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IMAGINARY DIALOGUES: IN-SERVICE TEACHERS' STEPS TOWARDS MATHEMATICAL ARGUMENTATION IN CLASSROOM DISCOURSE

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Abstract: *The purpose of this qualitative study was to explore in-service teachers' first experiences with imaginary dialogues – a form of mathematical writing where students are introduced to a written and unfinished dialogue between two imaginary persons discussing a mathematical problem. Students are supposed to continue working with the problem and to complete the initial dialogue between these persons. In-service teachers were enrolled in a continuing university education mathematics course. They were given the task to try out imaginary dialogues in their classes from grades 4 to 10. Based on in-service teachers' responses in open-ended self-evaluation forms, the study examined how the in-service teachers perceived imaginary dialogues as a tool to approach students' mathematical argumentation. The study also sought to investigate how they identified levels of argumentation in their students' written dialogues based on the background of Balacheff's levels of proofs in school mathematics practices.*

Keywords: argumentation, written imaginary dialogues, mathematical reasoning.

Introduction

Proofs in school mathematics has in Norway traditionally been linked to upper secondary education. However, many researchers claim that the corresponding activity of proving should become part of students' mathematical experiences throughout the grades, and students should be made familiar with explaining and reasoning their ideas. Expressing oneself orally and in writing are basic skills in the sense that they are fundamental to learning mathematics (Ministry of Education and Research, 2013). Teachers in primary and lower secondary education are committed to make students familiar with explaining and reasoning in dealing with these two basic skills, but how can teachers support students making up their mind for asking questions, arguing, and explaining a process of thinking using mathematics, and engage them in arguing and justifying their solutions? In teacher education, we were looking for a teaching approach that may motivate mathematical reasoning and students' learning of argumentation and proving, and at the same time let teachers know how students reason. In that way, imaginary dialogues may come into play and raise teaching possibilities.

Imaginary dialogues have been introduced as a method to approach students' mathematical thinking process (Wille, 2017a). The starting point is a written dialogue in which two imaginary people are facing a mathematical problem. Students are then asked to proceed on this initial dialogue, writing a continuation of the initial dialogue while they are investigating the problem further. In a broad number of studies during the last decade explored the potential of individual dialogue writing to support students' ability to build a mathematical argumentation on different topics in mathematics (Wille, 2017a, 2017b). This exploration was done in different German classrooms with students aged 10-16, most of them 10-14 years. She found the method to initiate reflection processes and argumentation. Askevold and Lekaus (2018) applied the method of imaginary dialogues for working in small groups of 2–4 students, aged 10-12, in Norwegian classes. Analyzing students' construction of arguments and their conceptions about proofs as expressed in their written texts, they found differences between the mathematical methods and representations applied by the students from different classrooms and grades. Also, while the method of imaginary dialogues was developed for school children, it turned out to be helpful to detect different aspects of mathematical conceptions when used on single student teachers in pre-service teacher education in Austria (Wille, 2017b). Results for Lekaus and Askevold (2015) and Wille (2017a) showed that writing imaginary dialogues may help students develop their mathematical ideas and that traces of their own thinking appear in the continued dialogues between the imaginary students.

Seeing aspects of students' reasoning become apparent through their written dialogue inspired us to apply the method as a tool in further mathematics education to help in-service teachers to gain more insight into their students' conceptions, arguments, and their line of argumentation. Imaginary dialogues have been tried out by teacher educator-researchers analyzing students' continued dialogues (Askevold & Lekaus, 2018; Wille, 2017a). However, there is a lack of research on in-service teachers' implementing the method of imaginary dialogues in their classrooms. It remains to be explored how useful imaginary dialogues will prove for in-service teachers when implementing the method in their classrooms, which types of obstacles they will experience and which types of argumentation they will find in students' written dialogues.

In continuing education courses in mathematics, we emphasize the issue of reasoning, focusing teachers' challenges and roles, and encouraging in-service teachers to work with arguments and proving them in their classes. After an introduction to the idea of imaginary dialogues as a teaching approach in mathematical reasoning, they have been assigned to try it out in practice. The initial start dialogue provided for the task was designed to introduce the handshake-problem (see Procedures in the Method section for description of this problem).

