{\vec{k}a}(y)}$ would be the kernel of the projector on $<\Psi_{\vec{k}a}>$, and the sum over all negative-frequency states would yield the projector on the whole Dirac sea. Unfortunately, the wave functions $\Psi_{\vec{k}a}$ are not normalizable. We could arrange normalizable states by considering the system in finite three-volume, but we do not want to do this here. It is more appropriate for our purpose to formally build up a projector on all negative-frequency states by integrating over the momentum parameter

$$P(x,y) \;=\; \sum_{a=1,2} \int_{\mathbb{R}^3} \Psi_{\vec{k}a}(x)\; \overline{\Psi_{\vec{k}a}(y)}\; d\vec{k} \quad , \tag{2.1.3}$$

which can be rewritten as the integral over the lower mass shell

$$= \int_{\mathbb{R}^3} \frac{d^4k}{(2\pi)^4}\; (k\!\!\!/ + m)\; \delta(k^2 - m^2)\; \Theta(-k^0)\; e^{-ik(x-y)}$$

(Θ denotes the Heavyside function $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ otherwise). $P(x,y)$ is a well-defined tempered distribution which solves the free Dirac equation $(i\partial\!\!\!/_x - m)\, P(x,y) =$

0. We can use it to characterize the Dirac sea in the vacuum. Our aim is to introduce a corresponding distribution $\tilde{P}$ for the Dirac equation with interaction (2.1.1). The construction of $\tilde{P}$ must be unique in a sense which we will discuss and specify later. We will assume the perturbation $\mathcal{B}$ to be a differential operator on the wave functions. Furthermore, it shall be Hermitian with respect to the (indefinite) scalar product

$$<\Psi \mid \Phi> \;=\; \int \overline{\Psi(x)}\, \Phi(x)\, d^4x \quad . \tag{2.1.4}$$

For an electromagnetic potential $\mathcal{B} = e\A$, these assumptions are satisfied because $\A = \gamma^0 \A^\dagger \gamma^0$. In addition, $\mathcal{B}$ can be composed of the scalar, pseudoscalar, pseudovector and bilinear potentials as e.g. discussed in [4]. According to [5], $\mathcal{B}$ also allows for the description of the gravitational field.

2.2 Non-Uniqueness of the Simple Perturbation Expansion

Our first idea for the construction of $\tilde{P}$ is to calculate solutions $\tilde{\Psi}_{\vec{k}a}$ of (2.1.1) with a perturbation expansion in $\mathcal{B}$ and to define $\tilde{P}$ in analogy to (2.1.3) by

$$\tilde{P}(x,y) \;=\; \sum_{a=1,2} \int_{\mathbb{R}^3} \tilde{\Psi}_{\vec{k}a}(x)\, \overline{\tilde{\Psi}_{\vec{k}a}(y)}\, d\vec{k} \quad . \tag{2.2.1}$$

We start with a discussion of this method in a perturbation calculation to first order. This is quite elementary and will nevertheless explain the basic difficulty. For the perturbation calculation, we need a Green's function $s(x,y)$ of the free Dirac operator, which is characterized by the distributional equation

$$(i\partial\!\!\!/_x - m)\, s(x,y) \;=\; \delta^4(x-y) \quad . \tag{2.2.2}$$

To first order, the perturbed eigenstates $\tilde{\Psi}_{\vec{k}a}$ are then given by

$$\tilde{\Psi}_{\vec{k}a}(x) \;=\; \Psi_{\vec{k}a}(x) \;-\; \int d^4y\, s(x,y)\, \mathcal{B}_y\, \Psi_{\vec{k}a}(y) \;+\; \mathcal{O}(\mathcal{B}^2) \quad , \tag{2.2.3}$$

as can be verified by substituting into (2.1.1). We insert this formula into (2.2.1) and obtain

$$\tilde{P}(x,y) \;=\; P(x,y) - \int d^4z\, \left[s(x,z)\, \mathcal{B}_z\, P(z,y) \;+\; P(x,z)\, \mathcal{B}_z\, s^*(z,y)\right] + \mathcal{O}(\mathcal{B}^2) \quad , \tag{2.2.4}$$

where we used that $\mathcal{B}$ is Hermitian with respect to the scalar product (2.1.4), and where $s^*(z,y)$ is given by $s^*(z,y) = \gamma^0\, s(y,z)^\dagger\, \gamma^0$. It is convenient to view the distributions $s(x,y), P(x,y)$ as integral kernels of corresponding operators s, P. Then we can write (2.2.4) with operator products

$$\tilde{P} \;=\; P \;-\; s\,\mathcal{B}\,P \;-\; P\,\mathcal{B}\,s^* \;+\; \mathcal{O}(\mathcal{B}^2) \quad , \tag{2.2.5}$$

where the superscript '*' denotes the adjoint with respect to the scalar product (2.1.4).

Equation (2.2.5) gives a possible definition for $\tilde{P}$. As apparent problem, the construction depends on the choice of the Green's function. For example, we could have chosen for

s either the advanced or the retarded Green's function $s^\vee_m$,$s^\wedge_m$, which are in momentum space as usual given by

$$s^\vee_m(k) \;=\; \lim_{0<\varepsilon\to 0} \frac{\not{k}+m}{k^2-m^2-i\varepsilon k^0} \quad , \qquad s^\wedge_m(k) \;=\; \lim_{0<\varepsilon\to 0} \frac{\not{k}+m}{k^2-m^2+i\varepsilon k^0} \quad . \tag{2.2.6}$$

More systematically, the arbitrariness of our construction is described as follows: According to (2.2.2), the difference between two Green's functions is a solution of the free Dirac equation. We can thus represent s in the form

$$s(x,y) \;=\; s^\vee_m(x,y) \;+\; a(x,y) \quad ,$$

where $a(x,y)$ is in the x-variable a linear combination of the plane-wave solutions, i.e.

$$a(x,y) \;=\; \sum_{a=1}^4 \int_{\mathbb{R}^3} \Psi_{\vec{k}a}(x)\, c_{\vec{k}a}(y)\, d\vec{k}$$

with (generally complex) functions $c_{\vec{k}a}(y)$, where $\Psi_{\vec{k}a}, a = 3,4$ denote the plane-wave solutions of positive frequency. We substitute into (2.2.5) and obtain

$$\tilde{P} \;=\; P \;-\; s^\vee_m\, \mathcal{B}\, P \;-\; P\, \mathcal{B}\, s^\wedge_m \;-\; (a\, \mathcal{B}\, P \;+\; P\, \mathcal{B}\, a^*) \;+\; \mathcal{O}(\mathcal{B}^2) \quad . \tag{2.2.7}$$

The expression in the brackets maps solutions of the free Dirac equation into each other and vanishes otherwise. We can thus write it in the form

$$(a\, \mathcal{B}\, P \;+\; P\, \mathcal{B}\, a^*)(x,y) \;=\; \sum_{a,b=1}^4 \int_{\mathbb{R}^3} d\vec{k}_1 \int_{\mathbb{R}^3} d\vec{k}_2 \;\Psi_{\vec{k}_1 a}(x)\, g_{ab}(\vec{k}_1,\vec{k}_2)\, \overline{\Psi_{\vec{k}_2 b}(y)} \tag{2.2.8}$$

with suitable functions $g_{ab}(\vec{k}_1,\vec{k}_2)$. This representation of $\tilde{P}$ can also be understood directly: The contribution (2.2.8) describes a mixing of the solutions $\Psi_{\vec{k}a}$ of the free Dirac equation. To the considered first order in $\mathcal{B}$, it vanishes in the Dirac equation $(i\partial\!\!\!/ + \mathcal{B} - m)\tilde{P} = 0$. Thus we cannot fix this contribution with the Dirac equation, it remains undetermined in our method. According to (2.2.7), this is the only arbitrariness of the construction; the other contributions to $\tilde{P}$ are unique.

In higher order perturbation theory, the non-uniqueness can be understood similarly, although the situation is more complicated: For a given Green's function s, we can construct a solution $\tilde{\Psi}_{\vec{k}a}$ of the Dirac equation (2.1.1) by the formal perturbation series

$$\tilde{\Psi}_{\vec{k}a} \;=\; \sum_{n=0}^\infty (-s\, \mathcal{B})^n\; \Psi_{\vec{k}a} \quad , \tag{2.2.9}$$

as is verified by substituting into (2.1.1). Actually, this is a very special ansatz. For example, we can use different Green's functions in every order of the perturbation calculation, which leads to the more general formula

$$\tilde{\Psi}_{\vec{k}a} \;=\; \Psi_{\vec{k}a} \;+\; \sum_{n=1}^\infty (-1)^n\; s^{(n)}\, \mathcal{B} \cdots s^{(2)}\, \mathcal{B}\, s^{(1)}\, \mathcal{B}\, \Psi_{\vec{k}a} \tag{2.2.10}$$

with a whole series of arbitrary Green's functions $s^{(1)}$, $s^{(2)}$, etc.. Once we have a formula for $\tilde{\Psi}_{\vec{k}a}$, the non-uniqueness of $\tilde{P}$ can again be discussed by substituting into (2.2.1). In

generalization of (2.2.8), the arbitrariness of the construction is described by a contribution to $\tilde{P}(x,y)$ of the form

$$\sum_{a,b=1}^{4} \int_{\mathbb{R}^3} d\vec{k}_1 \int_{\mathbb{R}^3} d\vec{k}_2 \; \tilde{\Psi}_{\vec{k}_1 a}(x)\, g_{ab}(\vec{k}_1, \vec{k}_2)\, \overline{\tilde{\Psi}_{\vec{k}_2 b}(y)} \quad ,$$

which mixes perturbed eigenstates $\tilde{\Psi}_{\vec{k}a}$ and vanishes in the Dirac equation $(i\partial\!\!\!/ + \mathcal{B} - m)\tilde{P} = 0$. The dependence of $g_{ab}(\vec{k}_1, \vec{k}_2)$ on $\mathcal{B}$ and on the Green's functions $s^{(n)}$ is rather involved, however, and we need not go into the details here.

To summarize, a simple perturbation expansion in $\mathcal{B}$ is not unique and therefore does not allow a meaningful definition of $\tilde{P}$. In the ansatz (2.2.10), for example, we should find a way to specify the Green's functions $s^{(n)}$. This cannot be done with the Dirac equation (2.1.1), and we must therefore look for additional input to completely determine $\tilde{P}$. Our basic idea is to apply some causality principle. For example, it might seem a reasonable condition to impose that $\tilde{P}(x,y)$ only depends on $\mathcal{B}$ in the "diamond" $(L_x^\vee \cap L_y^\wedge) \cup (L_x^\wedge \cap L_y^\vee)$, where

$$L_x^\vee \;=\; \Big\{ y \,|\, (y-x)^2 \geq 0,\; y^0 - x^0 \geq 0 \Big\} \quad , \qquad\qquad L_x^\wedge \;=\; \Big\{ y \,|\, (y-x)^2 \geq 0,\; y^0 - x^0 \leq 0 \Big\}$$

denote the future and past light cones around x, respectively. If we want to study conditions of this type, it is no longer useful to look at the perturbation expansion for the individual states $\Psi_{\vec{k}a}(x)$ (because these states only depend on one argument x). We must take into account for the perturbation expansion that P is composed of many states in a specific way.

2.3 The Causal Perturbation Expansion

In preparation, we first describe how the perturbation expansion for the advanced and retarded Green's functions can be performed uniquely: The support of the distribution $s_m^\vee(x,y)$ is in the future light cone $y \in L_x^\vee$ (this can be checked by calculating the Fourier transform of (2.2.6) with contour integrals). As a consequence, the perturbation operator $\mathcal{B}(z)$ only enters into the operator product

$$(s_m^\vee \,\mathcal{B}\, s_m^\vee)(x,y) \;=\; \int d^4z \; s_m^\vee(x,z)\, \mathcal{B}(z)\, s_m^\vee(z,y) \tag{2.3.1}$$

for $z \in L_x^\vee \cap L_y^\wedge$. In this sense, the expression (2.3.1) is *causal*. Especially, the support of (2.3.1) is again in the future light cone. It follows by iteration that the higher powers

$$s_m^\vee \,\mathcal{B}\, s_m^\vee \,\mathcal{B} \cdots \mathcal{B}\, s_m^\vee \,\mathcal{B}\, s_m^\vee$$

are also causal and have their support in the upper light cone. We define the perturbed advanced Green's function as the formal sum over these operator products,

$$\tilde{s}_m^\vee \;=\; \sum_{k=0}^{\infty} \left(-s_m^\vee \,\mathcal{B}\right)^k \; s_m^\vee \quad . \tag{2.3.2}$$

Accordingly, the perturbed retarded Green's function is defined by

$$\tilde{s}_m^\wedge \;=\; \sum_{k=0}^{\infty} \left(-s_m^\wedge \,\mathcal{B}\right)^k \; s_m^\wedge \quad . \tag{2.3.3}$$

These operators satisfy the defining equations for the perturbed Green's functions

$$(i\partial\!\!\!/ - m + \mathcal{B})\, \tilde{s}^{\vee}_m \;=\; \mathbb{1} \;=\; (i\partial\!\!\!/ - m + \mathcal{B})\, \tilde{s}^{\wedge}_m \quad , \tag{2.3.4}$$

as is verified directly.

Notice that the perturbation expansion for the Green's functions becomes unique by the condition that the contribution to $\tilde{s}^{\vee}_m, \tilde{s}^{\wedge}_m$ to every order has its support in the future and past light cones, respectively. We want to take this construction as the guiding line for the perturbation expansion of P. Unfortunately, the method cannot be directly applied to the Dirac sea, because the distribution $P(x,y)$ does not vanish for space-like $y-x$, and we thus have no notion of causality. As way out, we decompose the free Dirac sea in the form

$$P(x,y) \;=\; \frac{1}{2}\,(p_m(x,y) \;-\; k_m(x,y)) \tag{2.3.5}$$

with the tempered distributions

$$p_m(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\,(k\!\!\!/ + m)\,\delta(k^2 - m^2)\, e^{-ik(x-y)} \tag{2.3.6}$$

$$k_m(x,y) \;=\; \int \frac{d^4k}{(2\pi)^4}\,(k\!\!\!/ + m)\,\delta(k^2 - m^2)\,\epsilon(k^0)\, e^{-ik(x-y)} \tag{2.3.7}$$

(ϵ denotes the step function $\epsilon(x) = 1$ for $x \geq 0$ and $\epsilon(x) = -1$ otherwise). We also consider $p_m(x,y)$ and $k_m(x,y)$ as integral kernels of corresponding operators p_m, k_m. The operator p_m is built up as a formal sum over the projectors on all solutions of the Dirac equation and can be viewed as a spectral projector of the free Dirac operator. The definition of k_m differs from p_m by a relative minus sign for the states on the upper and lower mass shell. As a consequence of this relative minus sign, the Fourier integral (2.3.7) vanishes if $y-x$ is space-like (this can be seen from Lorentzian invariance and a symmetry argument for $k = (0, \vec{k})$). Thus $k_m(x,y)$ is causal in the sense that it has the support in the light cone $y \in L^{\vee}_x \cup L^{\wedge}_x$. This makes it possible to uniquely express its perturbation expansion in terms of the perturbed Green's functions: We substitute the distributional equation

$$\lim_{0<\varepsilon\to 0} \left(\frac{1}{x - i\varepsilon} \;-\; \frac{1}{x + i\varepsilon} \right) \;=\; 2\pi i\, \delta(x)$$

into the formula for k_m in momentum space,

$$\begin{aligned} k_m(p) \;&=\; (p\!\!\!/ + m)\,\delta(p^2 - m^2)\,\epsilon(p^0) \\ &=\; \frac{1}{2\pi i}\,(p\!\!\!/ + m) \lim_{0<\varepsilon\to 0} \left[\frac{1}{p^2 - m^2 - i\varepsilon} \;-\; \frac{1}{p^2 - m^2 + i\varepsilon} \right] \epsilon(p^0) \\ &=\; \frac{1}{2\pi i}\,(p\!\!\!/ + m) \lim_{0<\varepsilon\to 0} \left[\frac{1}{p^2 - m^2 - i\varepsilon p^0} \;-\; \frac{1}{p^2 - m^2 + i\varepsilon p^0} \right] \quad , \end{aligned}$$

and obtain with (2.2.6) a simple relation between k_m and $s^{\vee}_m$, $s^{\wedge}_m$,

$$k_m \;=\; \frac{1}{2\pi i}\,(s^{\vee}_m \;-\; s^{\wedge}_m) \quad . \tag{2.3.8}$$

We extend this relation to the case with external fields:

Def. 2.3.1 *We define the operator $\tilde{k}_m$ by*

$$\tilde{k}_m \;=\; \frac{1}{2\pi i}\,(\tilde{s}^{\vee}_m - \tilde{s}^{\wedge}_m) \tag{2.3.9}$$

with the Green's functions (2.3.2),(2.3.3).

According to (2.3.4), $\tilde{k}_m$ really is a solution of the Dirac equation $(i\partial\!\!\!/ + \mathcal{B} - m)\,\tilde{k}_m = 0$.

In order to explain the significance of this construction, we point out that the factor $\epsilon(k^0)$ in (2.3.7) describes the splitting of the solutions of the free Dirac equation into solutions of positive and negative frequency. With the introduction of $\tilde{k}_m$, we were able to uniquely generalize this splitting to the case with external fields. This solves the basic problem in defining the Dirac sea. It remains to perform the perturbation expansion for $\tilde{p}_m$. On a formal level, this is very easy, because we can remove the relative minus sign for the positive and negative frequency states by taking the absolute value of $\tilde{k}_m$,

$$\tilde{p}_m \overset{\text{formally}}{:=} \sqrt{\tilde{k}_m^2} \quad . \tag{2.3.10}$$

This gives a unique definition for $\tilde{p}_m$. Since $\tilde{k}_m$ is composed of eigenstates of the perturbed Dirac operator with eigenvalue m, it follows automatically that $(i\partial\!\!\!/ + \mathcal{B} - m)\,\tilde{p}_m = 0$.

Unfortunately, it requires some effort to convert the formal relation (2.3.10) into a mathematical definition. The problem is that the square $\tilde{k}_m^2$ is ill-defined; furthermore we want to write $\tilde{p}_m$ as a power series in $\mathcal{B}$. These problems are solved in the following theorem. The reader who is not so interested in the technical details and the combinatorics of the expansion may skip the proof. For the statement of the theorem, we need some notation: We work with the Green's function

$$s_m \;=\; \frac{1}{2}(s_m^\vee + s_m^\wedge) \quad , \tag{2.3.11}$$

which has the advantage of being Hermitian (with respect to the scalar product (2.1.4)). Furthermore, we introduce the series of operator products

$$b_m^< \;=\; \sum_{k=0}^\infty (-s_m\,\mathcal{B})^k \quad , \qquad b_m \;=\; \sum_{k=0}^\infty (-\mathcal{B}\,s_m)^k\,\mathcal{B} \quad , \qquad b_m^> \;=\; \sum_{k=0}^\infty (-\mathcal{B}\,s_m)^k$$

and set for $Q \subset N$

$$F_m(Q,n) \;=\; \begin{cases} p_m & \text{if } n \in Q \\ k_m & \text{if } n \notin Q \end{cases} \quad .$$

Theorem 2.3.2 *The relations (2.3.9),(2.3.10) uniquely determine the perturbation expansion for k_m and p_m. We have the explicit formulas*

$$\tilde{k}_m \;=\; \sum_{\beta=0}^\infty (-i\pi)^{2\beta}\; b_m^<\, k_m\, (b_m\, k_m)^{2\beta}\; b_m^> \tag{2.3.12}$$

$$\tilde{p}_m \;=\; \sum_{\beta=0}^\infty \sum_{\alpha=0}^{\left[\frac{\beta}{2}\right]} c(\alpha,\beta)\; G_m(\alpha,\beta) \tag{2.3.13}$$

with the coefficients

$$c(0,0) \;=\; 1 \quad , \tag{2.3.14}$$

$$c(\alpha,\beta) \;=\; \sum_{n=\alpha+1}^{\beta} (-1)^{n+1}\, \frac{(2n-3)!!}{n!\,2^n} \begin{pmatrix} \beta-\alpha-1 \\ n-\alpha-1 \end{pmatrix} \qquad \textit{for } \beta \geq 1 \tag{2.3.15}$$

and the operator products

$$\begin{aligned} G_m(\alpha,\beta) \;=\; & \sum_{Q \in \mathcal{P}(\beta+1),\;\#Q=2\alpha+1} (-i\pi)^{2\beta} \\ & \times b_m^<\, F_m(Q,1)\, b_m k_m b_m\, F_m(Q,2)\, b_m k_m b_m \cdots b_m k_m b_m\, F_m(Q,\beta+1)\, b_m^> \quad , \end{aligned} \tag{2.3.16}$$

where $\mathcal{P}(n)$ denotes the set of subsets of $\{1,\ldots,n\}$ (we use the convention $l!! = 1$ for $l \leq 0$).

Proof: Notice that $(i\partial\!\!\!/ + \mathcal{B} - m)\, b_m^< = 0$. Since all operator products in (2.3.12),(2.3.16) have a factor $b_m^<$ at the left, the operators $\tilde{p}_m$, $\tilde{k}_m$ are solutions of the Dirac equation

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{p}_m \;=\; 0 \;=\; (i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{k}_m \quad .$$

Thus the theorem gives a possible perturbation expansion for p_m and k_m. We must verify that the conditions (2.3.9),(2.3.10) are satisfied and show uniqueness.

According to (2.3.8), the advanced and retarded Green's function can be written in the form

$$s_m^\vee \;=\; s_m + i\pi\, k_m \quad , \qquad\qquad s_m^\wedge \;=\; s_m - i\pi\, k_m \quad . \tag{2.3.17}$$

We substitute the sums (2.3.2),(2.3.3) into (2.3.9),

$$\tilde{k}_m \;=\; \frac{1}{2\pi i} \sum_{k=0}^\infty \Big((-s_m^\vee\, \mathcal{B})^k\, s_m^\vee \;-\; (-s_m^\wedge\, \mathcal{B})^k\, s_m^\wedge \Big) \quad , \tag{2.3.18}$$

insert (2.3.17) and expand. This gives a sum of operator products of the form

$$C_1\, \mathcal{B}\, C_2\, \mathcal{B} \cdots \mathcal{B}\, C_{l+1} \qquad \text{with} \qquad C_j = k_m \text{ or } C_j = s_m \quad .$$

The contributions with an even number of factors k_m have the same sign for the advanced and retarded Green's function and cancel in (2.3.18). The contributions with an odd number of k_m's occur in every Green's function exactly once and have opposite sign. Using the notation

$$C_m(Q,n) \;=\; \begin{cases} k_m & \text{if } n \in Q \\ s_m & \text{if } n \notin Q \end{cases} \quad , \qquad\qquad Q \subset \mathbb{N} \quad ,$$

we can thus rewrite (2.3.18) in the form

$$\begin{aligned} \tilde{k}_m \;=\; & \sum_{l=0}^\infty (-1)^l \sum_{Q \in \mathcal{P}(l+1),\ \#Q \text{ odd}} (i\pi)^{\#Q-1} \\ & \times C_m(Q,1)\, \mathcal{B}\, C_m(Q,2)\, \mathcal{B} \cdots \mathcal{B}\, C_m(Q,l)\, \mathcal{B}\, C_m(Q,l+1) \quad . \end{aligned}$$

After reordering the sums, this coincides with (2.3.12).

Next we want to give the relation (2.3.10) a mathematical sense. For this, we consider $m \geq 0$ as a variable mass parameter. Then we can form products of the operators p_m, k_m by manipulating the arguments of the distributions in momentum space. For example, we have with (2.3.6)

$$\begin{aligned} p_m(k)\, p_{m'}(k) \;&=\; (k\!\!\!/ + m)\, \delta(k^2 - m^2)\, (k\!\!\!/ + m')\, \delta(k^2 - (m')^2) \\ &=\; (k^2 + (m+m')k\!\!\!/ + mm')\, \delta(m^2 - (m')^2)\, \delta(k^2 - m^2) \\ &=\; (k^2 + (m+m')k\!\!\!/ + mm')\, \frac{1}{2m}\, \delta(m - m')\, \delta(k^2 - m^2) \\ &=\; \delta(m-m')\, p_m(k) \quad , \end{aligned} \tag{2.3.19}$$

and similarly with (2.3.7),

$$p_m\, k_{m'} \;=\; k_{m'}\, p_m \;=\; \delta(m-m')\, k_m \tag{2.3.20}$$

$$k_m\, k_{m'} \;=\; \delta(m-m')\, p_m \quad . \tag{2.3.21}$$

This formalism has some similarity with the bra/ket notation in quantum mechanics, if the position variable $\vec{x}$ is replaced by the mass parameter m. Equation (2.3.19) can be understood directly from the fact that p_m are the spectral projectors of the free Dirac operator; the relations (2.3.20),(2.3.21) reflect the relative minus sign in k_m for the states on the upper and lower mass shell. Especially one sees that $k_m\, k_{m'} = p_m\, p_{m'}$. This relation can be extended to the case with interaction,

$$\tilde{p}_m\, \tilde{p}_{m'} \;=\; \tilde{k}_m\, \tilde{k}_{m'} \quad , \tag{2.3.22}$$

and gives a meaningful square of (2.3.10) (we will see in a moment that $\tilde{k}_m\, \tilde{k}_{m'}$ vanishes for $m \neq m'$). If our construction ensures that $\tilde{p}_m$ is a positive operator, (2.3.22) is even equivalent to (2.3.10).

We calculate the product $\tilde{k}_m \tilde{k}_{m'}$ explicitly: The definitions (2.3.6), (2.3.7) and (2.3.11), (2.2.6) yield in analogy to (2.3.19) the formulas

$$p_m\, s_{m'} \;=\; s_{m'}\, p_m \;=\; \mathrm{PP}\left(\frac{1}{m-m'}\right) p_m \tag{2.3.23}$$

$$k_m\, s_{m'} \;=\; s_{m'}\, k_m \;=\; \mathrm{PP}\left(\frac{1}{m-m'}\right) k_m \tag{2.3.24}$$

$$s_m\, s_{m'} \;=\; \mathrm{PP}\left(\frac{1}{m-m'}\right) (s_m - s_{m'}) \quad , \tag{2.3.25}$$

where $\mathrm{PP}(x^{-1}) = \frac{1}{2}\, \lim_{0<\varepsilon\to 0}[(x+i\varepsilon)^{-1} + (x-i\varepsilon)^{-1}]$ denotes the principal value. As a consequence, the operator products with factors $s_m, s_{m'}$ are telescopic, i.e.

$$\sum_{p=0}^{n} k_m\, (\mathcal{B}\, s_m)^p\, (s_{m'}\, \mathcal{B})^{n-p}\, k_{m'} \;=\; 0 \qquad \text{for } n \geq 1. \tag{2.3.26}$$

This allows us to easily carry out the product $b_m^{>}\, b_m^{<}$ in the expression

$$k_m\, b_m^{>}\, b_{m'}^{<}\, k_{m'} \;=\; \delta(m-m')\, p_m \quad . \tag{2.3.27}$$

With this formula, we can calculate the square of (2.3.12) to

$$\tilde{k}_m\, \tilde{k}_{m'} \;=\; \delta(m-m') \sum_{\beta_1,\beta_2=0}^{\infty} (-i\pi)^{2\beta_1+2\beta_2}\, b_m^{<}\, (k_m\, b_m)^{2\beta_1}\, p_m\, (b_m\, k_m)^{2\beta_2}\, b_m^{>} \quad . \tag{2.3.28}$$

We could continue the proof by verifying explicitly that the product $\tilde{p}_m\, \tilde{p}_{m'}$ with $\tilde{p}_m$ according to (2.3.13) coincides with (2.3.28). This is a straightforward computation, but it is rather lengthy and not very instructive. We prefer to describe how the operator products (2.3.16) and the coefficients (2.3.15) can be derived. In order to keep the proof better readable, we make some simplifications: Since the factors $b_m^{<}$, $b_m^{>}$ cancel similar to (2.3.27) in telescopic sums, we can omit them in all formulas without changing the multiplication rules for the operator products. Then all operator products have k_m or p_m as their first and last factor, and we can multiply them with the rules (2.3.19),(2.3.20), and (2.3.21). Since all these rules give a factor $\delta(m-m')$, we will in any case get the prefactor $\delta(m-m')$ in (2.3.28). Therefore we can just forget about all factors $\delta(m-m')$ and consider all expressions at the same value of m. Furthermore, we will omit the subscript '$_m$' and write the intermediate factors b as a dot '.'. After these simplifications, we end up with formal products of the form

$$F_1\, .\, F_2\, .\, F_3\, .\, \cdots\, .\, F_n \qquad \text{with} \qquad F_j = k \text{ or } F_j = p \tag{2.3.29}$$

and have the multiplication rules

$$p^2 \;=\; k^2 \;=\; 1 \quad , \qquad\qquad p\,k \;=\; k\,p \;=\; k \quad . \tag{2.3.30}$$

We must find a positive operator $\tilde{p}$ being a formal sum over operator products (2.3.29) such that

$$\tilde{p}^2 \;=\; \sum_{\beta_1,\beta_2=0}^{\infty} (-i\pi)^{2\beta_1+2\beta_2}\,(k\,.)^{2\beta_1}\,p\,(.\,k)^{2\beta_2} \quad . \tag{2.3.31}$$

In this way, we have reduced our problem to the combinatorics of the operator products. As soon as we have found a solution $\tilde{p}$ of (2.3.31), the expression for $\tilde{p}_m$ is obtained by adding the subscripts '$_m$' and by inserting the factors $b_m^<$, b_m, $b_m^>$. Relation (2.3.22) follows as an immediate consequence of (2.3.31).

The basic step for the calculation of $\tilde{p}$ is to rewrite (2.3.31) in the form

$$\tilde{p}^2 \;=\; p + A \qquad \text{with} \qquad A \;=\; \sum_{(\beta_1,\beta_2)\neq(0,0)} (-i\pi)^{2\beta_1+2\beta_2}\,(k\,.)^{2\beta_1}\,p\,(.\,k)^{2\beta_2} \quad . \tag{2.3.32}$$

The operator p is idempotent and acts as the identity on A, $Ap = pA = A$. Therefore we can take the square root of $p + A$ with a formal Taylor expansion,

$$\tilde{p} \;=\; \sqrt{p+A} \;=\; p \;+\; \sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{(2n-3)!!}{n!\,2^n}\,A^n \quad , \tag{2.3.33}$$

which uniquely defines $\tilde{p}$ as a positive operator.

It remains to calculate A^n. If we take the nth power of the sum in (2.3.32) and expand, we end up with one sum over more complicated operator products. We first consider how these operator products look like: The operator products in (2.3.32) all contain an even number of factors k and exactly one factor p. The factor p can be the 1st, 3rd,... factor of the product. Each combination of this type occurs in A exactly once. If we multiply n such terms, the resulting operator product consists of a total odd number of factors p, k. It may contain several factors p, which all occur at odd positions in the product. Furthermore, the total number of factors p is odd, as one sees inductively. We conclude that A^n consists of a sum of operator products of the form

$$(k\,.\,k\,.)^{q_1}\,p\,.\,k\,.\,(k\,.\,k\,.)^{q_2}\,p\,.\,k\,.\,(k\,.\,k\,.)^{q_3}\,\cdots\,(k\,.\,k\,.)^{q_{2\alpha+1}}\,p\,(.\,k\,.\,k)^{q_{2\alpha+2}} \tag{2.3.34}$$

with $\alpha, q_j \geq 0$. We set $\beta = 2\alpha + \sum_j q_j$. Notice that the number of factors p in (2.3.34) is $2\alpha + 1$; the total number of factors p, k is $2\beta + 1$. The form of the operator product gives the only restriction $0 \leq 2\alpha \leq \beta$ for the choice of the parameters α, β.

Next we count how often every operator product (2.3.34) occurs in the sum: The easiest way to realize (2.3.34) is to form the product of the $\alpha + 1$ factors

$$\Big[(k.k.)^{q_1}\,p\,(.k.k)^{q_2+1}\Big]\;\Big[(k.k.)^{q_3+1}\,p\,(.k.k)^{q_4+1}\Big]\;\cdots\;\Big[(k.k.)^{q_{2\alpha+1}+1}\,p\,(.k.k)^{q_{2\alpha+2}}\Big] \quad . \tag{2.3.35}$$

However, this is not the only possibility to factorize (2.3.34). More precisely, we can apply to each factor in (2.3.35) the identities

$$\begin{aligned}
(k\,.\,k\,.)^q\,p\,(.\,k\,.\,k)^r &= \big[(k\,.\,k\,.)^q\,p\big]\;\big[p\,(.\,k\,.\,k)^r\big] \\
(k\,.\,k\,.)^q\,p\,(.\,k\,.\,k)^r &= \big[(k\,.\,k\,.)^s\,p\big]\;\big[(k\,.\,k\,.)^{q-s}\,p\,(.\,k\,.\,k)^r\big] \\
(k\,.\,k\,.)^q\,p\,(.\,k\,.\,k)^r &= \big[(k\,.\,k\,.)^q\,p\,(.\,k\,.\,k)^{r-s}\big]\;\big[p\,(.\,k\,.\,k)^s\big] \quad .
\end{aligned}$$

By iteratively substituting these identities into (2.3.35), we can realize every factorization of (2.3.34). Each substitution step increases the number of factors by one. The steps are independent in the sense that we can fix at the beginning at which positions in (2.3.35) the product shall be split up, and can then apply the steps in arbitrary order. There are $(\alpha+1)+(q_1-1)+\sum_{j=2}^{2\alpha+1} q_j + (q_{2\alpha+2}-1) = \beta-(\alpha+1)$ positions in (2.3.35) where we could split up the product (in the case $q_1=0$ or $q_{2\alpha+2}=0$, the counting of the positions is slightly different, but yields the same result). Since we want to have n factors at the end, we must choose $n-(\alpha+1)$ of these positions, which is only possible for $\alpha+1 \leq n \leq \beta$ and then gives $(\beta-\alpha-1)!/((n-\alpha-1)!\,(\beta-n)!)$ possibilities.

Combining these combinatorial factors with the constraints $0 \leq 2\alpha \leq \beta$, $\alpha+1 \leq n \leq \beta$ gives for $n \geq 1$

$$\begin{aligned} A^n \;=\; \sum_{\beta=n}^{\infty} \sum_{\alpha=0}^{\min\left(n-1,\left[\frac{\beta}{2}\right]\right)} \binom{\beta-\alpha-1}{n-\alpha-1} \sum_{Q\in\mathcal{P}(\beta+1),\;\#Q=2\alpha+1} \\ \times\,(-i\pi)^{2\beta}\, F(Q,1)\,.\,k\,.\,F(Q,2)\,.\,k\,.\,\cdots\,.\,k\,.\,F(Q,\beta+1) \end{aligned} \tag{2.3.36}$$

with $F(Q,n)=p$ for $n\in Q$ and $F(Q,n)=k$ otherwise. Notice that the last sum in (2.3.36) runs over all possible configurations of the factors p,k in the operator product (2.3.34) for fixed α,β. We finally substitute this formula into (2.3.33) and pull the sums over α,β outside. This gives the desired formula for $\tilde{p}$. ∎

In order to illustrate the derived formulas for $\tilde{p}$ and $\tilde{k}$, we give the contribution up to third order in more detail:

$$\begin{aligned}
\tilde{k}_m \;=\;& k_m \;-\; k_m\,\mathcal{B}\,s_m \;-\; s_m\,\mathcal{B}\,k_m \\
&+s_m\,\mathcal{B}\,s_m\,\mathcal{B}\,k_m \;+\; s_m\,\mathcal{B}\,k_m\,\mathcal{B}\,s_m \;+\; k_m\,\mathcal{B}\,s_m\,\mathcal{B}\,s_m \;-\; \pi^2\,k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m \\
&-s_m\,\mathcal{B}\,s_m\,\mathcal{B}\,s_m\,\mathcal{B}\,k_m \;-\; s_m\,\mathcal{B}\,s_m\,\mathcal{B}\,k_m\,\mathcal{B}\,s_m \\
&\qquad -s_m\,\mathcal{B}\,k_m\,\mathcal{B}\,s_m\,\mathcal{B}\,s_m \;-\; k_m\,\mathcal{B}\,s_m\,\mathcal{B}\,s_m\,\mathcal{B}\,s_m \\
&+\pi^2\,s_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m \;+\; \pi^2\,k_m\,\mathcal{B}\,s_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m \\
&\qquad +\pi^2\,k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,s_m\,\mathcal{B}\,k_m \;+\; \pi^2\,k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,s_m \;+\; \mathcal{O}(\mathcal{B}^4) \\
\tilde{p}_m \;=\;& p_m \;-\; p_m\,\mathcal{B}\,s_m \;-\; s_m\,\mathcal{B}\,p_m \\
&+s_m\,\mathcal{B}\,s_m\,\mathcal{B}\,p_m \;+\; s_m\,\mathcal{B}\,p_m\,\mathcal{B}\,s_m \;+\; p_m\,\mathcal{B}\,s_m\,\mathcal{B}\,s_m \\
&-\frac{\pi^2}{2}\,p_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m \;-\; \frac{\pi^2}{2}\,k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,p_m \\
&-s_m\,\mathcal{B}\,s_m\,\mathcal{B}\,s_m\,\mathcal{B}\,p_m \;-\; s_m\,\mathcal{B}\,s_m\,\mathcal{B}\,p_m\,\mathcal{B}\,s_m \\
&\qquad -s_m\,\mathcal{B}\,p_m\,\mathcal{B}\,s_m\,\mathcal{B}\,s_m \;-\; p_m\,\mathcal{B}\,s_m\,\mathcal{B}\,s_m\,\mathcal{B}\,s_m \\
&+\frac{\pi^2}{2}(s_m\,\mathcal{B}\,p_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m \;+\; p_m\,\mathcal{B}\,s_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m \\
&\qquad +p_m\,\mathcal{B}\,k_m\,\mathcal{B}\,s_m\,\mathcal{B}\,k_m \;+\; p_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,s_m) \\
&+\frac{\pi^2}{2}(s_m\,\mathcal{B}\,k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,p_m \;+\; k_m\,\mathcal{B}\,s_m\,\mathcal{B}\,k_m\,\mathcal{B}\,p_m \\
&\qquad +k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,s_m\,\mathcal{B}\,p_m \;+\; k_m\,\mathcal{B}\,k_m\,\mathcal{B}\,p_m\,\mathcal{B}\,s_m) \;+\; \mathcal{O}(\mathcal{B}^4) \quad .
\end{aligned}$$

The theorem gives precise formulas for the perturbation expansion of the Dirac sea. Both the combinatorics of the factors k_m, p_m, s_m and the numerical prefactors are a non-trivial result and, as far as the author knows, cannot be understood intuitively.

