Processes of mathematical inquiry
in kindergarten

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Preface

Before I started working as a PhD research fellow at the University of Agder and within the Agder Project, I was working as a mathematics teacher in upper secondary school in Kristiansand in Norway. I had few experiences with kindergarten children's ways of engaging in mathematical activities, besides playing mathematical games with my friends' children now and then. However, I have always loved to play with children, and been fascinated by their spontaneous and open-minded way of encountering life. Moreover, as a mathematics teacher, I have always been curious about how people actually learn mathematics, and how 'it all' starts in early childhood.

When the University of Agder and the Agder Project offered a 4-years full time position as a PhD research fellow, I saw a great opportunity to learn more about how young children learn mathematics. During the process of investigating mathematical inquiry processes in kindergarten, from a cultural-historical perspective, I realised that I didn't only get insights into processes of mathematics teaching and learning. I also got insights into what it means to be human in a more general sense, which I am deeply grateful for.

There are many people who deserve to be acknowledged for their support during the 4-years process of conducting this research study and writing up this thesis. First, I would like to thank the kindergarten teachers and the children who participated in my study and welcomed me into their kindergartens. I also want to thank the Agder Project research team for trusting me to be part of the team, and for support and encouragement during the process. Moreover, I want to thank my fellow PhD students at UiA, with whom I have discussed various topics related to the process of doing a PhD in mathematics education. I want to thank all research fellows in the Mathematics Education Research Group at UiA for a warm and supportive research environment. I also want to thank the teachers at the doctoral courses that I attended; Said Hadjerrouit, Pauline Vos, John Monaghan, Simon Goodchild, Reinhard Siegmund-Schultze, Sven Arntzen and Hans Herlof Grelland. Thank you for giving me a theoretical, methodological and ethical sound foundation before I started on my research project.

I would especially like to thank my supervisors, Martin Carsen, Per Sigurd Hundeland, John Monaghan and Ingvald Erfjord, for their excellent guidance and support during the whole process. Without you I would never have been able to do this PhD. I also wish to give warm thanks to Luis Radford for insightful and inspiring conversations when I visited him in October 2017.

To my mum, dad, sister, boyfriend and friends: Thank you so much for always being supportive and above all patient. I promise that you will see me more in the future!

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Abstract

This thesis reports from a research study which investigates the processes that unfolds when kindergarten children (age 5-years-old), together with their kindergarten teachers and peers inquire into and solve mathematical problems. The study is situated within a research and development project called the Agder Project. The project aims to investigate the effects of a preschool intervention programme in Norway, which focuses on four sets of competences: social-emotional, self-regulation, language, and mathematics. This research study uses a case study design where five kindergarten teachers and their groups of 5-year-old children, who were part of the focus group of the Agder Project, were chosen as participants. To explore the processes that unfolds when kindergarten children and their kindergarten teachers inquire into mathematical problems, the study adopts a cultural-historical perspective on learning and development and draws in particularly on Radford's (2013) theory of knowledge objectification. Moreover, a qualitative approach to data collection and data analysis was adopted, and the empirical material was collected through ethnographic field notes, observations and interviews.

The results of this study provide support for recognising mathematical inquiry in kindergarten as multimodal, dialectical and 'ethical' processes. It is multimodal because both children's and kindergarten teachers' contributions to the inquiry processes are heavily based on other modalities than language. In the segments analysed in this research study, some children provide quite sophisticated argumentations and explanations, but their mathematical reasoning is to a great extent materialised through gestures and other bodily actions, in combination with words. Moreover, the kindergarten teachers engage children in the mathematical activities by their multimodal participation. For example, instead of asking a lot of verbal questions they guide the children by their bodily actions and tone of voice towards the aim of the activity and consequently give children a space of freedom to contribute. The results also provide support for recognising mathematical inquiry in kindergarten as dialectical processes. The kindergarten teachers play significant roles for creating mathematical inquiry in kindergarten. They are central sources to cultural ways of thinking mathematically, and to cultural ways of collaborating. However, children's contributions are also crucial. This research study shows how children are able to take responsibility for moving mathematical activities forward and how they guide kindergarten teachers in mathematical inquiry activities. Both kindergarten teachers and children are mutually dependent on each other to carry out mathematical inquiry activities, and thus both are teachers and learners of each other. Moreover, the results indicate that mathematical inquiry in kindergarten

must be regarded as 'ethical' (Radford, 2008b) processes. Ethical considerations as trust, responsibility and respect lay the ground for the mathematical inquiry segments examined in this study. Based on the above findings, this research study argues for an ethical, multimodal and dialectical conception of mathematical inquiry in kindergarten. Ethical inquiry may be accomplished on the basis that children find it meaningful to be with others in mathematical activities and to solve problems together with others.

The research study further suggests that these processes of mathematical inquiry may prepare 5-year-old children for school. The results show that ethical inquiry, which challenges the children to argue for and explain their ideas, may, I hold, lay the ground for children to become critical and autonomous thinkers and problem solvers. Through ethical inquiry children can learn to collaborate, which includes respecting each other, trusting each other and working together to accomplish a task, and may help children to adjust to the new school context.

This research study illustrates that it is important that kindergarten teachers facilitate children's opportunities to solve mathematical problems together with others in a variety of settings and with a variety of available artefacts. This research study also shows that it is important that kindergarten teachers pay attention to children's contributions (verbal and non-verbal) and try to understand their perspectives, that is, position themselves as learners and let the children guide them in the activities. This may promote children to contribute with their ideas and explanations and to take responsibility for carrying out the mathematical learning activities. Moreover, this research study emphasises that KTs may benefit from being consciously aware of the affect their bodily actions have on children's mathematical reasoning and for engaging them in mathematical discourse without having to 'teach' (i.e., tell) them mathematical concepts and relations.

Sammendrag

Denne avhandlingen rapporterer fra en studie av femåringer som arbeider med matematiske problemer i barnehagen. Studien er en del av et større forsknings- og utviklingsprosjekt kalt Agderprosjektet. Prosjektet har som mål å undersøke effekten av et førskoleopplegg i Norge som fokuserer på sosiale ferdigheter, selvregulering, språk og matematikk. Studien, som er rapportert her, er designet som en kasusstudie der fem barnehagelærere og deres førskolebarn fra Agderprosjektets fokusgruppe deltok. For å undersøke prosessene som foregår når førskolebarn og barnehagelærere undersøker og løser matematiske problemer, benytter studien et sosiokulturelt perspektiv på læring og utvikling og bygger særlig på Radford's (2013) teori om 'kunnskapsobjektifisering'. Studien anvender en kvalitativ tilnærming til datainnsamling og dataanalyse, og det empiriske materialet ble samlet inn gjennom etnografiske feltnotater og ved observasjoner og intervjuer som ble lagret som videoopptak.

Resultatene fra studien belyser betydningen av å se på de matematiske problemløsningsprosessene som foregår i barnehagen som multimodale prosesser. Både barnas og barnehagelærernes bidrag til prosessene er i stor grad basert på andre modaliteter enn språk. I segmentene som analyseres i denne studien, bidrar noen barn med ganske sofistikerte argumentasjoner og forklaringer, men deres matematiske resonnementer er i stor grad materialisert gjennom gester og andre kroppslige handlinger. Videre benytter barnehagelærerne seg av multimodale spørsmålsstrategier for å engasjere barna i de matematiske aktivitetene. I stedet for å stille mange verbale spørsmål, veileder de barna ved hjelp av sitt kroppsspråk og tonefall og dermed gi barna rom til å bidra inn i aktiviteten. Videre viser resultatene hvordan barnehagelærerne og barna spiller på hverandres bidrag, noe som belyser betydningen av å se på de matematiske problemløsningsprosessene som dialektiske prosesser. Barnehagelærerne spiller en viktig rolle for å introdusere barna for matematiske begreper og problemstillinger. I tillegg er barnehagelærerne sentrale kilder til kulturelle måter å tenke matematisk på, og til kulturelle måter å samarbeide med andre på. Men barna er også vesentlige bidragsytere til de matematiske problemløsningsprosessene. Denne studien viser hvordan barna tar ansvar for å drive den matematiske aktiviteten fremover og hvordan de veileder barnehagelærerne i de matematiske aktivitetene. Slik sett er både barna og barnehagelærerne elever og lærere av hverandre. Dessuten indikerer resultatene at disse matematiske problemløsningsprosessene kan betraktes som 'etiske' prosesser, hvor samhørighet er en sentral komponent. Etiske hensyn som tillit, ansvar og respekt legger grunnlaget for de matematiske problemløsningsprosessene som er identifisert i

denne studien. Dette oppnås på grunnlag av at barna synes det er meningsfullt å være med andre i matematiske aktiviteter og å løse problemer sammen med andre.

Videre viser resultatene at de matematiske problemløsningsprosessene, som er basert på etiske prinsipper, potensielt kan hjelpe førskolebarn å tilpasse seg til den nye hverdagen de møter i skolen. Ved å delta i slike prosesser, som utfordrer barna til å forklare og argumentere for sine ideer, og som fokuserer på gjensidig respekt, tillit og ansvar, kan hjelpe førskolebarn til å håndtere fremtidige læringssituasjoner i skolen.

Denne studien illustrerer viktigheten av at barnehagelærere legger til rette for at førskolebarn får muligheter til å løse matematiske problemer sammen med andre i ulike situasjoner og ved hjelp av ulike artefakter. Denne studien viser også at det er viktig at barnehagelærere lytter til og imøtekommer barns bidrag, både verbale og ikke verbale, samt prøver å forstå barna sine perspektiver i matematiske problemløsningsaktiviteter. Barnehagelærerne kan med fordel forsøke å posisjonere seg som 'elev' i prosessene, og la barna veilede dem i aktivitetene. Dette kan understøtte at barna bidrar med sine ideer og forklaringer, samt at de villig tar delansvar for å drive de matematiske læringsaktivitetene fremover. Videre understreker denne studien at barnehagelærere bør være bevisst på hvordan de påvirker barnas matematiske tenkning med sitt kroppsspråk, og at de kan engasjere barna i matematiske samtaler uten å fortelle dem matematiske sammenhenger.

Resultatene fra denne studien påpeker kompleksiteten i å engasjere barn i matematiske problemløsningsaktiviteter i barnehagen. Dette understreker også viktigheten av å sikre høy kvalitet på barnehagelærerutdanningen i Norge og i læringsfelleskapene blant de voksne i barnehagene. Det er viktig å legge til rette for god utdanning av barnehagelærere, som retter oppmerksomhet mot noen av hovedpunktene som kommer frem i denne studien.

List of contents

1 Mathematical inquiry in kindergarten: an introduction

Observing young children exploring mathematical relationships in free play activities, it seems at first glance as if learning mathematics is as natural as breathing. Without formal instruction, it seems as if children all by themselves discover mathematical relationships in the environment and solve mathematical problems, just by help of their natural curiosity. However, if we take a closer look, this is not necessarily the case.

In the last year of kindergarten children stand on the border to a quite different tradition in school. They are in an institution where play, care and upbringing are the main focuses, and are about to enter an institution which focuses more on their academic development. Many children that enter first grade are only 5 years old, and research has shown that the transition between kindergarten and school is challenging for many children (Lillejord, Borte, Halvorsrud, Ruud, & Freyr, 2017; OECD, 2017; Peters, 2010). A smooth transition between kindergarten and school is decisive for how children will manage school both socially, emotionally and cognitively. To prepare for a smooth transition it is not enough to take into consideration the context and demands that the children are about to meet, it is also important to bear in mind the tradition that they come from and to consider the ways in which young children learn and develop.

This research study investigates the processes that unfold when 5-year-old children are introduced to and engaged in mathematical activities that aim to prepare them for school. This study therefore contributes to an understanding of what mathematics is for young children, how they engage in mathematical activities, and what can be expected, or not expected, from these young children that are about to enter school. In particular, this research study examines the processes that unfold when 5-year-old children together with their kindergarten teacher (KT) inquire into and solve mathematical problems, and therefore contributes to the literature on mathematical inquiry as a theoretical construct.

In this introductory chapter the background and the rationale for the present study is elaborated in Section 1.1, before the aims and research issues are presented in Section 1.2. The chapter ends with an overview of the thesis in Section 1.3.

1.1 Background and rational for the study

This research study on processes of mathematical inquiry in kindergarten is situated within a research and development project called the Agder

Project (AP)[1]. The AP aims to develop an intervention programme that prepares Norwegian 5-year-olds for school and to investigate the effects of the programme. In addition, the AP aims to investigate the processes of teaching and learning that unfolds when the participating KTs implement the mathematical activities designed in the project.

The intervention programme in the AP aims to prepare children for school, and simultaneously fit the Norwegian kindergarten tradition which emphasises care, play, upbringing and learning. Two main design principles were used to design the mathematical activities: inquiry and playful learning. The term playful learning aims to merge play and learning and takes into consideration that for a child play and learning are two sides of the same coin (Hirsh-Pasek, Golinkoff, Berk, & Singer, 2009). Inquiry is seen as "a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (Wells, 1999, p. 121). Inquiry and playful learning[2], were used as design principles in the AP with the intention to make the mathematical activities meaningful, exiting and engaging for the children and to encourage the children to work together to solve mathematical problems. The participating KTs were supposed to guide the children into meaningful problems by asking questions and presenting the mathematical content in an exciting way.

The mathematical activities in the AP were designed in a collaboration between the researchers in the project, including myself, and the participating KTs during the developmental phase of the project. Playful learning seemed to be a familiar construct to the KTs and they had few problems to understand the pedagogical principle. Inquiry on the other hand, which has emerged as an educational approach in school, was a novel construct to the KTs and a construct they had to be familiarised with. The construct arguably fits the kindergarten context well because it coincides with children's 'natural curiosity' and with the practice of Norwegian kindergartens where children and adults frequently interact in play and other practical activities (Breive, Carlsen, Erfjord, & Hundeland, 2018). The meaning of inquiry as an approach to teaching[3] and learning of mathematics in kindergarten was carefully discussed

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[1] The research design, and content in the AP will be elaborated in Chapter 5

[2] The terms playful learning and inquiry will be further elaborated in Section 2.3 and in Chapter 3 respectively.

[3] In the Norwegian kindergarten tradition, the term teaching is rarely used for the kindergarten teachers' practice. Teaching is associated with school where the teacher 'teaches' the students. In kindergarten the children are guided into learning activities, and some instruction occurs. In this research study the terms teaching and learning are used in line with Vygotsky's construct 'obuchenie' which indicates that there is a dialectic relationship between teaching and learning, and where the participants are teachers and learners of each other.

with the KTs in the project and the mathematical inquiry activities were designed to fit the Norwegian kindergarten context.

Despite the collaboration between the KTs and the researchers in trying to make sense of adopting an inquiry approach to mathematics in a kindergarten setting, some issues arose. During the design process and the implementation of the activities a contradiction between the theoretical conception of inquiry and the empirical experiences from practice emerged. Some of the KTs expressed that they somehow understood the idea, theoretically, but it was difficult to implement. In addition, I experienced a contradiction between my conception of the construct and how it unfolded in real life when the KTs implemented the pedagogical principle in their kindergarten groups. A need for practical understanding of inquiry as an educational approach in kindergarten arose.

There are several possible reasons for why these contradictions arose. First, the approach to inquiry used in the AP draws on conceptions of inquiry developed by scholars who have investigated processes of inquiry at primary school level (Wells, 1999) and at primary and secondary school level (Jaworski, 2005). In addition, the term inquiry as we know it today, emerged as a reaction against 'traditional' teaching methods, and is, perhaps, treated a bit ideally. There is a latent meaning underneath the concept, I believe, that if students or children are given agency and are allowed to, they will (automatically) investigate problems because it awakens their 'natural curiosity' and their desire to know and make sense. However, as Lipman (2003) argues, to make change requires theory, but it is worthless without connection to real life and practical understanding. It is exactly this tension between theory and practice that has been the source of my desire to understand and thus investigate inquiry as a phenomenon in kindergarten. The rational for the study is therefore to narrow the divide between the theoretical conception of inquiry (the ideal form) and the empirical experiences from practice (the real form) and to comply with Wells' (1999) request to investigate what inquiry might be at all levels of education.

This research study contributes to the substantial literature that exists on inquiry related to mathematics education. However, the literature on mathematical inquiry in kindergarten from a dialogical[4] perspective is sparse, at least literature which try to characterise the nature of inquiry in line with Wells' (1999) conception of the term. There is, however, research on, for example, argumentation and problem solving in kindergarten (e.g., Dovigo, 2016; Tsamir, Tirosh, Tabach, & Levenson, 2010), which are important elements of inquiry. The focus of my study diverges

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[4] The dialogical approach to inquiry is based on a sociocultural perspective and will be elaborated in Section 3.2.

from the other literature on inquiry in the sense that it focuses on mathematical inquiry in kindergarten, where the mathematics is not as formalised as in school and the approach to education is different. The aim is to disclose how we may understand inquiry in the kindergarten context and to get insight into the processes of interaction, and thinking, that unfolds when young children, together with their KT, inquire into and solve mathematical problems.

1.2 Aims and research issues

As indicated above, a purpose of this research study is to bring forth new practical understanding about inquiry as a theoretical construct. Since inquiry is closely linked to issues about learning, the research study also aims to reveal important aspects of mathematical teaching-learning processes in kindergarten. Through inquiry and problem solving the participants, I posit, encounter historically constituted cultural forms of reasoning and co-create new knowledge (Radford, 2013b). A study of inquiry processes will, hopefully, illuminate aspects that influence teaching-learning processes and contribute to the overall understanding of mathematical teaching and learning in kindergarten. Moreover, the purpose is to investigate processes and illuminate issues that arise when young children are introduced to and engaged in mathematical activities that aim to prepare them for school. Thus, a purpose of the study is to contribute to an understanding of what mathematics is for young children, how they engage in mathematical activities, and what can be expected, or not expected from these young children that are about to enter school. From the background of this the following research issues were formulated to guide the study:

- What characterises processes of mathematical inquiry in kindergarten?
- What enables processes of mathematical inquiry to occur?
- How, if at all, do these processes of mathematical inquiry prepare kindergarten children for school?

To deal with the empirical material the two former research issues were operationalised into specific research questions or aims in each of the papers accompanying this thesis. The latter research issue arose during the research process and is seen as an exploratory research issue. The aim is to use results emerging from the two former research issues and consider them in light of literature on important aspects for a smooth transition between kindergarten and school.

1.3 Overview of the thesis

Chapter 2, which is titled 'Early childhood mathematics education' (ECME) describes the theoretical foundations for understanding the context in which this research study is situated within and consists of four sections. Section 2.1 considers how ECME is influenced by social and political trends. Section 2.2 presents the Norwegian kindergarten tradition and discusses how this tradition stands in contrast to the school tradition. Section 2.3. provides insights into some of the literature on play which is central in kindergarten and discusses the link between play and learning. Section 2.4. considers issues associated with the transition that young children make from kindergarten to school, and how kindergartens and schools can facilitate a smooth transition. Section 2.5 is devoted to a literature review on mathematics teaching and learning in kindergarten and gives insight into the origin of children's mathematical reasoning (Sub-section 2.5.1), and then elaborates on early arithmetic learning (Sub-section 2.5.2).

Chapter 3 is devoted to literature on inquiry and consists of four sections. Section 3.1 gives a historical background for how we understand the term inquiry today. Section 3.2 considers literature on inquiry which comes from the sociocultural, dialogical tradition, which was taken as a point of departure for investigating inquiry in kindergarten. In Section 3.3 literature on some of the main features found in the dialogic approach is elaborated.

Chapter 4 elaborates on the cultural-historical perspective which is used as a theoretical lens to study processes of mathematical inquiry in kindergarten. Section 4.1 provides the rational for using a cultural-historical perspective to study processes of mathematical inquiry in kindergarten, and Section 4.2 is devoted to an elaboration of the cultural-historical perspective used in this research study. Section 4.2.1 is particularly devoted to Vygotsky's well-known concept, the zone of proximal development, which is a central construct used in this research study.

In Chapter 5 the Agder Project, which is the setting of this research study, is described. Section 5.1 describes the research design of the Agder Project and provides a timeline which gives insight into all the phases of the project. Section 5.2 describes the pedagogical foundations and the mathematical content used to design the mathematical activities in the project.

Chapter 6 elaborates on the methodological considerations used in this research study. Section 6.1 describes the research paradigm and the research strategy used in this research study, and Section 6.2 elaborates on the case study design of this research study. In Section 6.3 the pilot study and experiences from it are elaborated. Section 6.4 provides background information about the five KTs participating in the case study.

Section 6.5 and 6.6 elaborate on the processes of data collection and data analysis, respectively. And Section 6.7 and 6.8 consider the trustworthiness of the research study and ethical considerations when studying 5-year-old children, respectively.

Chapter 7 provides a summary of the five papers produced during this research study.

Chapter 8 is devoted to a synthesis and a discussion of the findings which leads towards a conclusion about the characteristics of mathematical inquiry in kindergarten and some of the issues that arise when children are engaged in mathematical activities aimed to prepare them for school. Section 8.1 makes a synthesis of the findings of the five papers and discusses, in particular, children's contributions to the activities identified as inquiry (Sub-section 8.1.1), the KT's contributions to the activities (Sub-section 8.1.2), and the dialectical relationship between the KT's contributions and the children's contributions and the co-creation of the zone of proximal development (Sub-section 8.1.3). In addition, Sub-section 8.1.4 considers children's mathematical reasoning, and how their mathematical thinking is materialised in the joint activities. These considerations lead to a conclusion in Section 8.2 which focuses on the characteristics of mathematical inquiry in kindergarten, what enables these processes to occur and how, if at all, these processes of mathematical inquiry can prepare kindergarten children for school. And finally, Section 8.3 is devoted to critical reflections and implication of the research study. The section focuses particularly on what contributions this research study makes to the Agder Project (Sub-section 8.3.1), implications for practice and further research (Sub-section 8.3.2 and 8.3.3 respectively) and critical reflections the ZPD concept (Sub-section 8.3.4).

2 Early childhood mathematics education

Early childhood mathematics education (ECME) often refers to preparing and creating learning activities for young children (usually 3-6 years old) aimed at stimulating the development of mathematical knowledge (Van den Heuvel-Panhuizen, & Elia, 2014). Depending on the educational system in the country, ECME usually takes place in public or private preschool centres or kindergartens, but it may also include the informal education that takes place in children's home environment.

In this chapter, issues about how ECME is influenced by social and political trends is discussed in Section 2.1. Further, a brief description of the Norwegian kindergarten tradition is given in Section 2.2 and considerations about play and learning are provided in Section 2.3. Section 2.4 is devoted to considerations about the challenging transition between kindergarten and school and Section 2.5 provides a review of literature on mathematics teaching and learning in early childhood.

2.1 Early childhood mathematics education in light of social and political trends

ECME has gained increasing attention from researchers, educators, and policy makers both in Norway and around the world at least since the onset of the 21st century. The interest in ECME follows social and political trends. Today we live in a so-called 'global, knowledge-based economy' (Bereiter, 2002), where mathematics is argued to be important for all members of society to learn (Clements, Baroody, & Sarama, 2013; Ministry of Education and Research, 2015). The 'global, knowledge-based economy' is in a constant need for a mathematical literate workforce. This is recognised by policy makers around the world. These social and political challenges and trends create demands on the educational system. In Norway natural science and mathematics are regarded as key content areas in school and recently also in kindergarten. In the strategy plan 'tett på realfag', the Ministry of Education and Research (2015) regards natural science and mathematics as key components needed to manage environmental and technological challenges that the society is faced with. The aim of the strategy plan is to increase the focus on natural science and mathematics in kindergarten, primary and secondary level of education (Ministry of Education and Research, 2015). However, many students experience difficulties with mathematics as a school subject and lack of mathematics learning is therefore a major cause of poor progression through the school system. There is a continuing need, nationally and internationally, to explore how quality of mathematics education at all levels may improve.

Early mathematics learning is argued to provide an important foundation for later mathematics learning is school (Baroody & Purpura, 2017; Duncan et al., 2007). The informal mathematics that young children experience at home and in kindergarten serve as an important basis for later formal, symbolic, mathematical experiences. Learning difficulties in school is often related to the gap between informal experiences and formal instruction, where children with rich mathematical experiences before they enter school have advantages compared with children who do not have as rich experiences with informal mathematics (Baroody & Purpura, 2017). Research has shown that early mathematical knowledge is the strongest predictor not only for later mathematical achievement but also for achievement in other content areas in school (Claesens & Engel, 2013; Duncan et al., 2007), and for long-term success in education and in professional life (Duncan et al., 2007; Entwisle, Alexander, & Olson, 2005).

There are huge individual differences in informal mathematical experiences and knowledge among young children. The development of early mathematical knowledge is related to children's socioeconomical background where children who come from disadvantaged families seem to be at risk of falling behind already at an early age (Starkey, Klein, & Wakeley, 2004). However, it is not the background itself that seems to be the cause, rather to what extent parents engage in mathematical-related activities and talk with their children (Anders et al., 2012; Ramani, Rowe, Eason, & Leech, 2015). In an attempt to close the gap in young children's experiences with mathematics, both policy makers, educators and researchers have increasingly viewed organised early mathematics education as an important means for that purpose, and efforts have been made to develop and evaluate kindergarten curricula (Clements & Sarama, 2007b; Clements, Sarama, Wolfe, & Spitler, 2013; Lewis Presser, Clements, Ginsburg, & Ertle, 2015). This is also one of the aims in the AP: to develop an intensive preschool intervention programme for 5-year-olds and investigate effects of the programme and whether the curriculum may contribute to decrease the gap between children of advantaged and disadvantaged families.

2.2 The Norwegian kindergarten tradition

In Norway the term kindergarten refers to private or public childcare centres which give institutional care and education for young children from 10 months to 6 years of age. The last year of kindergarten, when children are about 5-years-old (more precise 4.5 – 6.5 years), the year before they enter school, is also sometimes referred to as preschool. Kindergartens in Norway are situated within a social pedagogical tradition, where the core enterprises are upbringing, care, play and learning. The

OECD has given positive evaluations of the Nordic approach to learning in kindergarten (OECD, 2001, 2006), which is more holistic than approaches in, for example, the UK or France which have a stronger emphasis on 'readiness for school', meaning that children are engaged in academic activities similar to those found in school. In Norway kindergartens are viewed as a place where children become 'prepared for life' in a broader sense.

In 2005, the responsibility for the Norwegian kindergartens was transferred from the Ministry of Children and Family Affairs to the Ministry of Education and Research, and thereby kindergartens became part of the formal educational system in Norway. This resulted in an increased focus on children's academic development, which is evident in the Framework Plan for contents and tasks in kindergartens (the curriculum) from 2006 (Ministry of Education and Research, 2006). In this version of the Framework Plan, mathematics was included as one amongst seven subject areas to be implemented in kindergartens. However, there is still a huge emphasis on safeguarding the institution's uniqueness, which focuses on care and security for children, as well as children's well-being and need for self-organised play. Norwegian KTs are trained to attend to the developmental needs of the children, and young children's physical and emotional development is at least as important as their academic development.

In 2017, 97% of all 3-5-year-old children in Norway attended kindergarten (Statistics Norway, 2019), which means that it is an ideal place to equally prepare children for school. The Norwegian Framework Plan is quite general and gives the KTs opportunities to adapt their teaching to children's needs, and to give most attention to the children that need it the most. This freedom may also be challenging for the KTs, because it requires them to be independent when they read and implement the ideas in the Framework Plan. However, research shows that there are huge differences in the way that the Framework Plan is implemented in Norwegian kindergartens (Østrem et al., 2009).

The kindergarten tradition in Norway (as in the rest of Europe) has its origin in the Enlightenment period which is also called the Age of Reason. The period was centred around ideas from several French philosophers, among them Rousseau, where people should question authorities and 'think for themselves' (Saracho & Spodek, 2009a). Friedrich Fröbel (1782–1852) may be regarded as the founder of the pedagogical institution called kindergarten. In 1837 he founded a care, play and activity institute for small children in Germany, which he later named 'kindergarten' and which literally means 'garden for the children'. He believed that children should be "nurtured and nourished like plants in a garden" (Kindergarten, 2019). Today, the term kindergarten is used in many

countries to describe a variety of educational institutions for young children.

Although we today consider Norwegian kindergartens to be situated within a social pedagogical tradition, the ideas from its origin are still somehow recognisable. The Norwegian kindergarten tradition stands in contrast to the school tradition that we know today, which arose in the early twentieth century, and which aimed to prepare students for the industrialised economy that arose at that time (Sawyer, 2005). At that time educators had limited knowledge about what learning was and the schools were designed around the assumption that knowledge was facts that the students should acquire. The teacher's mandate was to transmit these facts to the students, and the students should get these facts into their brains. To make sure that the schools worked properly the students were tested to see how many of the facts that the students had acquired. This way of teaching is referred to as 'instructionism' (Sawyer, 2005), and gave rise to what we today would call the 'traditional' teaching approach.

Although Norwegian kindergartens became part of the formal education system in 2005, they should continue to emphasise care, play and children's social-emotional development. On the one hand, KTs are expected to prepare children for school, and on the other hand safeguard the unique pedagogical approach which emphasises children's opportunities for self-organised play. These expectations may create tensions, but they may also create opportunities. There is an increasing focus on issues related to young children's transition from kindergarten and school (Lillejord et al., 2017; OECD, 2017). While most children experience no problems in the transition, some children find the transition difficult. Continuation in curriculum and pedagogy is argued to be one of the main factors for facilitating a smooth transition (Lillejord et al., 2017; OECD, 2017). But research also indicate that the responsibility for securing continuation in curriculum and pedagogy lays equally on kindergarten and school (OECD, 2017). Research on the transition between kindergarten and school, and on tensions that may arise between the two traditions, will be further elaborated in Section 2.4.

2.3 Play and learning in Norwegian kindergartens

As mentioned above, play is one of the main foci in the Norwegian kindergarten tradition, and where children's free play, or self-organised play, is particularly valued. This research study does not consider play in particular, but since play is an essential construct and part of the pedagogical foundation in the AP, a brief outline of play is therefore appropriate.

Vygotsky and Leont'ev considered play to be the 'leading activity' for young children (Veresov & Barrs, 2016), and regarded it as essential for children's social-emotional as well as cognitive development. In play, Vygotsky argues,

> a child is always above his average age. Above his daily behaviour; in play it is as though he were a head taller than himself. …; in play it is as though the child is trying to jump above the level of his normal behaviour" (Vygotsky, 2016, p. 18).

In play children can imagine things that they cannot do in real life and follow rules and structures that they find difficult in real life. Play, Vygotsky (2016) argues, "must always be understood as the imaginary, illusionary realisation of unrealisable desires" (p. 7). The interconnection between the imagined situation and the rules emerging from it is why play is such a fruitful space for development. Vygotsky explains this with two paradoxes of play. In play children can, without knowing it, separate meaning from an object, that is in the imaginary play the child can separate the visual field from the field of meaning, which the young child cannot do in real life. Second, children play because it is associated with pleasure. In play the child follows his/her own wishes and desires, and pleasure is the line of least resistance. But to gain maximum pleasure from the play the child must follow the rules of the play, which is the line of greatest resistance. The child must regulate spontaneous impulses to follow the maximum path to pleasure. Vygotsky compared the play-development relationship with the instruction-development relationship and argued that play is the source of development because it creates the zone of proximal development. However, "play provides changes in needs and in consciousness of a much wider nature" (Vygotsky, 2016, p. 18), than the instruction-development relationship.

Vygotsky (2016) talks about imaginary, self-organised play, where the child is not guided by an adult and where the goals emerge from the play itself and not initially set up by an adult. Today, there seems to be an agreement among researchers that play and learning goes hand in hand (in line with Vygotsky's idea), but additionally, activities where adults are involved and guides the play may also be considered as play. However, there is disagreement upon the degree of adult involvement in an activity before it must be seen as a traditional teaching practice rather than play. Winther-Lindqvist (2017) argues that

> One can disguise a didactic activity as a form of play by tapping into a typical play format, but if the activity has a narrowly defined end goal decided upon beforehand by the adults, it loses a central characteristic of what play is: open ended (without specific goals) and spontaneously generated (p. 6).

Although Vygotsky considered imaginary, self-organised play as the leading activity for kindergarten children, he argued that play did not disappear even in adult-led activities in school.

In school age play does not die away, but permeates the attitude toward reality. It has its own inner continuation in school instruction and work (compulsory activity based on rules). All examinations of the essence of play have shown that in play a new relationship is created between the semantic field – that is, between situations in thought and in real situations" (Vygotsky, 2016, p. 20).

As mentioned above, self-organised play is highly valued in the Norwegian kindergarten tradition, however learning of academic knowledge has gained increasing focus the past decades. Along with self-organised play, play-based curriculum in early childhood education is argued to be the basis for 'lifelong learning' (Fisher, Hirsh-Pasek, & Golinkoff, 2012; Hirsh-Pasek et al., 2009; Pellegrini, 2011).

The pedagogical principle 'playful learning', used in the AP is elaborated by Hirsh-Pasek et al. (2009), as a concept which aims to capture the interrelationship between play and learning in childhood. Playful learning encompasses both free play (child-initiated and child-directed play) and guided play (adult-initiated and child-directed play). In free play children both initiates the play and direct the movement and direction of the play themselves without interference from adults. In guided play the KT initiates the play and organises the environment and guides the play in specific directions with respect to certain learning goals. For playful learning to take place, the children must be physically and cognitively engaged and the KT must be sensitive to children's interests and participation. It is important that the KT makes room for children´s self-directed explorations. "Playful learning, and not drill-and-practice, engages and motivates children in ways that enhance developmental outcomes and lifelong learning" (Hirsh-Pasek et al., 2009, p. 4). Weisberg, Kittredge, Hirsh-Pasek, Golinkoff, and Klahr (2015) emphasise that it is the balance between structure and freedom that makes guided play suitable for learning of academic skills. Similarly, van Oers (2014) argues that it is the balance and variation between instruction and self-directed exploration that is important for learning. He argues that playful learning activities should contain some elements of instruction, but children's self-directed play should always be the starting point. "The nature of the actions embedded in play can vary with respect to their degree of freedom allowed, as long as the activity as a whole remains a playful activity" (van Oers, 2014, p. 121).

Self-organised play is valued by many, not only for the individual child's development, but also for the development of democracy. Winther-Lindqvist (2017) argues that children's opportunities for self-organised imaginary play is a perfect arena for practicing important competences in a democracy, like autonomy and solidarity. In the Nordic countries these competences are also valued in the educational system and therefore Winther-Lindqvist (2017) argues, play should play a great role in early childhood and primary education.

2.4 The challenging transition from kindergarten to school

As indicated in the introduction, the transition from kindergarten to primary school is a big step for children. Most children manage this transition well, but for some children the transition is tough and may lead to anxiety and other social-emotional problems (Lillejord et al., 2017). Research indicate that high-quality early childhood education is important for children's early development as well as for their long-term success in school, education and professional life (Duncan et al., 2007; Entwisle et al., 2005; OECD, 2006, 2011, 2015). However, research has found that some of the positive effects of high-quality early childhood education may fade in primary school if the transition between kindergarten and school is not well prepared (Lillejord et al., 2017; OECD, 2017; Peters, 2010). And this often affects children from disadvantaged backgrounds more than children from advantaged backgrounds (OECD, 2017). High quality early childhood education and well prepared transition activities between kindergarten and school are important aspects of children's experiences of the transition, and for how they will manage school, integrate in the society and make a successful transition into the labour market (Lillejord et al., 2017; OECD, 2017).

Several researchers argue that the transition between kindergarten and school must be seen as a process over a longer period of time, and not just as a single event (Chan, 2012; Lillejord et al., 2017; Peters, 2010). The transition is not only a physical move from kindergarten to school, it is also a longer social-emotional process. Kindergarten and school are two different educational practices. The school has different expectations and puts new demands on the children. In school the days are often longer, the classes are bigger, and classrooms are smaller. This often result in a pedagogy including more seat-work, less free time, and fewer child-directed activities (OECD, 2017). Children have to get used to new rules, new adults, new peers and new activities. Children become gradually aware of the transition, and often much earlier than the transition itself. Some children need a longer time to adjust than others, both before and after the physical move. As Lillejord et al. (2017) argues, the children are not only entering a new environment, they are also leaving something behind. "The children are not just becoming pupils – they also have to get used to no longer being kindergarten children" (ibid., p. 22).

There are several intertwined aspects that are important for making a smooth and successful transition between kindergarten and school. For example, establishing a collaboration between kindergarten and school, where they jointly create pedagogical transition activities, not only academic activities, which familiarise children with the school context and help them settle in (Lillejord et al., 2017; OECD, 2017; Peters, 2010). It

is also important to include parents, and establish good home-school relationships (OECD, 2017; Peters, 2010). However, research points to continuity in curriculum and pedagogy in kindergarten and primary school as one of the main aspects for making a smooth transition (Lillejord et al., 2017; OECD, 2017).

Although the pedagogy in kindergarten and in primary school increasingly have become more aligned (OECD, 2017), it is mostly the kindergarten curriculum that has changed (Lillejord et al., 2017). In their reports, 'Starting Strong', OECD (2006, 2011, 2015) point to a policy trend of focusing on 'school readiness'. As Lillejord et al. (2017) argues 'school readiness' is a vague term which has its origin in the Anglo-American areas in the US where children may start school one year later if they fail a 'school readiness test'. Although the Norwegian kindergarten tradition is based on a social pedagogical tradition and a more holistic view on early childhood education, there is an increasing trend from policy makers on making children 'ready for school'. Making children 'ready for school' often involves children in kindergarten being engaged in academic activities similar to those found in school.

However, OECD (2017) argues that the responsibility for facilitating a smooth transition rests on both kindergarten and school. There is a need for 'pushing up' play-based curriculum and pedagogical principles from kindergarten to school. It is not only the kindergarten that must prepare children for school, the schools must also be prepared for the children, which is one of the main 'key policy messages' from OECD (2017).

Lillejord et al. (2017) argue for developing a 'hybrid pedagogy' in both kindergarten and primary school, which involves an increased emphasis on academic learning opportunities in kindergarten and an increased emphasis on social-emotional and play activities in primary school. This would arguably ensure a continuity in curriculum and pedagogy, where pedagogical elements during the transition would remain the same, and the academic challenges gradually increase (OECD, 2017). However, there is a need, in most countries, for curricula and guidelines that reflects these combined pedagogical ideas (OECD, 2017). And there is also a need for a deeper understanding of the connection between play and learning (Lillejord et al., 2017).

The lack of pedagogical understanding between kindergarten and school is one of the main obstacles for a good collaboration and continuity in curriculum and pedagogy between the two traditions (OECD, 2017). As mentioned in Section 2.2, the kindergarten tradition and the school tradition have different origins and thus quite different approaches to education. Lillejord et al. (2017) argues that tradition - in itself - is not necessarily a problem, but it may cause problems if the KTs

and teachers in school are not aware of these differences and if they do not take into account what consequences it has for the children.

Although research suggests that continuity in curricula and pedagogy between kindergarten and school has positive effects on children's experiences of the transition and for later development, research on the impact of continuity curriculum and pedagogy is sparse (OECD, 2017). In addition, there is little research on characteristics of instruction which will insure a continuation in pedagogy and content between kindergarten and school. And there is a need for research on the impact of play-based curriculum, both in kindergarten and school, for children's experiences of the transition (Lillejord et al., 2017).

As mentioned in the introduction, the research study reported here is situated within the AP which aims to develop a school readiness intervention programme aimed to prepare children for school. The programme focuses on social-emotional skills, self-regulation, mathematics and language which are argued to be the most important content areas in a transition curriculum (Lillejord et al., 2017; OECD, 2017). Inquiry was used as a pedagogical approach in the mathematical activities and was argued to fit the kindergarten context as well as challenge children's mathematical thinking. This research study focuses on the processes that unfold when KTs and children inquire into mathematical problems in the context of these school readiness activities. One of the aims in this research study is to illuminate aspects of how, if at all, these processes prepare children for school and facilitate a smooth transition.

2.5 Research on mathematics teaching and learning in early childhood

How young children develop mathematical knowledge has been and still is an important issue for developmental psychology across the world. Therefore, a lot of the research on children's early mathematics learning is founded on an interest to understand the relationship between the child's acquisition of mathematical concepts and his or her psychological and cognitive development (Sophian, 2007). This does not always serve research on *how* children learn mathematical concepts and relations well.

This section is divided in two sub-sections. Sub-section 2.5.1 considers the origin of mathematical reasoning in early childhood and Sub-section 2.5.2 provides literature on the development of children's additive and multiplicative reasoning.

2.5.1 Origin of mathematical reasoning in early childhood

There has been a change in how researchers view children's abilities to learn mathematics, from believing that children had limited capacity to learn mathematics (Piaget, 1952; Thorndike, 1922) to contemporary re-

searchers recognising that children are able to learn substantial mathematical ideas from a very young age (e.g., Baroody, Lai, & Mix, 2006; Clements & Sarama, 2014; Perry & Dockett, 2002). My research study investigates 4-6-year-old children, where they have normally already acquired some mathematical knowledge. The study therefore does not focus on the beginning of mathematical learning in infancy. However, a glimpse into the research on the beginning of mathematical learning is important for understanding the further development, and what should be emphasised in ECME. In this section I limit myself to a brief overview of the literature on the beginning of children's mathematical learning. Further reading of this topic can be found in (Baroody & Purpura, 2017; Clements & Sarama, 2007a; Sophian, 2007). After a brief introduction of the first learning of mathematics I focus on children's learning of addition, multiplication, which is more relevant for this thesis.

There are different views on what kind of mathematics is the basis for children's mathematics learning. The majority of researchers focusing on ECME claim that numeracy[5] is the foundation for children's further mathematical learning (see overviews in Baroody & Purpura, 2017; Clements & Sarama, 2007a), others claim that measurement is the foundation for early childhood mathematics learning (Davydov, 1975; Sophian, 2007). Sophian (2007) discusses these two perspectives on the development of mathematical knowledge in childhood, which she calls the 'counting-based position' and the 'comparison-of-quantities position' respectively. These two strands represent different views on how mathematics should be introduced to young children and have implications for the further teaching of elementary mathematics.

Scholars representing the 'counting-based-position' have developed hypothetical learning trajectories for how children develop 'early number sense' (including recognition of numbers, counting, numerical relations and operations on number), which is argued to serve as a foundation for further mathematical learning, (Baroody & Purpura, 2017; Clements & Sarama, 2014). These hypothetical learning trajectories do not necessarily follow a linear path but are combined and intervened in a complex network. Clements, Baroody and Sarama (2013) argue that children first work with small numbers (one to three items) within each step in the learning trajectory, and then gradually start working on larger numbers.

Common to all learning trajectories from the 'counting-based-position' is that 'verbal subitising' is the basis for children's further development of number sense. According to Clements (1999) subitising is "the direct perceptual apprehension of the numerosity of a group" (p. 2). For

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[5] 'Numeracy' refers here to knowledge about numbers and counting.

the counting-based position, verbal subitising is argued to be fundamental for understanding the meaning of the first counting words (Baroody & Purpura, 2017; Clements, 1999; Clements & Sarama, 2007a). Subitising is also seen as a basic ability for addition and subtraction and how small collections of objects (easily subitised) may be composed and decomposed into larger or smaller collections. Clements (1999) distinguishes between two types of subitising, namely 'perceptual subitising' and 'conceptual subitising'. Perceptual subitising is "recognising a number without using other mathematical processes" (ibid., p. 2) and is argued to be a natural ability of infants. Conceptual subitising is when a child (or adult) 'just know' that there are for example eight dots on a domino. One explanation is that they 'just see' that the domino is composed of two sets of four dots and know that this is a set of eight.

Some researchers argue that infants, as young as 5 months old, are able to recognise numerosities through perceptual subitising, that is to discriminate different collections of objects or different sequences of events (Starkey, Spelke, & Gelman, 1990; Wynn, Bloom, & Chiang, 2002; Xu, Spelke, & Goddard, 2005). However, these studies on infants' abilities to discriminate numbers have been criticised by many researchers (e.g., Mix, Huttenlocher, & Levine, 2002), who question whether children's abilities to discriminate between different collections of objects are, in fact, based on recognition of other characteristics by the arrays rather than the ability to recognise discrete quantity (i.e., number). As Mix et al. (2002) state

> the extant literature provides no clear-cut evidence that infants use number to perform quantitative tasks. Instead, new research suggests that quantification is initially based on nonnumerical cues, such as area and contour length, whether or not a task involves discrete items (p. 278, abstract).

For the 'counting-based position' number words and related quantitative terms like "more" are important for learning verbal-based number concepts (Clements et al., 2013, p. 6). Likewise, the issue of equal (or fair) sharing can help children understand key aspects of fractions, division, and even and odd numbers. In the number sense view these terms, 'more', 'less', 'equal' etc., are seen as important means for learning to compare quantities and to broaden children's understanding of numbers, that is develop their 'number sense'. But Sophian (2007) would rather regard these concepts as the foundation of early mathematics learning.

When Sophian (2007) discusses the two perspectives (the counting-based-position and the comparison-of-quantities position), she emphasises that a key for understanding the differences between the positions is to make a distinction between numerosity and quantity. A number, she argues, is a symbol or a word, and quantity is a property of things that exist in the physical world. Numbers are then abstract representations of a quantitative collection.

> *Numeracy* refers to knowledge about numbers (which may or may not be used to represent physical quantities). … *Quantitative knowledge* refers to knowledge about physical quantities and relations between them (which may or may not be represented numerically) (Sophian, 2007, p. 11).

From a 'comparison-of-quantities position' Sophian (2007) argues the most fundamental for children's mathematical development is relations between quantities (rather than counting and other aspects of numerosity). Ideas of *equality*, *less than*, and *greater than* are particularly important and argued to be fundamental concepts for children's mathematical development. These ideas can be used both on continuous quantities (like length or volume) and discrete quantities (like the amount of physical objects).

> Numerosity is just one dimension along which quantities can be compared. Others are length, area, volume, and mass. Similarly, counting is just one of a class of procedures for comparing quantities. Indeed, both discrete and continuous quantities can be compared without any reference to number (Sophian, 2007, p. 42).

In addition, in the comparison-of-quantities perspective, the understanding of the *unit*-concept is a prerequisite for understanding the number concept. In the comparison-of-quantities position counting is seen as a fundamental means for comparing quantities and is therefore a measurement tool. Measuring, including counting, needs a unit (either discrete or continuous units). If you are going to count socks you must decide whether a pair is the unit, or a single sock is the unit, and likewise if you are going to measure the amount of sugar, you must decide whether a spoon of sugar is the unit, or a cup of sugar is the unit. This also lays the ground for understanding 'higher order units' (for example the base 10 system), or for understanding how a unit may be divided into smaller units or parts (fractions). So, the numerosity, or the numerical measure you get when you count or when you measure things is not a function of its physical composition alone, but also a result of what you decide to be the unit (Sophian, 2007).

Although Clements (1999) realises that to recognise units is important for subitising, he does not emphasise this as a separate ability, it is rather an ability incorporated into perceptual subitising. "Perceptual subitising also plays an even more primitive role, one that most of us do not even think about because we take it for granted. This role is making units, or single 'things' to count" (ibid., p. 2). According to Clements (1999), children use perceptual subitising "to make units for counting and to build their initial ideas of cardinality" (p. 3).

Another study that gives rise to questioning the counting-based position was done by Gordon (2004). He studied the counting system of the Pirahã tribe in Amazonia. They use a 'one-two-many' system of counting. Quantities above two items were not counted, only represented as

'many'. In his study Gordon (2004) asked tribe members to solve some arithmetic tasks to see if they could represent the quantities by their fingers, instead of using words. The study shows that, although the tribe members were allowed to represent the quantities by their fingers, their finger representations were highly inaccurate. Above two they represented the quantity randomly by 3 to 8 fingers, which shows that above two, the tribe members were not able to separate items. And as the paper also indicates, perhaps they did not count items at all. The word 'hoi' which they used when they represented one or two items meant 'roughly one' or 'small'. Their mathematical thinking is not based on counting the way we understand counting. They have no need in their daily cultural activities to count more than two and have not developed this way of thinking. Even with some instruction, they were not able to count in the manner that some three-year-olds in Norway are able to. Counting, as any other mathematical thinking, is not 'natural', it is cultural.

2.5.2 Development of additive and multiplicative reasoning

Regardless of whether one agrees with the counting-based position or the comparison-of quantity position, children in our culture eventually start to learn addition and subtraction, multiplication and division. However, the two positions would argue for different ways to introduce young children to these mathematical concepts.

Many studies have categorised addition and subtraction, multiplication and division problems according to their semantic structure which refers to the way in which the problem is formulated, either in writing text or verbally, before the children start to solve it. The semantic structure of addition, subtraction, multiplication, and division problems and children's related problem-solving strategies have been the focus in substantial empirical researches in the past four decades (see e.g., Baroody and Purpura (2017) and Clements and Sarama (2007a) for extensive overviews on empirical research on early childhood mathematics teaching and learning).

Before children become fluent in using symbolic addition and subtraction strategies, they use other strategies, like counting. Some of the pioneers in the research on how children understand and solve addition and subtraction problems are Fuson (1988, 1992), Carpenter, Moser, and Romberg (1982) and Carpenter and Moser (1984). Later also researchers like, for example, Sarama and Clements (2009) and Thompson (2010) have contributed to the understanding of children's development of early arithmetic (see Baroody and Purpura (2017) for an overview). First, children seem to learn 'count all' strategies for solving addition problems. In the beginning these strategies are usually carried out with use of building blocks or other physical artefacts. For example, given a problem like 4 + 5, represented by building blocks, children may first count the items and

find the cardinality in the first set, "one, two, three, <u>four</u>", then count the items and find the cardinality in the second set, "one, two, three, four, <u>five</u>", and then count the items in the two sets together as 'a whole', "one, two, three, four, five, six, seven, eight, nine" and report "nine". Children may also use the 'shortcut-version' of this 'count all' strategy, which is to solve the problem 4 + 5 without counting each collection first, but start directly to count all, "one, two, three, four, five, six, seven, eight, nine" and report "nine". These strategies are eventually developed into other and more sophisticated strategies like 'count-on' strategies. For example, given a problem 4 + 2 children may take the number of items in the first set as a point of departure, and then count further on the other set of items "four, five, six" and report "six". An even more sophisticated strategy is to take the number of the largest set as a point of departure and count further on from that. Carpenter and Moser (1982) observed that, even if the children are capable of using the counting-on procedure, the children nevertheless often used the counting-all procedure. They suggest that children can be prompted to use the counting-on procedure if there are no physical artefacts available for the children to manipulate.

These counting based strategies evolve from concrete strategies to more abstract counting strategies (Fuson, 1988). As concrete strategies children more or less directly model the addition process by representing both addends by concrete objects and afterwards count the sum. Later children can indirectly model the addition process by representing both addends and figure out the sum simultaneously. For example, if children are supposed to solve 3 + 4 they may use their fingers and count "one, two, three" and then raise one and one finger and count "four, five, six, seven, there are seven". In this case the children must know when to stop counting because the fingers do not directly represent each of the addends. This is what Fuson (1988) called the 'numerical chain level', which is the insight that the counting words themselves can be counted. The strategy also requires the children to understand that 4, for example, is part of the total of 7.

Similar strategies are found for subtraction, although they are not as clearly defined as those for addition (Thompson, 2008). As with counting strategies for addition, the counting strategies for subtraction evolve from concrete strategies with available physical artefacts to more abstract counting strategies without available physical artefacts (Fuson, 1988). As with the learning of counting, children first learn to add and subtract with small numbers. Later they extend these solution strategies for addition and subtraction to larger numbers and usually also with the use of subitising (Clements & Sarama, 2007a).

To perceive relations between numbers, like part-whole relations, is fundamental for developing more sophisticated strategies for solving addition and subtraction problems with memorised number facts (Baroody & Purpura, 2017; Carpenter & Moser, 1984; Fuson, 1992; Sophian & McCorgray, 1994; Verschaffel, Greer, & De Corte, 2007). Sophian (2007) argues that "all addition and subtraction problems can be understood as a matter of either finding the whole, given the parts, or finding a part, given the other part and the whole" (p. 86). This relationship is important for interpreting the problems at hand and is a conceptual basis for understanding additive commutativity, associativity, and inversion.

Some number relations, like, for example 'number after relations' (add 1 combinations), may serve as a foundation for more advanced number facts. Moreover, doubling of numbers such as 3 + 3 or 5 + 5 are usually easily learnt because children experience these on for example domino pieces or from their hands (5 fingers on one hand plus five fingers on the other hand is 10 fingers together). A combination of both number after relations and doubling of numbers may be used on for example the problem 5 + 6, by decomposing 6 into 5 + 1 and then retrieve 5 + 5 = 10 and then use 'number after rule' for adding 1 to retrieve 10 + 1 = 11. Such combinations have been found to be more difficult to learn when related to subtraction than to addition and multiplication (Baroody & Purpura, 2017)

Although the body of research concerning children's development of additive reasoning (addition and subtraction) is more extensive than the body of research on multiplicative reasoning (multiplication and division) there is an increasing amount of studies in this domain too (e.g., Anghileri, 1989; Clark & Kamii, 1996; Greer, 1992; Kouba, 1989; Mulligan & Mitchelmore, 1997; Vergnaud, 1983). Multiplicative reasoning is traditionally considered as more complex than additive reasoning (Anghileri, 1989; Clark & Kamii, 1996; Greer, 1992; Mulligan & Mitchelmore, 1997). The semantic structure of the problems is one reason why multiplicative reasoning is more complex than additive reasoning. In addition-problems quantities of the same type are added, for example 3 apples + 2 apples equals 5 apples. In multiplication problems quantities of different types are involved, for example 4 baskets with 3 apples in each basket equal 12 apples altogether. Although multiplicative problems often involve different types of quantities, they may be solved with additive thinking, where repeated addition is one common way to solve such problems. Several researchers have classified multiplication and division problems related to their semantic structure (Greer, 1992; Mulligan & Mitchelmore, 1997; Vergnaud, 1983), and (at least) four different semantic groups relevant for kindergarten and early school-years have been found: 1) equal groups (e.g., 4 baskets with 3 apples in each),

2) array (e.g., 4 rows with 3 chairs in each) 3) rate (e.g., 1 basket of 3 apples cost 2 kroner, how much does 2 baskets cost) or 4) cartesian product (e.g., possible combinations of 2 shirts and 3 skirts).

The relation between the semantic structure of the problem and children's solution strategies have been investigated by many scholars (e.g., Anghileri, 1989; Kouba, 1989; Mulligan & Mitchelmore, 1997), however research also shows that it is difficult to make an exact categorisation and reach a complete understanding of this relation. A given multiplicative problem can give rise to various types of multiplicative thinking, and various solution strategies.

The development of strategies to solve multiplication and division problems run in parallel with children's development of strategies to solve addition and subtraction problems. In a similar manner to children's addition and subtraction strategies, children start solving multiplication and division problems by combining direct modelling with counting and grouping strategies (Anghileri, 1989; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Mulligan & Mitchelmore, 1997; Verschaffel et al., 2007). First, children usually use concrete 'count all' strategies, later they progress to strategies related to additive thinking like repeated addition, doubling, rhythmic counting and skip counting (or step-counting) and diverse additive calculation, before they start using multiplicative calculation. All these strategies develop from concrete modelling with physical tools to more abstract reasoning without modelling with physical tools.

Although many scholars regard repeated addition as a gateway into multiplicative reasoning, and which is also often how multiplication is introduced in school (Sophian, 2007), others argue that multiplication is best seen as many-to-one mappings. The necessity for making a many-to-one correspondence arises in situations like for example three cones have to be sorted in four boxes each. To practically map three cones in four boxes each, is not that difficult for children, but "anticipating the numerical consequences of a series of many-to-one mappings is substantially more difficult" (Sophian, 2007, p. 98).

Clark and Kamii (1996) argue that multiplicative thinking is distinguished from additive reasoning because it involves two levels of abstraction, as opposed to addition which only involves one. This aligns with the 'comparison-of-quantity' perspective which regard multiplication as involving a 'higher order unit', that is one quantity is used as a unit for a group of another quantity (Sophian, 2007). For example, if the problem 3 x 4 is solved additively the groups of three items are just successively combined on one level as 3 + 3 more ones + 3 more ones + 3 more ones. The child perceives each group of three made of ones (three ones) which they can just combine with the other groups of three on one

level. If the problem is solved multiplicatively it requires that the child see that there is a many-to-one correspondence between the three units of one and the one unit of three and that the four 'new' units are then combined into the product.