This article is based on an analysis of in-service teachers' documentation of their first experience with implementation of imaginary dialogues in their classrooms in grades four to 12, and their analyses of mathematical texts written by their students. In contrast to the original method (Wille, 2017a) that required students to work individually, the students in the in-service teachers' classrooms worked in pairs.

Research Questions

This paper examines in-service teachers' work with the mission of stimulating and analyzing their students' argumentation and mathematical reasoning. In-service teachers' first experiences with implementation of imaginary dialogues may differ. This could, in turn, have important implications for teachers' possible subsequent use of the method. We therefore sought to answer the following research questions:

1. How did in-service teachers perceive their first experience with imaginary dialogues in their classroom?
2. Which types of obstacles did in-service teachers experience when implementing the method of imaginary dialogues?
3. Which types of argumentation could in-service teachers find in their students' written imaginary dialogues?

From answers to these questions teacher educators may learn possible pitfalls and opportunities teachers who new to the method may meet. In this way, the teacher educators can help them prepare for "look fors" and better succeed with the new teaching possibility.

Theoretical Framework

The issue of what mathematics education researchers mean by proof and proving, and the meanings that proof may have for learners have been widely studied in the last few decades (Hanna, 2000; Harel & Sowder, 2007). There is no common definition of argument and evidence and the relationship between them within mathematics education literature. Stylianides (2007) defined a mathematical argument as “a connected sequence of assertions intended to verify or refute a mathematical claim” (p. 2) emphasizing the argument structure and convincing power. Harel and Sowder (2007) stressed a subjective evidence perspective when they defined “a proof is what establishes truth for a person or a community” (p. 806), closely relating argumentation, reasoning and evidence to an interaction context and emphasizing what is felt as a compelling argument.

Focusing on students’ exposition to fully developed and logical deductive proofs is argued not to have the greatest potential in elementary school, and as such, teaching should focus on forming arguments and communicative aspects of the evidence in proof-similar activities throughout the grades (Stylianides, 2007; Yackel & Hanna, 2003). Facilitating a communication in mathematics class based on arguments and justifications has been found both to contribute overcoming the misconception that empirical arguments are proofs and to recognize the need for proof (Schwarz, Hershkowitz, & Prusak, 2010; Stylianides & Stylianides, 2009). A sufficient argument in class may meet certain criteria: (a) it is based on the established statements and definitions that are generally accepted in the community of the classroom; (b) it makes use of forms of reasoning that are valid, known to the students or within their conceptual reach; and (c) it communicates with forms of expression suitable and understandable to the students (Stylianides & Ball, 2008).

Within the setting of a learning community, Balacheff (1988) considered what a proof may entail. Using the term evidence in a broadest sense, since students feel the conjecture proved, he distinguished between pragmatic and conceptual forms. A proof is identified as pragmatic if depending on actions or visual representations. A conceptual evidence, however, rests on the formulation and the connections between the relevant properties of conjecture. He proposed four proof levels: (1) naive empiricism; (2) crucial example; (3) generic example; and (4) thought experiment. Their hierarchical relationship is based on the degree of generality and how much conceptualization of knowledge they require. The first three levels are all examples of pragmatic justifications. On the level of naive empiricism or “proof by example”, the learner concludes based on only a small number of cases, while on the level of crucial experiment, the learner tests the conjecture with an example well outside the range so far considered to explore the extent of its validity. While proofs on these levels do not establish the truth of an assertion, generic examples indicate the level in which the assertion is made explicit using a prototypical case where an object is chosen not on its own but as a characteristic representative of its class. A generic example, verbal or symbolic, involves properties and structures, and encompasses the justification of generality. Thus, while not being a strict mathematical proof, the term proof is used. The level of thought experiment then is a conceptual justification detached from any examples where the learner arrives at structured deductive logical forms based on the use of formalized symbolic expressions. Balacheff (2010) reasoned that learners during the proof-making process likely will go through several of these levels, become aware of the necessity to produce valid arguments, and over time slightly move in this direction while developing their language to become a tool for formal evidence. The four terms describe very specific mathematical approaches learners adopt in the proof process, making them useful in the analysis of students' reasoning and line of argumentation.

Method

The method section describes the procedures used for initiating and structuring in-service teachers' implementation of imaginary dialogues in their classrooms, including the nature of assignments they were given. It describes procedures for planning and receiving their feedback on their first experience with imaginary dialogue as well as on their first approach to identifying levels of argumentation in student pairs' written dialogue.

