AUTHOR QUERY FORM

ELSEVIER	**Journal:** LAA **Article Number:** 13143	**Please e-mail your responses and any corrections to:** **E-mail:** **corrections.esch@elsevier.vtex.lt**

Dear Author,

Please check your proof carefully and mark all corrections at the appropriate place in the proof (e.g., by using on-screen annotation in the PDF file) or compile them in a separate list. Note: if you opt to annotate the file with software other than Adobe Reader then please also highlight the appropriate place in the PDF file. To ensure fast publication of your paper please return your corrections within 48 hours.

For correction or revision of any artwork, please consult *http://www.elsevier.com/artworkinstructions*

Any queries or remarks that have arisen during the processing of your manuscript are listed below and highlighted by flags in the proof. Click on the 'Q' link to go to the location in the proof.

Location in article	**Query / Remark: click on the Q link to go** **Please insert your reply or correction at the corresponding line in the proof**
Q1	Your article is registered as belonging to the Special Issue/Collection entitled "ILAS2014 Conference". If this is NOT correct and your article is a regular item or belongs to a different Special Issue please contact <p.muthukumar@elsevier.com> immediately prior to returning your corrections. (p. 1/ line 1)
Q2	Please confirm that given names and surnames have been identified correctly and are presented in the desired order. (p. 1/ line 14)
Q3	Please check the address for the corresponding author that has been added here, and correct if necessary. (p. 1/ line 38)
Q4, Q5, Q6, Q7, Q8	The name "Murdock" (Murdoch) was spelled differently. Please check and amend if necessary. (p. 1/ line 31; p. 2/ line 33; p. 19/ line 26; p. 21/ line 41; p. 22/ line 23)
Q9, Q10	Please provide a grant numbers for the sponsors "GNCS-INDAM" and "Knut and Alice Wallenberg Foundation" (if available) and include this numbers into the main text where appropriate. (p. 22/ line 4,6)
Q11	Please check if sponsor names have been identified correctly and correct if necessary. (p. 24/ line 4)

Thank you for your assistance.

Location in article	Query / Remark: click on the Q link to go Please insert your reply or correction at the corresponding line in the proof
	Please check this box if you have no corrections to make to the PDF file ☐

Thank you for your assistance.

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications

E. Salinelli [a], S. Serra-Capizzano [b,c,*], D. Sesana [b]

[a] Dipartimento di Studi per l'Economia e l'Impresa, Università del Piemonte Orientale "Amedeo Avogadro" Alessandria, Novara, Vercelli, Via Perrone 18, 28100 Novara, Italy

[b] Dipartimento di Scienza e Alta Tecnologia, Università dell'Insubria, Via Valleggio 11, 22100 Como, Italy

[c] Department of Information Technology, Uppsala University, Box 337, SE-751 05 Uppsala, Sweden

A R T I C L E I N F O

Article history:
Received 12 November 2014
Accepted 11 March 2015
Available online xxxx
Submitted by L. Cvetkovic

MSC:
15B05
15A18
47B36
62H25

Keywords:
Schoenmakers–Coffey matrices
Green matrices
Tridiagonal matrices
Spectral distribution
GLT sequence
Eigenvector oscillation

A B S T R A C T

The paper is concerned with the spectral properties of Green matrices and of a special subclass of the latter, known as Schoenmakers–Coffey matrices, which have a role in financial applications. The main results are related to the eigenvalue distribution of sequences of Green matrices of increasing size, while for the subclass of interest mentioned above, we also study the eigenvector oscillation structure: interestingly enough, even if these matrices are not shift invariant (Toeplitz), the results are obtained by using tools coming from Toeplitz technology. Indeed, for the asymptotic spectral distribution analysis, we use the theory of Generalized Locally Toeplitz sequences, while techniques taken from the study of Kac–Murdoch–Szegö matrices (again connected to Toeplitz matrices) are employed for the eigenvector oscillation structure results of the Schoenmakers–Coffey

* Corresponding author at: Dipartimento di Scienza e Alta Tecnologia, Università dell'Insubria, Via Valleggio 11, 22100 Como, Italy.

E-mail addresses: ernesto.salinelli@eco.unipmn.it (E. Salinelli), stefano.serrac@uninsubria.it, stefano.serra@it.uu.se (S. Serra-Capizzano), debora.sesana@uninsubria.it (D. Sesana).

http://dx.doi.org/10.1016/j.laa.2015.03.017
0024-3795/© 2015 Published by Elsevier Inc.

matrices. Few numerical tests are reported in order to illustrate the theoretical findings.

1. Introduction

The paper is concerned with the spectral properties of Green matrices and of a special subclass of the latter, known as Schoenmakers–Coffey matrices. The main results concern the eigenvalue distribution of sequences of Green matrices of increasing size in the sense of Weyl [5]: in particular, despite its apparent complicate expression, we prove that these matrices can be seen as the product of diagonal matrices and inverse of Toeplitz matrices generated by linear trigonometric polynomials. Under the assumption that the diagonal matrices can be seen as samplings of Riemann integrable functions over the interval $[0, 1]$, we prove that the limit distribution of the spectra of the considered sequence is independent of such functions and indeed equals the constant zero. In other words, the Green matrix sequence $\{G_n\}$ shows a cluster at zero of the eigenvalues, i.e., for every ϵ, for the size n large enough, most of the eigenvalues of G_n have modulus bounded by ϵ, except $o(n)$ of them. In practice, as shown in the numerical experiments, the quantity indicated as $o(n)$ behaves just as $O(\sqrt{n})$, so it seems that there is room for improving the theoretical result.

Concerning the subclass of the Schoenmakers–Coffey matrices we also study the eigenvector oscillation behavior. We recall that these structures appear as special instances of correlation matrices and the oscillations of the first three eigenvectors provide useful information in trendy financial problems, associated with interest rates models and risk management/valuation (see Section 4 for a brief account and [12,14,17] for more details).

Even if these matrices are not shift invariant, that is they do not enjoy the Toeplitz structure, it is worth stressing that the results are obtained by using tools coming from Toeplitz technology. Indeed, concerning the asymptotic spectral results, we use the theory of Generalized Locally Toeplitz (GLT) sequences [23,24], while, for the second type of results, tools used in the study of Kac–Murdoch–Szegö matrices (again connected to Toeplitz matrices) are employed [10].

The paper is organized as follows. In Section 2 we report the definition of spectral distribution in the sense of Weyl and we briefly introduce Toeplitz matrices and GLT matrix sequences. In Section 3 we introduce the Green matrices and we use the tools of the previous section in order to identify the distribution symbol of Green matrix sequences: selected numerical tests are reported and commented. In Section 4 we define the subclass of the Schoenmakers–Coffey matrices and we study the oscillatory behavior of the eigenvectors, by including also some examples. Finally Section 5 is devoted to conclusions and to stress few open problems.

Please cite this article in press as: E. Salinelli et al., Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications, Linear Algebra Appl. (2015), http://dx.doi.org/10.1016/j.laa.2015.03.017

2. Asymptotic analysis of Green matrix sequences via the GLT theory

The section contains the mathematical tools that we use for studying the global behavior of the spectra of Green matrix sequences. In Subsection 2.1 we introduce the notion of spectral distribution in the Weyl sense [5,8] and in Subsection 2.2 we give the definition of Toeplitz matrices generated by a symbol (see [5]) and of the class of GLT sequences (see [23,24]), whose main properties are reported and discussed.

2.1. Definitions and distribution results

Before starting, let us introduce some notations. We denote by $\mathcal{C}_0(\mathbb{C})$ and $\mathcal{C}_0(\mathbb{R}_0^+)$ the set of continuous functions with bounded support defined over $\mathbb{C}$ and $\mathbb{R}_0^+ = [0, \infty)$, respectively. Given a function $F : \mathbb{C} \to \mathbb{C}$ and a matrix A of size m, with eigenvalues $\lambda_j(A)$ and singular values $\sigma_j(A)$, $j = 1, \ldots, m$, we set

$$\Sigma_\lambda(F, A) := \frac{1}{m}\sum_{j=1}^{m} F(\lambda_j(A)), \qquad \Sigma_\sigma(F, A) := \frac{1}{m}\sum_{j=1}^{m} F(\sigma_j(A)).$$

We use the notation $\|A\|_p$ for the Schatten p-norm of A, defined as the p-norm of the vector formed by the singular values of A. In symbols, $\|A\|_p = (\sum_{j=1}^{m} \sigma_j^p(A))^{1/p}$ for $1 \le p < \infty$ and $\|A\|_\infty = \max_{j=1,\ldots,m} \sigma_j(A) = \|A\|$ is the usual spectral norm [4].

Definition 2.1. Let $f : G \to \mathbb{C}$ be a complex-valued measurable function, defined on a measurable set $G \subset \mathbb{R}$ with finite and positive Lebesgue measure, $0 < \mu(G) < \infty$. Let $\{A_n\}$ be a matrix-sequence, with A_n of size d_n, $d_n < d_{n+1}$. We say that:

- $\{A_n\}$ is *distributed as the pair* (f, G) *in the sense of the eigenvalues,* in symbols $\{A_n\} \sim_\lambda (f, G)$, if for all $F \in \mathcal{C}_0(\mathbb{C})$ we have
$$\lim_{n\to\infty} \Sigma_\lambda(F, A_n) = \frac{1}{\mu(G)} \int_G F(f(t))\mathrm{d}t. \qquad (1)$$
- $\{A_n\}$ is *distributed as the pair* (f, G) *in the sense of the singular values,* in symbols $\{A_n\} \sim_\sigma (f, G)$, if for all $F \in \mathcal{C}_0(\mathbb{R}_0^+)$ we have
$$\lim_{n\to\infty} \Sigma_\sigma(F, A_n) = \frac{1}{\mu(G)} \int_G F(|f(t)|)\mathrm{d}t. \qquad (2)$$

Finally we say that two sequences $\{A_n\}$ and $\{B_n\}$ are *equally distributed* in the sense of the eigenvalues [30] if, $\forall F \in \mathcal{C}_0(\mathbb{C})$, we have

$$\lim_{n\to\infty} [\Sigma_\lambda(F, A_n) - \Sigma_\lambda(F, B_n)] = 0.$$

An analogous definition works for singular values with $F \in \mathcal{C}_0(\mathbb{R}_0^+)$ and σ in place of λ.