We call the perturbation expansion of this theorem the *causal perturbation expansion.* It allows to uniquely define the Dirac sea by

$$\tilde{P}(x,y) \;=\; \frac{1}{2}\,(\tilde{p}_m - \tilde{k}_m)(x,y) \quad .$$

2.4 Generalization to Systems of Dirac Seas

In the previous section, we defined the Dirac sea for a system of interacting fermions of mass m. A realistic model, however, is composed of several types of fermionic particles with masses $m_1, \ldots, m_f$. Furthermore, the fermions of zero mass may (like the neutrinos in the standard model) occur only as left or right handed particles. The perturbation $\mathcal{B}$ will in general mix up the eigenstates to different masses and will in this way describe an interaction of all the fermions. We will now extend the previous construction to this more general setting.

First we must generalize (2.1.3) and define a distribution $P(x,y)$ which describes the system in the vacuum: In order to distinguish the chirality of the zero-mass fermions, we introduce (4×4)-matrices $X_1, \ldots, X_f$. For the zero-mass fermions $m_j = 0$, they can be either $X_j = \mathbb{1}$, $X_j = \chi_L$, or $X_j = \chi_R$, where $\chi_{L/R} = \frac{1}{2}(1 \mp \gamma^5)$ are the chiral projectors. For $m_j \neq 0$, they must coincide with the identity $X_j = \mathbb{1}$. The Dirac seas of the individual types of fermions are then described by $X_j\, \frac{1}{2}(p_{m_j} - k_{m_j})(x,y)$. The remaining question is how to build up $P(x,y)$ from the individual Dirac seas. In view of the configuration and the interactions of the fermions in the standard model, one might want to use combinations of sums and direct sums

$$P(x,y) \;=\; \bigoplus_l \sum_\alpha X_{l\alpha}\, \frac{1}{2}\,(p_{m_{l\alpha}} - k_{m_{l\alpha}})(x,y) \tag{2.4.1}$$

(e.g. with $l = 1, \ldots, 8 = 2(3+1)$ running over the color, lepton, and isospin index, and with the index $\alpha = 1, \ldots, 3$ to distinguish the three fermion families. It seems reasonable to use the ordinary sum over α because the families show the same interactions). From the mathematical point of view, however, it is easier to use only direct sums

$$P(x,y) \;=\; \bigoplus_{l=1}^{f} X_l\, \frac{1}{2}(p_{m_l} - k_{m_l})(x,y) \quad . \tag{2.4.2}$$

This is no loss of generality, because the ansatz (2.4.1) can be obtained from (2.4.2) by taking a suitable partial trace over the l-index (in our example, by choosing $f = 24 = 3 \cdot 8$ and forming the trace over the three families). For the perturbation expansion, we can also restrict ourselves to the ansatz (2.4.2), because the perturbation expansion for (2.4.1) is obtained by taking the partial trace of $\tilde{P}(x,y)$ (see [8] for a more detailed discussion of this method). Therefore we must in the following only consider a $P(x,y)$ of the form (2.4.2); it is called the *fermionic projector of the vacuum.*

It is convenient to use a matrix notation in the direct sum: We set

$$p(x,y) \;=\; \oplus_{l=1}^{f}\, p_{m_l}(x,y) \quad , \qquad k(x,y) \;=\; \oplus_{l=1}^{f}\, k_{m_l}(x,y)$$

and define the matrices

$$X \;=\; \bigoplus_{l=1}^{f} X_l \quad , \qquad Y \;=\; \frac{1}{m} \bigoplus_{l=1}^{f} m_l \quad ,$$

which are called *chiral asymmetry matrix* and *mass matrix*, respectively (m is an arbitrary mass parameter; e.g. one can choose $m = \max_j m_j$). Then we can write the fermionic projector as

$$P(x,y) \;=\; X\,\frac{1}{2}\,(p(x,y) - k(x,y)) \quad . \tag{2.4.3}$$

Since $m_j = 0$ for $X \neq \mathbb{1}$ and because $p_{m=0}$, $k_{m=0}$ anti-commute with γ^5, we have alternatively

$$P(x,y) \;=\; \frac{1}{2}\,(p(x,y) - k(x,y))\,X^* \quad , \tag{2.4.4}$$

where $X^* = \gamma^0 X^\dagger \gamma^0$ is the adjoint with respect to the scalar product (2.1.4). The fermionic projector is a solution of the free Dirac equation

$$(i\partial\!\!\!/_x - mY)\,P(x,y) \;=\; 0 \quad .$$

In order to describe the interacting system, we again insert a differential operator $\mathcal{B}$ into the Dirac equation. Thus the fermionic projector $\tilde{P}(x,y)$ is supposed to be a solution of the Dirac equation

$$(i\partial\!\!\!/_x + \mathcal{B} - mY)\,\tilde{P}(x,y) \;=\; 0 \quad . \tag{2.4.5}$$

$\mathcal{B}$ may be non-diagonal in the "Dirac sea index" l; we assume it to be Hermitian with respect to the scalar product

$$<\Psi \,|\, \Phi> \;=\; \sum_{l=1}^{f} \int \overline{\Psi_l(x)}\,\Phi_l(x)\,d^4x \quad .$$

The perturbation expansion for k and p can be carried out exactly as in the previous section: We define the advanced and retarded Green's functions by

$$s^\vee(x,y) \;=\; \oplus_{j=1}^f\, s^\vee_{m_j}(x,y) \quad , \qquad s^\wedge(x,y) \;=\; \oplus_{j=1}^f\, s^\wedge_{m_j}(x,y) \quad .$$

Their perturbation expansion is, in analogy to (2.3.2),(2.3.3), uniquely given by

$$\tilde{s}^\vee \;=\; \sum_{k=0}^{\infty} (-s^\vee\,\mathcal{B})^k\,s^\vee \quad , \qquad \tilde{s}^\wedge \;=\; \sum_{k=0}^{\infty} (-s^\wedge\,\mathcal{B})^k\,s^\wedge \quad . \tag{2.4.6}$$

Theorem 2.4.1 *The perturbation expansion for p and k is uniquely determined by the conditions*

$$\tilde{k} \;=\; \frac{1}{2\pi i}\,(\tilde{s}^\vee - \tilde{s}^\wedge) \quad , \qquad \tilde{p} \;\overset{formally}{=}\; \sqrt{\tilde{k}^2} \quad . \tag{2.4.7}$$

We have the explicit formulas

$$\tilde{k} \;=\; \sum_{\beta=0}^{\infty} (-i\pi)^{2\beta}\; b^< \,k\,(b\,k)^{2\beta}\, b^> \quad , \qquad \tilde{p} \;=\; \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\left[\frac{\beta}{2}\right]} c(\alpha,\beta)\,G(\alpha,\beta)$$

with

$$\begin{aligned}
c(0,0) \;&=\; 1 \quad , \\
c(\alpha,\beta) \;&=\; \sum_{n=\alpha+1}^{\beta} (-1)^{n+1}\,\frac{(2n-3)!!}{n!\,2^n}\,\binom{\beta-\alpha-1}{n-\alpha-1} \qquad \textit{for } \beta \geq 1 \textit{ and} \\
G(f,g) \;&=\; \sum_{Q \in \mathcal{P}(\beta+1),\;\#Q = 2\alpha+1} (-i\pi)^{2\beta}\; b^<\,F(Q,1)\,bkb\,F(Q,2)\,bkb\,\cdots\,bkb\,F(Q,\beta+1)\,b^> \quad ,
\end{aligned}$$

where $\mathcal{P}(n)$ is the set of subsets of $\{1,\ldots,n\}$ and where we used the notation

$$s = \frac{1}{2}(s^\vee + s^\wedge) \quad , \qquad F(Q,n) = \begin{cases} p & \text{if } n \in Q \\ k & \text{if } n \notin Q \end{cases}$$

$$b^< = \sum_{k=0}^\infty (-s\,\mathcal{B})^k \quad , \qquad b = \sum_{k=0}^\infty (-\mathcal{B}\,s)^k\,\mathcal{B} \quad , \qquad b^> = \sum_{k=0}^\infty (-\mathcal{B}\,s)^k \quad .$$

Proof: Follows exactly as Theorem 2.3.2. ∎

After this straightforward generalization, we come to the more interesting question of how $\tilde{P}$ can be defined. Our first idea is to set in generalization of (2.4.3)

$$\tilde{P}(x,y) = X\,\frac{1}{2}\,(\tilde{p} - \tilde{k})(x,y) \quad . \tag{2.4.8}$$

This is not convincing, however, because we could just as well have defined $\tilde{P}(x,y)$ in analogy to (2.4.4) by $\tilde{P} = \frac{1}{2}(\tilde{p} - \tilde{k})\,X^*$, which does not coincide with (2.4.8) as soon as X, X^* do not commute with $\mathcal{B}$. It turns out that this arbitrariness in defining the Dirac sea reflects a basic problem of the causal perturbation expansion for systems with chiral asymmetry. In order to describe the problem in more detail, we consider the perturbation calculation for k to first order: According to (2.4.6),(2.4.7), we have

$$\begin{aligned} \tilde{k} &= k - \frac{1}{2\pi i}\,(s^\vee\,\mathcal{B}\,s^\vee - s^\wedge\,\mathcal{B}\,s^\wedge) + \mathcal{O}(\mathcal{B}^2) \\ &= k - s\,\mathcal{B}\,k - k\,\mathcal{B}\,s + \mathcal{O}(\mathcal{B}^2) \quad . \end{aligned} \tag{2.4.9}$$

This expansion is causal in the sense that $\tilde{k}(x,y)$ only depends on $\mathcal{B}$ in the "diamond" $(L_x^\vee \cap L_y^\wedge) \cup (L_y^\vee \cap L_x^\wedge)$, as is obvious in (2.4.9). It is not clear, however, how to insert the chiral asymmetry matrix into this formula. It seems most natural to replace all factors k by Xk,

$$(\tilde{Xk}) = Xk - s\,\mathcal{B}\,Xk - Xk\,\mathcal{B}\,s + \mathcal{O}(\mathcal{B}^2) \quad . \tag{2.4.10}$$

This formula really gives a possible perturbation expansion for the system of Dirac seas. Unfortunately, it cannot be expressed similar to (2.4.9) with the advanced and retarded Green's functions, which means that the causality of the expansion is in general lost. In order to avoid this problem, one might want to insert X at every factor s, k,

$$\begin{aligned} (\tilde{Xk}) &= Xk - Xs\,\mathcal{B}\,Xk - Xk\,\mathcal{B}\,Xs + \mathcal{O}(\mathcal{B}^2) \\ &= Xk - \frac{1}{2\pi i}\,(Xs^\vee\,\mathcal{B}\,Xs^\vee - Xs^\wedge\,\mathcal{B}\,Xs^\wedge) + \mathcal{O}(\mathcal{B}^2) \quad . \end{aligned} \tag{2.4.11}$$

Similar to (2.4.9), this expansion is causal. In general, however, it does not give a solution of the Dirac equation $(i\partial\!\!\!/ + \mathcal{B} - m)\,\tilde{k} = 0$, which does not make sense.

The only way out of this dilemma is to impose that the perturbation expansions (2.4.10) and (2.4.11) must coincide. This yields a condition on the perturbation operator $\mathcal{B}$, which can be characterized as follows: We demand that

$$Xs^\vee\,\mathcal{B}\,Xs^\vee = s^\vee\,\mathcal{B}\,Xs^\vee = Xs^\vee\,\mathcal{B}\,s^\vee \quad . \tag{2.4.12}$$

Since the operator $s^\vee_{m=0}$ anti-commutes with γ^5, we have $Xs^\vee = s^\vee X^*$. Substituting into the second equation of (2.4.12) yields the condition $X^*\,\mathcal{B} = \mathcal{B}\,X$. Since X is idempotent,

this condition automatically implies the first equation of (2.4.12). We formulate the derived condition for the whole Dirac operator $i\partial\!\!\!/ + \mathcal{B} - mY$ and thus combine it with the fact that chiral fermions are massless (i.e. $X^* Y = Y X = Y$) and that X is composed of chiral projectors (which implies that $X^* \partial\!\!\!/ = \partial\!\!\!/ X$).

Def. 2.4.2 *The Dirac operator is called* **causality compatible** *with X if*

$$X^* \,(i\partial\!\!\!/ + \mathcal{B} - mY) \;=\; (i\partial\!\!\!/ + \mathcal{B} - mY)\, X \quad . \tag{2.4.13}$$

In the perturbation expansion to higher order, the condition (2.4.13) allows to commute X through all operator products. Using idempotence $X^2 = X$, we can moreover add factors X to the product, especially

$$X\, C_1 \,\mathcal{B}\, C_1 \,\mathcal{B} \cdots \mathcal{B}\, C_n \;=\; X C_1 \,\mathcal{B}\, X C_1 \,\mathcal{B} \cdots \mathcal{B}\, X C_n \qquad \text{with} \qquad C_j = p,\; C_j = k \text{ or } C_j = s \quad .$$

This ensures the causality of the perturbation expansion. For a Dirac operator which is causality compatible with X, the *fermionic projector in the external field* is uniquely defined by (2.4.8).

2.5 Discussion, Outlook

In this paper, we gave the formal definition of the Dirac sea in the presence of external fields. The method differs considerably from earlier attempts to solve the external field problem (see e.g. [4] and the references therein). Namely, in these previous approaches, the Dirac sea was always constructed as the "negative frequency solutions" of the Dirac equation. The basic problem of this concept is that the notions of "positive" and "negative" frequency do not make sense in the case with general interaction. Therefore the construction was always limited to potentials which are either static or have an only adiabatic time dependence. As shown in this paper, the notion of "negative frequency states" is not essential for the description of the Dirac sea. For a general definition of the Dirac sea, it must be given up and must be replaced by a suitable causality condition. In this way, it becomes possible to define the Dirac sea in the presence of potentials with arbitrary time dependence. Although the details of the perturbation expansion are a bit complicated, the basic concept is very simple. The construction is explicitly covariant. It puts the usual "hole"-interpretation of the Dirac equation on a satisfying theoretical basis.

In order to clarify the connection to the usual definition of the Dirac sea, we describe how our definition simplifies in the limit of static potentials: If considered as multiplication operators, static potentials map functions of positive (negative) frequency into functions of positive (negative) frequency. Since p, k, and s are diagonal in momentum space, they clearly also preserve the sign of the frequency. Thus we have

$$[P^\pm, p] \;=\; [P^\pm, k] \;=\; [P^\pm, s] \;=\; [P^\pm, \mathcal{B}] \;=\; 0 \quad , \tag{2.5.1}$$

where $P^\pm$ denote the projectors on the states of positive and negative frequency, respectively. The operators p and k only differ by a relative minus sign for the states of positive and negative frequency,

$$P^\pm \, p \;=\; \pm\, P^\pm \, k \quad .$$

Using this relation together with (2.5.1), we can replace pairs of factors p by pairs of factors k, e.g.

$$\begin{aligned} \cdots p\,\mathcal{B} \cdots p\,\mathcal{B} \cdots &= \cdots p\,\mathcal{B} \cdots p\,\mathcal{B} \cdots\, (P^+ + P^-) \\ &= P^+ (\cdots k\,\mathcal{B} \cdots k\,\mathcal{B} \cdots) \;+\; P^- (\cdots (-k)\,\mathcal{B} \cdots (-k)\,\mathcal{B} \cdots) \\ &= \cdots k\,\mathcal{B} \cdots k\,\mathcal{B} \cdots \quad , \end{aligned}$$

where the dots '$\cdots$' denote any combination of the operators s, k, p, and $\mathcal{B}$. This allows us to simplify the formula for $\tilde{p}$ by only using exactly one factor p in every operator product. After going through the details of the combinatorics, one obtains the formula

$$\tilde{p} \;=\; \sum_{b=0}^{\infty} (-i\pi)^{2b}\, b^{<}\, p\, (b\,k)^{2b}\, b^{>} \quad .$$

Thus the Dirac sea (2.4.8) can be written as

$$\tilde{P}(x,y) \;=\; \sum_{b=0}^{\infty} (-i\pi)^{2b}\, b^{<} \left[\frac{1}{2}\, X\, (p-k)\right] (b\,k)^{2b}\, b^{>} \quad .$$

This equation shows that $\tilde{P}(x,y)$ is composed of the negative-frequency eigenstates of the Dirac operator (notice that the expression in the brackets $[\cdots]$ is the fermionic projector of the vacuum and that all other factors preserve the sign of the frequency). Thus, for static potentials, our definition is equivalent to the usual concept of "negative frequency states." On the other hand, this consideration illustrates in which way our definition goes beyond the usual picture.

In order to get a better understanding of the time-dependent situation, we next consider a scattering process. For simplicity, we use the elementary framework of [2], but our consideration also applies to the operator algebra and Fock space formalism as e.g. described in [9]. We first recall how a scattering process is commonly described in the classical Dirac theory. We assume the scattering to take place in finite time $t_0 < t < t_1$. This means that the external perturbation $\mathcal{B}$ in (2.1.1) vanishes outside this time interval,

$$\mathcal{B}(t,\vec{x}) \;=\; 0 \qquad \text{for} \qquad t \notin [t_0, t_1] \quad . \tag{2.5.2}$$

We consider a solution $\tilde{\Psi}$ of the Dirac equation with interaction (2.1.1). According to (2.5.2), $\tilde{\Psi}(t,\vec{x})$ is, for $t < t_0$, a solution of the free Dirac equation. We uniquely extend this free solution to the whole Minkowski space and denote it by $\tilde{\Psi}_{\text{in}}$, i.e.

$$(i\partial\!\!\!/ - m)\, \tilde{\Psi}_{\text{in}} \;=\; 0 \qquad \text{with} \qquad \tilde{\Psi}_{\text{in}}(t,\vec{x}) \;=\; \tilde{\Psi}(t,\vec{x}) \quad \text{for } t < t_0.$$

Similarly, $\tilde{\Psi}(t,\vec{x})$ is also for $t > t_1$ a solution of the free Dirac equation; we denote its extension by $\tilde{\Psi}_{\text{out}}$,

$$(i\partial\!\!\!/ - m)\, \tilde{\Psi}_{\text{out}} \;=\; 0 \qquad \text{with} \qquad \tilde{\Psi}_{\text{out}}(t,\vec{x}) \;=\; \tilde{\Psi}(t,\vec{x}) \quad \text{for } t > t_1.$$

The wave functions $\tilde{\Psi}_{\text{in}}$ and $\tilde{\Psi}_{\text{out}}$ are called the incoming and outgoing scattering states. The S-matrix S maps the incoming scattering states into the corresponding outgoing states, i.e.

$$\tilde{\Psi}_{\text{out}} \;=\; S\, \tilde{\Psi}_{\text{in}} \qquad \text{for every } \tilde{\Psi} \text{ with } (i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{\Psi} \;=\; 0 \quad .$$

As a consequence of the Dirac current conservation, S is a unitary operator (with respect to the scalar product (2.1.2)). Using the scattering states, one can build up asymptotic Dirac seas for $t < t_0$ and $t > t_1$. Namely, for an observer in the past $t < t_0$, the bosonic potentials are zero. Thus it is natural for him to describe the vacuum with the free Dirac sea (2.1.3). If this Dirac sea is extended to the whole Minkowski space with external field, one gets the object

$$\tilde{P}^\wedge(x,y) \;=\; \sum_{a=1,2} \int_{\mathbb{R}^3} \tilde{\Psi}^\wedge_{\vec{k}a}(x)\, \overline{\tilde{\Psi}^\wedge_{\vec{k}a}(y)}\, d\vec{k} \quad ,$$

where the wave functions $\tilde{\Psi}^\wedge_{\vec{k}a}$ are the solutions of the perturbed Dirac equation whose incoming scattering states are the plane wave solutions $\Psi_{\vec{k}a}$,

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{\Psi}^\wedge_{\vec{k}a} \;=\; 0 \qquad \text{with} \qquad (\tilde{\Psi}^\wedge_{\vec{k}a})_{\text{in}} \;=\; \Psi_{\vec{k}a} \quad .$$

Accordingly, an observer in the future $t > t_0$ describes the vacuum with the Dirac sea

$$\tilde{P}^\vee(x,y) \;=\; \sum_{a=1,2} \int_{\mathbb{R}^3} \tilde{\Psi}^\vee_{\vec{k}a}(x)\, \overline{\tilde{\Psi}^\vee_{\vec{k}a}(y)}\, d\vec{k} \quad ,$$

where

$$(i\partial\!\!\!/ + \mathcal{B} - m)\, \tilde{\Psi}^\vee_{\vec{k}a} \;=\; 0 \qquad \text{with} \qquad (\tilde{\Psi}^\vee_{\vec{k}a})_{\text{out}} \;=\; \Psi_{\vec{k}a} \quad .$$

The states $\tilde{\Psi}^\vee_{\vec{k}a}$ and $\tilde{\Psi}^\wedge_{\vec{k}a}$ have a more explicit form in terms of the perturbation series

$$\tilde{\Psi}^\wedge_{\vec{k}a} \;=\; \sum_{n=0}^{\infty} (-s^\wedge\, \mathcal{B})^n\, \Psi_{\vec{k}a} \qquad \text{and} \qquad \tilde{\Psi}^\vee_{\vec{k}a} \;=\; \sum_{n=0}^{\infty} (-s^\vee\, \mathcal{B})^n\, \Psi_{\vec{k}a} \quad ,$$

as is immediately verified with (2.5.2) using that the support of the advanced and retarded Green's functions is the future and past light cone, respectively. The asymptotics of the Dirac seas is completely described by the S-matrix; namely

$$\tilde{P}^\wedge_{\text{in}} \;=\; P \;=\; \tilde{P}^\vee_{\text{out}} \quad , \qquad \tilde{P}^\wedge_{\text{out}} \;=\; S\, \tilde{P}^\vee_{\text{out}}\, S^{-1} \quad , \qquad \tilde{P}^\vee_{\text{in}} \;=\; S^{-1}\, \tilde{P}^\wedge_{\text{in}}\, S \quad . \tag{2.5.3}$$

The physical scattering process is conveniently described with the two Dirac seas of the observers in the past and in the future: If the physical system is described by $\tilde{P}^\wedge$, for example, the observer in the past is in the vacuum. According to (2.5.3), $\tilde{P}^\wedge$ does in general not coincide with the Dirac sea $\tilde{P}^\vee$. This means that for the observer in the future, both positive frequency states are occupied and negative frequency states are unoccupied, so that for him the system contains both particles and anti-particles. This explains the physical effect of pair creation. Other scattering processes are described similarly.

The causal perturbation expansion yields a unique object $\tilde{P}$ describing the Dirac sea in the scattering process. $\tilde{P}$ coincides neither with $\tilde{P}^\vee$ nor with $\tilde{P}^\wedge$; since its construction involves both the advanced and retarded Green's functions, it can be considered as being an "interpolation" between $\tilde{P}^\wedge$ and $\tilde{P}^\vee$. At first sight, it might seem strange that the Dirac sea is now in both asymptotic regions $t < t_0$ and $t > t_1$ described by the same object. Namely, it was essential for our discussion of pair creation that the Dirac seas of the past and future observers were different. It might seem that by redefining the Dirac sea, we no longer have pair creation. Clearly, this is not the case; all physical effects occur in the same way regardless if one works with the asymptotic Dirac seas $\tilde{P}^\wedge$, $\tilde{P}^\vee$ or with $\tilde{P}$. This is because the S-matrix, which completely describes the physical scattering process, does

not depend on the definition of the Dirac sea. Thus the choice of the definition of the Dirac sea in the asymptotic regions is merely a matter of convenience. This may require some explanation: Suppose that we describe the Dirac sea with $\tilde{P}$. Then the asymptotic Dirac seas $\tilde{P}_{\text{in}}$ and $\tilde{P}_{\text{out}}$ consist of both positive and negative frequency states. As a consequence, they are not stable; they tend to decay into the Dirac sea P of all negative-energy states (this is clear physically from the fact that P has lower energy than $\tilde{P}_{\text{in}}$ and $\tilde{P}_{\text{out}}$). Taking this into account, one gets a consistent description of the physical observations. A further complication with $\tilde{P}$ is that the current and energy distributions in the asymptotic regions are in general not homogeneous. For these reasons, it is highly inconvenient to describe the scattering process only with $\tilde{P}$; it is much easier to work with $\tilde{P}^\wedge$ and $\tilde{P}^\vee$. But apart from these purely practical considerations, there is no reason against the description of the Dirac sea with $\tilde{P}$. The great advantage of the causal perturbation expansion is that it gives a unique definition of the Dirac sea, even in the region with interaction $t_0 < t < t_1$. The Dirac sea is not defined with reference to an observer, but becomes a global object of space-time.

Our definition of the Dirac sea is the starting point for the more technical analysis in [10], where all operator products are estimated and computed explicitly in an expansion around the light cone. In order to further clarify the definition of the Dirac sea, we now qualitatively anticipate some results of [10].

First of all, we explain what "causality" of the perturbation expansion for the Dirac sea precisely means: The expansion (2.3.9) for $\tilde{k}_m(x,y)$ is causal in the strict sense that the perturbation operator $\mathcal{B}(z)$ only enters for z in the "diamond" $z \in (L_x^\vee \cap L_y^\wedge) \cup (L_x^\wedge \cap L_y^\vee)$. Since $p_m(x,y)$ does not vanish for space-like $y-x$, its perturbation expansion, and consequently also the expansion of the Dirac sea, cannot be causal in this strict sense. As is shown in [10], the distribution $\tilde{P}(x,y)$ has singularities on the light cone (i.e. for $(y-x)^2 = 0$). It turns out that these singularities can be completely described in terms of $\mathcal{B}(z)$ and its partial derivatives along the convex line $z \in \overline{xy}$. Our perturbation expansion is causal in this weaker sense. It is even uniquely characterized by this "causality" of the singularities on the light cone.

Both the operator products and the perturbation series were only treated as formal expressions throughout this paper. We outline in which sense these expressions make mathematical sense: It is shown in [10] that all operator products are well-defined distributions if reasonable regularity conditions on $\mathcal{B}$ are assumed. The convergence of the perturbation expansion is a more difficult problem. For chiral and scalar/pseudoscalar potentials, convergence is shown in [10] for the formulas of the light-cone expansion by explicit calculation. For a gravitational field, the situation is more complicated, because the contributions to $\tilde{P}(x,y)$ of higher order in $\mathcal{B}$ become more and more singular on the light cone. With a Taylor expansion of the δ-distribution

$$\delta(x+a) \;=\; \delta(x) \;+\; a\,\delta'(x) \;+\; \frac{a^2}{2}\,\delta''(x) \;+\; \cdots \quad ,$$

these contributions can be understood as describing a "deformation" of the light cone (corresponding to the diffeomorphism invariance of General Relativity), but the convergence has not yet been established rigorously.

We finally remark that the fermionic projector $\tilde{P}(x,y)$ of section 2.4 is considered in [8] as the basic physical object. In this context, the above construction gives a unique characterization of $\tilde{P}$ by a perturbation $\mathcal{B}$ of the Dirac operator. This makes it possible to get a connection to the description of the interaction with classical potentials. It turns

out that this "classical limit" is completely determined by the singularities of $\tilde{P}(x, y)$ on the light cone. The "causality" of our perturbation expansion is then directly related to the locality and causality of the classical field equations.

Kapitel 3

Light-Cone Expansion of the Dirac Sea in the Presence of Chiral and Scalar Potentials

We study the Dirac sea in the presence of external chiral and scalar/pseudoscalar potentials. In preparation, a method is developed for calculating the advanced and retarded Green's functions in an expansion around the light cone. For this, we first expand all Feynman diagrams and then explicitly sum up the perturbation series. The light-cone expansion expresses the Green's functions as an infinite sum of line integrals over the external potential and its partial derivatives.

The Dirac sea is decomposed into a causal and a non-causal contribution. The causal contribution has a light-cone expansion which is closely related to the light-cone expansion of the Green's functions; it describes the singular behavior of the Dirac sea in terms of nested line integrals along the light cone. The non-causal contribution, on the other hand, is, to every order in perturbation theory, a smooth function in position space.

3.1 Introduction

In relativistic quantum mechanics with interaction, the fermionic wave functions Ψ are solutions of a Dirac equation of the form

$$(i\partial\!\!\!/ + \mathcal{B} - m)\,\Psi \;=\; 0 \quad , \tag{3.1.1}$$

where $\mathcal{B}$ is composed of the classical bosonic potentials. According to the common conception, the Dirac sea of the system is built up of all the negative-energy solutions of the Dirac equation. We can describe it with the so-called *fermionic projector* $\tilde{P}$ [11]. On the non-rigorous level of this introduction, the fermionic projector is given by the formal sum of the projectors on all these solutions; i.e.

$$\tilde{P}(x,y) \overset{\text{formally}}{=} \sum\!\!\!\!\!\!\int_a \Psi_a(x)\,\overline{\Psi_a(y)} \quad , \tag{3.1.2}$$

where the index 'a' runs over all the quantum numbers of the negative-energy states. We want to analyze how the fermionic projector depends on the bosonic potentials in (3.1.1). According to the decomposition (3.1.2) into the individual states, this dependence

can be regarded as a collective effect of all the fermions of the Dirac sea moving in the external potential $\mathcal{B}$. Following Dirac's original concept that the completely filled Dirac sea describes the vacuum, we can also say that we are interested in how the fermionic vacuum is influenced by the bosonic fields. Our aim is to describe this physical effect in a detailed and explicit way.

It turns out that the dependence of the fermionic projector on the external potential has a complicated non-local structure. In order to simplify the problem, we shall study $\tilde{P}(x,y)$ in an expansion about the light cone, which is called *light-cone expansion*. The light cone around a space-time point x consists of all points which can be reached from x with a light ray. In flat Minkowski space, which we will consider here, the light cone is given by all pairs of points (x,y) of Minkowski space whose Lorentzian distance $(y-x)^2 \equiv (y-x)_j\,(y-x)^j$ is zero. In the simplest case of a smooth function $A(x,y)$, the light-cone expansion is just an expansion in powers of $(y-x)^2$, i.e. a representation of the form

$$A(x,y) \;=\; \sum_{j=0}^{\infty} (y-x)^{2j}\; A_j(x,y) \tag{3.1.3}$$

with smooth functions $A_j(x,y)$. Since the expansion parameter $(y-x)^2$ vanishes on the light cone, the coefficients $A_j(x,y)$ give approximations of $A(x,y)$ in a neighborhood of the light cone (i.e., $A_0(x,y)$ coincides with $A(x,y)$ on the light cone, $A_1(x,y)$ gives the first order behavior of $A(x,y)$ for pairs (x,y) which are close to the light cone, etc.). The important point is that the $A_j(x,y)$ are approximations of $A(x,y)$ even for points x, y which are far apart. We only need that the pair (x,y) is close to the light cone, which is an unbounded hypersurface in $\mathbb{R}^4 \times \mathbb{R}^4$. In this sense, the light-cone expansion is a *non-local* expansion. The major advantage over local approximation techniques (like e.g. Taylor expansions in the space-time coordinates) is that the light-cone expansion gives a much more detailed description of the fermionic projector in position space. Furthermore, since the light cone is the boundary of the domain of causal dependence, all the effects related to the causality of the Dirac equation occur near the light cone. Thus the light-cone expansion describes the fermionic projector precisely in the region which is most interesting physically. In this paper, we will develop an efficient method for performing the light-cone expansion of the fermionic projector.

After this simplified and very qualitative introduction, we briefly discuss the difficulties and methods of the more detailed study. First of all, it is not obvious how to characterize the "negative-energy solutions" of the Dirac equation in the case with general interaction. In other words, one problem is to find the right quantum numbers for the a-summation in (3.1.2). As explained in [11], this problem can only be solved if the notion of "negative-energy states" is given up and replaced by a causality principle for the Dirac sea; this gives a unique definition of $\tilde{P}$ in terms of a power series in the external potential. Our task is to convert this formal definition into explicit formulas for the fermionic projector in position space. The basic technique is to construct solutions of the inhomogeneous Klein-Gordon and Dirac equations and to show that these solutions coincide with the contributions to the perturbation expansion of $\tilde{P}$. For the contribution to first order in the external potential, a similar technique was already used in [12], which also contains a general discussion of the method. In the following, we will first generalize this technique to higher order perturbation theory. Then we will explicitly sum up the light-cone expansions of all Feynman diagrams, which will finally yield exact formulas for the light-cone expansion of the fermionic projector without the restriction for the external potential to be (in any

sense) "small." We shall use the notation, the definitions, and the results of [11] throughout. Since the method of [12] had to be refined considerably for the analysis of higher order Feynman diagrams, we will develop the light-cone expansion from the very beginning. Thus this paper can be considered as being independent of [12]. Nevertheless, the more elementary approach in [12] is a preparation and might be helpful for the understanding. We will use the so-called *residual argument* to deduce the light-cone expansion of the Dirac sea from that of the advanced and retarded Green's functions. This allows us to bypass the explicit Fourier transformations in [12]. However, the residual argument has its limitations; making it mathematically precise leads to the decomposition of the Dirac sea into a causal and a non-causal contribution.

In the remainder of this section, we specify our problem in mathematical terms. Since a realistic physical system consists of several types of fermions, we describe empty space by the *fermionic projector of the vacuum*, which was introduced in [11] as the direct sum of $f \geq 1$ Dirac seas. The *chiral asymmetry matrix* X and the *mass matrix* Y are considered as a-priori given. The reader who is only interested in the light-cone expansion of a single Dirac sea may specialize to $f = 1$ and $X = \mathbb{1} = Y$. On the wave functions, we consider the indefinite scalar product

$$<\Psi \,|\, \Phi> \;=\; \int_{\mathbb{R}^4} \sum_{l=1}^{f} \overline{\Psi_l(x)}\, \Phi_l(x)\, d^4x \tag{3.1.4}$$

with the adjoint spinor $\overline{\Psi} = \Psi^* \gamma^0$. Similar to (3.1.1), the interaction is described by a perturbation $\mathcal{B}$ of the Dirac operator. We allow $\mathcal{B}$ to be composed of chiral and scalar/pseudoscalar potentials,

$$\mathcal{B}(x) \;=\; \chi_L \,\mathcal{A}\!\!\!/_R(x) \;+\; \chi_R \,\mathcal{A}\!\!\!/_L(x) \;+\; \Phi(x) \;+\; i\gamma^5\, \Xi(x) \quad , \tag{3.1.5}$$

where $\chi_{L/R} = \frac{1}{2}(\mathbb{1} \mp \gamma^5)$ are the chiral projectors and where we use a matrix notation in the Dirac sea index,

$$A_{L/R} \;=\; (A^l_{L/R\, m})_{l,m=1,\ldots,f} \quad , \qquad \Phi \;=\; (\Phi^l_m)_{l,m=1,\ldots,f} \quad , \qquad \Xi \;=\; (\Xi^l_m)_{l,m=1,\ldots,f} \tag{3.1.6}$$

(so the potentials may be non-diagonal on the Dirac seas and thus describe a general interaction of all the fermions). Furthermore, the perturbation $\mathcal{B}$ shall be Hermitian with respect to the scalar product (3.1.4). We assume the Dirac operator to be *causality compatible* with X, i.e.

$$X^* \,(i\partial\!\!\!/ + \mathcal{B} - mY) \;=\; (i\partial\!\!\!/ + \mathcal{B} - mY)\, X \quad .$$

This assumption is crucial; if it was violated, unbounded line integrals would occur in the light-cone expansion, making it impossible to carry out the sum over all Feynman diagrams (see the calculations in [13] for more details). The form of the chiral decomposition $\chi_L \mathcal{A}\!\!\!/_R + \chi_R \mathcal{A}\!\!\!/_L$ is useful because, as we will see later, the left and right handed components of the fermions couple to A_L and A_R, respectively. An interesting feature of our system is that, as a consequence of the non-diagonal form (3.1.6) of the potentials on the Dirac seas, the potentials and the mass matrix do in general not commute with each other,

$$[A_L(x), A_{L/R}(y)] \neq 0 \quad , \qquad [A_L(x), \Phi(y)] \neq 0 \quad , \qquad [A_L(x), Y] \neq 0 \;, \ldots \quad . \tag{3.1.7}$$

Compared to the situation in [12], the bilinear potential $H_{jk}\sigma^{jk}$ is missing in (3.1.5). It leads to complications when the sum over all Feynman diagrams is carried out. These

complications are not serious, but in order to keep the expansion reasonably simple, the bilinear potential was left out. Furthermore, we do not consider the gravitational field. The reason is that the higher order contributions in the gravitational potential become more and more singular on the light cone. This leads to technical problems which we will not deal here. Despite these simplifications, the considered ansatz for $\mathcal{B}$ includes arbitrary left and right handed Yang-Mills potentials and is general enough for a description of e.g. the interactions of the standard model. The *fermionic projector in the presence of external fields*, $\tilde{P}(x,y)$, is defined via the perturbation series in [11], which is a formal sum of operator products of the form

$$\tilde{P} \;=\; \sum_{n=0}^{\infty} \sum_{\ldots} \text{const}(n,\ldots)\; C_{n,\ldots}\, \mathcal{B}\, C_{n-1,\ldots}\, \mathcal{B} \cdots \mathcal{B}\, C_{0,\ldots} \quad ,$$

where the factors $C_{j,\ldots}$ coincide either with the spectral projectors k, p, or with the Green's function s (the index '$\ldots$' is a short notation for the different configurations of these factors).