Participants

Subjects are in-service teachers enrolled in a national program of continuing university mathematics education for teaching staff under the strategy Competence for Quality in Norway. The program applies to teachers with teaching certificates who already work as teachers, provides scholarships for taking further training, and an exam to meet new qualification requirements for teaching mathematics in primary and lower secondary school.

A purposive sample was used with all in-service teachers in one class being willing to participate in the study. Forty-three (43) of 53 in-service teachers gave their informed consent to participate in the study, while 10 did not. These teachers (27 male and 16 female) ranged in age from 27 to 64 (mean 43.6, median 42.0, mode 41.0) and attended Algebra, Number Theory, Geometry, and Didactics (15 ECTS) as the first part of the Year 1 mathematics program delivered entirely online, autumn 2016. Their classes ranged mainly from grade 4 to 10 with seven in upper primary, 35 in lower secondary level, plus one of grade 12**.** In terms of ethics, in-service teachers' written assignments were anonymized and cleared for any biographical data before analysis.

The Imaginary Dialogue Assignment

The basic mathematical handshake-problem is: How many handshakes will there be if each person in a group shakes the hand of every other person once? Several strategies may be applied to solving the problem. The problem allows to be adjusted to fit in primary and secondary school classes from an arithmetic problem to an algebraic problem by varying the number in the group. First primary graders may work on the task to figure out how many handshakes there would be if all students in class shook hands and act out the few simple cases with say two, three, and four people to find the path to a pattern. Progressing to the algebraic generalization would fit for secondary school level when asked for the number of handshakes for any group of *n* people.

The start dialogue, "Shaking hands", was designed to introduce the handshake-problem and to stimulate entering the dialogue and developing it further along with exploring the problem and arguing for findings. It is as follows:

Knut: Imagine how many handshakes it would be if everybody shook hands.

Idunn: That would be a lot of handshakes!

Knut: If you and I shake hands, it would be one handshake.

Idunn: Yes. Let us shake hands with one more person. Both of us shake hands with him. That makes another two handshakes. How many handshakes all together?

Knut: 2 + 1 = 3. Ok, we do the same with one more. How many handshakes all together now?

Idunn: I believe it is 6 handshakes.

Knut: What about five persons shaking hands? Or ten? Can we find out how many handshakes that will be? How many handshakes will it be when 100 persons shake hands?

Idunn: Oh - may be drawings or tables can help us.

In-service teachers were instructed to let their students work in pairs and collaboratively, without teacher involvement, in a problem-solving and reasoning context. The strategy suggested each of the student pairs had to continue the initial written dialogue in form of an imaginary dialogue, and help Knut and Idunn to come with a solution. Students were instructed to write without removing any once-written text, with deletions only indicated by striking through, so that the in-service teacher could read how the students were reasoning.

Setting and Procedures

The authors, in the role of in-service teacher trainers, gave the teachers a mandatory assignment and then used the assignments as data in a study. In-service teachers were assigned to plan and conduct a teaching session where they attempt to apply a modified method of imaginary dialogues in their classroom, presenting the “Shaking hands” - dialogue between Knut and Idunn and letting students continue working on the mathematical problem in pairs. Students writing and working in pairs and collaboratively, not individually, makes it a modification of the method of imaginary dialogues used by Wille (2017a).

The assignment included inviting a colleague to observe and video record parts of the session, and the in-service teachers self-reporting on their teaching experience, taking into consideration the feedback and insights offered by the peer. Observing, analyzing, evaluating, reflecting, and reporting entailed communication between observed in-service teacher and observer. The process was based upon Self-Assessment of Teaching Statement [SATS] (Spicer-Escalante & deJonge-Kanna, 2016), a guided teacher observation model which combines both self- and peer-observations. The SATS-approach is chosen to open for and force discussions and reflections between colleagues about the implementation of imaginary dialogues.

After the session as part one of the task, in-service teachers had to report on their experience, and as part two of the task, to pick up one of their students’ dialogues and identify any mathematical arguments based on Balacheff’s four proof levels.