Please cite this article in press as: E. Salinelli et al., Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications, Linear Algebra Appl. (2015), http://dx.doi.org/10.1016/j.laa.2015.03.017

Notice that if two sequences are equally distributed and one of them has a distribution function, then the other necessarily has the same distribution function.

A matrix sequence $\{A_n\}$ is distributed in the eigenvalue sense as the pair (f, G) if and only if the sequence of linear functionals $\{\phi_n\}$ defined by $\phi_n(F) = \Sigma_\lambda(F, A_n)$ converges weak-$*$ to the functional

$$\phi(F) = \frac{1}{\mu(G)} \int_G F(f(t))\mathrm{d}t,$$

as in (1), the same is true for the singular values. A useful tool for the study of the spectral distribution of a matrix sequence is the notion of approximating class of sequences.

Definition 2.2. Suppose a sequence of matrices $\{A_n\}$, A_n of size d_n, $d_n < d_{n+1}$, is given. We say that $\{\{B_{n,m}\} : m \in \mathbb{N}\}_m$ is an approximating class of sequences (a.c.s.) for $\{A_n\}$ if, for all sufficiently large $m \in \mathbb{N}$, the following splittings hold:

$$A_n = B_{n,m} + R_{n,m} + N_{n,m},$$

with

$$\mathrm{rank}(R_{n,m}) \le d_n c(m), \qquad \|N_{n,m}\| \le \omega(m), \qquad \forall n > n_m,$$

where n_m, $c(m)$ and $\omega(m)$ depend only on m and

$$\lim_{m\to\infty} c(m) = 0, \qquad \lim_{m\to\infty} \omega(m) = 0.$$

Proposition 2.1. *(See [22].) Suppose a sequence of matrices $\{A_n\}$, A_n of size d_n, $d_n < d_{n+1}$, is given and let $\{\{B_{n,m}\} : m \in \mathbb{N}\}_m$ be an a.c.s. for $\{A_n\}$ in the sense of Definition 2.2. Suppose that, for all sufficiently large $m \in \mathbb{N}$ we have $\{B_{n,m}\} \sim_\sigma (f_m, G)$, and $\lim_{m\to\infty} f_m = f$. Then it holds $\{A_n\} \sim_\sigma (f, G)$.*

Similarly, if the matrices A_n and $B_{n,m}$ are eventually Hermitian, and for all sufficiently large $m \in \mathbb{N}$ we have $\{B_{n,m}\} \sim_\lambda (g_m, G)$, with $\lim_{m\to\infty} g_m = g$, then it also holds $\{A_n\} \sim_\lambda (g, G)$.

Finally we introduce the definition of sparsely vanishing sequence: a sequence of matrices $\{A_n\}$, A_n of size d_n, is said to be sparsely vanishing if, for each $M > 0$ there exists an n_M such that for $n \ge n_M$ we have

$$\#\{i : \sigma_i(A_n) < M^{-1}\} \le r(M)d_n, \qquad \lim_{M\to\infty} r(M) = 0.$$

Please cite this article in press as: E. Salinelli et al., Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications, Linear Algebra Appl. (2015), http://dx.doi.org/10.1016/j.laa.2015.03.017

2.2. Toeplitz, Locally Toeplitz and Generalized Locally Toeplitz sequences

Given an integrable complex-valued function f, $f \in L^1(Q)$, $Q = (-\pi, \pi)$, from the Fourier coefficients of f

$$a_j = \frac{1}{2\pi} \int_Q f(x) e^{-\mathbf{i} jx} \mathrm{d}x, \qquad j \in \mathbb{Z}, \quad \mathbf{i}^2 = -1,$$

we can build the sequence of Toeplitz matrices $\{T_n(f)\}$ as follows:

$$T_n(f) := [a_{i-j}]_{i,j=1}^n.$$

The eigen/singular values asymptotic distribution of a sequence of Toeplitz matrices, started in a famous theorem by Szegö [8], has been studied by many authors (see [5,26, 31] and the references reported therein). The results are reported below.

Theorem 2.1. *(See [31].) If $f \in L^1(Q)$ and $\{T_n(f)\}$ is the sequence of Toeplitz matrices generated by f, then*

$$\{T_n(f)\} \sim_\sigma (f, Q).$$

Moreover, if f is also real-valued, then each matrix $T_n(f)$ is Hermitian and

$$\{T_n(f)\} \sim_\lambda (f, Q). \tag{3}$$

Now we introduce the notion of (unilevel) Locally Toeplitz matrix-sequence that leads to a generalization of (unilevel) Toeplitz sequences. We mention that, with respect to the original paper by Tilli [27], the definitions will take into account very minor improvements (as discussed in Remark 1.1 of [23]). We recall that, given two matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{m\times m}$, their direct sum is defined as

$$A \oplus B = \begin{bmatrix} A & O \\ O & B \end{bmatrix} \in \mathbb{C}^{(n+m)\times(n+m)},$$

where O is the null matrix, and the tensor product $A \otimes B \in \mathbb{C}^{nm\times nm}$ is defined as the $n \times n$ block matrix with $m \times m$ blocks, whose block (i, j), $i, j = 1, \ldots, m$, is given by $a_{i,j}B$.

Definition 2.3. A sequence of matrices $\{A_n\}$, A_n of size n, is said to be Locally Toeplitz with respect to a pair of functions (a, f), with $a : [0, 1] \to \mathbb{C}$ and $f : Q \to \mathbb{C}$, if f is Lebesgue integrable and, for all sufficiently large $m \in \mathbb{N}$, the following splitting holds:

$$A_n = LT_n^m(a, f) + R_{n,m} + N_{n,m}, \tag{4}$$

Please cite this article in press as: E. Salinelli et al., Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications, Linear Algebra Appl. (2015), http://dx.doi.org/10.1016/j.laa.2015.03.017

with

$$\operatorname{rank}(R_{n,m}) \leq c(m), \qquad \|N_{n,m}\|_1 \leq \omega(m)n, \qquad \forall n > n_m,$$

where n_m, $c(m)$ and $\omega(m)$ are functions of m and $\lim_{m\to\infty} \omega(m) = 0$. The matrix $LT_n^m(a, f)$ in (4) is defined as

$$LT_n^m(a, f) = D_m(a) \otimes T_{\lfloor n/m \rfloor}(f) \oplus O_{n \, mod \, m},$$

where $\lfloor n/m \rfloor$ is the integer part of m/n and $n \, mod \, m = n - m \lfloor n/m \rfloor$ (it is understood that the zero block $O_{n \, mod \, m}$ is not present if n is a multiple of m). Moreover $D_m(a)$ is the $m \times m$ diagonal matrix whose entries are given by $a(j/m)$, $j = 1, \ldots, m$, $T_k(f)$ denotes the Toeplitz matrix of order k generated by f and O_q is the null matrix of order q. In this case we write in short $\{A_n\} \sim_{\mathrm{LT}} (a, f)$.

The topological closure of the space of Locally Toeplitz sequences is that formed by Generalized Locally Toeplitz sequences.

Definition 2.4. A sequence of matrices $\{A_n\}$, A_n of size n, is approximated by unilevel Locally Toeplitz sequences with respect to a measurable function κ if, for every $\epsilon > 0$,

- there exist pairs of functions $\{(a_{i,\epsilon}, f_{i,\epsilon})\}_{i=1}^{N_\epsilon}$ with $f_{i,\epsilon}$ polynomial and $a_{i,\epsilon}$ defined over $\Omega = [0, 1]$ such that $\sum_{i=1}^{N_\epsilon} a_{i,\epsilon} f_{i,\epsilon} - \kappa$ converge in measure to zero over $\Omega \times Q$ as ϵ tends to zero,
- there exist matrix sequences $\{\{A_n^{(i,\epsilon)}\}\}_{i=1}^{N_\epsilon}$ such that $\{A_n^{(i,\epsilon)}\} \sim_{\mathrm{LT}} (a_{i,\epsilon}, f_{i,\epsilon})$
- $\{\{\sum_{i=1}^{N_\epsilon} A_n^{(i,\epsilon)}\} : \epsilon = (m + 1)^{-1}, m \in \mathbb{N}\}_m$ is an a.c.s. for $\{A_n\}$.

In this case the sequence $\{A_n\}$ is said to be a Generalized Locally Toeplitz sequence with respect to κ and we write in short $\{A_n\} \sim_{\mathrm{GLT}} \kappa$.

Some remarks are in order. When $\{A_n\} \sim_{\mathrm{LT}} (a, f)$, we call a the *weight function*, and f the *generating function*. Furthermore, in the splitting (4), the matrices $R_{n,m}$ are called *rank corrections*, while the matrices $N_{n,m}$ are called *norm corrections*.