In the language of Feynman diagrams, the perturbation series for $\tilde{P}$ only consists of tree diagrams. These tree diagrams are all finite; this is not difficult to prove if we assume suitable regularity of the potential:

Lemma 3.1.1 *Let (C_j), $0 \leq j \leq n$, be a choice of operators $C_j = k,\ p,$ or s. If the external potential (3.1.5) is smooth and decays so fast at infinity that the functions $\mathcal{B}(x)$, $x^i \mathcal{B}(x)$, and $x^i x^j \mathcal{B}(x)$ are integrable, then the operator product*

$$(C_n\, \mathcal{B}\, C_{n-1}\, \mathcal{B} \cdots \mathcal{B}\, C_0)(x,y) \tag{3.1.8}$$

is a well-defined tempered distribution on $\mathbb{R}^4 \times \mathbb{R}^4$.

Proof: Calculating the Fourier transform of (3.1.8) gives the formal expression

$$\begin{aligned} M(q_2,q_1) \;:=\; & \int \frac{d^4p_1}{(2\pi)^4} \cdots \int \frac{d^4p_{n-1}}{(2\pi)^4} \\ & \times C_n(q_2)\, \tilde{\mathcal{B}}(q_1 - p_{n-1})\, C_{n-1}(p_{n-1})\, \tilde{\mathcal{B}}(p_{n-1} - p_{n-2}) \cdots C_1(p_1)\, \tilde{\mathcal{B}}(p_1 - q_1)\, C_0(q_1) \;, \end{aligned} \tag{3.1.9}$$

where we consider the C_j as multiplication operators in momentum space and where $\tilde{\mathcal{B}}$ denotes the Fourier transform of $\mathcal{B}$. It is more convenient to work in momentum space because the operators C_j then have a simpler form. We will show that $M(q_2,q_1)$ is a well-defined tempered distribution; our Lemma then immediately follows by transforming back to position space.

The assumptions on $\mathcal{B}$ yield that $\tilde{\mathcal{B}}$ is C^2 and has rapid decay at infinity, i.e.

$$\sup_{q \in \mathbb{R}^4,\, |\kappa| \leq 2} |q^{i_1} \cdots q^{i_n}\, \partial_\kappa \tilde{\mathcal{B}}(q)| \;<\; \infty$$

for all n, tensor indices $i_1, \ldots, i_n$, and multi-indices κ (so $\kappa = (\kappa^1, \ldots, \kappa^p)$, $|\kappa| := p$). As is verified explicitly in momentum space, the distributions k, p, and s are bounded in the Schwartz norms of the test functions involving derivatives of only first order, more precisely

$$|C(f)| \;\leq\; \text{const}\, \|f\|_{4,1} \qquad \text{with } C = k, p, \text{ or } s \text{ and } f \in \mathcal{S},$$

where the Schwartz norms are as usual defined by

$$\|f\|_{p,q} \;=\; \max_{|I| \leq p,\, |J| \leq q} \sup_{x \in \mathbb{R}^4} |x^I\, \partial_J f(x)| \quad .$$

As a consequence, we can apply k, p, and s even to functions with rapid decay which are only C^1. Furthermore, we can form the convolution of such functions with k, p, or s; this gives continuous functions (which will no longer have rapid decay, however). A convolution decreases the order of differentiability of the functions by one.

We consider the combination of multiplication and convolution

$$F(p_2) \;:=\; \int \frac{d^4p_1}{(2\pi)^4}\, f(p_2-p_1)\, C(p_1)\, g(p_1) \quad , \tag{3.1.10}$$

where we assume that $f \in C^2$ has rapid decay and $g \in C^1$ is bounded together with its first derivatives, $\|g\|_{0,1} < \infty$. For any fixed p_2, the integral in (3.1.10) is well-defined and finite because $f(p_2 - .)\, g(.)$ is C^1 and has rapid decay. The resulting function F is C^1 and bounded together with its first derivatives, more precisely

$$\|F\|_{0,1} \quad\leq\quad \text{const}\; \|f\|_{4,2}\, \|g\|_{0,1} \quad . \tag{3.1.11}$$

After these preparations, we can estimate the integrals in (3.1.9) from the right to the left: We choose two test functions $f, g \in \mathcal{S}(\mathbb{R}^4, \mathbb{C}^{4f})$ and introduce the functions

$$F_1(p_1) \;=\; \int \frac{d^4q_2}{(2\pi)^4}\, \tilde{\mathcal{B}}(p_1-q_1)\, C_0(q_1)\, g(q_1) \tag{3.1.12}$$

$$F_j(p_j) \;=\; \int \frac{d^4p_{j-1}}{(2\pi)^4}\, \tilde{\mathcal{B}}(p_j-p_{j-1})\, C_{j-1}(p_{j-1})\, F_{j-1}(p_{j-1}) \quad , \quad 1 < j \leq n\,. \tag{3.1.13}$$

The integral (3.1.12) is of the form (3.1.10) and satisfies the above considered assumptions on the integrand. Using the bound (3.1.11), we can proceed inductively in (3.1.13). Finally, we perform the q_2-integration,

$$M(f,g) \;=\; \int \frac{d^4q_2}{(2\pi)^4}\, f(q_2)\, C_n(q_2)\, F_n(q_2) \quad .$$

We conclude that M is a linear functional on $\mathcal{S}(\mathbb{R}^4, \mathbb{C}^{4f}) \times \mathcal{S}(\mathbb{R}^4, \mathbb{C}^{4f})$, which is bounded in the Schwartz norm $\|.\|_{4,1}$ of the test functions. ■

Clearly, the existence of the perturbation expansion to every order does not imply the convergence of the perturbation series, and we will not study this problem here. Our method is to first perform the light-cone expansion of the individual Feynman diagrams. For the resulting formulas, it will then be possible to sum up the perturbation series. Since the Feynman diagrams are only defined as distributions, we must generalize (3.1.3) in a way which allows for the possibility that $A(x,y)$ is singular on the light cone.

Def. 3.1.2 *A tempered distribution $A(x,y)$ is of the order $\mathcal{O}((y-x)^{2p})$, $p \in \mathbf{Z}$, if the product*

$$(y-x)^{-2p}\, A(x,y)$$

is a regular distribution (i.e. a locally integrable function). It has the **light-cone expansion**

$$A(x,y) \;=\; \sum_{j=g}^{\infty} A^{[j]}(x,y) \tag{3.1.14}$$

with $g \in \mathbf{Z}$ if the distributions $A^{[j]}(x,y)$ are of the order $\mathcal{O}((y-x)^{2j})$ and if A is approximated by the partial sums in the way that

$$A(x,y) \;-\; \sum_{j=g}^{p} A^{[j]}(x,y) \qquad \textit{is of the order} \qquad \mathcal{O}((y-x)^{2p+2}) \tag{3.1.15}$$

for all $p \geq g$.

The lowest summand $A^{[g]}(x,y)$ gives the leading order of $A(x,y)$ on the light cone. If A is singular on the light cone, g will be negative. The light-cone expansion (3.1.3) of a smooth function is recovered as a special case by setting $g=0$ and $A^{[j]}(x,y) = (y-x)^{2j}\, A_j(x,y)$. Notice that the definition of the light-cone expansion does not include the convergence of the infinite sum in (3.1.14), which is only a convenient notation for the approximation of $A(x,y)$ by the partial sums (3.1.15).

3.2 The Light-Cone Expansion of the Green's Functions

In this section, we shall perform the light-cone expansion for the advanced and retarded Green's functions $\tilde{s}^\vee$, $\tilde{s}^\wedge$, which are defined in [11] by the perturbation series

$$\tilde{s}^\vee \;=\; \sum_{k=0}^{\infty} (-s^\vee\, \mathcal{B})^k s^\vee \quad , \qquad \tilde{s}^\wedge \;=\; \sum_{k=0}^{\infty} (-s^\wedge\, \mathcal{B})^k s^\wedge \quad . \tag{3.2.1}$$

These perturbation expansions are causal in the sense that $\tilde{s}^\vee(x,y)$ and $\tilde{s}^\wedge(x,y)$ only depend on the potential $\mathcal{B}$ in the "diamond" $(L_x^\vee \cap L_y^\wedge) \cup (L_x^\wedge \cap L_y^\vee)$, where

$$L_x^\vee \;=\; \Big\{ y \,|\, (y-x)^2 \geq 0,\; y^0 - x^0 \geq 0 \Big\} \quad , \quad L_x^\wedge \;=\; \Big\{ y \,|\, (y-x)^2 \geq 0,\; y^0 - x^0 \leq 0 \Big\} \tag{3.2.2}$$

denote the future and past light cones around x, respectively. Since this "diamond" is (for fixed x and y) a bounded region of space-time, the assumptions of Lemma 3.1.1 on the decay of the external potential at infinity are not necessary in this section; it suffices to assume that $\mathcal{B}$ is smooth. Furthermore, the chiral asymmetry matrix X and the causality compatibility condition will not be needed in this section.

In order to get a first idea of how the light-cone expansion can be carried out, we look at the free advanced Green's function $s_m^\vee$ of a single Dirac sea in position space: It is convenient to pull the Dirac matrices out of $s_m^\vee$ by setting

$$s_m^\vee(x,y) \;=\; (i\partial\!\!\!/_x + m)\; S_{m^2}^\vee(x,y) \quad , \tag{3.2.3}$$

where $S_{m^2}^\vee$ is the advanced Green's function of the Klein-Gordon operator,

$$S_{m^2}^\vee(x,y) \;=\; \lim_{0<\varepsilon\to 0} \int \frac{d^4p}{(2\pi)^4}\, \frac{1}{p^2 - m^2 - i\varepsilon p^0}\, e^{-ip(x-y)} \quad . \tag{3.2.4}$$

This Fourier integral can be calculated explicitly; we expand the resulting Bessel function J_1 in a power series,

$$\begin{aligned} S_{m^2}^\vee(x,y) \;&=\; -\frac{1}{2\pi}\, \delta((y-x)^2)\, \Theta(y^0-x^0) \\ &\quad + \frac{m^2}{4\pi}\, \frac{J_1(\sqrt{m^2\,(y-x)^2}}{\sqrt{m^2\,(y-x)^2}}\, \Theta((y-x)^2)\, \Theta(y^0-x^0) \end{aligned}$$

$$= -\frac{1}{2\pi}\,\delta((y-x)^2)\,\Theta(y^0-x^0) + \frac{m^2}{8\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,(j+1)!}\,\frac{(m^2(y-x)^2)^j}{4^j}\,\Theta((y-x)^2)\,\Theta(y^0-x^0) \quad (3.2.5)$$

(Θ is the Heaviside function $\Theta(t)=1$ for $t\geq 0$ and $\Theta(t)=0$ otherwise). This computation shows that $S^\vee_{m^2}(x,y)$ has a $\delta((y-x)^2)$-like singularity on the light cone. Furthermore, one sees that $S^\vee_{m^2}$ is a power series in m^2. The important point for the following is that the higher order contributions in m^2 contain more factors $(y-x)^2$ and are thus of higher order on the light cone. More precisely,

$$\left(\frac{d}{dm^2}\right)^n S^\vee_{m^2\,|m^2=0}(x,y) \qquad \text{is of the order} \quad \mathcal{O}((y-x)^{2n-2}). \quad (3.2.6)$$

According to (3.2.3), the Dirac Green's function is obtained by computing the first partial derivatives of (3.2.5). Therefore $s^\vee_m(x,y)$ has a singularity on the light cone which is even $\sim \delta'((y-x)^2)$. The higher order contributions in m are again of increasing order on the light cone. This means that we can view the Taylor expansion of (3.2.3) in m,

$$s^\vee_m(x,y) \;=\; \sum_{n=0}^{\infty}(i\partial\!\!\!/+m)\,\frac{1}{n!}\left(\frac{d}{dm^2}\right)^n S^\vee_{m^2\,|\,m^2=0}(x,y) \quad ,$$

as a light-cone expansion of the free Green's function. Our idea is to generalize this formula to the case with interaction. More precisely, we want to express the perturbed Green's function in the form

$$\tilde{s}^\vee(x,y) \;=\; \sum_{n=0}^{\infty} F_n(x,y)\,\left(\frac{d}{dm^2}\right)^n S^\vee_{m^2\,|\,m^2=0}(x,y) \quad (3.2.7)$$

with factors F_n which depend on the external potential. We will see that this method is very convenient; especially, we can in this way avoid working with the rather complicated explicit formula (3.2.5). Apart from giving a motivation for the desired form (3.2.7) of the formulas of the light-cone expansion, the mass expansion (3.2.5) leads to the conjecture that even the higher order contributions in the mass to the *perturbed* Green's functions might be of higher order on the light cone. If this conjecture was true, it would be a good idea to expand the perturbation expansion for $\tilde{s}$ with respect to the parameter m. Therefore our basic strategy is to first expand (3.2.1) with respect to the mass and to try to express the contributions to the resulting expansion in a form similar to (3.2.7).

The expansion of (3.2.1) with respect to m gives a double sum over the orders in the mass parameter and in the external potential. It is convenient to combine these two expansions in a single perturbation series. For this, we first write the mass matrix and the scalar/pseudoscalar potentials together by setting

$$Y_L(x) \;=\; Y \;-\; \frac{1}{m}\,(\Phi(x)+i\Xi(x)) \quad , \qquad Y_R(x) \;=\; Y \;-\; \frac{1}{m}\,(\Phi(x)-i\Xi(x)) \quad . \quad (3.2.8)$$

The matrices $Y_{L/R}(x)$ are called *dynamic mass matrices*; notice that $Y_L^* = Y_R$. With this notation, we can rewrite the Dirac operator in the form

$$i\partial\!\!\!/ + \mathcal{B} - mY \;=\; i\partial\!\!\!/ + B \qquad \text{with} \quad (3.2.9)$$

$$B \;=\; \chi_L\,(A\!\!\!/_R - m\,Y_R) \;+\; \chi_R\,(A\!\!\!/_L - m\,Y_L) \quad . \quad (3.2.10)$$

For the light-cone expansion of the Green's functions, we will always view B as the perturbation of the Dirac operator. This has the advantage that the free theory consists only of zero-mass fermions; the Green's functions of the free Dirac operator have the simple form

$$s^\vee(x,y) \;=\; i\partial\!\!\!/_x \, S^\vee_{m^2=0}(x,y) \quad , \qquad s^\wedge(x,y) \;=\; i\partial\!\!\!/_x \, S^\wedge_{m^2=0}(x,y) \tag{3.2.11}$$

(to be very precise, we should write $s^\vee = i\partial\!\!\!/ S_0^\vee \otimes \mathbb{1}$, where $\mathbb{1} = (\mathbb{1}^l_m)_{l,m=1,\ldots,f}$ is the identity on the Dirac seas). The Green's functions with interaction are given by the perturbation series

$$\tilde{s}^\vee \;=\; \sum_{k=0}^\infty (-s^\vee\, B)^k s^\vee \quad , \qquad \tilde{s}^\wedge \;=\; \sum_{k=0}^\infty (-s^\wedge\, B)^k s^\wedge \quad . \tag{3.2.12}$$

The constructions of the following subsections are exactly the same for the advanced and retarded Green's functions. In order to treat both cases at once, we will in the remainder of this section omit all superscripts '$\vee$', '$\wedge$'. The formulas for the advanced and retarded Green's functions are obtained by either adding '$\vee$' or '$\wedge$' to all factors s, S.

3.2.1 Inductive Light-Cone Expansion of All Feynman Diagrams

In this subsection, we explain how the individual contributions to the perturbation expansion (3.2.12) can be written similar to the right side of (3.2.7) as a sum of terms of increasing order on the light cone. For the mass expansion of S_{m^2}, we set $a = m^2$ and use the notation

$$S^{(l)} \;=\; \left(\frac{d}{da}\right)^l S_{a|a=0} \quad . \tag{3.2.13}$$

In preparation, we derive some computation rules for the $S^{(l)}$: S_a satisfies the defining equation of a Klein-Gordon Green's function

$$(-\Box_x - a)\, S_a(x,y) \;=\; \delta^4(x-y) \quad .$$

Differentiating with respect to a yields

$$-\Box_x S^{(l)}(x,y) \;=\; \delta_{l,0}\, \delta^4(x-y) \;+\; l\, S^{(l-1)}(x,y) \quad , \qquad l \geq 0. \tag{3.2.14}$$

(For $l = 0$, this formula does not seem to make sense because $S^{(-1)}$ is undefined. The expression is meaningful, however, if one keeps in mind that in this case the factor l is zero, and thus the whole second summand vanishes. We will also use this convention in the following calculations.) Next, we differentiate the formulas for S_a in momentum space,

$$S^\vee_a(p) \;=\; \frac{1}{p^2 - a - i\varepsilon p^0} \quad , \qquad S^\wedge_a(p) \;=\; \frac{1}{p^2 - a + i\varepsilon p^0} \quad , \tag{3.2.15}$$

with respect to both p and a. Comparing the results gives the relation

$$\frac{\partial}{\partial p^k} S_a(p) \;=\; -2p_k \, \frac{d}{da} S_a(p) \quad ,$$

or, after expanding in the parameter a,

$$\frac{\partial}{\partial p^k} S^{(l)}(p) \;=\; -2p_k \, S^{(l+1)}(p) \quad , \qquad l \geq 0. \tag{3.2.16}$$

This formula also determines the derivatives of $S^{(l)}$ in position space; namely

$$\begin{aligned}
\frac{\partial}{\partial x^k} S^{(l)}(x,y) &= \int \frac{d^4p}{(2\pi)^4}\, S^{(l)}(p)\, (-ip_k)\, e^{-ip(x-y)} \\
&\overset{(3.2.16)}{=} \frac{i}{2} \int \frac{d^4p}{(2\pi)^4}\, \frac{\partial}{\partial p^k} S^{(l-1)}(p)\, e^{-ip(x-y)} \;=\; -\frac{i}{2} \int \frac{d^4p}{(2\pi)^4}\, S^{(l-1)}(p)\, \frac{\partial}{\partial p^k} e^{-ip(x-y)} \\
&= \frac{1}{2}\, (y-x)_k\, S^{(l-1)}(x,y) \quad , \qquad l \geq 1. \qquad (3.2.17)
\end{aligned}$$

We iterate this relation to calculate the Laplacian,

$$\begin{aligned}
-\Box_x S^{(l)}(x,y) &= -\frac{1}{2}\, \frac{\partial}{\partial x^k} \Big((y-x)^k\, S^{(l-1)}(x,y) \Big) \\
&= 2\, S^{(l-1)}(x,y) \,-\, \frac{1}{4}\, (y-x)^2\, S^{(l-2)}(x,y) \quad , \qquad l \geq 2.
\end{aligned}$$

After comparing with (3.2.14), we conclude that

$$(y-x)^2\, S^{(l)}(x,y) \;=\; -4l\, S^{(l+1)}(x,y) \quad , \qquad l \geq 0. \qquad (3.2.18)$$

Finally, $S^{(l)}(x,y)$ is only a function of $(y-x)$, which implies that

$$\frac{\partial}{\partial x^k} S^{(l)}(x,y) \;=\; -\frac{\partial}{\partial y^k} S^{(l)}(x,y) \quad , \qquad l \geq 0. \qquad (3.2.19)$$

The following lemma gives the light-cone expansion of an operator product which is linear in the external potential. We will later use it for the iterative light-cone expansion of more complicated operator products; in this case, the potential will be a composite expression in B and its partial derivatives. In order to avoid confusion then, we denote the external potential by V.

Lemma 3.2.1 (light-cone expansion to first order) *The operator product $S^{(l)}\, V\, S^{(r)}$ with $l, r \geq 0$ has the light-cone expansion*

$$\begin{aligned}
&(S^{(l)}\, V\, S^{(r)})(x,y) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y + (1-\alpha)x}\, d\alpha\; S^{(n+l+r+1)}(x,y) \quad . \qquad (3.2.20)
\end{aligned}$$

Proof: The method of the proof is to first compute the Laplacian of both sides of (3.2.20). The resulting formulas will be of similar structure, which allows us to proceed inductively.

On the left side of (3.2.20), we calculate the Laplacian with the help of (3.2.14) to

$$-\Box_x (S^{(l)}\, V\, S^{(r)})(x,y) \;=\; \delta_{l,0}\, V(x)\, S^{(r)}(x,y) \,+\, l\, (S^{(l-1)}\, V\, S^{(r)})(x,y) \quad . \qquad (3.2.21)$$

The Laplacian of the integral on the right side of (3.2.20) can be computed with (3.2.17) and (3.2.14),

$$\begin{aligned}
&-\Box_x \int_0^1 \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y + (1-\alpha)x}\, d\alpha\; S^{(n+l+r+1)}(x,y) \qquad (3.2.22) \\
&= -\int_0^1 \alpha^l\, (1-\alpha)^{r+2}\, (\alpha-\alpha^2)^n\, (\Box^{n+1} V)_{|\alpha y + (1-\alpha)x}\, d\alpha\; S^{(n+l+r+1)}(x,y) \\
&\quad -\int_0^1 \alpha^l\, (1-\alpha)^{r+1}\, (\alpha-\alpha^2)^n\, (\partial_k \Box^n V)_{|\alpha y + (1-\alpha)x}\, d\alpha\; (y-x)^k\, S^{(n+l+r)}(x,y) \\
&\quad +(n+l+r+1) \int_0^1 \alpha^l\, (1-\alpha)^r\, (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y + (1-\alpha)x}\, d\alpha\; S^{(n+l+r)}(x,y) \quad .
\end{aligned}$$

In the second summand, we rewrite the partial derivative as a derivative with respect to α and integrate by parts,

$$\begin{aligned}
&\int_0^1 \alpha^l\,(1-\alpha)^{r+1}\,(\alpha-\alpha^2)^n\,(\partial_k \Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,(y-x)^k \\
&= \int_0^1 \alpha^l\,(1-\alpha)^{r+1}\,(\alpha-\alpha^2)^n\,\frac{d}{d\alpha}(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha \\
&= -\delta_{n,0}\,\delta_{l,0}\,V(x) - (n+l)\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^{n-1}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha \\
&\quad +(n+r+1)\int_0^1 \alpha^l\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha \\
&= -\delta_{n,0}\,\delta_{l,0}\,V(x) \\
&\quad -n\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^{n-1}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha \\
&\quad +(n+l+r+1)\int_0^1 \alpha^l\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha \\
&\quad -l\int_0^1 \alpha^{l-1}\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha \quad .
\end{aligned}$$

We substitute back into the original equation and obtain

$$\begin{aligned}
(3.2.22) \quad &= \delta_{n,0}\,\delta_{l,0}\,V(x)\,S^{(r)}(x,y) \\
&\quad +l\int_0^1 \alpha^{l-1}\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,S^{(n+l+r)}(x,y) \\
&\quad -\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^n\,(\Box^{n+1} V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,S^{(n+l+r+1)}(x,y) \\
&\quad +n\int_0^1 \alpha^l\,(1-\alpha)^{r+2}\,(\alpha-\alpha^2)^{n-1}\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,S^{(n+l+r)}(x,y) \quad .
\end{aligned}$$

After dividing by $n!$ and summation over n, the last two summands are telescopic and cancel each other. Thus one gets

$$\begin{aligned}
&-\Box\sum_{n=0}^\infty \frac{1}{n!}\int_0^1 \alpha^l\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,S^{(n+l+r+1)}(x,y) \\
&= \delta_{l,0}\,V(x)\,S^{(r)}(x,y) \\
&\quad +l\sum_{n=0}^\infty \frac{1}{n!}\int_0^1 \alpha^{l-1}\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,S^{(n+l+r)}(x,y) \quad . \quad (3.2.23)
\end{aligned}$$

We now compare the formulas (3.2.21) and (3.2.23) for the Laplacian of both sides of (3.2.20). In the special case $l=0$, these formulas coincide, and we can use a uniqueness argument for the solutions of the wave equation to prove (3.2.20): We assume that we consider the advanced Green's function (for the retarded Green's function, the argument is analogous). For given y, we denote the difference of both sides of (3.2.20) by $F(x)$. Since the support of $F(x)$ is in the past light cone $x \in L_y^\wedge$, F vanishes in a neighborhood of the hypersurface $\mathcal{H} = \{z \in \mathbb{R}^4 \,|\, z^0 = y^0 + 1\}$. Moreover, the Laplacian of F is identically zero according to (3.2.21) and (3.2.23). We conclude that

$$\Box F \;=\; 0 \qquad \text{and} \qquad F_{|\mathcal{H}} \;=\; \partial_k F_{|\mathcal{H}} \;=\; 0 \quad .$$

Since the wave equation has a unique solution for given initial data on the Cauchy surface $\mathcal{H}$, F vanishes identically.

The general case follows by induction in l: Suppose that (3.2.20) holds for given $\hat{l}$ (and arbitrary r). Then, according to (3.2.21), (3.2.23), and the induction hypothesis, the Laplacian of both sides of (3.2.20) coincides for $l = \hat{l} + 1$. The above uniqueness argument for the solutions of the wave equation again gives (3.2.20). ∎

We recall for clarity that, according to (3.2.6), the higher a-derivatives of $S_a(x,y)$ are of higher order on the light cone. Thus the summands in (3.2.20) are of increasing order on the light cone, and the infinite sum makes mathematical sense in terms of Def. 3.1.2 via the approximation by the partial sums (3.1.15).

The representation (3.2.20) of an operator product as an infinite series of line integrals has some similarity with the formal light-cone expansion [12, Theorem 3.3]. In order to make it easier for the reader to see the connection between these two expansions, we briefly discuss the analogy and the differences: First of all, we are here considering the Green's functions instead of the negative-energy solutions of the Klein-Gordon equation. The causality of the Green's functions (i.e. supp $S(x,.) \subset L_x$) simplifies the construction considerably. We could use it to avoid the explicit Fourier transformations of the proofs in [12]; furthermore, it makes the resummation of non-local contributions unnecessary. In this paper, the complications related to the non-causality of the negative-energy states will reappear in the light-cone expansion of the Dirac sea in Section 3.3. We point out that the light-cone expansion in [12] is more general in the way that the mass parameter $a = m^2$ need not be zero. This is sometimes more convenient, because then the mass is just a parameter of the Green's functions instead of occurring in the line integrals of the light-cone expansion. With our concept of dynamic mass matrices, however, an expansion around $a = 0$ is more appropriate. The important generalization in (3.2.20) is that the two factors S on the left side may be derivatives with respect to a. On the right side of equation (3.2.20), this is taken into account by the additional factors $\alpha^l (1-\alpha)^r$ in the integrand and by the higher order $(n+l+r+1)$ of the a-derivative of S. This generalization is the basis for the following iterations.

Lemma 3.2.1 can be used for the light-cone expansion of more complicated operator products. To explain the method, we look at the simplest example of three factors $S^{(0)}$ and two potentials V, W,

$$(S^{(0)}\, V\, S^{(0)}\, W\, S^{(0)})(x,y) \;=\; \int d^4z\, S^{(0)}(x,z)\, V(z)\, (S^{(0)}\, W\, S^{(0)})(z,y) \quad . \tag{3.2.24}$$

Having split up the operator product in this form, we can apply Lemma 3.2.1 to the factor $S^{(0)} W S^{(0)}$,

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4z\, S^{(0)}(x,z) \left\{ V(z) \int_0^1 (\alpha - \alpha^2)^n\, (\Box^n W)_{|\alpha y + (1-\alpha) z}\, d\alpha \right\} S^{(n+1)}(z,y) \quad .$$

Now we rewrite the z-integral as the operator product $(S^{(0)} g_y S^{(0)})(x,y)$, where $g_y(z)$ is the function in the curly brackets. The y-dependence of g_y causes no problems because we can view y as a fixed parameter throughout the expansion. Thus we can simply apply Lemma 3.2.1 once again and obtain

$$= \sum_{m,n=0}^{\infty} \frac{1}{m!\, n!} \int_0^1 d\beta\, (1-\beta)^{n+1}\, (\beta - \beta^2)^m \int_0^1 d\alpha\, (\alpha - \alpha^2)^n$$

$$\times \Box_z^m \left(V(z)\, (\Box^n W)_{|\alpha y+(1-\alpha)z} \right)_{|z=\beta y+(1-\beta)x} \; S^{(m+n+2)}(x,y) \quad .$$

The Laplacian $\Box_z^m$ could be further computed with the Leibniz rule. Notice that the manipulations of the infinite sums are not problematic because the number of terms to every order on the light cone is actually finite (the situation would be more difficult if we studied the convergence of the sum (3.1.14), but, as pointed out earlier, the light-cone expansion is defined merely via the partial sums).

We want to iteratively perform the light-cone expansion of the operator products in (3.2.12). This is not directly possible with the method just described, because (3.2.12) contains the Dirac Green's function s (instead of S). We must think about how to deal with this problem. Relation (3.2.11) allows us to replace the factors s by S, but this gives additional partial derivatives in the operator product. These derivatives can be carried out after each iteration step using the Leibniz rule and the differentiation rule (3.2.17). In the simplest example, we have

$$\begin{aligned}
(s^{(0)}\, V\, S^{(0)})(x,y) \;&=\; (i\partial\!\!\!/_x)(S^{(0)}\, V\, S^{(0)})(x,y) \\
&= i\partial\!\!\!/_x \sum_{n=0}^\infty \frac{1}{n!} \int_0^1 (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y+(1-\alpha)x}\; S^{(n+1)}(x,y) \\
&= i \sum_{n=0}^\infty \frac{1}{n!} \int_0^1 (1-\alpha)\, (\alpha-\alpha^2)^n\, (\partial\!\!\!/ \Box^n V)_{|\alpha y+(1-\alpha)x}\; S^{(n+1)}(x,y) \\
&\quad + \frac{i}{2} \sum_{n=0}^\infty \frac{1}{n!} \int_0^1 (\alpha-\alpha^2)^n\, (\Box^n V)_{|\alpha y+(1-\alpha)x}\; (y-x)_j \gamma^j\; S^{(n)}(x,y) \quad .
\end{aligned}$$

The only problem with this method is that the partial derivatives might hit a factor $S^{(0)}$, in which case rule (3.2.17) cannot be applied. In order to resolve this problem, we extend our constructions in a way which allows us to use all previous formulas also in this special case. For this, we take (3.2.17) as the defining equation for $(y-x)_k\, S^{(-1)}(x,y)$,

$$(y-x)_k\, S^{(-1)}(x,y) \;:=\; 2\, \frac{\partial}{\partial x^k} S^{(0)}(x,y) \tag{3.2.25}$$

(notice that $S^{(-1)}$ itself remains undefined, only the combination $(y-x)_k\, S^{(-1)}(x,y)$ makes mathematical sense as the partial derivative of the distribution $S^{(0)}$). With this definition, the computation rule (3.2.18) becomes also valid for $l=-1$: According to (3.2.18), the distribution $(y-x)^2\, S^{(0)}(x,y)$ vanishes identically, and thus

$$\begin{aligned}
0 \;&=\; \Box_x \left((y-x)^2\, S^{(0)}(x,y) \right) \\
&= \left(\Box_x (y-x)^2 \right) S^{(0)}(x,y) \;-\; 4\,(y-x)^j\, \frac{\partial}{\partial x^j} S^{(0)}(x,y) \;+\; (y-x)^2\, \Box_x S^{(0)}(x,y) \quad .
\end{aligned}$$

In the last summand, we substitute (3.2.14), which gives $-(y-x)^2\, \delta^4(x-y) = 0$. Using the identity $\Box_x (y-x)^2 = 8$ and the definition (3.2.25), we conclude that

$$(y-x)^2\, S^{(-1)}(x,y) \;=\; 4\, S^{(0)}(x,y) \quad . \tag{3.2.26}$$

The following lemma extends the result of Lemma 3.2.1 to the case when the right factor of the operator product is $S^{(-1)}$.

Lemma 3.2.2 (light-cone expansion to first order for $r=-1$) *The operator product $(S^{(l)}\,.\,S^{(-1)})$, $l\geq 0$, has the light-cone expansion*

$$\int d^4z\; S^{(l)}(x,z)\,V(z)\,(y-z)_k\,S^{(-1)}(z,y) \;=\; \sum_{n=0}^\infty \frac{1}{n!}\int_0^1 \alpha^l\,(1-\alpha)^{-1}\,(\alpha-\alpha^2)^n$$
$$\times\,\Box_z^n\,(V(z)\,(y-z)_k)_{|z=\alpha y+(1-\alpha)x}\;d\alpha\;S^{(n+l)}(x,y) \quad . \tag{3.2.27}$$

Proof: We deduce the light-cone expansion for the left side of (3.2.27) from (3.2.20) by pulling one y-derivative outside,

$$\int d^4z\; S^{(l)}(x,z)\,V(z)\,(y-z)_k\,S^{(-1)}(z,y) \overset{(3.2.25),(3.2.19)}{=} -2\,\frac{\partial}{\partial y^k}(S^{(l)}\,V\,S^{(0)})(x,y)$$

$$= \quad -2\,\frac{\partial}{\partial y^k}\sum_{n=0}^\infty\frac{1}{n!}\int_0^1\alpha^l\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\;S^{(n+l+1)}(x,y)$$

$$\overset{(3.2.19),(3.2.17)}{=} \quad \sum_{n=0}^\infty\frac{1}{n!}\int_0^1\alpha^l\,(\alpha-\alpha^2)^n\,(\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\,(y-x)_k\;S^{(n+l)}(x,y) \tag{3.2.28}$$

$$-2\sum_{n=0}^\infty\frac{1}{n!}\int_0^1\alpha^{l+1}\,(\alpha-\alpha^2)^n\,(\partial_k\Box^n V)_{|\alpha y+(1-\alpha)x}\,d\alpha\;S^{(n+l+1)}(x,y) \tag{3.2.29}$$

In (3.2.29), we substitute the identity

$$\partial_k\Box^n V(z) \;=\; -\frac{1}{2(n+1)}\Box_z^{n+1}\,(V(z)\,(y-z)_k)+\frac{1}{2(n+1)}\left(\Box_z^{n+1}V(z)\right)(y-z)_k \quad . \tag{3.2.30}$$

The contribution of the second summand in (3.2.30) cancels all the summands $n=1,2,\ldots$ of (3.2.28), and we get

$$\int d^4z\; S^{(l)}(x,z)\,V(z)\,(y-z)_k\,S^{(-1)}(z,y) \;=\; \int_0^1\alpha^l\,V_{|\alpha y+(1-\alpha)x}\,d\alpha\,(y-x)_k\;S^{(l)}(x,y)$$
$$+\sum_{n=0}^\infty\frac{1}{(n+1)!}\int_0^1\alpha^{l+1}\,(\alpha-\alpha^2)^n\,\Box_z^{n+1}\,(V(z)\,(y-z)_k)_{|z=\alpha y+(1-\alpha)x}\;d\alpha\;S^{(n+1+l)}(x,y) \quad .$$

After shifting the summation index, this coincides with (3.2.27). ∎

Notice that the pole of the factor $(1-\alpha)^{-1}$ for $\alpha=1$ does not lead to problems in (3.2.27): In the case $n=0$, it disappears since $(1-\alpha)^{-1}(y-z)=y-x$; for $n>0$, it is compensated by the zero of the factor $(\alpha-\alpha^2)^n$.

After these preparations, we come to the light-cone expansion of general Feynman diagrams. For the line integrals of Lemma 3.2.1 and Lemma 3.2.2, we introduce the short notation

$$\int_x^y [l,r\mid n]\,dz\;f(z) \;:=\; \int_0^1 d\alpha\;\alpha^l\,(1-\alpha)^r\,(\alpha-\alpha^2)^n\;f(\alpha y+(1-\alpha)x) \quad . \tag{3.2.31}$$

If l, r, and n all vanish, we sometimes omit the bracket, i.e.

$$\int_x^y dz\;f(z) \;:=\; \int_0^1 d\alpha\;f(\alpha y+(1-\alpha)x) \quad .$$

Furthermore, we abbreviate the following products with multi-indices

$$\partial_z^J \;:=\; \frac{\partial}{\partial z^{j_1}}\cdots\frac{\partial}{\partial z^{j_l}} \quad , \quad (y-x)^J \;:=\; (y-x)^{j_1}\cdots(y-x)^{j_l} \quad , \quad \gamma^J \;:=\; \gamma^{j_1}\cdots\gamma^{j_l} \quad ,$$

where $J=(j_1,\ldots,j_l)$.

Theorem 3.2.3 (inductive light-cone expansion) *The light-cone expansion of the k^{th} order contribution $((-s\,B)^k\,s)(x,y)$ to the perturbation series (3.2.12) can be written as an infinite sum of expressions, each of which has the form*

$$\chi_c\, C\, (y-x)^I \int_x^y [l_1, r_1 \,|\, n_1]\, dz_1\, \partial_{z_1}^{I_1} \Box_{z_1}^{p_1} V^{(1)}_{J_1,c_1}(z_1) \int_{z_1}^y [l_2, r_2 \,|\, n_2]\, dz_2\, \partial_{z_2}^{I_2} \Box_{z_2}^{p_2} V^{(2)}_{J_2,c_2}(z_2)$$
$$\cdots \int_{z_{k-1}}^y [l_k, r_k \,|\, n_k]\, dz_k\, \partial_{z_k}^{I_k} \Box_{z_k}^{p_k} V^{(k)}_{J_k,c_k}(z_k)\, \gamma^J\, S^{(h)}(x,y) \quad . \tag{3.2.32}$$

In this formula, C is a complex number and the parameters l_a, r_a, n_a, and p_a are non-negative integers; the indices c and c_a can take the two values L or R. The functions $V^{(a)}_{J_a,c_a}$ coincide with any of the individual potentials in (3.2.10) with chirality c_a, i.e.

$$\begin{aligned} V^{(a)}_{c_a} &= A_{c_a} \qquad \text{(in which case } |J_a| = 1) \qquad \text{or} \\ V^{(a)}_{c_a} &= mY_{c_a} \qquad \text{(in which case } |J_a| = 0) \quad . \end{aligned} \tag{3.2.33}$$

The chirality c_a of the potentials is determined by the following rules:

(i) The chirality c_1 of the first potential coincides with the chirality of the projector χ_c.