In preparation, in-service teachers were introduced to Balacheff’s distinction of mathematical reasoning in school mathematics by a video providing characteristics for each level as well as exemplifying how students may argue for the sum of two odd numbers to be even on the respective level. In-service teachers were also introduced to the idea of imaginary dialogues as a method to get students started and develop students’ argumentative skills in the classroom. They were offered six examples of dialogues, called “start dialogues”, among them the one on the handshake problem to be used in the task.

Data Collection

The mandatory two-part assignment made up the data for this study. Data related to part one answered research questions one and two, and data related to part two answered research question three.

As part one of the assignment, the in-service teacher integrated self- and peer-observation notes responding to each point of a self-assessment form, called SATS-form:

1. What was happening in the teaching session?
2. What was agreed to be observed?
3. Aspects that went well.
4. Aspects that could be improved.
5. Lessons learnt from doing observations of teaching.

Point five of in-service teachers' responses related to the introduction and use of imaginary dialogues constituted the primary data for subsequent analysis. Other statements in their responses served as background information.

For part two of the assignment, in-service teachers' reflection notes, including the analysis of one of their student pairs' written dialogues based on Balacheff's four proof levels with the dialogues themselves attached, established the research data for the third research question. Both in-filled SATS-form and reflection notes were to be submitted in the learning management system by the deadline.

Data Analysis

NVivo software was used to help manage and organize the qualitative data from the SATS-form. Coding included identifying and classifying positive, negative, neutral, and mixed statements concerning aspects of the imaginary dialogues technique, or labelling "not shared" (N/A) if experiences and perceptions were not reported. Statements expressing favorableness were coded positive, and statements expressing skepticism or doubt were coded negative. Statements sharing experiences but not perceptions, saying that neither side of the experience is strong enough to sway to that side (neither to favorableness nor to skepticism or doubt) belonged in the neutral category. Statements of being indecisive as to whether their experience was positive or negative were coded to the mixed category. If an in-service teacher stated two different views, such as being positive about one aspect and negative about another, this was not coded as mixed but rather multiple perceptions. While obstacle was a code emerging directly from a research question, memorable was used to mark quotes found illustrative for aspects of the research. Data chunks associated with each code, or combination of codes, were then grouped in NVivo and exported to Excel for further formatting, reading, and analysis. Two researchers independently performed the coding of data, then comparing and aligning was used to improve the validity of coding.

Classifying by coding and identifying themes in-service teachers' analyses was performed in two steps. First, researchers labelled according to which of Balacheff's four levels the in-service teachers identified in their students' written dialogue. Upon this labelling, cross tabulation was used to provide a basic picture of the interrelation between students' grade and level of proof as revealed by their teachers. Second, both independent researchers looked for interesting and often reported findings from in-service teachers' analyses as well as exceptional findings. They wrote a summary of observations for each case and made coding suggestions. When discrepancies were found, they were discussed until consensus was reached.

Results

Perceptions

Even though 11 of 43 participating in-service teachers did not respond explicitly by sharing perceptions on imaginary dialogues on the SATS-form, as they and their observers had agreed upon other teaching aspects to be observed, most of the in-service teachers expressed positive experiences concerning aspects of the imaginary dialogues technique as to their first-time implementation. As Table 1 shows 62.5% of those who made any explicit statement about the method as being positive, 9.4% neutral, and 6.3% negative, while 21.9 % were indecisive as to whether their experience was positive or negative.

Table 1
Perceptions of Imaginary Dialogues

Statement	Responses		
	Frequency	Percentage	Percent Excluding N/A
Positive	20	46.5%	62.5%
Neutral	3	7.0%	9.4%
Negative	2	4.7%	6.3%
Mixed	7	16.3%	21.9%
N/A	11	25.6%	

N = 43; n = 32 for excluding the N/A

Qualitative Perception Data: Teachers' Written Statements

Positive:

