



## Research Article

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# Two properties of Müntz spaces

<https://doi.org/10.1515/dema-2017-0025>

Received June 14, 2017; accepted August 21, 2017

**Abstract:** We show that Müntz spaces, as subspaces of  $C[0, 1]$ , contain asymptotically isometric copies of  $c_0$  and that their dual spaces are octahedral.**Keywords:** Müntz space; Asymptotically isometric copy of  $c_0$ ; Octahedral space; Diameter 2 properties**MSC:** 46E15, 46B04, 46B20, 26A99

## 1 Introduction

Let  $\Lambda = (\lambda_k)_{k=0}^{\infty}$  be a strictly increasing sequence of non-negative real numbers and let  $M(\Lambda) = \overline{\text{span}}\{t^{\lambda_k}\}_{k=0}^{\infty} \subset C[0, 1]$  where  $C[0, 1]$  is the space of real valued continuous functions on  $[0, 1]$  endowed with the max-norm. We will call  $M(\Lambda)$  a Müntz space provided  $\sum_{k=1}^{\infty} 1/\lambda_k < \infty$ . The name is justified by Müntz' wonderful discovery that if  $\lambda_0 = 0$  then  $M(\Lambda) = C[0, 1]$  if and only if  $\sum_{k=1}^{\infty} 1/\lambda_k = \infty$ .

It is well known that  $C[0, 1]$  contains isometric copies of  $c_0$  (see e.g. [1, p. 86] how to construct them) and that its dual space is isometric to an  $L_1(\mu)$  space for some measure  $\mu$ . The aim of this paper is to demonstrate that Müntz spaces inherit quite a bit of structure from  $C[0, 1]$  in that they always contain asymptotically isometric copies of  $c_0$ , and that their dual spaces are always octahedral. (An  $L_1(\mu)$  space is octahedral. See below for an argument.) Let us proceed by recalling the definitions of these two concepts and put them into some context.

**Definition 1.1.** [2, Theorem 2] A Banach space  $X$  is said to contain an *asymptotically isometric copy* of  $c_0$  if there exist a sequence  $(x_n)_{n=1}^{\infty}$  in  $X$  and constants  $0 < m < M < \infty$  such that for all sequences  $(t_n)_{n=1}^{\infty}$  with finitely many non zero terms

$$m \sup_n |t_n| \leq \left\| \sum_n t_n x_n \right\| \leq M \sup_n |t_n|,$$

and

$$\lim_{n \rightarrow \infty} \|x_n\| = M.$$

R. C. James proved a long time ago (see [3]) that  $X$  contains an almost isometric copy of  $c_0$  as soon as it contains a copy of  $c_0$ . Note that containing an asymptotically isometric copy of  $c_0$  is a stronger property, see e.g. [2, Example 5].

**Definition 1.2.** A Banach space  $X$  is said to be *octahedral* if for any finite-dimensional subspace  $F$  of  $X$  and every  $\varepsilon > 0$ , there exists  $y \in S_X$  with

$$\|x + y\| \geq (1 - \varepsilon)(\|x\| + 1) \text{ for all } x \in F.$$

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This concept was introduced by G. Godefroy and B. Maurey (see [4, p. 118]), and in [5] the following result can be found on page 12:

**Theorem 1.3** (Deville-Godefroy). *Let  $X$  be a Banach space. Then  $X^*$  is octahedral if and only if every finite convex combination of slices of  $B_X$  has diameter 2.*

By a slice of  $B_X$  we mean a set of the form

$$S(x^*, \varepsilon) := \{x \in B_X : x^*(x) > 1 - \varepsilon, \varepsilon > 0, x^* \in S_{X^*}\}.$$

*Remark 1.4.* As we have mentioned, Theorem 1.3 can be found, but without proof, in [5]. Deville had proven in [4, Theorem 1 and Proposition 3] that if  $X$  is octahedral, every finite convex combination of  $w^*$ -slices of  $B_X$  has diameter 2. In the same paper he asks if the converse is true (Remark (c) on page 119). Since there is no proof included in [5], new proofs appeared, independently, in [6] and [7], in connection with a new study of spaces where all finite convex combination of slices of  $B_X$  has diameter 2.

When we show that the dual of Müntz spaces are octahedral we will use Theorem 1.3 and establish the equivalent property stated there. Note that an  $L_1(\mu)$  space is octahedral. Indeed, the bidual of such a space can be written  $L_1(\mu)^{**} = L_1(\mu) \oplus_1 X$  for some subspace  $X$  of  $L_1(\mu)^{**}$  (see e.g. [8, IV. Example 1.1]). From here the octahedrality of  $L_1(\mu)$  is a straightforward application of the Principle of Local Reflexivity.