As the example above illustrates, the unit-concept is fundamental for understanding the link between additive and multiplicative reasoning. Sophian (2007) argues that the need for multiplicative thinking arises as soon as one considers using one unit of quantification to describe another unit of quantification of the same measure. For example, if one considers using metre as a unit instead of centimetre as a unit to describe a length. In addition, multiplicative thinking, is deeply connected with proportional reasoning. In proportional reasoning, the operation results in a numerical value for the relation between the quantities.

What can be found from the substantial empirical research on children's development of additive and multiplicative thinking is that most research has focused on children's individual skills and individual problem-solving strategies where experiments or clinical interviews are often used as methods for collecting data (e.g., Anghileri, 1989; Baroody & Purpura, 2017; Greer, 1992; Lu & Richardson, 2017). These studies do not consider the influence of the broader cultural context in which children's quantitative and numerical thinking develop. Paper 4, in this research study, focuses on the multiplicative thinking that emerges from children's joint activity and their coordination of semiotic means. The paper focuses on the thinking processes that unfolds when children solve mathematical problems together, before they have become fluent in multiplicative thinking. The paper illustrates the influence of the contextual features on their reasoning process.

3 Inquiry and learning in early childhood mathematical activities

At least since the early 1900s, when John Dewey elaborated on his educational philosophy, the term inquiry has played a significant role in attempts to reform educational practice in school. Although there are diverse conceptions of the term in different theoretical and philosophical approaches, the ideas behind inquiry are somehow united in the idea that it is about seeking information or knowledge by investigation and questioning. A key point, however, is that the students should be (or become) the inquirers and seekers of information. This stands in contrast to what has been called the 'traditional' teaching approach, where the students listen to the teacher when he/she explains mathematical procedures and concepts, and where the students afterwards practice the teacher's procedures. In the 'traditional' approach the students become receivers of information and practitioners of rote learning (to put things bluntly). In an open inquiry approach, the students are given more freedom and responsibility, and the focus is on collaboration, discussions, argumentations and problem solving. The term has been used in diverse theoretical approaches from constructivist or sociocultural approaches to more pragmatic approaches with more or less the same goal – to make the (mathematics) education 'better'. What 'better' means is conceived differently in different perspectives but is always linked to (explicitly or implicitly) epistemological beliefs, that is what knowledge is and how it may be acquired.

Since inquiry in essence is understood as an educational approach which tries to meet the manner in which children learn, the concept was also recognised as an approach which could help to unite the aims of the intervention programme developed in the Agder Project (AP) of preparing children for school, but at the same time fit the Norwegian kindergarten tradition. However, since the conception of inquiry used in the AP has mainly emerged from research on teaching and learning in school contexts, and the school tradition is different from the kindergarten tradition, it has the potential of creating tensions.

In the following sections I first consider the development of inquiry as an educational approach (Section 3.1). Further I elaborate on the dialogical approach to inquiry which is the approach appropriated in the AP and which is taken as a point of departure in this research study (Section 3.2). And finally, in Section 3.3 research related to some of the main aspects found in the dialogical approach are considered. Section 3.3 also briefly considers inquiry as identified in the Norwegian Framework Plan for contents and tasks in kindergarten (Sub-section 3.3.1).

3.1 Development of inquiry as an educational approach

As mentioned above, a tremendous effort to define and understand the concept of inquiry in educational settings was done in the early 1900s by John Dewey, who represents a pragmatist tradition to education. 'Learning by doing' has become a slogan for Dewey's educational philosophy, where learning happens through action and 'reflective inquiry' (Dewey, 1998). According to Dewey, it is through sensory experiences of the world and from investigations of the environment that children learn. Thus, for education to be meaningful, it should be connected to real-life situations and situations that the students find interesting (Dewey, 1938). Dewey regarded inquiry as an integral part of human life and argues that "In everyday living, men examine; they turn things over intellectually; they infer and judge as "naturally" as they reap and sow, produce and exchange commodities" (Dewey, 1998, p. 170). Dewey (1998) regarded inquiry as a "progressive determination of a problem and its possible solution" (p. 174), it is *the controlled or directed transformation of an indeterminate situation into one that is so determinate in its constituent distinctions and relations as to convert the elements of the original situation into a unified whole*" (p. 171, emphasis in origin). In Dewey's view, the indeterminate situation is a pre-condition for inquiry. It is the indeterminate situation that provokes uncertainty and consequently questions to be inquired. It is not the questions itself, but the situation that has this quality. Dewey argues that the initial step in inquiry is to see that a situation requires examination, that is to identify a problem.

The interest for inquiry-based education has increased over the past two decades, which may be evident from the increasing use of the term in research and in school curriculums. One way to view the emergence of the inquiry-based approaches to mathematics education is to follow the traces from Dewey and the influence of his thoughts into science education and further into inquiry-based mathematics education (Artigue & Blomhøj, 2013). Another way to view the emergence of inquiry-based education in mathematics is to see the links to the problem-solving tradition, which was initiated in 1945 by George Polya and his book 'How to Solve It'. This book gave birth to new ideas in mathematics education. For Polya (1945) mathematical pedagogy had to reflect the 'nature' of mathematics. How students experienced mathematics had to be consistent with the way mathematics was done. Polya (1954) also emphasised the role of guessing (conjecturing) in mathematical problem solving. "To a mathematician, who is active in research, mathematics may appear something as a guessing game; you have to guess a mathematical theorem before you prove it, you have to guess the idea of the proof before you carry through all the details" (ibid., p. 158, in Schoenfeld, 1992, p. 339). The problem-solving tradition as an approach to understand the

acquisition of mathematical thinking was further developed by scholars like Halmos (1980), Mason, Burton, and Stacey (1982) and (Schoenfeld, 1985), among others.

In the 1980s-1990s Schoenfeld (1985, 1992) made a significant contribution to the problem-solving tradition by his investigation of student's mathematical behaviour, seeing that problem-solving behaviour was more than the ability to use a method or strategy. Mathematical problem-solving behaviour, he argued, is determined by several factors, like the person's mathematical resources and strategies, his/her ability to control and use the resources and the strategies, and his/her beliefs about mathematics and about him-/herself. In both the natural science tradition and the problem-solving tradition, the idea that students should experience the subject as mathematicians or scientists is central (Artigue & Blomhøj, 2013; Polya, 1945; Schoenfeld, 1992). Artigue and Blomhøj (2013) define inquiry-based pedagogy and as "a way of teaching in which students are invited to work in ways similar to how mathematicians and scientists work" (p. 797).

In the 1990s the focus turned to the teachers, and their professional development. Inquiry became a concept which was no longer only focusing on the experiences of the students and how problems could incite students' inquiry 'habit of mind'. To prepare for children's opportunities to inquire into mathematics the teaching had to be given more attention and the focus turned to teacher education and professional development. How could teachers acquire the knowledge that was needed to orchestrate lessons which would help students learn and experience mathematics as meaningful? This line of research was initiated by Lytle and Cochran-Smith (1992), and Cochran-Smith and Lytle (2009) from a constructivist perspective, and brought further by, for example, Jaworski (2004, 2005, 2006), Jaworski and Fuglestad (2010) and Jaworski and Goodchild (2006). The latter researchers represent a sociocultural perspective to inquiry, which the dialogic approach is situated within. In the next section, I introduce the dialogic approach to inquiry which, for example, Wells (1999) is a strong representative of, and which is the approach appropriated in the AP. The dialogic approach to inquiry was therefore also taken as a point of departure in this research study.

3.2 The dialogic approach to inquiry

In parallel with the social turn in late 1980s and 1990s (Lerman, 2000) the focus on dialogic interaction related to inquiry emerged. Dialogic interaction related to inquiry refers to the way problems are solved through a dialogue between two (or several) people, and not as a personal endeavour. Although many scholars (e.g., Socrates, Dewey, Polya) acknowledges the significance of the relation between the teacher and

the student(s), none of them emphasise the social nature of a dialogic interaction.

As a cultural-historical activity theorist, Wells (1999) aims to close the gap between the 'child-centred, personal constructive' approach to education and the 'teacher-directed, transmissive' approach to education by introducing 'dialogic inquiry – toward a sociocultural practice and theory of education'. Dialogic inquiry is here used as a normative approach to education, and Wells (1999) describes 'dialogic inquiry' as "a stance toward experiences and ideas – a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (p. 121). Wells draws on Bereiter's (1994) conception of 'progressive discourse'. Progress in Bereiter's conception is an 'attribute of discourse' and he argues that "there is no knowledge beyond discourse" (ibid., p. 5). Further he explains

> The importance of discourse to scientific progress … arises from a recognition that scientific theories cannot be verified; they can at the most be falsified. Progress therefore arises from continual criticism and efforts to overcome criticisms by modifying or replacing theories. Research, according to this view, does not generate progress directly, but does so by providing evidence that can be brought into the critical discourse, where it may lead to progress (ibid., p. 5).

Progressive discourse is for Bereiter what characterises knowledge-building and scientific communities. Dialogic inquiry in education must therefore be regarded as a discourse with specific 'qualities'. For collaborative knowledge building to be 'progressive', that is to reach new understanding which is new to everyone, "the discourse must involve more than simply the 'sharing' of opinions" (Wells, 1999, p. 112). It must involve sharing of ideas, questioning and revising of opinions to develop new insights. Although Wells (1999) may be regarded as the main representative for the dialogic approach to inquiry, others have similar approaches. Both Jaworski (2004, 2005, 2006), Alrö and Skovsmose (2004) and Lipman (2003) conceive inquiry as a discursive activity with certain qualities. And, although the term inquiry is not explicitly used by Linell (1998) or Alexander (2008), they have similar approaches to education.

As mentioned above, inquiry from a dialogic approach, seems to be a discourse with certain 'qualities'. One of these qualities may be identified as critical and reflective thinking, and includes questioning one's own and other's thoughts and ideas to reach new understanding (Jaworski, 2004, 2005, 2006; Mercer, 2000; Wells, 1999). The role of questions seems to have been at the heart of inquiry ever since John Dewey. Dewey (1998) argues that "Inquiry and questioning, up to a certain point, are synonymous terms. We inquire when we question; and we inquire when we seek for whatever will provide an answer to a question

asked" (p. 171). Furthermore, in the problem-solving tradition Halmos (1980) argues that

> I strongly recommend that students in a problem seminar be encouraged to discover problems on their own … and that they should be given public praise (and grade credit) for such discoveries. Just as you should not tell your students all the answers, you should also not ask them all the questions. One of the hardest parts of problem solving is to ask the right question, and the only way to learn to do so is to practice (p. 524).

He further argues that one cannot teach someone to ask (good) questions, but one can encourages students to practice asking questions.

In the dialogical approach, questions are also emphasised as essential elements to inquiry (e.g., Alrö & Skovsmose, 2004; Jaworski, 2006; Lindfors, 1999; Roth, 1996; Wells, 1999). Jaworski and Goodchild (2006) argue that "Fundamentally, inquiry and exploration are about questioning: asking and seeking to answer questions. Together, we ask and seek to answer questions to enable us to know more about mathematics teaching and learning" (p. 353). Lindfors (1999) defines students' questions as inquiry acts. However, she argues that inquiry questions are not 'perfect'. "It makes sense that inquiry utterances are often imperfectly formed – even downright messy sometimes, for they are acts of going beyond, not acts of having arrived" (ibid., p. 63).

Lindfors (1999) distinguishes between 'information-seeking' and 'wondering' as two different ways of 'turning to someone' to reach new insight and understanding of something. Lindfors says that information seeking is a kind of 'work' towards a specific goal, where the participants work towards a goal of building new understanding of something. Lindfors (1999) argues that

> *Working* is a key word. Information-seeking utterances sound very much like work – deliberate, effortful, focused, moving toward a specific end. It is work that has a goal. Information seeking is product oriented, if you like. It is oriented toward what one is wanting to know (p. 38).

Wondering is different from information seeking because the purpose is to keep the conversation open, not to find answers, confirm or explain something. Questions used as wondering acts have the purpose to reflect on and enlighten ideas from different perspectives, not to resolve problems. Lindfors' (1999) conception of inquiry as 'wondering' has similarities with Alrö and Skovsmose's (2004) and Lipman's (2003) approach to inquiry, where they emphasise the unpredictable nature of a dialogue. Lipman (2003) argues that "A community of inquiry attempts to follow the inquiry where it leads rather than be penned in by the boundary lines of existing disciplines" (p. 20). The direction of a dialogue should not be decided by an authority, rather follow its own path decided by all participants. Similarly, through 'wondering' an issue is considered from many

angles without any specific aim or desire to reach a conclusion. In addition, Lindfors understands wondering acts as playful.

On the other hand, information seeking acts are 'work-full' where the conversation has a goal that the participants work together in order to reach (Lindfors, 1999). This is similar to Wells' (1999) conception of inquiry as a cultural-historical object-oriented activity. Through joint activity the participants work together towards an object. This is also parallel to the problem-solving tradition where the goal is to solve a mathematical problem and reach a conclusion. However, the road toward the solution may have many facets and cannot be decided in advance. Lindfors is aware that her characteristics of wondering and information seeking inquiry must be seen as tendencies, not as an absolute definition.

Inquiry is more than asking questions and using language to think and solve problems together. Inquiry is also regarded as *a stance* toward knowledge (Schoenfeld, 1996; Wells, 1999) and *a way of being* in practice (Jaworski, 2005). Wells (1999) explains this stance as an identified 'will' to solve problems together with others. Similarly, Schoenfeld (1996) argues that the underlying stance toward knowledge is based on an eagerness to know and the stance toward each other is based on trust. Others have also identified that inquiry is based on a foundation of mutual respect and trust, where the participants listen to, consider and challenge one another's ideas (Alrö and Skovsmose's, 2004; Lipman, 2003; Schoenfeld, 1996; Wells, 1999; Jaworski, 2005).

From the literature above some features are repeated. Inquiry may be regarded as a discourse with certain qualities. Inquiry implies that there needs to be something to inquire into, that is there needs to be a problem to investigate. Second, participants in inquiry activities must be given opportunities ask questions. As argued above, questioning seems to be at the heart of inquiry. Another feature is that the children answer questions and solve problems through further questioning, argumentation and critical examination. A fourth aspect is that there must be identified a 'will' among the children to ask questions, to argue and explain their ideas and to solve these problems together with others. These aspects were taken as a point of departure for this research study and were used in the data processing to identify inquiry segments. The next section is devoted to literature related to some of these aspects. A lot of the literature below is taken from school setting where teaching-learning activities are often more structured than in kindergarten. Since the mathematical activities in the AP are designed to be structured around various mathematical areas and moreover to prepare children for school, it seems relevant to present literature considering teacher-led activities and literature that points to challenges that may arise in teacher-led activities.

3.3 Questioning, argumentation, and turn-taking in mathematical inquiry in kindergarten

As mentioned above, reflection and critical thinking are essential elements in the dialogic approach to mathematics teaching and learning (Wells, 1999; Mercer, 2000; Alrö & Skovsmose, 2002; Jaworski, 2005; 2006), where the use of questions may be argued to be at the core of the approach. And it has been especially emphasised that the students should get opportunities and be promoted to ask questions. The distribution of questions in a classroom discourse, which emphasises students' questioning is perhaps the most apparent contrast to the 'traditional' teaching where the teacher becomes the inquirer by asking a lot of questions in order to understand what knowledge the children has acquired, for example in order to give a mark. Although many scholars have emphasised that children should get opportunities to ask their own questions, and that the presence of children's or student's questions is an indicator for what we may call inquiry, other scholars have identified that children or students rarely ask questions themselves in teaching-learning situations (Dillon, 1988; Myhill & Dunkin, 2005). Some have argued that the absence of children's own questions may be a consequence of the extensive use of questions that teachers ask and thus the lack of opportunities that the children get to ask questions (Dillon, 1988).

Although Dillon (1988) is 'questioning the use of questions', that is he argues that teachers should strive for using other pedagogical means in the classroom interaction than questions, there is an extensive literature that emphasises that teachers' use of questions is important for engaging students in mathematical discussions and leads to positive learning outcomes (Carlsen, 2013; Franke et al., 2009; Kirby, 1996; Myhill & Dunkin, 2005; Rojas-Drummond & Mercer, 2003; Roth, 1996). Nevertheless, teachers should carefully consider their own use of questions. Some questions may help children to participate, other may hinder the children to participate in the mathematical discourse. Closed questions, which invite short factual or procedural responses are often found to initiate the well-known IRE-exchange (initiation-response-evaluation). This type of exchange often serves only to check pupils' knowledge, which can be counterproductive for learning (Wells, 1999; Wood, 1992). This exchange is perhaps one of the characteristics in the 'traditional' teaching approach mentioned earlier. Open questions on the other hand often invite longer and possibly more elaborated responses and are often found to initiate the IRF-exchange (initiation-response-follow up). Wells (1999) argues that IRF-exchanges can serve as a useful pedagogical tool to achieve co-construction of knowledge. The quality of this exchange relies on the underlying expectations and goals of the teacher. Similarly, Rojas-Drummond and Mercer (2003) argue that IRF-exchange can help

to guide children's learning, especially if the teacher follows up with 'why-questions' which promote students to reflect on their responses.

Teachers' careful question strategies are regarded as important for establishing inquiry environments. Roth (1996) investigates the question strategies of an experienced science teacher during an open-inquiry learning environment in a grade 4/5 science class. The teacher was concerned about and aware of how and why she asked questions. When Roth analysed the teacher's question strategies in light of the content matter, he found that the teacher was more concerned with processes than with facts. The teacher hardly evaluated the children's responses (according to the IRE-exchange). Rather, the teacher, by her well-planned questioning strategies, gave the students opportunities to start from their own experiences and, from that, develop their discourse about the content matter. The results also indicated that the students' discourse developed into exploratory talk (cf. Mercer, 2000), where the children seemed to start asking questions themselves.

Questions are also found as an often-used pedagogical tool in kindergarten (Carlsen, 2013; Carlsen, Erfjord, & Hundeland, 2010; Sæbbe & Mosvold, 2016). For example, in their investigation of a KT's participation in an everyday mathematical learning activity, Sæbbe and Mosvold (2016) found that question-posing was one out of two core components in the KT's discourse. Sæbbe and Mosvold studied what they called an "everyday situation", where the children and the KT were sitting around a table playing with Lego bricks. The KT had no formal learning goals, the aim was to play with the bricks and talk about what they were doing. The KT's question-posing was followed up by affirmation, which was the other core component in the KT's discourse. Through affirmation the KT confirmed or acknowledged the children's contributions. Sæbbe and Mosvold (2016) argue that these components (questioning and affirmation) were strongly related and served as means for reaching joint attention. It is interesting to notice how this communication pattern is similar to the IRE-exchange, commonly used in a 'traditional' teaching approach in school. However, the purpose of the questioning-affirmation dyad used by the KT was mainly not checking children's knowledge. Through the questioning-affirmation dyad the KT introduced new concepts, invited the children to participate in the mathematical discourse and encouraged them to think further on their ideas.

The evaluation or the follow-up in the IRE-/IRF-exchange, or the affirmation mentioned in the previous study, may take many forms, for example as re-phrasing or re-voicing of children's response. O'Connor and Michaels (1996) define re-voicing as "a particular kind of re-uttering (oral or written) of a student's contribution—by another participant in the discussion" (p.8). By re-phrasing, O'Connor and Michaels argue, the

teacher confirms that he or she has heard what has been said and may also help children to concentrate attention on key-points. If reformulated into a question the re-phrasing can also challenge an idea and invite the children to think further on their ideas.

As mentioned above, the teachers' question strategies may both hinder and facilitate children's learning possibilities in a mathematical activity. Children's learning possibilities are often related to the degree of participation, that is whether the KTs, through their question strategies, are able to give room for children's participation or not. In activities where the KT (or teacher) has a particular aim for the activity, and where he or she guides the conversations around the aim, it is difficult to balance teacher-talk and child-talk (Dovigo, 2016; O'Connor & Michaels, 1996). On the one hand, the teacher or KT is supposed to facilitate children's opportunities to take part in the conversation, and on the other hand organise the conversation around a specific aim. Dovigo (2016) investigated how preschool children's (age 3-5) participation in different types of conversations influenced their learning possibilities. He compared children's participation in peer-talk and child-teacher talk and found that children had richer opportunities to contribute in peer-talk, whereas in child-teacher talk, the KT talked more than the children. In peer-talk the children asked more questions (including open questions), however, the children's abilities to build argumentations were limited in peer-talk. In child-teacher talk, the most experienced teachers carefully facilitated children's explanations and helped them generate more sophisticated argumentations than what they managed on their own. In addition, the children showed increased responsibility in the conversation and improved their ability to collaborate and think critically. Children's participation in argumentative conversations with the KT had significant influences on children's learning.

Argumentation is, as mentioned above, another 'quality' of dialogic inquiry, however argumentations may also have different 'qualities'. Mercer (2000) describes three common ways in which the word 'argument' can be understood. An argument can mean the way that two (or several) people quarrel. Argumentation is then a 'fight' with words, where the participants defend their opinions. Mercer calls this type of collective argumentation 'disputational talk'. In disputational talk there is a lot of disagreement, and everyone makes their own decisions. There are few attempts to offer constructive criticism and the atmosphere is competitive rather than co-operative. This type of argumentation does not (necessarily) lead to new insights or new understanding of something. A second way to use 'argument' is to describe how one person use rhetoric to present evidence for a statement or an opinion. This argumentation often has a monologue nature and is usually recognised as written

texts or as political speeches. Similar characteristics may be applied to a collective argumentation where the participants build on each other's statements in order to construct support for a shared view. Mercer calls this type of collective argumentation 'cumulative talk'. In cumulative talk participants accept and agree with each other. Knowledge is shared, but in an uncritical way and ideas are repeated and not further elaborated. Cumulative talk has similarities with what Wells (1999) calls sharing of opinions, which does not really lead to new insight either (new to everyone).

A third way to understand 'argument' is when people together constructively debate around an issue in order to understand it better. Mercer calls this type of argumentation 'exploratory talk'. In exploratory talk everyone listens actively to one another, the participants ask questions and share opinions/ideas. Ideas and opinions might be challenged but, at the same time, are treated with respect. Everyone is encouraged to contribute, and there is an atmosphere of trust. There is a sense of shared purpose and the participants seek agreement and shared understanding. Mercer (2000) defines exploratory talk as

> Exploratory talk is that in which partners engage critically but constructively with each other's ideas. Relevant information is offered for joint consideration. Proposals may be challenged and counter-challenged, but if so reasons are given and alternatives are offered. Agreement is sought as a basis for joint progress. Knowledge is made publicly accountable and reasoning is visible in the talk (Mercer, 2000, p. 98).

According to Mercer (2000), exploratory talk is discourse that is characterised by knowledge-building communities, similar to progressive discourse in Wells' (1999) terminology. Both Wells (1999) and Mercer (2000) argue that one should promote progressive or exploratory discourse as the knowledge building discourse at all levels of education. This is similar to the ideas in the problem-solving tradition which strive for developing the classroom discourse so that the children may experience the mathematics (or another subject) as professional mathematicians or scientists do (cf. Polya, 1945; Schoenfeld, 1992).

Several studies have pointed to the role of mathematical argumentation for enabling young children's mathematical reasoning (Mercer & Sams, 2006; Tsamir, Tirosh, & Levenson, 2009; Yackel & Cobb, 1996). However, children have different foundations for providing elaborated argumentations. For example, Tsamir et al. (2009) illustrates how children (age 5-6) provide different types of justification while working with various number and geometry tasks. Their study shows that some children were able to justify their statements by using appropriate mathematical ideas. Other children, in contrast, used their 'visual reasoning' as a way to justify their statements. When the researcher asked the children to justify their solutions, some children answered, "because we see", and

they felt no further need to justify their answer or did not know how to do it.

In their research study Yackel and Cobb (1996) investigate how norms, specific to mathematical classrooms, are developed in inquiry-based activities. These norms are called sociomathematical norms, and account for how students develop their abilities to argue, explain and justify their mathematical ideas and consequently increase their autonomy in the mathematical classroom. Sociomathematical norms are different from social norms and can be regarded as tacit rules which concern what are culturally accepted mathematical ways of arguing, explaining and justifying. Social norms on the other hand, concern culturally accepted ways of doing things more generally, not directly relate to mathematics. For example, the children in Tsamir, Tirosh, and Levenson's (2009) study, may be aware of cultural rules that expect them to justify their claims. But they may not be aware of cultural rules about how to justify their claims. According to Yackel and Cobb (1996) the teacher plays a significant role for bringing in and establishing such norms, by representing the mathematical community.

Krummheuer (1995) regards argumentation as the practical business of producing an argument and an argument as the product of argumentation. The purpose of an argument is to convince others that their claim or statement is true, or to make a claim accountable for others (Krummheuer, 1995; Toulmin, 2003). Toulmin (2003) developed a model which aims to illuminate how statements are organised for the purpose of constituting an argument. In Toulmin's model the core of an argument is based on three elements: claim (C), data (D) and warrant (W). The claim is an initial statement or assertion about something, for example, *'birds can fly'*. If the claim is questioned, the arguer has to produce data that supports the claim for example, *'birds got wings'*. Data are facts or statements on which the claim can be grounded. If the claim is still questioned, the arguer has to produce a warrant which points to the relation between the data and the claim. A warrant is a justification of the data with regard to the claim and aims to tie the argument together. For example, *'because wings are what birds use to fly'*. In addition, Toulmin's (2003) model contains three other elements, backing (B), qualifier (Q) and rebuttal (R). A backing is a statement that supports the warrant. It is like a special case of data that is provided as evidence for the warrant. The purpose of a backing is to answer, "why in general this warrant should be accepted as having authority" (Toulmin, 2003, p. 95). For example, in relation to the claim above it could be *'I know an ornithologist, and he says that birds use their wings to fly'*. A backing connects the core of an argument (C-D-W) to collectively accepted assump-

tions. A qualifier says something about the extent to which the data confirm the claim. Words like 'probably', 'presumably' etc. are often used as qualifiers. Rebuttals refer to exceptions or conditions under which the claim is true and are often used subsequent to a qualifier. For example, *'birds can fly unless they are chickens'*.

The above literature provides various forms of conversation or characteristics of discourse. What most conversations have in common, if there are two or more people involved, is that they are built up of turn-taking. Sacks, Schegloff, and Jefferson (1978) describe two ways in which turn-taking in a conversation can be organised: 1) a current speaker may select the next speaker, or 2) a non-speaker may self-select in starting to talk. In self-selected turn-taking the potential next speaker must find a 'transitional-relevance place' where it is relevant with a transition in the conversation. It is clausal, phrasal and/or lexical principles which create conversational units, that determine such transitional-relevance places and indicate for the next speaker when it is relevant to construct a turn. If the current speaker selects the next speaker, there are many tacit or explicit ways of doing that. Lerner (2003) discusses a range of explicit and tacit methods for addressing the next speaker. The current speaker may select next speaker using address terms like 'you' or the next speakers name, or the current speaker may direct his or her gaze towards another participant while speaking. Lerner emphasises that these explicit methods for addressing next speaker are often used in concert with each other. For example, an address term may be used in concert with gaze and/or gestures. Tacit methods for addressing the next speaker, on the other hand, make evident who is being addressed without using explicit address terms or other explicit means. Tacit ways of addressing next speaker draws upon specific features of the current context or content in the conversation. For example, if the content in a conversation relates to a specific person in the group, he or she may be the only one who can contribute at a certain point. Lerner emphasises that both explicit and tacit ways of addressing next speaker are context sensitive, however, tacit addressing cannot be considered without it. Similarly, Mondada (2007) emphasises the situatedness of turn-taking. In her case study she investigates how participants in a conversation gradually establish themselves as next speakers through specific use of gestures. The participants are sitting round a table with diverse artefacts (maps, documents etc.) in the middle. And all the participants are engaged in reading, writing and considering those artefacts. The context promotes the participants to use pointing gestures to establish him-/herself as the next speaker while the current speaker is still talking. The turn-taking is primarily organised as a side-by-side exchange (as opposed to face-to-face exchange), where

the participants are not looking at each other (having eye contact), rather looking at the artefacts and their joint actions.

To summarise, this section has provided a review of literature which considers some of the key elements identified in the dialogic approach to inquiry, for example, children's possibilities to ask questions, explain and argue for their ideas. In addition, this section has considered challenges that may arise in teacher-led activities and how careful question strategies may facilitate children's opportunities to contribute in mathematical activities. Argumentation is another important element in dialogic inquiry, and this section has described diverse ways in which argumentation may be categorised and analysed. Furthermore, this section has provided a brief overview over different ways in which turn-taking may be organised in conversation, which served as an important analytical tool in one of the papers accompanying this thesis.

4 A cultural-historical perspective on researching early mathematics teaching and learning

To study teaching and learning of mathematics, or more precisely, processes of mathematical inquiry in kindergarten, I adopt a cultural-historical perspective on learning and development. Drawing on the cultural-historical ideas of Lev Vygotsky, A. N. Leont'ev and especially ideas by the contemporary mathematics educator Luis Radford, I embrace the idea of the social origin of the development of the human mind, which always takes place within a specific culture in a specific time of history.

This chapter elaborates on the theoretical and philosophical foundation of my research. The chapter consists of two sections. Section 4.1 elaborates on the rational for using a cultural-historical approach as the theoretical foundation of the study. In Section 4.2 the cultural-historical approach and Radford's (2013b) theory of knowledge objectification will be elaborated.

4.1 Rational for using a cultural-historical perspective to investigate mathematical inquiry processes in kindergarten

The 'choice' of using a cultural-historical approach was not really a choice (in a pragmatic sense). The 'choice' of theory had to fit my own understanding of the world (ontology). For me it was important to understand the processes of learning in relation to evolutionary issues. I had to see the link between the development of human consciousness and the consciousness of other species. Reading Vygotsky, Leont'ev and especially Radford I immediately felt that it resonated with my conception of the world. However, I have to admit that it was extremely hard reading in the beginning and I am still trying to grasp the whole sense of the perspective. In the process of making sense of the cultural-historical approach, I slowly understood that cultural-historical activity theory is not just a 'theory'. In my view it is a science, with a coherent argumentation about the development of human consciousness, which can be seen in relation to other sciences (neuroscience, evolutionary theories etc.). However, as any other theory of science, it cannot be proven, but its veracity is reinforced each time it is tested.

When Deacon (1998) discusses 'the co-evolution of language and brain' he argues that the difficulty of understanding the origin of the language development and the co-evolution of brain and language is the

way we conceive ourselves. "I think the difficulty of the language origins question is not to be blamed on what we don't know, but rather on what we think we already know" (ibid., p. 25). I agree with this statement, and I think that the way we are 'raised' to think about ourselves stands in the way of making sense of our own existence and development. I think we are 'raised' (at least I was) to think of our own consciousness as something we are born with, a kind of inside 'light' that we have. In a cultural-historical approach our consciousness is formed by our sensuous relation to the world. Consciousness is not a 'light' that we originally have and that we need in order to perceive the world. It is our sensuously perceptual 'picture' of the world deeply connected to the ways that we do things in the world.

A lot of the research on children's development of mathematical knowledge stems from research within educational psychology that has been influenced by the positivist tradition of medicine and natural sciences. In this tradition the researchers often use tests to investigate children's 'cognition', and where they try to understand 'what is in the heads' of the children (Hedegaard & Fleer, 2008). As I mentioned in Sub-section 2.5.3, a lot of the research on children's early development of mathematical thinking used clinical interviews as means to investigate how children reason about mathematical problems. The focus is again on what mathematical thinking can be found 'in children's heads'. But these studies fail to see that children's development is completely intertwined in their cultural and historical context and fail to capture how the environmental context influences children's mathematical thinking processes. Using a cultural-historical approach helps me to capture (in ways I shall elucidate on in Chapter 6) environmental influences on children's mathematical thinking and thus to get a more complete picture of children's mathematical learning and development.

In kindergarten children are constantly interacting with their peers and their KTs in free play activities and in more organised activities, for example during mealtimes. Through these activities, kindergarten children become familiarised with cultural and historical ways of thinking, among them ways of thinking mathematically. For example, during mealtime children learn to lay the table with as many plates as there are children in the kindergarten. Or they learn what it means to be first in line, or last in line when they are waiting for their turn to use the swing. In the last year of kindergarten, however, the familiarisation with cultural ways of thinking often takes a more structured form, where the KTs prepare activities with a certain aim. This is also the case in the AP, where the KTs are supposed to introduce children to mathematical ideas through pre-designed activities aimed to prepare children for school. But what happens when children are introduced to mathematical ways of

thinking through more structured activities than in an everyday situation in kindergarten? As mentioned above the cultural-historical approach is especially useful when the aim is to capture contextual features that influence children's thinking and learning. Radford's (2013b) semiotic approach and theory of knowledge objectification is particularly useful for making sense of children's mathematical reasoning as a process linked with the cultural and historical context that they live in. The cultural-historical perspective helps me to understand processes of mathematical inquiry in its own habitat (Radford, 2010a). Moreover, the cultural-historical approach helps me to see these processes from a historical point of view, and especially understand how tensions may occur when different traditions meet.

In the next section I elaborate on key ideas and some essential constructs in the cultural-historical perspective and Radford's (2013b) theory of knowledge objectification, beginning with a brief introduction of its founders.

4.2 A cultural-historical perspective and theory of knowledge objectification

Vygotsky's cultural-historical school of thought has its origin in Karl Marx and Friedrich Engels's dialectic materialism, a philosophy of science and nature, that is a way of understanding reality. In dialectic materialism 'human nature' (both consciousness and behaviour) is conceived as a product of historical changes in society and material life (Cole & Scribner, 1978).

Dialectic materialism had a real influence on psychology at the beginning of 1920s, when Vygotsky developed an historical approach to psychology as a science. His approach is a science of consciousness where 'higher' form of reflection of reality (specific for humans) has developed from 'lower' forms (specific for other species)[6], and where human activity (practice/labour) and its structures and dynamics has a central role, (Leont'ev, 1978). In Vygotsky's cultural-historical perspective humans develop their understanding of the world (including mathematical objects) through participation in social activities. Vygotsky's emphasis on the social origin of mind is evident in the general genetic law[7] which is formulated as follows

> Any function in the child's cultural development appears twice, or on two planes.
> First it appears on the social plane, and then on the psychological plane. First it
> appears between people as an interpsychological category, and then within the

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[6] Today we know a lot more about animal intelligence than what they did in the beginning of the 20th century.
[7] For a thorough analysis of Vygotsky's general genetic law see Veresov (2005).

child as an intrapsychological category. This is equally true with regard to voluntary attention, logical memory, the formation of concepts, and the development of volition (Vygotsky, 1981, p. 163).

These mental functions are mediated through social activity and the use of signs (semiotic means) where language is central. Summarising Vygotsky's core ideas, Leont'ev (1981) says

> higher psychological processes unique to humans can be acquired only through interaction with others, that is, through interpsychological processes that only later will begin to be carried out independently by the individual. When this happens, some of these processes lose their initial, external form and are converted into intrapsychological processes (p.56).

Leont'ev, who is recognised as (one of) the founder(s) of activity theory was a colleague of Vygotsky and developed cultural-historical ideas of psychology further. To study human activity as a scientific method for studying the development of human psyche/cognition was already proposed by Vygotsky. Leont'ev's contribution was to identify the object-orientedness of activity. In Leont'ev's activity theory, activity is conceived as a process, or a system of actions and relations, and which comprise both inner (cognitive) and outer (material) processes.

> Activity is a molar, not an additive unit of the life of the physical, material subject. In a narrower sense, that is, at the psychological level, it is a unit of life, mediated by psychic reflection, the real function of which is that it orients the subject in the objective world. In other words, activity is not a reaction and not a totality of reactions but a system that has structure, its own internal transitions and transformations, its own development (Leont'ev, 1978, p. 50).

Human object-oriented activity is always driven by a motive (or a need). It is the motive that gives rise to the structure of the activity and which drives the activity forward. According to Leont'ev (1978) it is the object that is the true motive of the activity.

> The main thing that distinguishes one activity from another, however, is the difference of their objects. It is exactly the object of an activity that gives it a determined direction. According to the terminology I have proposed, the object of an activity is its true motive. It is understood that the motive may be either material or ideal, either present in perception or existing only in the imagination or in thought. The main thing is that behind activity there should always be a need, that it should always answer one need or another (Leont'ev, 1978, p. 62).

Leont'ev distinguishes three levels of activity. He distinguishes object-oriented activity from goal-directed actions and he distinguishes goal-directed actions from operations, which are constrained by certain conditions. Goal-directed actions are what individuals consciously do in the process of participating in an object-oriented activity. Operations are generally sub-conscious actions, that is what individuals 'automatically' do to accomplishing goal-directed actions. These three levels are mutually constitutive. Activities are realised through conscious goal-directed actions, but goals are initiated by the object (motive) of the activity. Goal-directed actions are realised through sub-conscious operations, but

operations are initiated by goal-directed actions. Activities may be seen as chains of goal-directed actions.

Based on the lineage of Vygotsky-Leont'ev-Holzkamp[8], Radford developed the theory of knowledge objectification (Radford, 2013b), which especially focuses on mathematical activities. Radford represents an approach to cultural-historical activity theory that diverges from Leont'ev in the sense that he does not separate activity into the different components (actions-goals, operations-conditions) in the same manner as Leont'ev does. Radford conceives activity (cognitive and material) as being comprised of conscious (and sub-conscious) actions, but he does not separate goal-directed actions from operations in his analysis. Separating activity into goal-directed actions and operations 'forces' the conception of activity into a more static form. Inspired by Engels (1925) who argues that motion is a mode of existence, Radford, in my view, is more concerned with conceiving activity as an ever-changing whole than with separating it into parts. In Radford's theory of knowledge objectification, activity is conceived as a 'flux' or as a dynamic 'form of life' (Roth & Radford, 2011).

In the theory of knowledge objectification, Radford especially focuses on processes and nuances of mathematical thinking and learning. Thinking, in Radford's (2015b) conception, is thought put into motion. Thought does not exist, materially, it is pure possibility. Thought is the possibility of thinking, which is put into motion through human activity. It is through joint practical activity that mathematical thinking is brought to life, that is being materialised or actualised.

In Radford's (2013b) theory of knowledge objectification, learning is theorised as a social process of becoming critically aware of cultural and historical ways of thinking and doing. Through participating in joint activities, which comprises specific ways of doing things, students (and teachers) encounter and become critically aware of cultural and historical ideas and perspectives. Knowledge is seen as systems of ideas, perspectives or forms of thinking which in the process of knowledge objectification, become materialised in consciousness. Knowledge is real and exists, but conceptual objects do not exist as Platonic objects. The objects that Radford is talking about are human-made objects developed through human history (Radford, 2009), and which are constantly changing (also right now as we 'speak'). In the moment of interaction in object-oriented activities, knowledge is materialised through a complex coordination of gestures, bodily actions, artefacts, (mathematical) signs and speech,

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[8] Klaus Holzkamp was a German psychologist and a significant representative for Critical Psychology in the late 20th century. But since he is not prominent in my work, I will not go into further details about his work.

which all are termed semiotic means. In the process of knowledge objectification, "Each semiotic means of objectification puts forward a particular dimension of meaning (signification); the coordination of all these dimensions results in a complex composite meaning that is central in the process of objectification" (Roth and Radford, 2011, p. 78). Put in another way, through a complex coordination of semiotic means, each of which 'represents' a dimension of meaning, objects ('entities' of knowledge) are brought to life or materialised in the process of objectification, and which the subjects encounter and become critically aware of. It is important to notice, however, that the process of objectification does not happen all of a sudden. Instead there are layers of generality (Radford, 2010b) which the subjects gradually become aware of. Another feature of this gradual process of learning is what Radford (2008c) calls semiotic contraction. As learning happens the use of semiotic means contracts, that is, as the participants become more aware of the object they can focus on important aspects in the situation and ignore irrelevant things. They become more fluent and start to use more precise gestures, shorter statements and often fewer and better articulated words.

In activity theory and Radford's approach, activity is deeply connected to consciousness. According to Leont'ev (1978) "Consciousness in its directness is a picture of the world, opening up before the subject, in which he himself, his actions, and his conditions are included" (p. 75-76). This conscious 'picture' or 'image' that Leont'ev speaks about must be considered as a sensory, perceptual picture. It is through our senses that we experience the world, and which constitutes our relationship with the world. But consciousness is not a passive sensuous reflection of the world. In this process there are two intertwined elements. Reflection is a dialectic relationship between the objective and the subjective (the reflected object), where the objective and the subjective co-evolve in an intertwined process (Radford, 2013a).

> On the one hand, mind can only arise from the progressive complexity of processes of life; on the other hand, more complex conditions of life require organisms to have the capacity to reflect reality through more complex forms of sensation (ibid., p. 145).

It is through the process of activity that a sensory image of the world is formed. Activity as movement is the substance of consciousness or, put in another way, without activity there is no consciousness (Radford, 2015a).

According to Radford (2008b, 2013b, 2018) learning is not just about knowing it is also about becoming in the process of subjectification. Through the process of subjectification children (and KTs) are positioning themselves within the unfolding (mathematical) activity, through which they become their unique selves (Radford, 2013b). In this endless

process of becoming individuals re-produce themselves as they encounter the world. Again, it is a dialectic relationship between the subjective and the objective, where, in the process of learning, "the subject objectifies cultural knowledge and, in so doing, finds itself objectified in a reflective move that can be termed subjectification" (Radford, 2008b, p. 225).

The research study reported here focuses on processes of mathematical inquiry in kindergarten. From a cultural-historical perspective, inquiry cannot be a personal construction of mathematical relationships, nor can it be 'discovery learning' where the child him- or herself discover mathematical relations in the environment. In a cultural-historical perspective knowledge exists as human made, not as platonic objects already there to be discovered. From a cultural-historical perspective, inquiry processes in kindergarten are better conceived as joint activities between KTs and children where knowledge is re-constructed (from previous forms) through the participants actions. The child cannot discover or construct mathematical objects him- or herself, but the child encounters these objects as they are materialised in joint mathematical activities. In addition, from a cultural-historical perspective, we cannot assume that children naturally ask questions and solve problems. They learn cultural ways of asking questions and how to argue and explain mathematical ideas in order to solve problems by participating in cultural activities.

In the cultural-historical perspective it is meaningless to speak about learning and teaching as separated activities. Teaching and learning always stand in a dialectic relationship with each other. In Vygotsky's dialectical approach teaching and learning must be conceived as mutually constitutive moments, where both (all) participants are teachers and learners of each other. Vygotsky used the notion 'obuchenie' when he talked about a teaching-learning situation and it comprises the mutually constitutive relationship between teaching and learning (Roth and Radford, 2011). There is no learning without teaching and there is no teaching without learning. As a basis for such joint teaching-learning activities, where knowledge objectification takes place, the participants make a commitment to one another to carry through an event. This commitment, Radford and Roth (2011) call 'togethering'. "Togethering is a theoretical category in our theory of knowledge objectification that aims to account for the teacher-students embodied-, sign-, and artifact-mediated interaction that includes both co-knowing and co-being" (p. 244). The construct considers the way the participants engage and attune to one another, in an 'ethical' manner, in joint activities. The participants commit to one another despite their differences. Without such commitment and trust the movement of the activity cannot occur, and the object of activity cannot be realised.

The next section is devoted to an elaboration of Vygotsky's (1987) well-known concept, the zone of proximal development[9], which considers the dialectic relationship between teaching and learning.

4.2.1 The co-creation of the zone of proximal development

The zone of proximal development (ZPD) is one of the most noted concepts that Vygotsky (1987) introduced. The concept has been frequently cited, interpreted and elaborated in a variety of ways. Veresov (2017) notes that more than 200 articles on the ZPD were published in 2010-2016. There is not an extensive corpus of original literature from Vygotsky on the concept, at least not available for English readers, and the concept seems to suffer under poor translations of Vygotsky's texts (Verosov, 2017).

Scholars have different opinions about Vygotsky's original meaning of the concept, and his ideas behind it. van der Veer and Valsiner (1991) claim that Vygotsky did not think of his idea as original, rather as a continuation of contemporary scholars' suggestion that one should consider at least two levels of child development, namely, the child's prevailing level (what the child is already able to do) and the child's potential. Veresov (2017) translates Vygotsky's definition of the ZPD as follow: "the distance between the level of his (sic) actual development, as determined with the help of the tasks the child solves independently, and the level of possible development, as determined with the tasks the child solves under the guidance of adults and in cooperation with more intelligent peers" (Vygotskii, 1935, p. 42, in Veresov, 2017, p. 26).

Veresov (2004) emphasises that one should distinguish between the ZPD as a concept, which is interconnected to other concepts in Vygotsky's writings, and the ZPD as a definition. Vygotsky's definition (above) was presented in a lecture in 1933, where Vygotsky discussed the relationship between instruction (learning) and development, criticising contemporary psychologists' ways of measuring children's intelligence (van der Veer & Valsiner, 1991; Veresov, 2004). Reading Vygotsky's (1987) quote (above) as a definition, indicates that the ZPD is an attribute of the individual child which can be recognised by others. Wells (1999), for example, regards the ZPD as belonging to the child, and it is a task for the 'more knowledgeable other' to recognise it and to act appropriately to. Wells refers to what Vygotsky writes in Thinking and Speaking (1987), where Vygotsky emphasises the role of instruction for development of higher mental functions. Vygotsky (1987) writes "Instruction is only useful when it moves ahead of development … leading

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[9] In this research study, as in many other educational studies, the ZPD is used as a pedagogical construct. However, did Vygotsky regard the concept as a pedagogical construct, or did he in fact regard it as a mere psychological construct? This question will be considered in Sub-section 8.3.4.

the child to carry out activities that force him to rise above himself" (p. 212-213). Wells argues that it is in the above quote that Vygotsky illuminates a significant feature of the ZPD. "The significance of the ZPD is that it determines the lower and upper bounds of the zone within which instruction should be pitched" (Wells, 1999, p.314). To consider the definition this literally is problematic, I would argue, because it is extremely difficult (if not impossible) to determine the boundaries of the two levels of development. Although Wells conceives the ZPD as personal, he includes "all aspect[s] of the learner - acting, thinking and feeling" (p.331) when he talks about learning in the ZPD.

Many scholars emphasise that the ZPD, as a concept, needs to be interpreted and understood in relation to Vygotsky's overall view on mental development, especially in relation to 'the general genetic law' (Veresov, 2004; Wells, 1999; Wertsch, 1984), play and imitation (Veresov, 2004; Chaiklin, 2003; Holzman, 2010), and the idea of 'learning-leading-development' (Holzman, 2010; Levykh, 2008; Veresov, 2017), among others.

Wertsch (1984) argue that "the zone of proximal development is an instantiation of Vygotsky's general genetic law of cultural development" (p.12), which is defined as follow:

> Any function in the child's cultural development appears twice, or on two planes. First it appears on the social plane, and then on the psychological plane. First it appears between people as an interpsychological category, and then within the child as an intrapsychological category. This is equally true with regard to voluntary attention, logical memory, the formation of concepts, and the development of volition (Vygotsky, 1981, p. 163).

Veresov (2004) emphasises that it is important to recognise that Vygotsky never regarded the child's mental functions to appear *in* social relation, but *as* social relation, and he quotes Vygotsky: "every higher mental function, before becoming internal mental function, previously was a social relation between two people. … All mental functions are internalized social relations" (Vygotsky, 1983, p.145 -146, in Veresov, 2004, p. 5).

According to Veresov (2017) the concept of ZPD is a concretisation of the general genetic law *applied on* issues related to the relationship between learning and development. Vygotsky, (1987) highlighted that instruction (teaching-learning) and development are two different processes. He proposed that a fundamental feature of instruction is that:

> *Instruction is only useful when it moves ahead of development*. When it does, it *impels or wakens a whole series of functions that are in a stage of maturation lying in the zone of proximal development* (ibid., p. 212, emphasis in original).

So, to this way of conceiving the ZPD, the ZPD comprises maturing functions that initially lay in the social as social relations, which, through the developmental process, become individual. The dialectic relationship

between learning and teaching gives rise to the ZPD and consequently to the child's internal development. The learning-teaching activity is thus the source of development because it creates the ZPD. Learning is not the outcome, it is the point of departure for development.

The conceptualisation of the ZPD has changed from looking at the ZPD as a property or an attribute of an individual, toward a view that the ZPD as a collective process (Holzman, 2010; John-Steiner, 2000; Levykh, 2008), or a collective 'space' (Abtahi, Graven, & Lerman, 2017; Hussain, Monaghan, & Threlfall, 2013; Mercer, 2000; Roth & Radford, 2011). Although several scholars regard the ZPD as being collective, they have different approaches to what it means to be collective. Mercer (2000) considers the ZPD as being part of real-life activity where both participants (the teacher and the student) contribute in creating the ZPD. However, he still focuses on the asymmetrical relationship between the participants, one being the teacher (more knowledgeable) and one being the learner (less knowledgeable). Goos, Galbraith, and Renshaw (2002) on the other hand, argue that the ZPD always has a two-way character, because the teacher and the students always appropriate each other's ideas. They want to eradicate the expert-novice distinction and move toward equal status interaction, where the participants coordinate their actions in order to achieve progress. Similarly, Zack and Graves (2002) emphasise that both the teacher and the children are learning in problem-solving situations, and they argue for a conception of the ZPD as an intellectual space where the children's and the teacher's knowledge and identities are formed and transformed in moment to moment interaction. Zuckerman (2007) argues that ZPDs are created as establishments of interactions in a specific time and place, where different minds 'meet', and create new formations of interactions. 'Meeting' of different minds is a moment of understanding or harmonisation but also a moment of transformation. This 'meeting' of two minds (consciousnesses) is possible whenever their interaction is coordinated and in process of harmonisation with each other. Recently some scholars have extended the notion of the 'more knowledgeable other' to include artefacts. Abtahi et al. (2017) suggest that the ZPD should be considered as multi-directional, instead of a bi-directional, where the role of the more knowledgeable other shifts between the child, the artefact (the properties of the artefact), and the adult.

Roth and Radford (2011) regard the ZPD as a symmetrical space where the teacher and the students are teachers and learners of each other. They draw on Bakhtine [Volochinov], (1977) who argues that in a conversation every word has two sides, the speaker's and the listener's sides, which lay the ground for regarding the ZPD as a symmetrical space. Roth and Radford (2011) illustrate how a ZPD emerges from the

sympractical activity arising between a teacher and a student in a Canadian 4th grade, working on an algebra task. They highlight that both participants learn in the activity: one learning mathematics and the other learning pedagogy. In Roth and Radford's (2011) view, the ZPD is not a mental zone (like a hypothetical learning potential in the mind), it is in the social relation between the child and the KT. In Roth and Radford's view the participants, through their sympractical activity, create a space where they mutually work to expand each other's action possibilities in order to move the activity forward. When a participant's actions expand the other participant's action possibilities, that '(inter)action space' is called the zone of proximal development. The space does not exist in advance. The participants need to produce the space during activity:

> The space is actually something that arises in and from their societal relation and cannot be conceived apart from it. They have to develop this space together without knowing beforehand what it might look like and how to create it. … Because neither one knows what actions will make this developmental zone, it can only emerge without that this emergence could be anticipated. This means that the participants come to realise consciously the possibilities that lie in their actions a posteriori (Roth and Radford, 2011, p. 71).

The ZPD emerges when the KT interacts with the child and the child interacts with the KT. Both are responsible for the emergence of the ZPD, thus both needs to engage in the sympractical activity. The ZPD emerges from the joint labour of expanding each other's actions possibilities but, as Roth and Radford (2011) say, the participants can only be aware of the potentiality that lays in their actions in retrospect.

Following Vygotsky (1987) and Veresov (2017), this research study conceives the ZPD in terms of social relations that emerge from the participants' joint activity. Roth and Radford's (2011) approach is used to investigate the emergence of the zone. Paper 3, in this research study investigates how a KT and a child co-create the ZPD by expanding each other's action possibilities and focuses especially on the role of the child in the co-creation. Paper 3 is presented in Section 7.3.

5 The Agder project on supporting school readiness: the case of early mathematics teaching and learning

The present study is situated within a research and development project called the Agder Project[10] (AP), which was briefly introduced in Chapter 1. The overall aim of the AP is to develop an intervention programme that prepare Norwegian 5-year-olds for school and to investigate the effects of the programme. The school readiness intervention programme focuses on four sets of competences: social-emotional, self-regulation, language, and mathematics. Research provides evidence that these four sets of competences are important for long-term success in school, education and professional life, (Duncan et al., 2007; Entwisle et al., 2005). Children from disadvantaged or low socioeconomic backgrounds seem to be at risk of falling behind in these important skills already at an early age, (Sektnan, McClelland, Acock, & Morrison, 2010; Wanless, McClelland, Tominey, & Acock, 2011). As mentioned in Section 2.5, research has found that some of the positive effects of high-quality early childhood education may fade in primary school if the transition between kindergarten and school is not well prepared (Lillejord et al., 2017; OECD, 2017; Peters, 2010). And this often affects children from disadvantaged backgrounds the most (OECD, 2017). Therefore, high quality early childhood education and well-prepared transition activities between kindergarten and school are important aspects for how children will manage school, integrate in the society and make a successful transition into the labour market (Lillejord et al., 2017; OECD, 2017). The AP wants to investigate whether the school readiness intervention programme contributes to decrease the gap between children of advantaged and disadvantaged families. In addition, the AP aims to investigate whether positive effects, if found, will last through children's lives.

This chapter comprises two sections. Section 5.1 presents the research design of the AP including a timeline which illustrates both the development and the research process. In Section 5.2 the pedagogical principles and the four competence areas in which the intervention programme builds upon are presented.

5.1 The Agder Project: research design

As mentioned above, the AP aims to develop and investigate the effects of an intensive school readiness intervention programme for 5-year-old children in Norwegian Kindergartens. The AP uses a randomized controlled trial design (RCT) with 701 five-year-old children in 71 kindergartens in the Agder counties in Southern Norway. The research design of the AP comprises four phases: preparation, professional development, intervention, and a research and documentation phase. Figure 5.1 gives an outline of the research design.

In the academic year 2014/2015 preparations were made for the implementation by reviewing literature on early childhood education, planning the professional development course and drafting the content of the intervention programme. In addition, kindergartens in the Agder counties in Norway were invited to participate, and 71 voluntary kindergartens were randomised in a control and a focus group. The intervention programme was inspired by similar intervention programmes in the US and designed by a multi-disciplinary team of researchers, including myself, who each had expertise in one of the competence areas. How the intervention programme was inspired by similar programmes in the US and how it was designed (including pedagogical principles and content) will be elaborated in Section 5.2.
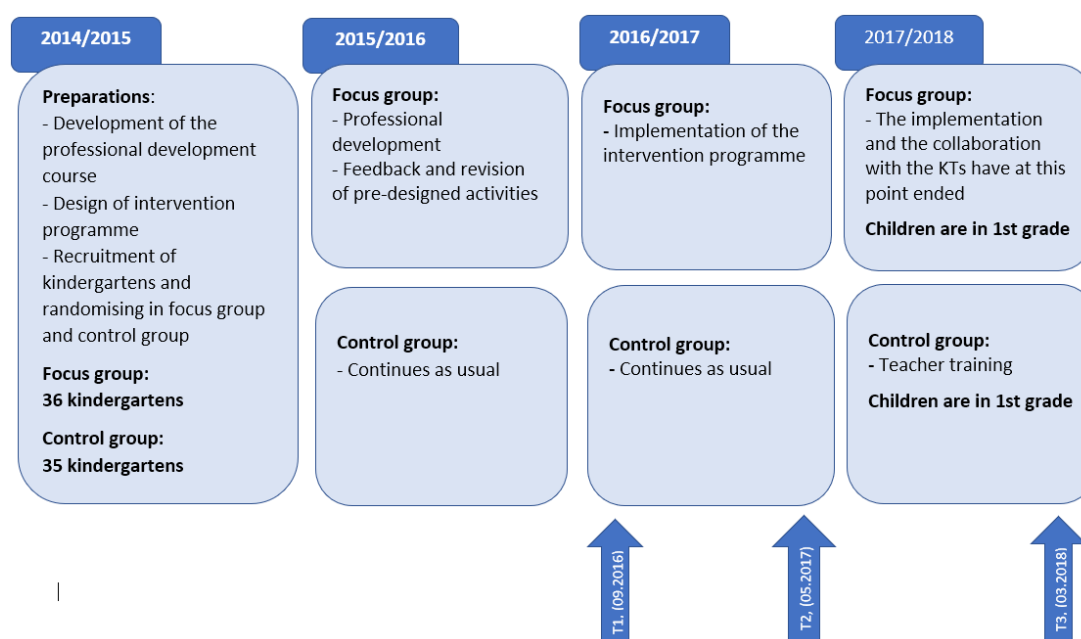

Figure 5.1: Research design of the AP, timeline

In the academic year 2015/2016 the KTs in the focus group participated in a 15 credit points professional development course, which served as preparation for the implementation of the intervention programme. The

professional development course was organised as four two-day gatherings. Each two-day gathering was organised as a combination of lecturers and workshops where relevant literature was presented, and the pre-designed activities within the four sets of competence areas were presented, discussed and improved. The lecturers were practice-oriented and focused on key elements drawn from the research literature on which the intervention programme built. The lecturers emphasised the importance of systematic early childhood education and the importance of school preparation activities. The lecturers also focused on the importance of child-adult relationship as well as key characteristics of playful learning and inquiry as pedagogical approaches in early childhood. Moreover, in the lecturers, key theoretical constructs and ideas from the literature on self-regulation, social-emotional competences, language and mathematics were presented.