- The method is very engaging for all students. Quite clear that the task and method were manageable for all students and challenging enough to stop them being bored. A bit surprised about the stamina of the students, even though some need extra encouragement and support along the way. [...] This is not a "routine" session, neither for the teacher nor student. The session needs good planning, especially regarding the summary. The students seemed to be very positive about sharing experiences regarding strategies, where the focus should be process, not result. [...] The students find a certain satisfaction in "discovering" the solution on their own. (grade 8)
- Imaginary dialogues seem to be a great way to get many students active. We have used thinking-writing previously, but not imaginary dialogues. Would absolutely try the method several times, in other subjects too. Many students took part in making their own proofs to a degree that they have never done before. We were both quite sure about the sum of engagement in the class was greater than in a "normal" session in mathematics. (grade 9)
- Trying out [the method] was interesting and informative, and an eye-opener when the best students in mathematics, also competent writers, got so little on paper. [It] requires good preparation and good knowledge by the teacher, and a willingness to encourage the students to go further if they are stuck. [But we] envision a great advantage for students getting used to this method, starting with shorter sessions. (grade 10)
- [The writing task] was open and exploratory, without any constraints on how to think or argue, and no solutions proposed. I felt engaged and curious about what students would spot. Presenting their dialogue in class, they likely felt a sense of mastery when getting a confirmation on their good and reasonable ideas. (grade 8).

Negative:

- The students had great problems creating dialogue, nothing productive came from it. They preferred to try without the dialogue. (grade 8)
- Very few or none, focused on the actual dialogue. (grade 10).

Neutral:

- The students understood well that the task was to think-write the dialogue between Knut and Idunn. I want to try the approach in grade 9. But, this method raises the “eternal” question: how do we engage all the students, and how do we ensure that everybody understands the solution? Have we managed to engage everybody if they all at least take part in testing though shaking hands? And has everybody understood the solution just because they have been participants in the group who together have used reason to arrive at the answer? (grade 7).

Mixed:

- The approach was quite demanding in the class with several unconcentrated students and some who had problems in receiving messages and easily to be distracted by irrelevant inputs or incidents, as to an upcoming test in a language course. What appeared most clearly, was that many students argued mathematically, but had problems writing the argumentation as text. One reason might have been that this [way of working] was new to the students. (grade 8)

In-service teachers being positive shared considerations to keep in mind when implementing the method. One of three suggested imaginary dialogues to be a variant of process-oriented writing pedagogy, where emphasis is on how writing can help us reason and develop. Referring to this first glimpse in a mathematics context and with many students responding well, their most frequent consideration was that more practice will be needed to be confident and proficient with the method. Some in-service teachers, in grade 8, also considered imaginary dialogues to give great room for differentiated teaching.

Obstacles

A content analysis of the responses reveals four categories of obstacles to using imaginary dialogues: (a) time and workload pressure, (b) lack of experience and training, (c) talking instead of writing, and (d) misconception. These categories are listed in descending order of the frequency with which they were mentioned in point five in the SATS-form.

Time and workload pressure (a) referred to general time squeeze and the amount of the teaching hours that could be devoted to inquiry and writing imaginary dialogues throughout the school year. Examples included, “This takes time and is at the same time very important for the student’s [sic] comprehension” (grade 8); and “Many competence [sic] aims have to be achieved in the three years in secondary school. This is the reason why it often feels hectic in the mathematics education because we feel that we do not have the time to stop and wonder, elaborate and reflect” (grade 8).

Lack of experience and training (b) pertained to the method being new for both in-service teachers and students, or to any considerable prior approaches to inquiry and reasoning in the class. Utterances in this category leave the impression of in-service teachers seeking to explain results they perceived less successful than expected. Most of these teachers also expressed their expectation on doing better next time.

Talking instead of writing (c) referred to in-service teachers conceiving of students’ written dialogue as being short compared to the dialogue the students had when talking to each other or not focused on by the students. For instance, in-service teachers described, “There were many good dialogues, but they were never put on paper” (grade 10); “The students became too focused on the task and forgot the dialogue” (grade 10); “Even on as high a level as grade 10, this was more difficult than I had imagined, or maybe just because of this. We are so drilled in how we work with mathematical problems, that a task like this with writing added, is perceived as giving extra work. The students (and the teacher) are so focused on solution" (grade 10); "Had a bit too much focus on the students talking together, and the thinking-writing got less priority. Believe the learning outcome of the lecture was just as good, but would have liked to see more how the students thought and wrote it down" (grade 9).