If $\{A_n\} \sim_{\mathrm{GLT}} \kappa$, it is evident that the unique function κ has simultaneously the role of *weight function* and of *generating function*: we call κ the *kernel function* or *symbol*.

For the class of Generalized Locally Toeplitz sequences the following Szegö-like result holds.

Theorem 2.2. *(See [23].) Assume that $\{A_n\}$, A_n of size n, is a sequence of complex matrices. Let κ be measurable over $\Omega \times Q$. Then*

$$\{A_n\} \sim_\sigma (\kappa(x, s), \Omega \times Q),$$

holds whenever $\{A_n\}$ is a GLT sequence with respect to κ as in Definition 2.4 and the functions $a_{i,\epsilon}$ involved in Definition 2.4 are Riemann integrable over Ω. If in addition the matrices A_n are eventually Hermitian, then the relation is true also for the eigenvalues.

The GLT sequences form a $*$-algebra. More precisely, the GLT sequences are stable under linear combinations, product, pseudo-inversion, and adjoint.

Theorem 2.3. *(See [23,24].) If $\{A_n\} \sim_{\mathrm{GLT}} \kappa_A$ and $\{B_n\} \sim_{\mathrm{GLT}} \kappa_B$, then we have*

- $\{\alpha A_n + \beta B_n\} \sim_{\mathrm{GLT}} \alpha\kappa_A + \beta\kappa_B$;
- *if the functions $a_{i,\epsilon}^{(A)}$ and $a_{i,\epsilon}^{(B)}$ involved in Definition 2.4 are Riemann integrable over Ω, then $\{A_n B_n\} \sim_{\mathrm{GLT}} \kappa_A \kappa_B$;*
- *if the function $a_{i,\epsilon}^{(A)}$ involved in Definition 2.4 is Riemann integrable over Ω and if $\{A_n\}$ is sparsely vanishing then $\{A_n^+\} \sim_{\mathrm{GLT}} \kappa_A^{-1}$, where A_n^+ is the pseudo inverse of A_n (we can replace the superscript $+$ with -1 if A_n is also invertible);*
- $\{A_n^*\} \sim_{\mathrm{GLT}} \kappa_A^*$, *where A_n^* denote the conjugate transpose of A_n.*

The class of GLT sequences contains all Toeplitz sequences, generated by L^1 symbols, and diagonal sequences, obtained as a uniform sampling of Riemann integrable functions.

Theorem 2.4. *(See [23].) If $\{T_n(f)\}$ is a sequence of Toeplitz matrices generated by $f \in L^1(Q)$, then $\{T_n(f)\}$ is a GLT sequence with respect to the function f.*

Theorem 2.5. *(See [23].) If $\{D_n(a)\}$ is a sequence of diagonal matrices $[D_n(a)]_{i,i} = a(i/n)$, $i = 1, \ldots, n$, generated by a Riemann integrable function a, then $\{D_n(a)\}$ is a GLT sequence with respect to the function a.*

3. Green matrices

A Green matrix $G_n = [g_{ij}]_{i,j=1}^n$ [16] is defined by

$$g_{ij} = \begin{cases} a_i c_j & \text{if } \; i \le j, \\ a_j c_i & \text{if } \; i > j, \end{cases} \qquad a_i, c_j \in \mathbb{R} \setminus \{0\}. \tag{5}$$

By construction, G_n is symmetric. Furthermore, it can be proved that G_n is a nonsingular Green matrix if and only if its inverse G_n^{-1} is a symmetric tridiagonal matrix with nonzero superdiagonal elements, whose explicit form is (see [3,2,11] and [15] for a study of the conditioning)

$$\left(G_n^{-1}\right)_{ij} = \begin{cases} \frac{1}{a_i c_{i+1} - a_{i+1} c_i} & i = j-1, \\ \frac{1}{a_{i-1} c_i - a_i c_{i-1}} & i = j+1, \\ \frac{a_{i+1} c_{i-1} - a_{i-1} c_{i+1}}{(a_i c_{i-1} - a_{i-1} c_i)(a_{i+1} c_i - a_i c_{i+1})} & i = j \neq 1, n, \\ \frac{a_2}{a_1 (a_2 c_1 - a_1 c_2)} & i = j = 1, \\ \frac{c_{n-1}}{c_n (a_n c_{n-1} - a_{n-1} c_n)} & i = j = n, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Hence, the pair $(\lambda, \mathbf{h})$, $\lambda \in \mathbb{R}$ and $\mathbf{h} \in \mathbb{R}^n$, $\mathbf{h} = [h_1, \ldots, h_n]^T$, is an eigenpair of G_n^{-1} if and only if it satisfies the discrete Sturm–Liouville Problem (SLP)

$$\begin{cases} -\Delta \left(p_{i-1} \Delta h_{i-1}\right) = \left(\mu - q_i\right) h_i & i = 1, \ldots, n, \\ h_0 = 0; \ h_{n+1} = 0 \end{cases} \tag{7}$$

with $\mu = 1/\lambda$, and

$$\Delta h_i = h_{i+1} - h_i,$$
$$p_i = \frac{1}{a_{i+1} c_i - a_i c_{i+1}}, \qquad i = 0, \ldots, n,$$
$$q_i = \frac{a_{i-1}\left(c_i - c_{i+1}\right) + a_i\left(c_{i+1} - c_{i-1}\right) - a_{i+1}\left(c_i - c_{i-1}\right)}{(a_i c_{i-1} - a_{i-1} c_i)(a_{i+1} c_i - a_i c_{i+1})}, \qquad i = 1, \ldots, n,$$

where, by convention, we have set $a_0 = c_{n+1} = 0$ and $a_{n+1} = c_0 = 1$.

Regarding the spectral properties of Green matrices, we recall a result on the ordering of the eigenvalues and on the oscillatory properties of the eigenvectors that passes through the theory of totally positive matrices [11,16]. We first introduce a definition and two theorems.

Definition 3.1. An $n \times n$ matrix A_n is called: (*strictly*) *totally positive*, denoted by TP (STP), if all its minors are nonnegative (positive); *oscillatory* if it is TP and there exists a $q \in \mathbb{N} \setminus \{0\}$ such that A^q is STP.

Of course, an STP matrix is also oscillatory with $q = 1$.

Theorem 3.1. *(See [11], Theorem 3.1 of Chapter 3.) A Green matrix G_n is TP if and only if $\{a_i\}$ and $\{c_i\}$ have the same strict sign and $\{a_i/c_i\}$ is increasing. Moreover, G_n^{n-1} is STP if and only if $\{a_i\}$ and $\{c_i\}$ have the same strict sign and $\{a_i/c_i\}$ is strictly increasing.*

The importance of being TP is well-illustrated by the following well-known result.

Theorem 3.2. *An $n \times n$ oscillatory matrix A_n has eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$ and $S\left(\mathbf{v}_k\right) = k - 1$ for each $k \in \{1, \ldots, n\}$, where $\mathbf{v}_k$ is the k-th eigenvector of A_n.*

Regarding the previous result, we recall that the symbol $S(\mathbf{v}_k)$ denotes the *number of sign variations* of $\mathbf{v}_k$ that coincides with the common value $S^-(\mathbf{v}_k) = S^+(\mathbf{v}_k)$, where $S^-(\mathbf{v}_k)$ is the number of sign changes of $\mathbf{v}_k$ when zero terms are discarded, and $S^+(\mathbf{v}_k)$ is the maximum number of sign changes of $\mathbf{v}_k$ when values $+1$ or -1 are arbitrarily assigned to zero terms. Notice that $S^+(\mathbf{v}_k)$ and $S^-(\mathbf{v}_k)$ can coincide only if the first and the last components of $\mathbf{v}_k$ are not zero and if for every zero component the preceding and the following components are not zero and of different sign.

In the following, we will be interested in characterizing the asymptotic spectral distribution of specific sequences of Green matrices and in analyzing, when possible, the monotonicity properties of the components of the eigenvectors; the latter analysis has an impact from an economic viewpoint (see [12,14,17]).

3.1. Spectral distribution of sequences of Green matrices

In this section we use the theory reported in Section 2 in order to show that a sequence of Green matrices is a GLT sequence: as a noteworthy result we obtain that the spectral distribution is independent of the possible parameters and in fact it is equal to zero.

Theorem 3.3. *Suppose that a_j and c_j in (5) are equally spaced sampling of Riemann integrable real valued functions $a(x)$ and $c(x)$, respectively, on the interval $[0,1]$, i.e. $a_j = a(j/n)$ and $c_j = c(j/n)$, $j = 1, \ldots, n$, then $\{G_n\} \sim_\sigma 0$ and $\{G_n\} \sim_\lambda 0$.*

Proof. The matrix G_n in (5) can be written as

$$G_n = D_n(a)D_n(c) + D_n(a)T_n^{(1)}D_n(c) + D_n(c)T_n^{(2)}D_n(a), \tag{8}$$

where

$$D_n(a) = \operatorname{diag}\{a_1, a_2, \ldots, a_n\}, \qquad D_n(c) = \operatorname{diag}\{c_1, c_2, \ldots, c_n\},$$

and

$$T_n^{(1)} = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}, \qquad T_n^{(2)} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 \end{bmatrix}.$$

We study separately the four sequences $\{D_n(a)\}$, $\{D_n(c)\}$, $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$.

From Theorem 2.5 we have $\{D_n(a)\} \sim_{\mathrm{GLT}} a$ and $\{D_n(c)\} \sim_{\mathrm{GLT}} c$.