In the workshops the pre-designed activities were distributed between the KTs where the KTs were asked to pilot the activities with their current groups of 5-year-olds between the gatherings. Since this was the year before the implementation, the children were not part of the focus group in the AP. Moreover, the KTs were asked to provide written reports of their implementation where they reflected on their experiences with implementing the activities that they piloted. The KTs sent the reflection notes to us researchers before the next two-day gathering. In the subsequent workshop the KTs were organised in groups and asked to share and discuss their experiences with implementing the activities and how the activities aligned with their understanding of the literature presented in the lecturers. In addition, the KTs provided suggestions for revision of the activities. They also suggested alternative activities which were integrated into the intervention programme. The programme of one of the two-day gatherings is provided as an example in Appendix 1.

The developmental part of the project had parallels to design research (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003), because the activities were developed through iterative cycles during the project. The first draft of some of the activities was piloted during the preparation phase in 2014/2015, in a voluntary kindergarten in the Agder-region which was not participating in the AP. Later the activities were piloted during the professional development course in 2015/2016, as elaborated above. Figure 5.2 gives an outline of the cyclic organisation of lecturers, workshops and piloting of activities, which resulted in a continuous revision of the activities.

The refined activities within the four sets of competence areas and outline of the theoretical foundation of the AP were comprised in a book

called 'Lekbasert læring'[11] (Størksen et al., 2018). The book was written by the researchers in the AP which had developed the content of the intervention programme. The front page of the book is presented in Appendix 2.
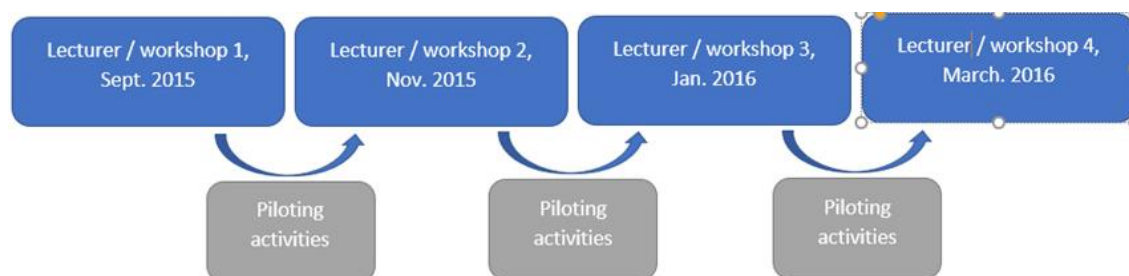
Figure 5.2: An outline of the cyclic organisation of lecturers, workshops and piloting of activities, 2015/2016.

In the academic year 2016/2017, the intervention was carried out in the 36 kindergartens in the focus group. The 35 kindergartens in the control group continued as usual. A pre-test (T1) was conducted before the implementation (08.2016) in all 71 kindergartens and similar a post-test (T2) was conducted after the implementation (06.2017) in all kindergartens. A follow-up-test (T3) was conducted in the spring 2018 when the children were at the end of their first grade. All three tests used the same established assessments, which were validated in the Norwegian context. Each test took approximately 40 minutes for each child and were conducted by trained and certified testers who were blind to treatment status. The tests assessed mathematics, language, working memory and inhibitory control. Inhibitory control and working memory are important components of executive functioning (which is also referred to as self-regulation). Social-emotional competences were not assessed due to lack of validated tests in the Norwegian context. In addition, registry data from Statistics Norway was collected, including the following variables: the child's gender, birth month, mother's and father's education, mother's and father's earnings and the child's immigrant status.

During the intervention year the KTs were invited to two two-day gatherings (September 2016 and March 2017), where they reflected on their experiences with implementing the intervention programme, and where they got feedback from us researchers for how to address possible issues or concerns. In addition, all KTs had scheduled phone meetings with one of the researchers each semester during the intervention. The scheduled phone meetings were conducted as semi-structured interviews

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[11] 'Lekbasert læring' is the Norwegian translation of playful learning.

which focused on the KT's experiences with implementing the intervention programme, how the centre administration supported them in the implementation and whether they succeeded in adopting a playful and inquiry-based approach to the learning activities.

The documentation and research phase is ongoing, and the results from the main study, on possible effects of the programme has not yet been published. However, from an initial examination of the data from the T1 test, the data shows an interesting interrelationship between children's performances (test scores) and their mothers' education. Figure 5.3 illustrates children's test scores on the T1 test, where the oldest children were 5,5 years old and the youngest children were 4,5 years old. The red line represents the children having mothers with higher education (university), and the blue dotted line represents the children having mothers with lower education (completed upper secondary school or less). Figure 5.3 illustrates that the oldest children, having mothers with lower education, are performing similar to six months younger children, having mothers with higher education. This shows that children's performances depend on their mothers' education, and in this case, it made a 6 months difference. It is important to notice that Figure 5.3 does not illustrate the variation within the two groups and how it develops over time.
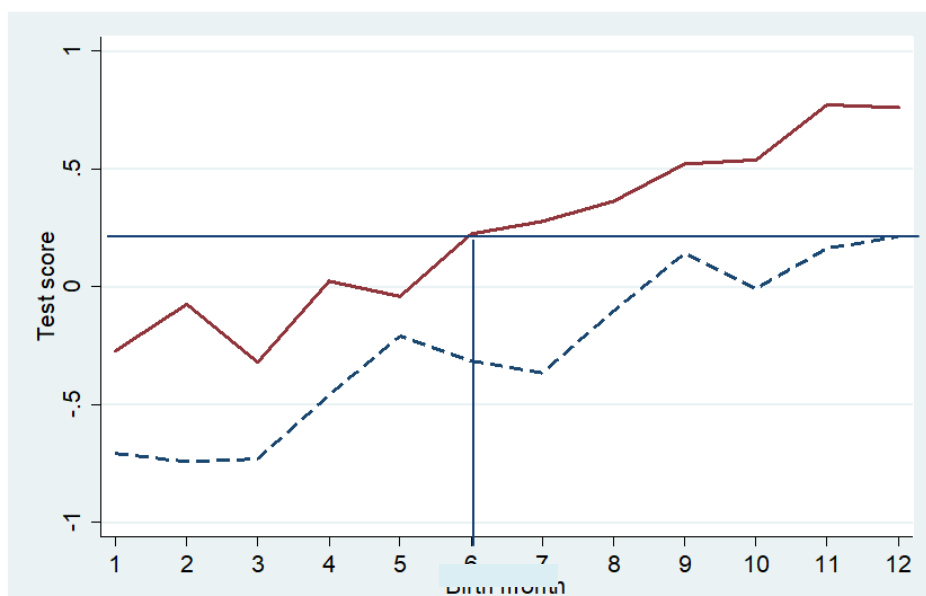
4,5 years old   5,5 years old

Figure 5.3: An illustration of the interrelationship between children's test scores on the T1 test and their mothers' education. The y-axis represents test scores, where 0 is the average, and 1 is standard deviation. The x-axis represents the increase of age in months.

Moreover, there has been published a study using data from the AP which focuses on differences in kindergarten quality and its effect on children's school readiness (Rege, Solli, Størksen, & Votruba, 2018). The study uses data from the T1 test among 627 five-year-olds in 67 different kindergartens participating in the AP. By controlling for children's family background (for example their mothers' education), the study shows that there is significant variation in school readiness across kindergartens in the project. Since the study controls for family background, the study claims that the differences can be explained by kindergarten quality.

5.2 Pedagogical principles and content in the Agder Project

The pedagogical principles used to design the intervention programme are based literature on important aspects for young children's academic as well as social-emotional development and for school readiness (which was elaborated in Section 2.4). The fundamental pedagogical principles are based on literature on teacher-child relationships (Pianta, 1999) and playful learning (Hirsh-Pasek et al., 2009). In addition, the mathematical activities are based on inquiry as an additional pedagogical principle (Jaworski, 2005; Wells, 1999). The pedagogical approach in the AP is illustrated in Figure 5.4 below.
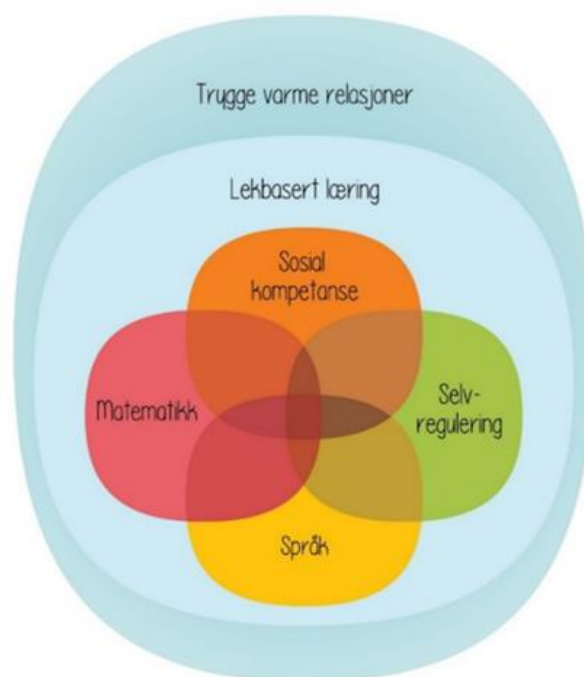

Figure 5.4: An illustration of the pedagogical model in the AP (Størksen et al., 2018). The outermost circle represents teacher-child relationships which is the fundamental pedagogical principle in the AP. The next circle represents playful learning. The four circles in the middle represent the four competence areas in the AP: social-emotional, self-regulation, language, and mathematics.

The principles of playful learning and inquiry, as were elaborated in Section 2.3 and in Chapter 3, respectively, appear in the activities through suggesting practical organisation, materials, questions etc. Concrete materials (building blocks, toys etc.) that the children are familiar with are often used as a point of departure in the activities and which aim to be a source of play and inquiry. In the activities there are examples of questions that might be posed by the teacher and ways to orchestrate the activities which aim to guide the children into playful and investigative situations. However, it is heavily emphasised that these suggestions are not meant to be followed to the letter. The KTs are encouraged to implement the activities in a sensitive manner focusing on children's engagement and children's own initiatives to inquire into mathematical problems. Breive et al. (2018) propose that there is an interrelationship between playful learning and inquiry, where both concepts highlight the balance between freedom and structure. Figure 5.5, below, illustrates this relationship. Whenever there is free play, there are potentially high degree of inquiry, however not necessarily related to mathematics. If the KT involves him-/herself in the play and guides the children into mathematical problems, the potentiality for inquiry into mathematics increases. On the other hand, if the guided play turns into direct instruction, the degree of inquiry also decreases and at some point, there is no inquiry left.
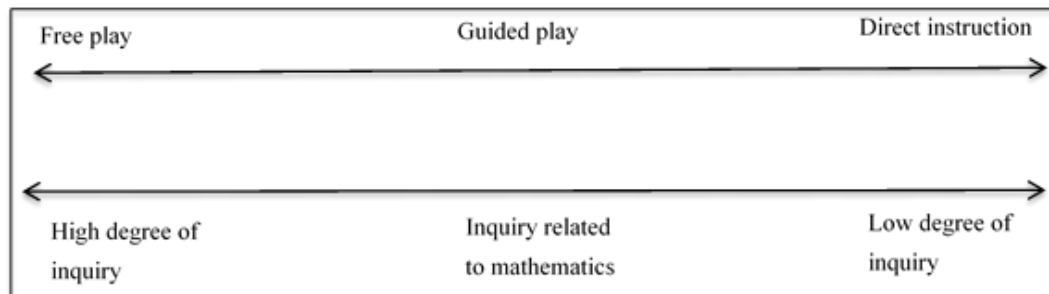
Figure 5.5: An illustration of the interrelationship between playful learning and inquiry found in (Breive et al. 2018).

As mentioned above, the intervention programme was inspired by similar intervention programmes and preschool curriculums in the US such as I Can Problem Solve (Shure, 2000), Interactive Book Reading (Mol, Bus, & de Jong, 2009), Building Blocks (Sarama & Clements, 2009), California Preschool Curriculum Framework (Californian Department of Educaton, 2010), Tools of the Mind (Bodrova & Leong, 2007), and Red Light, Purple Light (McClelland & Tominey, 2015). In addition, the intervention programmes had to fit the Norwegian kindergarten tradition and the Norwegian Framework Plan, and therefore, playful learning and inquiry as pedagogical principles were found suitable.

Inspired by the above-mentioned intervention programmes and pedagogical principles, about 130 activities within the four competence areas were drafted. The text in each activity was organised by intention, preparation, materials needed and implementation. An example of one of the mathematical activities is provided in Appendix 3. The aim of the activities was to promote the KTs to implement the activities thoughtfully and intentionally, and to implement a more structured practice than what is common in Norwegian kindergartens. However, the activities were designed as suggestions, not as strict manuscripts which the KTs had to follow to the letter. The KTs were encouraged to make the activities 'their own' and fit the activities to their kindergarten practice. In addition, the KTs were encouraged to use the activities in a flexible manner to meet the needs of each child. These ideas were later explicitly communicated in the introduction to the activities when the intervention programme was comprised in a book called 'Lekbasert læring' (Størksen et al., 2018), as mentioned in Section 5.1. In addition to the 130 playful learning activities, the book contains an introduction to the theoretical foundation on which the intervention programme builds. In particular, it contains theoretical considerations on the four competence areas: social-emotional, self-regulation, language and mathematics. Furthermore, the book considers the importance of a positive and stimulating adult-child relationship and gives an introduction to playful learning and inquiry as pedagogical approaches. Moreover, the book contains suggestions for how to plan each day, each week, and each month, and it contains semester plans for the autumn and the spring. The semester plans suggest relevant activities during the academic year. During the implementation of the intervention the KTs were required to use at least eight hours per week divided on the four competence areas (40minutes on mathematics 40 minutes on language, 20 minutes on self-regulation and 20 minutes on social-emotional skills each day, four days a week). An example of the semester plan and monthly structure is provided in Appendix 4, and a suggestion for how to prepare one day is provided in Appendix 5.

As mentioned earlier, mathematics is one out of four competence areas in the AP. The activities in the AP aims at helping the KTs implement the national curriculum, and to prepare children for school. Thus, the mathematical content in the activities are based on both the Framework Plan which is the national curriculum for kindergartens, (Ministry of Education and Research, 2006)[12], and the national school-curriculum for grades 1-2 (Ministry of Education and Research, 2013). The Framework Plan emphasises three main areas of mathematics: numbers, spaces

[12] A new Framework Plan for contents and tasks in Norwegian kindergartens was developed in 2017, but during the developmental phase of the AP it was the Framework Plan from 2006 that was the prevailing plan.

and shapes. The school-curriculum for grades 1-2 emphasises numbers, geometry, measuring and statistics as the main content areas. In addition, the mathematical content in the activities is also inspired by the Building Blocks material Sarama and Clements (2009), which consist of a comprehensive collection of mathematical activities sorted in three main areas of mathematics: number and quantitative thinking, geometry and spatial thinking, and geometric measurement. Based on the above-mentioned literature the mathematical activities in the AP focused on four areas: numbers, geometry, measuring and statistics. Numbers was essential in all three sources of inspiration. Geometry comprises both spaces and shapes in the Framework Plan, and spatial thinking and elements from geometrical measurement in the Building Blocks. Measuring is not explicitly mentioned in the Framework Plan but emphasised in the school-curriculum for grades 1-2 and can be identified as geometrical measurement in the Building Blocks. Statistics was only mentioned in the school-curriculum for grades 1-2. However, we regard statistics in early years as 'meaningful counting' where the children collect information from their surroundings, and later sort and compare the information. For example, children may collect different toys in the kindergarten, sort them in categories, count how many toys are in each category and compare the information to make some conclusions. Thus, we found statistics as suitable for the content in the activities in the AP. The Building Blocks programme builds on hypothetical learning trajectories within the above-mentioned content areas, which became too structured for the Norwegian kindergarten context. Therefore, we used some of the mathematical problems, and not the structure of the Building Blocks programme, as inspiration for the mathematical activities in the AP.

Within numbers, the activities focused on counting, identifying numerocities, addition and subtraction. Moreover, the activities could be extended to involve beginning multiplication and division. Within geometry, the activities focused on identifying 2-dimentional and 3-dimentional shapes and the relationships between them. Geometry also included activities that focused on geometrical patterns or symmetries. Within measuring, the activities focused on comparing lengths, areas and volumes by direct and indirect comparisons. As mentioned above, statistics was described in the intervention programme as a 'meaningful way of counting'. The activities focused on various ways of collecting and sorting data, representing the data in diagrams or columns (for example as towers of building blocks), and then comparing the information involving counting, identifying numericities, addition and subtraction. The activities also involved combinatorial problems, for example how many unique ways can you dress a doll with two different sweaters and two

different skirts. The children were allowed and prompted to use carton dolls, and paper sweaters and skirts to solve the problem.

Since this research study considers children's mathematical reasoning processes, I do not go into details about the activities in the three other competence areas, rather just briefly explain what they include. First, language involved activities that mainly focused on principles of interactive book reading (Mol et al., 2009), and related activities such as pre-reading with children to create engagement for the book, identifying focus words in the book during reading and retelling the story after the reading session. Moreover, it included book related activities such as drawing and drama. The activities also involved rhymes and identifying letters and words.

Self-regulation relies on underlying executive function processes such as attentional or cognitive flexibility, working memory, and inhibitory control (Blair, 2002). Thus, this area involved activities that focused on children's ability to use attentional flexibility, working memory, and inhibitory control. For example, in the activity called 'ready, steady, go' the children stand on a line and are supposed to run if the KT says "ready, steady, go!". But if the KT says "ready, steady, gorilla!" the children are supposed to inhibit their impulse to run and stay put.

In the AP, social-emotional competences include self-control, assertiveness, responsibility, collaboration and empathy, and the activities related to social-emotional competences focused on these 'skills'. For example, in an activity called 'the gingerbread man' children are encouraged to express their emotions by colouring a gingerbread man outlined on a paper. The children are then encouraged to talk about their drawing and consequently their emotions.

6 Methodology: accessing and analysing processes of inquiry in kindergarten mathematics

The purpose of this chapter is to present methodological approaches developed for investigating processes of mathematical inquiry in kindergarten. The aim is to comply with Burton's (2002) quest for transparency and clarity of choices made when planning and carrying out an empirical research, and to give reasons for those choices. Wellington (2000) defines methodology as "the activity or business of choosing, reflecting upon, evaluating and justifying the methods you use", (p.22). The aim is therefore to elaborate on the strategy and design of my research, including the methods used for data collection and data analysis. In other words, my aim is to discuss how I planned and conducted my study and why I think this was an appropriate way to do it.

In what follows, an elaboration on the research paradigm and research strategy is provided in Section 6.1. The section is followed by an outline of the research design in Section 6.2, and an outline of the pilot study in Section 6.3. Section 6.4 gives brief background information of the five KTs participating in the research study. Section 6.5 and 6.6 is devoted to an elaboration on the data collection and the data analysis conducted in this research study, respectively. The chapter ends with considerations about trustworthiness in Section 6.7 and ethical considerations in Section 6.8.

6.1 Research paradigm and research strategy

The research study reported here, which aims to capture dynamics of teaching and learning mathematics and processes of mathematical inquiry in kindergarten, is conducted within a cultural-historical paradigm. A paradigm is a set of ideas, or a world view which consists of three interrelated elements: ontology, epistemology and methodology (Denzin & Lincoln, 2013; Lincoln & Guba, 1985). Entangled in these three terms are the researcher's ideas and beliefs about reality and how reality can be known (Denzin & Lincoln, 2013). My world view aligns with the cultural-historical perspective and theory of knowledge objectification (Radford, 2013b). The object of study is a consequence of my ontological and epistemological beliefs and the theoretical platform I am standing on. This also influences considerations about how to formulate research issues and research questions and how to investigate them. This aligns with Radford and Sabena (2015) who argue that research methods must be viewed as incorporated in theory, with its epistemological and ontological foundations: the action of using a theory (theorizing reality)

also includes using methods that are embedded in, or fit, the theory. Therefore, to study mathematical teaching and learning processes in kindergarten from a cultural-historical perspective I take a qualitative approach to data collection and data analysis.

This research study, as part of the AP, has parallels to design research (Cobb et al., 2003), because the activities were developed through iterative task-design cycles during the project (cf. Section 5.1). The pre-designed activities were implemented by the KTs in the project where the aim was to investigate processes of mathematics teaching and learning within these kindergartens. This coincides with what Cobb et al. (2003) argue that "Design experiments have both a pragmatic bent - "engineering" particular forms of learning - and a theoretical orientation - developing domain specific theories by systematically studying those forms of learning and the means of supporting them" (p. 9).

On the other hand, this particular case study[13] which is conducted within a cultural-historical paradigm, may be regarded as falling under a naturalistic approach (Lincoln & Guba, 1985). The naturalistic approach is based on the assumptions that realities are holistic and multiple and cannot be understood in isolation from their context. The aim of the investigation is not to develop 'truth' or generalisable knowledge independent of time and context, but to develop an understanding of the phenomenon within its context, which may illuminate and inform similar phenomenon in different contexts. In naturalistic approaches, the reality is holistic and thus cannot be fragmented into parts and understood separately or as causes and effects. Since reality in naturalistic approaches is seen as holistic, the object of inquiry is influenced by the researcher (and via versa). The research is dependent on the researcher and his/her values which is noticeable through his/her choices of problems, theory, methods etc. (Lincoln & Guba, 1985).

As this study falls under a naturalistic approach, the aim is to carry out the study of the phenomenon (mathematical inquiry) in its natural setting (cf. Lincoln & Guba, 1985). Since this study is part of the AP, where teaching and learning mathematics is studied within an intervention, one may question whether this study meets Lincoln & Guba's requirement. However, all teaching and learning activities in kindergarten (and in school) may be argued to have interference from others through the curriculum and/or through suggestions about how to carry out mathematical activities. Thus, the designed activities in the AP (as a parallel to the curriculum in ordinary kindergartens), may be considered as an important part of the context referred to in the naturalistic approach. The

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[13] The case study design is elaborated in the next section (Section 6.2).

case study reported here aims to capture processes of mathematical inquiry within its context, being the intervention programme in the AP in this case.

6.2 Case study design

Every empirical research study needs or has, even if it is unintended, a design, that is a plan for getting from the initial research issues to a conclusion (Yin, 1994). The plan guides the entire process which includes elaboration of research issues, consideration about what could be relevant data, how to collect data and considerations about data analysis. A thorough design provides a sustainable frame and useful guidelines for conducting a thorough research study.

This research study uses a case study design to investigate processes of mathematical inquiry in kindergarten. In a naturalistic approach a case study is the most common design, because it gives opportunities for giving thorough description of the holistic reality and focus on capturing the phenomenon in its natural setting (Lincoln & Guba, 1985). According to Yin (1994) a case study is an empirical research study that "investigates a contemporary phenomenon (the 'case') in depth and within its real-world context, especially when the boundaries between phenomenon and context may not be clearly evident" (p. 16). According to Bryman (2012) a case study design implies an intensive and detailed analysis of a single case or a few cases. The analysis often focuses on how the case develops over time related to a particular context (Bryman, 2012). Both definitions emphasise that case studies focus on an 'in depth' investigation of a phenomenon, within its specific context. Flyberg (2013) argues that "The decisive factor in defining a study as a case study is the choice of the individual unit of study and the setting of its boundaries" (p. 169).

Based on a cultural-historical perspective, this research study takes 'processes of mathematical inquiry in kindergarten' as the object of study. The object of study was selected because of personal curiosity and from a knowledge gap in the research field. The three research issues guiding the study, stated in Section 1.2, were formulated having this object of study in focus. To capture the object of study, I needed a concrete case in which phenomenon to be studied could be materialised. Processes of mathematical inquiry in kindergarten, from a cultural-historical perspective, unfolds through joint mathematical activities, involving KTs and children. The case is therefore decided to be 'specific segments of interaction where a KT and children work with (inquire into) a mathematical problem'. However, I am not interested in *any* processes of mathematical inquiry, I am interested in the processes that unfold within the pre-designed mathematical activities in the AP which aim to prepare children for school, which is then the boundaries of my case.

This case study may also be regarded as a 'theory testing' case study (Bassey, 1999), because it investigates how inquiry, as a theoretical construct, may be conceived in a kindergarten context. The aim is to investigate whether the empirical findings support the initial understanding of inquiry as a theoretical construct and a normative educational approach, or whether modifications must be made with respect to the particular context in which this study is situated.

The above considerations led me to consider what data I needed, how much data I needed and how to analyse the data. These elements in the design have been revised several times during the research process and are elaborated in Section 6.5 and 6.6. According to Yin (1994), a case study design cannot be considered as an initial plan only, but also as a part of the practice of carrying out the investigation. This accord with this research study, where the plan has been continuously revised during the investigation.

6.3 Pilot study

The main study took place during the implementation-year of the AP, the academic year 2016/2017. As part of preparations for the main study I conducted a pilot study in the end of February 2016. The pilot study had several aims. One of the aims was to get practical experiences of how to conduct a case study, including collecting and analysing data. The second aim was to get indications of whether modifications had to be done with respect to the research issues, the theoretical framework, or the methodology. The pilot served as an important step in the further development of the research design.

The pilot study took place in one of the kindergartens in the focus group of the AP. At the time of the pilot study the KT was attending the professional development course in the project and were piloting some of the pre-designed mathematical activities developed in the project. I visited the kindergarten on two occasions, 22.02.2016 and 29.02.2016 where the KT worked with reflection-symmetry on both occasions. With respect to the research focus, which was stated in the previous section, I took a participant observation approach (Bryman, 2012) to data collection and collected data by video-recordings and field notes. In addition, I had an unstructured talk with the KT between the first and second lesson, where we discussed his experiences of implementing the activities. During the first visit I observed that the children were quite silent during the activity. The children eagerly participated in the activity, but mainly with physical contributions. In other words, the children were physically 'doing' mathematics, like drawing reflection symmetrical pictures or identifying reflection symmetrical objects in the room. Those experiences re-

sulted in an email to the KT in which I asked him to try to get the children more involved in the mathematical conversation. During the second visit, the KT seemed more concerned about promoting the children to talk about and explain their mathematical ideas.

From the pilot study I produced a paper focusing on structural aspects of children's argumentation related to reflection symmetry. With structural aspects I mean the way in which the children structurally build, step by step, an argument for a claim. The aim was to characterise structural aspects of the ongoing argumentation and to examine what role various semiotic means (e.g., artefacts, linguistic devices and gestures) played in the ongoing argumentation. Children's argumentation is considered as one of the main features of inquiry and important for children's learning (Dovigo, 2016; Mercer & Sams, 2006; Yackel & Cobb, 1996). The results from the pilot study are reported in Paper 1, which has the title, 'Kindergarten children's argumentation in reflection symmetry: The role of semiotic means'. The data presented in the paper is taken from the second visit, 29.02.2016.

From the experiences of conducting the pilot study, the following modifications were made in the main study:

- In the pilot study I had only one camera during the data collection. This was not sufficient to capture both the KT's participation and the children's participation in the activity, simultaneously. In addition, the children were sitting around a table, which made it hard to capture all actions of the children sitting with their back to the camera. In the main study I therefore decided to use at least two cameras.
- In the pilot study I had a conversation with the KT after the first session, which was unintentional. However, I found this conversation insightful. It helped me to get insight into the KT's initial intentions and his experiences of implementing the activities, which helped me in the overall interpretation of the sessions. I therefore decided to use some time after each session in the main study to talk to the KTs, and to make field notes from the conversations. These conversations were intended to be unstructured, informal conversations, where I and the KTs could talk unaffected of any video or audio recordings.
- In the pilot study I realised that inquiry processes could take many forms and occur in a variety of settings. I also assumed that the nature of inquiry processes would vary from kindergarten to kindergarten. It was important, for me, to capture a variety of situations, and therefore to collect a quite extensive amount of data material in different kindergartens in the main study.

- In addition, the data analysis in the pilot study helped me to plan the data analysis in the main study. The results of the pilot study (which is presented as Paper 1 in Section 7.1) indicate the significant role of semiotic means, other than language, for understanding children's argumentation. In the main study I therefore increased the focus on children's (and the KTs') multimodal participation in the activities. The data analysis in the main study therefore focuses on identifying actions, both verbal and especially non-verbal, which play an important role for constituting activities identified as mathematical inquiry.

6.4 Participants in the main study

In the main study, five volunteer KTs from the focus group of the AP participated. All KTs were experienced kindergarten teachers, with at least 3 years higher education. During the professional development course, I had time to get to know the KTs well, and it was not difficult to get some of them to voluntarily participate. The children participating in this research study are also part of the focus group of the AP. In the first observation period[14] the children were 4,9-5,9 years old and in the second observation period they were 5,3-6,3 years old. I have no further background information about the children. In the following I will briefly describe the five participating KTs' education and working experience.

KT1: Trained for 3 years in higher education, comparable to bachelor training, to become a kindergarten teacher. In addition, she has 30 credit points in preschool pedagogy. KT1 has worked with children from all ages, but from 2015 she has only worked with the oldest children (age 3-6). KT1 works in a public 'nature and theatre kindergarten', where arts and drama have great emphasis in the daily practice. Outdoor life and experience of nature are also valued. KT1 has worked within the current kindergarten since 2007, where she started working as a pedagogical leader in 1999.

KT2: Trained for 3 years in higher education, comparable to bachelor training, to become a kindergarten teacher and graduated in 1999. In addition, she has 30 credit points in counselling, and 30 credit points in infant-pedagogy. In 2000 she established the kindergarten where she is currently working, together with colleagues. It is a cooperative kindergarten, where parents take a share in the cooperative when their children enter kindergarten. Outdoor life and experience of sea and nature have great emphasis in the kindergarten's daily practice. The last 3 years KT2

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[14] The data collection in this research study was conducted over two observation periods, which will be further elaborated in the next section (Section 6.5).

has had responsibility for the oldest children (age 4-6) in the kindergarten.

KT3: Trained for 3 years in higher education, comparable to bachelor training, to become a kindergarten teacher and graduated in 2004. He has worked as a KT in 15 years, of which 9 of them within the current kindergarten. In his whole career he has had responsibility for the oldest children (age 3-6). KT3 works in a private kindergarten which is owned by an international company that has kindergartens in Germany, Sweden and Norway. The kindergarten focuses on research-oriented knowledge about what is important content for high-quality early childhood education and care.

KT4: Trained for 3 years in higher education, comparable to bachelor training, to become a kindergarten teacher and graduated in 2001. In addition, she has 90 credit points in special needs education, 30 credit points in pedagogical development work, and 30 credit points in counselling. KT4 works in a public kindergarten which equally focuses on free play- and outdoor activities as well more structured learning activities. The kindergarten is also part of another project called 'Være Sammen' [being together], focusing on how to build adult-child relationships in the kindergarten. KT4 has worked in the current kindergarten since 2004 and has worked with the oldest children (age 3-6) in her whole career.

KT5: Trained for 3 years in higher education, comparable to bachelor training, to become a kindergarten teacher. I have no further information about KT5, because she didn't respond to the email where I asked for her background information. KT5 works in the same kindergarten as KT4.

All of the KTs expressed that they were grateful to be part of the AP, and that they had positive experiences with implementing the activities designed in the project. This may be one of the reasons for why they eagerly shared their experiences and invited me into their kindergartens.

6.5 Data collection in the main study

In this research study I take a participant observation (ethnographic) approach to data collection. (Bryman, 2012) argues that participant observation and ethnography refer to similar (if not identical) approaches to data collection, because the researcher, to various degrees, is engaged in the social setting where the research study takes place. The rational for using a participant observation approach to data collection is the way in which I may address the research issues posed in Section 1.2 and the specific research questions posed in the five papers. I agree with Punch (2009) who argue that participant observation (ethnography) gives us a unique (and perhaps the only) access to understand human behaviour as a continually changing process in its cultural environment. This also accords with how Radford conceive activity as a dynamic 'flux' (Roth &

Radford, 2011). Ethnography, as a powerful naturalistic approach (Eisenhart, 1988), has developed within anthropology as a way to study culture and the people within the culture (Punch, 2009). The core enterprise of ethnography is to get first-hand experience with a research setting, that is the culture, and the interpretations thereof. The aim of this research study is to characterise processes of mathematical inquiry in kindergarten and what enables it to occur, and a participant observation approach to data collection enables me to get insight into these interactional processes and the culture in which mathematical inquiry unfolds. Processes of mathematical inquiry in kindergarten can only be disclosed from observing real-life activity between the children and KTs.

A participant observer can take many roles depending on the degree of involvement in the social setting. In this research study the main focus was to capture the participants interaction working on mathematical problems in their 'natural' setting, and thus observations of the participants activity without my interference was the main approach to data collection. However, in order to understand the activity from the KTs points of views, and for triangulation of data, semi-structured interviews and conversations with the KTs before and after each observed session, were also part of the data collection. My role as a participant observer may thus be regarded as a combination between 'observer-as-participant' and 'complete observer' (Bryman, 2012). The interference with the participants were minimal whenever the pre-designed mathematical activities were implemented. However, having conversations with the KTs and conducting interviews with the KTs required a higher degree of participation. In addition, I had close contact with the KTs during their professional development course during the academic year 2015/2016, the year before the intervention.

As mentioned in Section 6.2, a part of the case study design was to consider what data I needed, how much data I needed and within what contexts I should collect data to capture the object of study and address the research issues. To capture the object of study the case was decided to be 'specific segments of interaction where the KT and children work with (inquire into) a mathematical problem' within the boundaries of the mathematical activities in the AP intervention. Initially, I didn't limit the case to any particular mathematical content area. However, as it turned out, after the data collection and the second phase of data processing (which will be elaborated in Section 6.6), I limited the case to addition and beginning multiplication. With respect to the first research issue about the characteristics of mathematical inquiry in kindergarten, I needed a broad range of segments. As indicated in Section 6.3, which reports from the pilot study, mathematical inquiry processes may occur in a variety of settings. It was important, for me, to capture the variety, and

therefore to collect quite extensive data material in different kindergartens. To answer the second research issue about what enables mathematical inquiry in kindergarten to occur, I needed to capture events that happened both before and after the salient segments. Moreover, I considered it relevant to conduct interviews with the KTs to get insight into their views on inquiry. I decided to collect observations, field notes and interviews in four kindergartens in the AP to capture differences and similarities in the KTs' implementation of the activities and to capture a variety of inquiry segments. The third research issue, which considers how, if at all, processes of mathematical inquiry may prepare kindergarten children for school, is regarded as an exploratory issue. To answer this issue, I used results emerging from discussion of the first (and second) research issue and consider them in light of literature on important aspects for facilitating a smooth transition from kindergarten to school (cf. Section 2.4).

The empirical material was collected through ethnographic field notes, observations and interviews. The data collection was conducted in five kindergartens in the focus group of the AP over two observation periods during the intervention year. In the initial design I planned to observe four KTs (and their groups of children), 4 sessions in autumn 2016 and 4 sessions in spring 2017. However, during the time of data collection, I was invited to visit the KTs more than the agreed four sessions, which resulted in a data set of video-recordings (and associated field notes) containing 4-7 sessions in each kindergarten in each observation period. In addition, the KT in one of the kindergartens (K4) was sick one day, and without informing me in advance, she made an appointment with another KT in the same kindergarten to implement the activity. Thus, I ended up observing 5 KTs and their groups of children. After each observation period I conducted a semi-structured interview with each of the five KTs. An overview of all data sets in this research study is provided in Appendix 6.

During the observations I used two video-cameras. In the pilot study I observed that the KT and the children were sitting around a table, and as I assumed, the KTs and the children in most of the observed sessions were either sitting around a table or in a circle on the floor. During these observations, I used one video-camera focusing on the KT and the children nearby, and one focusing on the children sitting opposite the KT. In some sessions, especially in outdoor activities, the children were more spread out, often working in groups. During these observations, I had to make ad-hoc decisions of where to place the cameras, and I tried to focus on one or two groups of children.

Although video-recordings capture more of human interaction than for example audio-recordings, they have their limitations. It does not

capture everything that an observer sees. To meet this concern, I made field notes during each observation. Field notes are an often used method for collecting data in ethnographic research (Walford, 2009). In many cases the researcher does not have time to do a lot of written work during observation or during field work. Sometimes the researcher is fully occupied with participating or paying attention to the ongoing interaction, and thus 'jotting' down key words is the only possibility for collecting written material. Jottings are brief written records about impressions of the interplay between the participants and information about the context (Emerson, Fretz, & Shaw, 2011). These jottings help the researcher remember significant events when he/she is working on his/her field notes later. In this research study jottings were written during and/or after each observed session. To take notes distracted me from paying attention to the rich interplay between the KTs and the children and in many cases, it was also important to be flexible with the cameras. For example, when the KTs implemented outdoor activities (which many of them did) it was difficult to both write notes and handle the camera, thus jottings were the only possibility for collecting any written material and were made during or straight after each session. The jottings from the observed sessions were supplemented by conversations with the KT after each session. Jottings from the interviews were made after the interviews and were extended when I later watched the video-recordings of the interviews.

After each observation, when coming back to the office, the jottings were turned into field notes. Emerson et al. (2011) recommend that researchers write field notes immediately or soon after the fieldwork to ensure "fresher, more detailed recollections" (p. 40). I did not write 'full fieldnotes' according to Emerson et al. (2011), which implies a coherent, step-by-step story of what was observed. I rather turned my jottings into 'more comprehensive jottings' or 'basic field notes'.

Jottings and field notes should not, ideally, include interpretations and judgements of the observed interaction and the researcher should not try to identify (interpret) motives for the participants actions (Emerson et al., 2011). The researcher should rather try to capture bits of talk and action and describe the ongoing activity without interpretations. However, as Emerson et al. (2011) argue

> in turning jottings and headnotes into full notes, the fieldworker is already engaged in a sort of preliminary analysis whereby she orders experience, both creating and discovering patterns of interaction. This process involves deciding not simply what to include but also what to leave out (p. 51).

Field notes include descriptions of the situation, but these descriptions are not identical with what actually happened. These descriptions are 'coloured' by the researcher's beliefs, earlier experiences, values and theoretical lenses. In addition, there is always more going on than the re-

searcher notices. In this research study the object of study and the research issues were already made before conducting the field work. And my initial understanding of inquiry, described in the end of Section 3.2, guided what I was especially looking for. In the jottings and the field notes I included events (actions) that I thought were relevant and left out things that I thought were not that relevant. Thus, I was not just observing the ongoing activity with an 'open mind', I was observing with a specific purpose – to understand the nature of mathematical inquiry in kindergarten. Both jottings and field notes were therefore influenced by the object of study, the research issues asked and my initial understanding of inquiry and thus have to be regarded as interpretative and analytical. As Emerson et al. (2011) argue

> as the field researcher participates in the field, she inevitably begins to reflect on and interpret what she has experienced and observed. As previously noted, writing fieldnotes heightens and focuses these interpretative and analytic process; writing up the day's observations generates new appreciation and deeper understanding of witnessed scenes and events (p. 100).

The process of writing jottings and field notes helped me to understand the situation, and thus an analytic process had already started during observations.

As mentioned above, semi-structured interviews (Bryman, 2012; Kvale, 1996) were conducted with all five KTs after each observation period, which means that I ended up with two interviews with each KT in my data set. The semi-structured interviews were designed to inform the second research issue about what enables mathematical inquiry in kindergarten to occur. The aim was to get insights into the KTs' views on inquiry as an educational approach to mathematics in kindergarten, their epistemological beliefs and their stance towards the children and their learning. Besides informing the second research issue, the KTs' views could help me in my interpretation of the observed sessions to inform the first research issue about the characteristics of mathematical inquiry in kindergarten.

To guide the semi-structured interview an interview guide was devised. According to Kvale (1996) each question in a semi-structured interview guide has a thematic and a dynamic dimension. The thematic dimension relates to the object of study, and the dynamic dimension relates to the interpersonal relationship in the interview. The researcher should, in advance, consider how the questions, in the interview guide, relate to the research topic, the theoretical foundation of the research and how the analysis will be conducted afterwards. Moreover, the questions should consider how the questions can promote the interviewee to share his/her views on the topic. Therefore, the questions should facilitate a positive interaction between the interviewee and the researcher to keep the flow

of the conversation. In this respect the order of the questions is also important.

All semi-structured interviews in this research study started with a broad opening question about the KTs' experiences of implementing the mathematical activities in the AP. This was followed up by a question focusing on the KTs' experience of the activity in which the KTs had just implemented, allowing the KTs to talk about whatever they felt relevant. Sometimes the KTs had concerns or problems they wanted to discuss, in which I tried to help them with. These two questions were intended to create a trustful relationship. The interview then continued, focused around six issues: 1) what, in your view, are the main characteristics of an inquiry session in kindergarten 2) what do you think about your role (as a KT) in an inquiry session 3) what role do children's own questions play for children's mathematical inquiry 4) what kind of questions do children ask 5) how do you see the relation between inquiry and play 6) on a scale from 1-6 how much inquiry and how much play would you consider the activity just implemented to be? When the interview was about to end, I asked whether the KT had other issues he/she wanted to raise or discuss.

As mentioned above these questions were particularly conducted to inform the second research issue about what enables mathematical inquiry in kindergarten to occur. The aim was to get insights into the KTs' views on inquiry as an educational approach to mathematics. In addition, the aim was that the KTs' views could help me in my interpretation of the observed sessions to inform the first research issue. The KTs eagerly answered the questions and shared their views on inquiry. As mentioned above, the interviews were semi-structured allowing the KTs to talk about their concerns and interests, which often carried the conversation in other directions than intended. As (Bryman, 2012) agues, allowing the interviewee to 'ramble off' gives insights into what the interviewee sees as relevant, and therefore often valued in qualitative research. During the interviews I made 'jottings' (Emerson et al., 2011), which were turned in to more comprehensive jottings or field notes immediately after coming back to the office, similar to how I made field notes from the observations. Each interview was also video-recorded which gave me the opportunity to study the interviews in retrospect.

6.6 Data processing and data analysis

This research study seeks to investigate and understand processes of mathematical inquiry in kindergarten. Using cultural-historical activity theory as a theoretical frame, the unit of analysis is activity (Roth & Radford, 2011). Activity, as elaborated in Section 4.2, is the process that

unfolds through human actions. These actions are historical accumulations of social relations. What we can observe in activity is always social relations, which we later attribute to individuals as their actions (Roth and Radford, 2011). It is the process that unfolds through human actions (relations) that is the unit of my analysis. Incorporated in that is the movement of activity, its structures and dynamics. As also mentioned in Section 4.2, activity and consciousness are related. Activity is the substance of human consciousness. Consciousness as a sensory, perceptual 'picture' of the objective world, or as our relationship with the objective world, is deeply connected with activity (Radford, 2015a). Thus, in a deep sense, the nature of human consciousness is part of what the study is after.

As mentioned in Section 6.2, this research study takes 'processes of mathematical inquiry in kindergarten' as the object of study. To capture the object of study, I needed a concrete case in which the phenomenon could be materialised. The case was therefore 'specific segments of interaction where the KT and children work with (inquire into) a mathematical problem'. These segments are in the following called 'salient segments'. To filter out such salient segments from the extensive data material, the data had to go through two phases of data processing before the identified segments were analysed in depth in a third phase. These three phases will be elaborated in Sub-section 6.6.1 below.

6.6.1 Processing and analysing the empirical material

As mentioned above, the process of processing and analysing data went through three phases. In the first phase the fieldnotes from observations and interviews were contracted into profiles of the five KTs, focusing on their orchestration of the pre-designed activities and their interaction with the children. Instead of using the field notes as 'raw material' from which I started a coding process, I worked directly with the field notes and contracted them, through an iterative process, into profiles of the KTs. Since the field notes were already, to some extent, analytical (as described in Section 6.5), they served as a useful starting point for this purpose. The profiles were contracted and organised around four key points which were based on my initial understanding of inquiry: 1) The children's opportunities or freedom to participate with questions, mathematical ideas and argumentations: 2) The KTs intention[15] to consider children contributions, 3) Children's intention to participate with ques-

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[15] Here, and in 3 below, 'intention' could be qualified as 'my ascription of intention', in as much as I cannot be 100% certain about people's intentions. It is used in line with how Wells (1999) uses 'will' in his definition of inquiry: "a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them" (p. 121).

tions, ideas and argumentations, and 4) The degree of inquiry or problem-solving interactions in the session. These key points are not mutually exclusive, rather their relationship is of a dialectic character. The profiles constitute an initial, general impression of the five KTs and their interaction with the children and serve as important background information when the results are discussed, and conclusions are made in Chapter 8. The profiles are especially important when considering the second research issue about what enables mathematical inquiry to occur. These profiles were continuously and critically evaluated and refined during the next two phases of data processing (which will be elaborated below). The profiles of the five KTs are attached in Appendix 7.

The profiles of four of the five KTs were used as data in the multiple-case study reported in Paper 5, which focuses on the KTs' orchestration of the mathematical activities with respect to the degree of freedom given by the KTs for children's participation. The profile of the fifth KT was not included in the study, because of her limited participation. KT5 was a substitute for KT4 when KT4 was sick one day during the first data collection period, autumn 2016. In addition, KT5 voluntarily participated a couple of times during the second data collection period, spring 2017. However, her orchestration of the mathematical activities was not significantly different from the other four KTs' orchestration, and therefore didn't contribute with any significant insights to the issues investigated in the multiple-case study reported in Paper 5. A summary of Paper 5 is provided in Section 7.5.

In the second phase of processing the empirical material, the video data from all observed sessions were watched, and a 'rough transcription' of each session was made. These rough transcriptions are similar to what Emerson et al. (2011) would call 'full fieldnotes', that is a coherent, step-by-step story of what was observed. The transcripts were separated into parts (segments) and organised in tables with columns, containing time and description of the interaction (included some utterances) and were supplemented by video stills of observable actions. The transcriptions aimed to be as objective as possible, avoiding interpretations and analytical statements. However, as mentioned above, to be completely objective when writing descriptions is not possible (Emerson et al., 2011). Again, choices were made for what to include and what to exclude in the descriptions. During the transcription process side-notes were made and salient segments which seemed to indicate processes of mathematical inquiry were marked. Two examples from the rough transcriptions are presented in Appendix 8.

The process of making these rough transcriptions helped me to get an overview and insight into the empirical material that I had. In addition, the transcripts and the side-notes served as a useful tool when I selected

inquiry segments to be analysed in the third phase of the data processing, which I elaborate below. To identify inquiry segments, I used my initial understanding of inquiry (elaborated in Section 2.4), that is I was mainly looking for longer stretches of activity where the participants solved mathematical problems together through sharing, questioning, and explaining ideas, and by argumentation and reflection. Moreover, the children had to show 'eagerness' to participate with questions, mathematical ideas and argumentations. Once inquiry segments were identified, salient segments were selected on their suitability to inform the research questions formulated in the five papers.

In the third phase of data processing, a fine-grained analysis of the salient segments was carried out through an iterative process of four subphases: 1) watching videos, 2) transcribing (including non-verbal actions), 3) analysing and interpreting data related to the research question at hand, from a multimodal perspective, and 4) conducting intercoder reliability checks (this was not included in each cycle). This iterative process is illustrated in Figure 6.1.

While watching the videos of the selected segments interpretative transcriptions were made. In the first cycle, the transcriptions only contained utterances and interpretation of utterances (for example whether the utterances were questions or statements etc.). In the next cycles the interpretative transcriptions were progressively extended to include gestures, facial expressions and other bodily actions, tone of voice, pauses etc., and were supplemented by video stills of observable actions. These multimodal interpretative transcriptions were then, again, analysed and interpreted in light of the research question(s) at hand. Appendix 9 gives an example of the transcripts made in the second cycle of the process, where multimodal aspects of the interaction were included and where the transcripts were supplemented by video stills.

The transcriptions were then analysed and interpreted in light of the research question(s) at hand. Based on Radford's (2013b) semiotic approach, the analysis involved identifying actions (verbal and non-verbal), which seemed significant for constituting the mathematical inquiry activities and moving the activities forward. These actions may also be termed semiotic means (cf. Section 4.2), and includes the participants use of spoken words, gestures, facial expressions, tone of voice, other bodily actions and the use of artefacts and (mathematical) signs. The significance of these semiotic means was then interpreted in light of the specific research question(s) at hand. Vygotsky's dialectic approach oriented the analysis and interpretation to social interaction (relations) and therefore two subsequent turns were always considered in relation to one another. A turn gets its meaning through what has already been said and from how the next turn informs the previous turn in retrospect. The

meaning of a turn is thus always co-created in a joint activity. This also corresponds to what Bakhtine [Volochinov], (1977) says, that in a conversation every word has two sides, the speaker's side and the listener's side. This, I hold, is equally true for all actions, both verbal and non-verbal actions. The analysis and the interpretation of identified actions were then critically reflected on in light of the videos, and the analytical cycle was again repeated.

On regular occasions (approximately every fourth month) I met with research colleagues (1-4 colleagues) at the University of Agder, to discuss the research process and to do intercoder reliability checks of the processed data material. Intercoder reliability refers to the degree of agreement between two or more independent coders on the coding of the same item (Bryman, 2012), and are used with the purpose to increase the reliability in qualitative based data analysis. In this research study the 'coding' involved identifying actions (verbal or non-verbal) which seemed important for constituting the inquiry activity and moving the activity forward. During such an intercoder reliability check, videos, transcriptions and related analysis of the salient segments were critically watched, read and discussed by me and the colleagues. Before watching the videos and/or reading the transcripts I informed the research colleagues about the current research focus and provided them with information about how I planned to do the analysis. Then the colleagues critically watched the videos and/or read the transcripts and related analysis and provided critical comments to the processed data material. These critical comments were then used in the next cycles of data analysis. The iterative process of watching videos, transcribing and analysing data continued until a final satisfactory interpretation, to me and my research colleagues, of the salient segments as a whole was made. For publications, some parts of the highly processed data were selected to inform the results of the analysis related to the current research question(s).
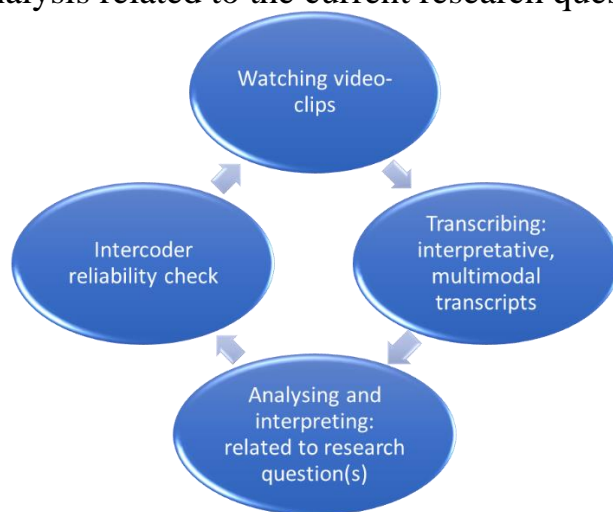

Figure 6.1: Illustration of the iterative analytical process

6.7 Trustworthiness

Qualitative research is based on context dependent data and subjective interpretations, which makes the findings more likely to be questioned. In a positivist paradigm issues like reliability and validity are addressed to ensure quality of the research. Reliability refers to the degree to which there is consistency and accuracy in the measurement, that is if the research will produce the same results if repeated (Wellington, 2000). Validity concerns the degree to which a test, method or an instrument measures what it claims to be measuring, that is the accuracy or correctness of the findings (Wellington, 2000). In qualitative research it is equally important to consider to what degree that data, methods and interpretation of the data is trustworthy to ensure quality of the research. Instead of using reliability and validity Lincoln and Guba (1985) outline four criteria for ensuring trustworthiness in qualitative research: credibility, dependability, confirmability, and transferability.

Credibility is, perhaps, the most important criterion for ensuring quality and trustworthiness in qualitative research. Credibility is similar to construct validity and internal validity in a positivist paradigm (Yin, 1994), and refers to the degree to which the findings corresponds with reality. Since qualitative research rely on subjective interpretations, it is the way in which reality is interpreted and presented in a trustworthy manner that determines its acceptance by others (Bryman, 2012). One way to address the issue of credibility is to adopt well established research methods (Lincoln & Guba, 1985; Yin, 1994), and to triangulate these methods. Triangulation is accomplished by investigating the same research issues using different data sources and different methods (Bryman, 2012). This research study used several data sources to investigate the three research issues; sessions where the five KTs and their groups of children were working with the mathematical activities designed in AP, interviews with the five KTs and conversations with the five KTs. Moreover, three different methods for data collection were used; observations, field notes and interviews (cf. Section 6.5). The research rests mainly on observations of sessions as the main method for data collection, because the object of study considers processes of mathematical inquiry, which manifests itself in real-life activity between the children and the KTs. However, interviews with the KTs are also used to check whether or how observations correspond with the KTs understanding of the situation, and thus help me to interpret the ongoing interaction.

Another way to address the issue of credibility is to carry out what Johnson (1997) call 'investigator triangulation', which involves using several different investigators in the analysis process. In this research study regular intercoder reliability checks of the processed data material were conducted to increase trustworthiness of findings (cf. Sub-section

6.6.1). A high degree of agreement between me and research colleagues at the University of Agder was reached with respect to data analysis and the final interpretation of data. Moreover, video clips, transcripts and related data analysis have been watched, read and discussed with other experts in the field of mathematics education. Thus, critical examination of data and data analysis by colleagues and other experts has been an important part of the whole research process.

Dependability, transferability and confirmability are less relevant to consider in a qualitative case study like this, because they parallel with reliability, generalisability and objectivity in quantitative research and focus on whether the research can be replicated and if the results are generalisable. This is arguably difficult to ensure in a case study like this, but some key point may nevertheless be useful to consider. Dependability parallels with reliability in a positivist paradigm (Yin, 1994), and refers to the degree to which similar results would be obtained if the research was repeated (with same data sources, same methods and in the same context). As argued above, this is difficult (if not impossible) to ensure in this type of research. It is impossible to capture the exact same circumstances of the initial research study and then replicate it. However, to keep a record, as I have done, of the entire research process and the choices made in planning and carrying out the empirical research, ensure that peers can get access to and check whether proper procedures have been followed (Bryman, 2012; Lincoln & Guba, 1985).

Transferability parallels with external validity or generalisability in a positivist paradigm (Yin, 1994) and refers to the degree to which the findings may be applied in other settings. As mentioned above, it is impossible to replicate a qualitative research study like this. However, qualitative studies aim to display in depth understanding of a phenomenon and the uniqueness of the phenomenon related to its context (Bryman, 2012), which may indicate the transferability of the results to other contexts. One way to address this issue, is to display the uniqueness of the study by providing sufficient contextual information (Bryman, 2012; Lincoln & Guba, 1985). This research study is situated within the AP, and the data collection was conducted within the focus group while the KTs implemented the intervention programme. In addition, I was one of the researchers that developed the intervention programme, and the KTs were familiar with me and the aims of the project. These contextual circumstances make this research study quite unique and may be regarded as a strength to the study. In Chapter 5, details about the design of the AP and how the project was carried out are provided and aim to give sufficient contextual information to display the uniqueness of the study.