Misconception (d) related to the nature of the task or the type of response expected. For one in-service teacher students' interpretations of the task were the biggest obstacle she faced. "Focus was declared to be on presenting the process, not the answer. [However,] One should not underestimate the possibility of misunderstandings or alternative interpretations of the tasks" (grade 8). One obstacle noted is on students' perception of what is an adequate response: "It took time to get the students to understand. I had [...] several times to stop for a moment in the session to explain better some information. [Some] were too intent on getting an answer [...] They immediately worked out a formula [...] and did not move on to prove anything else. The formula worked and that was enough" (grade 10).

Types of Argumentation

As recorded in their statements (SATS-form) and reflection notes, all in-service teachers encouraged their students to explain and validate their findings in their written continued dialogue, though none explicitly demanded a conclusive proof. In-service teachers found their students establishing and entering a reasoning process, and identified one or more of Balacheff's levels of proof in the imaginary dialogue they picked for the task, as the cross tabulation, Table 2, shows.

Table 2
Balacheff's Levels of Proof in the Samples

Grade	n^a	Balacheff's Levels			
		1	2	3	4
4	1	1	1	0	0
5	2	2	0	0	0
6	1	1	0	0	0
7	3	2	1	0	0
8	13	10	9	1 (+1)	0
9	10	6	4	4 (+3)	1
10	12	8	4	2 (+1)	3
12	1	0	0	1	0

Note: n^a = number of dialogue samples. Count identification is listed under each level; counts with reservations are in parentheses.

Types of levels used in all grades, except grade 12, are the lowest two: naive empiricism (level 1) and crucial example (level 2). Generic example (level 3) was identified in eight student pairs' dialogues from grade eight and higher. In addition, in-service teachers made some reservation in identifying level 3 in five additional student pairs' dialogues in grades 8-10 because argumentation on one side comprised both finding a pattern and using empirical methods to generate a hypothesis and formulated in their own words or visualized;

while, on the other side was no justification for why the pattern holds. Those in-service teachers stated that the dialogue moved towards level 3, but not fully reaching this level. Four in-service teachers claimed that the dialogue they picked included argumentation on level 4, a thought experiment, all written by students of grade 9 or 10. These dialogues included formulas for handshakes, $H(n) = 1 + 2 + 3 + \cdots + (n-1)$ or $H(n) = \frac{n(n-1)}{2}$.

Most of the dialogues collected follow a typical pattern: In an initial part Knut and Idunn established a basis of knowledge, followed by exploring. Due to this basis, some stated a hypothesis and explored it, while others got stuck. In-service teachers found numerical, graphical, and/or symbolic representations used in students' argumentations. They found arguments and hypotheses expressed numerically, diagrammatically, graphically, and/or symbolically, and some students reasoning out a formula for handshakes. In-service teachers expressed that the dialogues in some cases let them know how students reason; however, in other cases the dialogues did not, as in the Figure 1 example.

The in-service teacher placed this dialogue on proof level 2. He justified it with the students "testing against an example and illustrating their attempt with a clear figure." Furthermore, "a formula is presented, but the students do not show how they think." The in-service teacher believed that students are "well on the way toward level 3" because they showed general mathematical relations, but also "used concrete examples."

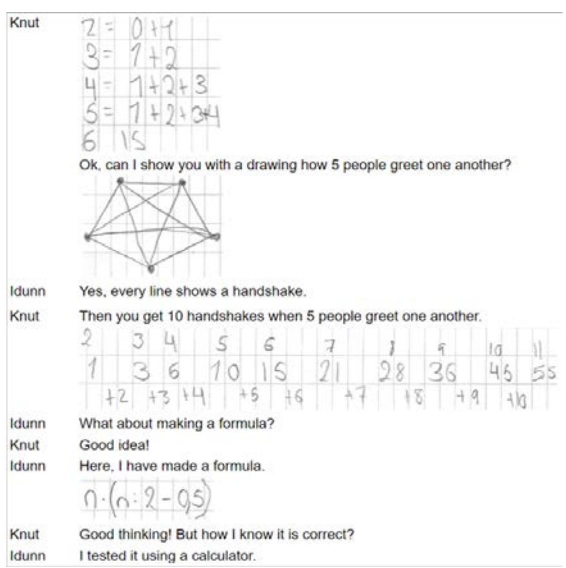

Figure 1. An imaginary dialogue, grade 10 (translated into English)