Consider the Toeplitz matrix $T_n^{(1)}$, we can write

$$T_n^{(1)} = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & & 0 \\ 0 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & 1 \end{bmatrix}^{-1} - I = T_n(\widetilde{f}_1)^{-1} - I,$$

with $\widetilde{f}_1(\theta) = 1 - e^{-\mathbf{i}\theta} \in L^1(Q)$. By Theorem 2.4 $\{T_n(\widetilde{f}_1)\} \sim_{\mathrm{GLT}} \widetilde{f}_1$. Since the sequence $\{T_n(\widetilde{f}_1)\}$ is sparsely vanishing ($T_n(\widetilde{f}_1)$ is invertible), from Theorem 2.3 we obtain $\{T_n(\widetilde{f}_1)^{-1}\} \sim_{\mathrm{GLT}} \widetilde{f}_1^{-1}$.

The same reasoning can be repeated verbatim for $T_n^{(2)}$:

$$T_n^{(2)} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & -1 & 1 \end{bmatrix}^{-1} - I = T_n(\widetilde{f}_2)^{-1} - I,$$

with $\widetilde{f}_2(\theta) = 1 - e^{\mathbf{i}\theta} \in L^1(Q)$. Again we can conclude that $\{T_n(\widetilde{f}_2)^{-1}\} \sim_{\mathrm{GLT}} \widetilde{f}_2^{-1}$.

Using Theorem 2.3, putting together the previous results we have

$$\begin{aligned}\{G_n\} &\sim_{\mathrm{GLT}} a(x)c(x) + a(x)\left(\frac{1}{1-e^{-\mathbf{i}\theta}} - 1\right)c(x) + c(x)\left(\frac{1}{1-e^{\mathbf{i}\theta}} - 1\right)a(x) \\ &= a(x)c(x)\left(1 + \frac{1}{1-e^{-\mathbf{i}\theta}} - 1 + \frac{1}{1-e^{\mathbf{i}\theta}} - 1\right) = 0,\end{aligned}$$

that is $\{G_n\} \sim_{\mathrm{GLT}} 0$. Theorem 2.2 ensures that $\{G_n\} \sim_\sigma 0$ and $\{G_n\} \sim_\lambda 0$ since G_n is also Hermitian. □

Remark 3.1. We can prove Theorem 3.3 using a result on the spectral distribution of nonscaled sampling matrices obtained in [1], but again with the support of GLT theorems.

The Green matrix G_n in (5) can be obtained as a nonscaled sampling of the Green function (kernel) defined as

$$\begin{aligned} K(x,y) &= \begin{cases} a(x)c(y) & 0 < x \le y \le 1, \\ a(y)c(x) & 0 < y \le x \le 1, \end{cases} \\ &= a(x)c(y) + \begin{cases} 0 & 0 < x \le y \le 1, \\ a(y)c(x) - a(x)c(y) & 0 < y \le x \le 1, \end{cases} \end{aligned} \tag{9}$$

Please cite this article in press as: E. Salinelli et al., Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications, Linear Algebra Appl. (2015), http://dx.doi.org/10.1016/j.laa.2015.03.017

that is, $K(x, y)$ is the sum of a separable function $a(x)c(y)$ and a function $\phi(x, y)$ whose associated nonscaled sampling matrix (using the notation in (8)) is given by

$$\Phi_n := D_n(c)T_n^{(2)}D_n(a) - D_n(a)T_n^{(2)}D_n(c).$$

If $a(x)$ and $c(y)$ are Riemann integrable real valued functions, using the GLT theorems as in the proof of Theorem 3.3, it follows that $\{\Phi_n\} \sim_\sigma 0$, so, from Theorem 5 in [1], $\{G_n\} \sim_\sigma 0$, and $\{G_n\} \sim_\lambda 0$ since G_n is also Hermitian.

Remark 3.2. If $B_n - G_n = E_{n,m}$, where G_n is the Green matrix defined in (5) and, for all sufficiently large $m \in \mathbb{N}$, $E_{n,m} = R_{n,m} + N_{n,m}$ (where $R_{n,m}$ and $N_{n,m}$ are as in Definition 2.2), that is, if B_n is a perturbation of G_n, then $\{B_n\} \sim_\sigma 0$ and $\{B_n\} \sim_\lambda 0$ if $E_{n,m}$ are Hermitian (see Proposition 2.1); that is, $\{G_n\}$ and $\{B_n\}$ have the same null distribution.

Remark 3.3. The decomposition (8) of G_n is interesting from the computational point of view. Starting from the two sets $\{a_j\}$ and $\{c_j\}$, the product $G_n x$, $x \in \mathbb{C}^n$, requires only n multiplications for each diagonal matrix and $n-2$ sums for each Toeplitz matrix. The total cost is $6n + 2(n-2) = 8n - 4$ operations.

3.2. Few numerical experiments

We start by giving the notion of proper (or strong) cluster and the notion of weak cluster.

Definition 3.2. A matrix sequence $\{A_n\}$ of size d_n, $d_m < d_{m+1}$ for each m, is *properly (or strongly) clustered at* $s \in \mathbb{C}$ (in the eigenvalue sense), if for any $\epsilon > 0$ the number of outliers, that is the eigenvalues of A_n off the disk

$$B(s, \epsilon) := \{z : |z - s| < \epsilon\},$$

can be bounded by a pure constant q_ϵ possibly depending on ϵ, but not on n. In other words

$$q_\epsilon(n, s) := \#\{i : \lambda_i(A_n) \notin B(s, \epsilon)\} = O(1), \quad n \to \infty.$$

If every A_n has only real eigenvalues (at least for all n large enough), then s is real and the disk $B(s, \epsilon)$ reduces to the interval $(s - \epsilon, s + \epsilon)$. Finally, the term “properly (or strongly)” is replaced by “weakly”, if

$$q_\epsilon(n, s) = o(d_n), \qquad n \to \infty,$$

in the case of a point s (a closed set S), respectively.

If the number of outliers grows as a function $\phi(n)$, we speak about $\phi(n)$-clustering.

Please cite this article in press as: E. Salinelli et al., Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications, Linear Algebra Appl. (2015), http://dx.doi.org/10.1016/j.laa.2015.03.017

Table 1
Number of eigenvalues of G_n, defined in (5) with $a_i = (i/n+1)^{-1}$, $c_j = \ln(j/n+2)$, greater than $\epsilon_1 = 10^{-1}$, $\epsilon_2 = 10^{-2}$, and $\epsilon_3 = 10^{-3}$.

n	$q_{\epsilon_1}(n,0)$	$q_{\epsilon_2}(n,0)$	$q_{\epsilon_3}(n,0)$	$\frac{q_{\epsilon_1}(n,0)}{n}$	$\frac{q_{\epsilon_2}(n,0)}{n}$	$\frac{q_{\epsilon_3}(n,0)}{n}$
20	4	16	20	0.2000	0.8000	1.0000
40	6	19	40	0.1500	0.4750	1.0000
80	8	25	80	0.1000	0.3125	1.0000
160	11	35	142	0.0688	0.2188	0.8875
320	16	49	174	0.0500	0.1531	0.5437
640	22	68	226	0.0344	0.1063	0.3531
1280	31	96	311	0.0242	0.0750	0.2430
2560	43	136	433	0.0168	0.0531	0.1691
5120	61	192	609	0.0119	0.0375	0.1189

It is immediate to see that a sequence of matrices is weakly clustered at s if and only if it admits a distribution in the sense of Definition 2.1 and the distribution function is equal to the constant s (almost everywhere on a reference domain $[0,1]$). Therefore Theorem 3.3 tells us that any Green matrix sequence, with Riemann integrable coefficients $a(\cdot)$ and $c(\cdot)$, is weakly clustered at zero.

In the following we report selected experiments and we also discuss the fact that the cluster is often strong, depending on the nature of the considered functions. This shows that there is room for improving the theoretical findings in Theorem 3.3.

We choose the functions $a(x)$ and $c(x)$, $x \in [0,1]$, compute the number of eigenvalues of the matrix G_n in (5) greater, in modulus, of a certain tolerance $\epsilon \in \{10^{-1}, 10^{-2}, 10^{-3}\}$, for various sizes n.

First of all, we observe that, if we set $c(x) = ka(x)$, with $k \in \mathbb{C}$, we obtain a rank-one matrix. Indeed, if we use the decomposition in (8) we have

$$G_n = k^2 D_n(a)(I_n + T_n^{(1)} + T_n^{(2)})D_n(a) = k^2 D_n(a)\mathbf{1}_n D_n(a),$$

where $\mathbf{1}_n$ is the matrix of size n with all elements equal to one. Now, since $\operatorname{rank}(\mathbf{1}_n) = 1$ and $\operatorname{rank}(AB) \le \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$, where A, B are matrices of size n, the result follows. In this case, obviously, the cluster is strong.

In Tables 1–3 we fix the function $a(x) = (x+1)^{-1}$ and we choose the functions $c(x)$ with different growth: logarithmic: $c(x) = \ln(x+2)$ (Table 1); polynomial $c(x) = x+1$ (Table 2); and exponential: $c(x) = e^x$ (Table 3). The experiments seem to suggest a deterioration of the cluster when the slope of $c(x)$ increases and $a(x)$ is a decreasing function, moreover, in all three cases, we can note that $q_{\epsilon_2}(n,0) \approx 3q_{\epsilon_1}(n,0)$ and $q_{\epsilon_3}(n,0) \approx 10q_{\epsilon_1}(n,0)$. Finally, we can observe that the growth of the outliers is proportional to the square root of the dimension n of the matrix.