Although the unique context may be regarded as a strength to the case study, there are also issues related to the contextual circumstances

that may be regarded as weaknesses of the study. For example, both the five KTs and the children knew they were part of a large research project, which might have affected their participation. In addition, because the KTs knew that I had pre-designed the activities (together with the other researchers in the project), they might have tried to impress me by implementing the activities the way they thought I wanted them to be implemented. Moreover, because the KTs knew me quite well, through the professional developmental course, they might have felt obliged to participate in my study. All these issues are elements of uncertainty related to how authentic the observations made in this study are. However, I think that the relationship that I established with the KTs, during the developmental course, helped to ensure that the KTs trusted me, and thus implemented the activities in an authentic manner. By 'authentic manner', I mean the way the KTs would have implemented the activities in the AP without my presence. Therefore, I regard the contextual circumstances, all in all, as strengths because they make this research study unique. The uniqueness of the case, together with the related results, may be used in a normative manner and indicate possible actions that could or should be initiated with respect to similar practices in the future.

Transferability may also be obtained if theory is used properly. Instead of trying to generalise to other cases the case study researcher should aim to relate findings to theory (Yin, 1994). In this research study the cultural-historical perspective and theory of knowledge objectification (Radford, 2013b) have been used as a foundation to study mathematical inquiry processes in kindergarten and are elaborated in Chapter 4. The theoretical approach has guided the whole process and the results in Chapter 8 are also considered in light of the theoretical perspective. In addition, inquiry as a theoretical construct has been carefully considered and further investigated. The aim of the study is to investigate how inquiry may be conceived in a kindergarten context. As mentioned in Section 6.2, this case study may therefore be regarded as a 'theory testing' case study (Bassey, 1999), because it investigates whether the empirical findings support theory (in this case the theoretical conception of inquiry) or whether modifications must be made to theory with respect to the specific context.

Confirmability is similar to objectivity in a positivist paradigm (Yin, 1994), and refers to the degree to which the findings may be considered objective. Complete objectivity is impossible to ensure in qualitative research (Bryman, 2012; Lincoln & Guba, 1985). In qualitative research where findings are obtained by subjective interpretation, confirmability is best met by showing that the research is conducted by the researcher in 'good faith' (Bryman, 2012). In addition, to get research colleagues to critically examine data collection and data analysis and check that proper

methods have been used, may be another way to address confirmability in qualitative research. In this research study confirmability is, I hold, addressed by the way that it has been designed, carried out and documented in this research report. Through this report I have tried to comply with Burton's (2002) quest for transparency by giving insights into the choices made when planning and carrying out the study. In addition, research colleagues have regularly and critically examined the methods used for data collection and data analysis during the research process. Moreover, I assure that the study has been conducted in 'good faith', that is, I have tried to act in an ethically sound manner when carrying out the research study. Ethical issues will be further elaborated in the next section.

6.8 Ethical considerations

Ethics refers to the knowledge about what is right and wrong and prescribes what humans morally ought to do. To be ethical is, in my opinion, to try to be fair and honest, try to reflect upon what is right or wrong and to make, what you ethically believe, is the right choices of actions in different situations. In mathematics education research, which is related to social science, ones conduct research involving humans and it is therefore of particular importance that the research is ethically sound.

The Norwegian National Research Ethics Committees have formulated some general guidelines for research ethics. In the introduction to the guidelines, ones can read that "Research is of great importance - to individuals, to society and to global development. Research also exercises considerable power at all these levels. For both these reasons, it is essential that research is undertaken in ways that are ethically sound", (The Norwegian National Research Ethics Committees, 2014, introduction). This research study, which focuses on mathematical teaching and learning processes in kindergarten, provides me with power to influence how children will meet mathematics in kindergarten, what mathematical content in kindergarten will be and what requirements KTs will meet in their practice in the future. This power should be (and has been, I would argue), carefully and thoughtfully considered.

Bryman (2012, p. 118) describes four groups of ethical principles that are important to consider when doing social research: whether there is harm to participants, whether there is informed consent, whether there is an invasion of privacy, and whether deception is involved. The first three principles concern the participants in the research study and overlap in many ways. The last principle is related to issues considered the people one is doing research for. All these issues are to some extent relevant to consider for this research study, when investigating teaching and learning processes in kindergarten. The first principle, whether there is harm

to participants, is somewhat hard to consider. The word harm is difficult to define but indicates that the participants should not be uncomfortable in any way as a consequence of the research, neither during the research process nor after the research is completed (Bryman, 2012). Throughout the research process I have tried to act respectfully and thoughtfully towards the participants. I think that a positive relationship between me and the participating KTs was established during the professional development course in the AP. This laid the foundation for the further collaboration and hopefully ensured that the KTs felt comfortable when I observed how they implemented the activities.

The second principle, whether there is informed consent, focuses on whether the participants get all the relevant information they need to decide whether they will participate in the study or not. According to Israel and Hay (2006) the consent should both be informed and voluntary, which is not always a straightforward procedure and has proven to be quite difficult for many researchers. In this research study most of the participants are children. The children do not make the choice to participate themselves. It is their parents that make this decision. Therefor it was important to provide the parents with all the relevant information they needed to make a deliberate choice for their children. Fortunately, this issue was to a large extent taken care of by the Norwegian social science data service (NSD). NSD gives permission to collect data on the basis on an application which includes approval of an information-letter sent to all participants in the research study. NSD has to approve the letter before it can be sent to the participants, and the participants have to sign the letter before data collection can start. The information-letter composed in this research study can be found in Appendix 10. The letter was sent to the KTs and parents and provides information about the research process, data storage and how the participants will be made anonymous. The letter gives, however, only formal information, and one may ask whether this letter gives the participants sufficient information to decide whether they will participate in the study or not. To address this issue, I arranged an informal conversation with the KTs before the data collection to inform about the research study and the process of data collection. This conversation gave me an opportunity to provide relevant information to the KTs. Through this conversation I provided all information that I thought was relevant, and the KTs could ask for whatever information they wanted (Faden & Beauchamp, 1986, in Israel & Hay, 2006). In addition, the KTs, to the best of my knowledge, passed this information onto parents when they distributed and collected the informed consent from the parents. As mentioned in the previous section, the KTs may have felt that they were obliged to participate in the research study, because I was their teacher in the professional development course.

However, I tried to ask them to participate in my study in a careful manner and I think they felt the opportunity to decline if they didn't want to participate.

The third principle, whether there is invasion of privacy, is related to the second principle and to some extent to the first principle. The idea of privacy is subjective, that is what one person finds private, another person may find acceptable to bring into public (Israel & Hay, 2006). Most people feel uncomfortable if someone invades their privacy, but at the same time, if the participants decide to take part in a research study, they invite the researcher into their lives to some extent. Again, to address this issue, it was important that the participants were well informed, so they could make a deliberated choice to participate. The issue was also addressed by the way that the research study was conducted; observations and interviews were conducted respectfully and thoughtfully, and likewise the research report aims to present the results in a respectful manner and with good intention.

The fourth principle is related to the people ones are doing research for. Deception may be described as the act of deliberately misleading or misrepresenting. Deception occurs if the research report gives an incorrect illustration of the conducted research study. It is difficult for others to get access to the data and to check the validity of the research, because of protection of the participants right to anonymity. Fabricating, falsifying or withholding data is thus a relevant issue in social science. This issue is directly related to the integrity of the research, the integrity of researchers' profession and the integrity of the research community. This principle challenges me as a researcher to be critical to my own work. It is a fine balance to provide sufficient information, but at the same time safeguard the participants' anonymity. Through this research report I have tried to address this issue by transparency of the process, and I have conducted the study in 'good faith' (Bryman, 2012).

7 Summary of research papers arising from this research

In this chapter I introduce five papers that qualify me to submit a 'PhD by papers' and which constitute the empirical core of the study. The papers are presented in the order they were written. The first paper investigates structural aspects of children's argumentation related to reflection symmetry and in addition highlights the importance of various semiotic means for understanding children's communication. The second and third paper are closely linked. They draw on the same empirical material and focus on related issues. The second paper considers a KT's multi-modal participation in a teaching-learning activity involving addition and counting, and how the KT engages children in a mathematical discourse and supports their opportunities for learning. The third paper investigates how the KT and a child co-create a ZPD and emphasises the role that the child plays in this co-creation. The fourth paper focuses on children's turn-taking in small-group collaboration. Together, the four papers aim to capture processes of mathematical inquiry in a variety of collaborative settings. The fifth paper investigates four of the five participating KTs' orchestrations of the mathematical activities designed in the AP and aims to capture issues about what enables mathematical inquiry in kindergarten to occur. Together, the five papers aim to constitute the foundation on which conclusions can be drawn to the three main research issues guiding this research study. In Section 8.1 a synthesis of the findings in the five papers will be made and from that possible answers to the three research issues will be provided in Section 8.2.

7.1 Paper 1: Kindergarten children's argumentation in reflection symmetry: The role of semiotic means

This paper reports from a case study which investigates structural aspects of children's argumentation related to reflection symmetry and examines what role semiotic means (e.g., artefacts, linguistic devices and gestures) play in the ongoing argumentation. The study uses Toulmin's (2003) model for substantial argumentation to reveal structural aspects of children's argumentation. The core of Toulmin's model consists of three elements: claim, data and warrant. In addition, the model contains three other elements: backing, qualifier and rebuttal. These elements are described in the paper (in Paper 1 attached to this thesis).

One kindergarten teacher (KT) and a group of six 5-year-old children engaged in mathematical activities about reflection symmetry were observed. Six episodes were identified as sequences of argumentation.

These episodes had more than two turns and more than two argumentative utterances from the children. All six episodes were analysed in accordance with Toulmin's (2003) model, from a multimodal perspective, to identify the argumentative structures, what kind of semiotic means children used with respect to the different elements in the argumentation and what role they played in constituting the argument.

In the results one example was presented to illustrate the structure of children's argumentation. The example is taken from an activity where the KT asks the children to find 'things' in the room which they think has reflection symmetry (or 'are equal on two sides', as the KT says), and bring it back to the table. One of the boys (John) picks a trolley (a doll's pram). In the example, the KT asks John why he thinks the trolley is equal on two sides. John starts to argue, but he is accompanied by Elias, and together they provide an argument for why the trolley is equal on two sides.

In the paper I argue that John *claim*s that the trolley is reflection symmetrical just by choosing the trolley. To support his claim, he lifts the trolley in a straight forward position, nods his head and says "there", which I interpret as *data*, for his claim (the trolley itself is used as a visual evidence). Then there is a little pause, before John continues with his *warrant*. First, he shows one side of the trolley and says "there", then he turns the trolley 180 degrees and shows the other side of the trolley and says "there". I hold that John provides an example which connects the claim to the data. Then Elias joins the argumentation and produces several *warrants* of the same kind as John, however he is more accurate in his explanation. Elias points, with his index finger, to pairs of corresponding points on the trolley while saying "there, there [pause] there, there and there, there". In the end, I hold, Elias produces a *backing* for the warrants, by generalising all the examples that he has given; he swipes his hand over the trolley while he says "everywhere". Then he produces what I interpret as a *qualifier*, which indicates that the trolley is 100% reflection symmetrical because "even the wheels" are reflection symmetrical.

The results show that some children were able argue for a claim in a quite complex manner and to use several of the elements in the Toulmin model in their argumentation. The findings also suggest that it is crucial to consider other semiotic means than linguistic ones to understand children's communication. The main corpus of the argumentation was based on deixis (deictic terms + additional contextual information). Both nodding, pointing gestures and the trolley were used to give contextual information to the deictic terms "there". Repetition and rhythm were other important means that Elias used in his warrant. By repeating "there and there" with corresponding pointing gestures Elias indicated that every

point had a corresponding point on the other side of the symmetry line. When Elias said "everywhere", he swiped his hand over the trolley, which played a significant role in his generalisation.

The case study illustrates that children's reasoning is multimodal and that their flow of thinking is materialised through a complex coordination of diverse semiotic means (cf. Radford, 2009). It points to the significance of recognising how children make use of various semiotic means when investigating children's mathematical thinking. In addition, it points to the importance of recognising and acknowledging children's multimodal thinking when introducing children to mathematics in kindergarten (and school).

7.2 Paper 2: Engaging children in mathematical discourse: a kindergarten teacher's multimodal participation[16]

The paper reports from a case study which investigates a kindergarten teacher's (KT) multimodal participation in a teaching-learning activity involving addition and counting, and how the KT engages children in a mathematical discourse and supports their opportunities for learning. The paper addresses the following research questions: What characterises the KT's multimodal participation in a teaching-learning activity involving addition and counting? How does the KT's multimodal participation supports discourse and children's opportunities for learning?

The segment examined in the paper was selected from an activity where one KT and a group of nine children (age 4-5 years) worked with an addition problem. The examined segment was selected because the children got ample opportunities to suggest and explain strategies to solve the problem and the children eagerly participated in the discourse. The salient segment was transcribed and then analysed from a multimodal perspective.

The findings suggest that the KT's multimodal participation was 'dynamic', which was based on three interrelated features: Her contributions 1) changed in relation to the children's contributions, 2) were oriented towards the aim of the activity, and 3) were based on her underlying 'stance' toward the children and their learning. These three features are deeply linked to basic ideas in the cultural-historical approach. The first feature is incorporated in Vygotsky's dialectic perspective: it is a participant's response to the previous utterance that informs the meaning of the utterance in retrospect. The second feature relates to a fundamental idea in activity theory: actions are initiated by the motive of the activity. It is the motive (the KT's aim of the activity) that initiates the KT's actions

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[16] This paper will be revised after submitting the thesis.

and thus orients the KT's actions towards the aim of the activity. The findings in this study illustrates how the KT's emotions, which are mediated by her questioning look and her excited facial expression, orients the activity. Whenever the children are on their way toward the object of the activity the KT changes her facial expression from a questioning look into excitement (and vice versa). The third feature considers how the KT's multimodal participation mediates her underlying stance toward the children and their learning. In this episode, the KT's question strategies, her non-verbal responses to the children and the way she sensitively uses building blocks, may indicate the KT's underlying stance toward the children and their learning. The KT appreciates every child contribution, and the children were given ample opportunities to contribute to the conversation. But simultaneously, she questions their contributions and promotes the children to argue for or reconsider their claims.

The final discussion point in the paper considers the KT's subjectification process (cf. Radford, 2008b, 2013b). As illustrated above, the KT's contributions (her verbal and non-verbal actions) were always balanced between her earlier experiences, which in this case is mediated by her underlying stance, children's contributions and the aim of the activity. The tensions that were created between the past, present and future is, I hold, what constitutes the KT's moment to moment acting and her way of becoming in the activity. Who the KT was when she entered the activity was transformed in the encounter with the children. This process is particularly salient in the part of the segment where the KT expresses that she does not understand. The KT is positioning herself within the unfolding activity, trying to understand the children. The KT shows a genuine interest in understanding the children which puts her in a vulnerable position, and therefore at one point must be led by one of the children. The way that the KT continuously transforms her unique participation in the moment in relation to children's contributions, and how she positions herself within the unfolding activity, trying to understand the children, illustrates how she becomes her unique self in the encounter with the children.

7.3 Paper 3: Student-teacher dialectic in the co-creation of a zone of proximal development: an example from kindergarten mathematics[17]

This paper reports on a case study which explores the teaching-learning interaction between a five-year-old girl (Ada) and a kindergarten teacher (KT) working on an addition problem. The paper aims to 1) investigate

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[17] This paper will be revised after submitting the thesis.

the role of Ada's (choices of) actions in keeping the mathematical oriented activity moving and for co-creating the zone of proximal development (ZPD), and 2) investigate Ada's mathematical learning in the joint activity. Following Vygotsky (1987) and Veresov (2017), Paper 3 emphasises that the ZPD must be seen in terms of social relations that emerge from the joint activity. Using Roth and Radford's (2011) approach to analyse the emergence of the ZPD this paper investigates how a KT and a child co-create a ZPD by expanding each other's action possibilities.

The findings of the paper provide insight into the significant role that Ada (the presumed less knowledgeable participant) plays in the sympractical activity and for co-creating a ZPD. Both Ada and the KT show perseverance to participate in the sympractical activity and to expand each other's action possibilities. However, Ada takes a special responsibility for moving the activity forward and in the co-creation of the ZPD. When the KT gets confused (which she expresses verbally and non-verbally), Ada works intensively to create new action possibilities for the KT to keep her in the joint activity. The KT also show persistence in the activity. Although she expresses at one point that she does not understand, she continues to stay in the activity. But because she does not understand she is not able to productively participate in the activity. It is Ada that, through her concrete actions, enables the KT to again participate productively in the joint activity.

Through the sympractical activity, the KT and Ada create a special relationship – both are dependent on each other to continue to participate in the activity. Through the activity Ada and the KT keep adjusting to one another. They listen to one another and observe one another's actions, and they do not give up the co-creation of the ZPD. This relationship is created through mutual commitment and trust and may be labelled as 'togethering' (cf. Roth and Radford, 2011). It is an ethical engagement that glues the KT and Ada together and enables the activity to move forward.

The second research question aims to investigate Ada's mathematical learning in the joint activity. In this study Ada is challenged to explain her idea without direct access to the building blocks. Instead of operating directly on the building blocks she explains how to do it with an iconic gesture and language, and the explanation becomes (with a lot of effort) short and precise. The situation required Ada to use more refined and sophisticated semiotic means which may be regarded as semiotic contraction (cf. Radford 2008c). The findings in this paper indicate that there might be a dialectic relationship between fluency and the practice of semiotic contraction. By practicing semiotic contraction (being 'forced' to

refine her articulation of the idea), Ada became more fluent and perhaps reached a deeper level of consciousness by this exercise.

7.4 Paper 4: Organisation of children's turn-taking in small-group interaction in kindergarten[18]

This paper reports from a case study which explores the characteristics of children's turn-taking and the movement of their joint activity in two small groups of 5-6-years-old children while they work on addition problems. The paper addresses the following research questions: 1) What characterises children's turn-taking while they work in small groups to solve an addition problem? 2) What role does turn-taking play in the movement of joint activity when children work in small groups to solve an addition problem?

This case study focuses on the coordination of turn-taking within two small-groups of kindergarten children (age 5-6) working on addition problems. In K1 the children worked on the following problem, formulated by the KT: "Run around the nearest located tree three times each. How many times have you run around the tree altogether?" In K2 the children worked on the following problem, formulated by the KT: "Consider your hands, how many fingers have you altogether in the group?" The two episodes were transcribed and then analysed from a multimodal perspective.

The findings in this study suggest that children both self-select in starting to talk and they address next speaker (cf. Sacks et al., 1978). These techniques are accomplished using gaze, word emphasis and gestures and appear to be context dependent (cf. Mondada, 2007). The turn-taking in the two episodes are quite different. In the first episode, from K1, there is mainly turn-taking by addressing next speaker. In the second episode, from K2, turn-taking is mainly achieved through self-selection. In K1, the children stand a bit apart from each other. To address next speaker, the current speaker turns his/her gaze (and emphasises the last counting word) to the next speaker. This may be regarded as face-to-face interaction (cf. Mondada, 2007), because the current speaker and the next speaker keep eye contact in the transition of turns. In K2, the children are standing closer to each other while they solve the problem. The children in this group have the possibility to count each other's hands and/or fingers (not only one's own fingers as in K1), and they may touch one another's hands for addressing next speaker. This way of taking turns may be considered as side-by-side interaction (cf. Mondada, 2007), since the children do not (or very seldomly) keep eye contact in the transition of turns (they usually kept their gaze on their hands/fingers).

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[18] This paper will be revised after submitting the thesis.

The children's turn-taking ensures that the activity moves towards a solution, and in some cases the children also attune the direction of the activity by taking or addressing turns. This is only accomplished by trust and responsibility. By prompting another participant to take the next turn the selected next speaker must accept taking the next turn in order for the activity to move forward. This requires both trust and responsibility and reflects the process of togethering (cf. Radford and Roth, 2011). The current speaker must trust the selected next speaker to take the turn, and the next speaker must repay this trust and take responsibility for acting. By self-selecting turns they take responsibility for moving the activity forward and is particularly used to adjust the direction of the activity. The responsibility that the participants show, may also be context dependent. In both groups the problem is formulated so that all children must participate. In K1 all children are asked to run around the tree, and they are all responsible for representing and counting their runs. In K2 the children are asked to count their fingers, and each child is responsible for bringing their fingers into the joint activity of counting.

Although the two problems that are posed by the KTs may be considered as addition problems they give possibilities for multiplicative thinking since the problems may be categorised as 'equal groups problems', (cf. Anghileri, 1989; Greer, 1992; Mulligan & Mitchelmore, 1997). The results show how layers of multiplicative structure arises from the children's actions. From children's coordinated turn-taking in K1, a rhythmic counting arose, which revealed the fundamental group-structure of multiplication (cf. Anghileri, 1989; Mulligan and Mitchelmore, 1997). In K2 the children were concerned about the group structure of their fingers and by counting by five and tens. The children's coordinated turn-taking gave rise to their multiplicative thinking, and seems to influenced by contextual circumstances, like how the problem at hand is formulated, available artefacts and children's positional location in space.

7.5 Paper 5: Kindergarten teachers' orchestration of mathematical learning activities: the balance between freedom and structure[19]

This paper reports on a multiple-case study which focuses on four kindergarten teachers' (KT) orchestration of mathematical learning activities pre-designed in the AP. The study focuses on the four KTs' orchestration of the mathematical activities with respect to the degree of freedom, and what impact their orchestrations had for children's mathematical learning possibilities. The study draws on Valsiner's (1987) zone theory to investigate the relationship between zone of free movement

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[19] This paper will be revised after submitting the thesis.

(ZFM) and zone of promoted action (ZPA) which the KTs set up to canalise children's actions and thinking and thus development.

The empirical material was collected through ethnographic field notes (Emerson et al., 2011) from observations of the four KTs' implementation of the mathematical activities and interviews with the four KTs. (Observations, interviews and field notes are the same as in the overarching research study on mathematical inquiry in kindergarten). The field notes were contracted into profiles of the four KTs' orchestration, focused around four key points[20]: (1) Children's access to different areas in the environment, (2) children's freedom to act (physically) within the accessible area, (3) children's freedom to participate with questions, mathematical ideas and argumentations etc., and (4) the degree to which problem-solving interaction was promoted (that is the degree to which the children were promoted to ask questions and explain and argue for their ideas to solve mathematical problems). Key points 1-3 concern the ZFM, that is the degree of freedom given by the KT in the activities. Key point 4 concerns the ZPA, that is how the KTs promote children to ask questions, explain and argue for mathematical ideas to solve mathematical problems. As Valsiner (1987) argues, ZFM and ZPA are related and work as a unit to canalise children's development, and in this case, learning and development related to mathematics.

The results show that the four KTs' orchestration differed quite significantly. KT1 seemed to be concerned about freedom, and the ZFM was relatively wide both with respect to what the children were allowed to talk about and what they were allowed to do (physically). KT2 restricted the ZFM to mathematics, however, she gave the children freedom to talk about and work with other mathematical objects than what she initially introduced. KT3 restricted the mathematical talk to specific mathematical themes, however the ZFM was relatively wide with respect to what the children were allowed to do (physically). KT4 was the most controlling of the four KTs, and the ZFM was relatively narrow both with respect to what the children are allowed to do (physically) and what the children were allowed to talk about. KT4 decides, to a large degree, *who* was allowed to talk (or 'do' something), *when* the children were allowed to talk (or 'do' something) and *what* the children are allowed to talk about.

The results show that in the kindergarten (KT2) where the ZFM was gently set up and limited to mathematics, and where the KT sensitively set up the ZPA and promoted the children to share, argue for and explain

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[20] The four key points used in this multiple case study are similar to the original key points used to organise the KTs' profiles in the first phase of data processing, described in Sub-section 6.6.1. However, in this research study the four key points were modified to fit the research questions and the theoretical framework.

their mathematical ideas, and explicitly promoted the children to collaborate, was where most problem-solving interaction occurred and where the KT facilitated children's learning possibilities the most. How a ZFM/ZPA complex canalise children's development is not only related to the boundaries of the zones itself, I hold, but how the ZFM and ZPA are set up. The KT2 was relatively mild in her way of setting up the ZFM, and instead of turning the ZPA into ZFM she acted in an exciting way which promoted the children to accept the ZPA. The KT2 'advertised' for the ZPA by the way she presented the mathematical problems and made the children want to pay attention and accept the ZPA.

The profiles are of course tendencies not absolute characteristics. The KTs' orchestrations, with respect to the degree of freedom, changed dynamically during each session and from session to session. In addition, the profiles are relative, which means that the degree of freedom in one kindergarten is relative to the three other kindergartens and cannot provide an indication for how it relates to other KTs' orchestration in other contexts more generally.

8 Mathematical inquiry in kindergarten: discussion, conclusions and implications

I begin this chapter by reviewing the purpose(s) of this research study and the research issues guiding the study and then outline the three substantive sections which follow. The purpose of this research study is to investigate processes of mathematical inquiry in kindergarten and bring forth new practical insights about inquiry as a theoretical construct. Since inquiry is closely linked to issues about learning, the research study also aims to reveal important aspects of mathematical teaching-learning processes in kindergarten. Moreover, the study aims to investigate processes and illuminate issues that arise when young children are introduced to and engaged in mathematical activities that aim to prepare them for school. Thus, a purpose of the study is to contribute to an understanding of what mathematics is for young children, how they engage in mathematical activities, and what can be expected, or not expected from these young children that are about to enter school. Based on these purposes the following research issues were formulated to guide the study:

- What characterises processes of mathematical inquiry in kindergarten?
- What enables processes of mathematical inquiry to occur?
- How, if at all, do these processes of mathematical inquiry prepare kindergarten children for school?

As mentioned in Section 1.2, the two former research issues were operationalised into specific research questions or aims in each of the papers accompanying this thesis, to deal with the empirical material. The latter research issue arose during the research process and is seen as an exploratory research issue. The aim is to use results emerging from the two former research issue and consider them in light of the literature on important aspects for a smooth transition between kindergarten and school. In this chapter I return to the above-mentioned research issues and attempt to make a holistic evaluation of them.

Before I return to the research issues, the findings in the five papers will be synthesised and discussed in Section 8.1. In Section 8.2 I return to the purposes and research issues of the study in order to provide possible answers to them. Finally, Section 8.3 is devoted to critical reflection and implications for practice and further research.

8.1 Synthesis of findings and discussion

The findings in the five papers are synthesised and organised around four discussion points: children's contributions to the activities (Sub-section 8.1.1), the KT's contributions to the activity (Sub-section 8.1.2), the dialectic relationship between the KT's and children's contributions and the co-creation of the zone of proximal development (Sub-section 8.1.3), and children's mathematical reasoning in the five papers (Sub-section 8.1.4). The three first discussion points are indeed interrelated but will be discussed separately to make a linear and coherent discussion. All four discussion points will serve as a foundation for making conclusions in Section 8.2.

8.1.1 Children's contributions to mathematical inquiry processes

This sub-section is devoted to a discussion about children's contributions to the segments of activity which, in this study, are identified as inquiry. First, it discusses how children contribute with multimodal claims and argumentations, later it discusses how children contribute through taking and addressing turns and goes on to discuss how children contribute to create action possibilities for the KTs in the activities. The sub-section ends with a discussion about the lack of questions asked by the children in these mathematical learning activities.

There is an agreement among researchers that mathematical explanations, justifications and argumentations are important in young children's mathematical reasoning (Mercer & Sams, 2006; Yackel & Cobb, 1996), but the literature is still sparse with respect to characteristics of young children's mathematical argumentations. Paper 1 investigates the multimodal nature of children's argumentation related to reflection symmetry. The paper illustrates how two 5-year-old boys argue for why a 'trolley' (a doll's pram) is reflection symmetrical. Using Toulmin's (2003) model for substantial argumentation the study illuminates the complex structural pattern of the two boys' argumentation. This is similar to the findings of Pontecorvo and Sterponi (2002), who found that children's reasoning in preschool activity unfolded "through complex argumentative patterns" (p. 133). Paper 1 also illustrates that the two boys' argumentation unfolds through a complex coordination of various semiotic means, where words (language) were just one of the components. In fact, children's argumentation and reasoning were mainly mediated by other semiotic means than language. Both boys argued for why the trolley was reflection symmetrical by the deictic term 'there' and related gestures and bodily actions.

Although both boys, in Paper 1, used the deictic term 'there' and related gestures and bodily actions, their argumentations were still different. The first boy (John) was more restricted in his argumentation. He 'showed' the whole trolley to argue that it was reflection symmetrical,

and then showing the sides of the trolley to further warrant his claim. The other boy (Elias) used more elaborated argumentation. He pointed to different points on the trolley with his index finger. Both boys used the deictic word 'there' and corresponding gestures and bodily actions, but Elias's argumentation was still more refined because he was more precise with his gestures and paid attention to more details. In addition, Elias seemed more confident than John. In their study on children's (age 5-6) different types of justification Tsamir et al. (2009) showed that young children are able to justify their statements by using what they call 'appropriate mathematical ideas'. However, there were differences in children's ways of reasoning. Some children used their 'visual reasoning' as a way to justify their statements, for example some children justified their statements by saying "because we see". Others argued by more sophisticated verbal arguments. The findings in Paper 1 are similar to the findings in the study by Tsamir et al. (2009). However, the differences in children's argumentation (the level of abstractness or the level of details) are not related to the use of words (or not), rather to different ways of using various other semiotic means like gestures and other bodily actions.

An important element in Toulmin's (2003) model is the claim – without a claim, there is nothing to build an argument around. And making claims seems salient for children's contributions in the segments identified as inquiry. In Paper 1, John claims that the trolley has reflection symmetry; he later builds an argument around this claim. In Paper 2 and Paper 3 most of the children contribute with claims and suggestions about the solution of the problem, only a few children contribute with more elaborated argumentations for how to solve the problem. This show that different children contribute with different 'things' (e.g., claims and arguments) in the activity, and it seems that the children who are more aware of mathematical ideas and the culturally ways of expressing them, are the children who are able to contribute with more sophisticated and refined argumentations and explanations.

Paper 2 illustrates how the KT, by her question strategies and her multimodal contributions, promotes children to contribute to the activity and to explain and argue for their ideas. The children eagerly (and quite spontaneously) contribute with ideas and claims about the solution, and some children are able to explain their ideas. Polya (1945) regarded guessing as a key element in problem solving. He argued that mathematics could be seen as a 'guessing game', and it seems as if some of the children in the activities in this research study are practicing such 'guessing games'. Similarly, however not identically, to what Polya (1945) argues, these claims are valuable contributions to the activities. According to Radford (2015b), thinking is thought put into motion, and it is through joint practical activity that mathematical thinking is brought to life. In

Paper 1 John initiates the argumentation claiming that the trolley is reflection symmetrical. John gets the idea when he picks the trolley and on that stage, the idea is pure possibility, which is put into motion when he starts to argue. Elias takes up Johns idea and contributes to move the thinking further. Similarly, in Paper 2 and 3, when the children are asked to suggest a strategy for how to solve the addition problem (8 + 5 + 7), Leo gets an idea in line 93 (in Paper 2). He points to the tower with eight building blocks but is not able to move the idea further. Later in line 111 (which is not part of the transcripts in Paper 2 or 3) Mia gets an idea (or, perhaps, takes up Leo's idea) and says 'eight'. The KT has at this point taken up Leo's and Mia's idea and in line 112 she prompts Mia to explain further and says, "Eight… Here it was eight ((points to the red tower)), and then, what do we have to do?", but Mia does not know how to move on. This seems to awaken the idea in Lea and Ada, because they eagerly raise their hands. It is when Lea and Ada explain their ideas (in interaction with the KT) that the thought is accelerated, and the collective thinking is materialised through the activity. When Ada explains her idea (in Paper 3) it was not Ada's idea solely – the idea was initiated by Leo (in line 93) and Mia (in line 111) and materialised in interaction with the KT. This illustrates that, although some children do not engage in elaborated argumentation, their suggestions are still valuable to the joint activity and to the flow of thinking. It is at the point where Ole points to the tower with eight building blocks that the collective thinking is carefully put into motion. When Ada later explains the idea, it helps the other children to become aware of the cultural and historical patterns of interaction which constitutes what we recognise as a mathematical idea.

Children make claims and argue for their ideas, but they also take and address turns which is part of the materialisation of mathematical ideas and part of the materialisation of activities which I identify as inquiry. To productively collaborate, some children must listen and observe, and others must talk and act and thus turn-taking is an important part of how the roles as listeners/observers and as speakers/actors are coordinated between the children. Sacks et al. (1978) describe two ways in which turn-taking may be organised. Either by the current speaker who select the next speaker or by 'self-selected turn-taking' where the non-speaker self-selects in starting to talk. Paper 4 illustrates how children use these two 'turn allocating techniques' to organise turn-taking in small groups, and how their mathematical thinking is materialised through the ways in which the children organise their turn-taking. Children both take turns and address turns in order to move the activity further. In particular, the paper illustrates how turn-taking is organised, and

thus how mathematical ideas are materialised, through children's use of gaze, gestures and word emphasis.

Paper 4 also illustrates that children's turn-taking is context sensitive, which is similar to what both Lerner (2003) and Mondada (2007) argue. However, in Paper 4 it is not only the tasks at hand and the available artefacts that influences children's turn-taking. Children's positional location in space also affects their turn-taking and thus the materialisation of their thinking. Whether the turn-taking is organised as a face-to-face exchange or as a side-by-side exchange (cf. Mondada, 2007) is to a great extent related to children's positional location in space. This illustrates how the flow of thinking, and in this case the flow of additive and multiplicative thinking, is deeply context dependent. In Sub-section 8.1.4 children's mathematical reasoning will be elaborated, including how their thinking is intertwined in the contextual circumstances.

Children's organisation of turn-taking in Paper 4 also illustrates that different children contribute with different things in the collective thinking. In segment 1, Pia seems to get the initial idea for how to solve the problem and is further trusted to take an organising role. It is actually not Pia that takes the role as an organiser, but the turns come back to Pia, which indicates that the other children trust Pia as an organiser. Pia accepts the role as an organiser and plays a key role for the materialisation of the thinking.

As part of the turn-taking, the children also contribute with actions that have the intention to keep the other participants productive in the activity. Roth and Radford's (2011) argue that a significant part of learning activities (for the co-creation of a ZPD) is the way that the participants create action possibilities for each other. Paper 3 illustrates how Ada contributes to create action possibilities for the KT to keep the KT in the activity. When the KT is confused, Ada guides the KT by creating action possibilities so the KT can productively participate in the activity again. Ada shows persistence and responsibility for maintaining the joint activity and to move the activity further. This illustrates the importance of making room for children's participation and of trusting the children to take responsibility in mathematical activities.

In an inquiry approach to education, children should get opportunities to ask their own questions (cf. Alrö & Skovsmose, 2004; Jaworski, 2006; Lindfors, 1999; Roth, 1996; Wells, 1999). However, many scholars have identified that children or students rarely ask questions themselves in teaching-learning situations (Dillon, 1988; Myhill & Dunkin, 2005), and that the reason may be the lack of opportunities that the children get to ask questions because of the extensive use of questions that teachers ask (Dillon, 1988). In kindergarten, which originates in a tradition other than school, and where children's freedom to participate is emphasised in

learning activities, one could expect that children would ask a lot of questions. However, this was not the case in this research study. From all the observations that were conducted in the study, children seldomly asked questions. It is of course important to consider that the activities in the AP were designed as structured learning activities, where the conversation should be focused around mathematics. However, as Paper 5 illustrates, even in the kindergarten where the KT gave children a lot of room to contribute with ideas and questions, there were seldom mathematical questions asked by the children. This is similar to the findings of Dovigo (2016), who found that children contributed with less questions in child-teacher interaction than in peer-interactions, however in child-teacher interactions children's argumentations were often more elaborated than in peer-interactions.

There may be several reasons for why there were not so many questions asked by the children, and one of the reasons may be the nature of the pre-designed activities. Perhaps the way that the activities were designed, or the way that they were orchestrated, didn't promote children to ask question. Another possible explanation, which is in line with the cultural-historical approach, is that children have not yet learned to ask mathematical questions. I think Dewey (1938), who represents a pragmatist view, has a point when he says that, "a problem well put is half-solved" (p. 173). Dewey's claim indicates that to be able to ask questions, at least well-defined questions which may serve as problems that the children can investigate further and solve together, requires that the children are, to some degree, aware of the object that they ask questions about. In addition, it requires that the children are aware of the culturally and historically ways of acting and thinking, which is culturally identified as mathematical questions. Moreover, it requires that they are aware of cultural norms, that asking questions is part of what we do in mathematical activities. From a cultural-historical perspective, questioning as any other cultural means must be learnt. In this research study it seems that the children have limited experience in asking mathematical questions or learned that asking questions is part of participating in mathematical learning activities.

To summarise, in the segments identified as inquiry in this research study, the children contribute with multimodal claims and argumentations, they also contribute with taking and addressing turns and creating action possibilities for other participants to move the activity further. However, different children contribute with different things, which indicates that some children have limited experiences in arguing and explaining their ideas yet. Although children contribute with different things, all contributions are still important for the movement of the math-

ematical activities. Another key finding is that children seldomly contribute with questions, not even in the activities where they have a lot of freedom to do so. Apart from children's lack of questions, the above findings indicate that the children are highly involved in the mathematical activities, both in initiating ideas, influencing the characteristics and structure of the activities and in involving the other participants.

8.1.2 Kindergarten teachers' contributions to mathematical inquiry processes

This subsection is devoted to the KTs' contributions to the segments identified as inquiry, where the KTs' mandate is to try to focus the activity around mathematics, which is quite different to everyday dialogues in kindergarten. In this research study 5 KTs in the focus group of the AP participated. Observations were made while they implemented the pre-designed activities. In the first phase of data analysis, most of the segments identified as inquiry segments were from kindergarten 2, where KT2 was involved. This KT is also most present in the papers, and the discussion below is therefore mainly, but not solely, focused on KT2 and her contributions to the activities identified as inquiry. First the discussion focuses on how the KTs initiate or set up needs for argumentation and problem solving. Then it discusses how the KTs follow up the initiation of the activity by appreciating and challenging children's contributions by asking questions in a multimodal manner. Later the discussion focuses on how the KTs bring structure into the activity, and what this contributes to the mathematical inquiry. The KTs also listen to the children, which gives room for children's participation. The section ends with a consideration of the norms that the KTs bring in to the activity.

First, Paper 1, 2, 3, and 4 illustrate how the KTs, with help of the pre-designed activities, set up needs for children's argumentation and problem solving. This relates to a basic principle in cultural-historical activity theory, that all human activities are motivated by cultural (or natural) needs (objects/motives) (Leont'ev, 1978). The need that the KTs are setting up may be considered on several levels. In Paper 1 the KT simply asks for an argument for why the trolley (doll's pram) is reflection symmetrical. In Paper 2 and 3 the KT asks if the children can help Super Sigurd with his problem. Here the problem itself (formulated by the KT) sets up a need for argumentation, but there is also a deeper social dimension, where the children are asked to help Super Sigurd. In Paper 4, in segment 1, the children participate in an activity called 'treasure hunt'. Each problem initiates a need for problem solving or argumentation, but the motivation for solving these problems are also related to finding the treasure at the end. In Paper 4, segment 2, the children are participating in a 'balloon play' where part of the fun is to burst balloons, but they

have to solve the problems inside the balloons before they can burst the next balloon.

The KTs are not only setting up needs for argumentation or explanations, but they also follow up the initiation in diverse ways. In activities where the KT is organising learning activities where the conversation is supposed to be focused around a mathematical object it may be difficult to balance teacher-talk and child-talk (O'Connor and Michaels, 1996; Dovigo, 2016). On the one hand, the KT aims to nurture children's talk and promote them to take part in the discursive activity. On the other hand, the KT aims to focus the discursive activity around a specific mathematical aim. Paper 2 illustrates how the KT follows up the initiation by appreciating and further challenging children's contributions. The KT responds to and appreciates almost every child's contribution. Through her multimodal question strategies, she promotes the children to argue for and explain their ideas. Dillon (1988) is critical of teachers' extensive use of questions in teacher-led activities. He argues that teachers should strive to use alternative pedagogical means. Others also point to the significance of teacher's careful questioning strategies (Franke et al., 2009; Myhill & Dunkin, 2005; Rojas-Drummond & Mercer, 2003; Roth, 1996). Although there is an agreement that teachers' questions may be important for engaging students in mathematical discussions and lead to positive learning outcomes, some questions may help children to participate, other questions may hinder children's participation in mathematical discourse. Paper 2 illustrates how a KT carefully uses various multimodal question strategies to engage the children in the discursive activity. However, instead of asking a lot of verbal questions, which may take up a lot of time, she uses various other means to guide the children in the discourse, for example facial expressions and other bodily actions. Each time the children move 'away' from what she is aiming for, she adopts a questioning look, and each time the children move towards what she is aiming for, she adopts an excited facial- and bodily expression. In this manner she questions and challenges the children's contributions, but she does not take up a lot of time in the conversation by asking a lot of verbal questions. Instead she makes room for children's participation. Paper 2 particularly illustrates how the KT manages to engage the children in the mathematical discourse without 'teaching' (i.e., telling) children mathematical relations and concepts.

The activities in the AP are supposed to be structured around a mathematical objective, which gives another context than a spontaneous floating conversation that often occurs in kindergarten. However, the activity between the KT and the children, described in Paper 2, is more 'open' than what Sæbbe and Mosvold (2016) found as the most common communication pattern in an everyday situation in kindergarten. Sæbbe and

Mosvold found that questioning and affirmation were the two core components in a KT's communication pattern in an everyday mathematical situation in kindergarten. The questioning-affirmation dyad was compared with the IRE-sequence (Wood, 1992), which is often found in a 'traditional' teaching approach in school. However, in kindergarten the questioning-affirmation dyad was used to introduce new concepts, invite the children to participate in the mathematical discourse and encourage them to think further on their ideas. In Paper 2 the KT never (explicitly) evaluated children's contributions by saying 'good' or 'correct' etc. The turn-taking identified in Paper 2 was more like an IRF-sequence and could also be identified as exploratory talk (Mercer, 2000), where the children got ample opportunities to explain and argue for their ideas. The KT continuously encouraged the children to consider and explain their ideas further, and thus facilitated children's opportunities for mathematical learning. This is similar to the results in Dovigo's (2016) study, where some KTs were able to guide the conversation in a careful manner which contributed to the development of children's argumentation. In addition, the KT in Paper 2 seldomly gave the turn to any particular child, by using address terms like 'you' or the names of the children (cf. Lerner, 2003). The conversation was open for anyone to participate in.

The openness (or freedom) found in the activities in some of the papers may also be attributed to the KTs' abilities to listen to the children or their effort in trying to understand the children. Wells (1999) found that the most important teacher contribution for creating an inquiry environment was to listen to the student's contributions. Similarly, John-Steiner (2000) argues that the capability to listen to others lays at the heart of the co-creation of a ZPD. Paper 3 emphasises that the KT's ability to listen to (and observe) Ada and letting Ada explain her idea, is a crucial part of the co-creation of the ZPD. Through the way that the KT positions herself as a learner, she carefully pays attention to Ada and wants to understand her. In the activity between Ada and the KT, the KT needed to listen to, and observe, Ada, in as much as Ada needed to listen to, and observe, the KT in order to create the ZPD and move the activity forward. The co-creation of ZPD will be further discussed in Sub-section 8.1.3.

Although the activity in Paper 2 (and 3) is more 'open' than in the conversations that Sæbbe and Mosvold (2016) exemplify, the KTs also brings in structure to the activities. The structure may be identified in various forms and on various levels. For example, Paper 2 indicates that the structure that the KT brings in by organising the turn-taking as an IRF-sequence creates an opportunity for exploratory talk and is valuable for children's flow of thinking. Paper 4 illustrates how the structure that the KT brings in by organising children's turn-taking is not only valuable

but also necessary for the children's flow of thinking. The structure helps the children to re-establish a joint attention and focus on a common strategy. Structure was also brought into the activity in Paper 2 by limiting the children's access to the building blocks. By limiting children's access to the building blocks, the KT 'forced' the children towards a more abstract form of reasoning. However, the KT was sensitive to children's contributions and their needs for using the building blocks in their explanations. Paper 5, which illustrates and discusses four KTs' orchestration of mathematical learning activities, also indicates that some structure is valuable, or necessary, for the activities identified as inquiry. It seems that KT2, who focuses the conversation around mathematics but allows the children to contribute with various mathematical ideas, is the KT that facilitates most problem-solving interaction among the children, compared to the three other KTs. The results also illustrate that KT2 gave more structured tasks to the children who had problems paying attention, which helped them focus and take part in the activity. This indicates that some structures set up by the KT are valuable for facilitating mathematical learning possibilities. This is in line with Hirsh-Pasek et al. (2009), Weisberg et al. (2015), and van Oers (2014), who also argue that playful learning activities can benefit from interactions containing some elements of instruction. This is also in line with Dovigo (2016), who found that children provided more elaborated argumentations in interaction with their KTs than in interaction with their peers. However, as Paper 5 indicates, which is similar to the arguments from Hirsh-Pasek et al. (2009) and Van Oers (2014), the structure must be carefully balanced with freedom.

Through the above-mentioned contributions, the KTs are also bringing in norms, both social norms and sociomathematical norms (cf. Yackel and Cobb, 1996). From a cultural-historical approach, norms can be seen as tacit rules that structure culturally ways of doing things. The KTs are the sources of, and representatives for, these tacit rules. In Paper 2 (and 3), the KT respects and values all children's contributions, and she encourages the children to listen to and respect each other's contributions. For example, the KT promotes the other children to listen to Ada's suggestion by explicitly saying "Did you hear what she suggested?" The KT also challenges the children to explain their ideas mathematically. To suggest a solution to the problem is not sufficient, and the children are expected to provide mathematical explanations for their ideas. Furthermore, Paper 5 emphasises how KT2 sets up pre-activities to the mathematical problem-solving activities by explicitly asking the children to reflect on what it means to collaborate. By these contributions the KT brings in norms which lay the ground for the problem-solving activities and are an important part of what is identified as inquiry interaction.

To summarise the key points in this sub-section, the KTs participate with various contributions to the activities identified as inquiry. First the KTs initiate activities that promote children to argue and explain mathematical ideas and solve mathematical problems. The KTs, with the help of the pre-designed activities, set up needs for argumentation and problem solving. In Paper 2, the KT follows up her initiation by multimodal question strategies which promote the children to continue to contribute in the activities. These multimodal question strategies combined with the way that the KT carefully pays attention to the children and positions herself as a learner within the activity creates a space of freedom for children to contribute and to take responsibility in the activities. This space of freedom is carefully combined with the KT's sensitivity for children's need of structure. Whenever it is needed, or suited, the KT structures children's turn-taking or access to building blocks which both opens up a space for children's contributions and simultaneously structures and organises their flow of thinking. In addition, the KT brings in important norms to the activities. To solve problems together the children must learn to pay attention to each other, value each other's contributions, and respect each other. It is the KTs that are the sources of and representatives for these tacit cultural rules.

As mentioned in the introduction to this chapter, the KTs' contributions are strongly related to children's contributions and vice versa. But KTs' contributions are also related to the aim of the activity and their underlying stance towards the children and their learning. In the next sub-section, I discuss the dialectic relationship between the KTs' and children's contributions in the co-creation of the ZPD. Moreover, I discuss how the KTs' contributions balance the past, present and future and how this constitutes the KTs' way of becoming in the activity.

8.1.3 The dialectical relationship between the participants' contributions and the co-creation of the ZPD

As mentioned above, this sub-section focuses on the dialectic relationship between the KTs' contributions and children's contributions and how the ZPD is co-constructed by the participants. Paper 2 especially identifies how a KT's contributions can be dynamic and change in relation to children's contributions. All of the KT's actions in this paper are related to children's previous contributions, and vice versa. This resonates with Vygotsky's (1981) dialectic perspective and is also related to Bakhtine's [Volochinov], (1977) symmetrical perspective, which emphasises that, in a conversation, every word has two sides, the speaker's and the listener's sides. In paper 3 I hold that all external social actions have two (or several) sides, the actor's side and the side of the observer or listener and all actions are in the same '(inter)action space', which lays the foundation for regarding the ZPD as a symmetrical space with two (or

more) sides. Paper 3 particularly focuses on Ada's actions, how they relate to the KT's actions, and the role of Ada's actions (the presumed less knowledgeable) in co-creating the symmetrical ZPD.

As pointed out in Paper 3, the conceptualisation of the ZPD has changed from looking at the ZPD as a property or an attribute of an individual, towards a view of the ZPD as a collective process (John-Steiner, 2000; Levykh, 2008; Holzman, 2010), or a collective space (Abtahi et al., 2017; Hussain et al., 2013; Mercer, 2000; Roth & Radford, 2011). But there are different conceptions about what it means to be collective. Some researchers are still focusing on the asymmetrical relationship between the more knowledgeable and the less knowledgeable (e.g., Mercer, 2000; Wells, 1999), while others recognise that the ZPD has a symmetrical character (Abtahi et al., 2017; Goos, Galbraith and Renshaw, 2002; Roth & Radford, 2011; Zack and Graves, 2002; Zuckerman, 2007).

Following Vygotsky (1987) and Veresov (2017), Paper 3 emphasises that the ZPD must be seen in terms of social relations that emerge from the joint activity. Using Roth and Radford's (2011) approach to analyse the emergence of the ZPD, this paper investigates how a KT and a child co-creates a ZPD by expanding each other's action possibilities. The ZPD is not initially there, the ZPD emerges when the KT interacts with the child and the child interacts with the KT. The paper emphasises the significant role that the child (the presumed less knowledgeable participant) plays in the joint activity and for co-creating a ZPD. When the KT gets confused, which she expresses both verbally and non-verbally, the child works intensively to create new action possibilities for the KT to keep her in the joint activity. By her actions, the child creates new action possibilities for the KT so the KT can productively participate in the joint activity. In this way they, together, move the activity forward.

During the activity the child and the KT stand in a constant relation to each other. This relation is kept together by a commitment the child and the KT make, which Radford and Roth (2011) call 'togethering'. Through an ethical commitment based on trust the child and the KT engage and attune to one another in order to move the activity forward. Togethering is evident by the way that they both show persistence and responsibility for creating action possibilities for each other. They really work hard together to keep the activity moving and to co-create the ZPD. It seems that it is the way that they both are faced with challenges which gives rise to the strong bond. It is when they struggle that their relation is challenged and when they are dependent on each other to carry out the event. It seems as if it is the struggle that both experience that makes them both learners and teachers of each other. It is this struggle that is important, not to reach the objective as quickly as possible.

Through the joint activity the participants become aware of historically and culturally forms of reasoning in the process of objectification (cf. Radford, 2008b, 2013b). Knowledge is materialised through the participants' actions, where the child's actions in this case, are especially significant for the materialisation. Through the child's actions, intertwined with the KT's actions, the KT (and the child) 'sees' the object in a new materialised form. The object has never been objectified in exactly this manner before. The object, I hold, is objectified to the KT (and to the child) in a new way, and the materialised form reflects mostly Ada's perspective, which now the KT becomes aware of.

Paper 2 also discusses Radford's (2008b, 2013b) claim that learning in the theory of knowledge objectification is more than becoming aware of cultural ways of thinking and acting. It is also about becoming in the process of subjectification, which is a "processes of creation of a particular (and unique) self" (Radford, 2013b, p. 27). Paper 2 (and 3) illustrates how the KT, in this process, is positioning herself within the unfolding activity as a learner and shows that she wants to understand Ada. The KT shows a genuine interest to understand the children. This makes her vulnerable, because the KT might end up learning or she might not. In addition, as found in Paper 2, the KT's contributions in every turn-exchange-pair relate to children's contributions (present), the aim of the activity (future) and her underlying stance towards the children and their learning (past). It is the melting between the past, present and future, I hold, that constitutes the KT's moment to moment acting and her way of becoming in the activity. It is through the encounter with the children, by her earlier experiences as a backdrop and the aim of the activity as a motive, that the KT becomes her unique self. Who the KT was when she entered the activity is transformed in the encounter with the children (cf. Vygotsky, 1989, in Roth & Radford, 2011).

This sub-section has considered the dialectic relation between the KTs' contributions and the children's contributions, and how the ZPD is co-created by the KT and a child through the way that they both create action possibilities for each other. It has been emphasised how the child plays a significant role for creating the ZPD, and how they both are learners and teachers of each other. The sub-section has also considered how the KT in the encounter with the children and the way the KT positions herself as a learner vis-à-vis the children, which enables the KT become her unique self in the activity.

In this research study the ZPD is used as a pedagogical construct to understand the symmetrical interaction between the KT and the children, and the symmetrical quality of learning activities. However, is this in line with Vygotsky's original idea when he first introduced the construct? Did Vygotsky consider the ZPD as a pedagogical construct, or

did he consider the ZPD as a mere psychological construct? Sub-section 8.3.4 will provide a critical reflection on the ZPD concept and discuss whether the way it is used in this research study, as a pedagogical construct, is in line with Vygotsky's original idea.

8.1.4 Mathematical reasoning in kindergarten

In order to draw some conclusions about processes of mathematical inquiry in kindergarten in Section 8.2, it is useful to discuss children's mathematical reasoning in the segments identified as inquiry. Although the previous sub-sections (8.1.1-8.1.3) implicitly deal with children's mathematical reasoning in the joint activities, this section aims to deal with it more explicitly.

Children stand in relation to an ever-changing environment. It is argued that to notice differences and similarities in an ever-changing environmental context and to recognise patterns and structures (generalities) from these differences and similarities is the essence of mathematical thinking (e.g., Mulligan & Mitchelmore, 2013; Radford, 2010b). Paper 1 illustrates how two boys argue for why a trolley (doll's pram) is reflection symmetrical. Both boys are pointing to similarities on the trolley. John is recognising and pointing to more distinct similarities, while Elias is recognising more details. Their argumentation is mediated by a rhythmic use of the word 'there' and correspond pointing gestures ('there-there and there-there and there-there') and signifies pairs of similarities. Similarly, in Paper 4, segment 1, children's coordinated turn-taking exhibits a rhythmic counting. The children use gaze and word emphasis to address the next speaker (on every third number) which brings to life the rhythm: 1, 2, $\underline{3}$ – 4, 5, $\underline{6}$ – 7, 8, $\underline{9}$ – 11, 12, $\underline{13}$ (which should have been 10, 11, $\underline{12}$). The rhythm mediates a pattern and that 'something' is repeated, which, in this case, is three counting words. This 'something' is a group of three counting words and is similar each time, but the counting words themselves are different each time. Each child must identify what is similar and what is different when it is their turn to count, and this is one of the things that the rhythm helps them to recognise.

Moreover, the materialisation of children's mathematical reasoning is deeply connected with contextual features, like how the problem at hand is formulated, available artefacts and positional location in space. For example, Paper 4 illustrates how the mathematical problems engage all children. Both problems considered in Paper 4, invite all children to participate. In the problem in segment 1, the children are invited to count their runs around the tree, and they are all responsible for their own runs. In the problem in segment 2, the children are asked to count the fingers on their hands, and thus all children must contribute with their own hands.

The materialisation of children's mathematical reasoning is also dependent on available artefacts. Paper 3 illustrates how the KT limits Ada's access to the building blocks and consequently 'forces' Ada to use more refined semiotic means. Instead of operating directly on the building blocks Ada explains how to perform the addition strategy with an iconic gesture and language, and the explanation is short and precise. Ada shows, I hold, the ability to contract her thoughts (cf. Radford, 2008c). Similarly, Paper 4 illustrates how children's flow of thinking is materialised by available artefacts. In segment 1, the children use their fingers to solve the problem at hand, and each child is responsible for counting his or her own fingers. This results in a rhythmic counting to solve the problem. Afterwards, the KT asks the children to find as many cones as they ran around the tree, and they gathered them in a pile and counted them all together. This resulted in a completely new materialisation of their thinking. The rhythmic counting disappeared due to the lack of group structure, which illustrates the significance of the available (or not available) artefacts, and how the available artefacts are used.