Discussion

Most in-service teachers were positive when implementing imaginary dialogues, envisioned the potential and will use further. Notwithstanding, this teaching approach needs getting used to, careful preparation, and classroom time. While Lekaus and Askevold (2015) found and discussed obstacles in the students' process as to what hampered students to come to a conclusion in their texts, this paper looked into obstacles that teachers met when implementing imaginary dialogue in their classes. The analysis reveals four categories of obstacles to using imaginary dialogues. The first two, time and workload pressure and lack of experience and training, are best understood as a kind of excuse teachers use for why a lesson did not go as expected. The two others, however, talking instead of writing and misconception, are important to bring along when implementing the method. Teachers may expect challenges in keeping students on the track to achieve the best possible yield. Though verbal communication between students may be constructive, without writing it will not help their teacher in gaining insight into their thinking. Also, teachers may have to deal with students' task understanding to avoid unnecessary constraints.

We found evidence that the handshake problem and the approach using imaginary dialogues allowed in-service teachers to discuss possible student strategies and mistakes they might make. In-service teachers found that fragments of mathematical reasoning and argumentation could be traced in some of the written dialogues, which is consistent with Wille (2017a) and Lekaus and Askevold (2015). In the samples, teachers identified students' path of argumentation and one or more of Balacheff's levels of proof in the imaginary dialogue. Our identifications, however, do not always concur with the teachers' identifications. One example is the teacher placing the imaginary dialogue in Figure 1 on Balacheff's proof level 2: crucial experiment. While the students test on a batch of examples, they did not explore the validity of an example well outside the range of the series. Another example is four teachers identifying Balacheff's level 4, when students came up with formulas though we do not see neither any evidence of verbal argumentation, nor by complete induction, a competence goal in upper secondary education.

Although we admit that the reasons for the difference of level identification are not clear, it could be argued that these differences might reflect that in-service teachers seem to note an expectation to find higher levels of mathematical argumentation as their students' progress through school. It might also reflect diverse teaching approaches applied in the different classrooms. The students' process requires the practice of mathematical reasoning and a specific state of knowledge. This result is largely mirrored by Table 2 and consistent with previous studies (i.e. Balacheff, 1988). According to Balacheff (1988), the practice of a level 4 proof would involve a commitment to a rigorous, theoretical problem-solving approach, not only the use of formulas, which is a level not intended for students in Norwegian primary and lower secondary education.

More room for reasoning, argumentation, and proving in the classroom does not mean teachers merely cultivating formal arguments but strengthening reasoning that may bring out mathematical proof ideas. In elementary school, we consider generic and generalizing arguments as a form of evidence. The focus is on exploring problems and statements, arguing for findings and developing students' capacity to discuss and communicate mathematical ideas, and understanding mathematical concepts more deeply. This continuum begins gradually then by the time students are in upper secondary education, informal reasoning may be expanded to formal proofs.

Generic thinking involving characteristic properties and structures may help seeing through the particular to the general. We have seen that the handshake problem opened for figurative and numerical generalization. Arguments for how the number of handshakes increases for groups of five, six, seven, eight ... people (i.e. for a series of single examples) could be used on any number of persons and handshakes between them. The way of thinking will be transferable from a "generic evidence" with prototypical, generalizable structure to an algebraic proof.

A limitation of this work is considered collecting data with the SATS-form, as students could leave the issue of imaginary dialogues unobserved, thus lowering the number of informants for research question one and two. However, all in-service teachers analyzed one dialogue from their class.

Conclusion

We conclude the method can be potentially useful for teachers, not only on individual students (Wille, 2017a), but also in a pair context, as also found in a similar research (Lekaus & Askevold, 2015). Taking a longer view, we see in-service teachers benefit in being introduced to imaginary dialogues as a tool in their continuing university education mathematics courses. The lecturer's individual feedback to their analyses may contribute to in-service teachers gaining more insight into mathematic-didactical theories on reasoning and proving.

Future Work

Based on the data sample of this work, a number of students' imaginary dialogues are suggested to be paired with their in-service teachers' analysis, to analyze in more detail how these dialogues let the in-service teachers know how students reason. Further, future research may include case studies undertaken on in-service teachers' first implementation of imaginary dialogues in their classroom. A suggestion is to observe a small number of in-service teachers and follow them more closely, for instance, by interviews, both after their intervention activities and after having identified levels of argumentation in their students' dialogues.

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