In Table 4 we consider a combination of a sinusoidal function $a(x)$ with a linear function $c(x)$ with positive and negative values, in detail $a(x) = \sin\left(\frac{\pi}{2}x + \frac{\pi}{4}\right)$ and $c(x) = x - 1/3$. Also in this case the same considerations of the previous cases on the proportionality of $q_\epsilon(n,0)$ and on the growth of the outliers as the square root of n are valid.

Please cite this article in press as: E. Salinelli et al., Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications, Linear Algebra Appl. (2015), http://dx.doi.org/10.1016/j.laa.2015.03.017

Table 2
Number of eigenvalues of G_n, defined in (5) with $a_i = (i/n + 1)^{-1}$, $c_j = j/n + 1$, greater than $\epsilon_1 = 10^{-1}$, $\epsilon_2 = 10^{-2}$, and $\epsilon_3 = 10^{-3}$.

n	$q_{\epsilon_1}(n,0)$	$q_{\epsilon_2}(n,0)$	$q_{\epsilon_3}(n,0)$	$\frac{q_{\epsilon_1}(n,0)}{n}$	$\frac{q_{\epsilon_2}(n,0)}{n}$	$\frac{q_{\epsilon_3}(n,0)}{n}$
20	6	20	20	0.3000	1.0000	1.0000
40	8	31	40	0.2000	0.7750	1.0000
80	11	37	80	0.1375	0.4625	1.0000
160	15	50	160	0.0938	0.3125	1.0000
320	22	68	290	0.0688	0.2125	0.9063
640	30	96	338	0.0469	0.1500	0.5281
1280	43	135	445	0.0336	0.1055	0.3477
2560	60	190	612	0.0234	0.0742	0.2391
5120	85	268	854	0.0166	0.0523	0.1668

Table 3
Number of eigenvalues of G_n, defined in (5) with $a_i = (i/n + 1)^{-1}$, $c_j = e^{j/n}$, greater than $\epsilon_1 = 10^{-1}$, $\epsilon_2 = 10^{-2}$, and $\epsilon_3 = 10^{-3}$.

n	$q_{\epsilon_1}(n,0)$	$q_{\epsilon_2}(n,0)$	$q_{\epsilon_3}(n,0)$	$\frac{q_{\epsilon_1}(n,0)}{n}$	$\frac{q_{\epsilon_2}(n,0)}{n}$	$\frac{q_{\epsilon_3}(n,0)}{n}$
20	7	20	20	0.3500	1.0000	1.0000
40	9	40	40	0.2250	1.0000	1.0000
80	13	45	80	0.1625	0.5625	1.0000
160	18	59	160	0.1125	0.3688	1.0000
320	25	81	320	0.0781	0.2531	1.0000
640	36	113	422	0.0563	0.1766	0.6594
1280	50	158	533	0.0391	0.1234	0.4164
2560	71	223	724	0.0277	0.0871	0.2828
5120	100	314	1007	0.0195	0.0613	0.1967

Table 4
Number of eigenvalues of G_n, defined in (5) with $a_i = \sin(\pi i/(2n) + \pi/4)$, $c_j = j/n - 1/3$, greater than $\epsilon_1 = 10^{-1}$, $\epsilon_2 = 10^{-2}$, and $\epsilon_3 = 10^{-3}$.

n	$q_{\epsilon_1}(n,0)$	$q_{\epsilon_2}(n,0)$	$q_{\epsilon_3}(n,0)$	$\frac{q_{\epsilon_1}(n,0)}{n}$	$\frac{q_{\epsilon_2}(n,0)}{n}$	$\frac{q_{\epsilon_3}(n,0)}{n}$
20	5	20	20	0.2500	1.0000	1.0000
40	7	25	40	0.1750	0.6250	1.0000
80	10	32	80	0.1250	0.4000	1.0000
160	14	44	160	0.0875	0.2750	1.0000
320	19	61	246	0.0594	0.1906	0.7688
640	27	85	291	0.0422	0.1328	0.4547
1280	38	120	391	0.0297	0.0938	0.3055
2560	54	169	542	0.0211	0.0660	0.2117
5120	76	238	759	0.0148	0.0465	0.1482

4. Eigenvectors of Schoenmakers–Coffey matrices

In this section we prove a result on the "shape" of the eigenvectors of a general subclass of Green matrices, the so-called Schoenmakers–Coffey matrices, motivated by a financial problem that we now briefly illustrate.

We recall first that $\Sigma_n = [\sigma_{ij}]_{i,j=1}^n$ is a *covariance matrix* if it is symmetric and (for simplicity) positive definite. Its corresponding *correlation matrix* $R_n = [\rho_{ij}]_{i,j=1}^n$ has elements $\rho_{ij} = \sigma_{ij}/\sqrt{\sigma_{ii}}\sqrt{\sigma_{jj}}$ that is $R_n = D\Sigma_n D$ where $D = \mathrm{diag}\,(\Sigma_n)^{-1/2}$. Thus, a correlation matrix is symmetric, positive definite with 1 on the main diagonal.

Please cite this article in press as: E. Salinelli et al., Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications, Linear Algebra Appl. (2015), http://dx.doi.org/10.1016/j.laa.2015.03.017

Covariance and/or correlation matrices play a crucial role in *multifactor models of interest rates* where changes in the shape of the *yield curve* are largely attributed to some unobservable factors. Their estimation on real data through the multivariate statistic technique of *principal component analysis* highlighted the importance of the first three factors, formally captured by the first three eigenvectors of the covariance (or correlation) matrix of yields: for details refer to [12,14,17]. These three eigenvectors were respectively called *shift*, *slope* and *curvature* (of the yield curve), hereafter SSC, because of the behavior of their elements. Approximately:

- a shift has constant sign and an "humped shape": when it is positive, it is first increasing then decreasing (see e.g. [7]);
- a slope is monotone, with a change of sign;
- a curvature has a one-peaked shape with two changes of sign.

These features are formally captured in the following definition (that resumes the ones in [18] and [20]) in terms of changes of sign of vectors $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} = [v_1, \ldots, v_n]^T$, and $\Delta\mathbf{v} \in \mathbb{R}^{n-1}$ defined by $(\Delta\mathbf{v})_i = v_{i+1} - v_i$ for $i = 1, \ldots, n-1$.

Definition 4.1. Let Γ_n be an $n \times n$, $n \geq 3$, correlation (or covariance) matrix having its first three eigenvalues simple, whose corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ have, by convention, non negative first element. We define:

- $\mathbf{v}_1$ *weak shift* if $S^-(\mathbf{v}_1) = 0$, *shift* if it is weak shift and $S^-(\Delta\mathbf{v}_1) = 1$ where the first no zero element of $\Delta\mathbf{v}_1$ is positive, *pure shift* if it is constant;
- $\mathbf{v}_2$ *weak slope* if $S^-(\mathbf{v}_2) = 1$, *slope* if it is weak slope and $S^-(\Delta\mathbf{v}_2) = 0$;
- $\mathbf{v}_3$ *weak curvature* if $S^-(\mathbf{v}_3) = 2$, *curvature* if it is weak curvature and $S^-(\Delta\mathbf{v}_3) = 1$.

In the empirical literature both cases of SSC and SSC in a weak form can be found (see Figures 3.16 and 3.17 in [14], Exhibit 5 in [7], Figures 1 and 2 in [12,13]). Anyway, in all the weak cases, in each interval where the elements of the first three eigenvectors are of constant sign, there is at most one hump, that is the conjecture $S(\Delta\mathbf{v}_1) \leq 1$, $S(\Delta\mathbf{v}_2) \leq 2$ and $S(\Delta\mathbf{v}_3) \leq 3$ appears reasonable.

We will show that this is the case for a class of Green matrices, those of Schoenmakers–Coffey, denoted by $SC_n = [r_{ij}]_{i,j=1}^n$ where [21]:

$$r_{ij} = \frac{\min\{b_i, b_j\}}{\max\{b_i, b_j\}}, \qquad i, j = 1, \ldots, n, \tag{10}$$

and

H1) $\{b_i\}$ is real and strictly positive;
H2) $\{b_i\}$ is strictly increasing;
H3) $\{b_i/b_{i+1}\}$ is strictly increasing, that is $\{b_i\}$ is log-concave.

Please cite this article in press as: E. Salinelli et al., Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications, Linear Algebra Appl. (2015), http://dx.doi.org/10.1016/j.laa.2015.03.017

We prove now that SC_n is a correlation matrix that possesses the following standard properties of the correlations ρ_{ij} of interest rates:

a) $\rho_{ij} > 0$ for all i, j;
b) $\{\rho_{ij}\}$ is increasing in i and decreasing in j for $i < j$;
c) $\{\rho_{i,i+1}\}$ is increasing.

In fact, assumptions H1) and H2) ensure that SC_n satisfies a) and b); furthermore, H1) and H3) imply c) in a strict sense. Moreover, SC_n is Green's with $a_i = b_i$ and $c_j = 1/b_j$ for $i < j$, and since $\{\frac{a_i}{c_i}\} = \{b_i^2\}$ is positive and strictly increasing, by Theorem 3.1 SC_n is oscillatory (see also Corollary 4 of [13]), so it is a correlation matrix. Theorem 3.2 ensures that the first three eigenvectors of SC_n are SSC in a weak form (obviously, the existence of a weak shift eigenvector can be justified via the Perron–Frobenius Theorem too).

From what we have just shown it follows that SC_n is a nonsingular symmetric Green matrix and then has tridiagonal inverse SC_n^{-1} as in (6). The corresponding SLP (7) has coefficients

$$p_i = \frac{b_i b_{i+1}}{b_{i+1}^2 - b_i^2}, \qquad i = 0, \dots, n,$$

$$q_i = \frac{b_i\left(b_{i+1} - b_{i-1}\right)}{\left(b_i + b_{i-1}\right)\left(b_i + b_{i+1}\right)}, \quad i = 1, \dots, n,$$

with $b_0 = 0$ and $b_{n+1} = \infty$.