The main reference concerning Müntz spaces is [9]. But there most of the phenomena that are studied are linked to spreading properties of  $\Lambda$  and not general results concerning all Müntz spaces.

We do not know of much research in the direction of our results. But we would like to mention a paper of P. Petráček ([10]), where he demonstrates that Müntz spaces are never reflexive and asks whether they can have the Radon-Nikodým property. Since the Radon-Nikodým property implies the existence of slices of arbitrarily small diameter, we now understand that Müntz spaces rather belong to the “opposite world” of Banach spaces.

See also Remark 2.9 for some more related results.

## 2 Results

**Definition 2.1.** We will say that a strictly increasing sequence of non-negative real numbers  $(\lambda_k)_{k=0}^\infty$  has the *Rapid Increase Property (RIP)* if  $\lambda_{k+1} \geq 2\lambda_k$  for every  $k \geq 0$ .

We will call a function of the form

$$p(x) = x^\alpha - x^\beta,$$

where  $0 \leq \alpha < \beta$ , a *spike function*.

*Remark 2.2.* If  $\alpha > 0$  it should be clear that any spike function  $p$  satisfies  $p(0) = p(1) = 0$ , attains its norm on a unique point  $x_p$ , is strictly increasing on  $[0, x_p]$ , and strictly decreasing on  $[x_p, 1]$ . To visualize the arguments that come, we think it is a good idea at this stage to draw the graphs of e.g.  $x^{100} - x^{200}$  and  $x^{1000} - x^{20000}$ .

We will need the following result below.

**Lemma 2.3.** *Let  $(\lambda_k)_{k=0}^\infty$  be an RIP sequence and  $(p_k)_{k=0}^\infty$  the sequence of corresponding spike functions  $p_k(x) = x^{\lambda_k} - x^{\lambda_{k+1}}$ . Then  $\inf_k \|p_k\| \geq 1/4$ . Moreover, the sequence  $(p_k/\|p_k\|)_{k=1}^\infty$  converges to 0 weakly in  $M(\Lambda)$ .*

*Proof.* We want to find the norm of the spike function defined by

$$p_k(x) = x^{\lambda_k} - x^{\lambda_{k+1}}.$$

Observe that  $r_k(x) := x^{\lambda_k} - x^{2\lambda_k} \leq p_k(x)$  for all  $x \in [0, 1]$ . Now, by standard calculus,  $r_k$  attains its maximum at  $x_k$  where  $x_k^{\lambda_k} = \frac{1}{2}$ . Thus

$$\|p_k\| \geq r_k(x_k) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

As  $(p_k)_{k=1}^\infty$  converges pointwise to 0 and  $\inf_k \|p_k\| \geq 1/4$ , the sequence  $(p_k/\|p_k\|)_{k=1}^\infty$  converges pointwise to 0 and thus weakly to 0 as it is bounded. □

*Remark 2.4.* By standard calculus one can show that the point at which  $p_k$  in Lemma 2.3 obtains its norm is  $\bar{x}_k = (\lambda_k/\lambda_{k+1})^{1/(\lambda_{k+1}-\lambda_k)}$ . For sufficiently large  $\lambda_k$  it is straightforward to show that

$$y_k := 1/(\lambda_{k+1} - \lambda_k)^{1/(\lambda_{k+1}-\lambda_k)} \leq \bar{x}_k,$$

that  $y_k$  is strictly monotone, and that  $y_k$  converges to 1 ( $\lambda_k \geq 3$  is sufficient).

**Theorem 2.5.** *The dual of any Müntz space is octahedral.*

*Proof.* Let  $M(\Lambda)$  be a Müntz space. Let

$$C = \sum_{j=1}^n \mu_j S(x_j^*, \varepsilon_j),$$

where  $\sum_{j=1}^n \mu_j = 1$ ,  $\mu_j > 0$ , and  $S(x_j^*, \varepsilon_j)$ ,  $1 \leq j \leq n$ , is a slice of  $B_{M(\Lambda)}$ . We will show that the diameter of  $C$  is 2 (cf. Theorem 1.3). To this end, start with some  $f \in C$  and write  $f = \sum_{j=1}^n \mu_j g^j$ , where  $g^j \in S(x_j^*, \varepsilon_j)$ . Let  $(\lambda_k)_{k=0}^\infty$  be an RIP subsequence of  $\Lambda$  (which is possible as  $\sum_{k=1}^\infty 1/\lambda_k < \infty$ ) and put