(ii) The chirality of the potentials is reversed at every mass matrix, i.e.

$$c_a \text{ and } c_{a+1} \quad \begin{cases} \text{coincide} & \text{if } V^{(a)}_{c_a} = A_{c_a} \\ \text{are opposite} & \text{if } V^{(a)}_{c_a} = mY_{c_a} \end{cases} \quad .$$

The tensor indices of the multi-indices in (3.2.32) are all contracted with each other, according to the following rules:

(a) No two tensor indices of the same multi-index are contracted with each other.

(b) The tensor indices of the factor γ^J are all contracted with different multi-indices, in the order of their appearance in the product (3.2.32) (i.e., for $J = (j_1, \ldots, j_l)$ and $1 \le a < b \le l$, the multi-index with which j_a is contracted must stand to the left of the multi-index corresponding to j_b).

The parameter h is given by

$$2h \;=\; k - 1 - |I| + \sum_{a=1}^k (|I_a| + 2p_a) \quad . \tag{3.2.34}$$

The number of factors $(y-x)$ is bounded by

$$|I| \;\le\; k + 1 - \sum_{a=1}^k |I_a| \quad . \tag{3.2.35}$$

Basically, this theorem states that the light-cone expansion of the k^{th} order Feynman diagrams can be written with k nested line integrals. Notice that, according to (3.1.7), the potentials $V^{(a)}(z_a)$ do in general not commute with each other, so that the order of multiplication is important in (3.2.32). In order to avoid misunderstandings, we point out that the derivatives $\partial_{z_a}^{I_a}$ and $\Box_{z_a}^{p_a}$ do not only act on $V^{(a)}(z_a)$, but also on all the following factors $V^{(a+1)}(z_{a+1})$, $V^{(a+2)}(z_{a+2})$,... (note that the variables z_{a+1}, z_{a+2},...

implicitly depend on z_a via the inductive definition of the line integrals). Clearly, these derivatives could be carried out further with the Leibniz rule, but it is easier not to do this at the moment. The restrictions *(a)* and *(b)* on the possible contractions of the tensor indices were imposed in order to avoid an abuse of our multi-index notation. More precisely, *(a)* prevents factors $(y-x)^2$ in $(y-x)^I$, an unnecessary large number of γ-matrices in γ^J, and "hidden" Laplacians in the partial derivatives $\partial_{z_a}^{I_a}$. The rule *(b)*, on the other hand, prevents factors $(y-x)^2$ and hidden Laplacians in combinations of the form $(y-x)_i\,(y-x)_j\,\gamma^i\,\gamma^j$ and $\partial_{ij} V_{J_a}^{(a)}\,\gamma^i\,\gamma^j$, respectively. Our ordering condition for the γ-matrices is just a matter of convenience. Relation (3.2.34) is very useful because it immediately tells for any configuration of the line integrals and potentials in (3.2.32) what the corresponding order on the light cone is. Notice that (3.2.34) and (3.2.35) imply the inequality

$$h \;\geq\; -1 + \sum_{a=1}^{k} |I_a| + p_a \quad . \tag{3.2.36}$$

Especially, one sees that $h \geq -1$. In the case $h=-1$, (3.2.34) yields that $|I|>0$, so that (3.2.32) must contain at least one factor $(y-x)$. Therefore the factor $S^{(h)}$ in (3.2.32) is always well-defined by either (3.2.13) or (3.2.25).

We point out that, although the total number of summands (3.2.32) is infinite, the number of summands for any given value of the parameter h is finite. This is clear because, for fixed h, the relations (3.2.34) and (3.2.35) only allow a finite number of possibilities to choose the parameters $|I|$, $|I_a|$, and p_a; thus one only gets a finite combinatorics for all expressions of the form (3.2.32). Since, according to (3.2.6), the contributions for higher values of h are of higher order on the light cone, we conclude that the number of summands (3.2.32) is finite to every order on the light cone. Therefore the light-cone expansion of Theorem 3.2.3 makes mathematical sense in terms of Def. 3.1.2.

Proof of Theorem 3.2.3: We proceed inductively in k. For $k=0$, the assumption is true because we can write the free Dirac Green's function in the required form (3.2.32),

$$s(x,y) \overset{(3.2.11),(3.2.25)}{=} (\chi_L + \chi_R)\,\frac{i}{2}\,(y-x)^j \gamma_j\, S^{(-1)}(x,y) \quad . \tag{3.2.37}$$

The conditions *(i)*, *(ii)*, *(a)*, *(b)*, and the relations (3.2.34), (3.2.35) are clearly satisfied.

Assume that the theorem holds for a given k. With the formula

$$((-s\,B)^{k+1}\,s)(x,y) \overset{(3.2.11)}{=} -i\partial\!\!\!/_x \int d^4z\; S^{(0)}(x,z)\,B(z)\,((-s\,B)^k\,s)(z,y) \quad , \tag{3.2.38}$$

we can express the $(k+1)^{\text{st}}$ order contribution to the perturbation series (3.2.12) in terms of the k^{th} order contribution. We must show that (3.2.38) can again be written as a sum of expressions of the form (3.2.32) (with k replaced by $k+1$), and that *(i)*, *(ii)*, *(a)*, *(b)*, and (3.2.34), (3.2.35) are satisfied. This is done in several construction steps:

1) Chiral decomposition: We substitute the induction hypothesis (3.2.32) into (3.2.38). This gives a sum of expressions of the form

$$\begin{aligned} C\, i\partial\!\!\!/_x \int d^4z\; S^{(0)}(x,z) \Big\{ &(y-z)^I\, B(z)\, \chi_c \int_z^y [l_1, r_1 \,|\, n_1]\, dz_1\, \partial_{z_1}^{I_1} \Box^{p_1}\, V_{J_1,c_1}^{(1)}(z_1) \\ &\cdots \int_{z_{k-1}}^y [l_k, r_k \,|\, n_k]\, dz_k\, \partial_{z_k}^{I_k} \Box^{p_k}\, V_{J_k,c_k}^{(k)}(z_k)\, \gamma^J \Big\}\, S^{(h)}(z,y) \quad . \end{aligned} \tag{3.2.39}$$

We insert the special form of the potential B, (3.2.10), and expand. Using the commutation rule $\gamma^i \chi_{L/R} = \chi_{R/L} \gamma^i$, we bring all chiral projectors to the very left, where they can be combined with the formula $\chi_c \chi_d = \delta_{cd} \chi_c$ to a single chiral projector. Next, we bring the γ-matrices of B to the right and write them together with the factor γ^J in (3.2.39) (notice that the Dirac matrices commute with the potentials $V^{(a)}_{c_a}$, which are only non-diagonal in the Dirac sea index). Denoting the individual potentials of the factor B in (3.2.39) by $V^{(0)}_{J_0,c_0}$, we thus get for (3.2.39) a sum of expressions of the form

$$\chi_c\, C\, i\partial\!\!\!/_x \int d^4z\; S^{(0)}(x,z) \left\{ (y-z)^I\; V^{(0)}_{J_0,c_0}(z) \int_z^y [l_1, r_1 \mid n_1]\, dz_1\; \partial_{z_1}^{I_1}\, \Box^{p_1}\; V^{(1)}_{J_1,c_1}(z_1) \right.$$
$$\left. \cdots \int_{z_{k-1}}^y [l_k, r_k \mid n_k]\, dz_k\; \partial_{z_k}^{I_k}\, \Box^{p_k}\; V^{(k)}_{J_k,c_k}(z_k)\, \gamma^J \right\} S^{(h)}(z,y) \quad . \tag{3.2.40}$$

The chiral decomposition (3.2.10) and the induction hypothesis *(i)* yield that the chirality c_0 coincides with c, whereas c_1 coincides with c_0 if and only if $V^{(0)}_{c_0} = A_{c_0}$. Since the chiralities c_2, c_3,... satisfy the induction hypothesis *(ii)*, we conclude that the rules *(i)* and *(ii)* are also satisfied in (3.2.40) (after relabeling the indices in an obvious way). The chirality of the potentials will not be affected in all the following construction steps; to simplify the notation, we will omit the indices c_a from now on.

2) Light-cone expansion with Lemma 3.2.1, Lemma 3.2.2: Since y can be considered as a fixed parameter, we can in (3.2.40) apply either Lemma 3.2.1 or Lemma 3.2.2 with V given by the expression in the curly brackets,

$$\begin{aligned} (3.2.40) \;&=\; \chi_c\, C\, i\partial\!\!\!/_x \sum_{n=0}^\infty \frac{1}{n!} \int_x^y [0, h \mid n]\, dz \\ &\quad \times \Box_z^n \left((y-z)^I\; V^{(0)}_{J_0}(z) \int_z^y [l_1, r_1 \mid n_1]\, dz_1\; \partial_{z_1}^{I_1}\, \Box^{p_1}\; V^{(1)}_{J_1}(z_1) \right. \\ &\qquad \left. \cdots \int_{z_{k-1}}^y [l_k, r_k \mid n_k]\, dz_k\; \partial_{z_k}^{I_k}\, \Box^{p_k}\; V^{(k)}_{J_k}(z_k) \right) \gamma^J\; S^{(n+h+1)}(x,y) \quad . \end{aligned} \tag{3.2.41}$$

3) Computation of the Laplacian $\Box_z^n$: We carry out the z-derivatives in (3.2.41) inductively with the Leibniz rule. Each derivative can act either on the factors $(y-z)^I$ or on the functions $V^{(a)}$. In the first case, one of the factors $(y-z)$ disappears. Thus we get a sum of expressions of the form

$$\chi_c\, C\, i\partial\!\!\!/_x \int_x^y [0, h \mid n]\, dz\; (y-z)^{\hat I}\; \partial_z^{I_0}\, \Box_z^{p_0}\; V^{(0)}_{J_0}(z) \int_z^y [l_1, r_1 \mid n_1]\, dz_1\; \partial_{z_1}^{I_1}\, \Box^{p_1}\; V^{(1)}_{J_1}(z_1)$$
$$\cdots \int_{z_{k-1}}^y [l_k, r_k \mid n_k]\, dz_k\; \partial_{z_k}^{I_k}\, \Box^{p_k}\; V^{(k)}_{J_k}(z_k)\, \gamma^J\; S^{(n+h+1)}(x,y) \tag{3.2.42}$$

with $|\hat I| \leq |I|$ and

$$2n = |I| - |\hat I| + |I_0| + 2p_0 \quad . \tag{3.2.43}$$

We can assume that no tensor indices of $\partial_z^{I_0}$ are contracted with each other (otherwise we rewrite the corresponding partial derivatives as additional Laplacians). Then all the partial derivatives ∂_z in (3.2.42) were generated in the case when one derivative

of a Laplacian $\Box_z$ in (3.2.41) hit a factor $(y-z)$ whereas the other derivative acted on the $V^{(a)}$. Thus the number of factors $(y-z)$ which disappeared by carrying out the Laplacians in (3.2.41) is larger or equal than the number of partial derivatives ∂_z,

$$|I| - |\hat{I}| \;\geq\; |I_0| \quad . \tag{3.2.44}$$

4) Extraction of the factors $(y-x)$: In (3.2.42), we iteratively apply the identity

$$\int_x^y [0, r \mid n]\, dz\, (y-z) \cdots \;=\; (y-x) \int_x^y [0, r+1 \mid n]\, dz\, \cdots \quad . \tag{3.2.45}$$

This gives $(k+1)$ nested line integrals of the form

$$\begin{aligned} (3.2.42) \quad &= \chi_c\, C\, i\partial\!\!\!/_x (y-x)^{\hat{I}}\; S^{(\hat{h})}(x,y) \int_x^y [l_0, r_0 \mid n_0]\, dz_0\, \partial_{z_0}^{I_0} \Box^{p_0}\, V_{J_0}^{(0)}(z_0) \\ &\quad \cdots \int_{z_{k-1}}^y [l_k, r_k \mid n_k]\, dz_k\, \partial_{z_k}^{I_k} \Box^{p_k}\, V_{J_k}^{(k)}(z_a)\, \gamma^J \end{aligned} \tag{3.2.46}$$

with

$$l_0 \;=\; 0 \quad , \qquad r_0 = h + |\hat{I}| \quad , \qquad n_0 = n \tag{3.2.47}$$

$$0 \;\leq\; 2\hat{h} \;=\; 2(n+h+1) \overset{(3.2.43)}{=} 2h + 2 + |I| - |\hat{I}| + |I_0| + 2p_0 \quad . \tag{3.2.48}$$

We can arrange that the parameters l_0, r_0, and n_0 are all positive: The only parameter which might be negative is r_0; in this case, $h = -1$, $|\hat{I}| = 0$, and thus $r_0 = -1$. The induction hypothesis (3.2.34) yields that $|I| > 0$. Thus $|I| > |\hat{I}|$, and relation (3.2.43) gives that $(n_0 =) n > 0$. Therefore we can apply the identity

$$[l_0, r_0 \mid n_0] \;=\; [l_0 + 1, r_0 + 1 \mid n_0 - 1]$$

to make all the parameters in this bracket positive.

5) Computation of the partial derivative $\partial\!\!\!/_x$: The x-derivative in (3.2.46) can act on the factors $S^{(\hat{h})}$, $(y-x)^{\hat{I}}$, or $V^{(a)}(z_a)$. The first case can be computed with the rules (3.2.17) or (3.2.25); it decreases $\hat{h}$ by one and gives one additional factor $(y-x)$. In the second case, one factor $(y-x)$ disappears, and thus $|\hat{I}|$ is decremented. The last case can be handled with the rule

$$\frac{\partial}{\partial x^k} \int_x^y [l, r \mid n]\, dz\, f(z,y) \;=\; \int_x^y [l, r+1 \mid n]\, \frac{\partial}{\partial z^k} f(z,y) \quad , \tag{3.2.49}$$

which increases $|I_0|$ by one. As is immediately verified in each of these cases, equation (3.2.48) transforms into

$$2\hat{h} \;=\; 2h + 1 + |I| - |\hat{I}| + |I_0| + 2p_0 \quad , \tag{3.2.50}$$

whereas inequality (3.2.44) must be weakened to

$$|\hat{I}| \;\leq\; 1 + |I| - |I_0| \quad . \tag{3.2.51}$$

Finally, we combine the γ-matrix of the factor $\partial\!\!\!/_x$ with γ^J.

After these transformations, the $(k+1)^{\text{st}}$ order Feynman diagram consists of a sum of terms of the form

$$\chi_c\, C\, (y-x)^{\hat{I}} \int_x^y [l_0, r_0 \mid n_0]\, dz_0\; \partial_{z_0}^{I_0} \Box_{z_0}^{p_0}\, V_{J_0}^{(0)}(z_0)$$
$$\cdots \int_{z_{k-1}}^y [l_k, r_k \mid n_k]\, dz_k\; \partial_{z_k}^{I_k} \Box_{z_k}^{p_k}\, V_{J_k}^{(k)}(z_k)\, \gamma^J\; S^{(\hat{h})}(x,y) \quad . \tag{3.2.52}$$

Notice that the parameters I_a, p_a, $a = 1, \ldots, k$, were not changed by the above construction steps; they are still the same as in the induction hypothesis (3.2.32). After renaming the indices and the integration variables, (3.2.52) is of the required form (3.2.32). The conditions *(a)* and *(b)* for the contractions of the tensor indices, however, will in general be violated. Therefore we need two further computation steps:

6) Simplification of the γ-matrices: If any two of the tensor indices of the factor γ^J are contracted with each other, we reorder the γ-matrices with the anti-commutation relations

$$\{\gamma^i,\, \gamma^j\} \;=\; 2\, g^{ij}\; \mathbb{1} \tag{3.2.53}$$

until the corresponding matrices are next to each other. Applying the identity $\gamma^i \gamma_i = 4\,\mathbb{1}$, both Dirac matrices disappear. We iterate this procedure until no tensor indices of γ^J are contracted with each other (notice that the iteration comes to an end because the number of γ-factors is decreased by two in each step). Again using the anti-commutation rule (3.2.53), we reorder the Dirac matrices until they are in the same order in which the factors to which their tensor indices are contracted appear in the product (3.2.52). If any two of the γ-matrices are contracted with the same multi-index, these γ-matrices are next to each other, and we can use the symmetry in the tensor indices to eliminate them both, more precisely

$$(y-x)_i\, (y-x)_j \cdots \gamma^i \gamma^j \;=\; (y-x)^2 \cdots \mathbb{1} \tag{3.2.54}$$
$$\partial_{ij} V^{(a)} \cdots \gamma^i \gamma^j \;=\; \Box V^{(a)} \cdots \mathbb{1} \quad . \tag{3.2.55}$$

After all these transformations, condition *(b)* is satisfied.

Notice that the parameters $|I_a|$ and p_a are in general changed in this construction step. More precisely, each transformation (3.2.55) modifies the parameters according to

$$|I_a| \;\to\; |I_a| - 2 \quad , \qquad p_a \;\to\; p_a + 1 \quad . \tag{3.2.56}$$

7) Handling of the new contractions: If any two tensor indices of a factor $\partial_{z_a}^{I_a}$ are contracted with each other, we rewrite the corresponding partial derivatives as a Laplacian; this changes the parameters $|I_a|$ and p_a according to (3.2.56). If two tensor indices of the factor $(y-x)^{\hat{I}}$ are contracted with each other, this gives a factor $(y-x)^2$. Using the identity (3.2.18) or (3.2.26), we inductively absorb the factors $(y-x)^2$ into $S^{(\hat{h})}(x,y)$, which transforms $\hat{h}$ and $|\hat{I}|$ as

$$\hat{h} \;\to\; \hat{h} + 1 \quad , \qquad |\hat{I}| \;\to\; |\hat{I}| - 2 \quad . \tag{3.2.57}$$

After these transformations, condition *(a)* is also satisfied.

After all these construction steps, the $(k+1)^{\text{th}}$ order Feynman diagram is a sum of terms of the form (3.2.52) satisfying the conditions *(a)* and *(b)*. It remains to show that the relations (3.2.34) and (3.2.35) remain valid in our inductive construction: As mentioned earlier, the parameters I_a, p_a, $a = 1, \ldots, k$ are not changed in the construction steps *1)* to *5)*. In the steps *6)* and *7)*, the transformations (3.2.56) and (3.2.57) preserve both the induction hypothesis (3.2.34),(3.2.35) and the relations (3.2.50),(3.2.51), as is immediately verified. By substituting (3.2.50) and (3.2.51) into (3.2.56),(3.2.57), we conclude that

$$2\hat{h} \;=\; (k+1) - 1 - |\hat{I}| + \sum_{a=0}^{k} |I_a| + 2p_a \quad , \qquad\qquad |\hat{I}| \;\leq\; (k+1) + 1 - \sum_{a=0}^{k} |I_a| \quad . \qquad \blacksquare$$

Note that this proof is constructive in the sense that it gives a procedure with which the light-cone expansion of every Feynman diagram can be carried out explicitly. The disadvantage of this procedure, however, is that the resulting formulas become more and more complicated to higher order perturbation theory. Therefore it is essential to rearrange and collect the contributions of all Feynman diagrams in a way which makes it clear how $\tilde{s}(x,y)$ looks like to every order on the light cone. In preparation for this analysis, which will be the task of the following Subsection 3.2.2, we shall now simplify the light-cone expansion of Theorem 3.2.3 a little bit. More precisely, we want to eliminate from (3.2.32) the partial derivatives in direction of the line integrals, i.e. those derivatives ∂_{z_l} which are contracted with a factor $(y-x)$. We call these derivatives *tangential*. The following combinatorial lemma controls the number of tangential derivatives.

Lemma 3.2.4 *The contributions (3.2.32) to the light-cone expansion of Theorem 3.2.3 satisfy for $a = 1, \ldots, k$ the inequalities*

$$l_a + n_a \;\geq\; t_a - 1 \qquad \text{and} \qquad r_a + n_a \;\geq\; \sum_{b=a}^{k} t_b \quad , \tag{3.2.58}$$

where t_a denotes the number of tensor indices of the multi-index I_a which are contracted with the factor $(y-x)^I$.

Proof: As in the proof of Theorem 3.2.3, we proceed inductively in the order k of the perturbation theory. For $k = 0$, the inequalities (3.2.58) are trivially satisfied according to (3.2.37). Assume that (3.2.58) is true for a given k. We go through the construction steps *1)* to *7)* of Theorem 3.2.3 and check that the inequalities (3.2.58) then also hold in (3.2.52) for $a = 0, \ldots, k$.

We first consider the case $a > 0$. The parameters l_a, r_a, and n_a remain unchanged in all the construction steps of Theorem 3.2.3. Furthermore, it is obvious that the parameters t_a are not affected in the steps *1)*, *2)*, *4)*, and *7)*. In the steps *3)* and *5)*, the computation of the derivatives $\Box_z^n$ and $\not{\partial}_x$ might annihilate some of the factors $(y-x)$ which were contracted with the factors $\partial_{z_a}^{I_a}$; this may decrease the parameters t_a. For the analysis of step *6)*, note that all γ-matrices which are contracted with factors $(y-x)$ stand to the left of those γ-matrices which are contracted with the $\partial_{z_a}^{I_a}$, $a = 1, \ldots, k$ (this follows from the ordering condition *(b)* in the induction hypothesis and the fact that additional factors $(y-x)^j \cdots \gamma_j$ are only generated during the construction if the partial derivative $\not{\partial}_x$ hits $S^{(\hat{h})}$ in step *5)*; in this case, the corresponding γ-matrix stands at the very left in γ^J). Therefore the commutations of the Dirac matrices do not lead to additional contractions between factors $(y-x)$ and $\partial_{z_a}^{I_a}$, which implies that the parameters t_a remain unchanged

in step *6)*. We conclude that the l_a, r_a, and n_a remain unchanged whereas the t_a may only decrease, and thus (3.2.58) holds for $a = 1, \ldots, k$ throughout all the construction steps.

It remains to show that the inequalities (3.2.58) hold in (3.2.52) for $a = 0$. We first look at the situation after step *4)* in (3.2.46): The values (3.2.47) for l_0, r_0, and n_0 give in combination with (3.2.43) the equations

$$l_0 + n_0 \;=\; \frac{1}{2}\Big(|I| \;-\; |\hat{I}| \;+\; |I_0| \;+\; 2p_0\Big) \tag{3.2.59}$$

$$r_0 + n_0 \;=\; h \;+\; \frac{1}{2}\Big(|I| \;+\; |\hat{I}| \;+\; |I_0| \;+\; 2p_0\Big) \quad . \tag{3.2.60}$$

Moreover, the number of tangential derivatives t_0 at the first potential is clearly bounded by the total number of derivatives there,

$$|I_0| \;\geq\; t_0 \quad . \tag{3.2.61}$$

Furthermore, the total number of tangential derivatives is smaller than the number of factors $(y - x)$,

$$|\hat{I}| \;\geq\; \sum_{a=0}^{k} t_a \quad . \tag{3.2.62}$$

Substituting (3.2.44) and (3.2.61) into (3.2.59) yields the inequalities

$$l_0 + n_0 \;\geq\; |I_0| + p_0 \;\geq\; t_0 \quad . \tag{3.2.63}$$

In order to get a bound for $r_0 + n_0$, we must distinguish two cases. If $h \geq 0$, we substitute (3.2.44) into (3.2.60) and get with (3.2.62) the inequality

$$r_0 + n_0 \;\geq\; |\hat{I}| \;+\; |I_0| \;+\; p_0 \;\geq\; |\hat{I}| \;\geq\; \sum_{a=0}^{k} t_a \quad . \tag{3.2.64}$$

In the case $h = -1$, (3.2.36) shows that $|I_a|$, and consequently also t_a, vanish for $1 \leq a \leq k$. Furthermore, (3.2.34) yields that $|I| \neq 0$. Thus (3.2.60) and (3.2.61), (3.2.62) give the bound

$$r_0 + n_0 \;\geq\; h \;+\; \frac{|I|}{2} \;+\; \frac{1}{2}\sum_{a=0}^{k} t_a \;+\; \frac{1}{2}\,t_0 \;\geq\; \frac{1}{2}\sum_{a=0}^{k} t_a \;+\; \frac{1}{2}\,t_0 \quad ,$$

where we used in the last inequality that $h + |I|/2 \geq -1/2$ and that all the other terms are integers. Since $t_0 = \sum_{a=0}^{k} t_a$, we conclude that inequality (3.2.64) also holds in the case $h = -1$.

We finally consider how the bounds (3.2.63) and (3.2.64) for $l_0 + n_0$ and $r_0 + n_0$ must be modified in the subsequent construction steps. In step *5)*, the partial derivative $\partial\!\!\!/_x$ may annihilate a factor $(y - x)$, in which case the parameters t_a might decrease. On the other hand, the partial derivatives $\partial\!\!\!/_x$ may produce an additional factor ∂_{z_0}; in this case, r_0 is incremented according to (3.2.49). In step *6)*, only this additional factor ∂_{z_0} may be contracted with $(y - x)^{\hat{I}}$. Step *7)* does not change l_0, r_0, n_0, and t_0. Putting these transformations together, we conclude that the inequality (3.2.63) for $l_0 + n_0$ must be weakened by one, whereas the bound (3.2.64) for $r_0 + n_0$ remains valid as it is. This gives precisely the inequalities (3.2.58) for $a = 0$. ∎

Theorem 3.2.5 (partial integration of the tangential derivatives) *Every contribution (3.2.32) to the light cone expansion of Theorem 3.2.3 can be written as a finite sum of expressions of the form*

$$\chi_c\, C\,(y-x)^K\, W^{(0)}(x) \int_x^y [l_1, r_1 \mid n_1]\, dz_1\, W^{(1)}(z_1) \int_{z_1}^y [l_2, r_2 \mid n_2]\, dz_2\, W^{(2)}(z_2)$$
$$\cdots \int_{z_{\alpha-1}}^y [l_\alpha, r_\alpha \mid n_\alpha]\, dz_\alpha\, W^{(\alpha)}(z_\alpha)\, \gamma^J\, S^{(h)}(x,y) \quad , \ \alpha \leq k, \tag{3.2.65}$$

where the factors $W^{(\beta)}$ are composed of the potentials and their partial derivatives,

$$W^{(\beta)} \;=\; (\partial^{K_{a_\beta}} \Box^{p_{a_\beta}} V^{(a_\beta)}_{J_{a_\beta}, c_{a_\beta}}) \cdots (\partial^{K_{b_\beta}} \Box^{p_{b_\beta}} V^{(b_\beta)}_{J_{b_\beta}, c_{b_\beta}}) \tag{3.2.66}$$

with $a_1 = 1$, $a_{\beta+1} = b_\beta + 1$, $b_\beta \geq a_\beta - 1$ (in the case $b_\beta = a_\beta - 1$, $W^{(\beta)}$ is identically one), and $b_\alpha = k$. The parameters l_a, r_a, and n_a are non-negative integers, C is a complex number, and $c = L/R$, $c_a = L/R$ are chiral indices. The potentials $V^{(a)}$ are again given by (3.2.33); their chirality is determined by the rules (i) and (ii) of Theorem 3.2.3. The tensor indices of the multi-indices J, K, J_a, and K_a are all contracted with each other, according to the rules (a),(b) of Theorem 3.2.3 and

(c) The tensor indices of $(y-x)^K$ are all contracted with the tensor indices of the factors $V^{(a)}_{J_a}$ or γ^J (but not with the factors ∂^{K_a}).

We have the relation

$$2h \;=\; k - 1 - |K| + \sum_{a=1}^k (|K_a| + 2p_a) \quad . \tag{3.2.67}$$

Proof: The basic method for the proof is to iteratively eliminate those partial derivatives $\partial_{z_a}^{I_a}$ in (3.2.32) which are contracted with a factor $(y-x)$. This is done with the partial integration formula

$$(y-x)^j \int_x^y [l, r \mid n]\, dz\, \partial_j f(z) \overset{(3.2.31)}{=} \int_0^1 d\alpha\, \alpha^l\, (1-\alpha)^r\, (\alpha - \alpha^2)^n\, \frac{d}{d\alpha} f(\alpha y + (1-\alpha)x)$$
$$= \delta_{r+n,0}\, f(y) \;-\; \delta_{l+n,0}\, f(x)$$
$$-(l+n) \int_x^y [l-1, r \mid n]\, dz\, f(z) \;+\; (r+n) \int_x^y [l, r-1 \mid n]\, dz\, f(z) \quad . \tag{3.2.68}$$

In order to see the main difficulty, consider the example of two nested line integrals with two tangential derivatives

$$(y-x)^j\, (y-x)^k \int_x^y [0,1 \mid 0]\, dz_1\, V^{(1)}(z_1) \int_{z_1}^y [0,1 \mid 0]\, dz_2\, \partial_{jk} V^{(2)}(z_2) \tag{3.2.69}$$
$$= \quad (y-x)^j \int_x^y [0,0 \mid 0]\, dz_1\, V^{(1)}(z_1)\, (y-z_1)^k \int_{z_1}^y [0,1 \mid 0]\, dz_2\, \partial_{jk} V^{(2)}(z_2)$$
$$= \quad -(y-x)^j \int_x^y dz_1\, V^{(1)}(z_1)\, \partial_j V^{(2)}(z_1) \tag{3.2.70}$$
$$+(y-x)^j \int_x^y dz_1\, V^{(1)}(z_1) \int_{z_1}^y dz_2\, \partial_j V^{(2)}(z_2) \quad . \tag{3.2.71}$$

Although the line integrals in (3.2.69) satisfy the conditions of Theorem 3.2.3, the expression cannot be transformed into the required form (3.2.65). Namely in (3.2.70), we cannot

eliminate the remaining tangential derivative (because partial integration would yield a term $(y-x)^j\,\partial_j V^{(1)}(z_1)$). In (3.2.71), on the other hand, we can successfully perform a second partial integration

$$(3.2.71) \;=\; \int_x^y [0,-1\,|\,0]\,dz_1\;V^{(1)}(z_1)\,(V^{(2)}(y)-V^{(2)}(z_1)) \quad ,$$

but then the second parameter in the bracket $[.,.\,|\,.]$ becomes negative. More generally, we must ensure that the boundary terms contain no tangential derivatives, and that the parameters l_a, r_a, and n_a stay positive in the construction.

Since the chirality of the potentials is not affected by the partial integrations, it is obvious that the rules *(i)* and *(ii)* of Theorem 3.2.3 will remain valid. For simplicity in notation, we will omit the indices c_a in the following.

First of all, we split up the factor $(y-x)^I$ in (3.2.32) in the form $(y-x)^I=(y-x)^K(y-x)^L$, where L are those tensor indices which are contracted with the partial derivatives $\partial_{z_a}^{I_a}$, $a=1,\ldots,k$. Setting $b=1$ and $z_0=x$, the first line integral in (3.2.32) can be written as

$$\cdots\;(y-z_{b-1})^L \int_{z_{b-1}}^y [l_b,r_b\,|\,n_b]\,dz_b\;\partial_{z_b}^{I_b}\,\Box_{z_b}^{p_b}\,V_{J_b}^{(b)}(z_b)\;\cdots \quad . \tag{3.2.72}$$

We rewrite the tangential derivatives in this line integral as derivatives in the integration variable,

$$=\;\cdots\;(y-z_{b-1})^N \int_0^1 d\alpha\;\alpha^l\,(1-\alpha)^r\;\left(\frac{d}{d\alpha}\right)^q \partial_{z_b}^{K_b}\,\Box_{z_b}^{p_b}\,V_{J_b}^{(b)}(z_b)\;\cdots \tag{3.2.73}$$

with $|L|=|N|+q$ and $l=l_b+n_b$, $r=r_b+n_b$. Lemma 3.2.4 gives the bounds

$$l\;\geq\;q-1 \quad , \qquad r\;\geq\;q+|N| \quad . \tag{3.2.74}$$

More generally, we use (3.2.73),(3.2.74) as our induction hypothesis, whereby the left factor '$\cdots$' stands for all previous line integrals (which contain *no* tangential derivatives), and the right factor '$\cdots$' stands for subsequent line integrals. The tensor indices of the factor $(y-z_{a-1})^N$ must all be contracted with the partial derivatives $\partial_{z_a}^{I_a}$ for $a>b$ and thus give tangential derivatives in the subsequent line integrals. The induction step is to show that all the α-derivatives in (3.2.73) can be eliminated, and that we can write the resulting expressions again in the form (3.2.73),(3.2.74) with b replaced by $b+1$. Under the assumption that this induction step holds, we can eliminate all tangential derivatives in k steps. The resulting expressions are very similar to (3.2.65),(3.2.66). The only difference is that the derivatives ∂^{K_a} and $\Box^{p_a}$ in the resulting expressions are differential operators acting on all the following factors $V^{(a)}$, $V^{(a+1)}$,...; in (3.2.66), on the other hand, the partial derivatives act only on the adjacent potential $V^{(a)}$. In order to bring the resulting expressions into the required form, we finally carry out all the derivatives with the Leibniz rule and the chain rule (3.2.49).

For the proof of the induction step, we integrate in (3.2.73) q times by parts (if q is zero, we can skip the partial integrations; our expression is then of the form (3.2.76)). Since the powers of the factors α and $(1-\alpha)$ are decreased at most by one in each partial integration step, (3.2.74) implies that the boundary values vanish unless in the last step for $\alpha=0$. We thus obtain a sum of terms of the form

$$\cdots(y-z_{b-1})^N\;\partial_{z_b}^{K_b}\,\Box_{z_b}^{p_b}\,V_{J_b}^{(b)}(z_b)\;\cdots_{|z_b\equiv z_{b-1}} \qquad \text{and} \tag{3.2.75}$$

$$\cdots(y-z_{b-1})^N \int_{z_{b-1}}^y [l,r\,|\,n=0]\,dz_b\;\partial_{z_b}^{K_b}\,\Box_{z_b}^{p_b}\,V_{J_b}^{(b)}(z_b)\;\cdots \qquad \text{with } l\geq 0,\; r\geq |N| \tag{3.2.76}$$

In (3.2.76), we iteratively use the relation

$$(y-x)^j \int_x^y [l,r \mid n]\, dz \;\cdots \;=\; \int_x^y [l,r-1 \mid n]\, dz\; (y-z)^j \;\cdots \tag{3.2.77}$$

to bring all factors $(y-z_{b-1})$ to the right,

$$(3.2.76) \;=\; \cdots \int_{z_{b-1}}^y [l,r \mid n=0]\, dz_b\; (y-z_b)^N\; \partial_{z_b}^{K_b} \Box_{z_b}^{p_b} V_{J_b}^{(b)}(z_b) \cdots \qquad \text{with } l,r \geq 0. \tag{3.2.78}$$

It might be convenient to reorganize the polynomials in this line integral with the identity

$$\int_x^y [l,r \mid n]\, dz \cdots \;=\; \int_x^y [l-1,r-1 \mid n+1]\, dz \cdots \quad ,$$

but this is not relevant for the statement of the theorem.

In both cases (3.2.75) and (3.2.78), we have an expression of the form

$$\cdots (y-z_b)^N\; \partial_{z_b}^{K_b} \Box_{z_b}^{p_b} V_{J_b}^{(b)}(z_b) \cdots \quad , \tag{3.2.79}$$

where the first factor '$\cdots$' stands for line integrals without tangential derivatives, and where none of the factors $(y-z_b)$ are contracted with $\partial_{z_b}^{K_b}$. Applying the "inverse Leibniz rules"

$$(y-x)^j\, \frac{\partial}{\partial x^k} \;=\; \frac{\partial}{\partial x^k}\, (y-x)^j \;+\; \delta_k^j \tag{3.2.80}$$

$$(y-x)_j\, \Box_x \;=\; \Box_x\, (y-x)_j \;+\; 2\frac{\partial}{\partial x^j} \quad , \tag{3.2.81}$$

we iteratively commute all factors $(y-z_b)$ in (3.2.79) to the right. This gives a sum of expressions of the form

$$\cdots \partial_{z_b}^{K_b} \Box_{z_b}^{p_b} V_{J_b}^{(b)}(z_b)\; (y-z_b)^L \cdots \quad , \tag{3.2.82}$$

where the factors $(y-z_b)$ are all contracted with the partial derivatives $\partial_{z_a}^{I_a}$, $a=b+1,\ldots,k$. The Leibniz rules may have annihilated some factors $(y-z_b)$ (i.e., $|L|$ might be smaller than $|N|$); in this case, the parameters t_a, $a=b+1,\ldots,k$ have decreased. As a consequence, the inequalities of Lemma 3.2.4 are still valid for all expressions (3.2.82). If we write (3.2.82) in the form (3.2.72) with b replaced by $b+1$, we can thus split up the tangential derivatives in the form (3.2.73),(3.2.74), which concludes the proof of the induction step.

It remains to derive equation (3.2.67): Note that each partial integration decreases both the number of factors $(y-z_{a-1})$ and the total number of partial derivatives by one. If we carry out the remaining derivatives with the Leibniz rule (in the last step of the proof), this does not change the total order $\sum_{a=1}^k |K_a| + 2p_a$ of the derivatives. Therefore relation (3.2.34) in Theorem 3.2.3 transforms into (3.2.67). ∎

3.2.2 Resummation, Reduction to the Phase-Free Contribution

In the previous subsection, we gave a procedure for performing the light-cone expansion of every summand of the perturbation series (3.2.12). In order to obtain formulas for the light-cone expansion of $\tilde{s}$, we must sum up the light-cone expansions of all Feynman

diagrams. Collecting the contributions of all Feynman diagrams gives, to every order on the light cone, an infinite number of terms. To get control over all these terms, we shall reorder the sums and partly carry them out. In the end, the light-cone expansion for $\tilde{s}(x,y)$ will, to any order on the light cone, consist of only a finite number of summands. This rearrangement and simplification of the sums is called *resummation* of the light-cone expansion.