Another significant finding in this research study is that the materialisation of children's thinking is also related to children's positional location in space. Paper 4 illustrates how children's mathematical thinking is materialised through the ways they organise their turn-taking and which, in turn, is dependent on their positional location in space. In segment 1 the children stand a bit apart from each other when they are adding up their runs, and therefore they use gaze and word emphasis to address next speaker. This way of taking turns may be regarded as face-to-face interaction (cf. Mondada, 2007), because the current speaker and the next speaker keep eye contact in the transition of turns. This face-to-face interaction arises from the children's positional location in space. The mathematics is embodied in the children's use of word emphasis, gaze, gestures and their positional location in space and are essential components for the flow of thinking, which in this case materialises repeated addition. In segment 2, children are positioned differently, and thus their reasoning is also different. In segment 2 the children are standing closer to each other and have the opportunity to touch each other's hands, which gives rise to a side-by-side interaction (cf. Mondada, 2007). The children have access to each other's hands and can count each other's fingers. This affects children's organisation or turn-taking and thus their flow of thinking. In this case a multiplicative structure through step counting is realised.

The above considerations about the situatedness of children's mathematical thinking and how children identify differences and similarities, tell us something about what mathematics is for young children and how they encounter mathematical concepts and relations. First, it seems that

children perceive underlying argumentative structures or patterns of mathematical ways of reasoning, for example rhythm, before they utilise a lot of words. Furthermore, it seems that the pattern is perceived by identifying similarities and differences in the situation. This aligns with the comparison-of-quantity position (cf. Sophian, 2007), which emphasises that children perceive other characteristics in their environment before they perceive numerosities. How the children in segment 1 in Paper 4, seem to be satisfied with the (wrong) result because they managed to keep the rhythm, that is they overlooked the numerosity, underlines this point. Moreover, the similarities and differences and the pattern that the children recognise emerge from what they and the KTs do. Mathematics emerges from the participants' actions, for example from their turn-taking by gaze and word emphasis. But what they do is intertwined with contextual circumstances. In line with Radford's (2013b) theory of knowledge objectification, this research study shows that mathematics is patterns of interaction emerging from a coordinated use of semiotic means, where language is just one component. Mathematics emerges from what the participants do physically and is dependent on contextual circumstances. Moreover, mathematical thinking is also organised by emotions, which is exemplified in Paper 2 about how the KT's excitement or uncertainty guides the activity.

As already mentioned, Paper 4 illustrates how multiplicative structures emerge from children's turn-taking. In the two cases that are examined in the paper, the children were given addition problems (considering the semantic structure of the problems). However, these problems and children's organisation of turn-taking while solving these problems promoted rhythmic counting of groups (in segment 1) and step counting (in segment 2), which have been considered as key steps towards multiplicative reasoning (Anghileri, 1989; Mulligan and Mitchelmore, 1997). Although rhythmic counting and step counting have been seen as steps towards multiplicative reasoning, others have argued that multiplication is best seen as one-to-many mappings (e.g., Clark & Kamii, 1996; Sophian, 2007). For example, Sophian (2007) argues that multiplication must be seen as involving a 'higher order unit', that is one quantity is used as a unit for a group of another quantity. Similarly, Clark and Kamii (1996) argue that multiplicative thinking is distinguished from additive reasoning because it involves two levels of abstraction, as opposed to addition which only involves one.

Although the problems and the situations in the two segments in Paper 4 initiate rhythmic counting of groups (in segment 1) and step counting (in segment 2), the children are probably not fully aware of the multiplicative structure that lies as a potential in their actions. For example, in segment 1 the children may not be aware of the relationship between

the three fingers on each hand and the four groups of three fingers (represented by themselves). Similarly, in segment 2 the children may not be aware of the relationship between the five fingers on each hand and the six groups of five fingers (represented by their hands). At least the children are not yet able to see that there are six groups of five, and then calculate 6 x 5 = 30. Nevertheless, the results show that the children are aware of some layers of generality (Radford, 2010b), that is some layers of the multiplicative structure emerging from their actions. In segment 2 the children know that there are five fingers on one hand and that the quantity can be represented numerically by the word 'five'. They also know that if they step count by fives, they will come to the right answer (although they are not fully able to use this strategy). Moreover, in segment 2, the children are able to follow the same pattern of interaction, that is rhythmic counting of groups. However, the children in segment 2 seems to be more aware of the underlying multiplicative structure than the children in segment 1. This research study indicates that to reach for example multiplicative reasoning (as any other form of reasoning) does not happen as a distinct step, and that rhythmic counting is not solely a means to reach this 'other form' of reasoning. Instead multiplicative structures seem to slowly emerge in children's consciousness, through layers of generality, until a level of generality is obtained that the 'mathematical community' would identify as multiplication.

As mentioned in 2.5.3, a lot of the research on children's solution strategies on addition or multiplication problems are often carried out by clinical interviews with individual children. However, as this research study reveals, in light of a cultural-historical perspective, children's thinking processes and solution strategies are deeply dependent on and interconnected with contextual circumstances. Both available artefacts, the formulation of the problem, and their positional location in space influence their mathematical strategies. Therefore, to consider children's solution strategies under clinical interviews, which can be considered as a limited context, equally limits children's thinking, and therefore also limits the conclusions that can be drawn from the results of such studies. It is important to notice that children's thinking is deeply connected with contextual features. What is revealed under clinical interviews is only those forms of thinking that are possible under such circumstances.

8.2 Mathematical inquiry in kindergarten: conclusions

In this section I return to the three research issues in order to provide possible answers to them. Sub-section 8.2.1 considers the first two research issues, about the characteristics of mathematical inquiry processes in kindergarten and what enables processes of mathematical inquiry to occur. Sub-section 8.2.2 is devoted to a discussion about the challenging

transition from kindergarten to school, and how, if at all, processes of mathematical inquiry can prepare kindergarten children for school.

8.2.1 Towards an ethical, multimodal and dialectical conception of mathematical inquiry in kindergarten

With respect to the first research issue about the characteristics of mathematical inquiry in kindergarten I pull out some characteristics that emerged from the discussion section above. In the attempt to answer the first research issue, issues related to the second research issue about what enables mathematical inquiry in kindergarten to occur, will simultaneously be illuminated.

The discussion above emphasises that children contribute with mathematical claims, argumentations and explanations to the segments identified as inquiry in this research study. Children also contribute with taking and addressing turns and creating action possibilities for other participants to move the activity further. The discussion also reveals that the KTs contribute with setting up needs for argumentation and problem solving. They follow up by multimodal question strategies and listen to the children which create a space of freedom for the children to participate. The KTs also bring in various forms of structure which affects children's flow of thinking, and they bring in various forms of norms which are important for creating a collaborative environment. The KTs' multimodal participation promotes the children to contribute, guides the children's flow of thinking and facilitates the enculturation of making children's claims warranted.

As Sub-section 8.1.1 and 8.1.2 reveal, both children's and the KTs' contributions are heavily based on other modalities than language. Kindergarten children do provide quite sophisticated argumentations, but their mathematical reasoning is, to a great extent, materialised through gestures and other bodily actions, in combination with words. This statement is not made to argue that our communication is multimodal, because this is accepted by most people. Rather, this statement is made to emphasise the importance of recognising the multimodal nature of mathematical teaching-learning activities and its role for children's mathematical reasoning. It seems that children perceive underlying argumentative structures or patterns of mathematical ways of reasoning, for example rhythm, before they utilise a lot of words. It seems that the pattern is perceived by identifying similarities and differences in the situation through various semiotic means. Moreover, the KTs enable to engage children in the mathematical activities by their multimodal participation, without teaching (i.e., telling) children mathematical concepts and relations. The dialogical approach to inquiry (e.g., Jaworski, 2005; Mercer, 2000; Wells, 1999) rightfully emphasises the role of language in abstract

thinking. However, this research study indicates that processes of mathematical inquiry in kindergarten is, to a great extent, based on other modalities than language, and that these other modalities are significant stepping stones in the transition towards more abstract reasoning. Moreover, the participants' contributions are deeply connected with contextual features (cf. Lerner, 2003; Mondada, 2007), like the problem at hand, available artefacts, and the participants' positional location in space. Processes of mathematical inquiry must be understood in their own habitat (cf. Radford, 2010a), where children's and the KTs' thinking emerge from their joint activities intertwined in the specific context and in the specific time of history.

It is also important to recognise that different children contribute with different things to the activity, but all children take part in the activity in their unique ways. Some children are able to provide quite elaborated argumentation, others provide more restricted argumentation and some only contribute with claims. The children that are most salient in the segments selected in this research study are children that are able to make elaborated argumentations and explanations. For example, the segment in Paper 2 and 3 was recognised as an inquiry segment because Ada (and Lea) provided elaborated argumentation(s). Although the discussion above emphasises that the other children's claims play an important part of the flow of thinking, and thus play an important part of the inquiry process, the core of the segment was Ada's explanation. In addition, the discussion emphasises that children in this research study seldomly asked mathematical questions. There may be several reasons for this, for example the nature of the pre-designed activities. But even if there are contextual reasons for the absence of questions asked by the children, it is still an important observation. These observations have implications for what we may expect from kindergarten children in mathematical activities and for what we may consider inquiry in kindergarten to be.

The dialogical approach to inquiry focuses on creating a discourse (language in use) with certain qualities that promote learning, where reflection and critical thinking are fundamental elements (Wells, 1999; Mercer, 2000; Alrö & Skovsmose, 2002; Jaworski, 2005; 2006). Both Wells (1999) and Mercer (2000) argue that one should promote progressive or exploratory discourse as the knowledge building discourse at all levels of education. This is similar to the ideas in the problem-solving tradition which strive for developing the classroom discourse so that the children may experience the mathematics (or another subject) as professional mathematicians or scientists do (cf. Schoenfeld, 1992). But to what extent can we expect young children to think like mathematicians, and to what extent can we expect young children to be critical thinkers? Observations made in this research study show that the children eagerly

participate with claims, and some contribute with elaborated argumentations, but they do not act, I hold, like mathematicians. Moreover, they are not particularly critical to each other's contributions. Instead they act, quite spontaneously, in the moment, related to the KTs and the other children's contributions. From a cultural-historical perspective, children are not naturally born critical thinkers, but they learn it by participating in cultural activities that practise critical thinking. The KT is principally the critical voice in the activities examined in this research study. It seems that kindergarten children, at least the children in this research study, are taking their first steps towards becoming mathematical problem solvers and critical thinkers. Although they are not critically engaging with each other's ideas, they do provide different ideas to the problem.

Although we cannot expect kindergarten children to have become autonomous problem solvers and critical thinkers, this does not mean that they do not find these mathematical activities meaningful. In this study, children show eagerness to solve problems that they were introduced to, although they did not ask a lot of questions. Instead they made a lot of claims and guesses about what the solutions might be, and some children made elaborated arguments for their claims. It seems that they find it meaningful to *be with others* and to solve problems *together with others*. Radford and Roth (2011) introduce the term 'togethering', which is a theoretical construct that aims to account for the commitment that the participants make to one another to carry through an event. Togethering is both a condition and an outcome of a joint 'ethical' activity. 'Ethical' means the manner in which the participants 'ethically' tune to one another, with trust and respect. Togethering is then a key component in an ethical activity. In the segments identified as inquiry in this research study, the participants commit to one another despite their differences, and they show responsibility and persistence to carry out the activities.

The KTs play a significant role for creating ethical inquiry in kindergarten. They play a key role in introducing the children to problems which create a need for collaboration and joint problem solving. The KTs bring in norms where the children learn to listen to each other and respect each other and the KTs may introduce pre-activities where the children discuss what it means to collaborate. The KTs organise children's turn-taking as needed or suited and bring in other structures which help to organise children's mathematical reasoning. But the KTs cannot create ethical inquiry alone. The KTs' contributions are deeply connected with children's contributions, and the children are as important contributors for carrying out the activities as the KTs. All participants are mutually dependent on each other to continue to participate in the activity and they need to respect and trust one another to keep the activity

moving. This research study shows that kindergarten children are able to take responsibility for carrying out the mathematical activities, and it is important that the KTs try to understand the children, that is position themselves as learners and let the children guide them in the activities.

A dialogic approach to inquiry regards inquiry as a stance toward knowledge (Wells, 1999) and a way of being in practice (Jaworski, 2005). Wells (1999) explains this stance as an identified 'will' to solve problems together with others. Similarly, Schoenfeld (1996) argues that the underlying stance toward knowledge is based on an eagerness to know. In this research study, the children show eagerness to solve problems that the KTs introduce. But rather than having a deep 'will' to know, it seems that the eagerness is directed towards the 'will' to accomplish (mathematical) activities together with others. The children show responsibility and persistence to carry out the initiated activities, however, in the moment the activities are accomplished, the children do not ask questions in order to know more, at least not explicitly. This argument is not made to say that children in general do not have a desire to know, but it is made to say that in these activities the identified eagerness seems to be more related to accomplish activities together with others, than the 'will' to know something for themselves.

As mentioned in the introduction, and in the beginning of Chapter 3, inquiry has emerged as a reaction against the 'traditional' teaching approach, and perhaps treated a bit ideally. A central part of the reaction was that the students or children were brought to the centre of their own learning processes. Although the sociocultural approach made impact on the inquiry concept by introducing dialogic inquiry (e.g., Wells, 1999), there is still huge emphasis on children's agency and autonomy in asking their own questions and solving problems that come from their own interests. From a cultural-historical approach autonomy is something children do not initially have, it is an outcome of learning. Kindergarten children are perhaps not as autonomous as dialogic inquiry requires (cf. Jaworski, 2005; Wells, 1999).

To summarise the above considerations, it seems that mathematical inquiry in kindergarten must be regarded as multimodal processes, where other modalities than words are significant stepping stones towards more abstract reasoning, including critical and reflective thinking. These multimodal processes of mathematical inquiry are deeply connected with contextual features, as for example available artefacts or the participants positional location in space. Moreover, in mathematical inquiry in kindergarten the KTs play significant roles. They are key sources to cultural ways of thinking mathematically and sources for bringing in rules for cultural ways of collaboration. But the KTs cannot accomplish this alone, and they cannot teach (i.e., tell) children mathematical concepts

and relations. By initiating problems and bringing in diverse forms of structures the KTs influence and canalise children's flow of thinking. However, they must also pay attention to children's contributions, and try to understand the children, that is position themselves as learners and let the children guide them in the activities. All participants are mutually dependent on each other to carry out mathematical inquiry activities, which means that mathematical inquiry must therefore be regarded as dialectic processes.

Rather than seeing mathematical inquiry in kindergarten as critical and reflective activities where the children ask a lot of questions, I suggest that inquiry should be seen as 'ethical' (Radford, 2008b) activities, where togethering is a key component. Ethical considerations as trust, responsibility and respect lay the ground for a critical dialogue to be fruitful. I think, children must learn to be ethical, before they can learn to be critical. Ethical inquiry may be accomplished on the basis that children find it meaningful to *be with others* in mathematical activities and to solve problems *together with others*.

Ethical inquiry is then a dialectic, multimodal process, where the participants encounter mathematical relations and concepts that emerges from their joint activity. In ethical inquiry both KTs and children are learners and teachers of each other, and it is particularly important that the KTs position themselves as learners and let the children guide them in the activities. Ethical inquiry is based on mutual trust, respect, responsibility and the participants 'will' to accomplish events together with others. Ethical inquiry then lies the ground for becoming critical and autonomous thinkers and problem solvers later, and it seems that in kindergarten we see the 'buds' of mathematical *dialogic* inquiry.

8.2.2 How processes of ethical inquiry may facilitate kindergarten children's transition to school

As mentioned in Section 2.4, children in the last year of kindergarten are about to make a big step from kindergarten to school. The transition is not only a physical move from kindergarten to school, it involves longer social-emotional processes. School has different expectations and puts new demands on the children. The children have to get used to new rules, new adults, new peers and new activities. They come from an institution that emphasises play, care, upbringing and learning, and are about to enter an institution which focuses more on their academic development. For many children the transition between kindergarten and school can be challenging. A smooth transition is decisive for how children will manage school, integrate in the society and make a successful transition into the labour market (Lillejord et al., 2017; OECD, 2017).

Research indicates that well prepared transition activities between kindergarten and school, and in particular activities which provide continuity in curriculum and pedagogy, are important for making a smooth transition (Lillejord et al., 2017; OECD, 2017; Peters, 2010). Research suggests that a transition curriculum and pedagogy should focus on children's well-being, social-emotional development and need to play as well as focusing on children's self-regulation and academic development.

This research study has focused on teaching-learning processes that unfold when 5-year-olds are introduced to mathematical activities ostensibly aimed to prepare them for school. The activities emphasise playful learning and inquiry as main pedagogical principles to ensure children's participation, collaborative engagement and learning. The activities meet many of the suggestions set forth by research to facilitate a smooth transition. However, the activities themselves do not prepare children for school. There are many potential tensions that may arise when KTs, who are enculturated in a tradition that originally, at least in Norway, has focused more on children's self-organised play and care, are required to implement activities focusing on children's academic development.

The literature presented in Section 3.3, indicates that there are several issues that may arise in teacher-led activities where the KT or teacher has an initial aim. For example, in teacher led activities, it is difficult to balance teacher-participation and child-participation (Dovigo, 2016; O'Connor & Michaels, 1996). This research study also points to some of these challenges. As mentioned earlier, Paper 5, for example, discusses the difficult fine-tuned balance between structure and freedom in teacher-led activities. The paper indicates that an 'optimal' balance is not easy to achieve. Moreover, Paper 2 (and 3) illustrate some of the complexity of introducing children to mathematical thinking through semi-structured activities. How the activities unfold depends both on the KT and on the children.

The activities in the AP are designed in line with Wells (1999) conception of inquiry which focuses on the participants 'will' to know and solve problems together with others through questioning and critical thinking. However, as mentioned in the previous section, kindergarten children are perhaps not as autonomous as the dialogic approach to inquiry requires. Instead, ethical inquiry, which takes children's 'will' to be together with others and do things together with others (including solving mathematical problems) as a point of departure, may be more suitable for kindergarten children. Perhaps ethical inquiry also suits the KTs better, because it takes a more holistic approach to children's mathematical learning, including social-emotional aspects (cf. Lillejord et al.,

2017; OECD, 2017). This is more in line with the tradition that the KTs are enculturated into.

In school children will, most likely, be engaged in more structured activities than what they are used to in kindergarten. In addition, they will be introduced to activities which challenge their abilities to argue, explain, think critically and solve mathematical problems together with others. Moreover, children will most likely be part of classes with a larger proportion of peers. Their abilities to collaborate with peers and work in more autonomous ways will be important for how they manage future learning situations with less adult involvement. Ethical inquiry within mathematics seems to meet some of these issues. Ethical inquiry in kindergarten, which focuses on togethering, and challenges the children to argue for and explain their ideas, may, I hold, lay the ground for children to become critical and autonomous thinkers and problem solvers in school. Through ethical inquiry children can learn to collaborate which includes respecting each other, trusting each other and working together to accomplish a task, and may help children to adjust to the school context.

Further to this, OECD (2017) argues that readiness should include 'readiness for life' or 'readiness for lifelong learning'. Ethical inquiry does not only focus on elements that are important for learning mathematics, it also includes elements that are important for making friends and for integrating into society. Therefore, ethical inquiry seems like a fruitful way to introducing children to early mathematics, to prepare them for school and to 'make them ready for life'.

Play is valued in the Norwegian kindergarten tradition and is emphasised in the literature as an important ingredient in the transition from kindergarten to school (Lillejord et al., 2017; OECD, 2017; Peters, 2010). Moreover, play is regarded as essential for children's social-emotional as well as cognitive development. Vygotsky and Leont'ev considered play to be the leading activity for young children (Veresov & Barrs, 2016). Vygotsky considered mainly self-organised imaginary play, where children are not guided by an adult who has an initial aim for the activity. However, according to many scholars, playful learning activities may also involve adults as long as the activity itself remains playful (e.g., van Oers, 2014; Weisberg et al., 2015).

Ethical inquiry seems to be compatible with playful learning. Through ethical inquiry the KTs appreciate and pay attention to children's contributions and trust children's abilities to move the activity forward. Moreover, in ethical inquiry the KTs position themselves as learners and let the children guide them in the activities. This opens up a way for children's participation and agency in the activities, which is important in playful learning (cf. van Oers, 2014; Weisberg et al., 2015). In

ethical inquiry, the KTs carefully balance freedom with structure. Some children may need more structured guidance from the KTs to be able to pay attention and participate. The relation between freedom and structure given by the KTs, with respect to children's participation and exploration is, perhaps, a central feature that relates playful learning and ethical inquiry (cf. Breive et al., 2018). Although playful learning and ethical inquiry may be fruitful ways to introduce children to mathematics, I agree with Vygotsky (2016) and Winther-Lindqvist (2017), among others, that children's self-organised play should still be valued and given considerable room in Norwegian kindergartens.

In their report, Starting Strong V, OECD (2017) argues that the responsibility for facilitating a smooth transition lays on both kindergarten and school. However, until now it is mostly the kindergarten curriculum that has changed and integrated ideas from school curriculum (Lillejord et al., 2017). According to OECD (2017), schools must be prepared for the children in as much as the children should be prepared for school. Kindergarten and school should collaborate and share ideas to ensure a well-prepared transition pedagogy. Research suggests an increased emphasis on social-emotional development in primary school and an increased emphasis on academic learning opportunities in kindergarten (Lillejord et al., 2017; OECD, 2017). But tensions between the two traditions (kindergarten and school) may be an obstacle for collaboration and agreement on a common pedagogy. The tensions may be results of the historical origins of the two traditions, but it is important for the children's well-being, learning and development that the two traditions overcome their divergency. Ethical inquiry may help KTs and teachers to coordinate ideas and practices from their traditions, where children's engagement in mathematical activities should be based on ethical considerations and focus on children's explanations and argumentations to solve mathematical problems. Moreover, ethical inquiry, based on cultural-historical ideas, tries to overcome the dichotomy between the child-centred approach, still present in the kindergarten tradition, and 'instructionism,' still present in the school tradition. Thus, ethical inquiry may address some of the tensions that arise between the two traditions. Whether ethical inquiry in mathematics would fit a primary school context, is a hypothesis that requires further investigation.

8.3 Critical reflections and implications

This section is devoted to critical reflections and implications of the study. Sub-section 8.3.1 considers how this research study contributes to the results emerging from the randomised control trials (RCT) in the AP. Sub-section 8.3.2 and 8.3.3 discuss implications for practice and further

research, respectively. Finally, Sub-section 8.3.4 makes critical reflections on the ZPD concept as used in this research study.

8.3.1 Contributions to the Agder Project

As mentioned in Chapter 5, the AP uses an RCT design to study the effects of implementing a school readiness programme in Norwegian kindergartens. In the RCT design, measures of children competences are made on three occasions before and after the intervention. There are clearly paradigmatic tensions between the RCT study, which is situated within a positivistic paradigm and my research study which falls under a naturalistic paradigm. However, I do not intend to go into a discussion about the incompatibility of these two research paradigms. Instead I want to say something about how my research study may complement the RCT in the AP.

First, the purpose of the AP is to investigate whether the intervention programme improves children's competences within the four content areas or not. This is accomplished by comparing test scores from the children in the focus group and the children in the control group on three occasions. This captures a quite 'static picture' of children's competences and learning progression (cf. Bryman, 2012). My research study, on the other hand, documents what was going on while the children participated in the mathematical activities designed in the AP. Put in another way, my research study illustrates the processes that (possibly) lead to the results in the AP. It documents the processes of children's mathematical thinking during specific implementations and in what ways the intervention programme supports children's possibilities for learning. Although, we can assume that the KTs implemented the activities designed in the project and that practice changed as a consequence of the implementation, the test scores say nothing anything about how the intervention programme relates to the test scores. This is what my research study may contributes to.

A common critique of quantitative research is how the relationship between the variables are explained (Bryman, 2012). My research study may contribute to explain the correlation between the variables, for example, between language and mathematics, which is a possible line of investigation in the AP. In addition, my research study explains, I hold, the relation between mathematics and social competences (including respect, responsibility and collaboration), which was the only competence area that was not assessed in the AP. My research study indicates a close relation between social competences and children's mathematical thinking.

One of the aims in the AP is to investigate whether the intervention programme prepares children for school or not. This will be accom-

plished by obtaining scores from standardised test in school and investigating whether the children in the focus group performs better on the tests than the children in the control group. The children's performances will arguably indicate how the children have managed to integrate in school and are able to follow the required progression. My research study illustrates the processes that may lead to children's progression and ability to integrate in school. Moreover, it indicates how children may be prepared for situations that they most likely will meet in school, and in their lives in general, by participating in ethical inquiry activities within mathematics.

8.3.2 Implications for practice

My research study illustrates how children use a variety of semiotic means in their mathematical thinking processes, and it seems that these means, deeply connected with contextual features, enable them to perceive similarities, differences and patterns, that is mathematical structures, emerging from their activities. It is important that KTs pay attention to children's multimodal ways of arguing and explaining, and value these as important stepping stones toward more abstract reasoning. It is also important that KTs facilitate children's opportunities to solve mathematical problems together with others in a variety of settings and with a variety of available artefacts. However, sometimes, and especially related to some children, it may be productive to limit children's access to artefacts and promote them to verbally explain their ideas, which may move them toward more abstract ways of reasoning.

Furthermore, the KTs can benefit from being aware of the ramifications of their own multimodal contribution in the mathematical activities, and especially in relation to their question strategies. Instead of asking a lot of verbal questions they can guide the children by their bodily actions and tone of voice towards the aim of the activity and consequently give children opportunities to contribute. This research study emphasises that KTs may benefit from being consciously aware of the affect their bodily actions may have on children's mathematical reasoning and for engaging them in mathematical discourse without having to 'teach' (i.e., tell) them.

This research study also shows that it is important that KTs pay attention to children's contributions (verbal and non-verbal) and try to understand their perspectives. This may promote children to contribute with their ideas and explanations and to take responsibility for carrying out the mathematical learning activities. Therefore, it is important that the KTs trust children's abilities to take responsibility in the activities and dare to go along with their choices of direction. However, this does not mean that the KTs should leave the children alone, all by themselves. Mathematical reasoning in kindergarten is materialised through their

joint activities where the KTs are important contributors. It is the KTs that are the central sources to cultural and ethical ways of thinking and collaborating. Even when children work in small groups, the KT should be available and contribute whenever needed. Moreover, this research study indicates that there is huge variation in what kindergarten children can (or want to) contribute with in mathematical activities. It is therefore important that KTs are sensitive to children's needs for guidance in the activities. Some children need more guidance and structure than others. But it is also important to value all children's (mathematical) contributions, because, even those contributions, which in the first place seem unimportant, may play an important part in children's flow of thinking.

When KTs implement mathematical learning activities in kindergarten, this research study indicates that they should focus on promoting ethical forms of collaboration. The focus should be on mutual trust, respect and responsibility, rather than on being critical to each other's ideas. Pre-activities where children discuss what it means to collaborate, may be valuable for promoting trustful and respectful forms of collaboration. However, the KTs may, beneficially, challenge children's contributions and make them reflect on their ideas, but the children should not be expected to be critical to one another.

The results from this research study indicate the complexity of engaging children in mathematical activities in kindergarten. The nature of the activities depends on the unique composition of individuals participating in the activities and is impossible to (fully) anticipate from the outset. However, KTs, who are aware of their influential role in the unique composition and are able to flexibly act in the moment, with respect to children's contributions, may create fruitful learning possibilities for the children (and themselves). This requires reflective and well-educated KTs which puts demands on the kindergarten-teacher education and policy makers in Norway. It is important to ensure high-quality kindergarten-teacher education and facilitate professional developmental courses which direct attention to some of the key elements found in this research study.

I also hold that both KTs and teachers in school may benefit from the insights in this study. Teacher's in the first grade must take into account the tradition that children entering first grade come from and furthermore what can be expected from them when they enter school. This is particularly important for facilitating a smooth transition from kindergarten to school, but also to ensure that children in first grade are engaged in mathematical activities that are meaningful to them.

8.3.3 Limitations and implications for further research:

This research study has focused on children in the last year of kindergarten, before they enter school, and how processes of ethical inquiry may

prepare them for school. However, as research suggest, the responsibility for facilitating a smooth transition is shared between kindergarten and school. Both kindergarten and school should implement transition activities, which focuses on children's well-being, as well as social-emotional and academic development. OECD (2017) argues that there is a need for curricula and guidelines which reflects this. In Sub-section 8.2.2 I conjecture that these ethical inquiry-based activities, if implemented in both kindergarten and school, could help to create a bridged pedagogy between kindergarten and school. Therefore, it is highly relevant to do a follow-up research study which develops similar playful and inquiry-based activities for first grade and investigates how these may facilitate a smooth transition from kindergarten to school. Moreover, it could be relevant to investigate children's progression from their last year in kindergarten through first grade, while they participate in playful and inquiry-based mathematical activities.

As this research suggests, children's participation in ethical inquiry-based mathematical activities seems fruitful for their further mathematical learning and integration in school and society. However, engaging children in ethical inquiry-based activities is a complex task for KTs. It is therefore interesting to investigate how KTs may develop their profession and their abilities to guide children in ethical inquiry-based mathematical activities. It is further relevant to investigate how to improve kindergarten teacher education and professional development courses that can provide high-quality education and focus on some of the key points in this research study.

Although this research study suggests a relation between play and inquiry, there is a need for a deeper understanding of this relationship. Further research, which considers this relationship and consequences for children's learning possibilities may be interesting to conduct.

This research study found that children seldomly asked questions in the mathematical activities, at least not verbal question. However, Paper 1 illustrates that children do perceive argumentative structures, although they do not use a lot of words in their argument. Perhaps children ask pre-verbal questions, for example by using 'questioning look'? There is a need to investigate how children learn to ask verbal questions, and what could be identified as pre-verbal questions when children inquiry into mathematics.

8.3.4 Critical reflection on the ZPD concept

In October 2017, I had the privilege to visit Luis Radford and to discuss the ZPD concept, among other things, with him. I really enjoyed the insightful discussions that we had. At least the discussions were insightful to me. Radford really made me think twice on the ZPD-concept. He cri-

tiqued[21] his own previous conception of the ZPD, and others interpretation and use of ZPD as a pedagogical construct. Radford pointed to the origin of the concept. As indicated in Sub-section 4.2.1, Vygotsky developed the ZPD concept as part of a critique against contemporary psychologists' ways of measuring children's intelligence. The ZPD was introduced as a construct to talk about the development of psychological functions (memory, perception etc.). However, as also mentioned in Sub-section 4.2.1, the ZPD must be understood in relation to other concepts and ideas developed by Vygotsky, especially 'the general genetic law' (Veresov, 2004; 2017; Wells, 1999; Wertsch, 1984), and the idea of 'learning-leading-development' (Levykh, 2008; Holzman, 2010; Veresov, 2017). Vygotsky (1978) proposed that a fundamental feature of learning is that it creates the ZPD, because

> …learning awakens a variety of internal developmental processes that are able to operate only when the child is interacting with people in his environment and in cooperation with his peers. Once these processes are internalised, they become part of the child's independent developmental achievement (p. 90).

Through this quote, Vygotsky points to the significant role that instruction (teaching-learning) plays in a child's development, which easily lead to a conception of the ZPD as a pedagogical construct.

Verosov (2017) emphasises that the ZPD must be regarded *as* social relations emerging in interaction with others. Taking this into account, the ZPD is then not a mental state of the child, it is a social phenomenon. But the question then becomes, how does this connect instruction (teaching-learning) and the child's development of psychological functions? In one of the conversations that I had with Radford, he said that the term 'social relations' (as introduced by Marx) is very general, and almost impossible to make (practical) sense of. If we regard mental functions (e.g., memory or perception) as initial social relations, in line with Veresov (2017), how can we for example differentiate between perception and memory? And further, if we think of it in relation to education, how can we account for the differences in for example geometric and arithmetic thinking? Radford argued that the ZPD concept must be regarded as a psychological construct, not an instructional or pedagogical construct, although the concept points to the connection between instruction and development.

I agree that it is difficult to make sense of ZPD as social relations, and especially in order to consider how children learn mathematical relations and concepts in teaching-learning activities. Perhaps, Vygotsky initially used the ZPD as a mere psychological construct to talk about how children's mental functions develop based on interaction with others.

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[21] I want to emphasise that the references I make to Radford in this section are based on conversations with him and my interpretations thereof and thus cannot be conceived as citations.

However, the way that many scholars have developed and used the ZPD, as a pedagogical construct in line with Vygotsky's overall ideas, is in my view useful. Especially the way that recent scholars have conceived it as a symmetrical space emerging from joint activities (e.g. Abtahi, Graven, & Lerman, 2017; Goos, Galbraith, and Renshaw, 2002; Roth & Radford, 2011). There is, I hold, something symmetrical and relational recognised in teaching-learning activities that are qualitatively different from activities where the participants do not learn (or where learning is not expansive), that is where the participants instead of reaching intersubjectivity creates a distance and become alienated from each other's ideas.

Whether we should continue to consider the ZPD as a pedagogical construct, or whether the construct should be replaced with another construct to account for the symmetrical relation in teaching-learning activities is an open question which I do not intend to consider her.

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10 Appendices

Appendix 1: Program, two-day gathering 3. Jan. 2016

Example of the program from one of the two-day gatherings in the professional developmental course:

Wednesday 20.01 Vest-Agder county		Thursday 21.01 Vest-Agder county	
10.00-10.15	Welcome	09.00-09.30	Mathematics: Groupwork (workshop) – feed-back on the math. activities Per Sigurd Hundeland
10.15-11.00	Semesterplan IGP work Svanaug Lunde	09.30-10.15	Mathematics Discussion in plenum
11.15-12.00	Discussion in plenum	10.30-11.30	Mathematics Combinatorics, probability, statistics Martin Carlsen
		11.45-12.00	Mathematics Presentation of try-out activities Ingvald Erfjord
12.00-13.00	Lunch	12.00-13.00	Lunch
13.00-13.45	Self-regulation + assesment Ingunn Størksen Dieuwer Ten Braak	13.00-13.45	Language Groupwork (workshop) – feed-back on the activities
14.00-14.45	Mathematics Geometrical shapes Ingvald Erfjord	15.00-15.45	Language Awareness of sounds + letters Ragnhild Lenes
15.00-15.45	Mathematics Groupwork (workshop) Per Sigurd Hundeland	15.45-16.00	Closing session
15.45-16.00	Mathematics Discussion in plenum Per Sigurd Hundeland		Vel hjem!
16.15-17.00	Mathematics Patterns and symmetry Svanhild Breive		

Appendix 2: The frontpage of 'Lekbasert læring'

'Lekbasert læring' is the Norwegian translation of playful learning. The frontpage of the book, which contains about 130 playful learning activities is illustrated below:
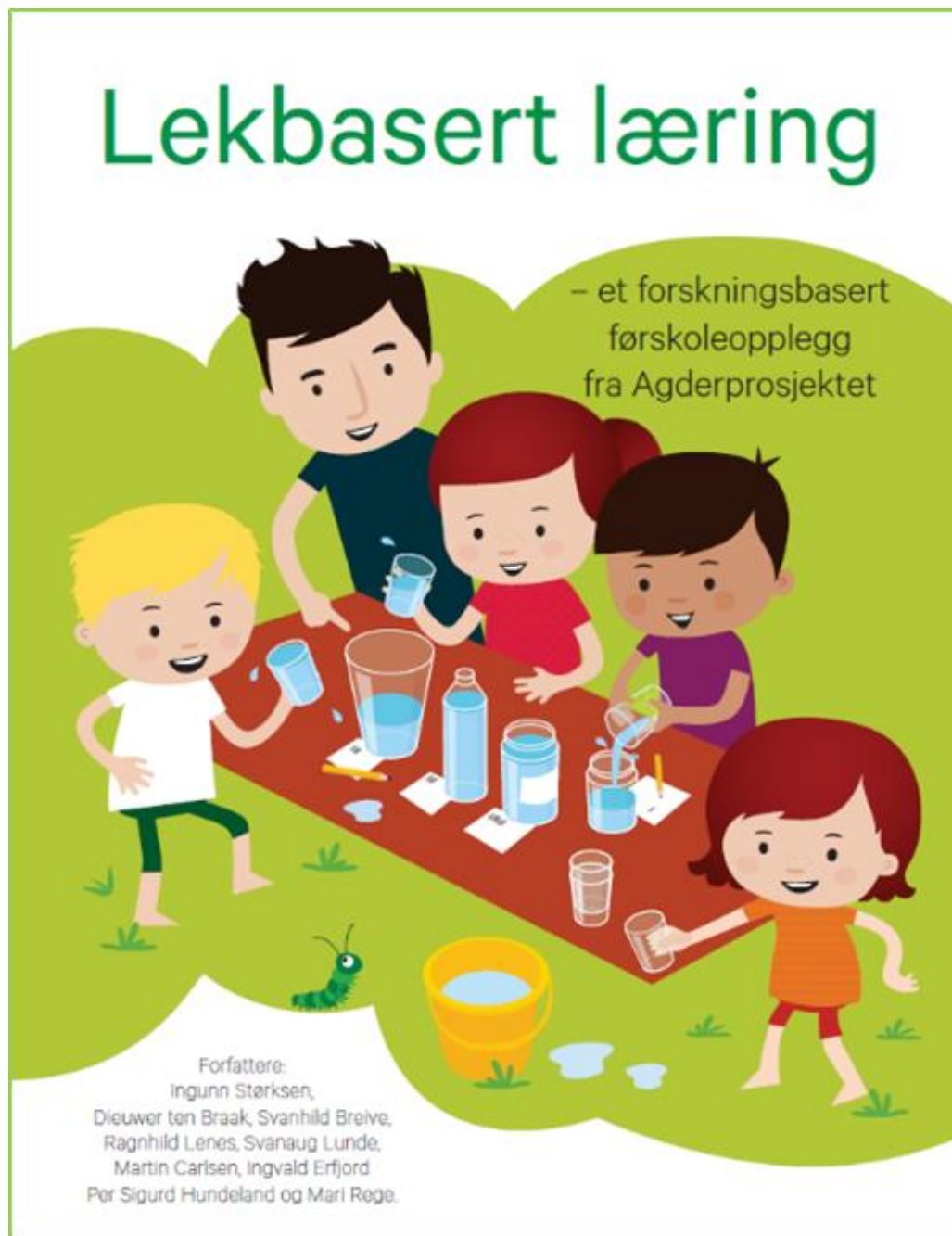
Appendix 3: Example of the mathematical activities

An example of one of the mathematical activities designed in the AP is provided below. The text in each activity was organised by intention, preparation, materials needed and implementation.

Marve Larve

UTSTYR

Perler med ulike farger, snor (til larver med 1 til 10 perler), store fyrstikkesker, papir, tusj, stjerneklistremerker, en pose og terning.

Forberedelse

Til denne aktiviteten kan du velge om du ønsker å lage larver med fyrstikkeskehus på forhånd, eller om barna skal få gjøre det selv. I begge tilfeller må du forberede en «invitasjon» eller en historie som fanger barnas interesse og leder dem inn på muligheter for matematiske oppdagelser. Her viser vi hvordan du kan legge det opp hvis barna lager egne larver.

Intensjon

Barna får erfaring med parkobling opp til 15. I tillegg får barna erfaring med separasjon og kombinasjon som prinsipper for å sette sammen og dele opp antall (grunnlag for addisjon og subtraksjon).

Gjennomføring

- Samtal med barna om larver.
  Har de sett en larve noen gang?
  Har de holdt en i hånden?
- Kast terning, og gi dem i oppgave å lage hver sin larve med dette antallet perler på.
  Larven lages ved å tre perlene på en snor.
- Senere skal barna lage hus til larvene hvor tallet til larven (antall perler) er påført både med tallsymbol (assister barna slik at tallene blir tydelige for alle) og som en mengde (prikker eller stjerner). Barna kan gjerne få lage egne navn på «sin» larve. (For eksempel Arve, Narve, Marve, eller ... Gustav.)
- La barna undersøke hvor mange perler de ulike larvenes kropper består av.
  o Hvilken larve er lengst, og hvilken larve er kortest?
- La dem samarbeide om å sortere larvene etter lengde i stigende rekkefølge.
- Samle inn hus og larver, og legg larvene i posen din og husene på gulvet. Barna får trekke hver sin larve fra posen. Deretter skal barna finne larvens hus. Larven med fem perler skal bo i fyrstikkesken med tallet 5 på osv.
- Velg en av larvene, og lat som om larven er på vei til å krype inn i huset sitt. Be barna lukke øynene mens du lukker igjen fyrstikkesken midt på larven. Når barna åpner øynene kan dere undre dere sammen over flere spørsmål:
  o Hvor mange perler av larvens kropp er utenfor huset?
  o Hvor mange perler av larvens kropp er inni huset nå?
  o Hvordan kan man vite det?
- Nå viser du fram en larve som er på vei inn i esken der noen perler er synlig på utsiden. Så titter du inn i esken (uten at barna ser det) og sier hvor mange perler du ser inni esken. Hvor mange perler har denne larven (kombinasjon av antall)?
- Bruk gjerne larvene videre i samlinger for å snakke om tall.

"Det var en fin progresjon i vanskelighetsgraden på oppgaven. Barna var nysgjerrige fra første stund. Utfordringen min lå i å finne de riktige spørsmålene når jeg skulle få barna til å finne løsninger. Det krever trening i å snakke 'matematikkspråk'. Jeg ser at det også skal trening til for barna for å kunne ordlegge seg på rett måte. For eksempel på spørsmålet 'Hvordan kan du vite det?' Svaret blir fort: 'Fordi det bare er sånn'. For barna var dette en gøy lek som også ga læring. Gjennomføringen av aktiviteten ble gjort uten andre tilpasninger enn å dele gruppa. Barna ble også oppfordret til å telle på fingrene sine når de skulle finne løsninger på oppgavene.

Hilde Helene Zikalla, Aust-Agder

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Appendix 4: Semester plan, monthly structure, AP

Example of the semester plan (autumn) which gives a monthly structure:

	SEPTEMBER	OKTOBER	NOVEMBER	DESEMBER
Sosial kompetanse	Etablering av førskolegruppa Årshjulet (s. 24) Du og jeg – Portrettbilder av alle i førskolegruppen (s. 26) Venneregler (s. 27)	Arbeid med vennskap En klem i posten (s. 30) Fest for førsteklassingene (s. 31)	Arbeid med følelser Hva er en følelse? (s. 32) Mim en følelse (s. 33)	Om å være god mot andre Den største gleden du kan ha (s. 38)
Språk	Dialogisk lesing (s. 44) Bøker om vennskap Språklig bevissthet – særlig fokus på rim Lakrismus og røde brus – oppdag rim (s. 59) Bokstaver (fyll inn de dere skal jobbe med): Aktuelle aktiviteter: Oppdag første lyd! (s. 70) Bokstavjakt (s. 73) Finn gjenstander hvor første lyd er ... (s. 74)	Dialogisk lesing (s. 44) Bøker om vennskap Valgfrie bøker Språklig bevissthet – særlig fokus på rim Is-gris-ris, rim med bilder (s. 60) Rim tull og tøys! (s. 61) Hvem skal ut? (s. 62) Tegn og skriv dikt! (s. 63) Bokstaver (fyll inn de dere skal jobbe med): Aktuelle aktiviteter: Oppdag første lyd! (s. 70) Bokstavjakt (s. 73) Finn gjenstander hvor første lyd er ... (s. 74)	Dialogisk lesing (s. 44) Bøker om følelser Valgfrie bøker Språklig bevissthet – særlig fokus på setninger Oppdag setninger (s. 64) Jeg er et ord i setningen (s. 65) Bokstaver (fyll inn de dere skal jobbe med): Aktuelle aktiviteter: Oppdag første lyd! (s. 70) Bokstavjakt (s. 73) Finn gjenstander hvor første lyd er ... (s. 74)	Dialogisk lesing (s. 44) Valgfrie bøker Språklig bevissthet – særlig fokus på ord Jeg er et ord i setningen (s. 65) En kort, en lang (s. 66) Bokstaver (fyll inn de dere skal jobbe med): Aktuelle aktiviteter: Oppdag første lyd! (s. 70) Bokstavjakt (s. 73) Finn gjenstander hvor første lyd er ... (s. 74)
Selvregulering	Ferieleken (s. 80) Hvem har / hvem er? (s. 91)	Hermegåsa (s. 82) Klar-ferdig-gå-leken (s. 83)	Rødt lys – Grønt lys (s. 86) Min hatt den har tre kanter (s. 81)	Bjørnen sover, sneglen sover (s. 89)
Matematikk	Tall Tripp, trapp, tresko (s. 102) Hopp over bekken (s. 104) Geitekillingen (s. 105) Bondegården (s. 106) Geometri Skjulte former (s. 124) Geometrisk fotosafari (s. 126) Mønster overalt (s. 127) Perling og symmetri (s. 128)	Måling Skjulte gjenstander (s. 144) Steintroll (s. 145) Statistikk Skoundersøkelse (s. 154) Tall Bygge fargede tårn (s. 156) Tårnprinsessas første brev (s. 108) Tårnprinsessas andre brev (s. 109) Lurebamsene (s. 110) Bamsebussen (s. 112)	Geometri 2D-detektiver (s. 129) Titt-tei, her er jeg! (s. 130) Pyramidene i Egypt (s. 131) Hemmelighetsposen (s. 132) Måling Skomåling (s. 146) Papirfly (s. 148) Statistikk Sortere og kombinere bamser eller biler (s. 158) Værestasjon (s. 160)	Tall Bondegården (s. 106) Marve Larve (s. 114) Tårn og terning (s. 115) Telletriks (s. 116)

Appendix 5: Daily plan, daily structure, AP

Example of the daily structure, which should take up 2 hours a day, 4 days a week:

Sosial kompetanse
(Se side 20 og utover)

Språk – ca. 1/3
Lek med språket
(Se side 40 og utover)

Dialogisk lesing
(Se side 44)

Selvreguleringslek eller frilek
(Se side 76 og utover)

Matematikk – ca. 1/3
Matematikkaktiviteter
(Se side 94 og utover)

Sosial kompetanse – avslutningsritualer
(Se side 23)

Appendix 6: Overview over data sets

Overview data sets, autumn 2016:

	KG 1, KT1	KG 2 KT2	KG 3 KT3	KG 4 KT4 and KT5
Observations	26.09: kam1 + kam2	26.09: kam1 + kam2	20.09: kam1 + kam2	20.09: kam1 + kam2 KT4
	27.09: kam1 + kam2	30.09: kam1 + kam2	22.09: gr1 kam1	21.09: kam1 + kam2 KT4
	30.09: kam1 + kam2	10.10: kam1 + kam2	22.09: gr2 kam1	04.10: kam1 + kam2 KT4
	10.10: kam1 + kam2	13.10: kam1 + kam2	14.10: gr1 kam1	06.10: kam1 + kam2 **KT5**
	13.10: kam1 + kam2		14.10: gr2 kam1	
Interviews	13.10: KT1	10.10: KT2	14.10: KT3	04.10: KT4
				06.10: **KT5**
Jottings/Field-notes (including the small conversations)	26.09	26.09	20.09	20.09 KT4
	27.09	30.09	22.09	21.09 KT4
	30.09	10.10	14.10	04.10 KT4
	10.10	13.10		06.10: **KT5**
	13.10			

Overview data sets, spring 2017:

	KG 1, KT1	KG 2 KT2	KG 3 KT3	KG 4 KT4 and KT5
Observations	14.03: kam1 + kam2	14.03: kam1 + kam2		04.04: gr1 kam1 + kam2 **KT5**
	17.03: kam1 + kam2	17.03: kam1 + kam2		04.04: gr2 kam1 + kam2 KT4
	21.03: kam1 + kam2	21.03: kam1 + kam2		07.04: kam1 + kam2 KT4
	24.03: kam1 + kam2 KT1	24.03: kam1 + kam2		20.04: gr1 kam1 + kam2 KT4
	28.03: kam1	28.03: kam1 + kam2		20.04: gr2 kam1 + kam2 **KT5**
	30.03: kam1 + kam2	30.03: kam1 + kam2		06.05: kam1 + kam2 KT4
		06.04: kam1 + kam2		
Interviews	05.04: KT1	06.04: KT2		09.05: KT4
				09.05: **KT5**
Jottings/Field-notes (including the small conversations)	14.03	14.03		04.04 **KT5**
	17.03	17.03		04.04 KT4
	21.03	21.03		07.04 KT4
	24.03	24.03		20.04 KT4
	28.03	28.03		20.04 **KT5**
	30.03	30.03		06.05 KT4
		06.04		

Appendix 7: Profile of the five kindergarten teachers

The profiles are organised around the following four key points: 1) children's opportunities or freedom to participate with questions, mathematical ideas and argumentations, 2) the KTs intention to consider children contributions, 3) children's intention to participate with questions, ideas and argumentations, and 4) the degree of inquiry or problem-solving interactions in the session. These key points are not mutually exclusive, rather their relationship is of a dialectic character. The profiles are of course tendencies not absolute characteristics. These profiles were used as data in Paper 5.

KT	Profile:
KT1 in K1	1) KT1 orchestrated the mathematical activities with a relatively high degree of freedom, which is based on the way that the children were allowed to move around in the room (and even walk out of the room) and to talk about almost whatever they wanted, like birthday parties or their parents' occupation etc. The KT never told the children to sit down and pay attention, instead she promoted the children to do so by the way she enthusiastically presented the activities, which captured the children's attention. For example, in an activity about reflection symmetry, the KT introduced the activity having diverse reflection symmetrical objects in a plastic bag without telling what was inside. She shook the bag and whispered, "Listen!", which made the children curious and created joint attention.
	2) Another characteristic of the KTs orchestration was that the KT listened to almost every child's contribution (not only related to mathematics). In one of the conversations with the KT, she expressed that her desire to listen to and appreciate every child's contribution could be a hinder for her, because her attention became very shifty. She rapidly turned her attention from one child to another.
	3) The children eagerly participated in the activities, however, as mentioned above, they often contributed with other ideas than mathematics. The KT expressed that she had a challenging group of children but their ability to pay attention to mathematics grew during the intervention. The children seldom asked mathematical

	questions themselves. Most of the questions that were asked in the activities came from the KT. 4) There were a lot of 'golden moments' for problem-solving interaction. The KT and the children initiated a lot of interesting 'topics' for investigation, but few ideas were thoroughly discussed. Mathematical questions were often (not always) considered briefly, and the children seldom had to ponder about problems and to express mathematical ideas, argue for and explain their ideas in order to solve the problem. Because the KT gave the children a lot of freedom to talk and payed attention to almost every contribution, the conversations moved quickly from one topic to another.
KT2 in K2	1) 2) KT2 gave the children relatively high degree of freedom to talk, however she often restricted children's talk to mathematics by ignoring some of the contributions that were about the children's everyday experiences. The KT also restricted the children's freedom to act (physically) to areas or with objects relevant to the mathematical activity. Although the KT for the most part focused attention to mathematics, she gave the children freedom to suggest other mathematical issues than what she initially introduced. Similarly as KT1, the KT2 presented the activities in an exciting way, by use of for example excited facial expressions and whispering, which captured the children's attention and promoted the children to contribute. But sometimes she also asked questions directly to children to capture their attention. For example, when Carl was distracted by something else, she said: "Carl, do you know how many building blocks there are in the red tower?" When Carl said that he didn't know, the KT further asked "Would you like to help me count?" This helped Carl, who often had difficulties paying attention, to focus his attention on mathematics. 3) 4) The conversations between the KT and the children were almost always mathematical, and sometimes the KT and the children had longer conversations about mathematical problems. The children had to argue for and explain their ideas in order to solve the problems, and the children eagerly participated with mathematical ideas and explanations. In addition, the KT seemed to

	focus on collaboration. For example, the KT had a conversation with the children about the meaning of collaboration, and the KT promoted the children to help each other if needed. She also promoted the children to listen to each other by for example asking the group of children: "Did you hear what Ada suggested?". Although the children eagerly participated with mathematical ideas, they seldom asked mathematical questions. Most of the questions that were asked in the activities came from the KT.
KT3 in K3	1) 2) KT3 was a football trainer in his spare time, which was somehow recognisable from his orchestration of the activities. He gave the children relatively high degree of freedom to act (physically) and focused on 'doing' mathematics, which for him was when the children got opportunities to use their hands, their body and various artefacts to solve mathematical tasks. In one of the conversations with the KT he expressed that 'doing' mathematics was for him an important feature of mathematics in kindergarten and therefore he especially liked physical outdoor activities. In addition, he was giving short 'missions' for the children to perform. For example, in the 'Sorting Shoes' activity, when the children had to figure out how many shoes there were in each category, the KT gave each child a 'mission' to draw equally many lines in the bottom of the diagram as there were shoes in each category. The KT expressed several times that it was important to give the children challenging but manageable tasks, so they felt they succeeded. He often encouraged the children, in an enthusiastic manner, with comments like "good" or "great" etc. It seemed that the children enjoyed the activities and the way that the KT encouraged. 3) 4) The children eagerly participated and were having fun. There was relatively little problem-solving interaction and the children often solved tasks without having to explain or argue for their ideas. For example, in the activity called 'Tripp, Trapp', where the children should count stairs in a staircase and find out what number each stair had, the KT made A4 papers with numbers from 1-24 on and the children, one by one, had to pick an A4 sheet and place it on the correct stair. (Stair number 15 should have the A4 sheet with the

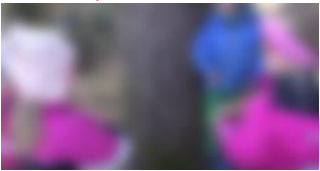|	
	number 15 on). The children just performed the tasks, without having to explain what they did, and why they did what they did. Sometimes the KT promoted the children to reflect on their solution strategies in retrospect, however the children's explanations were seldom helping them to solve problems in the first place. The children seldom asked mathematical questions themselves. Most of the questions that were asked in the activities came from the KT
KT4 in K4	1) 2) KT4 gave the children relatively little freedom to act (physically) or talk which is based on the way that she, to a large degree, controlled who was going to talk (or 'do' something) and when. For example, in an activity called 'The Farm' the children were, at one point in the activity, supposed to find how many animals there were on the farm. First the KT asked a girl, "Helene, can you figure out how many animals there are all together?". After Helene had counted and answered the KT asked a boy, "John, can you find how many different animals there are?". The KT continued to give similar 'missions' to each child. The KT made sure that each child got the opportunity to answer or 'do' something mathematically, and she appreciated children's contributions by comments like 'that's correct', 'very good' etc. The KT expressed in one of the conversations that it was important that the children learnt to respect the other children and to wait for their turn in an activity. The KT also expressed that some activities were difficult to implement as outdoor activities, because the children often got disturbed by other things. These characteristics are of course tendencies, and sometimes the activities were a lot more open where the children had a lot more freedom to act. But, as she also expressed in one of the interviews, she thought it was difficult to 'hold back' and give room for the children to figure out the problems themselves without too much interference. 3) It is difficult to state how 'eager' the children were to participate, because they seldom answered or did something without being asked. They accepted the KTs request to sit and wait for their turn. In some activities, like when they measured how much water there were

	room for in a tank, the children laughed and were having fun and showed eagerness to participate. Still they were asked to wait for their turn and respect that each child got the same opportunity to fill water. The children seldom asked mathematical questions themselves. Most of the questions that were asked in the activities came from the KT. 4) There were few incidents where the children together solved problems by expressing ideas and arguing for solutions. The children were waiting for their turn to answer or to perform 'missions'. The KT sometimes asked the children to explain what they did when they solved a task, but this explanation did not help the children to solve the problem, but to reflect on their strategy in retrospect.
KT5 in K4	I visited this KT once in the autumn (first observation period) and twice in the spring (second observation period). The first time I was invited into her session, was because K4 in the same kindergarten (KT4) was sick. 1) 2) KT5 orchestrated the mathematical activities with a relatively high degree of structure. She expressed that it was important that the children learned to wait for their turn. However, if the children 'spontaneously' contributed with ideas, the KT listened to and appreciated their contributions. KT4's orchestration varied more between structure and freedom than the other four KTs' orchestration. Some parts of her sessions were strictly organised, but other parts were open, where the children got freedom to explore on their own. The KT was presenting the activities in an excited way, which seemed to help children to keep joint attention. 3) The children were eagerly participating whenever they were allowed, but they were also patient waiting for their turn whenever that was required. The children seldom asked questions themselves. The KT asks most of the questions. 4) There were some episodes where the children and the KT got into deeper conversations and discussed mathematical problems.