Since $\{b_i\}$ is positive and increasing, both $\{p_i\}_{i=1}^{n-1}$ and $\{q_i\}_{i=1}^{n-1}$ are positive. Further insight are given in the following.

Lemma 4.1. *The sequence $\{p_i\}_{i=1}^{n-1}$ is (strictly) increasing if and only if $\{b_i/b_{i+1}\}$ is (strictly) increasing.*

The sequence $\{q_i\}_{i=2}^{n-1}$ is bounded from above by 1 and (strictly) decreasing if $\{b_i/b_{i+1}\}$ is (strictly) increasing.

Proof. The monotonicity statements straightforwardly follows by

$$p_i - p_{i-1} = \frac{b_i^2 b_{i+1}\left(b_{i-1} + b_{i+1}\right)\left(\frac{b_i}{b_{i+1}} - \frac{b_{i-1}}{b_i}\right)}{\left(b_i^2 - b_{i-1}^2\right)\left(b_{i+1}^2 - b_i^2\right)}, \qquad i = 2, \dots, n-1,$$

$$q_i - q_{i-1} = \frac{b_{i-1} b_{i+1}\left(\frac{b_{i-2}}{b_{i-1}} - \frac{b_i}{b_{i+1}}\right)}{\left(b_{i-1} + b_{i-2}\right)\left(b_i + b_{i+1}\right)}, \qquad i = 3, \dots, n-1,$$

and properties H1)–H3) of $\{b_i\}$. Furthermore, since $b_i > 0$ for any $i = 1, \dots, n$, we have:

$$q_i < \frac{b_{i+1} - b_{i-1}}{b_i + b_{i+1}} < 1. \qquad \square$$

Please cite this article in press as: E. Salinelli et al., Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications, Linear Algebra Appl. (2015), http://dx.doi.org/10.1016/j.laa.2015.03.017

We recall now a definition and a theorem due to Hartman [9].

Definition 4.2. A solution $\mathbf{h} = [h_1, \ldots, h_n]^T$ of Eq. (7) has a generalized zero at i_0 provided $h_{i_0} = 0$ if $i_0 = 1$ and if $i_0 > 1$ either $h_{i_0} = 0$ or $h_{i_0-1} h_{i_0} < 0$.

Theorem 4.1. *(Rolle's) Assume the sequence of real numbers $v_1, \ldots, v_n$ has N_j generalized zeros and $\Delta v_1, \ldots, \Delta v_{n-1}$ has M_j generalized zeros. Then $M_j \geq N_j - 1$.*

Given a vector $\mathbf{h} \in \mathbb{R}^n$, set

$$\Omega_i = -\Delta \left(p_{i-1} \left(\Delta \mathbf{h}\right)_{i-1}\right) = p_{i-1} \left(h_i - h_{i-1}\right) - p_i \left(h_{i+1} - h_i\right).$$

Obviously if $h_{i-1} = h_i = h_{i+1}$ then $\Omega_i = 0$. Furthermore, if in i there is a strict maximum, that is $(\Delta \mathbf{h})_{i-1} > 0$ and $(\Delta \mathbf{h})_i < 0$, then $\Omega_i > 0$; analogously, if in i there is a strict minimum then $\Omega_i < 0$.

We are now able to prove the main result of this section.

Theorem 4.2. *The k-th eigenvector $\mathbf{h}_k$ of a Schoenmakers–Coffey matrix SC_n of size n, $n \geq 4$, has exactly $k-1$ changes of sign; between two consecutive changes of sign of $\mathbf{h}_k$ there is exactly one change of monotonicity and $k - 2 \leq S^- (\Delta \mathbf{h}_k) \leq k$.*

Proof. The first statement follows by Theorem 3.2.

Assume now $\mathbf{h}_k$ has two consecutive generalized zeroes in $i_*, i_{**} \in \{2, \ldots, n\}$ with $i_* < i_{**}$. Since SC_n is oscillatory, if $h_{k,i_*} = h_{k,i_{**}} = 0$ there exists an index i^* such that $i_* < i^* < i_{**}$ and $h_{i^*} \neq 0$. By Rolle Theorem, $\Delta \mathbf{h}_k$ has (at least) a generalized zero between i_* and i_{**}. Let i^+ between i_* and i_{**} the minimum index for which $\Delta \mathbf{h}_k$ has a generalized zero, so $(\Delta \mathbf{h}_k)_{i^+} = 0$ or $(\Delta \mathbf{h}_k)_{i^+-1} (\Delta \mathbf{h}_k)_{i^+} < 0$. If, for example, $(\Delta \mathbf{h}_k)_{i^+-1} > 0$ then $h_{k,i^+} > 0$ and in both the previous cases we obtain $\Omega_{i^+} > 0$. Since $\{\mu - q_i\}_{i=2}^{n-1}$ is strictly increasing, by (7) it follows $\Omega_i > 0$ for all $i > i^+$ for which $h_i > 0$ and this prevent the existence of other generalized zero of $\Delta \mathbf{h}_k$ between i_* and i_{**}. Assuming alternatively $(\Delta \mathbf{h}_k)_{i^+-1} < 0$ (and then $h_{k,i^+} < 0$) we get the same conclusion.

With the same argument, it is possible to show that before the first (the minimum) and after the last (the maximum) generalized zeroes of $\mathbf{h}_k$ there exists at most a change of sign of $\Delta \mathbf{h}_k$ and the last conclusion follows. □

If one removes the assumption that $\{b_i\}$ is log-concave, then the sequence $\{\mu - q_i\}_{i=2}^{n-1}$ is no longer strictly increasing and the statement of Theorem 4.2 on the changes of monotonicity does not necessarily hold. In this last case, by the proof of the previous theorem it emerges that the first eigenvector presenting "internal" humps is the first one, as illustrated by the following example.

Example 4.1. The sequence $\{b_i\}$ defined by $b_i = \exp\left\{\frac{6}{5}i + \sin \frac{6}{5}i\right\} + 1$ is strictly increasing but not log-concave. One can verify that the first eigenvector of the corresponding matrix SC_n has two humps for $n \geq 9$.

Theorem 4.2 tells us that the first three eigenvectors of a Schoenmakers–Coffey matrix SC_n are SSC in a strict sense if and only if they do not present an hump before their first and after their last zero. The following corollary formalizes this idea.

Corollary 4.1. *Let $(\lambda, \mathbf{h})$, with $\mathbf{h} \in \mathbb{R}^n$, $\mathbf{h} = [h_1, \dots, h_n]^T$, be an eigenpair of a Schoenmakers–Coffey matrix SC_n with, for example, $h_1 > 0$. Then, $\mathbf{h}$ does not present an hump before (after) its first (last) zero if an only if $\lambda < 1 + \frac{b_1}{b_2}$ ($\lambda < 1 + \frac{b_{n-1}}{b_n}$).*

Proof. $\mathbf{h}$ does not present a hump before its first zero if and only if $h_1 > h_2$. The first row of $\left(I - \lambda \left(SC_n\right)^{-1}\right) \mathbf{h} = \mathbf{0}$ gives:

$$h_2 = \left(\lambda \frac{b_2^2}{b_1^2 - b_2^2} + 1\right) \left(\frac{\lambda b_1 b_2}{b_1^2 - b_2^2}\right)^{-1} h_1,$$

therefore

$$h_1 - h_2 = \frac{h_1}{\lambda b_1 b_2} \left(b_2 - b_1\right) \left(b_1 + b_2 - \lambda b_2\right),$$

and the claim follows by $0 < b_1 < b_2$, $h_1 > 0$ and $\lambda > 0$, as SC_n is oscillatory. The second statement is obtained in the same way. □

Given a Schoenmakers–Coffey matrix SC_n, two conclusions immediately follow by the previous corollary. First, if an eigenvector $\mathbf{h}$ of SC_n does not have a hump before its first zero, then it does not have anyone after its last zero. Second, since SC_n has positive elements and $\{b_i/b_{i+1}\}$ is increasing, by the classical inequalities

$$\min_j \sum_i r_{ij} \le \lambda_1 \le \max_j \sum_i r_{ij},$$

we obtain $\lambda_1 > 1 + b_1/b_2$, namely, in financial terms, the first eigenvector of a Schoenmakers–Coffey matrix is always shift.

For what concerns the second and third eigenvector of SC_n, we must notice that the conditions for the non-existence of an initial and/or final hump for a given eigenvector $\mathbf{h}$ of Corollary 4.1 are based on knowledge (at least in terms of estimation) of the corresponding eigenvalue λ. In the absence (as usual) of such information, it is still possible to obtain some partial conclusions, as illustrated by the following example.

Example 4.2. If $b_s = s^\alpha$ with $\alpha > 0$, then:

$$r_{ij} = \begin{cases} \left(\frac{i}{j}\right)^\alpha & i \le j, \\ \left(\frac{j}{i}\right)^\alpha & i > j. \end{cases}$$

Please cite this article in press as: E. Salinelli et al., Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications, Linear Algebra Appl. (2015), http://dx.doi.org/10.1016/j.laa.2015.03.017

Since $\lim_{\alpha\to 0^+} r_{ij} = 1$ and $\lim_{\alpha\to+\infty} r_{ij} = 0$, we obtain:

$$\lim_{\alpha\to 0^+} \lambda_2 = 0; \qquad \lim_{\alpha\to+\infty} \lambda_2 = 1.$$

On the other hand

$$\lim_{\alpha\to 0^+} 1 + \frac{b_1}{b_2} = \lim_{\alpha\to 0^+} 1 + \frac{b_{n-1}}{b_n} = 2,$$
$$\lim_{\alpha\to+\infty} 1 + \frac{b_1}{b_2} = \lim_{\alpha\to+\infty} 1 + \frac{b_{n-1}}{b_n} = 1.$$

Therefore for α in a suitable right neighborhood of 0 there are no initial and final humps in $\mathbf{h}_2$. Similar considerations apply to λ_3 and $\mathbf{h}_3$.