In order to get a first impression of what needs to be done, we consider the leading singularity on the light cone (more precisely, we neglect all terms of the order $\mathcal{O}((y-x)^{-2})$). This corresponds to taking only the contributions with $h=-1$ in Theorem 3.2.3. According to the bound (3.2.36), the multi-indices I_a and the parameters p_a must all vanish. Furthermore, equation (3.2.34) yields that $|I| = k+1$. The only possibility to satisfy the rules *(a)* and *(b)* is to contract one factor $(y-x)$ with each potential $V^{(a)}$, $a=1,\ldots,k$, and the remaining factor $(y-x)$ with a γ-matrix. Thus the potentials $V^{(a)}$ must all be chiral potentials $A_{L/R}$. According to the rules *(a)* and *(b)*, the chirality of all potentials must coincide with the chirality of the projector χ_c. We conclude that the leading order of $((-sB)^k s)(x,y)$ on the light cone consists of a sum of expressions of the form

$$\chi_c\, C\, (y-x)_{j_1} \cdots (y-x)_{j_k} \int_x^y [l_1, r_1 \,|\, n_1]\, dz_1\; A_c^{j_1}(z_1) \\ \ldots \int_{z_{k-1}}^y [l_k, r_k \,|\, n_k]\, dz_k\; A_c^{j_k}(z_k)\; (y-x)^j \gamma_j\; S^{(-1)}(x,y)$$

with $c=L$ or $c=R$. Thus Theorem 3.2.3 makes a precise statement on the mathematical structure of all the contributions to the light-cone expansion. However, it does not give information about the values of the parameters C, l_a, r_a, and n_a. In order to see more precisely how the leading order on the light cone looks like, we perform the light-cone expansion directly with Lemma 3.2.2. To first order in the external potential, we obtain

$$\begin{aligned}
(-s\,B\,s)(x,y) &\overset{(3.2.11),(3.2.25)}{=} \frac{1}{2}\, \partial\!\!\!/_x \int d^4z\; S^{(0)}(x,z) \Big(B(z)\, (y-z)^k \gamma_k \Big)\, S^{(-1)}(z,y) \\
&\overset{(3.2.27)}{\asymp} \frac{1}{2}\, \partial\!\!\!/_x \int_x^y [0,-1 \,|\, 0]\, dz\; B(z)\, (y-z)^k \gamma_k\; S^{(0)}(x,y) \\
&\overset{(3.2.25)}{\asymp} \frac{1}{4} \int_x^y dz\; (y-x)^j \gamma_j\, B(z)\, (y-x)^k \gamma_k\; S^{(-1)}(x,y) \\
&\asymp \frac{1}{2} \int_x^y dz\; (\chi_L\, A_L^j(z) + \chi_R\, A_R^j(z))\, (y-x)_j\, (y-x)^k \gamma_k\; S^{(-1)}(x,y) \\
&\overset{(3.2.25),(3.2.11)}{=} -i\, (y-x)_j \int_x^y dz\; (\chi_L\, A_L^j(z) + \chi_R\, A_R^j(z))\; s(x,y) \quad , \qquad (3.2.83)
\end{aligned}$$

where '$\asymp$' denotes that we only take the leading order on the light cone. Since (3.2.83) is a product of a smooth function with a single factor $s(x,y)$, this formula can be immediately iterated. We obtain for the left and right handed component of the k^{th} order Feynman diagram

$$\chi_c\, ((-s\,B)^k s)(x,y) \;\asymp\; \chi_c\, (-i)^k\, (y-x)_{j_1} \cdots (y-x)_{j_k} \int_x^y [0,k-1 \,|\, 0]\, dz_1\; A_c^{j_1}(z_1) \\ \times \int_{z_1}^y [0,k-2 \,|\, 0]\, dz_2\; A_c^{j_2}(z_2) \cdots \int_{z_{k-1}}^y [0,0 \,|\, 0]\, dz_k\; A_c^{j_k}(z_k)\; s(x,y) \quad . \qquad (3.2.84)$$

(Notice that, according to Lemma 3.2.1, the higher order terms which were neglected in (3.2.83) also give contributions of the order $\mathcal{O}((y-x)^{-2})$ in the iteration. Therefore

it is really sufficient to take only the leading contribution on the light cone in every step.) The line integrals in (3.2.84) are particularly simple. Namely, they are the k^{th} order contributions to the familiar Dyson series. From this we conclude that, to leading order on the light cone, the sum over all Feynman diagrams converges absolutely. We can carry out the sum and obtain

$$\chi_c \, \tilde{s}(x,y) \;\asymp\; \chi_c \, \mathrm{Pexp}\left(-i \int_0^1 A_{c\,|\alpha y+(1-\alpha)x}^j \, (y-x)_j \, d\alpha\right) \, s(x,y) \quad , \tag{3.2.85}$$

where Pexp is the usual ordered exponential. For completeness, we give its definition:

Def. 3.2.6 *For a smooth one-parameter family of matrices $F(\alpha)$, $\alpha \in \mathbb{R}$, the ordered exponential* $\mathrm{Pexp}(\int F(\alpha)\, d\alpha)$ *is given by the Dyson series*

$$\begin{aligned} \mathrm{Pexp}\left(\int_a^b F(\alpha)\, d\alpha\right) \;&=\; \mathbb{1} \;+\; \int_a^b dt_0 \, F(t_0)\, dt_0 \;+\; \int_a^b dt_0 \, F(t_0) \int_{t_0}^b dt_1 \, F(t_1) \\ &\quad + \int_a^b dt_0 \, F(t_0) \int_{t_0}^b dt_1 \, F(t_1) \int_{t_1}^b dt_2 \, F(t_2) \;+\; \cdots \quad . \end{aligned}$$

As is verified by a direct calculation, the ordered exponential is a solution of the ordinary differential equation

$$\frac{d}{da}\mathrm{Pexp}\left(\int_a^b F(\alpha)\, d\alpha\right) \;=\; -F(a)\, \mathrm{Pexp}\left(\int_a^b F(\alpha)\, d\alpha\right) \tag{3.2.86}$$

with the boundary conditions

$$\mathrm{Pexp}\left(\int_b^b F(\alpha)\, d\alpha\right) \;=\; \mathbb{1} \quad . \tag{3.2.87}$$

Because of the uniqueness of the solutions of ordinary differential equations, one can alternatively take (3.2.86),(3.2.87) as the definition for the ordered exponential.

For the ordered exponential in (3.2.85), we also use the shorter notations

$$\mathrm{Pexp}\left(-i \int_x^y A_c^j \, (y-x)_j\right) \qquad \text{and} \qquad \mathrm{Pe}^{-i \int_x^y A_c^j \, (y-x)_j} \quad . \tag{3.2.88}$$

Notice that (3.2.88) is a unitary $(f \times f)$-matrix which depends only on the chiral potentials along the line segment $\overline{xy}$. Its inverse is[1]

$$\mathrm{Pexp}\left(-i \int_x^y A_c^j \, (y-x)_j\right)^\dagger \;=\; \mathrm{Pexp}\left(-i \int_y^x A_c^j \, (x-y)_j\right) \quad . \tag{3.2.90}$$

If the chiral potentials are commutative, i.e. $[A_c(x), A_c(y)] = 0$, then the ordered exponential coincides with the ordinary exponential (this is e.g. the case if one considers the system $f = 1$ of only one Dirac sea). For the ordered exponential (3.2.88) along a line

[1] We mention for clarity that this is in general not the same as

$$\mathrm{Pexp}\left(i \int_x^y A_c^j \, (y-x)_j\right) \quad , \tag{3.2.89}$$

because the exponentials in (3.2.90) and (3.2.89) are ordered in opposite directions.

segment in Minkowski space, the differential equation (3.2.86) can be written with partial derivatives as

$$(y-x)^k \frac{\partial}{\partial x^k} \mathrm{Pe}^{-i\int_x^y A_c^j\,(y-x)_j} \;=\; i(y-x)_k\, A_c^k(x)\; \mathrm{Pe}^{-i\int_x^y A_c^j\,(y-x)_j} \quad . \tag{3.2.91}$$

We conclude that, to leading order on the light cone, the special form of the formulas of the light-cone expansion allows us to immediately carry out the sum over all Feynman diagrams. Unfortunately, the situation to higher order on the light cone is more difficult, because the combinatorics of the partial derivatives and of the tensor contractions becomes very complicated. We cannot expect that the sum over all Feynman diagrams can then still be written in a simple, closed form. Nevertheless, ordered exponentials over the chiral potentials should be helpful. More precisely, it is promising to write the light-cone expansion with line integrals which contain intermediate ordered exponentials, like e.g. the line integral

$$\int_x^y dz\; \mathrm{Pe}^{-i\int_x^z A_L^j\,(z-x)_j}\; (\Box \A_L(z))\; \mathrm{Pe}^{-i\int_z^y A_L^k\,(y-z)_k} \tag{3.2.92}$$

(expressions of this form are also suggested in view of the behavior of the fermionic projector under local gauge transformations of the external potential). Our basic idea is to arrange the contributions to the light-cone expansion of Theorem 3.2.5 to any given order on the light cone in such a way that all infinite sums (which arise from the fact that we have an infinite number of Feynman diagrams) can be carried out giving ordered exponentials. We want to end up with a *finite* number of terms which are of the form (3.2.65) with the only exception that the nested line integrals contain, similar to (3.2.92), additional ordered exponentials.

Before we can make this idea mathematically precise, we must clarify the connection between the line integrals in (3.2.65) and the line integrals with intermediate ordered exponentials. For this, we consider the example (3.2.92). If we expand the ordered exponentials in a Dyson series and reparametrize the integrals, (3.2.92) goes over into an infinite sum of nested line integrals of the form as in (3.2.65), more precisely

$$\begin{aligned}
(3.2.92) \;=\; & \sum_{p,q=0}^{\infty} \int_x^y [l_1, r_1 \,|\, n_1]\, dz_1\, A_L^{j_1}(z_1)\, (y-x)_{j_1} \int_{z_1}^y [l_2, r_2 \,|\, n_2]\, dz_2\, A_L^{j_2}(z_2)\, (y-x)_{j_2} \\
& \cdots \int_{z_{p-1}}^y [l_p, r_p \,|\, n_p]\, dz_p\, A_L^{j_p}(z_p)\, (y-x)_{j_p} \int_{z_p}^y [l, r \,|\, n]\, dz\, \Box \A_L(z) \\
\times & \int_z^y [l_{p+1}, r_{p+1} \,|\, n_{p+1}]\, dz_{p+1}\, A_L^{j_{p+1}}(z_{p+1})\, (y-x)_{j_{p+1}} \int_{z_{p+1}}^y A_L^{j_{p+2}}(z_{p+2})\, (y-x)_{j_{p+2}} \\
& \cdots \int_{z_{p+q-1}}^y [l_{p+1}, r_{p+q} \,|\, n_{p+1}]\, dz_{p+q}\, A_L^{j_{p+q}}(z_{p+q})\, (y-x)_{j_{p+q}} \quad .
\end{aligned} \tag{3.2.93}$$

The formulas of our desired light-cone expansion must be such that, after expanding the ordered exponentials in this way, we get precisely all the contributions to the light-cone expansion of Theorem 3.2.5. We point out that we can view the expansion (3.2.93) as a power series in the functions $A_{L/R}^j\,(y-x)_j$. The leading contribution to this power series is simply the line integral

$$\int_x^y dz\, \Box \A_L(z) \quad ;$$

it is obtained from (3.2.92) by taking out the ordered exponentials there. In view of this example, we can hope to get the contributions to our desired expansion without the ordered

exponentials (i.e. with all the ordered exponentials removed from the formulas) by picking those contributions to the light-cone expansion of Theorem 3.2.5 which contain no factors $A^j_{L/R}\,(y-x)_j$. We take these contributions as the starting point for our construction.

Def. 3.2.7 *A contribution (3.2.65),(3.2.66) to the light-cone expansion of Theorem 3.2.5 is called* **phase-free** *if all the tangential potentials $V^{(a)}_{J_a}$ are differentiated, i.e.*

$$|K_a| + 2p_a > 0 \qquad \text{whenever} \qquad J_a \text{ is contracted with } (y-x)^K.$$

To leading order on the light cone, the only phase-free contribution is the free Green's function s (namely, according to (3.2.84), the contributions with $k \geq 1$ all contain factors $A^j_{L/R}\,(y-x)_j$). The restriction to the phase-free contribution also simplifies the situation in the general case. Namely, the following proposition shows that the phase-free contributions of the higher order Feynman diagrams involve higher mass-derivatives of the Green's functions.

Proposition 3.2.8 *For every phase-free contribution (3.2.65) to the light-cone expansion of the k^{th} order Feynman diagram $((-s\,B)^k s)(x,y)$, the parameter h satisfies the bound*

$$h \;\geq\; -1 + \left[\frac{k+1}{2}\right] \quad , \tag{3.2.94}$$

where $[.]$ denotes the Gauß bracket.

Proof: Consider a phase-free contribution (3.2.65) of the k^{th} order Feynman diagram. According to the rules for the possible contractions of the tensor indices, only one factor $(y-x)$ may be contracted with γ^J; the remaining $|K|-1$ factors $(y-x)$ must be contracted with the $V^{(a)}_{J_a,c_a}$. Thus at least $|K|-1$ potentials are tangential and must (according to Def. 3.2.7) be differentiated. This gives the inequality

$$|K| - 1 \;\leq\; \sum_{a=1}^{k} (|K_a| + 2p_a) \quad .$$

We substitute this bound into (3.2.67) and obtain $2h \geq -2 + k$; this is equivalent to (3.2.94). ∎

According to the explicit formula (3.2.5), the higher mass-derivatives of the Green's functions are of higher order on the light cone. More precisely, (3.2.94) and (3.2.6) yield that the phase-free contribution to the light-cone expansion of the k^{th} order Feynman diagram is of the order

$$\mathcal{O}((y-x)^{2g}) \qquad \text{with} \qquad g = -2 + \left[\frac{k+1}{2}\right] \quad . \tag{3.2.95}$$

This means that, to every order on the light cone, only a finite number of Feynman diagrams contribute. As a consequence, there are, to every order on the light cone, only a finite number of phase-free terms.

The phase-free contributions are useful because our desired light-cone expansion is obtained from them by inserting ordered exponentials into the line integrals. We do this "by hand," according to simple rules.

Def. 3.2.9 *For every phase-free contribution (3.2.65) to the light-cone expansion of Theorem 3.2.5, we introduce a corresponding* **phase-inserted contribution**. *It is constructed according to the following rules:*

(I) We insert one ordered exponential into each line integral and one ordered exponential at the very end. More precisely, the phase-inserted contribution has the form

$$\begin{aligned}
&\chi_c\, C\, (y-x)^K\, W^{(0)}(x) \int_x^y [l_1, r_1 \,|\, n_1]\, dz_1\, \mathrm{Pe}^{-i \int_x^{z_1} A_{c_1}^{j_1}\, (z_1-x)_{j_1}}\, W^{(1)}(z_1) \\
&\quad \times \int_{z_1}^y [l_2, r_2 \,|\, n_2]\, dz_2\, \mathrm{Pe}^{-i \int_{z_1}^{z_2} A_{c_2}^{j_2}\, (z_2-z_1)_{j_2}}\, W^{(2)}(z_2)\, \cdots \\
&\quad \times \int_{z_{\alpha-1}}^y [l_\alpha, r_\alpha \,|\, n_\alpha]\, dz_\alpha\, \mathrm{Pe}^{-i \int_{z_{\alpha-1}}^{z_\alpha} A_{c_\alpha}^{j_\alpha}\, (z_\alpha - z_{\alpha-1})_{j_1}}\, W^{(\alpha)}(z_\alpha) \\
&\qquad\qquad \times \mathrm{Pe}^{-i \int_{z_\alpha}^y A_{c_{\alpha+1}}^{j_{\alpha+1}}\, (y-z_\alpha)_{j_{\alpha+1}}}\; \gamma^J\; S^{(h)}(x,y) \quad . \qquad (3.2.96)
\end{aligned}$$

(II) The chirality c_β, $\beta = 1, \ldots, \alpha+1$ of the potentials in the ordered exponentials is determined by the number of dynamic mass matrices in the factors $W^{(\beta)}$; namely

$$c_{\beta-1} \text{ and } c_\beta \left\{ \begin{array}{c} \text{coincide} \\ \text{are opposite} \end{array} \right\} \text{if } W^{(\beta-1)} \text{ contains an } \left\{ \begin{array}{c} \text{even} \\ \text{odd} \end{array} \right\} \text{number of factors } Y,$$

where $c_0 := c$ is the chirality of the projector χ_c in (3.2.96).

To illustrate these insertion rules, we consider the example of two nested line integrals

$$\chi_L \int_x^y dz_1\, (\Box \not{A}_L)(z_1) \int_{z_1}^y dz_2\, m Y_L(z_2)\, S^{(1)}(x,y) \quad .$$

The corresponding phase-inserted contribution is

$$\begin{aligned}
&\chi_L \int_x^y dz_1\, \mathrm{Pe}^{-i \int_x^{z_1} A_L^j\, (z_1-x)_j}\, (\Box \not{A}_L)(z_1) \\
&\quad \times \int_{z_1}^y dz_2\, \mathrm{Pe}^{-i \int_{z_1}^{z_2} A_L^k\, (z_2-z_1)_k}\, m Y_L(z_2)\, \mathrm{Pe}^{-i \int_{z_2}^y A_R^l\, (y-z_2)_l}\, S^{(1)}(x,y) \quad .
\end{aligned}$$

Theorem 3.2.10 *The light-cone expansion of the Green's function $\tilde{s}(x,y)$ coincides with the sum of all phase-inserted contributions.*

Proof: A possible method for the proof would be to rearrange all the contributions to the light-cone expansion of Theorem 3.2.5 until recovering the Dyson series of the ordered exponentials in (3.2.96). However, this method has the disadvantage of being technically complicated. It is more elegant to use a particular form of "local gauge invariance" of the Green's function for the proof: First we will, for given x and y, locally transform the spinors. The transformation will be such that the light-cone expansion for the transformed Green's function $\hat{s}(x,y)$ consists precisely of all phase-free contributions. Using the transformation law of the Green's function, we then show that the light-cone expansion of $\tilde{s}(x,y)$ is obtained from that of $\hat{s}(x,y)$ by inserting unitary matrices into the line integrals. Finally, we prove that these unitary matrices coincide with the ordered exponentials in Def. 3.2.9.

In preparation, we consider the transformation law of the Dirac operator and the Green's function under generalized local phase transformations of the spinors: We let U_L and U_R be two unitary matrix fields acting on the Dirac sea index,

$$U_L \;=\; (U_L(x)^l_m)_{l,m=1,\ldots,f}\;, \quad U_R \;=\; (U_R(x)^l_m)_{l,m=1,\ldots,f} \qquad \text{with} \qquad U_L\, U_L^* = \mathbb{1} = U_R\, U_R^*,$$

and transform the wave functions $\Psi = (\Psi_l^\alpha(x))_{l=1,\dots,f}^{\alpha=1,\dots,4}$ according to

$$\Psi(x) \;\to\; \hat{\Psi}(x) \;=\; U(x)\,\Psi(x) \qquad \text{with} \qquad U(x) \;=\; \chi_L\,U_L(x) \;+\; \chi_R\,U_R(x) \quad .$$

Thus U_L and U_R independently transform the left and right handed component of the wave functions, respectively. Notice that the transformation U is *not* unitary with respect to our scalar product (3.1.4), because

$$\begin{aligned} V \;&:=\; U^{-1} \;=\; \chi_L\,U_L^{-1} \;+\; \chi_R\,U_R^{-1} \qquad \text{but} \\ U^* \;&=\; \gamma^0\,U^\dagger\,\gamma^0 \;=\; \chi_R\,U_L^{-1} \;+\; \chi_L\,U_R^{-1} \quad . \end{aligned}$$

Therefore we must in the following carefully distinguish between U, U^* and their inverses V, V^*. As an immediate consequence of the Dirac equation $(i\partial\!\!\!/ + \mathcal{B} - m)\Psi = 0$, the transformed wave functions $\hat{\Psi}$ satisfy the equation

$$V^*(i\partial\!\!\!/ + B)V\,\hat{\Psi} \;=\; 0 \quad .$$

A short computation yields for the transformed Dirac operator

$$V^*(i\partial\!\!\!/ + B)V \;=\; i\partial\!\!\!/ + \hat{B}$$

with

$$\hat{B} \;=\; \chi_L\,(\hat{A\!\!\!/}_R - m\,\hat{Y}_R) \;+\; \chi_R\,(\hat{A\!\!\!/}_L - m\,\hat{Y}_L) \quad , \tag{3.2.97}$$

whereby $\hat{A}_{L/R}$ and $\hat{Y}_{L/R}$ are the potentials

$$\hat{A}^j_{L/R} \;=\; U_{L/R}\,A^j_{L/R}\,U^{-1}_{L/R} \;+\; iU_{L/R}(\partial^j U^{-1}_{L/R}) \tag{3.2.98}$$

$$\hat{Y}_{L/R} \;=\; U_{L/R}\,Y_{L/R}\,U^{-1}_{R/L} \quad . \tag{3.2.99}$$

We denote the advanced and retarded Green's functions of the transformed Dirac operator $i\partial\!\!\!/ + \hat{B}$ by $\hat{s}$. They satisfy the equation

$$(i\partial\!\!\!/_x \;+\; \hat{B}(x))\,\hat{s}(x,y) \;=\; \delta^4(x-y) \quad . \tag{3.2.100}$$

Since we can view $\hat{B}$ as the perturbation of the Dirac operator, the Green's function $\hat{s}$ is, in analogy to (3.2.12), given by the perturbation series

$$\hat{s} \;=\; \sum_{n=0}^{\infty} (-s\,\hat{B})^n\,s \quad . \tag{3.2.101}$$

The important point for the following is that the Green's functions $\tilde{s}$ and $\hat{s}$ are related to each other by the local transformation

$$\hat{s}(x,y) \;=\; U(x)\,\tilde{s}(x,y)\,U(y)^* \quad . \tag{3.2.102}$$

This is verified as follows: The right side of (3.2.102) also satisfies the defining equation (3.2.100) of the Green's functions; namely

$$\begin{aligned} &(i\partial\!\!\!/_x + \hat{B}(x))\,U(x)\,\tilde{s}(x,y)\,U(y)^* \;=\; V(x)^*\,(i\partial\!\!\!/_x + B(x))\,V(x)\,U(x)\,\tilde{s}(x,y)\,U(y)^* \\ &\quad=\; V(x)^*\,(i\partial\!\!\!/_x + B(x))\,\tilde{s}(x,y)\,U(y)^* \;=\; V(x)^*\,\delta^4(x-y)\,U(y)^* \\ &\quad=\; V(x)^*\,U(x)^*\,\delta^4(x-y) \;=\; \delta^4(x-y) \quad . \end{aligned}$$

Furthermore, the supports of both sides of (3.2.102) lie (depending on whether we consider the advanced or retarded Green's functions) either in the upper or in the lower light cone. A uniqueness argument for the solutions of hyperbolic differential equations yields that both sides of (3.2.102) must coincide.

Next, we specify the unitary transformations U_L and U_R: We fix the points x and y. For any point z on the line segment $\overline{xy}$, we chose $U_{L/R}(z)$ as

$$U_{L/R}(z) \;=\; \mathrm{Pexp}\left(-i \int_x^z A^j_{L/R}\; (z-x)_j\right) \quad . \tag{3.2.103}$$

The differential equation (3.2.91) yields that

$$\begin{aligned}(y-x)^j\; U_c(z)\; (\partial_j U_c(z)^{-1}) \;&=\; \mathrm{Pe}^{-i \int_x^z A_c^k\, (z-x)_k}\; (y-x)^j \frac{\partial}{\partial z^j} \mathrm{Pe}^{-i \int_z^x A_c^k\, (x-z)_k} \\ &=\; \mathrm{Pe}^{-i \int_x^z A_c^k\, (z-x)_k}\; i(y-x)_j\; A_c^j(z)\; \mathrm{Pe}^{-i \int_z^x A_c^k\, (x-z)_k} \\ &=\; i(y-x)_j\; U_c(z)\; A_c^j(z)\; U_c(z)^{-1} \quad .\end{aligned}$$

Substituting into (3.2.98) gives

$$\hat{A}^j_{L/R}(z)\; (y-x)_j \;=\; 0 \qquad \text{for } z \in \overline{xy}. \tag{3.2.104}$$

Thus our choice of U_L and U_R makes the potentials $\hat{A}_L(z)$ and $\hat{A}_R(z)$, $z \in \overline{xy}$, orthogonal to $(y-x)$. Notice, however, that since $Y_{L/R}$ in (3.2.99) is arbitrary and independent of $U_{L/R}$, (3.2.103) gives no constraints for the dynamic mass matrices $\hat{Y}_{L/R}$.

We point out that we did not specify $U_{L/R}(z)$ outside the line segment $z \in \overline{xy}$; the unitary transformation $U_{L/R}$ may be arbitrary there. This means that also $\hat{A}_{L/R}$ is undetermined outside the line segment $\overline{xy}$. Especially, all the non-tangential derivatives of $\hat{A}_{L/R}(z)$, $z \in \overline{xy}$, are undetermined. However, equation (3.2.103) does give constraints for the tangential derivatives. For example, differentiating (3.2.104) in the direction $(y-x)$ yields

$$(y-x)^j\; (y-x)_k\; \partial_j \hat{A}^k_{L/R}(z) = 0 \qquad \text{for} \qquad z \in \overline{xy} \quad . \tag{3.2.105}$$

We now consider the perturbation expansion (3.2.101). The light-cone expansion of all Feynman diagrams according to Theorem 3.2.5 gives a sum of terms of the form

$$\begin{aligned}&\chi_c\; C\; (y-x)^K\; \hat{W}^{(0)}(x) \int_x^y [l_1, r_1 \mid n_1]\; dz_1\; \hat{W}^{(1)}(z_1) \int_{z_1}^y [l_2, r_2 \mid n_2]\; dz_2\; \hat{W}^{(2)}(z_2) \\ &\qquad \cdots \int_{z_{\alpha-1}}^y [l_\alpha, r_\alpha \mid n_\alpha]\; dz_\alpha\; \hat{W}^{(\alpha)}(z_\alpha)\; \gamma^J\; S^{(h)}(x,y) \quad ,\end{aligned} \tag{3.2.106}$$

where the factors $\hat{W}^{(\beta)}$ are given by

$$\hat{W}^{(\beta)} \;=\; (\partial^{K_{a_\beta}} \Box^{p_{a_\beta}} \hat{V}^{(a_\beta)}_{J_{a_\beta}, c_{a_\beta}}) \cdots (\partial^{K_{b_\beta}} \Box^{p_{b_\beta}} \hat{V}^{(b_\beta)}_{J_{b_\beta}, c_{b_\beta}}) \quad . \tag{3.2.107}$$

Because of condition (3.2.104), all the contributions which are not phase-free vanish. Furthermore, according to Theorem 3.2.5, the contributions (3.2.106),(3.2.107) contain no tangential derivatives. Clearly, the derivatives in these formulas may have a component in direction of $(y-x)$; but the contribution of the derivatives transversal to $(y-x)$ uniquely determines the form of each derivative term. Therefore all the phase-free contributions of

the form (3.2.106),(3.2.107) are independent in the sense that we have no algebraic relations between them. We conclude that, as long as the potentials $\hat{A}_{L/R}$ and $\hat{Y}_{L/R}$ are only specified by (3.2.98), (3.2.99), and (3.2.103), the light-cone expansion (3.2.106),(3.2.107) consists precisely of all phase-free contributions.

Next, we exploit the local transformation law (3.2.102) of the Green's functions: We solve this equation for $\tilde{s}$,

$$\tilde{s}(x,y) \;=\; V(x)\, \hat{s}(x,y)\, V(y)^* \quad . \tag{3.2.108}$$

The transformation $U_{L/R}$ does not enter on the left side of this equation. Thus the right side of (3.2.108) is also independent of $U_{L/R}$. Especially, we conclude that the light-cone expansion of $\hat{s}(x,y)$ must be independent of the derivatives of $U_{L/R}$ along the line segment $\overline{xy}$. At first sight, this might seem inconsistent because the individual contributions (3.2.106),(3.2.107) do depend on the derivatives of $U_{L/R}$ (this is obvious if one substitutes (3.2.98) and (3.2.99) into (3.2.107) and carries out the derivatives with the Leibniz rule). The right way to understand the independence of $\hat{s}(x,y)$ on the derivatives of $U_{L/R}$ is that all derivative terms of $U_{L/R}$ cancel each other to every order on the light cone if the (finite) sum over all contributions (3.2.106) to the light-cone expansion of $\hat{s}(x,y)$ is carried out. Since we will form the sum over all contributions to the light-cone expansion in the end, it suffices to consider only those contributions to the light-cone expansion which contain no derivatives of $U_{L/R}$. This means that we can substitute (3.2.98) and (3.2.99) into (3.2.107), forget about the derivative term $iU_{L/R}(\partial^j U_{L/R}^{-1})$ in (3.2.98), and pull the unitary transformations $U_{L/R}, U_{L/R}^{-1}$ out of the derivatives. In other words, we can replace $\hat{W}^{(\beta)}$, (3.2.107), by

$$\hat{W}^{(\beta)} \;=\; U_{c_{a_\beta}} (\partial^{K_{a_\beta}} \Box^{p_{a_\beta}} V^{(a_\beta)}_{J_{a_\beta}, c_{a_\beta}}) U^{-1}_{d_{a_\beta}} \;\cdots U_{c_{b_\beta}} (\partial^{K_{b_\beta}} \Box^{p_{b_\beta}} V^{(b_\beta)}_{J_{b_\beta}, c_{b_\beta}}) U^{-1}_{d_{b_\beta}} \tag{3.2.109}$$

with chiral indices $c_a, d_a = L/R$. The light-cone expansion for $\hat{s}(x,y)$ consists precisely of the sum of all phase-free contributions of the form (3.2.106), (3.2.109).

The chiralities c_a, d_a of the unitary transformations $U_{L/R}$, $U_{L/R}^{-1}$ in (3.2.109) are determined by the rules *(i)* and *(ii)* (in Theorem 3.2.3) and by (3.2.98) and (3.2.99). According to the rule *(ii)*, the indices c_a and c_{a+1} coincide iff $V^{(a)}$ is a chiral potential. According to (3.2.98) and (3.2.99), on the other hand, the indices c_a and d_a coincide iff $V^{(a)} = A_{L/R}$. We conclude that the indices d_a and c_{a+1} always coincide. Thus all the intermediate factors $U_{d_a}\, U_{c_{a+1}}$ give the identity, and (3.2.109) simplifies to

$$\hat{W}^{(\beta)} \;=\; U_{c_\beta}\, W^{(\beta)}\, U^{-1}_{d_\beta} \quad . \tag{3.2.110}$$

Furthermore, the chiralities c_β and d_β coincide if and only if $W^{(\beta)}$ contains an even number of dynamic mass matrices.

Finally, we substitute the light-cone expansion (3.2.106), (3.2.110) for $\hat{s}(x,y)$ into (3.2.108). This gives for the light-cone expansion of $\tilde{s}(x,y)$ a sum of expressions of the form

$$\begin{aligned} &\chi_c\, C\, (y-x)^K\, U_c^{-1}(x)\, (U_{c_0} W^{(0)} U_{d_0}^{-1})(x) \int_x^y [l_1, r_1 \mid n_1]\, dz_1\, (U_{c_1} W^{(1)} U_{d_1}^{-1})(z_1) \\ &\quad \cdots \int_{z_{\alpha-1}}^y [l_\alpha, r_\alpha \mid n_\alpha]\, dz_\alpha\, (U_{c_\alpha} W^{(0)} U_{d_\alpha}^{-1})(z_\alpha)\, U_{c_{\alpha+1}}(y)\, \gamma^J\, S^{(h)}(x,y) \quad , \end{aligned} \tag{3.2.111}$$

where the sum runs over all phase-free contributions of this type. Similar to the considerations before (3.2.110), one sees that adjacent unitary transformations always have the same chirality. Thus (3.2.111) can be simplified to

$$\begin{aligned}
&\chi_c\, C\, (y-x)^K\, W^{(0)}(x) \int_x^y [l_1, r_1 \mid n_1]\, dz_1\, U_{c_1}(x)^{-1}\, U_{c_1}(z_1)\, W^{(1)} \\
&\quad \cdots \int_{z_{\alpha-1}}^y [l_\alpha, r_\alpha \mid n_\alpha]\, dz_\alpha\, U_{c_\alpha}(z_{\alpha-1})^{-1}\, U_{c_\alpha}(z_\alpha)\, W^{(0)}(z_\alpha)\, U_{c_{\alpha+1}}(z_\alpha)^{-1} \\
&\quad \times U_{c_{\alpha+1}}(y)\, \gamma^J\, S^{(h)}(x,y) \quad ,
\end{aligned}$$

whereby the indices c_a satisfy the rule *(II)* of Def. 3.2.9. According to (3.2.103), the factors $U_c^{-1}(.)\, U_c(.)$ coincide with the ordered exponentials in (3.2.96), which concludes the proof. ∎

For clarity, we point out that all the constructions following Def. 3.2.9 are based on the light-cone expansion of Theorem 3.2.5. It is essential for the statement of Theorem 3.2.10 that the phase-free contributions contain no tangential derivatives. If we had worked with the light-cone expansion of Theorem 3.2.3 (instead of Theorem 3.2.5), the light-cone expansion of $\hat{s}(x,y)$ would not have consisted of all the phase-free contributions to the light-cone expansion. For example, a line integral containing the tangential derivative (3.2.105) would vanish, although it is phase-free. As a consequence of this problem, the whole construction would break down.

The introduction of the phase-free and phase-inserted contributions has simplified the light-cone expansion of the Green's functions considerably: Assume that we want to perform the light-cone expansion to some given order on the light cone. Then we first calculate the phase-free contribution to the light-cone expansion; according to Proposition 3.2.8, this gives only a finite number of terms. Using the rules of Def. 3.2.9, we can easily construct the corresponding phase-inserted contributions. According to Theorem 3.2.10, this finite number of phase-inserted contributions gives precisely the light-cone expansion of the Green's functions to the desired order on the light cone. This procedure is called the *reduction to the phase-free contribution.*

3.2.3 Calculation of the Phase-Free Contribution

According to the reduction to the phase-free contribution, it remains to calculate the phase-free contribution to any given order on the light cone. Although this is still a very complicated problem, we know from Proposition 3.2.8 that we only get a finite number of terms. This makes it possible to use a computer algebra program for the calculation. The author has developed the C++ program "class_commute" specifically for this problem. It generates explicit formulas for the phase-free contribution to any order on the light cone. We now outline how this program works, without entering implementation details[2].

All the objects occurring in the calculation (like integrals, partial derivatives, Laplacians, potentials, Dirac matrices, etc.) are described by different data structures (classes in C++). Formulas are built up as sums of lists of these data structures. The calculation is performed by manipulating the lists. More precisely, this works as follows: Each data structure carries an ordering number. At the beginning of the computation, the lists are

[2]The commented source code of the program "class_commute" is available from the author on request. It is an extension of the program used in [12] for the light-cone expansion to first order in the external potential.

disordered in the sense that their elements do not occur with increasing ordering numbers. For ordering the lists, the program iteratively commutes adjacent elements of a list. Each commutation is performed by a function of the program which is specific to the particular pair of data structures; i.e., there is a function for commuting a partial derivative with a potential, a function for commuting two Dirac matrices, etc. (this is easily implemented in C++ using virtual class functions). The data structures and commutation rules are designed in such a way that, after the ordering process has come to an end, the lists consist of the desired formulas of the light-cone expansion.

The main advantage of this implementation with commutation rules is that the programmer must only think of the calculation on a "local" scale by telling the computer the rules for commuting a given pair of data structures. Furthermore, this gives a convenient segmentation of the computer program into small, independent parts, which can be written and debugged separately. As soon as all commutation rules are specified correctly, the program can perform the whole calculation "globally" by recursively applying the commutation rules. Compared to standard computer algebra packages like e.g. Mathematica or Maple, this concept is very flexible and efficient.

For the more complex manipulations like partial integrations and the handling of tensor indices, the program uses a so-called "message pipe," through which the objects in the formulas can pass information to other objects (this is again implemented with virtual class functions). In this way, the elements of the lists can exchange data and give commands to each other. This allows a convenient coordination of the formula manipulations.

The calculation rules of the program "class_commute" are very similar to the construction steps described in the proofs of Theorem 3.2.3 and Theorem 3.2.5. The only difference is that, according to our implementation as commutation rules, the calculation does not follow the same strict and clear order as in Subsection 3.2.1. Basically, one may think of the construction steps of Theorem 3.2.3 and Theorem 3.2.5 as being performed simultaneously in a disordered way, whereby the program ensures that all rules are applied consistently.

Some formulas generated by the program "class_commute" are compiled in the appendix; they give a picture of the leading singularities of $\tilde{s}(x, y)$ on the light cone. We remark that "class_commute" was also a valuable tool for finding and checking the combinatorial results of Theorem 3.2.3, Lemma 3.2.4, and Theorem 3.2.5.

3.3 The Light-Cone Expansion of the Dirac Sea

In this section, we shall perform the light-cone expansion of the fermionic projector as defined in [11]. In Subsection 3.3.1, we establish a formal analogy between the light-cone expansions of the Dirac sea and of the Green's functions. This *residual argument* allows us to use the results of the previous section also for the fermionic projector. However, the analogy between the Dirac sea and the Green's functions cannot be extended beyond a purely formal level. The basic reason is that, in contrast to the Green's functions, the Dirac sea is a *non-causal* object. This is developed in detail in Subsections 3.3.2 and 3.3.3.

We point out that, in this section, we do not work with the dynamic mass matrices $Y_{L/R}(x)$, (3.2.8). The reason is that, for the Dirac sea, the regularity conditions of Lemma 3.1.1 are necessary for the contributions to the perturbation expansion to be well-defined. Working with the dynamic mass matrices, however, implies that we consider the potential B, (3.2.10), as the perturbation of the Dirac operator; but B does in general not go to zero at infinity. For our notation, the reader is referred to [11].

3.3.1 The Residual Argument

We begin by describing how the light-cone expansion of the Green's functions can be understood in momentum space. Apart from giving a different point of view, this allows us to get a connection to the light-cone expansion of the Dirac sea. For clarity, we begin with the special case $mY = 0$ of zero fermion mass. This case is particularly simple because then $B = \mathcal{B}$, so that the perturbation expansions (3.2.1) and (3.2.12) coincide. This is sufficient to explain the basic construction; the extension to $mY \neq 0$ will later be accomplished by a general argument. Furthermore, we only consider the advanced Green's function; for the retarded Green's function, the calculation is analogous.