Appendix 8: Two examples of the rough transcriptions

Example 1 from the rough transcriptions: 170328_K1_Treasure Hunt. The pictures are manipulated to ensure anonymity of the participants.

Time:	Situation:	Pictures and notes:
Camera 1.2	The KT introduces the activity by explaining what Treasure Hunt is and that they have to find 'post1' first. The children rejoice!	
00:00	The children start looking after 'post1'. The KT says, "Look to the left", and the children recognises 'post1'.	
00:25	The children explore the environment – there are boxes with pictures of squirrels on.	
00:50	The KT reads the task, "Three squirrels are collecting cones. They find 21 cones. How many cones do each of the squirrels get if the cones shall be shared equally?"	
01:04	The children start dividing the cones in the three boxes. First, they put some cones in each box, without counting. The the KT asks how many cones there are in each. The children count the cones in each box and find that there are seven in each box. They solved the problem a bit by luck…	
02:40	The KT wants the children to consider another way to share the cones. What if this was a family with a dad, a mother and a baby squirrel, how should they share the cones then? The children are not so interested in solving this type of task. They are looking for the next post. The children argue that they still have to have seven each, because they still are three squirrels and they should have equally many.	
03:47	A girl burst out "I found the next one!" and points to the next post. The children run towards the next post.	
04:35	The KT reads the problem, "Run around the nearest tree three times each. How many times have you run around the tree all together?" The children immediately start to run around the tree.	
05:05	A girl suggests that they must go one at the time. The other children continue to run.	

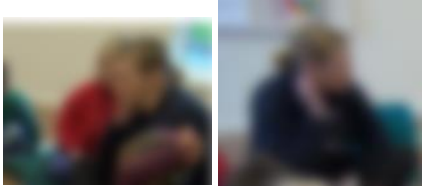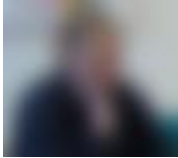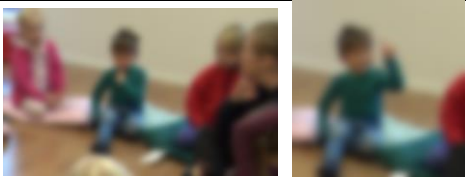|		
05:25	The KT asks, "How many times have you run around now?". One girl says that she has run four times. Others are not sure.	
05:49	The KT asks the children whether they should do as … suggested to run one and one. The girl that suggested that stops the other children, and the KT helps organising them so they can follow that strategy.	
06:21	… starts running and the other children count. Each child does the same, run around the tree, and the other children count.	
08:00	The KT reads the other part of the task, "How many times have you run all together?"	
08:08	The children start to solve the problem together. The KT interferes minimal. They have some problems, but they are able to eventually solve it. They do a mistake and find that they were running thirteen times each.	The children collaborate to solve the problem. They overcome challenges without the KT's help.
09:09	The KT solve the task one more time together with the children using cones. They put three cones each on the ground and counts all together. … Counts, and they find that there are twelve cones.	
12:22	They conclude that there are twelve cones. The KT wants to discuss why it became thirteen before but that it became twelve now, but the children want to go to the next post… The KT get the children to take three cones each, and the KT explains that twelve was equally shared by all children, there was probably twelve together. One of the girls hide behind the tree, and the other children start to look for her. They do not want to pay attention to the KT's explanation.	
14:00	The children and the KT moves to 'post3' which the children have recognised already.	

Example 2 from the rough transcriptions: 161013_K2_Tower Building. The pictures are manipulated to ensure anonymity of the participants.

Time:	Situation:	Pictures and notes:
00:00	The KT introduces the three towers that Super Sigurd has built. Kine spontaneously suggest, "Oh, it is greatest, middlemost, smallest!" Another boy responds: "greatest, middlemost, smallest!" Mia: "Maybe it is Tower Princess!" The KT responds: "Or, great, greater, greatest!" Hans: "Yes!" Kine: "smallest, middlemost, greatest!!"	
	The KT pretends that Super Sigurd whispers something in her ear. Super Sigurd thinks he has 20 building blocks in the three towers, but he is not quite sure. He asks if the children can help him figure this out. The children respond "yes!" and lean towards the towers to start counting. Ada suggests that they have to count.	
01:50	The KT holds up the yellow tower, and some children answers that the tower consists of five building blocks. The KT initiates a conversation about how they know that. Some say that they just saw it, some say that they counted inside themselves. The KT suggests that they can count together. The KT points at the building blocks and the children count.	
02:47	Then the KT asks how many there are in the red tower. Kine answers "eight!" The KT asks how she knew that, and Kine explains that she counted inside herself. The KT then asks another boy, Karl, whether he knows how many building blocks there are, but he is not sure. Karl gets the opportunity to count the building blocks together with the KT. They find out that there are eight building blocks.	
03:22	The KT asks, "What about the middle tower?" Kine raises her hand and wants to suggest, but the KT asks another girl to count. The girl counts seven building blocks.	
03:58	The KT finds a paper and pencil and writes the number of building blocks on three pieces of papers and puts the papers in front of each tower (5, 7, 8).	

05:00	The KT says "But, Super Sigurd wondered how many there were all together. He thinks there is twenty, but he is not quite sure. We know that there are eight red, five yellow and seven green, how may we figure out how many there are all together?" Ole puts his index finger to his mouth and says «Hmm», and then he raises his hand. The KT lets him answer, and Ole says «What if …?" but then he sits back again.	
	Other children start to suggest how many there are all together, some say 21, some say 26. The KT asks "Twentysix? But <u>how</u> may we figure out how many there are all together?"	
05:40	Some children suggest that they may count. The KT responds "But we have already counted, how may we count now?" Several children raise their hands. Mia wants to suggest a strategy but is not able to explain what she had in mind. Then Lea explains her strategy. She wants to count all.	Lea explains her stratgy
06:07	After, Ada gets the opportunity to explain her strategy. She suggests "Do not count eight, and then we just count further!" The KT helps Ada explain her idea. Some of the other children count the building blocks while Ada explains. Other children carefully watch Ada while she explains.	Ada explains her strategy
07:48	After Ada has explained her strategy, with help from the KT, the KT repeats Lea's and Ada's strategies. And she especially captures the attention to one boy, Karl, who easily loses focus. The KT says, "Karl, lock at this!" The KT first repeats Ada's strategy, and then she repeats Lea's strategy. The children closely watch what the KT does. From these two strategies they count twenty building blocks. Fia says, "Super Sigurd was right, it was twenty building blocks!" Other children agree.	
09:30	Hans asks the KT to writhe number twenty on a piece of paper, and the KT does as Hans asks. Some children confirm that it is two-cero.	
10:45	The KT then introduces the box with all the building blocks and asks the children to find twenty building blocks each, and then to build towers similar as Super Sigurd's. The children eagerly start to count twenty building blocks and build towers.	

Appendix 9: Multimodal interpretative transcriptions, example

Example from the multimodal interpretative transcriptions: 161013_K2_Tower Building. The pictures are manipulated to ensure anonymity of the participants.

Turn	Who	Utterance	
			The KT prepares the question. She scratches her head, behind her ear
92	KT	How can we figure out how many there are all together?	This picture is taken when the KT says "out"
93	Ole	Hmm… ((puts his finger to the mouth and suddenly he raises his hand))	Ole puts his finger to his mouth when he says "hmmm…"
94	KT	Ole	
95	Ole	Hmm… What about … ((he points at number eight))	Again, Ole puts his finger to his mouth when he says "hmm…", then he leans forward in order to explain, but then he moves back to his seat
96	Kine	Ah… ((puts her hand in the air))	Kine draws air quickly in her lounges when she raises her hand
97	KT	Kine	

Appendix 10: Letters of consent

Letter of consent to the parents:

UNIVERSITETET I AGDER

Deltakelse i en studie med fokus på barns læring av matematikk i barnehagen

Kjære foreldre til _____!

I forbindelse med Agderprosjektet, som barnet deres allerede er en deltaker i, ønsker vi i barnehageåret 2016/2017 å gjennomføre en kvalitativ studie som ser nærmere på barns læring av matematikk. Målet er å skaffe innsikt om hvordan barn lærer gjennom lekbaserte matematikkaktiviteter under veiledning av barnehagelærere. I den forbindelse ønsker vi å gjøre observasjoner av barna og barnehagelærerne mens de arbeider med matematikkaktiviteter i barnehagen. Vi ønsker også å gjennomføre korte samtaler med barna og barnehagelærerne. Til hjelp for vårt eget forskningsarbeid ønsker vi å gjøre opptak av observasjoner og samtaler med lyd- og videoutstyr. I tillegg kan det bli aktuelt å sammenligne funn i den kvalitative studien med andre data i Agderprosjektet begrenset til de testene som allerede skal gjennomføres. Alle observasjonene, samtalene og de kvantitative dataene vil kun bli brukt til forskningsmessig analyse og formidling.

Alle barn og ansatte vil få sin anonymitet ivaretatt i forskningspublisering. Lyd- og videoopptak vil bli lagret og kun tilgjengelig for forskerne i prosjektet. De vil siden bli slettet i tråd med instruks og tillatelse fra Personvernombudet for forskning og UiAs interne retningslinjer.

Dersom du er villig til at barnet ditt kan være med i denne studien, ber vi deg undertegne under, og levere svarslippen til personalet i barnehagen. Barnehagen med sine ansatte har allerede gitt tilsvarende samtykke til at studien kan gjennomføres.

Vi håper på positive tilbakemeldinger og er på forhånd svært takknemlig for hjelpen. Dersom du ønsker mer- informasjon om studien er det bare å ta kontakt med undertegnede eller barnehagen. Vi svarer gjerne på spørsmål. Observasjon, samtaler og forskning vil bli utført av en- prosjektgruppe av vitenskapelig ansatte på Universitetet i Agder: John Monaghan (professor), Martin Carlsen, Ingvald Erfjord og Per Sigurd Hundeland (førsteamanuenser) og Svanhild Breive (doktorgradsstipendiat).

Med vennlig hilsen

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UNIVERSITETET I AGDER

Deltakelse i en studie med fokus på barns læring av matematikk i barnehagen

Svarslipp

Jeg tillater herved at mitt barn i barnehageåret 2016/2017 deltar i den ovenfor nevnte studien.

Barnets navn: _____

Underskrift: _____ Sted/dato: _____|_____

Letter of consent to the KTs:

UNIVERSITETET I AGDER

Deltakelse i en studie med fokus på barns læring av matematikk i barnehagen

Kjære _____!

I forbindelse med Agderprosjektet, som du allerede er en deltaker i, ønsker vi i barnehageåret 2016/2017 å gjennomføre en kvalitativ studie som ser nærmere på barns læring av matematikk. Målet er å skaffe innsikt om hvordan barn lærer gjennom lekbaserte matematikkaktiviteter under veiledning av dere barnehagelærere. I den forbindelse ønsker vi å gjøre observasjoner av barna og av deg som barnehagelærer mens barna arbeider med Agderprosjektets matematikkaktiviteter i barnehagen. Det kan også bli aktuelt med korte samtaler med deg og barna i forbindelse med gjennomføring av noen av aktivitetene. Til hjelp for vårt eget forskningsarbeid ønsker vi å gjøre opptak av observasjoner og samtaler med lyd- og videoutstyr. I tillegg kan det bli aktuelt å sammenligne funn i den kvalitative studien med andre data i Agderprosjektet begrenset til de testene som allerede skal gjennomføres. Alle data vil kun bli brukt til forskningsmessig analyse og formidling.

Både du og barna vil få deres anonymitet ivaretatt i forskningspublisering. Lyd- og videoopptak vil bli lagret og kun tilgjengelig for forskerne i prosjektet. De vil siden bli slettet i tråd med instruks og tillatelse fra Personvernombudet for forskning og UiAs interne retningslinjer.

Dersom du er villig til å være med i denne studien, ber vi deg undertegne under, og levere svarslippen til oss.

Vi håper på positive tilbakemeldinger og er på forhånd svært takknemlig for hjelpen. Dersom du ønsker mer informasjon om studien er det bare å ta kontakt med undertegnede. Vi svarer gjerne på spørsmål. Observasjon, samtaler og forskning vil bli utført av en- prosjektgruppe av vitenskapelig ansatte på Universitetet i Agder: John Monaghan (professor), Martin Carlsen, Ingvald Erfjord og Per Sigurd Hundeland (førsteamanuenser) og Svanhild Breive (doktorgradsstipendiat). Dersom du gir ditt samtykke til undersøkelsen, vil vi i eget brev til foreldre/foresatte be om deres samtykke for barnas deltakelse.

Med vennlig hilsen

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UNIVERSITETET I AGDER

Deltakelse i en studie med fokus på barns læring av matematikk i barnehagen

Svarslipp

Jeg samtykker til å delta i den ovenfor nevnte studien som foregår i barnehageåret 2016/2017.

Underskrift: _____ Sted/dato: _____

Papers 1-5

Paper 1: Kindergarten children's argumentation in reflection symmetry: The role of semiotic means

This paper is published in the proceedings of the CERME 10 conference.

Paper 2: Engaging children in mathematical dis-course: a kindergarten teacher's multimodal participation

This paper was submitted on 20.04.2018 to NOMAD Nordic Studies in Mathematics Education. The Editors decision on 14.08.2018 was that subsequent to revisions the paper will be accepted. The version included in this thesis is a revised version following the decision which was re-submitted on 10.01.2019.

Paper 3: Student-teacher dialectic in the co-creation of a zone of proximal development: an example from kindergarten mathematics

This paper was submitted on 15.01.2018 to Educational Studies in Mathematics. The Editors decision after review on 21.12.2018 was request for 'major revisions'. The version included in this thesis is the original version of the paper submitted on 15.01.2018.

Paper 4: Organisation of children's turn-taking in small-group interaction in kindergarten

This paper was submitted on 01.10.2018 and is a peer-reviewed chapter prepared for a scientific anthology to be published by Springer based on selected papers from the international 'POEM' conference held at the University of Agder, May 2018. The Editors decision after review on 20.12.2018 was that subsequent to revisions the paper will be accepted. The version included in this thesis is the version of the chapter submitted on 01.10.2018.

Paper 5: Kindergarten teachers' orchestration of mathematical learning activities: the balance between freedom and structure

This paper was presented at CERME 11 in 2019. The paper will be published in the proceedings of that conference.

Kindergarten children's argumentation in reflection symmetry: The role of semiotic means

Svanhild Breive

Kindergarten children's argumentation in reflection symmetry: The role of semiotic means

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In this paper I investigate the characteristics of children's argumentation when they work with reflection symmetry. Using Toulmin's (2003) model for substantial argumentation, I illuminate structural aspects of the ongoing argumentation. In addition, I analyse the children's argumentation with respect to their use of semiotic means. Results show that children are able to argue for a claim in a quite complex manner. The study also illustrates the extensive use of semiotic means in children's argumentation. In every element in the argumentative structure, children use gestures and other semiotic means to mediate their ideas. It is actually impossible to make sense of the ongoing argumentation without considering the use of semiotic means.

Keywords: Argumentation, kindergarten, gestures, semiotic means.

Introduction

How children communicate their mathematical ideas is an important aspect in the attempt to understand children's reasoning in mathematics. In kindergarten children experience mathematical concepts through play and interaction with others. In their communication they justify and explain their mathematical ideas and in return they need to consider other's ideas and arguments. Thus argumentation can be seen as important for fostering children's mathematical learning.

This study is situated within a research and development project called the Agder project[i] (AP). One of the aims in the project is to investigate how researcher designed mathematical activities, developed in the project, stimulate mathematical competences. In this case study I observed one kindergarten teacher (KT) and a group of six 5-year-old children engaged in mathematical activities about reflection symmetry. The aim of this paper is to examine the characteristics of children's argumentation when they work with reflection symmetry. Furthermore, I examine what role 'semiotic means' (e.g. objects, linguistic devices and signs) play in the ongoing argumentation.

Following Toulmin (2003) and Krummheuer (1995), I regard argumentation as the practical business of choosing statements that serve the purpose of making an initial assertion reasonable and accountable for others. Argumentation is the production of an argument. An argument is then "the final sequence of statements accepted by all participants" (Krummheuer, 1995, p. 247). In addition, several arguments can serve as units in an expanded argumentation which again constitutes a new and extended argument. Toulmin also recognises that single statements can contain argumentative features. Just by making a statement you put yourself in a position of potentially being questioned.

Argumentation, acknowledged as an important means for enabling young children's mathematical reasoning, can be promoted through a dialogic approach to teaching (Mercer & Sams, 2006; Yackel & Cobb, 1996). Despite the increased focus on the role of mathematical argumentation for enabling young children's mathematical reasoning, little research has focused explicit on the role and characteristics of mathematical argumentation at kindergarten level. Pontecorvo & Sterponi (2002) found that children's reasoning in preschool activity unfolded "through complex argumentative

patterns" (p. 133). They emphasised that teachers should pay attention to the different ways children argue in order to facilitate children's "possibilities to practice, enrich and refine argumentative resources they have already acquired" (p. 139). Tsamir, Tirosh, and Levenson (2009) investigated different types of justification given by children between five and six years old, working with number and geometry tasks. Their study shows that young children are able to justify their statements by using appropriate mathematical ideas. Some children, in contrast, used their 'visual reasoning' as a way to justify their statements. When the researcher asked how they could know which bunch of bottle caps had more they answered "because we see", and they felt no further need to justify their answer or did not know how to do it. Dovigo (2016) investigated how argumentation promoted collaboration and problem solving in preschool (age 3-5). By comparing different ways of how argumentation took place in teacher-talk and peer-talk they found that peer-talk contributed very positively for promoting collaboration and problem solving. But at the same time they emphasised that if the teachers were able to guide the debate in a careful and exploratory way the teacher guidance could be a positive contribution to the development of the argumentation.

Theoretical framework

My theoretical stance is rooted in a sociocultural paradigm where interaction is regarded as the very engine of learning and development, (Vygotsky, 1978). As a consequence of adopting this theoretical stance, I regard argumentation as a cultural and historical activity. Children are not naturally born with the ability to argue. Argumentation is a communicative pattern which they learn through interaction with more knowledgeable others.

Interaction, specific for human beings, is characterised by the use of tools and especially by the use of language (Vygotsky, 1978). In recent years there has been a growing interest to study the interplay between gestures, language and thought both in mathematics education and in other domains. McNeill (2005) developed a theory where he regarded gestures as an integral part of language, not merely as a support for language. He regarded gestures as having an active and inseparable role in language and thought.

Not only gestures have been recognised as important for human reasoning. Radford, Edwards, and Arzarello (2009) talk about the importance of the multimodal nature of cognition; how different sensorial modalities – tactile, perceptual, kinesthetic become integral parts of our cognitive learning processes. Radford's (2002; 2003) theory of knowledge objectification emphasises how gestures, bodily actions, artifacts, (mathematical) signs and speech in cooperation affect mathematical reasoning. A special category of semiotic means of objectification that Radford (2002) considers is deixis. Deictic terms are words that have the function "to point at something in the visual field of the speakers" (p. 17), and cannot be fully understood without additional contextual information (e.g. "here", "there" "that", "this" etc.). All semiotic means play a significant role in mathematical mediation and reasoning. "Each semiotic means of objectification puts forward a particular dimension of meaning (signification); the coordination of all these dimensions results in a complex composite meaning that is central in the process of objectification" (Roth & Radford, 2011, p. 78).

The concept of argumentation used in mathematics and mathematics education is often related to the production of proofs. It is nevertheless important not merely to connect the concept of argumentation to formal logic as found in mathematical proofs. Toulmin (2003) distinguishes

between analytic argumentation, which is used in production of mathematical proofs, and substantial argumentation which is informal argumentation used in everyday practices. Substantial argumentation does not necessarily have a strict logical structure. Substantial argumentation gradually supports a statement by presenting relationships, explanations, background information, etc. (Krummheuer, 1995). Toulmin (2003) strongly emphasises that substantial argumentation should not be regarded weaker as or less important than analytic argumentation.

Toulmin (2003) developed a model for analysing structural and functional aspects of substantial argumentation with the aim to illuminate how statements are organised for the purpose of constituting an argument, and how a conclusion is established through the production of an argument. In Toulmin's model the core of an argument is based on three elements: claim (C), data (D) and warrant (W). The claim is an initial statement, for example an assertion or an opinion about something. To support the claim, the arguer needs to produce data. Data are facts or statements on which the claim can be grounded. A warrant is a justification of the data with regard to the claim. The warrant holds the argument together. It points to the relation between the data and the claim.

In addition, Toulmin's (2003) model contains three other elements, backing (B), qualifier (Q) and rebuttal (R). A backing is a statement that supports the warrant. It is like a special case of data that is provided as evidence for the warrant. The purpose of a backing is to answer "why in general this warrant should be accepted as having authority" (Toulmin, 2003, p. 95). A qualifier says something about the extent to which the data confirm the claim. Words like 'probably', 'presumably' etc. are often used as qualifiers. Rebuttals refer to exceptions or conditions under which the claim is true, often used subsequent to a qualifier, exemplified as "The claim is true except/unless/only if …".

Method for data collection and data analysis

In this case study I observed one KT in the focus group of AP and his group of six 5-year-old children engaged in mathematical activities about reflection symmetry. The activities had been developed in the AP, and as part of an in-service program for the focus group the KTs were asked to implement a number of activities with their children. I visited the kindergarten on two occasions with a one week interval. It was the KT himself that decided to work with reflection symmetry activities on both occasions. The method for data collection was observations and the sessions were video recorded and field notes were made.

I regard argumentation as a sequence of statements (both verbal and non-verbal) that serve the purpose of supporting an initial claim. Thus one criterion for selecting episodes from the transcript was that they should contain verbal communication and have more than one utterance from the children. Another criterion was that the episodes I selected should contain mathematics, and they should be linked to the lesson aim (reflection symmetry). In total I found 11 episodes from the transcript using these criteria. Ten of these episodes had more or less an argumentative structure. Six of the episodes had more than two turns and more than two argumentative utterances from the children. These episodes were analysed in depth according to Toulmin's (2003) model to identify the argumentative structures. In addition, I analysed each of the six episodes from a multimodal perspective. In fact, I *had* to look at multimodal aspects in order to be able to differentiate between the elements in the argumentative structure. I did not focus on any specific semiotic means and their significance for children's reasoning. Rather I focused on what kind of semiotic means children

used with respect to the different elements in the argumentation, and what role they played in constituting the argument.

Results

In this section I will present the analysis of one of the six episodes to illustrate the structure of children's argumentation, and what role semiotic means play in the ongoing argumentation.

In advance of this episode the children have been asked, by the KT, to find things in the room which they think are symmetric or as the KT says "equal on both sides". Each child is then asked to explain why they think the toy they have chosen is equal on both sides. In this particular episode one of the boys (John), who has chosen a trolley, is being asked to explain why he thinks the trolley is equal on both sides (or more precise; he is being asked if he think the trolley *is* equal on both sides).

KT:	Maybe we should start with John, since he has a very large thing. John, is this equal on two sides?
John:	Mmm (2)[ii] There (2) ((He lifts his trolley up from the table, and holds it in a straight forward position. Then he says "there" and nods his head)).
John:	and there… ((He rotates the trolley 90 degrees, showing the side of the trolley and then nods his head while saying "there")).
KT:	Aha!
John:	and (2) there. ((He rotates the trolley 180 degrees, showing the other side of the trolley and nods his head again while saying "there")).
Elias:	And there and there. ((Elias has already paid attention to the situation)).
KT:	Can you see if this is equal Elias?
Elias:	Look…
Elias:	There, there (.) there, there (.) there, there and there, there. (.) And there and there, and (1) everywhere. ((He is pointing with his index finger to show where he thinks the trolley is equal. When he says "everywhere" he is letting his whole hand swipe over the trolley)).
KT:	((The KT lifts up the trolley and tries to show the symmetry line and explaining how the trolley is equal on both sides of that line)).
Elias:	Everything is equal on both sides, even the wheels.

The structure of children's argumentation

Before this episode each child was asked by the KT to choose a thing that they thought was equal on both (two) sides. In this episode John has picked a trolley and just by doing so he has implicitly made the claim that the trolley is equal on both (two) sides. When the KT asks John; "John, is this equal on two sides?" John's claim is being challenged. The KT actually asks a yes-no question, but the question is still a quest for explanation or justification.

To argue for his claim, John lifts the trolley up in the air, like he wants to make it visible to the others. The first "there" and the first nod is also a part of this visualization which together constitute

the *data* he presents. It is important to notice that at this moment he holds the trolley in a straight forward position from his own point of view. He is not referring to any particular equal points on the trolley. The trolley itself is being presented as the data that supports the claim.

Then he is trying to present a *warrant* for his data by rotating the trolley 90 degrees while saying "there" and then back again 180 degrees while saying "there" again. Each time he is saying "there" he is nodding his head. John is presenting the warrant as two particular sides that are equal, and exemplifies the equality. The warrant (the example) relates the data (the presentation of the whole trolley) to the claim (the trolley is equal on two sides). Considering the time John is using while presenting the warrant and the way he utters the second and third "there", John does not seem very confident in his presentation of the warrant. Nevertheless, when John was walking around in the kindergarten trying to find a thing that had two equal sides, he considered the trolley for some seconds before he took it back to the table. This indicates that his choice was not completely random. John seems certain that the trolley is equal, but he is not quite certain how to justify it.

After John has presented the second "there", the KT utters "aha" (with a rising intonation at the end). By this utterance the KT gently appreciates John's contribution, even if the two sides that John presented thus far were not equal. While presenting his data and his warrant, John waits several seconds, and it seems that the KT thinks that John has finished his explanation after the second "there". From the children's point of view, the KTs "aha" gives Johns contribution authority and can be regarded as a backing of John's warrant-attempt. But from the KTs point of view, the "aha" was not meant as a backing, only as a gentle appreciation of his contribution.

Elias then contributes to the argumentation. By the utterance "and there and there" and a pointing gesture he is presenting another warrant for John's data. Elias is talking faster and more concisely than John. Because he is using his index finger rather than nodding he is also more precise in his communication and is able to point on specific points on the trolley, like the joints and the handles. The way Elias communicates indicates that he is more confident and has more knowledge about reflection symmetry. Elias is actually presenting several warrants for the data. Every time he says "there and there" and points at different corresponding points, he gives a new warrant. By presenting particular corresponding points, each warrant is exemplifying exactly where the trolley is equal. By repeating several almost identical warrants (presenting several examples) it seems that he is trying to communicate that *every* point on one side has a corresponding point on the other side.

After presenting several warrants Elias says "everywhere" while he is swiping his whole hand over the trolley. I interpret this as a generalisation of his previous statements (warrants), and thus a backing for the warrants because it answers "why in general this warrant should be accepted as having authority" (Toulmin, 2003, p. 95). The warrants are not independent examples of equality rather examples of a more general property of reflection symmetry.

When the KT shifts his attention to Elias in the middle of this episode, Elias answers by saying "look". The intonation of the utterance indicates that he is only introducing his coming explanation. I interpret his utterance as synonymous to "let me explain".

At the end of this episode, Elias says "everything is equal on both sides, even the wheels". The use of the word "even" in this sentence is very interesting. The word "even" I interpret as a qualifier for the claim. It says something about to what degree the data confirm the claim. Usually words like

"probably" or "presumably" are used as qualifiers, but in this case Elias is indicating that he is *very* certain that everything is equal on both sides, by saying "even the wheels". It seems that the probability for everything being equal increases since 'the critical points', the wheels, are equal. Why Elias regards the wheels as important points is hard to tell.

Data (John): *There* [Lifts the trolley in a straight forward position and nods his head]	→	Claim (John): The trolley is equal on two sides	←	Qualifier (Elias): *even the wheels*

Warrant (John): [pause] *and there* [pause] *and there* [He is first showing one side, then the other side]

Warrant (Elias): *And there and there* [pointing with his index finger at equal points on two sides of the trolley]

Warrant (Elias): *There, there. There, there. There, there and there, there. And there and there* [pointing with his index finger at equal points on two sides of the trolley]

Backing (Elias): *Everywhere* [swiping his hand over the trolley]

Figure 1: The structure of children's argumentation

This example illustrates the complexity of children's argumentation. They are able to present more than only the core of an argument. In this episode I found that some children are able to present both data, warrant, backing and even qualifier for a claim. In another episode that is not provided in this paper (because of the limited space) Elias was also able to present a rebuttal. He was able to modify his claim by giving examples of exceptions.

Discussion

This study shows that young children are able to argue for a claim in a quite complex manner. Using Toulmin's (2003) model to illuminate structural aspects of the children's argumentation, the results show that some children are able to use several of the elements in the model in their argumentation.

This study also illustrates the extensive use of semiotic means in children's argumentation. In every element in the argumentative structure, children use gestures and other semiotic means to mediate their ideas. This illuminates the significant role that gestures and other semiotic means play in children's communication and especially in their argumentation. (cf. McNeill, 2005; Radford, 2002; 2003; Roth and Radford, 2011). Deixis, in particular, are extensively used in the argumentation above. Both the data that John presents and the warrants that John and Elias present are based on the deixis "there" and the related pointing and nodding gestures. Even if John and Elias use different signs for mediating their ideas, both the nodding and the pointing gestures serve the same purpose,

namely to give contextual information to the deixis "there". It is actually not possible to get the whole meaning of the words "there and there" without including the pointing and nodding gestures.

The deixis and the related pointing and nodding gestures are not the only significant semiotic means in this argumentation. To be able to distinguish between the data and the warrant that John provides I had to interpret his related actions. When he presents the data he holds the trolley in a straight forward position, he is not referring to any particular equal points, only presenting the trolley as a whole, as if he wants to show the equality. The way he presents his claim corresponds with one of the findings in Tsamir, Tirosh and Levensons (2009), that some children based their justification on 'visual reasoning'. The trolley itself is being presented as the fact that supports the claim. In the warrant he is presenting two corresponding sides, as if he wants to give an example of the equality. Without interpreting these actions, it is impossible to distinguish between the data and the warrant, and thus fully understand the structural aspects of the ongoing argumentation.

The repetitive presentation of Elias' warrants and the swiping hand that generalises the repetitive warrants are other important semiotic means in the argumentation. By repeating "there and there" with corresponding pointing gestures Elias indicates that *every* point has a corresponding point on the other side of the symmetry line. When Elias says "everywhere", he swipes his hand over the trolley. This swiping gesture plays a significant role in the generalisation process of the points.

The results from this study point to significant features of children's argumentation and give important insights into *how* children argue. I think teachers could benefit from paying attention to the different ways children argue and being aware of the structural aspects in children's argumentation in order to provide opportunities for improving children's mathematical communication and reasoning (cf. Dovigo, 2016; Pontecorvo and Sterponi, 2002). But to be able to do so, the KTs also need to pay attention to how children make use of semiotic means in their argumentation. The Toulmin model revealed structural aspects of children's argumentation, but these structural aspects would not have emerged without considering the use of semiotic means. In line with Roth and Radford (2011) I would argue that all the different semiotic means play a significant role in the constitution of meaning.

In the example above we saw that Elias was able to use several elements in the model and demonstrated more confidence in his argumentation than John. A possible explanation could be that Elias is further in his appropriation of the properties of reflection symmetry than John. Maybe there is a correspondence between how far children have appropriated a certain subject and their ability to use several elements in the Toulmin model. This is thus a suggestion for further research on this interesting topic.

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[ii] (2) indicates approximately 2 second pause, (1) indicates approximately 1second pause and (.) indicates a pause less than one second

Engaging children in mathematical discourse: a kindergarten teacher's multimodal participation

Abstract

This article reports from a case study which investigates a kindergarten teacher's multimodal participation in a teaching-learning activity involving addition and counting. By her multimodal participation the kindergarten teacher engages children in a mathematical discourse and supports their opportunities for learning. Implications from the study for practice is that kindergarten teachers (and teachers) can benefit from being consciously aware of the affects their bodily actions have on children's mathematical reasoning, and how they can engage children in mathematical discourse without having to 'teach' (i.e., tell) children mathematical concepts and relations. Moreover, the article considers how kindergarten teachers can prepare for a smooth transition to school by introducing children to mathematics through semi-structured activities.

Introduction

In the last year of kindergarten, children (age 4,5-6,5) are about to make a transition from an institution where play, care upbringing and learning are the main focuses, into an institution which focuses more on their academic development. This transition may be challenging for many children (Lillejord, Borte, Halvorsrud, Ruud, & Freyr, 2017), and it is important to get insight into how to prepare for a smooth transition. This article reports on a case study which focuses on the manner in which a kindergarten teacher (KT) engages with 4.9-5.9-year-old children in a mathematical problem, involving addition and counting, in the context of pre-designed mathematical activities aimed to prepare children for school. The KT's mandate is to structure the discourse around mathematics, which is quite different from engaging in a spontaneous floating conversation in an 'everyday situation' in kindergarten.

Children's opportunities to take part in mathematical discourse[1] are found important in learning mathematics, both in kindergarten (Dovigo, 2016), and in school (Mercer, 2000; Rojas-Drummond & Mercer, 2003; Wells, 1999), but in activities where the teacher or KT have a pedagogical aim, it may be difficult to balance teacher-talk and child-talk (O'Connor & Michaels, 1996; Dovigo, 2016). Research considering how KTs may facilitate children's opportunities to participate in mathematical discourse is sparse. This article contributes to this area of research by presenting results from a case study which investigates how a KT engages children in a mathematical discourse and supports their opportunities for learning by her multimodal participation (i.e., her use of various semiotic means).

This paper addresses the following research questions:

> What characterises the KT's multimodal participation in a teaching-learning activity involving addition and counting?

> How does the KT's multimodal participation supports discourse and children's opportunities for learning?

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[1] In this article the terms discourse, dialogue and conversation are used interchangeably and in line with Gee's (2008, 2011) definition of discourse, which will be defined below, but which simply means the way two or several people are talking together.

The rational for the study is to investigate the processes that unfolds when a KT engages with kindergarten children in mathematical activities aimed to prepare them for school. And furthermore, to understand how KTs can facilitate children's learning opportunities and prepare for a smooth transition between kindergarten and school.

A cultural-historical perspective to investigate a kindergarten teacher's multimodal engagement in a mathematical activity in kindergarten

In this paper teaching and learning are conceived as an interconnected whole - a dialectical, mediated activity where mathematical ideas (among others) are mediated through semiotic means (e.g., language, artefacts, gestures and signs) (Radford, 2013; Vygotsky, 1987). Although language is essential for developing abstract thoughts, the multimodal nature of cognition has gained attention in 21st century research (Radford, 2009, 2013; Radford, Edwards & Arzarello, 2009; Roth, 2001). Radford's (2013) theory of knowledge objectification emphasises how gestures, bodily actions, artefacts, mathematical signs and speech work together in the constitution of mathematical reasoning. In Radford's theory, activity is conceived as a process, or a system of relations, which unfolds through human actions. Activity comprises both inner (cognitive) and outer (material) processes and is something 'real' that can be observed. In Radford's theory, learning is viewed as an objectification process (i.e. is related to the object of the activity) where knowledge is mutually constructed i.e. mediated through semiotic means. Learning is theorised as "social processes of progressively becoming critically aware of an encoded form of thinking and doing - something we gradually take note of and at the same time endow with meaning" (ibid., p. 26). This research study adopts Radford's (2013) cultural-historical perspective to study the dynamics of a mathematical teaching-learning activity in kindergarten.

The term 'discourse' is complex and used in a variety of ways. In this article the terms discourse, dialogue and conversation are used interchangeably and in line with Gee's (2008, 2011) definition of discourse, which is "language in use or connected stretches of language that make sense" (Gee, 2008, p. 154). My epistemological stance resides in Radford's (2013) cultural-historical perspective and theory of knowledge objectification, and I use Radford's definition of activity to describe the teaching-learning interaction that unfolds in the kindergarten group examined in this study. However, I see discourse, as defined above, as a subset of activity, and is used in this paper to describe how the children are promoted to use language when they engage in the activity (without ignoring their use of other semiotic means).

As mentioned above, the case study reported here is situated in the context of pre-designed mathematical activities aimed to prepare children for school, where the KT's mandate is to structure the discourse around a mathematical object which is quite different from a spontaneous floating conversation in an 'everyday situation' in kindergarten. Therefore, a lot of the literature below is taken from school setting where teaching-learning activities are often more structured than in kindergarten, and which points to challenges that may arise in such teacher-led activities.

How to engage children in mathematical discourse

Children's opportunities to take part in mathematical discourse are important in learning mathematics but distinct forms of discourse facilitate different learning opportunities. Dialogic inquiry (Wells, 1999) and exploratory talk (Mercer, 2000) are two (not dissimilar)

constructs which describe perceived effective ways for participants to interact, to reason and to solve problems together. Developing effective discourse for learning, however, is not a straightforward process. In teacher guided activities it is challenging to find a balance between teacher-talk and child-talk (O'Connor & Michaels, 1996; Dovigo, 2016). Teachers may experience tensions between overseeing the conversation and promoting the children to participate in the discourse. O'Connor and Michaels (1996) argue that an aim of teaching is to nurture children's talk and promote them to take part in the ongoing discourse, but simultaneously focus learning and discourse around a specific content.

Questions are regarded as important for engaging students in mathematical conversations and give positive learning outcomes (Kirby, 1996; Roth 1996; Rojas-Drummond & Mercer, 2003; Myhill & Duncan, 2005; Carlsen, 2013). However, some question strategies can help children to participate, others can limit children's participation in the ongoing discourse. If questions are used for checking children's knowledge or overemphasise factual or procedural knowledge, they can promote an unproductive discourse for learning, (Kirby, 1996; Roth, 1996). Questions can be classified in a number of ways, for example, as 'closed' (inviting short factual/procedural responses) or as 'open' (inviting longer and possibly elaborate responses, often with no predetermined answer). Both types of questions are found to initiate the well-known teacher led discourse pattern called the IRE-/IRF-exchange (initiation-response-evaluation/follow up), (Wood, 1992; Wells, 1999; Rojas-Drummond & Mercer, 2003). This exchange has both been criticised and appreciated as a pedagogical tool. When criticised (Wood, 1992) the initial teacher question is often a closed question with a 'correct answer'. This type of exchange often serves only to check pupils' knowledge, which is not necessarily productive for learning. Wells (1999) agrees that the triadic dialogue can be counterproductive for learning but argues that IRF exchanges can also serve as a useful pedagogical tool to achieve co-construction of knowledge. The quality of this exchange relies in the underlying expectations and goals of the teacher. Similarly, Rojas-Drummond and Mercer (2003) argue that IRF exchange can help to guide children's learning, especially if the teacher follows up with 'why-questions' which promote students to reflect on their responses. Another way to follow up children's contributions and promote them to think further on their ideas is by re-voicing (O'Connor & Michaels, 1996). Re-voicing is defined as when a participant repeats another participant's contribution (or parts of the contribution). O'Connor and Michaels argue for the usefulness of this type of follow up for engaging students in the classroom discourse, to focus attention to key-points and thus facilitate learning possibilities.

Mathematical learning activities in Norwegian kindergartens are often organised in ways where KTs and children interact in informal and semi-formal settings, and where mathematical ideas come to play through conversation and the use of artefacts, (Erfjord, Hundeland, & Carlsen, 2012). Dovigo (2016) investigated how participation in different types of conversations (peer-talk and child-teacher talk) influenced preschool children's learning opportunities. The study showed that the children had richer opportunities to contribute in peer-talk. In child-teacher talk, the KT talked more than the children. In peer-talk the children asked more questions (including open questions). However, the children's abilities to build arguments were limited in peer-talk. Teacher guidance helped the children to elaborate their argumentation and to improve their abilities to collaborate.

Questions are also found as an often-used pedagogical tool to initiate mathematical conversations in kindergarten. Sæbbe and Mosvold (2016) studied what they called an

"everyday situation", where a KT organised an activity where the children and the KT were sitting around a table playing with Lego bricks. The KT had no formal learning goals, the aim was to play with the bricks and talk about what they were doing. The authors found that questioning and affirmation of children's responses were the two most prominent discursive acts used by the KT. Through questions the KT promoted mathematical argumentation and helped the children to participate in the mathematical discourse. Through affirmation the KT confirmed or acknowledged the children's contributions. Sæbbe and Mosvold (2016) argue that questioning and affirmation are strongly related, as both serve as means for reaching joint attention and promote children's participation in mathematical discourse. They found clear similarities between the communication pattern of questioning and affirmation, and the IRE-exchange, although the purpose of the affirmation was mainly not checking the children's knowledge.

This research study investigates how a KT engages with kindergarten children in an addition problem, where the children use various counting strategies to solve the problem. Children's counting strategies to solve addition problems are well documented (Baroody & Purpura, 2017; Carpenter & Moser, 1982; Fuson, 1992), and at least three different counting strategies for addition have been identified: 1) The 'counting-all' strategy has been identified as the most common, and which children typically use first. After having identified the cardinality of the two sets (by counting each of the sets), children find the sum of the two sets by starting from the beginning and count all items together as 'a whole'. 2) A more sophisticated strategy is 'counting-on'. After having counted and found the cardinality of the two sets, children take the number of items in the first set as a point of departure, and then count further on the other set of items. 3) Using the 'counting-on from largest' strategy children take the number of items in the largest set as a point of departure and count further on from that. Carpenter and Moser (1982) observed that, even if children are capable to use the counting-on procedure, the children nevertheless often used the counting-all procedure. They argue that children can be encouraged to use the counting-on procedure if there are no physical artefacts available for the children to manipulate.

Setting

This case study is situated within a Norwegian research and development project called the Agder Project. The project aims to develop a curriculum that prepare Norwegian 5-year-olds for school, and to investigate the processes of teaching and learning that unfolds when the curriculum, in the form of pre-designed mathematical activities, are implemented by the participating KTs. The study reported here took place in one of the kindergartens in the project, where the KT implemented a mathematical activity called 'Tower Building'. In the Tower Building activity, the children are supposed to get experiences with counting, comparing sets of numbers and introductory addition. In the written activity description, the KT is requested to introduce the activity using a doll called Super Sigurd, which the children are familiar with. Super Sigurd has built three towers, and he thinks that he has 20 building blocks altogether, but he is not quite sure. (The three towers consist of 5+7+8 building blocks). The KT is requested to start the activity by asking the children if they can help Super Sigurd to figure out how many building blocks his towers consist of altogether. The excerpt examined in this paper is selected from the introduction of the activity, where a KT and a group of preschool children (age 4,9-5,9 years, three boys and six girls) are working with the

addition problem. The KT is an experienced KT and the children are familiar with similar whole-group learning activities.

Methodology

To capture the dynamics of teaching and learning mathematics in kindergarten I used qualitative methods within an interpretative paradigm. The segment examined in this paper is selected from data collected in four kindergartens at four occasions (16 sessions altogether) during the school year 2016/2017. All observed sessions were video recorded and field notes were written. The video data from all observed sessions were separated into parts (segments) and organised in tables with columns, containing time and description of the interaction. The description included some utterances and were supplemented by video stills of observable actions. For further analysis, segments were selected from these descriptions based on three main criteria: 1) problem-solving interactions 2) the children's contributions of mathematical ideas, mathematical arguments, explanations or reflections 3) observed eagerness from children to participate.

The segment examined in this study was selected because the children got ample opportunities to suggest and explain strategies to solve the problem and the children eagerly participated in the discourse. The selected segment was transcribed[2] and then analysed from a cultural-historical activity theory perspective, where *activity* (the process that unfolds when the participants interact) is the unit of analysis. Using Vygotsky's (1987) dialectic approach, I always considered two subsequent turns in relation to one another, or I considered a turn in relation to the following activity (several turns in a row). For example, when the KT asks "How can we figure out how many there are altogether?", I argue that the KT invites the children to contribute with different strategies to solve the problem because the children, in the following activity, eagerly contribute with different strategies. I also argue that the KT by her 'questioning look' (a term which is elaborated in the Results section) mediates how she wants the children to approach the problem. I interpret that the KT wants the children to think carefully about the problem to find a strategy to solve it, and I interpret the KT's actions in this manner because of Leo's subsequent response. Leo puts his index finger to the mouth and makes a 'questioning look' when he says "Hmmm…" which I argue show that he thinks carefully about the problem before he answers. His facial expression is in fact quite similar to the KT's facial expression. The analysis was accomplished through an iterative examination of the data (the segment). Videoclips from the whole-group session and extracts from the data analysis were watched and discussed with four research colleagues in the project and were important for the final interpretation of the segment.

In the Results section I briefly explain what happened just before the selected segment. Then I present extracts from the transcript (including sketches from the video stills) together with the analysis. I present a moment to moment analysis (interpretation) of the KT's contributions to the activity, that is I identify the KTs actions which seem important for engaging the children in the mathematical discourse and interpret what she mediates through them and how children respond to them.

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[2] Transcription codes: (( )) denotes non-verbal actions or contains explanations and interpretations necessary to understand the dialouge; _ denotes that the underlined word is emphasised; … denotes a pause in the verbal utterance; [ ] denotes that the utterance is cut off by another participant.

Results

Prior to the segment analysed below, the KT and the children count each of the three towers. In cooperation they conclude that the yellow, blue and red towers consist of, respectively, five, seven and eight blocks. The KT writes the numbers of blocks on small pieces of paper and lays them in front of each tower and with a questioning look initiates an interplay with the children:

92	KT	How can we figure out how many there are altogether? ((Questioning look))
93	Leo	((Puts his finger to his mouth)) Hmm … ((Then he raises his hand in the air. His facial expression changes from a questioning look into an 'understanding look' in the moment that he raises his hand))
94	KT	Leo
95	Leo	((Leans forward and puts his finger on top of the red tower)) What about … ((He stops and pauses for a while before he gets back to his place))
96	Lily	((Puts her hand in the air))
97	KT	Lily
98	Lily	Twenty-one
99	KT	Is it twenty-one? ((Questioning look))

The original problem was formulated as a question to the children on helping Super Sigurd to figure out how many building blocks there were altogether. In line 92 the KT asks "How can we figure out how many there are altogether?", which changes a closed question into an open question. This question invites the children into a conversation about different strategies they may use to solve the problem.

Figure 1: The KT's facial expression and body positioning when she asks; "How can we figure out how many there are altogether?" in line 92

Figure 1 illustrates the KT's facial expression and body positioning at the time of line 92 when she invites the children to contribute with different strategies to figure out how many building blocks there are altogether. The KT knits her eyebrows, tightens her mouth and rests her head in her hand. I interpret her facial expression as a 'questioning look', which mediates that the task is not easy, and that the children need to think carefully about how to solve the problem and suggest solutions.

Figure 2 illustrates Leo's stance when he says "Hmmm…" in line 93. Leo puts his index finger to his mouth and he knits his eyebrows when he says "Hmm…". His facial expression is similar to the KT's facial expression, and the utterance "Hmm…" may indicate that he thinks carefully about the problem. Leo does not orally express what he thinks, but his facial expression communicates that he ponders.

Figure 2: Leo's stance when he says "Hmmm…" in line 93.

A second later Leo's facial expression suddenly changes from a 'questioning look' to an 'understanding look' when he raises his hand in the air. The KT invites Leo to explain his idea (line 94), however Leo seems to forget his idea because he sits down again (line 95). Then Lily expresses that she wants to contribute (line 96) and the KT invites Lily to explain her idea (line 97). Lily suggests "Twenty-one" (line 98), and the KT responds "Is it twenty-one?" (line 99), without any marked rising intonation, which might indicate excitement, at the end. The KT takes Lily's suggestion, re-formulates it into a question and sends it back to the group for consideration. The KT continues to lean forward, with her hand in her face and with the same facial expression (questioning look). I interpret the KT's response as indicating that she does not want to explicitly evaluate Lily's suggestion. She does not say whether the suggestion is correct or incorrect. By re-voicing Lily's suggestion, the KT kindly appreciates Lily's contribution, and she also shares Lily's idea with the other children. However, by re-formulating the suggestion into a question, she sends it back to the group for a re-consideration.

After Lily has shared her idea, another girl (Fia) offers a different suggestion:

100	Fia	We can count.
101	KT	Maybe we can count? ((He moves her hand away from the face, down in her lap, she straightens her back and makes an excited facial expression. She emphasis "count" with a marked rising intonation))
102	Childr.[3]	Yes!
103	John	It is twenty-six
104	KT	((Questioning look)) Is it twenty-six? … How can we figure out how many building blocks there are altogether?
105	John	Count!
106	Lily	Count!
107	KT	Shall we count? … But we have already counted. How do we have to count <u>now</u>?

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[3] Several children respond simultaneously.

Figure 3: The KT's facial expression and body positioning in the time of line 101

In line 100 Fia says "We can count", which is a strategy to figure out how many building blocks there are altogether. Figure 3 illustrates the KT's stance at the time of line 101. Again, the KT takes Fia's contribution, re-formulates it into a question and sends it back to the group. But this time she also gives a clue. The KT moves her hand away from the face, down in her lap, she straightens her back and makes an excited facial expression. In addition, she emphasises "count" with a marked rising intonation. I contend that by her bodily action, her facial expression and her tone of voice, the question is no longer only an appreciation and a further challenge, it is also a clue. It mediates that the children are on a mathematically interesting (and correct) path. It is the sudden change in the KT's facial expression and body positioning, from a 'questioning look' in line 92-99, into excitement that gives the hint. However, the children (at least not all of them) do not seem to get the hint, because John continues to suggest *how many* building blocks there are (line 103).

When John suggests that there are twenty-six building blocks the KT changes her facial expression back to a 'questioning look' and she responds; "Is it twenty-six? ((Pause)) How can we figure out how many building blocks there are altogether?" (line 104). John and Lily immediately respond "Count!", which indicates that they realise that it is not the answer that the KT is aiming for. The KT still has the same 'questioning look' when she responds "Shall we count? ((Pause)) But we have already counted. How do we have to count now?" (line 107). To count is the basic strategy to solve the problem, however there are diverse ways to count in order to solve the addition problem. In the introduction the children, together with the KT, counted how many building blocks there were in each of the towers. By asking "How do we have to count now?" the KT mediates that there are other ways to count and that she wants the children to consider different strategies to solve the problem.

After the "How do we have to count now?" question, three other girls (Lily, Mia and Leah) contribute with ideas. (I do not present extracts from the transcripts of these three contributions due to space limitations). I now move on to the interplay between the KT and Ada, which follows after Leah's explanation (which was to put all three towers on top of each other and then count all building blocks together):

126	Ada	((Holds her hand in the air))
127	KT	Yes
128	Ada	I know … ehm … if we say … ehm … ((Moves her body up and down)) do <u>not</u> count eight and then we just count further
129	KT	((Excited facial expression)) Oh, did you hear what she suggested? ((Whispers)) Would you like to show us?

Ada explains her idea in line 128, and the KT immediately reacts positively to Ada's contribution and turns to the other children (line 129). I interpret the KT's excited reaction as indicating that she recognises a quite sophisticated counting strategy for solving the addition problem, which she wants to share with the rest of the children. The KT's excited reaction, then, is not only intended for Ada, but for all children.

Figure 4: The KT's facial expression and body positioning when she whispers "Oh, did you hear what she suggested?" in line 129

Figure 4 illustrates the KT's facial expression and body positioning at the time of line 129. The KT uses her tone of voice (whispering), her facial expression (eyes and mouth open) and her index finger to indicate her positive evaluation of the suggestion. I argue that through these actions the KT mediates that this is an interesting suggestion that it is worth listening to. She highlights the suggestion to get the other children's attention. The other children are quiet, and their attention is on the KT (and Ada). The KT does not only indicate that it is important that the other children listen to the strategy, but she also mediates that *she* thinks it is an interesting idea. Her sudden positive reaction is not only, I hold, (consciously) intended for the other children, it is also a genuine emotional action which is connected to her aim of the activity (to teach the children about counting strategies for addition). Considering the KT's profession as a pedagogue and her aim of the activity, Ada's contribution is probably interesting because she gets the opportunity to share a quite sophisticated strategy with the whole group. In addition, she gets the opportunity to learn more about Ada's reasoning.

Figure 5: Ada blushes

This KT's positive response results in a positive emotional valuation from Ada, she smiles and blushes, and it seems to make her proud (see Figure 5). Her contribution has been shared and evaluated as interesting. This encourages Ada to continue her explanation, and to continue to contribute in the ongoing discourse.

130   Ada   ((Starts to count the yellow tower)) One, two, three, four, five, count without eight

131   KT   OK. Now you counted one, two, three, four, five and then, what do you want to do next? ((She uses her index finger to count the five building blocks in the yellow tower and then she moves her finger to the blue tower when she says "and then")).

In line 129, the KT asks Ada "Would you like to show us?", and Ada moves over to the building blocks and starts explaining her idea (line 130). She skips counting the eight building blocks in the red tower and starts counting from one on the yellow tower of five building blocks. Then she moves back to her place and verbally explains how to continue by saying "count without eight" (line 130). Her explanation is partly done with use of the building blocks and partly verbally.

Figure 6: The KT's (left), Ada's (right) and Leah's (middle) stance when the KT says "…and then…" in line 131.

Figure 6 illustrates the KT's, Ada's and Leah's stance at the time of line 131 when the KT moves her index finger from the yellow tower to the blue tower and says "…and then…". The KT repeats Ada's actions (counts the five building blocks in the yellow tower), which helps Ada to focus attention to her previous actions. Then the KT moves her index finger to the blue tower when she says "…and then…", which I interpret as a hint for a possible next move, namely to count further on the blue tower. In addition, the KT asks "what do you want to do next?" which is a request for Ada to explain further.

132	Ada	Ehm … To count similar as we counted the yellow
133	KT	((Questioning look and pause)) Start to count from one at the bottom here? ((Points on the building block at the bottom of the blue tower))
134	Ada	Mm ((Agreement))
135	KT	One, two, three, four, five, six, seven, and then?
136	Ada	Ehm, we can just count like this all the time, without eight
137	KT	((Questioning look)) I think you have to show me, because I don't really understand what you mean. Maybe you can show [ ]
138	Ada	We count this one first ((Points at the yellow tower)) and then this one ((Points on the blue tower))
139	KT	Yes, maybe you can count it? Do as how you think, Ada
140	Ada	One, two, three, four five ((Counts the yellow tower)), six, seven, eight, nine, ten, eleven, twelve ((Continues to count the blue tower)). And then we just find it without counting.
141	KT	Oh, we have to continue? ((Excited facial expression))
142	Ada	Mm ((agreement))

Ada does not act in correspondence to the KT's hint in line 131, instead she answers "Ehm … To count similar as we counted the yellow" (line 132). Ada's response seems to confuse the KT, because she gives a questioning look again and, after a couple of seconds, she asks Ada "Start to count from one at the bottom here?" (line 133). The pause may indicate that the KT

considers Ada's suggestion before she responds to Ada. I interpret the KT's question as a request for confirmation, if she has understood Ada's suggestion correctly, and Ada confirms the KT has understood her correctly (line 134). Then the KT does exactly as Ada suggests, she counts from one at the bottom of the blue tower, before she asks, "and then?" (line 135). The KT does not suggest any further actions, rather she asks Ada what to do next. In line 136 Ada says "Ehm, we can just count like this all the time, without eight" (line 136), which is very similar to her explanation in line 130. The KT seems unsure what to do and expresses this verbally and non-verbally in line 137 through her questioning look and through her utterance "I don't really understand what you mean". To promote Ada to change her explanation, the KT invites her to come forth and use the building blocks in her explanation by asking "maybe you can show [ ]?" (line 137). Ada comes forth and carefully explains her idea, both verbally and by using the building blocks (line 140). Even though Ada is not able to fully complete her strategy (counting on from eight), she is able to count further from the yellow tower to the blue tower, and the KT is again able to understand Ada's counting strategy which she expresses in line 141 "Oh, we have to continue?", and Ada confirms that the KT has understood her correctly.

After this the KT and Ada together complete Ada's counting strategy (counting on from eight). And in the end of the segment the KT repeats Leah's and Ada's strategies with support from the other children.