$$\begin{aligned} h_k^{j+} &= g^j + (1 - g^j(x_k)) \frac{p_k}{\|p_k\|} \\ h_k^{j-} &= g^j - (1 + g^j(x_k)) \frac{p_k}{\|p_k\|} \end{aligned}$$

where  $(p_k)_{k=0}^\infty$  is the sequence of spike functions corresponding to  $(\lambda_k)_{k=0}^\infty$  and  $x_k$  the (unique) point where  $p_k$  attains its norm. We will prove that, for any  $\varepsilon > 0$ , there exists a  $K = K(\varepsilon)$  such that whenever  $k \geq K$  we have  $\frac{1}{1+2\varepsilon} h_k^{j+}, \frac{1}{1+2\varepsilon} h_k^{j-} \in S(x_j^*, \varepsilon_j)$  for every  $1 \leq j \leq n$ . Then, clearly

$$\frac{1}{1+2\varepsilon} \sum_{j=1}^n \mu_j h_k^{j\pm} \in C,$$

and

$$\left\| \frac{1}{1+2\varepsilon} \sum_{j=1}^n \mu_j h_k^{j+} - \frac{1}{1+2\varepsilon} \sum_{j=1}^n \mu_j h_k^{j-} \right\| \geq \frac{1}{1+2\varepsilon} \left( \sum_{j=1}^n \mu_j [h_k^{j+}(x_k) - h_k^{j-}(x_k)] \right) = \frac{2}{1+2\varepsilon}.$$

for all  $k \geq K$ . Since  $\varepsilon$  is arbitrary, we can thus conclude that  $C$  has diameter 2.

To produce the  $K = K(\varepsilon)$  above, note that  $h_k^{j\pm}$  converges to  $g^j$  pointwise, and thus weakly since the sequences are bounded. As  $U_j := \{x \in M(\Lambda) : x_j^*(x) > 1 - 2\varepsilon_j\}$  is weakly open, each sequence  $(h_k^{j\pm})_{k=0}^\infty$  enters  $U_j$  eventually. Since there are only a finite number of sets  $U_j$ , this entrance is uniform. So, what is left to prove is that for  $\varepsilon > 0$  there exists  $K$  such that  $\|h_k^{j\pm}\| \leq 1 + 2\varepsilon$  whenever  $k \geq K$ .

Now, let  $\varepsilon > 0$ . Combining Remark 2.2, Remark 2.4, that  $(p_k/\|p_k\|)_{k=1}^\infty$  converges pointwise to 0, and the continuity of  $g^j$ , we can find  $K \in \mathbb{N}$  such that for all  $k \geq K$  there are points  $0 < a_k < x_k < b_k < 1$  such that

$$\begin{aligned} \frac{p_k(x)}{\|p_k\|} > \varepsilon &\Leftrightarrow x \in (a_k, b_k), \\ \sup_{u,v \in (a_k, b_k)} |g^j(u) - g^j(v)| &< \varepsilon, \quad j = 1, \dots, n. \end{aligned}$$

We will see that this  $K$  does the job for the given  $\varepsilon > 0$ : Let  $k \geq K$  and suppose  $x \notin (a_k, b_k)$ . Then

$$|h_k^{j+}(x)| = \left| g^j(x) + (1 - g^j(x_k)) \frac{p_k(x)}{\|p_k\|} \right| \leq |g^j(x)| + 2\varepsilon \leq 1 + 2\varepsilon.$$

If  $x \in (a_k, b_k)$ , observe that

$$\begin{aligned} |h_k^{j+}(x)| &\leq \left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| + |g^j(x) - g^j(x_k)| \frac{p_k(x)}{\|p_k\|} \\ &< \left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| + \varepsilon. \end{aligned}$$

Now, if  $g^j(x) \geq 0$ , then

$$\left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| \leq g^j(x) + (1 - g^j(x)) = 1.$$

If  $g^j(x) < 0$  and  $g^j(x) + (1 - g^j(x))p_k(x)/\|p_k\| \geq 0$ , then

$$\left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| \leq g^j(x) + (1 - g^j(x)) = 1.$$

If  $g^j(x) < 0$  and  $g^j(x) + (1 - g^j(x))p_k(x)/\|p_k\| < 0$ , then

$$\left| g^j(x) + (1 - g^j(x)) \frac{p_k(x)}{\|p_k\|} \right| \leq |g^j(x)| \leq 1.$$