Suppose that we want to perform the light-cone expansion of the k^{th} order contribution to the perturbation series (3.2.1)=(3.2.12). We write the contribution as a multiple Fourier integral,

$$\begin{aligned}&((-s^\vee B)^k\, s^\vee)(x,y) \\ &= \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q_1}{(2\pi)^4} \cdots \int \frac{d^4q_k}{(2\pi)^4}\, \Delta s^\vee(p; q_1, \ldots, q_k)\, e^{-i(p+q_1+\cdots+q_k)x + ipy} \quad , \end{aligned} \tag{3.3.1}$$

where the distribution $\Delta s^\vee(p; q_1, \ldots, q_k)$ is the Feynman diagram in momentum space,

$$\begin{aligned}\Delta s^\vee(p; q_1, \ldots, q_k) \;=\; & (-1)^k\, s^\vee(p+q_1+\cdots+q_k)\, \tilde{B}(q_k)\, s^\vee(p+q_1+\cdots+q_{k-1})\, \tilde{B}(q_{k-1}) \\ & \cdots\; \tilde{B}(q_2)\, s^\vee(p+q_1)\, \tilde{B}(q_1)\, s^\vee(p)\end{aligned} \tag{3.3.2}$$

($\tilde{B}$ denotes the Fourier transform of the potential B, and $s^\vee(p)$ is the multiplication operator in momentum space). For the arguments of the Green's functions, we introduce the abbreviation

$$p_0 \;:=\; p \qquad \text{and} \qquad p_l \;:=\; p + q_1 + \cdots + q_l\,, \quad 1 \leq l \leq k. \tag{3.3.3}$$

Substituting the explicit formulas (3.2.3) and (3.2.15) into (3.3.2), we obtain

$$\begin{aligned}\Delta s^\vee(p; q_1, \ldots, q_k) \;&=\; (-1)^k\, \not{p}_k\, \tilde{B}(q_k)\, \not{p}_{k-1} \;\cdots\; \not{p}_1\, \tilde{B}(q_1)\, \not{p}_0 \\ &\quad \times \lim_{0<\varepsilon\to 0} \frac{1}{(p_k)^2 - i\varepsilon p_k^0}\, \frac{1}{(p_{k-1})^2 - i\varepsilon p_{k-1}^0} \cdots \frac{1}{(p_0)^2 - i\varepsilon p_0^0} \quad .\end{aligned} \tag{3.3.4}$$

We now expand the Klein-Gordon Green's functions in (3.3.4) with respect to the momenta $p_l - p$. If we expand the terms $i\varepsilon p_l^0$ with a geometric series,

$$\frac{1}{(p_l)^2 - i\varepsilon p_l^0} \;=\; \sum_{n=0}^\infty \frac{(i\varepsilon\,(p_l^0 - p^0))^n}{((p_l)^2 - i\varepsilon p^0)^{1+n}} \quad ,$$

all contributions with $n \geq 1$ contain factors ε and vanish in the limit $\varepsilon \to 0$. Therefore we must only expand with respect to the parameters $((p_l)^2 - p^2)$. This gives, again with geometric series,

$$\begin{aligned}\Delta s^\vee(p; q_1, \ldots, q_k) \;&=\; (-1)^k\, \not{p}_k\, \tilde{B}(q_k)\, \not{p}_{k-1} \;\cdots\; \not{p}_1\, \tilde{B}(q_1)\, \not{p}_0 \\ &\quad \times \sum_{n_1,\ldots,n_k=0}^\infty (p^2 - p_k^2)^{n_k} \;\cdots\; (p^2 - p_1^2)^{n_1} \lim_{0<\varepsilon\to 0} \frac{1}{(p^2 - i\varepsilon p^0)^{1+k+n_1+\cdots+n_k}} \quad .\end{aligned}$$

Rewriting the negative power of $(p^2 - i\varepsilon p^0)$ as a mass-derivative,

$$\begin{aligned}
&\frac{1}{(p^2 - i\varepsilon p^0)^{1+k+n_1+\cdots+n_k}} \\
&\quad = \frac{1}{(k+n_1+\cdots+n_k)!} \left(\frac{d}{da}\right)^{k+n_1+\cdots+n_k} \frac{1}{p^2 - a - i\varepsilon p^0}_{|a=0} \quad , \qquad (3.3.5)
\end{aligned}$$

we obtain a formula containing only one Green's function. Namely, using the notation (3.2.13),

$$\begin{aligned}
\Delta s^\vee(p; q_1, \ldots, q_k) \;&=\; (-1)^k \not{p}_k \, \tilde{B}(q_k) \not{p}_{k-1} \;\cdots\; \not{p}_1 \, \tilde{B}(q_1) \not{p}_0 \\
&\times \sum_{n_1,\ldots,n_k=0}^{\infty} \frac{1}{(k+n_1+\cdots+n_k)!} \, (p^2 - p_k^2)^{n_k} \;\cdots\; (p^2 - p_1^2)^{n_1} \; S^{\vee(k+n_1+\cdots+n_k)}(p) \; . \qquad (3.3.6)
\end{aligned}$$

This is the basic equation for the light-cone expansion of the Green's functions in momentum space. Similar to the light-cone expansion of the previous section, (3.3.6) involves the differentiated Green's functions $S^{\vee(\cdot)}$. It remains to transform the polynomials in the momenta $p_0, \ldots, p_k$ until getting a connection to the nested line integrals of, say, Theorem 3.2.3: Substituting (3.3.3), we rewrite (3.3.6) in terms of the momenta p, $q_1, \ldots, q_k$ and multiply out. Furthermore, we simplify the Dirac matrices with the anti-commutation rules (3.2.53). This gives for (3.3.6) a sum of terms of the form

$$\chi_c \, C \, \gamma^I \, q_k^{I_k} \cdots q_1^{I_1} \; \tilde{V}^{(k)}_{J_k,c_k}(q_k) \cdots \tilde{V}^{(1)}_{J_1,c_1}(q_1) \, p^L \; S^{\vee(h)}(p) \qquad (h \geq [|L|/2]), \qquad (3.3.7)$$

where the tensor indices of the multi-indices I, I_l, J_l, and L are contracted with each other (similar to the notation of Theorem 3.2.3, the factors $\tilde{V}^{(l)}_{J_l,c_l}$ stand for the individual potentials of $\tilde{B}$). If tensor indices of the power p^L are contracted with each other, we can iteratively eliminate the corresponding factors p^2 with the rule (3.2.14), more precisely

$$p^2 \; S^{\vee(h)}(p) \;=\; h \; S^{\vee(h-1)}(p) \qquad (h \geq 1). \qquad (3.3.8)$$

Thus we can arrange that the tensor indices of p^L in (3.3.7) are all contracted with tensor indices of the factors γ^I, $q_l^{I_l}$, or $\tilde{V}^{(l)}_{J_l,c_l}$. By iteratively applying the differentiation rule (3.2.16), we can now rewrite the power p^L in (3.3.7) with p-derivatives, e.g.

$$\begin{aligned}
p_j \, p_k \, S^{\vee(2)}(p) \;&=\; -\frac{1}{2} \, p_j \, \frac{\partial}{\partial p^k} S^{\vee(1)}(p) \;=\; -\frac{1}{2} \, \frac{\partial}{\partial p^k}(p_j \; S^{\vee(1)}(p)) \;+\; \frac{1}{2} \, g_{jk} \; S^{\vee(1)}(p) \\
&=\; \frac{1}{4} \, \frac{\partial^2}{\partial p^j \, \partial p^k} S^{(0)}(p) \;+\; \frac{1}{2} \, g_{jk} \; S^{(1)}(p) \quad .
\end{aligned}$$

In this way, we obtain for $\Delta s^\vee(p; q_1, \ldots, q_k)$ a sum of terms of the form

$$\chi_c \, C \, \gamma^I \, q_k^{I_k} \cdots q_1^{I_1} \; \tilde{V}^{(k)}_{J_k,c_k}(q_k) \cdots \tilde{V}^{(1)}_{J_1,c_1}(q_1) \, \partial_p^K \; S^{\vee(h)}(p) \quad , \qquad (3.3.9)$$

whereby no tensor indices of the derivatives ∂_p^K are contracted with each other. We substitute these terms into (3.3.1) and transform them to position space. Integrating the derivatives ∂_p^K by parts gives factors $(y-x)^K$. The factors $q_l^{I_l}$, on the other hand, can be written as partial derivatives ∂^{I_l} acting on the potentials $V^{(l)}$. More precisely, the term (3.3.9) gives after substitution into (3.3.1) the contribution

$$\chi_c \, C \, i^{|I_1|+\cdots+|I_k|} \, (-i)^{|K|} \; \gamma^I \; (\partial^{I_k} V^{(k)}_{J_k,c_k}(x)) \cdots (\partial^{I_1} V^{(1)}_{J_1,c_1}(x)) \, (y-x)^K \; S^{\vee(h)}(x,y) \quad , \qquad (3.3.10)$$

where the tensor indices of the factor $(y-x)^K$ are all contracted with tensor indices of the multi-indices I, I_l, or J_l. The Feynman diagram $((-sB)^k s)(x,y)$ coincides with the sum of all these contributions.

This expansion has much similarity with the light-cone expansion of Theorem 3.2.3. Namely, if one expands the nested line integrals in (3.2.32) in a Taylor series around x, one gets precisely the expansion into terms of the form (3.3.10). Clearly, the light-cone expansion of Theorem 3.2.3 goes far beyond the expansion (3.3.10), because the dependence on the external potential is described by non-local line integrals. Nevertheless, the expansion in momentum space (3.3.6) and subsequent Fourier transformation give an easy way of understanding in principle how the formulas of the light-cone expansion come about. We remark that, after going through the details of the combinatorics and rearranging the contributions (3.3.10), one can recover the Taylor series of the line integrals in (3.2.32). This gives an alternative method for proving Theorem 3.2.3. However, it is obvious that this becomes complicated and does not yield the most elegant approach (the reader interested in the details of this method is referred to [12], where a very similar technique is used for the light-cone expansion to first order in the external potential).

Our next aim is to generalize the previous construction. Since we must, similar to (3.3.5), rewrite a product of Green's functions as the mass-derivative of a single Green's function, we can only expect the construction to work if all Green's functions in the product (3.3.2) are of the same type (e.g. the construction breaks down for a "mixed" operator product containing both advanced and retarded Green's functions). But we need not necessarily work with the advanced or retarded Green's functions. Instead, we can use Green's functions with a different position of the poles in the complex p^0-plane: We consider the Green's functions

$$s^\pm(p) \;=\; \not{p}\, S^\pm_{a\,|\,a=0}(p) \qquad \text{with} \qquad S^\pm(p) \;=\; \lim_{0<\varepsilon\to 0} \frac{1}{p^2 - a \mp i\varepsilon} \tag{3.3.11}$$

and again use the notation (3.2.13),

$$S^{\pm\,(l)} \;=\; \left(\frac{d}{da}\right)^l S^\pm_{a\,|\,a=0} \quad .$$

The perturbation expansion for these Dirac Green's functions is, similar to (3.2.1) or (3.2.12), given by the formal series

$$\tilde{s}^+ \;:=\; \sum_{n=0}^\infty (-s^+\, B)^n s^+ \quad , \qquad \tilde{s}^- \;:=\; \sum_{n=0}^\infty (-s^-\, B)^n s^- \quad . \tag{3.3.12}$$

The light-cone expansion in momentum space is performed exactly as for the advanced and retarded Green's functions; we obtain in analogy to (3.3.1) and (3.3.6) the formula

$$\begin{aligned} &((-s^\pm\, B)^k\, s^\pm)(x,y) \\ &\quad = \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q_1}{(2\pi)^4} \cdots \int \frac{d^4q_k}{(2\pi)^4}\, \Delta s^\pm(p; q_1, \ldots, q_k)\, e^{-i(p+q_1+\cdots+q_k)x + ipy} \end{aligned}$$

with

$$\begin{aligned} \Delta s^\pm(p; q_1, \ldots, q_k) \;&=\; (-1)^k\, \not{p}_k\, \tilde{B}(q_k)\, \not{p}_{k-1} \cdots \not{p}_1\, \tilde{B}(q_1)\, \not{p}_0 \\ &\times \sum_{n_1,\ldots,n_k=0}^\infty \frac{1}{(k+n_1+\cdots+n_k)!}\, (p^2-p_k^2)^{n_k} \cdots (p^2-p_1^2)^{n_1}\, S^{\pm\,(k+n_1+\cdots+n_k)} \quad . \end{aligned}$$

Since $S^\pm$ are Green's functions of the Klein-Gordon equation, they clearly also satisfy the identity (3.3.8). Furthermore, the differentiation rule (3.2.16) is also valid for $S^\pm$; namely

$$\begin{aligned}\frac{\partial}{\partial p^j} S^{\pm\,(l)}(p) &= \left(\frac{d}{da}\right)^l \lim_{0<\varepsilon\to 0} \frac{\partial}{\partial p^j} \left(\frac{1}{p^2 - a \mp i\varepsilon}\right)_{|a=0} \\ &= \left(\frac{d}{da}\right)^l \lim_{0<\varepsilon\to 0} \frac{-2p_j}{(p^2 - a \mp i\varepsilon)^2}_{|a=0} \;=\; -2p_j\, S^{\pm\,(l+1)}(p) \quad .\end{aligned}$$

Therefore we can, exactly as in (3.3.9), rewrite the power p^L with p-derivatives. Thus the expansion (3.3.10) is valid in the same way for the Green's functions $s^\pm$ if one only replaces the index "$\vee$" in (3.3.10) by "$\pm$". As explained before, the expansion (3.3.10) is obtained from the light-cone expansion of Theorem 3.2.3 by expanding the potentials around the space-time point x. Since the formulas of the light-cone expansion are uniquely determined by this Taylor expansion, we immediately conclude that the statement of Theorem 3.2.3 is also valid for the k^{th} order contribution to the perturbation expansion (3.3.12) if the factor $S^{(h)}$ in (3.2.32) stands more generally for $S^{+\,(h)}$ or $S^{-\,(h)}$, respectively. This simple analogy between the formulas of the light-cone expansion of the Feynman diagrams $((-s^{\vee/\wedge}\, B)^k\, s^{\vee/\wedge})$ and $((-s^\pm\, B)^k\, s^\pm)$, which is obtained by changing the position of the poles of the free Green's functions in momentum space, is called the *residual argument*.

After these preparations, we come to the fermionic projector in the general case $mY \neq 0$. We want to extend the light-cone expansion to an object $\tilde{P}^{\text{res}}$ being a perturbation of the free fermionic projector. Our method is to define $\tilde{P}^{\text{res}}$ in such a way that it can be easily expressed in terms of the Green's functions $\tilde{s}^\vee$, $\tilde{s}^\wedge$, $\tilde{s}^+$, and $\tilde{s}^-$. The light-cone expansion of the Green's functions then immediately carries over to $\tilde{P}^{\text{res}}$. We denote the lower mass shell by T_a, i.e. in momentum space

$$T_a(q) \;=\; \Theta(-q^0)\, \delta(q^2 - a) \quad , \tag{3.3.13}$$

and set

$$T^{(l)} \;=\; \left(\frac{d}{da}\right)^l T_{a\,|\,a=0} \quad . \tag{3.3.14}$$

Furthermore, we introduce, exactly as in [11], the series of operator products

$$b^< \;=\; \sum_{k=0}^\infty (-s\,\mathcal{B})^k \quad , \qquad b \;=\; \sum_{k=0}^\infty (-\mathcal{B}\,s)^k\, \mathcal{B} \quad , \qquad b^> \;=\; \sum_{k=0}^\infty (-\mathcal{B}\,s)^k \quad .$$

Def. 3.3.1 *The* **residual fermionic projector** $\tilde{P}^{res}(x,y)$ *is defined by*

$$\tilde{P}^{res}(x,y) \;=\; \frac{1}{2}\, X\, (\tilde{p}^{res} - \tilde{k})(x,y) \quad , \tag{3.3.15}$$

where the operators $\tilde{p}^{res}$ *and* $\tilde{k}$ *are given by the perturbation series*

$$\tilde{p}^{res} \;=\; \sum_{\beta=0}^\infty (-i\pi)^{2\beta}\, b^<\, p\, (b\,p)^{2\beta}\, b^> \tag{3.3.16}$$

$$\tilde{k} \;=\; \sum_{\beta=0}^\infty (-i\pi)^{2\beta}\, b^<\, k\, (b\,k)^{2\beta}\, b^> \quad . \tag{3.3.17}$$

Proposition 3.3.2 (formal light-cone expansion of the residual fermionic projector) *The results of Section 3.2 also apply to the residual fermionic projector. More precisely, the light-cone expansion of Theorem 3.2.3 holds for $\tilde{P}^{res}(x,y)$ if we replace $S^{(h)}$ by $T^{(h)}$ and multiply the formulas of the light-cone expansion from the left with the chiral asymmetry matrix X. According to Theorem 3.2.5, all tangential derivatives can be integrated by parts. With Def. 3.2.7, Def. 3.2.9, and Theorem 3.2.10, the light-cone expansion can be reduced to the phase-free contribution. According to Proposition 3.2.8, the phase-free contribution consists, to every order $\sim T^{(h)}$, of only a finite number of terms.*

Proof: First of all, we must generalize the residual argument to the case $mY \neq 0$ of massive fermions. According to (3.2.1) and (3.2.12), there are two equivalent perturbation series for $\tilde{s}^\vee$,

$$\tilde{s}^\vee = \sum_{k=0}^\infty (-s^\vee \mathcal{B})^k \, s^\vee \tag{3.3.18}$$

$$= \sum_{k=0}^\infty (-s^\vee_{m=0} B)^k \, s^\vee_{m=0} \quad . \tag{3.3.19}$$

In both perturbation series, each summand is a well-defined tempered distribution (this follows from the smoothness of B, $\mathcal{B}$ and from the causality of the perturbation expansion). In Section 3.2, we developed the light-cone expansion from the series in (3.3.19). But by arranging the contributions to this light-cone expansion in powers of the potential $\mathcal{B}$, one also obtains formulas for the light-cone expansion of every Feynman diagram of the perturbation series (3.3.18). For the Green's functions $s^\pm$, we have similar perturbation expansions

$$\tilde{s}^\pm = \sum_{k=0}^\infty (-s^\pm \mathcal{B})^k \, s^\pm \tag{3.3.20}$$

$$= \sum_{k=0}^\infty (-s^\pm_{m=0} B)^k \, s^\pm_{m=0} \quad . \tag{3.3.21}$$

Since the support of the distributions $s^\pm(x,y)$ does *not* vanish outside the light cone, we now need the conditions of Lemma 3.1.1 on the decay of the potentials at infinity. According to our assumptions on $\mathcal{B}$, each summand of the perturbation expansion (3.3.20) is a well-defined distribution. The potential B, however, does in general not decay at infinity; thus the Feynman diagrams of the perturbation expansion (3.3.21) are ill-defined. This is a problem, especially because in our above consideration, the residual argument was derived for the Feynman diagrams of the expansions (3.3.19) and (3.3.21). The solution to this problem is an approximation argument using the "causality" of the formulas of the light-cone expansion: We consider a smooth function $\eta_R(x)$ which is equal to one inside the ball of radius R around the origin and vanishes outside the ball of radius $2R$ (in $\mathbb{R}^4$ equipped with the standard Euclidean metric). Then the potential $\eta_R B$ has compact support and, according to Lemma 3.1.1, the Feynman diagrams

$$\left((-s^\pm_{m=0} \, \eta_R B)^k \, s^\pm_{m=0} \right)(x,y) \tag{3.3.22}$$

are well-defined. We can apply the above residual argument for $mY = 0$; this yields formulas of the light-cone expansion in terms of the potential $(\eta_R B)$ and its partial derivatives.

Since the potential enters into the formulas of the light-cone expansion only along the convex line $\overline{xy}$, we can, by taking the limit $R \to \infty$, remove the cutoff function η_R from these formulas. This limiting process shows that the summands of the perturbation series in (3.3.21) make mathematical sense in terms of the light-cone expansion. By reordering the contributions, we immediately also get formulas for the light-cone expansion of the Feynman diagrams of the perturbation series (3.3.20). The analogy between the light-cone expansions of the Feynman diagrams of the perturbation series (3.3.18) and (3.3.20) finally yields the extension of the residual argument to a general mass matrix mY.

Evaluating the poles in (3.2.15) and (3.3.12) in the complex p^0-plane gives (using the formula $\lim_{0<\varepsilon\to 0}((x-i\varepsilon)^{-1} - (x+i\varepsilon)^{-1}) = 2\pi i \delta(x)$) the relations

$$s^\vee = s + i\pi k \quad , \qquad s^\wedge = s - i\pi k \tag{3.3.23}$$

$$s^+ = s + i\pi p \quad , \qquad s^- = s - i\pi p \quad , \tag{3.3.24}$$

where s denotes as in [11] the arithmetic mean of the advanced and retarded Green's functions,

$$s = \frac{1}{2}(s^\vee + s^\wedge) \quad .$$

We substitute (3.3.23) and (3.3.24) into the perturbation series (3.2.1), (3.3.20) and multiply out. After rearranging the sums, one sees that the series (3.3.16) and (3.3.17) can be written as

$$\tilde{p}^{\text{res}} = \frac{1}{2\pi i}(\tilde{s}^+ - \tilde{s}^-) \qquad \text{and} \qquad \tilde{k} = \frac{1}{2\pi i}(\tilde{s}^\vee - \tilde{s}^\wedge) \quad , \tag{3.3.25}$$

respectively (see [11, proof of Theorem 3.2] for the details of the combinatorics). According to the residual argument, all Green's functions have a light-cone expansion according to Theorem 3.2.3. By substituting into (3.3.25), this light-cone expansion immediately generalizes to $\tilde{p}^{\text{res}}$ and $\tilde{k}$. Using (3.3.15), we conclude that Theorem 3.2.3 is also valid for $\tilde{P}^{\text{res}}$ after the replacement $S^{(h)} \to T^{(h)}$ and multiplication with the chiral asymmetry matrix. Since the results of Theorem 3.2.5, Proposition 3.2.8, and Theorem 3.2.10 are obtained merely by manipulating and rearranging the formulas of the light-cone expansion, they also hold for the residual fermionic projector. ∎

We point out that the argumentation in this subsection was only formal in the sense that we did not care about the convergence of the infinite sums. Also, the approximation argument in the proof of Proposition 3.3.2 requires a mathematical justification. Furthermore, the derivative (3.3.14) is ill-defined because, for $a = 0$, the mass shell degenerates to the cone $\{q^2 = 0,\ q^0 < 0\}$, which is not differentiable at $q = 0$. We postpone the mathematical analysis of these problems to Subsection 3.3.3.

3.3.2 The Non-Causal High Energy Contribution

Unfortunately, the residual fermionic projector $\tilde{P}^{\text{res}}$ of the previous subsection does not coincide with the fermionic projector $\tilde{P}$ of [11],

$$\tilde{P}(x,y) = \frac{1}{2} X\, (\tilde{p} - \tilde{k})(x,y) \quad . \tag{3.3.26}$$

The difference is that, instead of the operator $\tilde{p}^{\text{res}}$ in the residual fermionic projector (3.3.15), the fermionic projector (3.3.26) involves the operator $\tilde{p}$, which is formally given

by

$$\tilde{p} \stackrel{\text{formally}}{=} \sqrt{\tilde{k}^2} \quad . \tag{3.3.27}$$

Using an operator calculus method, this formal definition is made mathematically precise in [11] in terms of a perturbation series for $\tilde{p}$. Similar to (3.3.16), this perturbation expansion consists of a sum of operator products. But the operator products are more complicated; they also contain operators k with some combinatorial factors (see [11] for details).

Before entering the mathematical analysis of the operator products, we point out that it is not just a matter of taste to take (3.3.26), and not (3.3.15), as the definition of the fermionic projector; only the definition (3.3.26) makes physical sense. This comes as follows: As explained in [11], the operator $\tilde{k}$ generalizes the splitting of the solutions of the Dirac equation into solutions of positive and negative frequency to the case with interaction. The "generalized positive and negative frequency solutions" are given by the eigenstates of $\tilde{k}$ with positive and negative eigenvalue, respectively. The construction (3.3.27),(3.3.26) projects out all positive eigenstates of $\tilde{k}$; the operator $\frac{1}{2}(\tilde{p} - \tilde{k})$ consists precisely of all eigenstates of $\tilde{k}$ with negative eigenvalue. The residual fermionic projector (3.3.15), however, consists of a mixture of positive and negative eigenstates of $\tilde{k}$, which is not a reasonable physical concept.

We begin by giving the difference between the fermionic projector and the residual fermionic projector a name.

Def. 3.3.3 *The* **non-causal high energy contribution** $\tilde{P}^{he}(x,y)$ *to the fermionic projector is given by*

$$\tilde{P}^{he}(x,y) \;=\; \tilde{P}(x,y) \;-\; \tilde{P}^{res}(x,y) \quad .$$

Clearly, this definition is only helpful if $\tilde{P}^{\text{he}}$ has some nice properties. The reason why the definition makes sense is that every contribution to the perturbation expansion of $\tilde{P}^{\text{he}}(x,y)$ is a smooth function. Thus the singular behavior of the fermionic projector on the light cone is completely described by the residual fermionic projector and its light-cone expansion, Proposition 3.3.2.

Theorem 3.3.4 *The non-causal high energy contribution* $\tilde{P}^{he}(x,y)$ *is, to every order in perturbation theory, a smooth function in* x *and* y.

Proof: The perturbation series [11, Theorem 4.1] defines $\tilde{p}$ as a sum of operator products of the form

$$C_n \,\mathcal{B}\, C_{n-1} \,\mathcal{B} \cdots \mathcal{B}\, C_0 \quad , \tag{3.3.28}$$

where the factors C_l coincide with either k, p, or s. The number of factors k in these operator products is always even. If one replaces all factors k by p, one gets precisely the perturbation series for $\tilde{p}^{\text{res}}$, (3.3.16) (this is verified using the details of the combinatorics in [11]). Therefore we can convert the perturbation series for $\tilde{p}$ into that for $\tilde{p}^{\text{res}}$ by iteratively replacing pairs of factors k in the operator products by factors p. Thus the difference $(\tilde{p} - \tilde{p}^{\text{res}})$ can, to every order in perturbation theory, be written as a finite sum of expressions of the form

$$\begin{aligned} C_n \,\mathcal{B} \cdots C_{b+1} \,\mathcal{B}\, (&p \,\mathcal{B}\, C_{b-1} \cdots C_{a+1} \,\mathcal{B}\, p \\ &- k \,\mathcal{B}\, C_{b-1} \cdots C_{a+1} \,\mathcal{B}\, k)\, \mathcal{B}\, C_{a-1} \cdots \mathcal{B}\, C_0 \quad , \end{aligned} \tag{3.3.29}$$

where the factors C_l again stand for k, p, or s. Since $\tilde{P}^{\mathrm{he}} = \frac{1}{2}\, X(\tilde{p} - \tilde{p}^{\mathrm{res}})$, it suffices to show that (3.3.29) is a smooth function in position space.

We first simplify our problem: Once we have shown that the bracket in (3.3.29) is smooth and bounded in position space, the additional multiplications to the very left and right can be carried out by iteratively multiplying with $\mathcal{B}$ and forming the convolution with C_l, which again gives a smooth and bounded function in each step (notice that, according to the assumptions of Lemma 3.1.1, $\mathcal{B}$ decays sufficiently fast at infinity). Thus we must only consider the bracket in (3.3.29). We rewrite this bracket with the projectors $\frac{1}{2}(p-k)$ and $\frac{1}{2}(p+k)$ on the lower and upper mass shells,

$$\begin{aligned} & p\,\mathcal{B}\,C_{n-1}\cdots C_1\,\mathcal{B}\,p \;-\; k\,\mathcal{B}\,C_{n-1}\cdots C_1\,\mathcal{B}\,k \\ &= \;\; \frac{1}{2}\,(p+k)\,\mathcal{B}\,C_{n-1}\cdots C_1\,\mathcal{B}\,(p-k) \;+\; \frac{1}{2}\,(p-k)\,\mathcal{B}\,C_{n-1}\cdots C_1\,\mathcal{B}\,(p+k) \quad . \end{aligned}$$

For symmetry reasons, it suffices to show that the first summand of this decomposition,

$$((p+k)\,\mathcal{B}\,C_{n-1}\cdots C_1\,\mathcal{B}\,(p-k))(x,y) \quad , \tag{3.3.30}$$

is smooth and bounded.

We proceed in momentum space. We say that a function $f(q)$ has *rapid decay for positive frequency* if it is C^1, bounded together with its first derivatives (i.e. $\sup|f|, \sup|\partial_l f| < \infty$), and satisfies for every $\alpha > 0$ the bounds

$$\sup_{\omega>0,\,\vec{k}\in\mathbb{R}^3} |\omega^\alpha\, f(\omega,\vec{k})|, \quad \sup_{\omega>0,\,\vec{k}\in\mathbb{R}^3} |\omega^\alpha\, \partial_l f(\omega,\vec{k})| \;<\; \infty \quad . \tag{3.3.31}$$

After setting $C_0 = p - k$ and $C_n = p + k$, the operator product (3.3.30) is of the form (3.1.8). We choose a function g with rapid decay for positive frequency and decompose the operator product in the form (3.1.12),(3.1.13). It follows by induction that the functions F_j all have rapid decay for positive frequency: The induction hypothesis is obvious by setting $F_0 = g$. The induction step is to show that for a function F_{j-1} with rapid decay for positive frequency, the convolution

$$F_j(\omega,\vec{k}) \;=\; \int \frac{d\omega'}{2\pi} \int \frac{d\vec{k}'}{(2\pi)^3}\; \tilde{B}(\omega-\omega',\vec{k}-\vec{k}')\; C_{j-1}(\omega',\vec{k}')\; F_{j-1}(\omega',\vec{k}') \tag{3.3.32}$$

also has rapid decay for positive frequency. In Lemma 3.1.1, it was shown that F_j is C^1 and bounded together with its first derivatives. As a consequence, we must only establish the bounds (3.3.31) for $\omega > 1$. Because of monotonicity $\omega^\alpha < \omega^\beta$ for $\alpha < \beta$ (and $\omega > 1$), it furthermore suffices to show that there are arbitrarily large numbers α satisfying the bounds (3.3.31); we only consider $\alpha = 2n$ with $n \in \mathbb{N}$. For $\omega > 1$ and $\omega' \in \mathbb{R}$, we have the inequality

$$\omega^{2n} \;\leq\; (2\omega')^{2n}\,\Theta(\omega') \;+\; (2(\omega-\omega'))^{2n} \quad ,$$

as is immediately verified by checking the three regions $\omega' \leq 0$, $0 < \omega' \leq \omega/2$, and $\omega' > \omega/2$. We combine this inequality with (3.3.32) and obtain for $\omega > 1$ the estimate

$$\begin{aligned} & |\omega^{2n}\, F_j(\omega,\vec{k})| \\ &\leq \left| \int \frac{d\omega'}{(2\pi)} \int \frac{d\vec{k}'}{(2\pi)^3}\, \tilde{B}(\omega-\omega',\vec{k}-\vec{k}')\, C_{j-1}(\omega',\vec{k}') \left[(2\omega')^{2n}\,\Theta(\omega')\, F_{j-1}(\omega',\vec{k})\right] \right| \qquad (3.3.33) \\ &\quad + \left| \int \frac{d\omega'}{2\pi} \int \frac{d\vec{k}'}{(2\pi)^3} \left[(2(\omega-\omega'))^{2n}\, \tilde{B}(\omega-\omega',\vec{k}-\vec{k}')\right]\, C_{j-1}(\omega',\vec{k}')\, F_{j-1}(\omega',\vec{k}) \right| . \qquad (3.3.34) \end{aligned}$$

According to the induction hypothesis, the square bracket in (3.3.33) is bounded together with its first derivatives. Since $\tilde{B}$ has rapid decay at infinity, the square bracket in (3.3.34) also has rapid decay at infinity. Thus both integrals in (3.3.33) and (3.3.34) satisfy the hypothesis considered in Lemma 3.1.1 for (3.1.10), and are therefore bounded. For estimating $|\omega^{2n}\, \partial_l F_j|$, we differentiate (3.3.32) and obtain similar to (3.3.33),(3.3.34) the inequality

$$\begin{aligned} &|\omega^{2n}\, \partial_l F_j(\omega, \vec{k})| \\ &\leq \left| \int \frac{d\omega'}{2\pi} \int \frac{d\vec{k}'}{(2\pi)^3}\, \partial_l \tilde{B}(\omega-\omega', \vec{k}-\vec{k}')\, C_{j-1}(\omega', \vec{k}') \left[(2\omega')^{2n}\, \Theta(\omega')\, F_{j-1}(\omega', \vec{k}) \right] \right| \\ &\quad + \left| \int \frac{d\omega'}{d\omega} \int \frac{d\vec{k}'}{(2\pi)^3} \left[(2(\omega-\omega'))^{2n}\, \partial_l \tilde{B}(\omega-\omega', \vec{k}-\vec{k}') \right] C_{j-1}(\omega', \vec{k}')\, F_{j-1}(\omega', \vec{k}) \right| \quad . \end{aligned}$$

This concludes the proof of the induction step.

We have just shown that for a function g with rapid decay for positive frequency, the function

$$F_n(q) \;=\; \int \frac{d^4 q_1}{(2\pi)^4}\, (\mathcal{B}\, C_{n-1}\, \mathcal{B} \cdots \mathcal{B}\, C_1\, \mathcal{B}\, C_0)\, (q, q_1)\, g(q_1) \tag{3.3.35}$$

has rapid decay for positive frequency. We now consider what this means for our operator product (3.3.30) in position space. For a given four-vector $y = (y^0, \vec{y})$, we choose

$$g(\omega, \vec{k}) \;=\; \eta(\omega)\, e^{-i(\omega y^0 - \vec{k}\vec{y})} \quad ,$$

where η is a smooth function with $\eta(\omega) = 1$ for $\omega \leq 0$ and $\eta(\omega) = 0$ for $\omega > 1$ (this choice of g clearly has rapid decay for positive frequency). Since the support of the factor $C_0 = (p-k)$ is the lower mass cone $\{q^2 \geq 0,\ q^0 \leq 0\}$, $g(\omega, \vec{k})$ enters into the integral (3.3.35) only for negative ω. But for $\omega \leq 0$, the cutoff function η is identically one. Thus the integral (3.3.35) is simply a Fourier integral; i.e., with a mixed notation in momentum and position space,

$$F_n(q) \;=\; (\mathcal{B}\, C_{n-1}\, \mathcal{B} \cdots \mathcal{B}\, C_1\, \mathcal{B}\, (p-k))\, (q, y) \quad .$$

Next, we multiply from the left with the operator $(p+k)$,

$$((p+k)\, \mathcal{B}\, C_{n-1}\, \mathcal{B} \cdots \mathcal{B}\, C_1\, \mathcal{B}\, (p-k))\, (q, y) \;=\; (p+k)(q)\, F_n(q) \quad . \tag{3.3.36}$$

Since F_n has rapid decay for positive frequency and $(p+k)$ has its support in the upper mass cone $\{q^2 \geq 0,\ q^0 > 0\}$, their product decays fast at infinity. More precisely,

$$\left| q^I\, (p+k)(q)\, F_n(q) \right| \;\leq\; \text{const}(I)\, (p+k)(q)$$

for any multi-index I. As a consequence, the Fourier transform of (3.3.36) is even finite after multiplying with an arbitrary number of factors q, i.e.

$$\left| \int \frac{d^4 q}{(2\pi)^4}\, q^I\, (p+k)(q)\, F_n(q)\, e^{-iqx} \right| \;\leq\; \text{const}(I) \;<\; \infty$$

for all x and I. This shows that our operator product in position space (3.3.30) is bounded and, for fixed y, a smooth function in x (with derivative bounds which are uniform in y). Similarly, one obtains that (3.3.30) is, for fixed x, a smooth function in y. We conclude

that the distribution (3.3.30) is a smooth and bounded function. ∎

We point out that $\tilde{P}^{\text{he}}(x,y)$ is a non-causal object in the sense that it does not only depend on the potential $\mathcal{B}$ in the "diamond" $(L_x^\vee \cap L_y^\wedge) \cup (L_x^\wedge \cap L_y^\vee)$ (with $L_x^\vee$ and $L_x^\wedge$ according to (3.2.2)), but on the external potential in the entire space-time. This becomes clear from the fact that the support of the operators $p(z_1, z_2)$ in the perturbation expansion for $\tilde{P}^{\text{he}}$ is the whole space $(z_1, z_2) \in \mathbb{R}^4 \times \mathbb{R}^4$. Especially, it is not possible to express $\tilde{P}^{\text{he}}(x,y)$ similar to the formulas of the light-cone expansion in terms of the potential and its partial derivatives along the convex line $\overline{xy}$.

The non-causal high energy contribution is an effect of higher order perturbation theory; it vanishes to first order in the external potential [12]. According to the decomposition into terms of the form (3.3.30), it comes about because states on the upper and lower mass shell are mixed through multiplication with the potential. Qualitatively speaking, this mixing only becomes an important effect if the energy (i.e. frequency) of the external potential is high enough to overcome the energy difference between the states on the upper and lower mass shell. This gives the justification for the name "high energy" contribution.

3.3.3 The Non-Causal Low Energy Contribution

In this subsection, we will put the residual argument and the formal light-cone expansion of Proposition 3.3.2 on a satisfying mathematical basis. In order to explain what we precisely need to do, we first recall how the light-cone expansion of the Green's functions makes mathematical sense: Theorem 3.2.3 gives a representation of every Feynman diagram of the perturbation series (3.2.12) as an infinite sum of contributions of the form (3.2.32). According to the bound (3.2.36), there are, for any given h, only a finite number of possibilities to choose I_a and p_a; as a consequence, we get, for fixed h, only a finite number of contributions (3.2.32). Thus we can write the light-cone expansion in the symbolic form

$$((-s\,B)^k\,s)(x,y) \;=\; \sum_{h=-1}^{\infty} \sum_{\text{finite}} \cdots\, S^{(h)}(x,y) \quad , \tag{3.3.37}$$

where '$\cdots$' stands for a configuration of the γ-matrices and nested line integrals in (3.2.32). According to the explicit formula (3.2.5), the higher a-derivatives of $S_a(x,y)$ contain more factors $(y-x)^2$ and are thus of higher order on the light cone. This makes it possible to understand the infinite sum in (3.3.37) in terms of Def. 3.1.2; we can give it a mathematical meaning via the approximation by the finite partial sums (3.1.15). In Subsection 3.2.2, it is shown that understanding the light-cone expansion via these partial sums even makes it possible to explicitly carry out the sum over all Feynman diagrams.