Discussion

In the result section I identified the KT's actions which seemed significant for children's engagement in the discourse. I will now return to the research questions which aims to 1) characterise the KT's multimodal participation in the teaching-learning activity involving addition and counting and 2) illustrate how the KT's multimodal participation supports discourse and children's opportunities for learning. With respect to the first research question, three main characteristics of the KT's multimodal participation were identified and will be discussed below: 1) The KT's multimodal participation changes from moment to moment in relation to children's contributions, 2) are oriented towards the aim of the activity, and 3) are informed by the KT's underlying stance toward the children and their learning. While discussing these three characteristics I also point to significances of the KT's multimodal participation for supporting discourse and children's opportunities for learning, which is the aim of the second research question.

An example that shows how the KT's contributions related to children's contributions and simultaneously oriented toward the aim of the activity is how the KT often re-voiced and reformulated children's suggestions (cf. O'Connor & Michaels, 1996). For example, when Lily says "Twenty-one" (line 98) the KT responds "Is it twenty-one?" (line 99), and similarly when John says, "It is twenty-six" (line 103) the KT responds "Is it twenty-six?" (line 104). Both contributions are suggestions for the solution of the problem, but they are not solely what the KT was aiming for. The KT takes the children's suggestions and reformulates them into questions and sends them back to the group. The KT keeps a questioning look and a 'neutral' tone of voice, which I interpret as indicating that she does not fully value the suggestions (explicitly), however she wants the children to continue to consider the problem. When Fia says "We can count" (line 100), which is a strategy for solving the problem (not a suggestion for the solution), the KT immediately responds "Maybe we can count?" (line 101). The KT re-voices Fia's suggestion and reformulates it into a question, however this time she

also clearly changes her body position, her facial expression and tone of voice. Her excitement indicates that the children are on their way to what she aims for.

The above examples show how the re-voicings are not only linked to children's contributions they are also oriented toward the aim of the activity. The KT reformulates the children's suggestions into questions which promotes the children to continue to consider the problem. She uses questions purposefully to help the children to move in the 'desired direction', that is towards the aim of the activity. In addition, the KT's revealed emotions, which are mediated through her actions, alternate between excitement, curiosity and uncertainty and indicate how children's contributions relate to the aim of the activity; if the children are moving in the 'desired direction' or not.

I found that all the KT's actions are related to the children's previous contributions. Even when the KT responds "Yes" in line 127 it fundamentally relates to Ada's gesture in line 126 (Ada holds her hand in the air). This result is in fact not surprising, because it is incorporated in Vygotsky's (1987) dialectic perspective. The KT's answer relates to Ada's gesture, and similar, Ada's hand gesture relates to the KT's utterance. It is the KT's response that informs the meaning of Ada's gesture in retrospect.

Similarly, how the KT's multimodal participation orients the activity (and thus the children's further actions) towards the aim of the activity, relates to a fundamental idea in activity theory; actions are initiated by the motive of the activity. It is the motive (the KT's aim of the activity) that initiates the KT's actions. Through every verbal and non-verbal action (especially through her questions, her questioning looks and excited facial expressions), the KT orients the activity toward the aim of the activity. Roth and Radford (2011) show how a child's emotions orients the activity towards the object of the activity, which is similar to the results in this study. My analytical findings, however, indicate that the KT's emotions, which is mediated by her questioning look and her excited facial expression, is part of the orientation of the activity. Whenever the children are on their way toward a counting strategy for addition, the KT's participation changes from a questioning look into excitement (and vice versa).

In addition, the KT's multimodal participation has some characteristics which may indicate her underlying 'stance' toward the children and their learning. Firstly, the KT uses a wide range of questions. Twenty-one out of thirty KT utterances in the class-time reported on above were questions. The other nine utterances were utterances like "Lily", "Leo" (naming the children), "Okey, you don't know", "Yes" etc. The KT never asked any closed, factual or procedural questions, which invite predetermined answers, (cf. Myhill & Duncan, 2005). The types of questions that the KT chose to use, invited the children to explain and reflect and thus the discourse can be considered as exploratory (cf. Mercer, 2000) or as inquiry (cf. Roth, 1996; Wells, 1999). And the KT kept this particular segment of activity moving for approximately five minutes through the use of open questions. The children were given ample opportunity to contribute to the conversation, which we know is difficult in teacher led activities and important for children's learning (cf. O'Connor & Michaels, 1996; Dovigo, 2016). Secondly, the KT never (explicitly) evaluated the children's contributions, that is she never said whether the suggestions were correct or incorrect, similar as an IRE-exchange (cf. Wood, 1992). By re-voicing children's suggestions, which is a common pedagogical tool (cf. O'Connor & Michaels, 1996), the KT appreciates their contributions. She appreciated almost

every contribution in this manner (except Lily's suggestion to count by twos), even if the children suggested the solution to the problem, which was not solely what the KT aimed for. At the same time, she re-formulated the suggestions into questions and sent them back to the group. The KT followed up children's contributions and promoted them to continue to consider the problem and suggest other ideas, which is similar to the IRF-exchange described by for example Wells (1999) or Rojas-Drummond and Mercer (2003).

If we compare the communication-pattern of the KT in this case study with the communication-pattern that Sæbbe and Mosvold (2016) found as the most common in an 'everyday situation' in their study, the results are not similar. Sæbbe and Mosvold argue that questioning and affirmation are two core components in the KT's communication, where affirmation is compared with the evaluation in the IRE-exchange. Sæbbe and Mosvold investigated the KT's communication-pattern in a twenty-two minutes episode while I have looked at a five minutes segment. However, my study shows that learning activities in kindergarten may be accomplished with an 'open conversation' where children are given ample opportunities to participate with, and argue for, their ideas. The KT never (explicitly) evaluated the children's contributions with utterances like "that's right" or "very good" etc. It's important to recognise that although the KT did not explicitly evaluate the children's suggestions, I argue that she implicitly did by her multimodal participation. The KT's questioning look or excitement oriented the children toward the aim of the activity, which can be considered as a type of evaluation because these implicitly informed the children if they were moving in the desired direction or not.

A third and important characteristics which indicates the KT's underlying stance toward the children and their learning is how she used the building blocks in the activity. Through the whole segment, the KT kept the three towers of building blocks close to her. The children did not have direct access to the building blocks, they only got access when the KT allowed them to come forward. Because the children did not have direct access to the building blocks, they needed to verbally explain, and direct what actions they wanted the KT to perform on the building blocks. By limiting the children's access to the building blocks, she 'forced' them to use other semiotic means (like language and gestures) to mediate their ideas. By 'force' I mean that she limited the children's agency, so they would go along a specific route or sequence of actions. Thus she 'forced' the children to move toward a more abstract form of reasoning. However, sometimes she gave the children access to the building blocks, and perhaps she did that because she realised that the children needed the building blocks to reason and explain their ideas. Carpenter and Moser (1982) argue that children can be encouraged to use the 'counting-on' procedure if there are no physical artefacts available for the children to manipulate. In this segment Ada suggested to use the 'counting-on' procedure to solve the problem, and perhaps she was promoted to use this strategy because the building blocks were not initially available for her to manipulate.

Roth and Radford (2011) argue that the most important aspect to understand a teaching - learning activity is to identify the underlying grounds that make the situation happen. They argue that words (and other semiotic means) belong to systems of ideas and are carriers of ideologies, and thus reflect "the social, political, and theoretical position of the person uttering it" (Roth & Radford, 2011, p. 104). Therefore, it is important to investigate the underlying 'tone' of the words, and to display the ideologies behind. In this segment, the KT's use of questions and non-evaluative response to the children may indicate the KT's underlying

stance toward the children and their learning. The KT's use of building blocks may be a result of the written activity description which prepares for a whole-group session with only three towers. However, the activity description says nothing about *how* the KT should use the building block, and thus her use can still be argued to be part of the KT's underlying position. Her sensitive way of using the building block is most likely influenced by her underlying stance toward the children and their learning and is revealed through her multimodal participation.

My final discussion point concerns Radford's (2008, 2013) claim that learning in the theory of knowledge objectification is more than becoming aware of cultural ways of thinking and acting, it is also about becoming in the process of subjectification, which is a "processes of creation of a particular (and unique) self" (Radford, 2013, p. 27). I hold that the KT's dynamic multimodal participation also illuminates the KT's way of becoming in the activity. The KT's actions (verbal and non-verbal) are always balanced between her earlier experiences, which in this case is mediated by her underlying stance, children's contributions and the aim of the activity. The tensions that are created between the past, present and future is what constitutes the KT's moment to moment acting. Who the KT was when she entered the activity is transformed in the encounter with the children. This process is particularly salient in the part of the segment where the KT expresses that she does not understand. The KT is positioning herself within the unfolding activity, trying to understand the children. The KT shows a genuine interest in understanding the children which puts her in a vulnerable position, and therefore at one point must be led by the children (she is led by Ada). Vygotsky (1989) said that "we become ourselves through others" (p. 56, in Roth & Radford, 2011, p. 87). The way that the KT continuously transforms her unique participation in the moment in relation to children's contributions, and how she positions herself within the unfolding activity, trying to understand the children, illustrates how she becomes her unique self in the encounter with the children.

The consequences of these observations for practice is that KTs, and teachers in school, can benefit from being consciously aware of the affect their bodily actions have on children's mathematical reasoning, and how they can engage children in mathematical discourse without having to 'teach' (i.e., tell) children mathematical concepts and relations. This article shows how mathematics emerges from the participants joint activity, that is through KT's multimodal engagement with the children. In addition, the article illustrates how KTs can prepare for a smooth transition from kindergarten to school by carefully introducing children to mathematical thinking through semi-structured activities. In school mathematical activities are usually more structured than what children are used to in kindergarten and there is an increasing emphasis on expressing mathematical ideas verbally and symbolically. This article shows how the KT encourages the children to participate with their ideas, and continuously promotes them to consider and explain their ideas further, by engaging with them in the semi-structured activity.

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Student-teacher dialectic in the co-creation of a zone of proximal development: an example from kindergarten mathematics

Abstract: *This paper reports on a case study which explores the teaching-learning dynamics between a five-year-old girl (Ada) and a kindergarten teacher working on an addition problem. The study draws on Roth and Radford's (2010; 2011) symmetrical view on the zone of proximal development to investigate Ada's role in the joint activity. The focus is on Ada's (choices of) actions and the findings suggest that Ada plays a key role in the joint activity. The kindergarten teacher is dependent on Ada's actions to continue to productively participate in the activity. Ada creates new action possibilities for the kindergarten teacher and thus moves the activity forward and co-creates the zone of proximal development. The paper provides insight into the significant role that a child (the presumed less knowledgeable participant) plays in co-creating a ZPD and insight into the understanding of the ZPD as a phenomenon.*

Introduction

This study draws on Roth and Radford's (2010; 2011) symmetrical view on the zone of proximal development (ZPD)[1] to investigate the dynamics of a teaching-learning activity[1] between a five-year-old girl (Ada) and a kindergarten teacher (KT). It explores the importance of Ada's (choices of) actions in the activity and her role in co-creating the ZPD. Research on mathematics teaching and learning in kindergarten tend to look at the KTs' (choices of) actions and their role in children's learning. In this study, I explore the importance of Ada's (choices of) actions and her role in moving the activity forward. This paper reports on a case study which is part of a larger study on mathematical inquiry in kindergarten and is situated within a research and development project called the Agder Project. The episode examined in this study is selected from a whole-group session in one of the kindergartens in the project where the participants work with an addition problem (5 + 7 + 8), and focuses particularly on a problem-solving interaction between Ada and the KT.

The aim of this study is two-fold:

> To investigate the role of Ada's (choices of) actions in keeping the mathematical oriented activity moving and for co-creating the ZPD;

> To investigate Ada's mathematical learning in the joint activity.

The paper advances knowledge in the field of mathematics education by illustrating the existence of a phenomenon, the significant role that a child (the presumed less

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[1] In the Norwegian kindergarten tradition, the term teaching is rarely used for the kindergarten teachers' practice. Teaching is associated with school where the teacher 'teaches' the students. In kindergarten the children are guided into learning activities, and some instruction occurs. In this research study the terms teaching and learning are used in line with Vygotsky's construct 'obuchenie' which to refers to the mutually constitutive relationship between teaching and learning (Roth & Radford, 2011).

knowledgeable participant) plays in co-creating a ZPD. In addition, the paper provides insight into the overall understanding of the ZPD as a phenomenon.

Theoretical framework

This study adopts Vygotsky's cultural-historical and dialectical perspective on mathematics teaching and learning. It draws on Radford's (2013) theory of knowledge objectification which resides in the activity theory of Leont'ev and Holzkamp. In addition, it draws on Roth and Radford's (2010; 2011) symmetrical view on the ZPD. In the lineage of Vygotsky-Leont'ev-Holzkamp, activity is conceived as a process, or a system of relations, which comprise both inner (cognitive) and outer (material) processes. Activity is something 'real' that can be observed, a process which unfolds through conscious human actions. Teaching and learning are conceived as mutually constitutive moments which are mediated by semiotic means (like language, artifacts, (mathematical) signs, gestures, bodily actions etc.). Learning happens through sympractical (i.e., joint practical (lived)) activity from where the child(rens)'s and the teacher's ZPD emerges (Roth & Radford, 2011).

In Radford's (2013) theory of knowledge objectification, learning is theorised as an process where knowledge (systems of ideas, cultural meanings, forms of thinking) is materialised through semiotic means in joint activity. Knowledge is encoded forms of internal (mental) and external (material) actions, and objectification is conceptualised as "those social processes of progressively becoming critically aware of an encoded form of thinking and doing—something we gradually take note of and at the same time endow with meaning" (ibid. p.26). Radford's theory emphasises the multimodal nature of cognition and how gestures, bodily actions, artifacts, signs and speech work together in the process of objectification. "Each semiotic means of objectification puts forward a particular dimension of meaning (signification); the coordination of all these dimensions results in a complex composite meaning that is central in the process of objectification" (Roth & Radford, 2011, p. 78). In the objectification process there may occur what Radford (2008) calls a 'semiotic contraction'; as the participants become more aware of the mathematical object, they can focus attention on relevant aspects and ignore things that are irrelevant. Semiotic contraction may be identified by the participants use of more precise gestures, shorter statements and perhaps fewer and better articulated words. Iconic gestures may be used in such semiotic contractions, they are gestures identified by McNeill (1992), which have a clear meaning, and which often provide more precise information than speech (e.g. illustrates relationships). As the participants become more fluent, contraction increases, which is a sign of a deeper level of consciousness of the object.

As a basis for the joint activity where knowledge objectification takes place, the participants make a commitment to one another to carry through an event, which Radford and Roth (2011) call 'togethering'. "Togethering is a theoretical category in our theory of knowledge objectification that aims to account for the teacher-students embodied-, sign-, and artifact-mediated interaction that includes both co-knowing and co-being" (p. 244). The construct makes us focus on the way the participants engage and attune to one another in the joint activity. They commit to one another despite their differences. Without such commitment

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and trust the movement of the activity cannot occur, and the object of activity cannot be realised.

The zone of proximal development

The zone of proximal development (ZPD) is one of the most noted concepts that Vygotsky (1987) introduced. The concept has been frequently cited, interpreted and elaborated on in a variety of ways. Veresov (2017) notes that more than 200 articles on the ZPD were published in 2010-2016. There is not an extensive corpus of original literature from Vygotsky on the concept, at least not available for English readers, and the concept seems to suffer under poor translations of Vygotsky's texts (Verosov, 2017).

Scholars have different opinions about Vygotsky's original meaning of the concept, and his ideas behind it. van der Veer and Valsiner (1991) claim that Vygotsky did not think of his idea as original, rather as a continuation of contemporary scholars' suggestion that one should consider at least two levels of child development, namely, the child's prevailing level (what the child is already able to do) and the child's potential. Veresov (2017) has translated Vygotsky's definition of the ZPD as the following: "...the distance between the level of his (sic) actual development, as determined with the help of the tasks the child solves independently, and the level of possible development, as determined with the tasks the child solves under the guidance of adults and in cooperation with more intelligent peers" (Vygotskii, 1935, p. 42, in Veresov, 2017, p. 26). Vygotsky's definition was presented in a lecture in 1933, where Vygotsky discussed the relationship between instruction (teaching-learning) and development, criticising contemporary psychologists' ways of measuring children's intelligence (van der Veer and Valsiner, 1991; Veresov, 2004).

To understand the ZPD, many scholars emphasise that the concept needs to be interpreted and understood in relation to Vygotsky's overall view on mental development, especially in relation to 'the general genetic law' (Veresov, 2004; 2017; Wells, 1999; Wertsch, 1984), play and imitation (Veresov, 2004; Chaiklin, 2003; Holzman, 2010), and the idea of 'learning-leading-development' (Levykh, 2008; Holzman, 2010; Veresov, 2017), among others. The zone of proximal development can be regarded as an instantiation of the general genetic law of cultural development (Veresov, 2017; Wertsch, 1984), which states that all mental functions appear twice in the child's development. "First it appears between people as an interpsychological category, and then within the child as an intrapsychological category" (Vygotsky, 1981, p. 163). All mental functions exist initially in the social sphere *as* social relations. In a child's development, all higher mental functions are reflections of the child's societal relations with others, (Veresov, 2004; Roth & Radford, 2011).

This relates to another fundamental idea in cultural-historical theory. According to Veresov (2017) the concept of ZPD is a concretisation of the general genetic law *applied on* issues related to the relationship between learning and development. Vygotsky, (1987) highlighted that instruction (teaching-learning) and development are two different processes. He proposed that a fundamental feature of instruction is that:

Instruction is only useful when it moves ahead of development. When it does, it *impels or wakens a whole series of functions that are in a stage of maturation lying in the zone of proximal development* (p. 212, emphasis in original).

So, the ZPD comprises maturing functions that initially lay in the social *as* social relations, which, through the developmental process, become individual. The dialectic relationship between learning and teaching gives rise to the ZPD and consequently to the child's internal development. The learning-teaching activity is thus the source of development because it creates the ZPD. Learning is not the outcome, it is the point of departure for development.

The conceptualisation of the ZPD has changed from looking at the ZPD as a property or an attribute of an individual, toward a view that the ZPD as a collective process (John-Steiner, 2000; Levykh, 2008; Holzman, 2010), or a collective 'space' (Hussain, Monaghan & Threlfall, 2013; Abtahi, Graven & Lerman, 2017; Mercer, 2000; Roth & Radford, 2010; 2011). Although several scholars regard the ZPD as being collective, they have different approaches to what it means to be collective. Mercer (2000) considers the ZPD as being part of real-life activity where both participants (the teacher and the student) contribute in creating the ZPD. However, he still focuses on the asymmetrical relationship between the participants; one being the teacher (more knowledgeable) and one being the learner (less knowledgeable). Goos, Galbraith and Renshaw (2002) on the other hand, argue that the ZPD always has a two-way character, because the teacher and the students always appropriate each other's ideas. They want to eradicate the expert-novice distinction and move toward equal status interaction. Similarly, Zack and Graves (2002) emphasise that both the teacher and the children are learning in problem solving situations, and they argue for a conception of the ZPD as an intellectual space where the children's and the teacher's knowledge and identities are formed and transformed in moment to moment interaction. Recently some scholars have extended the notion of the 'more knowledgeable other' to include artefacts. Abtahi et al. (2017) suggest that the ZPD should be considered as multi-directional, instead of a bi-directional, where the role of the more knowledgeable other shifts between the child, the artefact (the properties of the artifact), and the adult.

Roth and Radford (2010; 2011) regard the ZPD as a symmetrical space where the teacher and the students are teachers and learners of each other. Roth and Radford (2011) illustrate how a ZPD emerges from the sympractical activity arising between a teacher and a student in a Canadian 4th grade, working on an algebra task. They highlight that both participants learn in the activity: one learning mathematics and the other learning pedagogy. Roth and Radford (2010; 2011) draw on Bakhtine [Volochinov], (1977) who argues that in a conversation every word has two sides, the speaker's and the listener's sides. I hold that all external social actions have two (or several) sides, the actor's side and the side of the observer or listener and all actions are in the same '(inter)action space'. This lays the foundation for regarding the ZPD as a symmetrical space with two (or several) sides. In Roth and Radford's (2011) view of the ZPD, the participants, through the sympractical activity, create a space where they mutually work to expand each other's action possibilities.

Following Vygotsky (1987) and Veresov (2017), this research study conceives the ZPD in terms of social relations that emerge from the joint activity. Using Roth and Radford's (2011)

approach to analyse the emergence of the ZPD this paper investigates how a KT and a child co-create a ZPD by expanding each other's action possibilities. The paper especially focuses on the role of the child in the co-creation. Similarly to Roth and Radford (2011), I argue that the ZPD emerges when the KT interacts with the child and the child interacts with the KT; both are responsible for the emergence of the ZPD, thus both need to engage in the activity. The ZPD emerges from the joint labour of expanding each other's actions possibilities. But as Roth and Radford (2011) say, the participants can only be aware of these particular actions and the possibilities that lay in their actions, after they have happened.

Setting

The intervention of the Agder Project was based on researcher designed activities described in Størksen et al. (2016), a book containing one-page outlines of the activities. The mathematical activities suggest how to organise learning sessions, what materials to use and suitable questions to ask etc. The activities are meant as suggestions not as strict manuscripts which the KTs need to follow to the letter. The episode investigated in this study is taken from a learning session based on an activity called 'Tower Building'. The KT in this study implemented the mathematical activity with a group of nine children (age 4-5 years).

In the 'Tower Building' activity the children are supposed to get experience with sorting, counting and comparing sets of numbers. In the activity description, the KT is requested to start the session by using a doll called Super Sigurd as a pedagogical tool. Super Sigurd has built three towers and needs help to figure out how many building blocks the three towers consist of all together. The KT is requested to ask the children if they want to help Super Sigurd with his mathematical problem. After the introduction the children may freely sort and count building blocks each, and after a while the KT is requested to ask the children to compare their towers with their peers, which can give rise to fruitful discussions about which towers have more and which towers have less building blocks.

Learning to add is a complex process. Before children are able to add and subtract with memorised 'number facts' they need to use different counting strategies. Children's counting strategies for addition are well documented (Carpenter, Moser & Romberg, 1982; Fuson, 1992). At least three different strategies that children use to find the sum of the cardinalities of two sets have been identified; however, this study focuses on the strategy regarded as the most sophisticated, the 'counting-on' strategy. After having found the cardinality of each set (for example, by counting from one), the children take the number of items in the first set as a point of departure, and then count further on the other set of items. For example, if a child is asked to find the sum of 2 + 3 building block, the child starts on two and counts a further three.

Methodology

One of the aims of the Agder Project is to investigate how the mathematical activities designed in the project stimulate preschool children's mathematical learning. To capture the complexity of teaching and learning mathematics in kindergarten I use qualitative methods within an interpretative paradigm. The episode selected in this case study is part of a larger dataset containing sixteen sessions implemented by four KTs. All observed sessions were

video recorded and field notes were written. The audio and video data from all observed sessions was transformed into tables with columns: time; description of the interaction which includes utterances (with speaker) and observable actions (supplemented by video stills). Episodes were selected from these descriptions for further analysis based on three main criteria: 1) problem-solving interaction 2) the children contribute with mathematical ideas, mathematical argumentations, explanations or reflections 3) the children show eagerness to contribute.

The episode examined in this study focuses on a problem-solving interaction between a girl (Ada) and a KT, selected from the introduction of the 'Tower Building' session. The episode occurs within a whole-group session where the oral interaction is between the KT and Ada, while the other children are watching/listening to the ongoing interaction. The other children are participating with ideas before and after this particular episode. In this paper I limit the presentation of data to the interplay between the KT and Ada. The episode was selected because it illustrates Ada's significant role in the joint activity and the KT's dependency on Ada's contributions, and was thus particularly suitable for the focus of this paper, student-teacher dialectic in the co-creation of a ZPD.

The selected episode was transcribed[2] focusing on the sequence of utterances (incrementing the utterance number by one whenever a new speaker entered the discourse) but commenting on facial expressions, gestures (hand movements), bodily actions and tone of voice. The utterances include, strictly speaking, 'utterance or gesture' as I incremented the number when a child raised her hand.

In cultural-historical activity theory the unit of analysis is *activity* (a process or system of relations), thus it is the process that unfolds that I focus on in my analysis. According to Radford's (2013) theory the multimodal nature of the activity, which comprises both inner (cognitive) and outer (material) processes, is fundamental. To capture these important aspects of the process I bring in various semiotic means in my analysis (spoken words, gestures, facial expressions, other bodily actions and the use of artefacts). When I interpret verbal or non-verbal actions I focus on what the participants make available to one another (and to me) in the activity. Roth and Radford's (2011) say that the social actors always have reasons for their behaviour, even though they do not necessarily consciously think about their actions before they act. The participants always make available to the others whatever is needed to accomplish something. In addition, Vygotsky's dialectic approach orients my analysis to social interaction and therefore I always consider two subsequent turns in relation to one another, or I consider a turn in relation to the following activity (several turns in a row). For example, when the KT in line 133 says "Start to count from one at the bottom here?" I interpret this as a request for confirmation, even if the KT does not say "Do you want me to start from one on the bottom here?". I do this because Ada in line 134 responds with "Mmm" which I interpret as a confirmation because the KT in line 135 does exactly what Ada suggests. Analysis consisted of arguing verbal transcripts with other co-

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[2] Transcription codes: (( )) denotes non-verbal actions or contains explanations and interpretations necessary to understand the dialouge; _ denotes that the underlined word is emphasised; … denotes a pause in the verbal utterance.

temporal semiotic means (gestures, facial expressions etc.). This analysis was discussed (and refined) whilst watching the video with four research colleagues in the Agder Project. Extracts from the transcript were also discussed with an external expert in the field. This analysis was not an end in itself but was a step towards interpreting the episode as a whole.

Results:

I start with a brief explaination of what happened before the selected episode, to provide the context. Then I present the transcript of the episode followed by the analysis which contains sketches of the video stills to illustrate the interaction between the KT and Ada.

Prior to the episode analysed below, the KT introduces the session in accordance with the activity description by asking the children if they can help Super Sigurd figure out how many building blocks there are in the three towers all together. The children eagerly suggest that they can count, and the KT then suggests that they can start with the smallest tower. In cooperation, the children and the KT find out that there are five building blocks in the yellow tower, seven in the blue tower and eight in the red tower. Then the KT asks "How can we figure out how many there are all together?", which changes a closed question into an open question. She encourages the children to contribute with different strategies to figure out how many building blocks there are all together. The children start suggesting different strategies.

Before Ada explains her idea, several other children contribute with their ideas. One girl, Mia, suggests "eight", but she is not able to explain further. Even when the KT offers help, Mia does not want to (or is not able to) continue her explanation. However, it seems that the suggestion "eight" gives rise to ideas in several other children, because several children immediately raises their hands, and Ada is one of them. Before Ada is invited to speak, another girl, Lea, is invited to explain her way of thinking. In cooperation with the KT, Lea suggests that they can put all three towers on top of each other and then count all building blocks together. Then the KT invites Ada to explain her idea:

126	Ada	((Holds her hand in the air))
127	KT	Yes
128	Ada	I know … ehm … if we say … ehm … ((moves her body up and down)) do <u>not</u> count eight and then we just count further
129	KT	((Excited facial expression)) Oh, did you hear what she suggested? ((whispers)) Would you like to show us?
130	Ada	((Starts to count the yellow tower)) One, two, three, four, five, count without eight
131	KT	OK. Now you counted one, two, three, four, five and then, what do you want to do next? ((She uses her index finger to count the five building blocks in the yellow tower and then she moves her finger to the blue tower when she says "and then")).
132	Ada	Ehm … To count similar as we counted the yellow
133	KT	((Questioning look)) Start to count from one at the bottom here? ((points on the building block at the bottom of the blue tower))

134	Ada	Mm ((agreement))
135	KT	One, two, three, four, five, six, seven, and then…?
136	Ada	Ehm, we can just count like this all the time, without eight
137	KT	((Questioning look)) I think you have to show me, because I don't really understand what you mean. Maybe you can show …
138	Ada	We count this one first ((points at the yellow tower)) and then this one ((points on the blue tower))
139	KT	Yes, maybe you can count it? Do as how you think, Ada
140	Ada	One, two, three, four five ((counts the yellow tower)), six, seven, eight, nine, ten, eleven, twelve ((continue to count the blue tower)). And then we just find it without counting …
141	KT	Oh, we have to continue?
142	Ada	Mm ((agreement))

As explained above, Mia's suggestion "eight" seems to give rise to ideas in several other children. Ada is one of them, and after Lea has explained her idea, Ada is invited to explain. When Ada says, "I know … ehm … if we say … ehm …" in the beginning of line 128, she moves her body up and down as if she is getting ready to explain. Ada's bodily actions suggest that this is not easy, and that she tries hard to choose suitable actions to explain properly. The KT carefully watches and listens to what Ada does and says.

Figure 1 illustrates Ada's and the KT's stances when Ada says, "do not count eight" (line 128). The building blocks are placed in front of the KT, which means that Ada does not have direct access to the building blocks, and is thus 'forced' to use language and gestures to explain her idea. The figure illustrates how Ada kind of 'holds on to' the tower of eight with her right hand when she emphasises "not". The gesture can be interpreted as an iconic gesture, where Ada indicates the tower's height, and makes it 'a whole'. Further in line 128, when Ada says, "and then we just count further", she swipes her index finger from the first tower to the last tower. From the whole utterance in line 128 it seems that Ada says that one should not count each building block in the first tower, rather regard the tower as 'a whole', and then count further from the whole (from eight).

Figure 1: Ada's and the KT's stances when Ada says, "do not count eight" in line 128. (Ada sits with her back to the camera. The KT (and Lea) focus on Ada).

The KT closely watches and listens to Ada's explanation in line 128 (illustrated in figure 1). The KT does not know what strategy Ada will bring into the activity, and she needs Ada to explain it to her. In addition, Ada has taken a risk to explain a strategy and the pauses in her speech and her body movements suggest that she uses a lot of effort to explain. This makes her vulnerable and dependent on the KT's support in her further explanation. From this moment, the KT and Ada are (emotionally and cognitively) connected, and they both feel responsible to continue the activity. I base this interpretation on what Radford and Roth (2011) write about 'togethering'. The construct explains the mechanism that make the participants commit and attune to one another in the joint activity. The activity (the process of interactions) that follows (explained below) show that this mechanism is there. The participants show an enourmous precistence to continue and accomplish the activity.

In line 129, the KT reacts with excitement, and the utterance "Oh, did you hear what she suggested?" which I interpret as indicating several things. First it indicates that the KT recognises a quite sophisticated counting strategy (counting on) which she may see as an opportunity to share with the whole group. In addition, the KT (as a pedagogue) is dependent on children who want to contribute in such situations. By her excitement and her utterance, the KT appreciates Ada's suggestions (which in turn makes Ada blush), and she promotes the other children to pay attention. Further in line 129 the KT promotes Ada to share her idea with the other children by asking "Would you like to show us?". This time the KT wants Ada to use the building blocks and show her idea.

In line 130, Ada leans forward to the building blocks and tries to explain (show) her strategy. Ada starts to count from one on the yellow tower. She purposely does not count eight, however she does not continue from eight (starting on nine) on the yellow tower. Ada's explanation does not correspond (directly) with the counting strategy that the KT recognised in line 128, and the KT understands that she needs to guide Ada in her explanation. In line 131 the KT repeats Ada's actions (brings Ada's previous actions into her consciousness again) and when she asks Ada, "and then, what do you want to do next?", she moves her index finger from the yellow tower to the blue tower, which suggests a possible next move. The KT gives a hint to count further on the blue tower. However, the gesture I interpret as just a careful hint, not an explicit verbal suggestion for a next move. The KT's careful acting makes the situation very fragile, because she makes minimal contributions to move the joint activity forward.

Figure 2 illustrates the KT's stance when Ada explains the next move in line 132, "Ehm … To count similar as we counted the yellow tower". It also illustrates how the KT still holds on to the blue tower (which was her 'hint' for a possible next move), but Ada does not choose an action which corresponds to the KT's intention.

Figure 2: The KT's stance when Ada explains her next move in line 132. (The KT (and Lea) focus on Ada who sits with her back to the camera.).

Figure 2 illustrates how the KT closely watches Ada when Ada explains her thoughts. Ada chooses another direction than the KT suggested, and the KT needs to 'change attention'. The KT's eyes are fixed on Ada and her facial expression, 'thoughtful', suggests that she is concentrating on following Ada's explanation. At this moment it seems that the KT is confused and does not know how to contribute in the sympractical activity. In line 133 the KT reformulates Ada's suggestion, "Start to count from one at the bottom here?", but her reformulation is turned in to a question and I interpret that the question is asked to make sense of Ada's explanation, not to create anything new in the activity. Ada replies "Mm" (line 134), and confirms that this is what she wants the KT to do. The KT then does exactly as Ada suggests, counts from one on the bottom of the blue tower. The KT says "One, two, three, four, five, six, seven, and then…?" (line 135), and she does *exactly* as Ada suggests without giving any indications for a possible next move. The KT does not contribute with any *new* actions, which can move the activity forward. Through her question "and then…?" at the end of line 135, and her still thoughtful facial expression, it seems that the KT does not have any clear/conscious action possibilities and asks Ada to guide her actions.

In line 136 Ada does not respond in a manner which is helpful for the KT. Ada just repeats an earlier suggestion, "Ehm, we can just count like this all the time, without eight" (line 136). The suggestion is similar to the suggestion in line 130 and does not create new action possibilities for the KT. This appears to confuse the KT even more, and she expresses her confusion by her facial expression (she knits her eyebrows and tightens her mouth) and her utterances in line 137; "I think you have to show me, because I don't really understand what you mean. Maybe you can show …". Before the KT is finished asking for an explanation, Ada raises her body which may be a sign that she has already interpreted the KT's request for explanation before the KT has asked the question verbally. Ada responds, "We count this one first ((points at the yellow tower)) and then this one ((points on the blue tower))" (line 138). This explanation is different from the previous one. Now she considers the two towers,

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and explains how to move from one tower to the next. However, even if Ada has contributed with a completely new explanation, the KT once again asks Ada for a further explanation. The KT says, "Yes, maybe you can count it? Do as how you think, Ada" (line 139). The "yes" indicates that the KT now has an idea of Ada's suggestion again, but it seems that she is not quite sure. This time the KT asks Ada to use the building blocks (to count them), probably to make sure that it is easier for Ada to explain. The KT's continuing request for explanation and her recent request for Ada to use the building blocks in her explanation indicate that the KT really *wants* to understand Ada's idea.

Even though she does not understand Ada's idea the KT continues in the sympractical activity. Both in line 137 and 139 the KT asks Ada to use the building blocks in her explanation, not only to explain verbally. She probably realises that Ada needs to use the building blocks in her explanation, at least to be able to explain in a different way. The KT's request for an explanation *with* the building blocks, I hold, creates action possibilities for Ada.

In line 140 Ada comes forth and explains her idea by using the building blocks. Ada chooses actions which (hopefully) can create new action possibilities for the KT and bring the KT into the sympractical activity again.

Figure 3: Ada's and the KT's stances in line 140. (Ada sits to the right and the KT to the left).

Figure 3 illustrates Ada's stance in line 140. Ada carefully counts the yellow tower and then continues on the blue tower. She talks quite slowly, and she is precise when she moves her finger from one building block to the next. She carefully chooses her actions and concentrates to make sure that her actions are clear. Even if Ada does not count further from eight, starting on nine on the yellow tower, she is able to count further from the yellow tower to the blue tower; it is precisely those actions that are necessary to create new action possibilities for the KT. In line 141 the KT expresses that she thinks she once again understands, and she asks Ada for confirmation. The KT says "Oh, we have to continue?", and Ada confirms the KT's understanding by responding "Mm" in line 142.

Ada does not (and actually she cannot) know the influence of her actions in line 140 in advance, but she knew she needed to change her explanation (choose other actions than the previous ones), to bring the KT into the activity. Ada's actions are crucial for the KT to once again be able to participate in the sympractical activity. Figure 3 illustrates how the KT tilts her head and focuses on Ada's explanation, which indicate that she wants to understand and

continue to participate in the activity. However, she is dependent on Ada's explanation to do so. From line 141 and onward the KT is once again able to participate in the joint activity and to create new action possibilities for Ada. Together they complete the counting strategy, 'counting on from eight' and show it to the whole group.

Both the KT and Ada show perseverance in the activity. Even if the KT at one point, is not able to productively contribute in the activity with new actions she is part of the symplactical activity all the time, in the sense that she does not 'drop out' and that she constantly struggles to understand. Likewise, Ada shows great responsibility, she chooses different actions and works hard to create new action possibilities for the KT all the time. However, as indicated by the KT's response, many of her attempts do not serve that purpose, but she does not give up and she continues to work hard to bring the KT into the activity again. Finally, in line 140 when Ada puts a lot of effort into her explanation, she succeeds. From the effort that Ada makes, it seems that she knows that her actions are extremely important. And her actions *are* important. If she fails to bring the KT into the activity again, the whole activity could collapse. It is important to note that the activity *could* collapse, but we cannot know if it *would*. Maybe the KT would make another effort to restore the activity.

Discussion:
In this section I revisit the results to highlight the main findings, and I relate the findings to what other scholars have found. At the end of the section I highlight why this study is important and what implications the study may have for practice.

The main aim of this study was to investigate the role of Ada's (choices of) actions in keeping the activity moving and in co-creating the ZPD. Taking Roth and Radford's (2011) symmetrical approach on the ZPD as a point of departure, I illustrated how Ada and the KT, by their concrete actions, co-create an (action-) space where social relations are realised. In addition, the KT and Ada expand each other's action possibilities, which creates possibilities for new social relations to emerge and thus to maintain the ZPD through their symplactical activity. The ZPD is not initially there, the ZPD emerges when the KT interacts with Ada and Ada interacts with the KT. Although both show perseverance to participate in the activity, Ada plays a significant role in the teaching-learning activity illustrated above. Ada takes a special responsibility for maintaining the activity. From line 132/133 it is the KT that gets confused and needs guidance. Ada works hard to create new actions and keep the KT in the activity. As emphasised above, the KT is all the time part of the symplactical activity, in the sense that she never 'drops out' and that she wants to understand, however she is not always able to productively participate in the joint labour. It is Ada who, through her actions, creates possibilities for the KT to once again productively participate in the activity. Ada's cautious and responsible actions (however not necessarily 'conscious' actions) makes sure that the activity keeps moving and the ZPD continues to be maintained. It's possible that the responsibility that Ada takes is a consequence of the fact that she is the one who suggested what counting strategy to use to solve the addition problem.

The special relationship that Ada and the KT creates, lays the ground for the symplactical activity. Both are mutually dependent on each other to continue to participate in the

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activity. They need to respect and trust one another to keep the activity moving. Especially when the KT starts to get confused, the KT needs to trust Ada's 'capabilities' for bringing her into the joint activity again. This mutual commitment and trust is what Radford and Roth (2011) call 'togethering'. Ada and the KT make an implicit ethical engagement and commitment to one another and show responsibility to keep the activity moving. Without this trust the activity could collapse. The KT and Ada could have given up the conversation, but they did not! They kept on adjusting to one another.

The ability to listen to others and consider others' initiatives have been identified as crucial features for co-creating ZPDs. John-Steiner (2000) argues that it is important to 'see' the other in the activity and that 'mutual care-taking' lays the ground for learning. She also argues that sharing of ideas and constructive criticism are crucial factors for co-construction of new ideas. That's why it is important to create a 'safe environment' where the participants venture to share their ideas, even if they risk facing criticism. In this episode Ada takes the risk to share her ideas with the KT and the other children, and perhaps this is because the KT manages to create a 'safe environment'. At least, the KT clearly listens to Ada and wants to understand her, which also relates to what Wells (1999) and his colleagues found when they reflected on their research on dialogic inquiry. They found that the most important thing they did in their science class was to listen to the student's. "We did not know this was the shift we needed to make, nor did we anticipate it at the outset, but it was the most significant learning for us" (p. 310). The capability to listen (and observe) lays at the heart of the findings in this study too. The KT needed to listen to (and observe) Ada, in as much as Ada needed to listen to (and observe) the KT and trust one another in order to move the activity forward. This is clearly not one-way communication. The activity (and thus the ZPD) is dependent on both (all) participants contributions (c.f. John-Steiner, 2000; Levykh, 2008; Holzman, 2010; Hussain, Monaghan & Threlfall, 2013; Abtahi, Graven & Lerman, 2017; Mercer, 2000; Roth & Radford, 2010; 2011). Zack and Graves (2002) highlight that even if the teacher creates optimal conditions, a ZPD does not always emerge. They argue that "we cannot expect to orchestrate a desired scenario" (p. 265). They argue similar as I do, that the ZPD and the co-construction of knowledge is dependent on all participants engagement.

The second aim of this article was to investigate Ada's mathematical learning in the joint activity. When Ada explains her idea in line 128, she says, "do not count eight and then we just count further", and the KT expresses verbally and non-verbally that she understands Ada's strategy. Ada makes an understandable explanation, however not without effort - she shows with her whole body that it is challenging. Using quite refined and sophisticated semiotic means, Ada shows an ability to contract her thoughts (c.f. Radford, 2008). Instead of operating directly on the building blocks she explains how to do it with an iconic gesture and language, and the explanation is short and precise. I have observed Ada in similar (yet not identical) situations, where she counts further to find the sum of building blocks. She manages such problems quite well. However, in those situations Ada has direct access to the building blocks. In this episode the KT keeps the building blocks close to her (as illustrated in figure 1), and Ada is thus promoted to explain without direct access to the building blocks. Radford (2008) says that, as the participants become more fluent, contraction increases, and

a deeper level of consciousness is obtained. In this situation the semiotic contraction is not a direct result of Ada's increasing fluency, rather Ada is in a situation which requires her to contract her use of semiotic means. This indicates that there might be a dialectic relationship between fluency and the practice of semiotic contraction. By practicing semiotic contraction, she refines her articulation of the idea, becomes more fluent and perhaps reaches a deeper level of conscioussness.

Since I regard that the ZPD as symmetrical, I claim that both participants learn. Through the joint activity the participants become aware of historically and culturally forms of reasoning in the process of objectification (cf. Radford, 2013). Knowledge is materialised through the participants actions, where the child's actions in this case are especially significant for the materialisation. Through the child's actions, intertwined with the KT's actions, the KT (and the child) 'sees' the object in a new materialised form. The object has never been objectified in exactly this manner before. The object, I hold, is objectified to the KT (and to the child) in a new way, and the materialised form reflects mostly Ada's perspective, which now the KT becomes aware of.

The findings in this study advance knowledge in the field of mathematics education because it illustrates the significant role that Ada (a 5-year-old child and the presumed less knowledgeable participant) plays in co-creating the ZPD. Scholars tend to focus on the importance of the KTs' actions and their roles in children's learning. In this study, I have explored the role of Ada's actions in the joint activity. Ada takes a special responsibility for moving the activity further and thus for co-creating and maintaining the ZPD. Through her actions, Ada makes sure that the KT continues to be part of the sympractical activity. Ada creates new action possibilities for the KT, so she can continue to productively participate in the activity. The findings also contribute to the overall understanding of ZPD, and how a ZPD is co-created in kindergarten. These findings suggest that it is important to invite children to actively participate in mathematical learning activities, and above all, to acknowledge children's contributions and trust their abilities to take responsibility in such learning activities.

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THE MATERIALISATION OF CHILDREN'S MATHEMATICAL THINKING THROUGH ORGANISATION OF TURN-TAKING IN SMALL GROUP INTERACTIONS IN KINDERGARTEN

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This chapter reports from a case study which explores the characteristics of children's turn-taking while they work in small groups to solve addition problems. Further the study explores what role children's organisation of turn-taking plays in the movement of the joint activity and thus for the materialisation of children's mathematical thinking. The findings suggest that children's various ways of organising turn-taking give rise to different ways in which their mathematical thinking is materialised. In one of the examples provided in the study, children's coordinated turn-taking releases a rhythmic counting which materialises a multiplicative structure. The rhythmic structure seems to be important for the flow of children's thinking and valued higher than a correct use of counting words during the problem solving process. The study also discusses the conditions for children's turn-taking and emphasises that children's turn-taking, and thus children's mathematical thinking, seem dependent on contextual features like the formulation of the problem, children's understanding of the problem, available artifacts and children's positional location in space. Drawing on Radford and Roth's (2011) construct 'togethering', the chapter makes a final discussion point about how the children, despite challenges and disagreements, tune to one another and work persistent to create a joint action space and reach intersubjectivity, which is essential for the movement of the activity and for children's joint thinking.

INTRODUCTION

Children stand in a constant relation to an ever-changing environment. To notice differences and similarities in the ever-changing environmental context and to recognise patterns and structures (generalitis) from these differences and similarities are argued by many to be the essence of mathematical thinking (e.g. Mulligan & Mitchelmore, 2013; Radford, 2010). In kindergarten children may experience mathematical structures in both free-play situations and in organised activities. For example, when children work in small groups to solve mathematical problem they must coordinate and organise their actions in order to productively solve the problem. It is through this coordination of actions that mathematical structures and patterns (generalities) emerge in the activity (Radford, 2010; 2013; 2015).

To understand more about how mathematical structures may appear in young children's activities, this study examines the characteristics of children's turn-taking while they work in small groups to solve addition problems. The aim is to reveal how children coordinate and organise their actions to move the activity forward and solve the problems. The analysis focuses on children's use of various semiotic means like gaze, word emphasis and gestures to organise their turn-taking and what mathematical structures are revealed through their joint activity. Drawing on Radford and Roth's (2011) construct 'togethering', the chapter also discusses the conditions for the joint activity and thus the turn-taking, and how the children tune to one another despite challenges and disagreements.

This study addresses the following research questions:

•	What characterises children's organisation of turn-taking while they work in small groups to solve addition problems?

•	What role does children's organisation of turn-taking play in the materialisation of children's mathematical thinking in the joint activity?

In the following sections I first present the theoretical framework, which includes explanations of essential constructs used in the chapter and a literature review on turn-taking and on children's additive and multiplicative reasoning. Then I present the methodology which explains the method for data collection and data analysis. The chapter continues with the results, which focuses on children's use of various semiotic means like gaze, word emphasis and gestures to coordinate their turn taking. The chapter ends with a discussion focused on the characteristics of children's turn-taking and what role their turn-taking plays in the movement of the activity and thus in the materialisation of children's mathematical thinking.

THEORETICAL FRAMEWORK

In the research study reported here I draw on Radford's theory of knowledge objectification, a cultural-historical theory of mathematics teaching and learning, where learning is conceived as "social processes of progressively becoming critically aware of an encoded form of thinking and doing" (Radford, 2013, p. 26). It is through a complex coordination of semiotic means (language, artefacts, mathematical signs, gestures and other bodily actions) that mathematical ideas are mediated into our consciousness. Put another way, learning mathematics is to become critically aware of mathematical structures and patterns (generalities) in the environmental context. However, this process does not happen all of a sudden. Instead there are layers of generality (Radford, 2010) in which the subject gradually becomes more aware of. And it is through human activity and through a coordinated use of semiotic means that these generalities are materialised, that is brought into life and into our consciousness.

Roth and Radford (2011) use the term 'joint practical activity' to describe how humans jointly work together towards a mathematical object in the process of objectification. In their study they show how a teacher and a student work together toward a mathematical object (a specific algebraic pattern) and how the algebraic pattern is materialised (brought into life) through the two participants' actions. Through a complex coordination and tuning of different semiotic means a space of joint action and intersubjectivity is created, where thinking (appears) as a collective phenomenon (Radford and Roth, 2011). To explain the conditions under which a joint action space and intersubjectivity may occur, Radford and Roth (2011) introduce 'togethering', which comprises "the ethical manner in which individuals engage, respond, and tune to each other, despite their cognitive, emotional, and other differences" (p. 235). Togethering is a theoretical construct which aims to capture the conditions under which joint practical activity aimed at a collective motivated mathematical object, may occur. Radford and Roth (2011) emphasise that togethering is not just any type of coming together to do something. Togethering refers to the ethical engagement based on trust and responsibility which lays the ground for the movement of the activity towards the object.

This study investigates young children's joint practical activity working in small groups to solve addition problems. To understand more about the nature of the coordinated interaction, the movement

of the activity, and the materialisation of children's mathematical thinking, the study focuses on children's turn-taking and especially how children organise their turn-taking by coordinating various semiotic means. In their description of a 'simplest systematics for organisation of turn-taking', Sacks, Schegloff, Jefferson (1978) characterise organisational features for turn-taking in conversation and describe how turn-taking is organised by two main types of 'turn-allocation techniques': a current speaker may select the next speaker, or a 'non-speaker' may self-select in starting to talk. In self-selected turn-taking the potential next speaker must find a 'transitional-relevance place', that is a place where it is relevant for a transition in the conversation. Such transitional-relevance places are determined by clausal, phrasal and/or lexical principles which creates conversational units, and by which the speaker may construct a turn.

In his investigation on how the next speaker in turn-taking is addressed by the current speaker, Lerner (2003) discusses a range of explicit and tacit 'techniques' for addressing the next speaker. The current speaker may select next speaker using address terms (like 'you' or the next speakers name), or through gaze-directional addressing where the current speaker is directing his/her gaze to another participant while speaking. Although describing these ways of addressing next speaker independently, Lerner emphasises that these methods are often used in concert with each other (like the use of an address term in concert with gaze or gestures). Tacit addressing is another method for addressing the next speaker and which makes evident who is being addressed without using explicit address terms or other explicit means. Tacit addressing draws upon specific features of the current circumstances and through a specific composition of content and initiating actions the next speaker is being selected. Lerner emphasises that both explicit and tacit address 'techniques' are context sensitive, however, tacit addressing cannot be considered without it. Similarly, Mondada (2007) emphasises the situatedness or context-sensitivity of turn-taking. From a multimodal perspective she investigates how participants in a conversation gradually establish him-/herself as next speaker through specific use of gestures. By using pointing gestures, while the current speaker is still talking, the participants establish him-/herself as the next speaker. In her case study the participants are sitting round a table with diverse artifacts (maps, documents etc.) in the middle, and where everyone is engaged in reading, writing and considering these artifacts. In this context the interaction is not primarily organised as a face-to-face exchange of talk but as a side-by-side exchange where the participants are not looking at each other (having eye contact), rather looking at the arteacts and their joint actions.

In the two cases that are examined in this study, the children were given addition problems (considering the semantic structure[1] of the problems). However, as will be shown in the results, these problems and children's organisation of turn-taking while solving these problems promote rhythmic counting of groups and repeated addition which may be considered as key steps towards multiplicative reasoning. Multiplicative reasoning is distinguished from additive reasoning and traditionally considered as more complex (Anghileri, 1989; Greer, 1992; Mulligan & Mitchelmore, 1997). In additive reasoning quantities of the same type are added, for example 5 apples + 3 apples equals 8 apples. In multiplicative reasoning quantities of different types are involved, for example 4 baskets with 3 apples in each basket equal 12 apples altogether. The example also illustrates the group-structure which is characterised by multiplication.

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[1] 'Semantic structure' refers to the way in which the problem is formulated, either in writing text or verbally, before the children start to solve it.

From research on the semantic structure of multiplicative situations (Greer, 1992; Mulligan & Mitchelmore, 1997) there have been found at least four different types of problems relevant for kindergarten and early school-years, where 'equal groups problems' (e.g. 4 baskets with 3 apples in each), is considered as one of the basic sematic groups. To solve equal groups problems, rhythmic counting and repeated addition, with diverse use of tools, are found as two key strategies that children use, and key steps toward multiplicative reasoning with number facts (Anghileri, 1989; Mulligan and Mitchelmore, 1997).

What can be found from the substantial number of empirical research on the relationship between the sematic structure of addition and subtraction problems and children's strategies for solving these problems, is that most of the research have focused on children's individual skills and individual problem-solving strategies (see Baroody & Purpura, 2017 for an overview). Similarly, research on children's understanding of multiplication and their multiplicative problem solving strategies, have also focused on children's individual skills (e.g. Anghileri, 1989; Greer, 1992; Mulligan & Mitchelmore, 1997; Lu & Richardson, 2017) where clinical interviews are often used as method for collecting data. These studies fail, I hold, to see the contextual features of children's thinking.

In the literature above, rhythmic counting is seen as a means to reach multiplicative thinking. In Radford's (2013; 2015) theory, on the other hand, rhythm must be seen as an integral part of mathematical thinking. Thinking, in Radford's (2015) conception, is thought put into motion and it is through joint practical activity that mathematical thinking is brought to life (that is being materialised or actualised). Radford (2015) argues that "mathematical thinking happens in time. … Mathematical thinking not only happens in time but its most striking feature is *movement*" (p. 68). Rhythm is one structuring feature through which children's mathematical thinking may be materialised, and Radford define rhythm like this: "In its general sense, the concept of rhythm tries to characterise the appearance of something at regular intervals and attempts to capture the idea of regularity, alternation, or something oscillating between symmetry and asymmetry" (p. 68). An important feature of rhythm is then movement, and in accordance with Radford's conception of mathematical thinking, rhythmic counting (as referred to in the literature above) is not merely a means for multiplicative thinking, rather a part of the multiplicative thinking itself.

Rhythm mediates several things, and one of the most important elements of rhythm is what Radford call 'theme'. "Theme is the very important component of rhythm that moves us from memory to imagination and that provides us with the feeling of continuity of the phenomenon under scrutiny – the sense that something will happen next, or the expectation of a forthcoming event" (Radford, 2015, p. 81). Rhythm mediates that there is a regularity or a continuation of something and it gives the children possibilities for imagine what comes next. Another important element of rhythm is 'prolongation'. "Prolongation is the component of rhythm where a phenomenon is *expressed*" (Radford, 2015, p. 81). Through rhythm, a mathematical phenomenon may be expressed or materialised. In this case the rhythmic counting that emerges from children's turn-taking materialises a structure fundamental for multiplication. The different elements of rhythm help to organise thinking and are essential components for the flow of thinking.

METHODOLOGY

The case study (s) reported in this chapter is part a larger Study (S) on mathematical teaching and learning in kindergarten and is situated within a research and development project called the Agder Project[2]. One of the aims in the project is to investigate how kindergarten teachers (KT) implement pre-designed mathematical activities, and children's learning thereof. Four KTs and their groups of children participated in the Study. The focus of the Study was on processes of learning and teaching mathematics and qualitative methods within an interpretative paradigm were chosen as means of analysis. Data was collected from the four kindergartens at four occasions (16 sessions altogether) during the academic year 2016/2017 (the intervention year of the project). All observed sessions were video recorded and field notes were written.

This case study focuses on the joint activity and the coordination of turn-taking within two small-groups of kindergarten children (age 5-6) working on addition problems. The two segments[3] examined in this study were selected from the data set of 16 sessions, focusing on activities where children were challenged to solve addition problems in small groups, without extensive interference from the KT, and where the children showed willingness to solve the problems, that is they persevered in their effort to solve the problems. These criteria for selecting segments limited the data reported here to two segments from two different kindergartens (K1 and K2). In K2 the KT interfered in children's group work at the end of the segment, and therefore segment 2 is divided in two sub-segments (segment 2.1 and 2.2).

The KT in K1 implemented an activity called 'Treasure Hunt' where children searched for a treasure, and to get to the treasure the children had to solve mathematical problems en route. Each problem needed to be solved before the children could move to the next problem. One of the problems in the activity, and which is the focus in segment 1, was formulated as follows: "Run around the nearest located tree three times each. How many times have you run around the tree altogether?"

The KT in K2 implemented an activity called 'Balloon Play'. In 'Balloon Play', the KT placed several balloons on a wall, each containing a mathematical task or problem. The children chose which balloon to burst (with a drawing pin), and they worked in groups to solve the problem. One of the problems, which is the focus in segment 2, was: "Consider your hands, how many fingers have you altogether in the group?"

The two segments were transcribed[4] and then analysed from a multimodal perspective. The analysis was conducted (and refined) through iterative examination of the video recording and the corresponding transcripts focusing on verbal and non-verbal actions which the participants used and made available to others (and thus to me) for the purpose of moving the activity forward. The analysis focused on identifying verbal and non-verbal actions which seemed important for understanding the

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[3] A segment is here considered a self-contained part of a lesson with a distinct beginning and end.

[4] Transcription codes: (( )) denotes non-verbal actions or contains explanations and interpretations necessary to understand the dialouge; _ denotes that the underlined word is emphasised; … denotes a pause in the verbal utterance; [ ] denotes that the utterance is cut off by another participant.

ongoing interaction in light of the formulated research questions. This fine-grained iterative analysis served as a means for interpreting the segment as a whole.