A further consideration. As it is well-known, the eigenvectors of a covariance matrix are different from the ones of the corresponding correlation matrix. However, in [13] it has been shown that an invertible covariance matrix is oscillatory if and only if its correlation matrix is oscillatory. This means that all Green (covariance) matrices having a corresponding correlation matrix of Schoenmakers–Coffey type, are oscillatory and their first three eigenvectors are SSC in a weak sense. This raises the question of whether the results obtained here on the number of monotonicity changes of Schoenmakers–Coffey matrices extend in a natural way to the corresponding covariance Green matrices. The negative answer is given by the following example.

Example 4.3. Consider the Green matrix G_n defined by the sequences $\{a_i\}$ and $\{c_j\}$ with:

$$t_k = \frac{\pi}{2}\left(1 + \frac{k-1}{n-1}\right), \qquad k = 1, \dots, n,$$
$$a_i = t_i(\sin(4t_i) + 2), \qquad i = 1, \dots, n,$$
$$c_j = \sin(4t_j) + 2, \qquad j = 1, \dots, n.$$

The sequences $\{a_i\}$ and $\{c_j\}$ are positive but not monotone. However we have that $\{a_i/c_i\} = \{t_i\}$ is strictly increasing, therefore G_n is oscillatory and in particular is a covariance matrix. The sequence $\{b_i\} = \{\sqrt{a_i/c_i}\} = \{\sqrt{t_i}\}$ is positive, strictly increasing and such that $\{b_i/b_{i+1}\}$ is strictly increasing too:

$$\frac{b_i}{b_{i+1}} - \frac{b_{i-1}}{b_i} = \sqrt{1 - \frac{1}{n-1+i}} - \sqrt{1 - \frac{1}{n-2+i}} > 0.$$

Therefore, the corresponding correlation matrix SC_n is of Schoenmakers–Coffey type and its first three eigenvectors are SSC (Fig. 1(a)). Nevertheless the first eigenvector of G_n is not shift and presents two humps (Fig. 1(b)). If we substitute $\sin(4t_i)$ with $\sin(10t_i)$ also the second and the third eigenvector of G_n are not more slope and curvature (Fig. 1(c)).

Please cite this article in press as: E. Salinelli et al., Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications, Linear Algebra Appl. (2015), http://dx.doi.org/10.1016/j.laa.2015.03.017

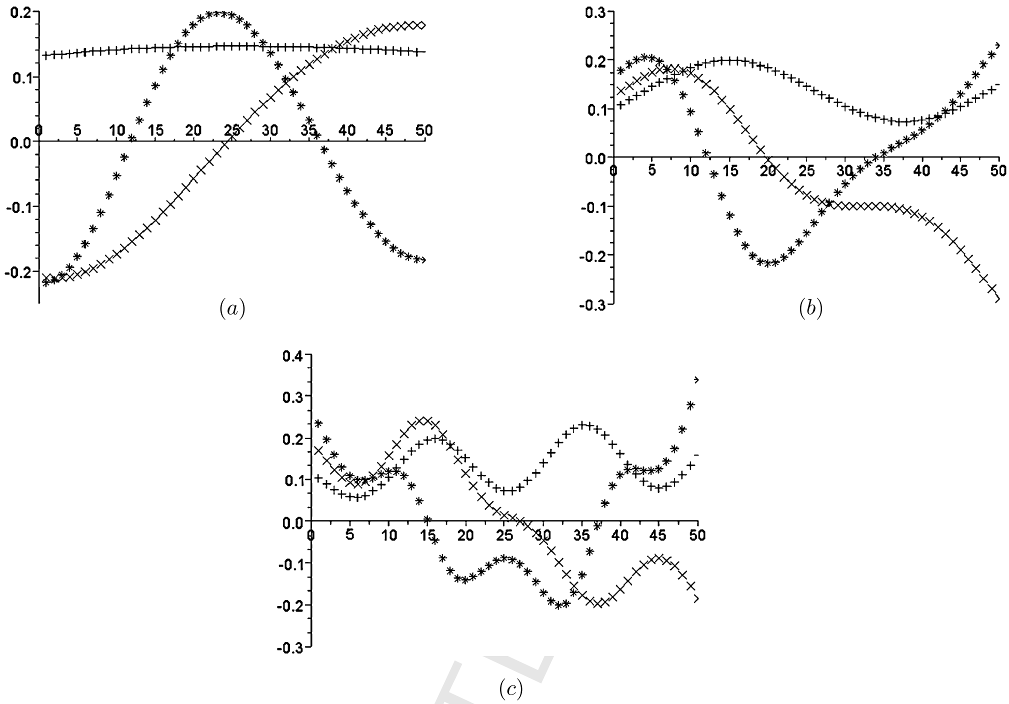

Fig. 1. First three eigenvectors, $\mathbf{h}_1$ $(+)$, $\mathbf{h}_2$ $(\times)$ and $\mathbf{h}_3$ $(*)$, of SC_{50} for $\sin(4t_i)$ (a), of G_{50} for $\sin(4t_i)$ (b), and of G_{50} for $\sin(10t_i)$ (c).

In the next example the estimation of the eigenvalues of the considered correlation matrix allows to obtain a complete result on the existence of slope and curvature.

Example 4.4. Consider the well-known Kac–Murdock–Szegö (KMS) matrices $R = \left[\rho^{|i-j|}\right]_{i,j=1}^{n}$, $\rho \in (0,1)$, [6,8,10,29]. They are Green's (see (6)) with

$$a_i = \rho^{1-i}, \qquad c_j = \rho^{j-1}, \tag{11}$$

and represent a limit case of the Schoenmakers–Coffey ones, as $\{b_i/b_{i+1}\}$ is constant. Indeed, by the proof of Lemma 4.1, $\{p_i\}_{i=1}^{n-1}$ is a constant sequence if and only if $\{b_i/b_{i+1}\}$ is constant. It is possible to show (see Theorem 3.1 in [21]) that under H1)–H3) (10) is equivalent to

$$(R)_{ij} = \exp\left\{-\sum_{l=i+1}^{n} \min\{l-i, j-i\}\,\widetilde{\Delta}_l\right\}, \qquad i<j, \tag{12}$$

where

$$\widetilde{\Delta}_l = \ln b_l - \ln b_{l-1} - (\ln b_{l+1} - \ln b_l), \qquad l = 2,\dots,n-1,$$
$$\widetilde{\Delta}_n = \ln b_n - \ln b_{n-1} = \ln b_2 - \sum_{l=2}^{n-1} \widetilde{\Delta}_l.$$

Hence, $\{b_i/b_{i+1}\}$ is constant if and only if one chooses $\widetilde{\Delta}_l = 0$ in (12) for $l = 2, \ldots, n-1$, obtaining (without any restrictions we assume $b_1 = 1$) $(R)_{ij} = \exp\{-\min\{n-i, j-i\}\Delta_n\} = b_2^{i-j}$ for $i < j$. Therefore this is the unique case where the main diagonal of SC_n^{-1} (except for the first and the last elements) and the super-diagonals are constant.

KMS matrices are oscillatory [18], so their first three eigenvectors are SSC in a weak sense. The authors in [19] have shown that these matrices admit shift for all $\rho \in (0, 1)$ and slope and curvature if and only if ρ is greater than a threshold value. We briefly show how to extend and precise these results for $n \geq 4$ (if $n = 3$ all is obvious, see Example 15 in [19]).

As it can be easily verified, the conclusions of Theorem 4.2 apply despite being in a borderline case. The same result can be obtained recalling that KMS matrices are Toeplitz's, with the $\lceil \frac{n}{2} \rceil$ symmetric and $\lfloor \frac{n}{2} \rfloor$ skew-symmetric eigenvectors $\mathbf{h}_k$ whose expressions (see [28]):

$$h_{tk} = \cos\left(\left(t - \frac{n+1}{2}\right)\theta_k\right), \qquad k \text{ odd}, \tag{13}$$

$$h_{tk} = \sin\left(\left(t - \frac{n+1}{2}\right)\theta_k\right), \qquad k \text{ even}, \tag{14}$$

are obtained by solving problem (7) with $p_i = \frac{\rho}{1-\rho^2}$ and $q_i = \frac{1-\rho}{1+\rho}$. As a consequence, the fact that between two consecutive zeros there is one and only one change of monotonicity follows by noting that each eigenvector is an equispaced sampling of sine or cosine functions.