According to Proposition 3.3.2, all the results of Section 3.2 are, on a formal level, also valid for the residual Dirac sea. We begin by considering the light-cone expansion of the individual Feynman diagrams in more detail. Similar to (3.3.37), the k^{th} order contribution ΔP^{res} to the residual Dirac sea has an expansion of the form

$$\Delta P^{\text{res}}(x,y) \;=\; \sum_{h=-1}^{\infty} \sum_{\text{finite}} \cdots\, T^{(h)}(x,y) \quad , \tag{3.3.38}$$

where $T^{(h)}$ is the a-derivative (3.3.14) of the lower mass shell T_a, (3.3.13). In position space, T_a has the explicit form

$$T_a(x,y) \;=\; -\frac{1}{8\pi^3} \lim_{0<\varepsilon\to 0} \frac{1}{\xi^2 - i\varepsilon\xi^0}$$

$$+\frac{a}{32\pi^3}\lim_{0<\varepsilon\to 0}\Big(\log(a\xi^2-i\varepsilon\xi^0)+i\pi+c\Big)\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,(j+1)!}\,\frac{(a\xi^2)^j}{4^j}$$
$$-\frac{a}{32\pi^3}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,(j+1)!}\,\frac{(a\xi^2)^j}{4^j}\,(\Phi(j+1)+\Phi(j)) \tag{3.3.39}$$

with $\xi \equiv (y-x)$, $c = 2C - \log 2$ with Euler's constant C, and the function

$$\Phi(0) = 0 \quad , \qquad \Phi(n) = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for} \quad n \geq 1 \quad .$$

The logarithm in (3.3.39) is understood in the complex plane which is cut along the positive real axis (so that $\lim_{0<\varepsilon\to 0} \log(x+i\varepsilon) = \log|x|$ is real for $x > 0$). Alternatively, one can avoid the complex logarithm using the formula

$$\lim_{0<\varepsilon\to 0} \log(a\xi^2 - i\varepsilon\xi^0) \;+\; i\pi \;=\; \log|a\xi^2| \;+\; i\pi\,\Theta(\xi^2)\,\epsilon(\xi^0)$$

(ϵ is the step function $\epsilon(x) = 1$ for $x \geq 0$ and $\epsilon(x) = -1$ otherwise); thus the complex logarithm describes both a logarithmic pole on the light cone and a constant contribution in the interior of the light cone. The basic difference between the light-cone expansions (3.3.37) and (3.3.38) is related to the logarithmic pole $\log|a|$ in (3.3.39). Namely, as a consequence of this logarithm, the higher a-derivatives of T_a are *not* of higher order on the light cone. To the order $\mathcal{O}((y-x)^2)$, for example, one has

$$\left(\frac{d}{da}\right)^n T_a(x,y) \;=\; \frac{1}{32\pi^3}\left(\frac{d}{da}\right)^n (a\,\log|a|) \;+\; \mathcal{O}((y-x)^2) \qquad (, n \geq 2). \tag{3.3.40}$$

This means that the infinite sum in (3.3.38) cannot be understood in terms of Def. 3.1.2; the number of summands is already infinite to a given order on the light cone. In our context of an expansion around $a = 0$, the situation is even worse, because the a-derivatives of T_a are singular for $a \to 0$ (as one sees e.g. in (3.3.40)). Thus not even the individual contributions to the light-cone expansion make mathematical sense. These difficulties arising from the logarithm in (3.3.39) are called the *logarithmic mass problem* (see [12] for a more detailed discussion in a slightly different setting). Since we know from Lemma 3.1.1 that the Feynman diagrams are all well-defined, the logarithmic mass problem is not a problem of the perturbation expansion, but shows that something is wrong with the light-cone expansion of Proposition 3.3.2.

In order to resolve the logarithmic mass problem, we first "regularize" the formal light-cone expansion by taking out the problematic $\log|a|$ term. By resumming the formal light-cone expansion, we then show that the difference between the residual Dirac sea and the "regularized" Dirac sea is a smooth function in position space. We introduce the notation

$$T_a^{\text{reg}}(x,y) \;=\; T_a(x,y) \;-\; \frac{a}{32\pi^3}\,\log|a|\,\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,(j+1)!}\,\frac{(a\xi^2)^j}{4^j} \tag{3.3.41}$$

$$T^{\text{reg}\,(l)} \;=\; \left(\frac{d}{da}\right)^l T^{\text{reg}}_{a\,|\,a=0} \quad . \tag{3.3.42}$$

Def. 3.3.5 *The* **causal contribution** $\tilde{P}^{causal}$ *to the fermionic projector is obtained from the residual Dirac sea* $\tilde{P}^{res}$ *by replacing all factors* $T^{(h)}$ *in the formal light-cone expansion*

by $T^{reg\,(h)}$. The **non-causal low energy contribution** $\tilde{P}^{le}$ *to the fermionic projector is given by*

$$\tilde{P}^{le}(x,y) \;=\; \tilde{P}^{res}(x,y) \;-\; \tilde{P}^{causal}(x,y) \quad .$$

By the replacement $T^{(h)} \to T^{\text{reg}\,(h)}$, the formal light-cone expansion of Proposition 3.3.2 becomes mathematically meaningful in terms of Def. 3.1.2. Thus we can restate this result as a theorem, leaving out the word "formal."

Theorem 3.3.6 (light-cone expansion of the causal contribution) *The results of Section 3.2 also apply to the causal contribution to the fermionic projector. More precisely, the light-cone expansion of Theorem 3.2.3 holds for $\tilde{P}^{causal}$ if we replace $S^{(h)}$ by $T^{reg\,(h)}$, (3.3.42), and multiply the formulas of the light-cone expansion from the left with the chiral asymmetry matrix X. According to Theorem 3.2.5, all tangential derivatives can be integrated by parts. With Def. 3.2.7, Def. 3.2.9, and Theorem 3.2.10, the light-cone expansion can be reduced to the phase-free contribution. According to Proposition 3.2.8, the phase-free contribution consists, to every order $\mathcal{O}((y-x)^{2g})$ on the light cone, of only a finite number of terms.*

We come to the analysis of the low energy contribution. In the following lemma, we reformulate the light-cone expansion of Theorem 3.2.3 in a way where the infinite sums are handled more explicitly.

Lemma 3.3.7 *The light-cone expansion of the k^{th} order contribution $((-s\,B)^k\,s)(x,y)$ to the perturbation series (3.2.12) can be written as a finite sum of expressions of the form*

$$\begin{aligned}
&\sum_{n_1,\ldots,n_k=0}^{\infty} \frac{1}{n_1! \cdots n_k!}\, \chi_c\, C\, (y-x)^I \int_x^y [a_1, b_1+n_2+\cdots+n_k \,|\, n_1]\, dz_1\; \partial_{z_1}^{I_1}\, \Box_{z_1}^{n_1+q_1}\, V^{(1)}_{J_1,c_1}(z_1) \\
&\quad \times \int_{z_1}^y [a_2, b_2+n_3+\cdots+n_k \,|\, n_2]\, dz_2\; \partial_{z_2}^{I_2}\, \Box_{z_2}^{n_2+q_2}\, V^{(2)}_{J_2,c_2}(z_2) \\
&\qquad \cdots \int_{z_{k-1}}^y [a_k, b_k \,|\, n_k]\, dz_k\; \partial_{z_k}^{I_k}\, \Box_{z_k}^{n_k+q_k}\, V^{(k)}_{J_k,c_k}(z_k)\, \gamma^J\, S^{(r+n_1+\cdots+n_k)}(x,y) \quad . \qquad (3.3.43)
\end{aligned}$$

In this formula, we use for the chiral and tensor indices the same notation as in Theorem 3.2.3; a_l, b_l, and q_l are non-negative integers. The parameters r and b_l satisfy the bounds

$$r \;\leq\; k - |I| \qquad (3.3.44)$$

$$b_l \;\geq\; r - l + |I| \quad , \qquad 1 \leq l \leq k. \qquad (3.3.45)$$

Proof: The form of the expression (3.3.43) is straightforward if one keeps track of the infinite sums in the inductive construction of Theorem 3.2.3; it is also obvious that we only get a finite number of such expressions. The only point which needs an explanation is how one can arrange that all infinite sums are of the form

$$\sum_{n=0}^{\infty} \frac{1}{n!}\, [.,. \,|\, n]\, \Box^n \;\cdots \quad . \qquad (3.3.46)$$

For this, we must manipulate the sums when then Laplacian $\Box^n$ is carried out after (3.2.41). Whenever a Laplacian acts on a factor $(y-x)^I$, we shift the summation index

by one. More precisely, in the case of one factor $(y-x)$, we use the transformation

$$\sum_{n=0}^{\infty} \frac{1}{n!}\,[a,b\mid n]\,\Box_x^n\,(y-x)^i\,f_{(n)}(x) \;=\; \sum_{n=0}^{\infty} \frac{1}{n!}\,[a,b\mid n]\,\Big((y-x)^i\,\Box_x^n\,f_{(n)} - 2n\,\partial^i \Box^{n-1} f_{(n)}\Big)$$

$$=\; (y-x)^i \sum_{n=0}^{\infty} \frac{1}{n!}\,[a,b\mid n]\,\Box_x^n\,f_{(n)}(x) \;-\; 2\sum_{n=0}^{\infty} \frac{1}{n!}\,[a+1,b+1\mid n]\,\Box_x^n \partial^i f_{(n+1)}(x) \qquad (3.3.47)$$

(where $f_{(n)}$ denotes a function depending on n); in the general case of several factors $(y-x)$, we inductively apply (3.3.47).

It remains to show that the parameters a_l, b_l, and q_l are non-negative, and that the inequalities (3.3.44) and (3.3.45) hold. For this, it suffices to consider the leading summand $n_1 = \cdots = n_k = 0$ of (3.3.43). Since this is a (special) contribution of the form (3.2.32), we can apply Theorem 3.2.3 with $a_i = l_i + n_i$, $b_i = r_i + n_i$, and $p_i = q_i$. It follows that a_l, b_l, and q_l are non-negative. For the proof of the inequalities (3.3.44) and (3.3.45), we proceed inductively in the order k of the perturbation theory. For $k=0$, we have $r=-1$ and $|I|=1$, so that the inequalities are satisfied. Assume that (3.3.44),(3.3.45) hold for a given k. We go through the construction steps of Theorem 3.2.3 using the index shift (3.3.47) and verify that (3.3.44) and (3.3.45) are also valid to $(k+1)^{\text{st}}$ order:

For the proof of (3.3.44), we note that additional factors $(y-x)$ are generated at most once in the construction; namely, if the derivative $\partial\!\!\!/_x$ acts on $S^{(\hat{h})}$ in step *5)*. The parameter r is only increased if either a Laplacian acts on the factor $(y-x)^I$ in step *3)* (leading to the index shift (3.3.47)) or if the derivative $\partial\!\!\!/_x$ does not act on $S^{(\hat{h})}$ in step *5)*. In both cases, one loses at least one factor $(y-x)$. This gives (3.3.44).

For the proof of (3.3.45), we take the contribution (3.3.43) with $n_1 = \cdots = n_k = 0$ and apply the construction of Theorem 3.2.3. When computing the Laplacian $\Box_z^n$ in step *3)*, we shift the index according to (3.3.47) whenever a Laplacian acts on a factor $(y-x)$. Denoting the number of index shifts by s, we get a finite number of terms of the form

$$\sum_{n=0}^{\infty} \frac{1}{n!}\,\chi_c\,C\,i\partial\!\!\!/_x\,S^{(\hat{r}+n)}(x,y) \int_x^y [s,r+s\mid 0]\,dz\;(y-x)^{\hat{I}}\,\partial_z^{I_0}\,\Box^n V_{J_0}^{(0)}(z)$$
$$\times \int_z^y [a_1,b_1\mid 0]\,dz_1\;\partial_{z_1}^{I_1}\,\Box_{z_1}^{q_1}\,V_{J_1,c_1}^{(1)}(z_1)\cdots \int_{z_{k-1}}^y [a_k,b_k\mid 0]\,dz_k\;\partial_{z_k}^{I_k}\,\Box_{z_k}^{q_k}\,V_{J_k,c_k}^{(k)}(z_k)\;\gamma^J \quad .$$

Each index shift decreases the number of factors $(y-z)$ and increments the order of the mass-derivative of the Green's function, thus

$$|\hat{I}| \;\le\; |I| - s \quad , \qquad \hat{r} \;=\; r+1+s \quad . \qquad (3.3.48)$$

It again suffices to consider the leading summand $n=0$; this is a contribution of the form (3.2.32). After extracting the factors $(y-x)$ in step *4)*, the parameter $b_0 = r_0 + n_0$ satisfies

$$b_0 \;=\; r + |\hat{I}| + s \;\overset{(3.3.48)}{\ge}\; \hat{r} + |\hat{I}| - 1 \quad .$$

The parameters b_l, $1 \le l \le k$, remain unchanged in the construction; they are still the same as in (3.3.45),

$$b_l \;\ge\; r - l + |I| \;\ge\; \hat{r} - (l+1) + |\hat{I}| \quad , \qquad l = 1,\ldots,k.$$

When the derivative $\partial\!\!\!/_x$ is carried out in step *5)*, either r is decremented and $|\hat{I}|$ increased by one, or $|\hat{I}|$ is decreased. In steps *6)* and *7)*, the transformations (3.2.56) and (3.2.57)

may only decrease the sum $\hat{r}+|\hat{I}|$. We conclude that, after performing all the construction steps,

$$b_l \;\geq\; \hat{r} - (l+1) + |\hat{I}| \quad , \qquad\qquad l = 0, \ldots, k.$$

The index shift $l \to l+1$ finally gives the inequalities (3.3.45) for $l = 1, \ldots, k+1$. ∎

Theorem 3.3.8 *The non-causal low energy contribution $\tilde{P}^{le}(x,y)$ is, to every order in perturbation theory, a smooth function in x and y.*

Proof: We first outline our strategy: According to Def. 3.3.1, Proposition 3.3.2, and Def. 3.3.5, the k^{th} order contribution to $\tilde{P}^{\text{le}}(x,y)$ is obtained from (3.3.37) by the replacement $S^{(h)} \to (T^{(h)} - T^{\text{reg}\,(h)})$,

$$\Delta P^{\text{le}}(x,y) \;=\; X \sum_{h=-1}^{\infty} \sum_{\text{finite}} \cdots\, (T^{(h)} - T^{\text{reg}\,(h)})(x,y) \quad . \tag{3.3.49}$$

Because of the logarithmic mass problem, the infinite sum in (3.3.49) is ill-defined. In order to give (3.3.49) a mathematical meaning, we manipulate the infinite sum until recovering it as a formal Taylor series, which can be carried out explicitly. Finally, we show that the mathematical object $\Delta P^{\text{le}}(x,y)$ obtained in this way is a smooth function in x and y.

Consider the light-cone expansion of the Green's functions of Lemma 3.3.7. Since there are only a finite number of contributions of the form (3.3.43), we can restrict ourselves to one of them. In order to get the corresponding contribution to $\tilde{P}^{\text{le}}$, we replace the factor $S^{(h)}$ in (3.3.43) according to (3.3.49) by the operator $L^{(h)} := T^{(h)} - T^{\text{reg}\,(h)}$, i.e. more explicitly

$$L^{(h)}(x,y) \;=\; \left(\frac{d}{da}\right)^h L_{a\,|\,a=0}$$

with

$$L_a(x,y) \;=\; \frac{a}{32\pi^3}\, \log|a| \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,(j+1)!}\, \frac{(a\,(y-x)^2)^j}{4^j} \quad . \tag{3.3.50}$$

We can leave out the factor $(y-x)^I$ in (3.3.43) and disregard the chiral asymmetry matrix X in (3.3.49), because they are irrelevant as smooth functions. Furthermore, we can carry out the partial derivatives $\partial_{z_l}^{I_l}$ in (3.3.43) with the Leibniz rule. According to the chain rule (3.2.49), this may increase the parameters b_l; nevertheless, the inequalities (3.3.44) and (3.3.45) remain valid. We conclude that it suffices to consider the formal series

$$\begin{aligned} \Delta P^{\text{le}}(x,y) \;=\; & \sum_{n_1,\ldots,n_k=0}^{\infty} \frac{1}{n_1! \cdots n_k!} \int_x^y [a_1, b_1 + n_2 + \cdots + n_k \,|\, n_1]\, dz_1\, \Box_{z_1}^{n_1}\, W_1(z_1) \\ & \cdots \int_{z_{k-1}}^{y} [a_k, b_k \,|\, n_k]\, dz_k\, \Box_{z_k}^{n_k}\, W_k(z_k)\, L^{(r+n_1+\cdots+n_k)}(x,y) \end{aligned} \tag{3.3.51}$$

together with the bounds (3.3.44) and (3.3.45), where W_l stand for partial derivatives of the potential $V^{(l)}_{J_l,c_l}$. Our task is to give this series a mathematical meaning and to show that it is a smooth function in x and y.

We first consider the case that the potentials W_l are plane waves,

$$W_l(x) \;=\; e^{-iq_l x} \quad .$$

This allows us to explicitly carry out the Laplacians in (3.3.51), and we obtain

$$\begin{aligned}\Delta P^{\mathrm{le}}(x,y) \;&=\; \int_x^y [a_1,b_1\,|\,0]\,dz_1\,e^{-iq_1z_1}\;\cdots\;\int_{z_{k-1}}^y [a_k,b_k\,|\,0]\,dz_k\,e^{-iq_kz_k}\\ &\times\sum_{n_1,\ldots,n_k=0}^{\infty}\frac{1}{n_1!\cdots n_k!}\left((\alpha_1^2-\alpha_1)\,p_1^2\right)^{n_1}\left((1-\alpha_1)(\alpha_2^2-\alpha_2)\,p_2^2\right)^{n_2}\\ &\cdots\left((1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{k-1})(\alpha_k^2-\alpha_k)\,p_k^2\right)^{n_k}\;L^{(r+n_1+\cdots+n_k)}(x,y) \qquad (3.3.52)\end{aligned}$$

where α_l denote the integration variables of the line integrals (all running from zero to one), and where p_l are the momenta

$$p_l \;=\; q_l+(1-\alpha_l)\,q_{l+1}\;+\;\cdots\;+\;(1-\alpha_l)(1-\alpha_{l+1})\cdots(1-\alpha_{k-1})\,q_k\quad . \qquad (3.3.53)$$

For fixed values of the parameters $\alpha_1,\ldots,\alpha_k$, (3.3.52) is a product of k formal Taylor series. For example by using the continuation of the logarithm into the complex plane

$$\log|a| \;=\; \frac{1}{2}\lim_{0<\varepsilon\to 0}\left(\log(a+i\varepsilon)\;+\;\log(a-i\varepsilon)\;+\;2i\pi\right)$$

(which is, as in (3.3.39), cut along the positive real axis), we can carry out these formal Taylor series and obtain

$$\Delta P^{\mathrm{le}}(x,y) \;=\; \int_x^y [a_1,b_1|0]dz_1e^{-iq_1z_1}\cdots\int_{z_{k-1}}^y [a_k,b_k|0]dz_ke^{-iq_kz_k}\left(\frac{d}{da}\right)^r L_a(x,y) \quad (3.3.54)$$

with

$$\begin{aligned}a \;&=\; (\alpha_1^2-\alpha_1)\,p_1^2\;+\;(1-\alpha_1)(\alpha_2^2-\alpha_2)\,p_2^2\\ &\qquad+\;\cdots\;+\;(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{k-1})(\alpha_k^2-\alpha_k)\,p_k^2 \qquad (3.3.55)\end{aligned}$$

and the momenta p_l according to (3.3.53). This construction is helpful in two ways: all infinite sums have disappeared, and the mass parameter a is now in general non-zero, so that the a-derivatives of L_a are no longer singular.

After this preparation, we consider the case that the potentials $V^{(l)}$ in (3.3.43) are smooth and satisfy the conditions of Lemma 3.1.1 on the decay at infinity. Then the Fourier transform $\tilde{V}^{(l)}$ is C^2 and has rapid decay at infinity. Since the potentials $W^{(l)}$ are partial derivatives of $V^{(l)}$, their Fourier transforms $\tilde{W}_l$ are also C^2 and have rapid decay at infinity. The low energy contribution is obtained from (3.3.54) by integrating over the momenta q_l. More precisely,

$$\begin{aligned}\Delta P^{\mathrm{le}}(x,y) \;&=\; \left(\frac{d}{db}\right)^r\int\frac{d^4q_1}{(2\pi)^4}\;\cdots\;\int\frac{d^4q_k}{(2\pi)^4}\;\tilde{W}_1(q_1)\cdots\tilde{W}_k(q_k)\\ &\times\int_x^y[a_1,b_1\,|\,0]\;dz_1\,e^{-iq_1z_1}\;\cdots\;\int_{z_{k-1}}^y[a_k,b_k\,|\,0]\;dz_k\,e^{-iq_kz_k}\;L_{a+b}(x,y)\;{}_{|\,b=0}\quad , \quad (3.3.56)\end{aligned}$$

where the parameter a depends on the momenta q_l via (3.3.55) and (3.3.53). We must show that (3.3.56) is well-defined and depends smoothly on x and y. Qualitatively speaking, we can view the q_l-integrations as a multiple convolution in the parameter a. These convolutions mollify L_{a+b} in such a way that the b-derivatives can be carried out giving a smooth function in x and y. Unfortunately, this "mollifying argument" is quite delicate.

Complications arise from the fact that the q_l-integrals are multi-dimensional and that we must handle the additional line integrals over α_l. The main problem is that the dependence of a on the momenta q_l becomes singular when the parameters α_l approach zero or one (see (3.3.55)). Because of these difficulties, we give the mollifying argument in detail. Equivalently to analyzing the regularity in the parameter a, one can take its Fourier transform,

$$L_a(x,y) \;=\; \int_{-\infty}^{\infty} \frac{d\tau}{2\pi}\, \tilde{L}_\tau(x,y)\, e^{-i\tau a} \quad , \tag{3.3.57}$$

and study the decay properties in τ for $\tau \to \pm\infty$. We prefer working with the parameter τ, because this is a bit easier and makes our argument clearer.

In the first step of the mollifying argument, we transform the q_l-integrals into integrals over the momenta p_l. Since the transformation (3.3.53) is volume preserving, we get

$$\begin{aligned} \Delta P^{\text{le}}(x,y) \;&=\; \left(\frac{d}{db}\right)^r \int \frac{d^4p_1}{(2\pi)^4} \cdots \int \frac{d^4p_k}{(2\pi)^4} \\ &\times \int_x^y [a_1, b_1 \,|\, 0]\, dz_1 \cdots \int_{z_{k-1}}^y [a_k, b_k \,|\, 0]\, dz_k\; f^{\{\alpha\}}(p_1,\ldots,p_k)\; L_{a+b}(x,y)_{\,|\,b=0} \end{aligned} \tag{3.3.58}$$

with

$$f^{\{\alpha\}}(p_1,\ldots,p_k) \;:=\; \tilde{W}_1(q_1) \cdots \tilde{W}_k(q_k)\; e^{-i(q_1 z_1 + \cdots + q_k z_k)} \quad ,$$

where p_l and q_l are related to each other via (3.3.53). The explicit inverse of (3.3.53),

$$q_l \;=\; p_l - (1-\alpha_l)\, p_{l+1} \;, \quad 1 \leq l \leq k \qquad \text{and} \qquad q_k = p_k \quad ,$$

shows that this transformation is regular for all values of the parameters α_l. As a consequence, the function $f^{\{\alpha\}}(p_1,\ldots,p_k)$ is C^2 and has rapid decay at infinity, both uniformly in α_l (more precisely, the Schwartz norms $\|f^{\{\alpha\}}\|_{p,2}$, $p \geq 0$, are all uniformly bounded in α_l). Because of this uniformity, we need not care about the dependence of $f^{\{\alpha\}}$ on the parameters α_l in the following.

According to the bound (3.3.44), $r \leq k$. Thus we can insert into (3.3.58) the identity

$$1 \;=\; \int_{-\infty}^{\infty} da_1 \cdots \int_{-\infty}^{\infty} da_r\; \delta(a_1 - p_1^2) \cdots \delta(a_r - p_r^2) \quad ,$$

pull the a_l-integrals outside and carry out the integrations over $p_1,\ldots,p_r$. This gives

$$\begin{aligned} \Delta P^{\text{le}}(x,y) \;&=\; \left(\frac{d}{db}\right)^r \int_{-\infty}^{\infty} da_1 \cdots \int_{-\infty}^{\infty} da_r \int \frac{d^4p_{r+1}}{(2\pi)^4} \cdots \int \frac{d^4p_k}{(2\pi)^4} \\ &\times \int_x^y [a_1, b_1 \,|\, 0]\, dz_1 \cdots \int_{z_{k-1}}^y [a_k, b_k \,|\, 0]\, dz_k\; U(a_1,\ldots,a_r,p_{r+1},\ldots,p_k)\; L_{a+b}(x,y)_{\,|\,b=0} \end{aligned} \tag{3.3.59}$$

with

$$U \;=\; \int \frac{d^4p_1}{(2\pi)^4} \cdots \int \frac{d^4p_r}{(2\pi)^4}\; \delta(a_1 - p_1^2) \cdots \delta(a_r - p_r^2)\; f^{\{\alpha\}}(p_1,\ldots,p_k) \quad . \tag{3.3.60}$$

In (3.3.59), the parameter a depends on a_l, p_l, and α_l via

$$\begin{aligned} a \;=\;& (\alpha_1^2 - \alpha_1)\, a_1 \;+\; \cdots \;+\; (1-\alpha_1)\cdots(1-\alpha_{r-1})(\alpha_r^2 - \alpha_r)\, a_r \\ &+ (1-\alpha_1)\cdots(1-\alpha_r)(\alpha_{r+1}^2 - \alpha_{r+1})\, p_{r+1}^2 \\ &\qquad + \cdots + (1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{k-1})(\alpha_k^2 - \alpha_k)\, p_k^2 \quad . \end{aligned}$$

Since $f^{\{\alpha\}}$ is C^2 and has rapid decay at infinity, it follows by evaluating the integrals over the mass shells in (3.3.60) that U is C^1 in the variables a_l and has rapid decay at infinity.

Next, we take the Fourier transform in the variables a and $a_1, \ldots, a_r$ by substituting (3.3.57) and

$$\begin{aligned} &U(a_1, \ldots, a_r, p_{r+1}, \ldots, p_l) \\ &\quad = \int_{-\infty}^{\infty} \frac{d\tau_1}{2\pi} \cdots \int_{-\infty}^{\infty} \frac{d\tau_r}{2\pi}\, \tilde{U}(\tau_1, \ldots, \tau_r, p_{r+1}, \ldots, p_k)\, e^{-i(\tau_1 a_1 + \cdots + \tau_r a_r)} \end{aligned}$$

into (3.3.59). The a_l-dependence of the resulting expression for ΔP^{le} has the form of plane waves; thus the a_l-integrals give δ-distributions. We can then also carry out the τ_l-integrations. Finally, the b-derivatives in (3.3.59) give a factor $(-i\tau)^r$, and we obtain

$$\begin{aligned} \Delta P^{\text{le}}(x,y) \;&=\; \int_x^y [a_1, b_1 \,|\, 0]\, dz_1 \cdots \int_{z_{k-1}}^y [a_k, b_k \,|\, 0]\, dz_k \int \frac{d^4 p_{r+1}}{(2\pi)^4} \cdots \int \frac{d^4 p_k}{(2\pi)^4} \\ &\quad \times \int_{-\infty}^{\infty} \frac{d\tau}{2\pi}\, \tilde{U}(\tau_1, \ldots, \tau_r, p_{r+1}, \ldots, p_k)\, (-i\tau)^r\, \tilde{L}_\tau(x,y) \quad , \qquad (3.3.61) \end{aligned}$$

where the parameters τ_l are given in terms of τ and α_l by

$$\tau_l \;=\; (1-\alpha_1) \cdots (1-\alpha_{l-1})\, (\alpha_l - \alpha_l^2)\, \tau \quad .$$

The Fourier transform $\tilde{L}_\tau(x,y)$ is an integrable function in τ which depends smoothly on x and y (this can e.g. be verified by writing the infinite sum (3.3.50) with the Bessel functions J_1 and K_1 which decay at infinity). Since U is C^2 in the parameters a_l and has rapid decay, its Fourier transform $\tilde{U}$ is a function in τ_l which decays at infinity at least like $\mathcal{O}(\tau_l^{-1})$.

We now estimate the α_l-integrals in (3.3.61) for $l = 1, \ldots, r$, from the right to the left. Since $\tilde{U}$ decays in τ_r at infinity like $\mathcal{O}(\tau_r^{-1})$, we have the bound

$$\int_0^1 d\alpha_r \;\cdots \tilde{U}_{|\tau_r = (1-\alpha_1)\cdots(1-\alpha_{r-1})(\alpha_r - \alpha_r^2)\tau} \;\leq \sup_{\alpha_r \in [0,1]} \cdots ((1-\alpha_1)\cdots(1-\alpha_{r-1})\tau)^{-1}\; g \quad ,$$

where we have for clarity only written out the α_r-integral; g is a function depending on the variables $\tau_1, \ldots, \tau_{r-1}$ and $p_{r+1}, \ldots, p_k$. The inequality (3.3.45) implies that all factors $(1-\alpha_l)^{-1}$ cancel against corresponding factors $(1-\alpha_l)$ in the nested line integrals. The decay properties in the remaining parameters $\tau_1, \ldots, \tau_{r-1}$ remain unchanged by our estimate of the α_r-integral. Therefore we can proceed inductively in the same way for the integrals over $\alpha_{r-1}, \alpha_{r-2}, \ldots, \alpha_1$. The bounds (3.3.45) ensure that all factors $(1-\alpha_l)^{-1}$ drop out. Since we get a factors τ^{-1} in each step, the factor $(-i\tau)^r$ in (3.3.61) disappears. Using that $\tilde{L}_\tau$ is integrable and that we have fast decay at infinity in the variables $p_{r+1}, \ldots, p_k$, all the remaining integrals are finite. We conclude that $\Delta P^{\text{le}}(x,y)$ is well-defined.

If we take partial derivatives of (3.3.56) with respect to x and y, the derivatives act either on the exponentials, yielding additional factors α_l, $(1-\alpha_l)$, and q_l, or they act on $L_a(x,y)$. Since L_a and its Fourier transform $\tilde{L}_\tau$ depend smoothly on x and y, we can repeat the above mollifying argument and conclude that $\Delta P^{\text{le}}(x,y)$ is smooth. ∎

For a very rigid mathematician, it might not seem quite satisfying that the light-cone expansion of the residual fermionic projector was first performed on a formal level and later made rigorous by resumming the formal expansion. We remark that one could avoid all formal series in intermediate steps of the construction by already "regularizing"

the logarithmic mass terms in the Green's functions $s^\pm$ after (3.3.12). However, this has the disadvantage of becoming quite technical. Our procedure is easier to understand because we could introduce the residual argument in Subsection 3.3.1 without entering the mathematical details right away.

Similar to the high energy contribution, the low energy contribution $\tilde{P}^{\text{le}}(x,y)$ is non-causal in the sense that it depends on the external potential in the entire Minkowski space. This can be understood from the fact that the operator products in the perturbation expansion (3.3.16) contain factors $p(z_1, z_2)$, whose support is the whole space $(z_1, z_2) \in \mathbb{R}^4 \times \mathbb{R}^4$. We point out that, although the statements of Theorem 3.3.4 and Theorem 3.3.8 are very similar, their proofs are completely different. This shows and illustrates that the high and low energy contributions describe two different physical effects. In contrast to the high energy contribution, the potential in the low energy contribution need not overcome an "energy gap;" as a consequence, the low energy contribution plays an important role even if the energy of the external potential is small.

This concludes our analysis of the Dirac sea. We briefly summarize our main results: According to Def. 3.3.1, Def. 3.3.3, and Def. 3.3.5, we decompose the fermionic projector in the form

$$\tilde{P}(x,y) \;=\; \tilde{P}^{\text{causal}}(x,y) \;+\; \tilde{P}^{\text{le}}(x,y) \;+\; \tilde{P}^{\text{he}}(x,y) \quad . \tag{3.3.62}$$

The causal contribution $\tilde{P}^{\text{causal}}(x,y)$ has singularities on the light cone, which are completely described by the light-cone expansion of Theorem 3.3.6. The non-causal low and high energy contributions $\tilde{P}^{\text{le}}(x,y)$ and $\tilde{P}^{\text{he}}(x,y)$, on the other hand, are, to every order in perturbation theory, smooth functions in x and y (see Theorem 3.3.4 and Theorem 3.3.8).

We finally point out that, in contrast to $\tilde{P}^{\text{causal}}$, both non-causal contributions $\tilde{P}^{\text{le}}$ and $\tilde{P}^{\text{he}}$ were only studied to every order in perturbation theory, but we did not consider the convergence of their perturbation series. For $\tilde{P}^{\text{le}}$, this convergence problem could be studied by resumming all phase-free contributions to the formal light-cone expansion of Proposition 3.3.2. However, there seems to be no easy method at the moment to control the convergence of the perturbation expansion of the high energy contribution $\tilde{P}^{\text{he}}(x,y)$. Nevertheless, Theorem 3.3.4 and Theorem 3.3.8 give a strong indication that the non-causal contribution is a smooth function. Even if singularities or divergences occurred when carrying out the sum over the perturbation series (which seems unlikely), these singularities would be of very different nature than the singularities of $\tilde{P}^{\text{causal}}(x,y)$. More precisely, they could not be expressed in terms of the external potential and its derivatives along the convex line $\overline{xy}$, because singularities of this type necessarily show up in finite order perturbation theory. Thus we can say that, although the non-causal contributions $\tilde{P}^{\text{le}}$ and $\tilde{P}^{\text{he}}$ require further study in order to get a complete understanding, the decomposition (3.3.62) is well-established. The light-cone expansion gives a method to explicitly calculate the causal contribution $\tilde{P}^{\text{causal}}(x,y)$.

Anhang A

Some Formulas of the Light-Cone Expansion

In this appendix, we give a compilation of explicit formulas of the light-cone expansion. More precisely, we list the phase-free contribution to the light-cone expansion of the Green's functions (cf. Def. 3.2.7). According to the reduction to the phase-free contribution, the light-cone expansion of the Green's functions is immediately obtained by inserting ordered exponentials into the line integrals, see Def. 3.2.9 and Theorem 3.2.10. Furthermore, using Theorem 3.3.6, the formulas can be directly applied to the fermionic projector; they then describe the singularities of $\tilde{P}(x,y)$ on the light cone.

All the following formulas were generated by the C++ program "class_commute" (see Subsection 3.2.3). Our listings are not intended to be in any sense complete; we made a selection in order to give the reader a first impression of the form of the singularities. The more detailed formulas to higher order on the light cone can be easily obtained with "class_commute." Without loss of generality, we restrict ourselves to the left handed component of the Green's functions; for the right handed component, the formulas are analogous.

We begin with the perturbation by a chiral perturbation to first order. The phase-free contribution to the order $\mathcal{O}((y-x)^2)$ on the light cone is

$$\chi_L \left(-s \left(\chi_L \A_R + \chi_R \A_L\right) s\right)(x,y) \overset{\text{phase-free}}{\asymp} \mathcal{O}((y-x)^2)$$

$$+\chi_L\, S^{(0)}(x,y)\, \xi^i \int_x^y dz\, [0,1 \mid 0]\, (\partial\!\!\!/ A_{Li}) \tag{A.0.1}$$

$$-\chi_L\, S^{(0)}(x,y) \int_x^y dz\, [0,0 \mid 0]\, \A_L \tag{A.0.2}$$

$$+\chi_L\, S^{(0)}(x,y)\, \A_L(x) \tag{A.0.3}$$

$$+\frac{1}{2}\, \chi_L\, S^{(0)}(x,y)\, \xi\!\!\!/ \int_x^y dz\, [0,0 \mid 0]\, (\partial\!\!\!/ \A_L) \tag{A.0.4}$$

$$-\chi_L\, S^{(0)}(x,y)\, \xi\!\!\!/ \int_x^y dz\, [1,0 \mid 0]\, (\partial^i A_{Li}) \tag{A.0.5}$$

$$+\frac{1}{2}\chi_L\, S^{(0)}(x,y)\, \xi\!\!\!/\, \xi^i \int_x^y dz\, [0,0 \mid 1]\, (\Box A_{Li}) \tag{A.0.6}$$

$$+\chi_L\, S^{(1)}(x,y)\, \xi^i \int_x^y dz\, [0,1 \mid 1]\, (\partial\!\!\!/ \Box A_{Li}) \tag{A.0.7}$$

$$+\chi_L\, S^{(1)}(x,y) \int_x^y dz\, [0,2 \mid 0]\, (\Box \A_L) \tag{A.0.8}$$

$$-2\chi_L\, S^{(1)}(x,y) \int_x^y dz\, [0,0 \,|\, 1]\, (\not\partial \partial^i A_{Li}) \tag{A.0.9}$$

$$+\frac{1}{2}\, \chi_L\, S^{(1)}(x,y)\, \not\xi \int_x^y dz\, [0,0 \,|\, 1]\, (\not\partial \Box \not{A}_L) \tag{A.0.10}$$

$$-\chi_L\, S^{(1)}(x,y)\, \not\xi \int_x^y dz\, [1,0 \,|\, 1]\, (\partial^i \Box A_{Li}) \tag{A.0.11}$$

$$+\frac{1}{4}\, \chi_L\, S^{(1)}(x,y)\, \not\xi\, \xi^i \int_x^y dz\, [0,0 \,|\, 2]\, (\Box^2 A_{Li}) \quad , \tag{A.0.12}$$

where we used the abbreviation $\xi \equiv (y-x)$. This formula has the disadvantage that it contains ordinary partial derivatives of the chiral potential; it would be better for physical applications to work instead with the Yang-Mills field tensor and the Yang-Mills current. Therefore, we introduce left and right handed gauge-covariant derivatives $D^{L/R}$,

$$D_j^L \;=\; \frac{\partial}{\partial x^j} \;-\; i A_{Lj} \quad , \qquad D_j^R \;=\; \frac{\partial}{\partial x^j} \;-\; i A_{Rj} \quad ,$$

and define the corresponding field tensor and current as usual by the commutators

$$F_{jk}^c \;=\; i \left[D_j^c,\, D_k^c \right] \quad , \qquad j_l^c \;=\; \left[D^{c\,k},\, F_{lk}^c \right] \qquad (c = L \text{ or } R). \tag{A.0.13}$$

In the Abelian case of a single Dirac sea (i.e. $f = 1$), (A.0.13) reduces to the familiar formulas for the electromagnetic field tensor and current,

$$F_{jk}^c \;=\; \partial_j A_{c\,k} - \partial_k A_{c\,j} \quad , \qquad j_l^c \;=\; \partial_{lk} A_c^k - \Box A_{c\,l} \quad .$$

Notice, however, that in the general case of a system of Dirac seas, (A.0.13) involves quadratic and cubic terms in the potential.