RESULTS

In this section three examples are presented to illustrate diverse ways in which turn-taking may be organised in small groups in kindergarten. The first example (segment 1) illustrates, primarily, how gaze in concert with word emphasis is used to address next speaker, and move the activity forward. From this way of organising turn-taking a rhythmic counting is released and a multiplicative pattern emerges. The second example (segment 2.1) illustrates, primarily, how children self-select turns in the organisation of turn-taking. In this example the children have different ideas for how to solve the problem, but during the activity they tune to one another and reach a compromise for how to solve the problem which they all support. In the third example (segment 2.2) the KT interferes in children's group work, and the turn-taking take yet another form. The KT now strongly structures the turn-taking and helps the children re-establish joint attention and focus on a common strategy to solve the problem.

Segment 1, from kindergarten 1 (K1):

This example is provided to illustrate how turn-taking as a structuring feature in the joint activity moves the activity forward and gives rise to children's mathematical thinking - in this case a rhythmic counting of groups. Before the segment presented below, the KT and the children have reached post 2 in the 'Treasure Hunt' activity. The KT reads the problem for the children ("Run around the nearest located tree three times each. How many times have you run around the tree altogether?"), and the children immediately starts to run around the tree. The children run around the tree three times each and then represent their runs by their fingers (each child shows three fingers to the KT and the other children). Then the KT initiates the second part of the task which is to figure out how many runs they have run around the three altogether. Figure 1 illustrates how the children are positioned when they try to solve the problem.

115	KT	But, how many times have you run alltogether?
116	Pia	((Pia shows three fingers))
117	KT	All of you ((swipes her hand over the children, while keeping her gaze on Pia))
118	Pia	Aaah, we have to count! … One, two, <u>three</u>. ((Turns her gaze to May))
119	May	Four, five, <u>six</u>. ((Turns her gaze back at Pia))
120	Pia	((Turns her gaze towards Amy)) Amy, it's your turn to count! ((Points towards Amy))
121	Amy	One, two, three, fo[ ]

Figure 1 illustrates how the children in K1 are positioned when they try to solve the problem.

In line 115, the children keep their gaze at the KT while she asks how many times they have run around the tree alltogether. Pia shows three fingers to the KT and the KT looks back at Pia (line 116). The KT then, in line 117, emphasises "all of you" and swipes her hand over the children, however she is still looking at Pia. In line 118 it seems that Pia gets an idea for how to solve the problem, because she immediately turns her gaze to May and says "Aaah, we have to count". To use Radford's (2015) terminology, Pia gets an idea (a thought) which is still pure possibility (the 'feeling' of a possible counting strategy), and which she has to put into motion. To put the thought into motion (that is to transform the idea into materialised thinking) she must interact with the other children. The idea includes the other children, because each child represents their runs around the tree and the mathematical thinking can (only) be materialised through their joint activity. Since the idea is pure possibility there is a risk to fail, and to succeed Pia is dependent on the other children's loyalty and their persistence to 'work out' the idea. Pia then, still in line 118, turns her gaze to her own fingers and starts to count "one, two, three". As she counts her third finger, she holds on to it and simultaneously moves her gaze to May. Both the gaze and the word emphasis address May as the next speaker. In line 119, May continues the initiated pattern, which indicates that she has got the 'feeling' of the idea/thought, but which is still pure possibility that is about to be materialised. May looks down at her fingers while she counts, and as she counts her third finger she holds on to it and simultaneously moves her gaze back to Pia. May's gaze and word emphasis addresses Pia as the next speaker. Pia has already counted and therefore she turns her gaze further to Amy (line 120), but Amy has not paid attention to the ongoing interaction. She has been examining something on the ground and does not recognise that Pia has turned her gaze to her. Pia then approaches Amy verbally and says, "Amy, it's your turn to count!" and simultaneously points eagerly towards Amy. Amy then recognises that it is her turn to count and responds, "One, two, three, fou[ ]" in line 121. Since Amy has not paid attention to Pia and May's previous interaction she has not recognised the pattern of the counting. Neither has she recognised that she has to continue on seven, nor has she recognised the rhythm in the counting. Amy starts to count from one and is about to continue further from three.

Pia recognises that Amy do not continue the same counting pattern as she and May initiated, and she interrupts Amy.

122	Pia	No, not like that! … One, two, three! ((Pia counts slowly and keeps her gaze at Amy while she counts))
123	Amy	One, two, three. ((Amy keeps her gaze at Pia while she counts))
124	Pia	Ahrr. … ((Pia sounds a bit irritated, and then she turns her gaze to May's fingers and points at May's index finger))
125	May	But I have already counted! … ((May sounds a bit irritated, she looks back at Pia with a resigned face expression)) …
126	Pia	But wait. Then we have to count one more time, since Amy counted one, two, three.
127	May	One, two, three. ((then she turns her gaze at Pia and pokes Pia's hands)) Your turn!
128	Pia	((Pia looks down at her fingers, and holds on to her index finger for a while, before she starts counting)) … Four, five, six. ((Pia turns her gaze further to Amy))
129	Amy	Seven, eight, nine. ((Amy turns her gaze back to Pia))

130	Pia	((Pia turns her gaze further to Adam))
131	Adam	((Adam blushes, and looks down at his thumb)) …
132	Pia	Your turn! ((Pia points at Adam))
133	Adam	OK … Eleven, twelve, <u>thirteen</u>.
134	May	Thirteen all together! ((May turns her gaze to the KT and smiles))
135	Pia	Yes. Thirteen!

In line 122 Pia interrupts Amy and says "No, not like that!", and expresses both verbally and non-verbally (with an resigned face expression), that Amy did not count as anticipated (in accordance with the initial ides). Then, still in line 122, Pia makes an attempt to correct Amy. She counts her fingers slowly and distinct, "one, two, <u>three</u>", with a marked stress on "three", while she keeps her gaze at Amy. By counting slowly and distinct Pia emphasises the rhythm in her counting - Amy is not supposed to count or say more than three counting words. It seems that Pia also tries to promote Amy to count further, by keeping her gaze at Pia while she counts, but she does not know how to express this explicitly. In line 123 Amy imitates Pia's actions. It seems that Amy understands the importance of the rhythm, however she does not recognise that she has to count further. She counts slowly "one, two, <u>three</u>" while she keeps her gaze at Pia as if she needs Pia to confirm, accept or correct her.

Amy's counting is still not in line with Pia's initial idea, and in line 124 Pia expresses her frustration both verbally and non-verbally. In frustration she points at May's index finger (which is difficult to understand why she did; perhaps it was just an attempt to keep the activity moving somehow). In line 125 May expresses, also a bit frustrated, that she has already counted and turns her gaze back to Pia with a resigned face expression. This action does not really move the activity further. There is a pause in the interaction, where none of the children do anything, and the activity could have stopped at this point. However, it seems that Pia understands that something needs to be done, and she suggest, in line 126, that they start over. May accepts the idea and immediately starts to count from one, in line 127, and in the same manner as earlier she turns her gaze to Pia while she holds on to her third finger and says "<u>three</u>". Again, the word emphasis and gaze address Pia as the next speaker. In line 128, Pia looks down at her fingers and holds on to her index finger for a while before she starts counting further from three, that is she counts "four, five, <u>six</u>". She seems concentrated, as if she wants to do it right and ensure that the activity moves forward in the desired direction, that is in accordance with the initial idea. As Pia counts her third finger she holds on to it and moves her gaze further to Amy while she says "<u>three</u>". This time Amy has payed attention to May's and Pia's counting strategy; she recognises the counting pattern (counting further, but not more than three numbers) and, in line 129, Amy counts further without hesitation. When she counts her third finger she turns her gaze back to Pia, and addresses Pia as the next speaker again. In line 130 Pia recognises that it is her turn, but without speaking she just turns her gaze further to Adam and addresses Adam as the next speaker. Adam has paid attention to the ongoing activity, but he has not yet contributed. Adam blushes as if he feels pressured. All the others have counted, and it is only him left. In addition, he knows that this is the second attempt to solve the problem, and the others would probably be disappointed if he failed. In line 132 Pia says "Your turn!" and expresses that she is impatient for him to count. In line 133 Adam says "OK" and after a little pause he counts "eleven, twelve <u>thirteen</u>". Although he skipped

counting 'ten', he still counted further, and the other children seem satisfied and accept thirteen as the final answer. May and Pia state, in line 134 and 135, that thirteen is the correct answer to the problem.

In most of the segment, gaze is used to organize turn-taking by addressing next speaker (in 15 out of 21 turns). In many of the turns gaze is used in concert with word emphasis (line 118, 119, 122, 123, 127, 128 and 129), which is typically in the turns where the current speaker counts and then promotes another person to count further. In some cases, gaze is also used in concert with a direct verbal prompt (line 120, 127 and 132), for example when the person being addressed does not pay attention or when the current speaker is impatient. Although turn-taking in the example above is mainly organised by addressing next speaker, turn-taking is also organised by self-selecting in some cases (in line 116, 122, 127, 132, 134 and 135). Although these turns have different reasons for being self-selected, they are still used to move the activity forward in some way. Or, as in line 122, the self-selected turn is used to move the activity in a different direction. In line 122 Pia interrupts Amy because Amy is not following the anticipated direction of the activity, that is Amy does not act in accordance with the initial idea. Pia must re-direct Amy, and try to make her realise the initial idea.

The turn-taking is important for the structure of the activity and thus for the materialisation of the thinking embedded in the activity. In the activity above the use of gaze and word emphasis to address next speaker (on every third number), is especially important for materialising a specific feature of the thinking, namely rhythm. First of all rhythm indicates that there is a regularity and a continuation. In this case the rhythm promotes the children to continue to count. Already in the beginning, in line 118, Pia sets out with a rhythmic counting of three ("one, two, three"), which is a regular sequence of three counting words. However, the stress on "three" indicates that "three" is the end of the sequence and which may be followed by another regular sequence (of three counting words). The rhythmic counting indicates that *something* should re-appear, which in this case is three counting words. Through the following turn-taking a rhythmic counting sequence is then released: 1, 2, 3 – 4, 5, 6 – 7, 8, 9 – 11, 12, 13 (which should have been 10, 11, 12). The phenomenon that is materialised through this rhythmic counting sequence is repeated addition of three (3+3+3+3) and the number sequence 3, 6, 9, (13 – which should have been 12), which both are elements of multiplication. During the activity, the children face challenges in 'seeing' the same structure appearing from their actions. Especially Amy does not realise the rhythmic counting in the beginning. Pia then re-starts the activity, in line 126, and in the activity that follows all the children realises the rhythmic counting. Put another way, the rhythmic counting is then objectified in the consciousness of all.

Another key point that underlines the importance of rhythm in children's thinking is what happens in line 133, 134 and 135. In line 133 Adam is able to stick to the rhythm, but he does not use the correct sequence of counting words. But the children seem satisfied with the solution even though the sequence (and thus the answer) is incorrect. This indicates the strong position that rhythm has in children's thinking. The correct sequence of words seems not that prioritised for solving the problem, and is actually ignored in line 133, 134, 135. An important aspect in this interpretation is that all the children in this group are able to count 'nine, ten, eleven, twelve' in other settings, so to skip 'ten' is not a common problem for the children when they count. In this situation Adam is true to the rhythm and the rhythm seems prioritised over the correct sequence of words, because the other children accept the last counting word as the answer, although it is incorrect.

After this segment, the KT makes another attempt to solve the problem together with the children and ensures that they get the correct answer. The KT asks the children to find as many cones as they have run around the tree and then to put them in a pile. To solve the task they count the cones which lies in the pile. This solution strategy does not materialise the group structure of multiplication. The rhythmic counting disappeares. So the problem itself is not enough for bringing forth the multiplicative pattern (rhythm). The way that the children are placed and the way that the children organice their turn-taking is an important part of the materialisation of multiplication.

Segment 2.1 and 2.2, from kindergarten 2 (K2):

The example from K2 is devided into two sub-segments (segment 2.1 and 2.2). Both segment 2.1 and 2.2 are provided to illustrate diverse ways in which children (and the KT) structures their turn-taking, and what mathematics is materialised through their joint activity.

Segment 2.1

In K2 the children play 'The Balloon Play". The children are working in small groups to solve the problems. In one of the problems the children are supposed to count how many fingers they have all together on their hands. A girl named Lily immediately starts to solve the problem. Figure 2 illustrates how the children are positioned when they try to solve the problem.

6	Lily	Ten, twenty, thirty, fourty, fifty … We have to count as well. … One, two [ ]
7	Eva	That goes a lot slower
8	Lily	[ ] three, four, five then you have to count mine. ((Touches Mia's hand))
9	Eva	But we know that this is five ((points at Mia's right hand))
10	Mia	This is ten all together ((Mia puts both her hands out to each side))
11	Eva	Yes
12	Lily	Ehm … this is ten … ((Lily counts Mia's right hand together with her own left hand as ten, and then she continues on Mia's left hand and further to Eva's hands)), eleven, twelve, thirteen, ((she continues counting from thirteen to twenty-nine)), twenty-nine
13	Mia	Twenty-nine! … It's twenty-nine! ((She turns her gaze to the KT))
14	Lily	Or, maybe not… ((Starts to count mentally her own fingers))
15	KT	Is it twenty-nine? ((The KT, who stands a little aside, recognises that the children do not find the correct answer))
16	Eva	Five, ten, twenty, thirty, forty, fifty ((Eva counts "five, ten" on Mia's hands and then "twenty, thirty" on Lily's hands and then "forty, fifty" on her own hands)).
17	Lily	((Lily continues to count by ones, and she counts her own ten fingers three times)) It's twenty-eight! We got twenty-eight!! ((She turns her gaze to the KT))

Figure 2 illustrates how the children in K2 are positioned when they try to solve the problem.

In line 6 Lily starts quite 'spontaneously', however a bit careless, to count by tens. She uses the correct counting words, but she does not really point at any fingers or hands when she counts. But then she changes her mind, and from the utterance "we have to count as well", it seems that she doesn't really think of counting by tens as a satisfactory strategy to solve the problem. Perhaps she just 'plays' with the counting words (ten, twenty, thirty, fourty, fifty) without really trying to solve the problem. But when she considers how to solve the problem she chooses to count by ones. In line 7, Eva self-select her turn by interrupting Lily's counting, and comments that Lily's strategy goes a lot slower. Lily ignores Eva's comment and continues her counting by one strategy in line 8. Lily counts five fingers on Mia's right hand, and then she promotes Mia to count hers. She addresses Mia by the address term 'you' and a corresponding touch on Mia's hand. In line 9 Eva again self-selects her turn, this time in a suitable transitional-relevance space, and comments that they know that there are five fingers on Mia's right hand. Although Mia was addressed by Lily in line 8 to continue her strategy, she does not follow Lily's suggestion. In line 10, Mia states that it is ten fingers altogether on her two hands. She puts both her hands out to the side which indicate that it is 'obvious' for her. Eva agrees with Mia in line 11 and confirms that it is ten fingers on two hands. In line 12 it seems that Lily accepts Eva's and Mia's statements, because she confirms that there are ten fingers on two hands (her left hand and Mia's right hand). And then she uses that derived fact to count further by ones. Lily continues with eleven on Mia's left hand, then she continues from sixteen on her own right hand, and then she continues from twenty on Eva's fingers. Eva and Mia are watching Lily's hands while she counts, and thus maintain joint attention. This indicate that everyone is now satisfied and support the strategy, and thus is it *their* strategy not only Lily's strategy, although it is Lily that counts.

None of them recognises that Lily makes a mistake, that is, she skips a finger when she counts eighteen. This results in an incorrect answer, they end up with twenty-nine but should have had thirty. Mia accepts twenty-nine as the solution in line 13, however Lily seems to doubt that the solution is correct. It is difficult to say whether she doubts the solution because she has an idea of what the answer should be or because she doubts the strategy that they used. Anyhow, in line 14 she starts to mentally count by ones as if she wants to check the answer.

The KT, who has helped another group of children and therefore stands a little aside, recognises that the children do not get the correct answer. In line 15 the KT asks "is it twnty-nine?". Lily continues to count by ones using her own fingers, and Eva, in line 16 tries to use counting by fives or tens, but she mixes the two counting sequences. Eva counts a bit 'sloppy' without actually pointing at any hands or fingers. Again, it seems that she 'plays' with the words, but she is not really able to use it as a strategy to solve the problem. Mia partly focuses on what Eva does and partly focuses on what Lily does.

In the segment above, all turns are self-selected turns. In line 8 Lily is addressing Mia as the next speaker by using "you" and by touching Mia's hand, however Mia does not respond to Lily's request. Instead Eva takes the turn in line 9. Except from Eva's interference in line 7, the other self-selected turns are taken in transitional-relevance spaces. The joint activity, at least in the beginning, is characterised by disagreement which is identified by the way that children interrupt each other. But the disagreement is not necessarily unproductive disagreement. The turn-taking is nevertheless moving the activity forward and gives possibilities to recognise diverse ways to solve the problem.

It seems that Lily's perspective is to solve the problem. In line 6 she 'plays' with the words in the counting sequence of counting by tens, however it seems that she realises that she is not able to use that strategy to solve the problem and she changes strategy. Lily wants to use counting by ones, which probably is the strategy that they have used the most and which is then the 'safest' strategy to solve the problem. Eva and Mia's perspectives are perhaps a bit different. It seems that they are more concerned about counting by fives or tens or at least use derived facts to solve the problem then by taking the 'safest journey' to the solution. During the activity the children struggle to tune to one another to reach a common strategy, and in line 12 they compromise and combine the two ideas. In line 12 Lily accepts Eva and Mia's perspectives and she uses the derived fact that there are ten fingers on two hands, and then she counts further (by ones) from ten. Eva and Mia seem satisfied with Lily's use of the derived fact, and support Lily's counting strategy. The way that the children compromise in line 12, illustrate the flexibility in their thinking. Instead of accepting one of the suggested strategies (and discard the other), they compromise and combine the strategies into one common strategy.

Segment 2.2

In line 14 in the segment above, Lily starts mentally to count by ones using her own fingers. Simultaneously as Lily counts her own fingers, Eva tries to count by fives or tens, and Mia is partly focusing on what Eva does and partly focusing on what Lily does. The KT recognises that the group has problems to keep joint attention and to collaborate to solve the problem and thus she interferes:

18	KT	Hmm … if you Lily, put your hands out. And you Mia. And then I. Maybe you can count how many fingers we have all together Eva?
19	Eva	One … Emm … Five, ten, fifteen, twenty … No …
20	Lily	Yes. ((Lifts her left hand a bit up in the air)) twenty
21	KT	Twenty, and then … ((Turns her gaze to Eva))
22	Eva	Thirty, forty ((points at Mia's right and left hand respectively))
23	KT	Is it thirty after twenty? … Twenty-one … ((she points at Mia's little finger when she says "twenty-one" and then moves her pointing finger to Mia's ring finger))
24	All	twenty-two, twenty-three, twenty-four, twenty-five, twenty-six, twenty-seven, twenty-eight, twenty-nine, thirty ((the KT points at each of Mia's fingers respectively))
25	KT	Thirty ((whispers)) … If we take away the thumbs? If you take away your thumbs, how many fingers have you got then?

In line 18 the KT physically (but gently) takes Lily and Mia's hands and organises them so they are easy to operate on. Then she asks Eva to count. The KT organises whose hands should be counted,

how the hands should be placed and who is going to count. The KT addresses Eva as 'the counter' and in line 19 Eva starts to count from one, but then she changes her mind and starts over counting by fives. First, she counts the KT's hands (five, ten) then Lily's hands (fifteen, twenty), and she is about to continue on Mia's hands, but then she stops, and says "no". Probably she stops because she can't remember what comes after twenty in the counting sequence. Lily has paid attention to Eva's actions, and in line 20 she interferes and says "yes" and then lifts her hand and says "twenty". Lily confirms that she agrees with Eva's way of counting until that point and promotes Eva to continue to count from twenty. Lily does not offer any suggestion for how to continue, so Eva does not respond to Lily's actions. In line 21 the KT also repeats "twenty", while she holds on to Lily's left hand, and then she moves her hand to Eva's right hand and says, "and then …", which prompts Eva to continue counting. The KT prompts Eva to continue, but Eva still needs to figure out what comes after twenty. In line 22, Eva continues to count, but she is not consistent with her previous counting strategy, which was counting by fives. Instead she continues counting by tens. The KT interferes in line 23 and asks, "is it thirty after twenty?". This might be a confusing question because thirty comes after twenty if you think of the number line, and it comes after twenty if you count by tens. However, thirty does not come directly after twenty if you count by fives, and this is, I think, what the KT means. The KT has a little pause, which might indicate that she considers how to continue, and then she initiates counting by ones by saying "twenty-one" and then points to the next finger which is about to be counted. The way that the KT initiates counting by ones is illustrated in Figure 3. Then, in line 24, they all count the rest of the fingers by ones and solve the task together. In line 25 the KT confirms the answer as a correct answer, and then she initiate another problem for the children to work on, which is of the same type.

Figure 3 illustrates how the KT in line 23 initiates counting by ones.

The nature of the turn-taking changes completely in line 18 when the KT interferes in children's group work. In segment 2.1 (line 6-17) the children self-select turns. From line 18 it is the KT that organises the turn-taking (except from line 20, where Lily self-select her turn). The KT both self-select her turns and she addresses the next speaker, and the activity becomes quite structured. In line 18 the KT organises whose hands should be counted, how the hands should be placed and she addresses Eva as the next speaker. The KT gives Eva the role as 'the counter'. Eva accepts being addressed as the counter, and starts counting by fives in line 19. Lily self-selects her turn in line 20 and then she invites Eva to continue, but Eva does not accept Lily's invitation. Then the KT self-select her turn in line 21 in order to move the activity further and again she addresses Eva as next speaker by turning her gaze to Eva. Eva accepts being addressed and tries to continue counting. In

line 23 the KT again self-select her turn, but this time she does not turn to any particular child. Instead she promotes everyone to count together.

Again the turn-taking does not materialise mathematics in the same manner as the segment from K1 where the turn-taking itself gives rise to rhythmic counting and thus materialise a multiplicative structure. However, the segment from line 18 to line 25 gives children other possibilities to realise ways of organising themselves in group work. In line 16 and 17 the children have problems in keeping joint attention and participating in a joint activity, and therefore, it seems, the KT interferes. From line 18 the activity becomes quite structured, where the KT organises most of the turn-taking and where there is little room for disagreement and negotiation. However, the structure that the KT brings in by taking and addressing turns, helps the children to re-establish joint attention and to work in a joint activity again. Although it is Eva that is given the role as 'the counter', both Lily and Mia pay attention to Eva's actions and the activity must therefore be recognised as a joint activity and materialised thinking as their joint thinking.

In line 19 Eva starts to count by ones, but then she changes her mind and tries to count by fives. The way that she changes her mind indicates her 'fascination' for this type of counting (counting equal groups). This time she is a lot more focused then she was in line 16 where she mixed counting by fives and counting by tens. In line 16 it seemed like she was (just) 'playing' with the counting words. The structure that the KT brings into the activity, seems to help Eva to concentrate and to be more accurate in her counting. Again rhythm seems to be an important part of the flow of the mathematical thinking. In line 19 Eva counts hands by a rhythmic counting by fives. In line 22 Eva continues the rhythmic counting, however this time she counts by tens (but still refers to hands with five fingers). In line 23 the KT problematise Eva's counting strategy after twenty, and then she invites all the children to count together, but now she goes back to count by ones. Again they show flexibility in their thinking by the way that they combine two strategies, but this time strongly led by the KT.

DISCUSSION

As mentioned above, turn-taking is used by the participants to move and adjust the activity further. Since thinking, in Radford's (2015) conception, is thought put into motion, turn-taking is one way that children put ideas into motion. To investigate the organisation of children's turn-taking is therefore a way to understand how children's (and the KT's) mathematical thinking is materialised through their joint activity. The discussion is organised around two issues: 1) The characterisation of children's turn-taking in the three segments reported in the result section, and possible reasons for the various ways in which children (and the KT) organise their turn-taking, and 2) The role of children's organisation of turn-taking in the materialisation of children's mathematical thinking in the joint activity.

The characterisation of children's turn-taking: similarities and differences in the three segments

The examples provided in the result section illustrate diverse ways that children (and the KT) organise turn-taking in small groups to solve addition problems. The turn-taking in the three segments are quite different. In the example from K1 there are mainly turn-taking by addressing next speaker and the children seem to agree on a common strategy. In segment 2.1 there are mainly self-selected turn-taking and the children do not immediately agree on which strategy to use. In segment 2.2 it is mainly

the KT that organises the turn-taking by taking and addressing turns. The structure that the KT brings into the activity by organising the turn-taking helps the children to focus on a common strategy and to re-establish a joint attention.

There are probably several reasons for the differences in the turn-taking, however one reason may be the way that the children are positioned and how the problems are formulated, which indicate that the turn-taking is strongly dependent on the context (cf. Mondada, 2007; Lerner, 2003). In the first example from K1, the children are working on a problem that invites all children to participate. All children are asked to run around the tree. The problem does not explicitly ask the children to represent their runs by their fingers, but this is, I hold, a likely strategy for modelling the problem. When the children are adding up their runs they stand a bit apart from each other, and this may be a reason why they address next speaker by gaze and word emphasis. The children use the fingers on one hand to represent their runs and the fingers one the other hand to count. And because they stand a bit apart from each other they lift their gaze and emphasise the last counting word to address next speaker. This way of taking turns may be regarded as face-to-face interaction (cf. Mondada, 2007), because the current speaker and the next speaker keep eye contact in the transition of turns.

In segment 2.1 and 2.2 the children are also working on a problem that invites all children to participate. The problem asks the children to count the fingers on all children's hands. In segment 2.1 the children are standing quite close to each other while they solve the problem, which makes it possible to touch one another's hands for addressing next speaker. Since they are standing quite close they also have the possibility to count each other's hands and/or fingers (not only one's own fingers as in K1). This way of taking turns may be considered as side-by-side interaction (cf. Mondada, 2007), since the children do not (or very seldom) keep eye contact in the transition of turns (they usually kept their gaze at their hands/fingers). In segment 2.2 the KT interferes in children's group work. Just before the KT interferes, in line 16 and 17, the children are not working together in a joint activity, rather they participate in separate activities. The KT recognises that the children have problems to collaborate, and she interferes to re-establish the joint activity. To re-establish the joint activity there need to be some structure to build the joint activity around, and the KT brings the necessary structure into the activity, and the children are again able to focus on a common strategy and act in a joint activity.

Another reason for the different ways in which children (and the KT) organise their turn-taking may be children's understanding of the problem. In segment 1 it seems that Pia gets a special organising role. Turns are often coming back to Pia, even when it is not her turn to count. Perhaps this is because Pia was the one who had the original idea for how to solve the problem. The idea was, in the beginning, pure possibility and Pia needed the other children to participate in a joint activity to put the idea into motion (cf. Radford, 2015). When the idea is put into motion, it seems that all children take up Pia's initial idea, however with various awareness of it, and through the joint activity children's thinking becomes materialised. Because Pia is most likely the one who is most aware of the idea, the other children trust Pia to organise the turn-taking to increase the possibility for the idea to be actualised. In segment 2.1 there are disagreement, at least in the beginning, in how to solve the problem. Lily focuses on the answer and the 'safest' strategy to solve the problem. But Eva and Mia seem to be focused on counting by fives or tens or by using derived facts to solve the problem. The turn-taking is influenced by the struggle to tune to one another in order to focus on a common strategy.

In the end of segment 2.1 the children compromise, and combine the two strategies. They use a derived fact (there are ten fingers on two hands) and then they count further by ones. All children seem satisfied with the strategy, but in line 14 Lily is still not sure that they have got the correct answer and she starts her own activity to check the answer. In segment 2.2 the KT tries to re-establish the joint activity. In this segment the KT takes a leading role for organising the turn-taking. The KT does not decide what idea should be materialised (at least not initially), but she is capable of taking up whatever idea that the children want to put into motion and help them materialise their thinking. The KT gives Eva the role to initiate an idea, and Eva initiates to count by fives.

In all segments, whenever there is disagreement, we find that children interrupt each other, and it seems that interruption is not only moving the activity forward, but also adjusting the direction of the activity so the children may focus on the same strategy.

The role of children's organisation of turn-taking for the materialisation of children's mathematical thinking in the joint activity

The previous paragraph pointed to ways in which children organise turn-taking and some possible reasons for the various ways in which the children (and the KT) organised their turn-taking. This paragraph is devoted to a discussion about what role children's organisation of turn-taking play in the materialisation of children's mathematical thinking. Since movement is the most striking feature of the (mathematical) thinking (Radford, 2015), the way that the children organise their turn-taking (through a complex coordination of various semiotic means) in order to move the activity forward reveals children's joint mathematical thinking and the way mathematical ideas are put into motion (Radford, 2013; 2015).

The sematic structure of the two problems may be considered as additive because they consider only one quantity (the number of runs around the tree or the number of fingers). However, the problems give possibilities for multiplicative thinking because the children are asked to add equal groups (cf. Anghileri, 1989; Mulligan and Mitchelmore, 1997), and as argued in the result section children's joint activity (their complex coordination of various semiotic means) in the two examples do bring to life multiplicative thinking (cf. Radford, 2013; 2015).

Segment 1 is particularly interesting for understanding how multiplicative thinking may be materialised through a joint activity. In segment 1 children's coordinated turn-taking releases a rhythmic counting. The children use gaze and word emphasis to address next speaker (on every third number) which brings to life the rhythm: 1, 2, 3 – 4, 5, 6 – 7, 8, 9 – 11, 12, 13 (which should have been 10, 11, 12). Such rhythmic counting is emphasised as important in the transition from additive reasoning to multiplicative reasoning because it reveals the fundamental group-structure of multiplication (Anghileri, 1989; Mulligan and Mitchelmore, 1997). In Radford's (2013; 2015) conception however the (mathematical) thinking is embedded the activity itself, and not merely a means for reaching 'another form' of thinking (in this case from additive reasoning into multiplicative reasoning). The turn-taking is a way to structure the activity and to move the joint activity forward. It is the turn-taking itself (children's coordination of various semiotic means) that releases the rhythmic counting which materialises the joint multiplicative thinking. To what extent children are aware of the multiplicative structure in their joint activity is of course an important consideration to make. The children are not yet able to multiply with number facts, that is, to see that there are four

groups of three, and then calculate 4 x 3 = 12. However, the way that the children are able to follow the same pattern indicates that they are aware of some layers of generality (Radford, 2010), that is, some layers of the multiplicative structure in their interaction.

From the analysis of segment 1 there are different elements of rhythm that is mediated through the activity. First, the rhythm mediates that there is a continuation. When, Pia starts to count "one, two, three" in line 118 the rhythm in her counting mediates that the sequence could be continued. The rhythm gives possibilities for the other children to imagine what comes next. This component of rhythm is what Radford calls 'theme' (Radford, 2015). In addition, rhythm mediates that *something* is repeated, which in this case is three counting words. The way that Pia the stresses "three" indicates that "three" is the end of the sequence and which may be followed by another regular sequence (of three counting words). And third, the rhythm mediates the phenomenon itself. Through the ongoing activity the multiplicative structure (at least some layers of the multiplicative structure) is mediated by children's joint rhythmic counting. This element of rhythm is what Radford calls 'prolongation' (Radford, 2015). And as Radford (2015) argues "They [elements of rhythm] are central features of the mediation of thought and the manner in which it becomes actualised in the students' reflections and actions. They are part of the materiality of thinking" (p. 78).

Above I argued that the initial problem gives rise to possibilities for multiplicative thinking, because the children are asked to add equal groups. But the problem itself is not enough to materialise multiplicative thinking. After segment 1 the KT asks the children to find as many cones as they run around the three. Each child find three cones each, and on the request of the KT they gather them in a pile. To solve the task they count all the cones which lies in the pile together. This solution strategy does not materialise the group structure of multiplication. The rhythmic counting disappeares. So the problem itself is not enough to bring forth a multiplicative pattern. The way that the children are placed (their positional location in space), the available artifacts and the way that the children organise the turn-taking is important for the materialisation of their multiplicative thinking. The mathematics is embodied in the children's use of word emphasis, gaze, gestures and their positional location in space and are essential components for the flow of thinking.

In segment 2.1 and 2.2 the children seem concerned about the group structure of their fingers. They know that there are five fingers on one hand and ten fingers on two hands. In the beginning of segment 2.1 there is disagreement on what strategy to use for solving the problem. The disagreement in not necessary unproductive disagreement. The way that Eva, May and Lily compromise at the end of segment 2.1 illustrates the flexibility in these children's mathematical thinking. The children flexible combine two strategies into one joint strategy. In segment 2.2, the KT helps the children to re-establish their joint activity (which in line 16 and 17 was splitted), and to organise their thinking. Eva, who was pointed to as 'the counter' initiates to count by fives. Again rhythm seems to be an important part of the flow of thinking. Rhythm helps Eva to count by fives (in line 19) and later count by tens (in line 22). In line 23 the KT problematise Eva's counting after twenty, and she invites all the children to count together, but she also changes counting strategy. Eva initiated counting by fives, but in line 23 the KT initiates counting by ones. Again, the joint activity (strongly organised by the KT) materialise a flexibility. Together they flexibly combine two strategies and end up with a satisfactory solution.

As emphasised above, rhythm is an essential part of children's mathematical thinking in both segment 1 and segment 2.1 and 2.2, and perhaps valued or prioritised higher than a correct sequence of number words or a correct solution. In segment 1 Adam is not able to use the correct counting words, but he is true to the rhythm. It seems that because Adam is true to the rhythm, the other children accept the last counting word as the answer. Rhythm seems more valued by the other children than the answer itself. It is as if the children trust the rhythm and the regularity that it gives, and therefore they trust the answer that it gives (although it is incorrect).

Although the turn-taking itself does not materialise rhythmic counting in segment 2.1 and 2.2, rhythm is still an important feature in children's mathematical thinking. Already in line 6 Lily 'plays' with the counting sequence "Ten, twenty, thirty, fourty, fifty". Lily does not point at any specific hands or fingers when she counts, however she still, somehow, rhythmically points to imaginary objects while she counts. After this both Mia and Eva seem concerned about using the group structure of their fingers and their hands to solve the problem. In line 16 Eva makes another attempt to count by fives and tens. She 'plays' with the counting words "five, ten, twenty, thirty, forty, fifty" while she rhythmically points to imaginary objects. Again it seems that she is 'playing' with the counting words and not emphasising the correct use of words. The way that Eva changes her mind in line 19 (she starts counting by ones and then she changes her mind and starts counting by fives) indicates the 'fascination' she has for this type of counting. Although the children do not know exactly how to solve the problem with counting by fives or tens, the way that they 'play' with the rhythmic counting illustrates the importance of rhythm in their counting.

My final discussion point is devoted to the way that the children overcome challenges and disagreements that emerge in their joint activity by constantly tuning to one another with mutually trust and respect and reflects the process of 'togethering' (cf. Radford and Roth, 2011). The analysis of segment 1 from K1 the turn-taking is mainly organised by 'current speaker addresses next speaker'-strategy. It is the current speaker that promote another participant to act, and thus move the activity by promoting another participant to do something. However, the activity does not move unless the selected next speaker accepts taking the next turn. This requires both trust and responsibility. The current speaker must trust the selected next speaker to take the turn, and the next speaker must repay this trust and take responsibility for acting. In K2 the movement of the activity happens through the participants self-selected turn-taking. By self-selecting turns the children take responsibility for moving the activity forward, and/or adjusting the direction of the activity. Although the children initially have different ideas (or different wishes) for how to solve the problem they tune to one another despite their differences and manages to compromise in how to solve the problem which is important for creating a space of joint action and intersubjectivity (cf. Radford and Roth, 2011).

The responsibility that the participants feel, I hold, is to some extent context dependent. In both cases the problem is formulated so that all children must participate. In the first segment all children are asked to run around the tree, and they are responsible for representing and counting their runs. In the second segment the children are asked to count their fingers, and each child is responsible for bringing their fingers into the joint activity of counting. The nature of the problems seems to make the children feel responsible for participating which is a crucial feature for 'togethering' (cf. Radford and Roth, 2011).

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Kindergarten teachers' orchestration of mathematical learning activities: the balance between freedom and structure

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This paper reports on a multiple-case study which focuses on four kindergarten teachers' orchestration of mathematical learning activities with respect to the degree of freedom, and what impact their orchestration has for children's mathematical learning possibilities. The study draws on Valsiner's (1987) zone theory to investigate the relationship between zone of free movement (ZFM) and zone of promoted action (ZPA) which the kindergarten teachers set up to canalise children's actions and thinking and thus development. The results show that in the kindergarten where the ZFM is gently set up and limited to mathematics, and where the kindergarten teacher sensitively sets up the ZPA and promotes children to share, argue for and explain their mathematical ideas, and explicitly promotes the children to collaborate, is where most problem-solving interaction occur and thus facilitate children's learning possibilities the most.

Keywords: Kindergarten, mathematics, orchestration, zone of free movement, zone of promoted action.

Introduction

This paper reports on a multiple-case study which aims to investigate four kindergarten teachers' (KTs') orchestration of pre-designed mathematical learning activities and what impact their orchestration has for children's learning possibilities. The current debate about mathematics in kindergarten is seldom about whether or not mathematics should be part of the curricula in kindergarten, rather on how mathematical activities in kindergarten should be orchestrated[1] (Gasteiger, 2012; Hirsh-Pasek, Golinkoff, Berk, & Singer, 2009; Van Oers, 2010). Children's opportunities to take part in mathematical discourses are important in learning mathematics, but in activities where the kindergarten teacher has a pedagogical aim, it may be difficult to balance teacher-talk and child-talk (Dovigo, 2016). The study reported here focuses on four KTs' orchestration of mathematical learning activities with respect to the degree of freedom[2] and aims to:

- Investigate the characteristics of four kindergarten teachers' orchestration of pre-designed mathematical activities with respect to the degree of freedom, and;
- Investigate what impact the four kindergarten teachers' orchestration of the mathematical activities has for children's mathematical learning possibilities.

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[1] The term 'orchestration' is used in accordance with Kennewell (2001), as a broad metaphor for how the KTs structure or organise the activity through use of questions, cues, prompts, information, demonstrations etc.

[2] The term 'degree of freedom' is used in accordance with van Oers (2014) as a characteristic of the way an activity (in cultural-historical activity theory) is carried out and refers to the "*degrees of freedom* allowed to the actor in the choice of goals, tools, or rules" (p. 113, emphasis in origin), which in turn initiates the actor's (choices of) actions.

Theoretical framework

The study reported here draws on Valsiner's (1987) zone theory to investigate the relationship between freedom and structure in four KTs' orchestration of mathematical learning activities. The balance between freedom and structure is the main focus when Hirsh-Pasek et al. (2009) and Weisberg, Kittredge, Hirsh-Pasek, Golinkoff, and Klahr (2015) discuss 'playful learning' as an educational approach in kindergarten. Playful learning captures both 'free play' where children play without interference from adults, and 'guided play' where the KT organises the learning environment and guides the play in desired directions with respect to a learning aim. Free play is child-initiated and child-directed play, whereas guided play is adult-initiated and child-directed play. Weisberg et al. (2015) argues that, although the KTs initiate and guide the activity in guided play, the KTs must make room for children's self-directed exploration. It is this balance between freedom and structure that makes guided play such an effective teaching tool. "Playful learning, and not drill-and-practice, engages and motivates children in ways that enhance developmental outcomes and lifelong learning" (Hirsh-Pasek et al., 2009, p. 4). Similar van Oers (2014) argues that playful learning activities should contain some elements of instruction. "The nature of the actions embedded in play can vary with respect to their degree of freedom allowed, as long as the activity as a whole remains a playful activity" (van Oers, 2014, p. 121). The learning activity must be engaging and give possibilities for the players freedom to explore the (mathematical) objects in their own manner.

In his study on preschool children's argumentation, Dovigo (2016) investigates children's learning opportunities in different types of conversations (peer-talk and child-teacher talk). Dovigo found that children had richer opportunities to participate and asked more questions in peer-talk than in child-teacher talk. It was a clear tendency that in child-teacher talk, the KT talked more than the children. However, the children's abilities to build arguments were limited in peer-talk and were facilitated in child-teacher talk. The KTs facilitated children's explanations and helped them to elaborate their argumentations, which again improved children's critical thinking and abilities to collaborate.

Through his zone-theory, Valsiner (1987) explores the development of children's actions and thinking through organisation of person-environment relationships. The physical environment of the child is the cultural frame which the child is acting within and thus develop its thinking. Valsiner's theory emphasis that both the developing child and the environment are structurally organised, however the structuring nature of the child and the environment is continuously and dynamically transformed. Valsiner (1987) uses three zone concepts to conceptualise the dynamic environmental structures that organise the child's development: Zone of Free Movement (ZFM); Zone of promoted Action (ZPA); and Zone of Proximal Development (ZPD). The ZFM is co-constructed by the child and the adult and organises the "child's (1) access to different areas in the environment, (2) availability of different objects within an accessible area, and (3) ways of acting with available objects in the accessible area" (p. 97). The ZFM canalises the child's development in culturally accepted ways.

The ZPA may be activities or objects which the child is promoted to engage with. An important characteristic of the ZPA is its non-binding nature. The child does not need to follow the ZPA and can act with other objects (in other ways) within the ZFM. The child cannot be 'forced' to accept the ZPA unless the ZPA is turned into ZFM. The ZFM and ZPA must be considered as a unit and Valsiner

(1987) labels it the 'ZFM/ZPA complex'. The ZFM/ZPA-complex work as a mechanism to canalise the child's actions and thinking and thus development. In addition, Valsiner (1987) discusses how ZPD relates to the ZFM/ZPA-complex, but due to space limitations, this study focuses primarily on the relationship between ZFM and ZPA in four kindergartens with respect to mathematics.

Methodology

The study reported here is a multiple-case study (Yin, 2014), which aims to characterise four KT's orchestration of pre-designed mathematical activities and its impact for children's mathematical learning possibilities. It is part of a larger study on mathematical teaching and learning in kindergarten and situated within a Norwegian research and development project called the Agder Project[3] (AP). The intervention of the AP was based on mathematical activities which were pre-designed in collaboration between researchers (including myself) and the KTs in the focus groups of AP. The activities are described in Størksen et al. (2018), a book containing one-page outlines of the activities. This study takes a qualitative approach to data collection and data analysis and the empirical material was collected over two observation periods (autumn 2016 and spring 2017) during the intervention of the AP. Observations were conducted of approximately 40 minutes sessions where the four KTs, who were part of the focus groups of the project, implemented the pre-designed activities. Interviews were conducted in each kindergarten after each observation period. Overview over data sets (observations and interviews) in each kindergarten is illustrated in Table 1 below.

	KT1	KT2	KT3	KT4
Autumn 2016	5 obs. + 1 interv.	4 obs. + 1 interv.	5 obs. + 1 interv.	3 obs. + 1 interv.
Spring 2017	6 obs. + 1 interv.	7 obs. + 1 interv.	0 obs. + 0 interv.	4 obs. + 1 interv.

Table 1: Overview over data sets (observations and interviews) in the four kindergartens

The empirical material was collected through ethnographic field notes (Emerson, Fretz, & Shaw, 2011) from observations and interviews. Field notes were written during and/or straight after each observed session and were supplemented when having conversations with the KTs after each session. Field notes from the interviews were made straight after the interviews and were extended when the video-recordings of the interviews later were watched. Field notes should not, ideally, include interpretations and judgements of the observed interaction (Emerson et al., 2011). However, as Emerson et al. (2011) argues, when the fieldworker starts to work with the field notes it automatically involves a preliminary analysis where the fieldworker order patterns of interactions and decides what to include and leave out from the field notes. In this research study the research questions were made before conducting the field work, which guided what I was especially looking for. In the field notes I included events (actions) that I thought were relevant and left out things that I thought were not that relevant and the field notes must therefore be regarded as interpretative or analytical.

Processing and analysing data went through an iterative process. Instead of using the field notes as 'raw material' from which I started a coding process, I worked directly with the field notes and carefully contracted them into profiles of the KTs orchestration and their interaction with the children. Since the field notes were already to some extent analytical, they served as a useful starting point for this purpose. The profiles concern four key points: (1) Children's access to different areas in the environment; (2) children's freedom to act (physically) within the accessible area; (3) children's freedom to participate with questions, mathematical ideas and argumentations etc.; and (4) the degree to which problem-solving interaction was promoted (that is the degree to which the children were promoted to ask questions and explain and argue for their ideas to solve mathematical problems). Key points 1-3 concern the KTs' orchestration of the mathematical activities with respect to the degree of freedom and intends the ZFM. Since the ZFM is co-constructed by the child and the adult, children's eagerness to participate is also used to identify ZFM. Key point 4 concerns how the KTs promote children to ask questions, explain and argue for mathematical ideas to solve mathematical problems, which intends the ZPA. As Valsiner (1987) argues, ZFM and ZPA are related and work as a unit to canalise children's development, and in this case, learning and development related to mathematics.

The profiles, which will be presented below, are of course tendencies not absolute characteristics. The KTs' orchestration with respect to the degree of freedom changes dynamically during each session and from session to session. In addition, the profiles are relative, which means that the degree of freedom in one kindergarten is relative to the three other kindergartens and cannot provide an indication for how it relates to other KTs' orchestration in other contexts.

Profiles of four kindergarten teachers' orchestrations

Kindergarten teacher 1

KT1 orchestrated the mathematical activities with a relatively high degree of freedom, which is based on the way that the children were allowed to move around in the room (and even walk out of the room) and to talk about almost whatever they wanted, like birthday parties or their parents' occupation etc. The KT never told the children to sit down and pay attention, instead she promoted the children to do so by the way she enthusiastically presented the activities, which captured the children's attention. For example, in an activity about reflection symmetry, the KT introduced the activity having diverse reflection symmetrical objects in a plastic bag without telling what was inside. She shook the bag and whispered, "Listen!", which made the children curious and created joint attention. Another characteristic of the KTs orchestration was that the KT listened to almost every child's contribution (not only related to mathematics). In one of the conversations with the KT, she expressed that her desire to listen to and appreciate every child's contribution could be a hinder for her, because her attention became very shifty. She rapidly turned her attention from one child to another. The children eagerly participated in the activities, however, as mentioned above, they often contributed with other ideas than mathematics. The KT expressed that she had a challenging group of children but their ability to pay attention to mathematics grew during the intervention.

There were a lot of 'golden moments' for problem solving interaction. The KT and the children initiated a lot of interesting 'topics' for investigation, but very few ideas were thoroughly discussed. Mathematical questions were often (not always) considered briefly, and the children seldom had to

ponder about problems and to express mathematical ideas, argue for and explain their ideas in order to solve the problem. Because the KT gave the children a lot of freedom to talk and payed attention to almost every contribution, the conversations moved very quickly from one topic to another.

Kindergarten teacher 2

KT2 gave the children relatively high degree of freedom to talk, however she often restricted children's talk to mathematics by ignoring some of the contributions that were about the children's everyday experiences. The KT also restricted the children's freedom to act (physically) to areas or with objects relevant to the mathematical activity. Although the KT for the most part focused attention to mathematics, she gave the children freedom to suggest other mathematical issues than what she initially introduced. Similar as KT1, the KT2 presented the activities in an exciting way, by use of for example excited face expressions and whispering, which captured the children's attention and promoted the children to contribute. But sometimes she also asked questions directly to children to capture their attention. For example, when Carl was distracted by something else, she said: "Carl, do you know how many building blocks there are in the red tower?" When Carl said that he didn't know, the KT further asked "Would you like to help me count?" This helped Carl, who often had difficulties paying attention, to focus his attention on mathematics.

The conversations between the KT and the children were almost always mathematical, and sometimes the KT and the children had longer conversations about mathematical problems. The children had to argue for and explain their ideas in order to solve the problems, and the children eagerly participated with mathematical ideas and explanations. In addition, the KT seemed to focus on collaboration. For example, the KT had a conversation with the children about the meaning of collaboration, and the KT promoted the children to help each other if needed. She also promoted the children to listen to each other by for example asking the group of children: "Did you hear what Ada suggested?".

Kindergarten teacher 3

KT3 was a football trainer in his spare time, which was somehow recognisable from his orchestration of the activities. He gave the children relatively high degree of freedom to act (physically) and focused on 'doing' mathematics, which for him was when the children got opportunities to use their hands, their body and various artifacts to solve mathematical tasks. In one of the conversations with the KT he expressed that 'doing' mathematics was for him an important feature of mathematics in kindergarten and therefore he especially liked physical outdoor activities. In addition, he was giving short 'missions' for the children to perform. For example, in the 'Sorting Shoes' activity, when the children had to figure out how many shoes there were in each category, the KT gave each child a 'mission' to draw equally many lines in the bottom of the diagram as there were shoes in each category. The KT expressed several times that it was important to give the children challenging but manageable tasks, so they felt they succeeded. He often encouraged the children, in an enthusiastic manner, with comments like "good" or "great" etc. It seemed that the children enjoyed the activities and the way that the KT encouraged. The children eagerly participated and were having fun.

There was relatively little problem-solving interaction and the children often solved tasks without having to explain or argue for their ideas. For example, in the activity called 'Tripp, Trapp', where the children should count stairs in a staircase and find out what number each stair had, the KT made

A4 papers with numbers from 1-24 on and the children, one by one, had to pick an A4 sheet and place it on the correct stair. (Stair number 15 should have the A4 sheet with the number 15 on). The children just performed the tasks, without having to explain what they did, and why they did what they did. Sometimes the KT promoted the children to reflect on their solution strategies in retrospect, however the children's explanations were seldom helping them to solve problems in the first place.

Kindergarten teacher 4

KT4 gave the children relatively little freedom to act (physically) or talk which is based on the way that she, to a large degree, controlled who was going to talk (or 'do' something) and when. For example, in an activity called 'The Farm' the children were, at one point in the activity, supposed to find how many animals there were on the farm. First the KT asked a girl, "Helene, can you figure out how many animals there are all together?". After Helene had counted and answered the KT asked a boy, "John, can you find how many different animals there are?". The KT continued to give similar 'missions' to each child. The KT made sure that each child got the opportunity to answer or 'do' something mathematically, and she appreciated children's contributions by comments like 'that's correct', 'very good' etc. The KT expressed in one of the conversations that it was important that the children learnt to respect the other children and to wait for their turn in an activity. The KT also expressed that some activities were difficult to implement as outdoor activities, because the children often got disturbed by other things. These characteristics are of course tendencies, and sometimes the activities were a lot more open where the children had a lot more freedom to act. But, as she also expressed in one of the interviews, she thought it was difficult to 'hold back' and give room for the children to figure out the problems themselves without too much interference.

It is difficult to state how 'eager' the children were to participate, because they seldom answered or did something without being asked. They accepted the KTs request to sit and wait for their turn. In some activities, like when they measured how much water there were room for in a tank, the children laughed and were having fun and showed eagerness to participate. Still they were asked to wait for their turn and respect that each child got the same opportunity to fill water. There were few incidents where the children together solved problems by expressing ideas and arguing for solutions. The children were waiting for their turn to answer or to perform 'missions'. The KT sometimes asked the children to explain what they did when they solved a task, but this explanation did not help the children to solve the problem, but to reflect on their strategy in retrospect.

Discussion

From the results above, it seems that KT1 is very concerned about freedom, and the ZFM is relatively wide compared with the ZFM the other KTs set up, both with respect to what the children are allowed to talk about and what they are allowed to do (physically). The children are even allowed to walk out of the room if they want to, and they can talk about whatever they want. KT2 restricts the ZFM to mathematics, both what the children can physically do and what the children are allowed to talk about. However, KT2 gives the children freedom to talk about and work with other mathematical objects than what she initially promotes. KT3 restricts the mathematical talk to specific mathematical areas, however the ZFM is relatively wide when it comes to what the children are allowed to do (physically). The KT3 gives the children freedom to move physically and to make loud voices when they solve the

mathematical tasks. KT4 is the most controlling of the four KTs, and the ZFM is relatively narrow both with respect to what the children are allowed to do (physically) and what the children are allowed to talk about. KT4 decides, to a large degree, who is allowed to talk (or 'do' something), when the children are allowed to talk (or 'do' something) and what the children are allowed to talk about.

Considering the ZPA, the results illustrate how the KTs promote children to ask questions, explain and argue for their ideas in order to solve mathematical problems. The children do not need to accept the ZPA set up by the KT, but instead of turning the ZPA into ZFM the KT may, I hold, 'advertise' for the ZPA to promote the child to act in a desired manner. In K1 the KT promotes children to act mathematically by acting in an exciting way, and by introducing the activities in a manner which makes the children curious. The children sometimes accept the ZPA, but sometimes they do not. The KT1 is carefully promoting the children to think mathematically, but the ZPA (related to mathematics) is never turned into ZFM. The KT2 also promotes children to think mathematically by acting in an exciting way. The ZPA is related to specific mathematical areas, however the ZFM is related to mathematics in general. Sometimes, especially related to some children, the KT2 carefully turns the ZPA into ZFM, that is the KT limits the ZFM to specific mathematical tasks whenever the children do not pay attention. In addition, the KT explicitly promotes the children to help each other and to collaborate which, I hold, is important for the way that the children solve problems together. KT3 promotes children to think mathematically or 'do' mathematics by a quite tight ZFM related to mathematics. The ZPA is often turned into ZFM by asking the children to perform 'missions'. However, the ZFM is relatively wide related to physical actions. KT4 almost always turns the ZPA into ZFM. What the KT promotes the children to do is also what the KT allows the children to do.

Considering the characteristics of ZFM/ZPA complex in each kindergarten which according to Valsiner (1987; 1997) canalise children's actions and thinking and thus development, it seems that the KTs who limit children's actions to mathematics, but where the children's freedom is relatively wide related to what the children may talk about within mathematics and who is allowed to speak, promotes most problem-solving situations (KT2), and thus children's opportunities for learning. How the ZFM/ZPA complex canalise children's development is not only related to the boundaries of the zones itself, but *how* the ZFM and ZPA are set up. The KT2 is relatively mild in her way of setting up the ZFM, and instead of turning the ZPA into ZFM she acts in an exciting way which promotes the children to accept the ZPA. The KT2 'advertise' for the ZPA by the way she presents the mathematical problems and makes the children want to pay attention and accept the ZPA. The KT4 is not that enthusiastic, and perhaps that is why she must turn the ZPA into ZFM to make the children pay attention and to accept the ZPA. The KT1 is also enthusiastic in setting up the ZPA, but since the ZFM is relatively wide, the children often choose to act in other ways than what the KT promotes. The results support Hirsh-Pasek et al. (2009), Weisberg et al. (2015) and van Oers (2014) who emphasise that playful learning activities should have some structure as long as the activity as a whole remains a playful activity and as long as the KTs gives freedom for children's self-directed play, so they may explore the content in their own manner. The results also supports Dovigo's (2016) results in the sense that whenever the KT is structuring the environment around mathematics, but opens up for children's own exploration around mathematical ideas, the children are canalised into more problem solving activity and thus create possibilities for children's mathematical learning.

The results indicate that instruction which structures children's actions around mathematics but introduces the mathematics in an 'exciting' way and allows and promotes children to contribute with various mathematical ideas not necessarily related to the aimed subject area, captures children's attention and promotes their voluntarily participation in the problem-solving activity and thus facilitates children's possibilities for mathematical learning.

Limitations of the study

This study focuses on the KTs' orchestration and its consequences for children's learning possibilities, and do not consider how children's (choices of) actions influence the KTs orchestration. Taking into consideration Vygotsky's (1987) dialectic approach on teaching and learning the children's choices of actions influence as much the KTs' action possibilities as the KTs' actions influence the children's action possibilities and their learning. KT1, expresses that she has a demanding group which suggests a possible reason why the conversations seldom take a problem-solving form. KT2 seem to have several high achieving children which suggests why the conversations more often take problem-solving form. This study does not consider how the children, in light of for example their background, influence the nature of interaction in each kindergarten.

In this study the ZFM/ZPA complex is considered on a group level. It would be interesting to investigate the ZFM/ZPA related to each child in the groups, to see which children benefitted the most from the KTs different orchestrations. Although it seems that there is overall more problem-solving interaction in KT2 than in the other three kindergartens this study does not reveal which children benefit the most from this type of interaction. This issue must be further investigated.

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