For what concerns the presence of initial (and by symmetry, final) humps, we recall that the eigenvalues of R are (interlaced, with λ_1 even and) given by

$$\lambda_k = \frac{1-\rho^2}{1+\rho^2 - 2\rho\cos\theta_k}, \tag{15}$$

where, as proved in [25] improving a previous result of [8],

$$\frac{(k-1)\pi}{n} < \theta_k < \frac{k\pi}{n+1}, \qquad k = 1, \ldots, n. \tag{16}$$

As Corollary 4.1 holds with $1 + \frac{b_1}{b_2} = 1 + \frac{b_{n-1}}{b_n} = 1 + \rho$, the monotonicity of the "first positive part" (with $h_{1k} > 0$) of any eigenvector of R depends on the sign of $\lambda_k - (1+\rho)$. Apart from the obvious conclusions about λ_1 ($> 1+\rho$) (and $\lambda_n < 1+\rho$), we show now that:

i) for $n = 4, 5$ we have ($\lambda_3 <$) $\lambda_2 < 1+\rho$ for all $\rho \in (0,1)$; for $n \geq 6$ we have $\lambda_2 \gtreqless 1+\rho$ if and only if $\rho \lesseqgtr 2\cos\frac{\pi}{n-2} - 1$;

Please cite this article in press as: E. Salinelli et al., Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications, Linear Algebra Appl. (2015), http://dx.doi.org/10.1016/j.laa.2015.03.017

ii) for $n = 6, 7, 8$ we have $\lambda_3 < 1 + \rho$ for all $\rho \in (0, 1)$; for $n \geq 9$ we have $\lambda_3 \gtreqless 1 + \rho$ if and only if $\rho \lesseqgtr 2\cos\frac{2\pi}{n-2} - 1$.

In fact, observe preliminarily that

$$\lim_{\rho\to1^-} \lambda_k(\rho) = \lambda_k(1) = 0 < 2 = \lim_{\rho\to1^-} 1 + \rho, \qquad k = 2, 3, \tag{17}$$

hence, by continuity, $\lambda_k < 1 + \rho$ for "sufficiently large" ρ.

i) By (15) it follows $\lambda_2 \gtreqless 1 + \rho$ if and only if $2\cos\theta_2 \gtreqless 1 + \rho$, which is equivalent to $h_{12} \gtreqless h_{22}$.

From (14), $h_{12} = h_{22}$ if and only if (see (16))

$$\theta_2 \in \left(\frac{\pi}{n}, \frac{2\pi}{n+1}\right): \quad \sin\left(\theta_2\frac{n-1}{2}\right) = \sin\left(\theta_2\frac{n-3}{2}\right), \tag{18}$$

that is if and only if $\theta_2^* = \frac{r\pi}{n-2}$ where the integer r satisfies

$$1 - \frac{2}{n} < r < 2 - \frac{6}{n+1}.$$

Then for $(3 \leq)n \leq 5$ there are no solutions, whereas for $n \geq 6$ Eq. (18) admits the unique solution $\widetilde{\theta}_2 = \frac{\pi}{n-2}$. These conclusions jointly with (17) prove the statement with $\rho = 2\cos\frac{\pi}{n-2} - 1$.

ii) Operating as in the proof of i), for $n \geq 9$ we find a unique value $\widehat{\theta}_3 = \frac{2\pi}{n-2}$ such that $h_{13} = h_{23}$ and $\rho = 2\cos\frac{2\pi}{n-2} - 1$.

The "translation" in the financial language of these results is immediate.

5. Conclusions and remarks

We have identified some spectral properties of Green matrices, whose eigenvalue distribution is always equal to zero: this result is not surprising giving the fact that these matrices come from integral operators (see [1] and references therein), but we have obtained it in a wider generality, by using the theory of GLT sequences. Concerning this first part we observe that the zero spectral distribution is equivalent to the weak clustering: however, in our numerical tests, often we observed a $\sqrt{n}$-clustering in the average case and hence it seems that there is room for improving the theoretical results.

As a second step, we have considered a subclass of the Green matrices, called Schoenmakers–Coffey matrices, which have a role in financial context. For this class, in view of its importance in applications such as in the portfolio estimation, we have analyzed the eigenvector oscillation structure.

As already observed in the introduction, even if these matrices are not shift invariant (Toeplitz), the results are obtained by using tools coming from Toeplitz technology, Q7 related to GLT sequences and to Kac–Murdoch–Szegö matrices, connected to Toeplitz matrices.

Acknowledgements

The work of the second and third author has been partly financed by the Italian Q9 national group GNCS-INDAM. The work of the second author has been also partly supported by a grant associated to the Project "Becoming the Number One – 2014" of Q10 the Knut and Alice Wallenberg Foundation, Sweden.

References

[1] A.S. Al-Fhaid, S. Serra-Capizzano, D. Sesana, M. Zaka Ullah, Singular-value (and eigenvalue) distribution and Krylov preconditioning of sequences of sampling matrices approximating integral operators, Numer. Linear Algebra Appl. 21 (2014) 722–743.
[2] W.W. Barrett, A theorem on inverse of tridiagonal matrices, Linear Algebra Appl. 27 (1979) 211–217.
[3] M. Capovani, Sulla determinazione della inversa delle matrici tridiagonali e tridiagonali a blocchi, Calcolo 7 (1970) 295–303.
[4] R. Bhatia, Matrix Analysis, Springer-Verlag, New York, 1997.
[5] A. Böttcher, B. Silbermann, Introduction to Large Truncated Toeplitz Matrices, Springer-Verlag, New York, 1999.
[6] A. Cantoni, P. Butler, Eigenvalues and eigenvectors of symmetric centrosymmetric matrices, Linear Algebra Appl. 13 (1976) 275–288.
[7] B.W. Golub, L.M. Tilman, Measuring yield curve risk using principal components analysis, value at risk, and key rate durations, J. Portf. Manag. 23 (1997) 72–84.
[8] U. Grenander, G. Szegö, Toeplitz Forms and Their Applications, second ed., Chelsea, New York, 1984.
[9] P. Hartman, Difference equations: disconjugacy, principal solutions, Green's functions, complete monotonicity, Trans. Amer. Math. Soc. 246 (1978) 1–30.
Q8 [10] M. Kac, W.L. Murdock, G. Szegö, On the eigenvalues of certain Hermitian forms, J. Ration. Mech. Anal. 2 (1953) 767–800.
[11] S. Karlin, Total Positivity, vol. 1, Stanford University Press, Stanford CA, 1968.
[12] S. Lardic, P. Priaulet, S. Priaulet, PCA of yield curve dynamics: questions of methodologies, J. Bond Trading Management 1 (2003) 327–349.
[13] R. Lord, A. Pelsser, Level, Slope and curvature: art or artefact?, Appl. Math. Finance 14 (2007) 105–130.
[14] L. Martellini, P. Priaulet, S. Priaulet, Fixed-Income Securities: Valuation, Risk Management and Portfolio Strategies, Wiley, Chichester, 2003.
[15] D. Noutsos, S. Serra-Capizzano, P. Vassalos, The conditioning of FD matrix sequences coming from semi-elliptic differential equations, Linear Algebra Appl. 428 (2008) 600–624.
[16] A. Pinkus, Totally Positive Matrices, Cambridge University Press, Cambridge, 2010.
[17] R. Rebonato, Modern Pricing of Interest-Rate Derivatives, Princeton University Press, Princeton, 2002.
[18] E. Salinelli, C. Sgarra, Correlation matrices of yields and total positivity, Linear Algebra Appl. 418 (2006) 682–692.
[19] E. Salinelli, C. Sgarra Shift, Slope and curvature for a class of yields correlation matrices, Linear Algebra Appl. 426 (2007) 650–666.
[20] E. Salinelli, C. Sgarra, Some results on correlation matrices for interest rates, Acta Appl. Math. 115 (2011) 291–318.
[21] J. Schoenmakers, B. Coffey, Systematic generation of parametric correlation structures for the LIBOR market model, Int. J. Theor. Appl. Finance 6 (2003) 507–519.
[22] S. Serra-Capizzano, Distribution results on the algebra generated by Toeplitz sequences: a finite dimensional approach, Linear Algebra Appl. 328 (2001) 121–130.
[23] S. Serra-Capizzano, Generalized Locally Toeplitz sequences: spectral analysis and applications to discretized partial differential equations, Linear Algebra Appl. 366 (2003) 371–402.
[24] S. Serra-Capizzano, The GLT class as a generalized Fourier analysis and applications, Linear Algebra Appl. 419 (2006) 180–233.

[25] R.J. Stroeker, Approximations of the eigenvalues of the covariance matrix of a first-order autoregressive process, J. Econometrics 22 (1983) 269–279.
[26] P. Tilli, A note on the spectral distribution of Toeplitz matrices, Linear Multilinear Algebra 45 (1998) 147–159.
[27] P. Tilli, Locally Toeplitz sequences: spectral properties and applications, Linear Algebra Appl. 278 (1998) 91–120.
[28] W.F. Trench, Characteristic polynomials of symmetric rationally generated Toeplitz matrices, Linear Multilinear Algebra 21 (1987) 289–296.
[29] W.F. Trench, Interlacement of the even and odd spectra of real symmetric Toeplitz matrices, Linear Algebra Appl. 195 (1993) 59–68.
[30] E.E. Tyrtyshnikov, A unifying approach to some old and new theorems on distribution and clustering, Linear Algebra Appl. 232 (1996) 1–43.
[31] E.E. Tyrtyshnikov, N. Zamarashkin, Spectra of multilevel Toeplitz matrices: advanced theory via simple matrix relationships, Linear Algebra Appl. 270 (1998) 15–27.

Please cite this article in press as: E. Salinelli et al., Eigenvalue-eigenvector structure of Schoenmakers–Coffey matrices via Toeplitz technology and applications, Linear Algebra Appl. (2015), http://dx.doi.org/10.1016/j.laa.2015.03.017

Sponsor names

Do not correct this page. Please mark corrections to sponsor names and grant numbers in the main text.

Q11 **Knut and Alice Wallenberg Foundation**, *country*=Sweden, *grants*=