By substituting (A.0.13) into (A.0.1)–(A.0.12) and manipulating the line integrals with partial integrations, one can rewrite the phase-free contribution in a way where the linear terms in the potential are gauge invariant. For example, we can combine (A.0.1), (A.0.2), and (A.0.3) by transforming the line integrals as

$$\begin{aligned} \xi^k \int_x^y dz\, [0,1 \,|\, 0]\, (\not\partial A_{Lk}) \;&=\; \xi^k \int_x^y dz\, [0,1 \,|\, 0]\, (\gamma^j F_{jk}^L \;+\; \partial_k \not{A}_L) \;+\; \mathcal{O}(A_L^2) \\ &=\; \xi^k \int_x^y dz\, [0,1 \,|\, 0]\, \gamma^j F_{jk}^L \;-\; \not{A}_L(x) \;+\; \int_x^y dz\, [0,0 \,|\, 0]\, \not{A}_L \;+\; \mathcal{O}(A_L^2) \quad . \end{aligned} \tag{A.0.14}$$

However, this procedure yields (in the non-Abelian case) quadratic and cubic terms in the potential which are *not* gauge invariant. Fortunately, these gauge-dependent terms are all compensated by corresponding contributions of the higher order Feynman diagrams. More generally, it turns out that, after summing up the perturbation series for the chiral perturbation, we can arrange a gauge invariant phase-free contribution for which the insertion rules of Def. 3.2.9 and the statement of Theorem 3.2.10 hold. This is not astonishing in view of the behavior (3.2.102) of the Green's functions under local gauge transformations; we verified it explicitly term by term for all following formulas. We now list the gauge invariant phase-free contribution to the order $\mathcal{O}((y-x)^2)$ on the light cone. For simplicity, we omit all third order terms in the potential which are of the order $\mathcal{O}((y-x)^0)$ on the light cone and which have a prefactor $\not\xi$ (the combinatorics of the tensor contractions leads to many such terms, but they are not very instructive here).

$$\chi_L \sum_{k=0}^{\infty} ((-s\, (\chi_L \not{A}_R + \chi_R \not{A}_L))^k\, s)(x,y)$$

$$
\begin{aligned}
&\overset{\text{phase-free}}{\asymp} \quad \mathcal{O}((y-x)^2) \;+\; \not{g}\; A^i_L A^j_L A^k_L\, \mathcal{O}((y-x)^0) \\
&+\chi_L\, S^{(0)}(x,y)\, \xi^i \int_x^y dz\, [0,1 \mid 0]\, \gamma^l F^L_{li} \\
&+\frac{1}{4}\, \chi_L\, S^{(0)}(x,y)\, \not{g}\, \int_x^y dz\, [0,0 \mid 0]\, \gamma^j \gamma^k\, F^L_{jk} \\
&-\frac{1}{2}\, \chi_L\, S^{(0)}(x,y)\, \not{g}\, \xi^i \int_x^y dz\, [0,0 \mid 1]\, j^L_i \\
&+\chi_L\, S^{(1)}(x,y)\, \xi^i \int_x^y dz\, [0,1 \mid 1]\, (\not{\partial} j^L_i) \\
&+\chi_L\, S^{(1)}(x,y)\, \int_x^y dz\, [0,2 \mid 0]\, j^L_k\, \gamma^k \\
&-\frac{1}{2}\, \chi_L\, S^{(1)}(x,y)\, \not{g}\, \int_x^y dz\, [0,0 \mid 1]\, (\not{\partial} j^L_k)\, \gamma^k \\
&-\frac{1}{4}\, \chi_L\, S^{(1)}(x,y)\, \not{g}\, \xi^i \int_x^y dz\, [0,0 \mid 2]\, (\Box j^L_i) \\
&-i\chi_L\, S^{(0)}(x,y)\, \not{g}\, \xi_i \xi^j \int_x^y dz_1\, [0,1 \mid 1]\, F^L_{kj} \int_{z_1}^y dz_2\, [0,1 \mid 0]\, F^{ki}_L \\
&+i\chi_L\, S^{(1)}(x,y)\, \xi^i \xi^j \int_x^y dz_1\, [0,3 \mid 0]\, \gamma^k\, F^L_{kj} \int_{z_1}^y dz_2\, [0,0 \mid 1]\, j^L_i \\
&+i\chi_L\, S^{(1)}(x,y)\, \xi^i \xi^j \int_x^y dz_1\, [0,2 \mid 1]\, j^L_j \int_{z_1}^y dz_2\, [0,1 \mid 0]\, \gamma^l\, F^L_{li} \\
&-2i\chi_L\, S^{(1)}(x,y)\, \xi_i \xi^j \int_x^y dz_1\, [0,2 \mid 1]\, F^L_{mj} \int_{z_1}^y dz_2\, [0,2 \mid 0]\, (\not{\partial} F^{mi}_L) \\
&-2i\chi_L\, S^{(1)}(x,y)\, \xi_i \xi^j \int_x^y dz_1\, [0,2 \mid 1]\, (\not{\partial} F^L_{kj}) \int_{z_1}^y dz_2\, [0,1 \mid 0]\, F^{ki}_L \\
&+i\chi_L\, S^{(1)}(x,y)\, \xi^i \xi^j \int_x^y dz_1\, [0,2 \mid 1]\, \gamma^k F^L_{kj} \int_{z_1}^y dz_2\, [0,2 \mid 0]\, j^L_i \\
&-\frac{i}{2}\, \chi_L\, S^{(1)}(x,y)\, \xi^i \int_x^y dz_1\, [0,2 \mid 0]\, \gamma^j F^L_{ji} \int_{z_1}^y dz_2\, [0,0 \mid 0]\, \gamma^k \gamma^l F^L_{kl} \\
&-\frac{i}{2}\, \chi_L\, S^{(1)}(x,y)\, \xi^i \int_x^y dz_1\, [0,2 \mid 0]\, \gamma^j \gamma^k F^L_{jk} \int_{z_1}^y dz_2\, [0,1 \mid 0]\, \gamma^l F^L_{li} \\
&+2i\chi_L\, S^{(1)}(x,y)\, \xi_i \int_x^y dz_1\, [0,3 \mid 0]\, \gamma^j F^L_{jk} \int_{z_1}^y dz_2\, [0,1 \mid 0]\, F^{ki}_L \\
&-2i\chi_L\, S^{(1)}(x,y)\, \xi^j \int_x^y dz_1\, [0,1 \mid 1]\, F^L_{ij} \int_{z_1}^y dz_2\, [0,1 \mid 0]\, \gamma_k F^{ki}_L \\
&-\frac{i}{2}\, \chi_L\, S^{(1)}(x,y)\, \not{g}\, \xi^j \xi^k \int_x^y dz_1\, [0,2 \mid 1]\, j^L_j \int_{z_1}^y dz_2\, [0,0 \mid 1]\, j^L_k \\
&-\frac{i}{2}\, \chi_L\, S^{(1)}(x,y)\, \not{g}\, \xi^j \xi^k \int_x^y dz_1\, [0,1 \mid 2]\, j^L_j \int_{z_1}^y dz_2\, [0,2 \mid 0]\, j^L_k \\
&+i\chi_L\, S^{(1)}(x,y)\, \not{g}\, \xi^i \xi^j \int_x^y dz_1\, [0,2 \mid 1]\, F^L_{kj} \int_{z_1}^y dz_2\, [0,1 \mid 1]\, (\partial^k j^L_i) \\
&+i\chi_L\, S^{(1)}(x,y)\, \not{g}\, \xi^i \xi^j \int_x^y dz_1\, [0,1 \mid 2]\, F^L_{kj} \int_{z_1}^y dz_2\, [0,3 \mid 0]\, (\partial^k j^L_i) \\
&-i\chi_L\, S^{(1)}(x,y)\, \not{g}\, \xi_i \xi^j \int_x^y dz_1\, [0,1 \mid 2]\, (\partial_k F^L_{lj}) \int_{z_1}^y dz_2\, [0,2 \mid 0]\, (\partial^k F^{li}_L)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{i}{4}\,\chi_L\, S^{(1)}(x,y)\, \not{\xi}\, \xi^i \int_x^y dz_1\, [0,2 \,|\, 0]\, \gamma^j \gamma^k F^L_{jk} \int_{z_1}^y dz_2\, [0,0 \,|\, 1]\, j^L_i \\
&+\frac{i}{4}\,\chi_L\, S^{(1)}(x,y)\, \not{\xi}\, \xi^i \int_x^y dz_1\, [0,1 \,|\, 1]\, \gamma^j \gamma^k F^L_{jk} \int_{z_1}^y dz_2\, [0,2 \,|\, 0]\, j^L_i \\
&+\frac{i}{4}\,\chi_L\, S^{(1)}(x,y)\, \not{\xi}\, \xi^i \int_x^y dz_1\, [0,1 \,|\, 1]\, j^L_i \int_{z_1}^y dz_2\, [0,0 \,|\, 0]\, \gamma^j \gamma^k F^L_{jk} \\
&-\frac{i}{2}\,\chi_L\, S^{(1)}(x,y)\, \not{\xi}\, \xi^i \int_x^y dz_1\, [0,1 \,|\, 1]\, \gamma^j \gamma^k (\partial^l F^L_{jk}) \int_{z_1}^y dz_2\, [0,1 \,|\, 0]\, F^L_{li} \\
&-\frac{i}{2}\,\chi_L\, S^{(1)}(x,y)\, \not{\xi}\, \xi^i \int_x^y dz_1\, [0,1 \,|\, 1]\, F^L_{ji} \int_{z_1}^y dz_2\, [0,1 \,|\, 0]\, \gamma^k \gamma^l (\partial^j F^L_{kl}) \\
&+i\chi_L\, S^{(1)}(x,y)\, \not{\xi}\, \xi^i \xi^j \int_x^y dz_1\, [0,1 \,|\, 2]\, (\partial^k j^L_j) \int_{z_1}^y dz_2\, [0,1 \,|\, 0]\, F^L_{ki} \\
&-i\chi_L\, S^{(1)}(x,y)\, \not{\xi}\, \xi^j \int_x^y dz_1\, [0,1 \,|\, 1]\, F^L_{ij} \int_{z_1}^y dz_2\, [0,0 \,|\, 1]\, j^i_L \\
&-i\chi_L\, S^{(1)}(x,y)\, \not{\xi}\, \xi^j \int_x^y dz_1\, [0,0 \,|\, 2]\, F^L_{ij} \int_{z_1}^y dz_2\, [0,2 \,|\, 0]\, j^i_L \\
&+i\chi_L\, S^{(1)}(x,y)\, \not{\xi}\, \xi^i \int_x^y dz_1\, [0,1 \,|\, 1]\, j^j_L \int_{z_1}^y dz_2\, [0,1 \,|\, 0]\, F^L_{ji} \\
&+2i\chi_L\, S^{(1)}(x,y)\, \not{\xi}\, \xi_i \int_x^y dz_1\, [0,2 \,|\, 1]\, F^L_{kl} \int_{z_1}^y dz_2\, [0,2 \,|\, 0]\, (\partial^k F^{li}_L) \\
&-2i\chi_L\, S^{(1)}(x,y)\, \not{\xi}\, \xi^j \int_x^y dz_1\, [0,0 \,|\, 2]\, (\partial_k F^L_{lj}) \int_{z_1}^y dz_2\, [0,1 \,|\, 0]\, F^{kl}_L \\
&-\frac{i}{8}\,\chi_L\, S^{(1)}(x,y)\, \not{} \int_x^y dz_1\, [0,1 \,|\, 0]\, \gamma^j \gamma^k F^L_{jk} \int_{z_1}^y dz_2\, [0,0 \,|\, 0]\, \gamma^l \gamma^m F^L_{lm} \\
&+3i\chi_L\, S^{(1)}(x,y)\, \not{} \int_x^y dz_1\, [0,1 \,|\, 1]\, F^L_{kl} \int_{z_1}^y dz_2\, [0,1 \,|\, 0]\, F^{kl}_L \\
&-2\chi_L\, S^{(1)}(x,y)\, \xi_i \xi^j \xi^k \int_x^y dz_1\, [0,4 \,|\, 0]\, \gamma^l F^L_{lk} \int_{z_1}^y dz_2\, [0,1 \,|\, 1]\, F^L_{mj} \int_{z_2}^y dz_3\, [0,1 \,|\, 0]\, F^{mi}_L \\
&-2\chi_L\, S^{(1)}(x,y)\, \xi_i \xi^j \xi^k \int_x^y dz_1\, [0,3 \,|\, 1]\, \gamma^l F^L_{lk} \int_{z_1}^y dz_2\, [0,3 \,|\, 0]\, F^L_{mj} \int_{z_2}^y dz_3\, [0,1 \,|\, 0]\, F^{mi}_L \\
&-2\chi_L\, S^{(1)}(x,y)\, \xi_i \xi^j \xi^k \int_x^y dz_1\, [0,3 \,|\, 1]\, F^L_{mk} \int_{z_1}^y dz_2\, [0,3 \,|\, 0]\, \gamma^l F^L_{lj} \int_{z_2}^y dz_3\, [0,1 \,|\, 0]\, F^{mi}_L \\
&-2\chi_L\, S^{(1)}(x,y)\, \xi^i \xi_j \xi^k \int_x^y dz_1\, [0,3 \,|\, 1]\, F^L_{mk} \int_{z_1}^y dz_2\, [0,3 \,|\, 0]\, F^{mj}_L \int_{z_2}^y dz_3\, [0,1 \,|\, 0]\, \gamma^l F^L_{li}
\end{aligned}
$$

We call this formulation of the phase-free contributions purely in terms of the Yang-Mills field tensor and the Yang-Mills current the *gauge invariant form* of the light-cone expansion.

For clarity, we mention a subtlety in the transformation to the gauge invariant form: The gauge invariant phase-free contribution implicitly contains tangential derivatives of the chiral potential (as one sees by writing out F^c and j^c in in terms of the chiral potentials). Thus it is not a phase-free contribution in correspondence with Def. 3.2.7; as a consequence, it is not obvious that the statement of Theorem 3.2.10 holds. In other words, one must be very careful when transforming the line integrals in order to ensure that the insertion rules of Def. 3.2.9 and Theorem 3.2.10 remain valid. A safe method is to insert all ordered exponentials into the line integrals before performing the partial integrations. According to Theorem 3.2.10, the phase-inserted formulas coincide with the light-cone expansion of the Green's functions. Therefore the partial integrations become identical transformations

of the Green's functions; we need not worry about the insertion rules. In the final step, we again take out the ordered exponentials from the line integrals, verifying that they are still in accordance with Def. 3.2.9. In the example of the line integral (A.0.14), the partial integration of the phase-inserted contribution gives

$$\begin{aligned}
&\xi^k \int_x^y dz\, [0,1 \,|\, 0]\, \mathrm{Pe}^{-i\int_x^z A_L^l\,(z-x)_l}\, \not\partial A_{Lk}(z)\, \mathrm{Pe}^{-i\int_z^y A_L^m\,(y-z)_m} \\
&\quad = \xi^k \int_x^y dz\, [0,1 \,|\, 0]\, \mathrm{Pe}^{-i\int_x^z A_L^l\,(z-x)_l} \left(\gamma^j F_{jk}^L + i\,[\not A_L,\, A_{Lk}] + \partial_k \not A_L\right)_{|z} \mathrm{Pe}^{-i\int_z^y A_L^m\,(y-z)_m} \\
&\quad = \xi^k \int_x^y dz\, [0,1 \,|\, 0]\, \mathrm{Pe}^{-i\int_x^z A_L^l\,(z-x)_l} \left(\gamma^j F_{jk}^L \;+\; i\,[\not A_L,\, A_{Lk}]\right)_{|z} \mathrm{Pe}^{-i\int_z^y A_L^m\,(y-z)_m} && \text{(A.0.15)} \\
&\qquad -\not A_L(x)\, \mathrm{Pe}^{-i\int_x^y A_L^m\,(y-z)_x} && \text{(A.0.16)} \\
&\qquad + \int_x^y dz\, [0,0 \,|\, 0]\, \mathrm{Pe}^{-i\int_x^z A_L^l\,(z-x)_l} \not A_L(z)\, \mathrm{Pe}^{-i\int_z^y A_L^m\,(y-z)_m} && \text{(A.0.17)} \\
&\qquad + \xi^k \int_x^y dz\, [0,1 \,|\, 0]\, \mathrm{Pe}^{-i\int_x^z A_L^l\,(z-x)_l}\, i\,[A_{Lk},\, \not A_L]\,(z)\, \mathrm{Pe}^{-i\int_z^y A_L^m\,(y-z)_m} \quad . && \text{(A.0.18)}
\end{aligned}$$

Thus the correct transformation of the phase-free contribution is

$$\xi^k \int_x^y dz\, [0,1 \,|\, 0]\, (\not\partial A_{Lk}) \overset{\text{phase-free}}{\asymp} \xi^k \int_x^y dz\, [0,1 \,|\, 0]\, \gamma^j F_{jk}^L + \int_x^y dz\, [0,0 \,|\, 0]\, \not A_L - \not A_L(x) \quad .$$

Notice that (A.0.18) are the derivative terms of the ordered exponentials; they get lost if the phase-free contribution is transformed in a naive way.

It remains to consider the scalar/pseudoscalar perturbation; i.e., we must study how the dynamic mass matrices $Y_{L/R}(x)$ show up in the light-cone expansion. We begin with the case of a single mass matrix. To first order in the external potential, the corresponding Feynman diagram has the light-cone expansion

$$\begin{aligned}
&\chi_L\, m\, (-s\,(-\chi_L Y_R - \chi_R Y_L)\, s)(x,y) \overset{\text{phase-free}}{\asymp} \mathcal{O}((y-x)^2) \\
&\quad + \frac{1}{2}\, \chi_L\, m\, S^{(0)}(x,y)\, \not\xi \int_x^y dz\, [0,0 \,|\, 0]\, (\not\partial Y_L) \\
&\quad + \chi_L\, m\, S^{(0)}(x,y)\, Y_L(x) \\
&\quad + \chi_L\, m\, S^{(1)}(x,y) \int_x^y dz\, [0,1 \,|\, 0]\, (\Box Y_L) \\
&\quad + \frac{1}{2}\, \chi_L\, m\, S^{(1)}(x,y)\, \not\xi \int_x^y dz\, [0,0 \,|\, 1]\, (\not\partial \Box Y_L) \quad . && \text{(A.0.19)}
\end{aligned}$$

Similar to (A.0.1)–(A.0.12), this formula involves partial derivatives of the potential, which is not a gauge invariant formulation. Since the chirality changes at every mass matrix (see e.g. Def. 3.2.9), the correct way to make the light-cone expansion gauge invariant is to work with the *gauge-covariant mass derivatives* D, Δ given by

$$\begin{aligned}
(D_i Y_L) &= D_i^L\, Y_L \;-\; Y_L\, D_i^R \;=\; (\partial_i Y_L) \;-\; i A_{Li}\, Y_L \;+\; i Y_L\, A_{Ri} \\
(\Delta Y_L) &= D_i (D^i Y_L) \;=\; D_i^L\, (D^i Y_L) \;-\; (D^i Y_L)\, D_i^R \quad ,
\end{aligned}$$

and similarly for Y_R and higher derivatives. Rewriting (A.0.19) with gauge-covariant mass derivatives yields additional terms which are linear or quadratic in the chiral potentials and which are *not* gauge invariant. But, similar as described for the chiral perturbation above, all these terms cancel if the sum over the perturbation series for the chiral potentials

is carried out. To the order $\mathcal{O}((y-x)^2)$ on the light cone, we obtain in this way the gauge invariant phase-free contribution

$$
\begin{aligned}
&\chi_L\, m \sum_{n_1,n_2=0}^{\infty} ((-s\,(\chi_L \not{A}_R + \chi_R \not{A}_L))^{n_1}\, s\,(\chi_L Y_R + \chi_R Y_L)\, s\,((-\chi_L \not{A}_R - \chi_R \not{A}_L)\, s)^{n_2})(x,y) \\
&\quad = \chi_L \sum_{n_1,n_2=0}^{\infty} ((-s\, \not{A}_L)^{n_1}\, s\, Y_L\, s\, (-\not{A}_R\, s)^{n_2})(x,y) \overset{\text{phase-free}}{\asymp} \mathcal{O}((y-x)^2) \\
&\qquad + \frac{1}{2}\, \chi_L\, m\, S^{(0)}(x,y) \not{\xi} \int_x^y dz\, [0,0 \mid 0]\, \gamma^j\, (D_j Y_L) \\
&\qquad + \chi_L\, m\, S^{(0)}(x,y)\, Y_L(x) \\
&\qquad + \chi_L\, m\, S^{(1)}(x,y) \int_x^y dz\, [0,1 \mid 0]\, (\Delta Y_L) \\
&\qquad + \frac{1}{2}\, \chi_L\, m\, S^{(1)}(x,y) \not{\xi} \int_x^y dz\, [0,0 \mid 1]\, \gamma^j\, (D_j \Delta Y_L) \\
&\qquad + \frac{i}{2}\, \chi_L\, m\, S^{(1)}(x,y) \not{\xi}\, \xi^i \int_x^y dz_1\, [0,1 \mid 1]\, j_i^L \int_{z_1}^y dz_2\, [0,0 \mid 0]\, \gamma^j\, (D_j Y_L) \\
&\qquad - i\chi_L\, m\, S^{(1)}(x,y) \not{\xi}\, \xi^i \int_x^y dz_1\, [0,1 \mid 1]\, F_{ji}^L \int_{z_1}^y dz_2\, [0,1 \mid 0]\, \gamma_k\, (D^{kj} Y_L) \\
&\qquad - \frac{i}{4}\, \chi_L\, m\, S^{(1)}(x,y) \not{\xi} \int_x^y dz_1\, [0,1 \mid 0]\, \gamma^k \gamma^l F_{kl}^L \int_{z_1}^y dz_2\, [0,0 \mid 0]\, \gamma^j\, (D_j Y_L) \\
&\qquad - i\chi_L\, m\, S^{(1)}(x,y)\, \xi^i \int_x^y dz_1\, [0,2 \mid 0]\, \gamma^k F_{ki}^L \int_{z_1}^y dz_2\, [0,0 \mid 0]\, \gamma^j\, (D_j Y_L) \\
&\qquad - \frac{i}{2}\, \chi_L\, m\, S^{(1)}(x,y) \int_x^y dz\, [0,1 \mid 0]\, \gamma^i \gamma^j F_{ij}^L\, Y_L \\
&\qquad + \frac{i}{2}\, \chi_L\, m\, S^{(1)}(x,y) \not{\xi} \int_x^y dz\, [0,0 \mid 1]\, j_k^L\, \gamma^k\, Y_L \\
&\qquad - i\chi_L\, m\, S^{(1)}(x,y) \not{\xi} \int_x^y dz\, [0,0 \mid 1]\, \gamma^j F_{ij}^L (D^i Y_L) \\
&\qquad + \frac{i}{2}\, \chi_L\, m\, S^{(1)}(x,y) \not{\xi}\, \xi^i \int_x^y dz_1\, [0,2 \mid 0]\, \gamma^j\, (D_j Y_L) \int_{z_1}^y dz_2\, [0,0 \mid 1]\, j_i^R \\
&\qquad + \frac{i}{2}\, \chi_L\, m\, S^{(1)}(x,y) \not{\xi}\, \xi^i \int_x^y dz_1\, [0,1 \mid 1]\, \gamma^j\, (D_j Y_L) \int_{z_1}^y dz_2\, [0,2 \mid 0]\, j_i^R \\
&\qquad - i\chi_L\, m\, S^{(1)}(x,y) \not{\xi}\, \xi^i \int_x^y dz_1\, [0,1 \mid 1]\, \gamma_k\, (D^{kj} Y_L) \int_{z_1}^y dz_2\, [0,1 \mid 0]\, F_{ji}^R \\
&\qquad - 1\frac{i}{4}\, \chi_L\, m\, S^{(1)}(x,y) \not{\xi} \int_x^y dz_1\, [0,1 \mid 0]\, \gamma^j\, (D_j Y_L) \int_{z_1}^y dz_2\, [0,0 \mid 0]\, \gamma^k \gamma^l F_{kl}^R \\
&\qquad + i\chi_L\, m\, S^{(1)}(x,y)\, \xi^i \int_x^y dz_1\, [0,2 \mid 0]\, \gamma^j\, (D_j Y_L) \int_{z_1}^y dz_2\, [0,1 \mid 0]\, \gamma^k F_{ki}^R \\
&\qquad - 2i\chi_L\, m\, S^{(1)}(x,y)\, \xi^i \int_x^y dz_1\, [0,2 \mid 0]\, (D^j Y_L) \int_{z_1}^y dz_2\, [0,1 \mid 0]\, F_{ji}^R \\
&\qquad + i\chi_L\, m\, S^{(1)}(x,y) \not{\xi} \int_x^y dz\, [0,0 \mid 1]\, (D^j Y_L)\, \gamma^i F_{ji}^R \\
&\qquad - \frac{i}{2}\, \chi_L\, m\, S^{(1)}(x,y) \not{\xi} \int_x^y dz\, [0,0 \mid 1]\, Y_L\, j_k^R \gamma^k \\
&\qquad + i\chi_L\, m\, S^{(1)}(x,y)\, \xi^i\, Y_L \int_x^y dz\, [0,0 \mid 1]\, j_i^R
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{i}{2}\,\chi_L\,m\,S^{(1)}(x,y)\int_x^y dz\,[0,1\,|\,0]\,Y_L\,\gamma^j\gamma^k F_{jk}^R\\
&-\frac{i}{2}\,\chi_L\,m\,S^{(1)}(x,y)\,Y_L\int_x^y dz\,[0,0\,|\,0]\,\gamma^j\gamma^k F_{jk}^R\\
&+\chi_L\,m\,S^{(1)}(x,y)\,\not\xi\,\xi_i\xi^j\int_x^y dz_1\,[0,2\,|\,1]\,F_{kj}^L\int_{z_1}^y dz_2\,[0,2\,|\,0]\,F_L^{ki}\int_{z_2}^y[0,0\,|\,0]\,\gamma^l\,(D_lY_L)\\
&+\chi_L\,m\,S^{(1)}(x,y)\,\not\xi\,\xi^j\int_x^y dz_1\,[0,1\,|\,1]\,F_{mj}^L\int_{z_1}^y dz_2\,[0,1\,|\,0]\,\gamma_k F_L^{mk}\,Y_L\\
&+\chi_L\,m\,S^{(1)}(x,y)\,\not\xi\,\xi_i\xi^j\int_x^y dz_1\,[0,2\,|\,1]\,F_{kj}^L\int_{z_1}^y dz_2\,[0,2\,|\,0]\,\gamma^l\,(D_lY_L)\int_{z_2}^y[0,1\,|\,0]\,F_R^{ki}\\
&+\chi_L\,m\,S^{(1)}(x,y)\,\not\xi\,\xi^i\int_x^y dz_1\,[0,1\,|\,1]\,F_{ki}^L\int_{z_1}^y dz_2\,[0,1\,|\,0]\,Y_L\,\gamma_j F_R^{kj}\\
&+\chi_L\,m\,S^{(1)}(x,y)\,\not\xi\,\xi_i\int_x^y dz_1\,[0,1\,|\,1]\,\gamma^j F_{kj}^L\,Y_L\int_{z_1}^y dz_2\,[0,1\,|\,0]\,F_R^{ki}\\
&+\chi_L\,m\,S^{(1)}(x,y)\,\not\xi\,\xi_i\xi^j\int_x^y dz_1\,[0,3\,|\,0]\,\gamma^l\,(D_lY_L)\int_{z_1}^y dz_2\,[0,1\,|\,1]\,F_{kj}^R\int_{z_2}^y[0,1\,|\,0]\,F_R^{ki}\\
&+\chi_L\,m\,S^{(1)}(x,y)\,\not\xi\,\xi_i\xi^j\int_x^y dz_1\,[0,2\,|\,1]\,\gamma^l\,(D_lY_L)\int_{z_1}^y dz_2\,[0,3\,|\,0]\,F_{kj}^R\int_{z_2}^y[0,1\,|\,0]\,F_R^{ki}\\
&+\chi_L\,m\,S^{(1)}(x,y)\,\not\xi\,\xi_i\int_x^y dz_1\,[0,1\,|\,1]\,Y_L\,\gamma^j F_{mj}^R\int_{z_1}^y dz_2\,[0,1\,|\,0]\,F_R^{mi}\\
&-2\chi_L\,m\,S^{(1)}(x,y)\,\xi_i\xi^j\,Y_L\int_x^y dz_1\,[0,1\,|\,1]\,F_{kj}^R\int_{z_1}^y dz_2\,[0,1\,|\,0]\,F_R^{ki}\quad .
\end{aligned}
$$

The higher orders in the mass matrices are treated similarly. To the order $\mathcal{O}((y-x)^2)$ on the light cone, only the terms up to fourth order in $m\,Y_{L/R}$ contribute (see (3.2.95)). One gets the formulas

$$
\begin{aligned}
&\chi_L\,m^2\sum_{n_1,n_2,n_3=0}^{\infty}((-s\,\not A_L)^{n_1}\,s\,Y_L\,s\,(-\not A_R\,s)^{n_2}\,Y_R\,s\,(-\not A_R\,s)^{n_3})(x,y)\\
&\overset{\text{phase-free}}{\asymp}\ \frac{i}{2}\,\chi_L\,m^2\,S^{(0)}(x,y)\,\not\xi\int_x^y dz\,[0,0\,|\,0]\,Y_L\,Y_R\\
&-\frac{i}{2}\,\chi_L\,m^2\,S^{(1)}(x,y)\,\not\xi\int_x^y dz_1\,[0,1\,|\,0]\,\gamma^j(D_jY_L)\int_{z_1}^y dz_2\,[0,0\,|\,0]\,\gamma^k(D_kY_R)\\
&+\frac{i}{2}\,\chi_L\,m^2\,S^{(1)}(x,y)\,\not\xi\int_x^y dz\,[0,0\,|\,1]\,Y_L\,(\Delta Y_R)\\
&+i\chi_L\,m^2\,S^{(1)}(x,y)\,\not\xi\int_x^y dz\,[0,0\,|\,1]\,(D_iY_L)\,(D^iY_R)\\
&+\frac{i}{2}\,\chi_L\,m^2\,S^{(1)}(x,y)\,\not\xi\int_x^y dz\,[0,0\,|\,1]\,(\Delta Y_L)\,Y_R\\
&+i\chi_L\,m^2\,S^{(1)}(x,y)\int_x^y dz\,[0,1\,|\,0]\,Y_L\,\gamma^j(D_jY_R)\\
&+i\chi_L\,m^2\,S^{(1)}(x,y)\int_x^y dz\,[0,1\,|\,0]\,\gamma^j(D_jY_L)\,Y_R\\
&-i\chi_L\,m^2\,S^{(1)}(x,y)\,Y_L\int_x^y dz\,[0,0\,|\,0]\,\gamma^j(D_jY_R)\\
&-\frac{1}{2}\,\chi_L\,m^2\,S^{(1)}(x,y)\,\not\xi\,\xi^i\int_x^y dz_1\,[0,1\,|\,1]\,j_i^L\int_{z_1}^y dz_2\,[0,0\,|\,0]\,Y_L\,Y_R
\end{aligned}
$$

$$
\begin{aligned}
&+\chi_L\, m^2\, S^{(1)}(x,y) \not\xi\, \xi^i \int_x^y dz_1\, [0,1 \mid 1]\, F^L_{ji} \int_{z_1}^y dz_2\, [0,1 \mid 0]\, Y_L\, (D^j Y_R) \\
&+\chi_L\, m^2\, S^{(1)}(x,y) \not\xi\, \xi^i \int_x^y dz_1\, [0,1 \mid 1]\, F^L_{ji} \int_{z_1}^y dz_2\, [0,1 \mid 0]\, (D^j Y_L)\, Y_R \\
&+\frac{1}{4}\, \chi_L\, m^2\, S^{(1)}(x,y) \not\xi \int_x^y dz_1\, [0,1 \mid 0]\, \gamma^j \gamma^k F^L_{jk} \int_{z_1}^y dz_2\, [0,0 \mid 0]\, Y_L\, Y_R \\
&+\chi_L\, m^2\, S^{(1)}(x,y)\, \xi^i \int_x^y dz_1\, [0,2 \mid 0]\, \gamma^j F^L_{ji} \int_{z_1}^y dz_2\, [0,0 \mid 0]\, Y_L\, Y_R \\
&-\frac{1}{2}\, \chi_L\, m^2\, S^{(1)}(x,y) \not\xi\, \xi^i \int_x^y dz_1\, [0,2 \mid 0]\, Y_L\, Y_R \int_{z_1}^y dz_2\, [0,0 \mid 1]\, j^L_i \\
&-\frac{1}{2}\, \chi_L\, m^2\, S^{(1)}(x,y) \not\xi\, \xi^i \int_x^y dz_1\, [0,1 \mid 1]\, Y_L\, Y_R \int_{z_1}^y dz_2\, [0,2 \mid 0]\, j^L_i \\
&+\chi_L\, m^2\, S^{(1)}(x,y) \not\xi\, \xi^i \int_x^y dz_1\, [0,1 \mid 1]\, Y_L\, (D^j Y_R) \int_{z_1}^y dz_2\, [0,1 \mid 0]\, F^L_{ji} \\
&+\chi_L\, m^2\, S^{(1)}(x,y) \not\xi\, \xi^i \int_x^y dz_1\, [0,1 \mid 1]\, (D^j Y_L)\, Y_R \int_{z_1}^y dz_2\, [0,1 \mid 0]\, F^L_{ji} \\
&+\frac{1}{4}\, \chi_L\, m^2\, S^{(1)}(x,y) \not\xi \int_x^y dz_1\, [0,1 \mid 0]\, Y_L\, Y_R \int_{z_1}^y dz_2\, [0,0 \mid 0]\, \gamma^j \gamma^k F^L_{jk} \\
&+\chi_L\, m^2\, S^{(1)}(x,y)\, \xi^i \int_x^y dz_1\, [0,2 \mid 0]\, Y_L\, Y_R \int_{z_1}^y dz_2\, [0,1 \mid 0]\, \gamma^j F^L_{ji} \;+\; \mathcal{O}((y-x)^2)
\end{aligned}
$$

$$
\chi_L\, m^3 \sum_{n_1,n_2,n_3,n_4=0}^{\infty} ((-s\, \not{A}_L)^{n_1}\, s\, Y_L\, s\, (-\not{A}_R\, s)^{n_2}\, Y_R\, s\, (-\not{A}_L\, s)^{n_3}\, Y_L\, s\, (-\not{A}_R\, s)^{n_4})(x,y)
$$

$$
\begin{aligned}
&\overset{\text{phase-free}}{\asymp}\ \chi_L\, m^3\, S^{(1)}(x,y)\, Y_L \int_x^y dz\, [0,0 \mid 0]\, Y_R\, Y_L \\
&+\frac{1}{2}\, \chi_L\, m^3\, S^{(1)}(x,y) \not\xi \int_x^y dz_1\, [0,1 \mid 0]\, Y_L\, Y_R \int_{z_1}^y dz_2\, [0,0 \mid 0]\, \gamma^j (D_j Y_L) \\
&+\frac{1}{2}\, \chi_L\, m^3\, S^{(1)}(x,y) \not\xi \int_x^y dz_1\, [0,1 \mid 0]\, \gamma^j (D_j Y_L) \int_{z_1}^y dz_2\, [0,0 \mid 0]\, Y_R\, Y_L \;+\; \mathcal{O}((y-x)^2)
\end{aligned}
$$

$$
\begin{aligned}
&\chi_L\, m^4 \sum_{n_1,\ldots,n_5=0}^{\infty} ((-s\, \not{A}_L)^{n_1}\, s\, Y_L\, s\, (-\not{A}_R\, s)^{n_2} \\
&\qquad\qquad \times Y_R\, s\, (-\not{A}_L\, s)^{n_3}\, Y_L\, s\, (-\not{A}_R\, s)^{n_4}\, Y_R\, s\, (-\not{A}_L\, s)^{n_5})(x,y) \\
&\overset{\text{phase-free}}{\asymp}\ \frac{i}{2}\, \chi_L\, m^4\, S^{(1)}(x,y) \not\xi \int_x^y dz_1\, [0,1 \mid 0]\, Y_L\, Y_R \int_{z_1}^y dz_2\, [0,0 \mid 0]\, Y_L\, Y_R \;+\; \mathcal{O}((y-x)^2)
\end{aligned}
$$

These classes of Feynman diagrams completely characterize the Green's functions to the order $\mathcal{O}((y-x)^2)$ on the light cone; notice that we only get a finite number of contributions.

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Selbständigkeitserklärung

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Felix Finster, im Januar 1999

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