In any case we have for  $k \geq K$  and  $x \in [0, 1]$  that  $|h_k^{j+}(x)| \leq 1 + 2\varepsilon$ . The argument that  $\|h_k^{j-}\| \leq 1 + 2\varepsilon$  is similar.  $\square$

**Theorem 2.6.** *Müntz spaces contain asymptotically isometric copies of  $c_0$ .*

*Proof.* We will construct a sequence  $(f_n)_{n=1}^\infty \subset M(\Lambda)$  and pairwise disjoint intervals  $I_n = (a_n, b_n) \subset [0, 1]$  such that for all  $n \in \mathbb{N}$

- (i)  $f_n(x) \geq 0$  for all  $x \in [0, 1]$ ,
- (ii)  $\|f_n\| = 1 - 1/2^n$ ,
- (iii)  $b_n < a_{n+1}$
- (iv)  $f_n(x) > 1/2^{2n} \Leftrightarrow x \in I_n$ ,
- (v)  $f_n(x) < 1/2^{2m}$  whenever  $m \geq n$  and  $x \in I_m$ .

To this end choose a subsequence of  $\Lambda$  with the RIP. For simplicity denote also this subsequence by  $(\lambda_k)_{k=0}^\infty$ . Let  $(p_k)_{k=1}^\infty$  be its corresponding sequence of spike functions, and let  $x_k$  be the (unique) point in  $(0, 1)$  where  $p_k$  obtains its maximum.

Now, start by letting  $k_1 = 1$  and put

$$f_1 = (1 - 1/2) \frac{p_{k_1}}{\|p_{k_1}\|}.$$

Using continuity and properties of  $p_1$ , we can find an interval  $I_1 = (a_1, b_1)$  such that  $0 < a_1 < b_1 < 1$  and  $f_1(x) > \frac{1}{2^2} \Leftrightarrow x \in I_1$ . By construction  $f_1$  satisfies the conditions (i) - (iv).

To construct  $f_2$  we use Lemma 2.3 and Remarks 2.2 and 2.4 to find  $k_2 \in \mathbb{N}$  and an interval  $I_2 = (a_2, b_2)$  with  $b_1 < a_2 < b_2 < 1$  such that

$$\begin{aligned} x \in I_2 &\Leftrightarrow p_{k_2}(x) > \frac{1/2^4}{1 - 1/2^2} \|p_{k_2}\|, \\ x \in I_2 &\Rightarrow p_{k_1}(x) \leq \frac{1}{2^4}. \end{aligned}$$

Let

$$f_2 = (1 - 1/2^2) \frac{p_{k_2}}{\|p_{k_2}\|}.$$

By construction  $f_1$  now satisfies condition (v) for  $m \leq 2$  and  $f_2$  satisfies conditions (i) - (iv).

To construct  $f_3$  we use Lemma 2.3 and Remarks 2.2 and 2.4 again to find  $k_3 \in \mathbb{N}$  and an interval  $I_3 = (a_3, b_3)$  with  $b_2 < a_3 < b_3 < 1$  such that

$$\begin{aligned} x \in I_3 &\Leftrightarrow p_{k_3}(x) > \frac{1/2^6}{1 - 1/2^3} \|p_{k_3}\|, \\ x \in I_3 &\Rightarrow p_{k_j}(x) \leq \frac{1}{2^6} \quad \text{for } j = 1, 2. \end{aligned}$$

Let

$$f_3 = (1 - 1/2^3) \frac{p_{k_3}}{\|p_{k_3}\|}.$$

By construction  $f_1$  and  $f_2$  now satisfy condition (v) for  $m \leq 3$  and  $f_3$  satisfies conditions (i) - (iv). If we continue in the same manner we obtain a sequence  $(f_n)_{n=1}^\infty \subset M(\Lambda)$  and a sequence of intervals  $I_n = (a_n, b_n)$  which satisfies the conditions (i) - (v).

Now we will show that  $(f_n)_{n=1}^\infty$  satisfies the requirements of Definition 1.1. To this end we need to find constants  $0 < m < M < \infty$  such that given any sequence  $(t_n)_{n=1}^\infty$  with finitely many non zero terms

$$m \sup_n |t_n| \leq \left\| \sum_n t_n f_n \right\| \leq M \sup_n |t_n| \tag{1}$$

and

$$\lim_{n \rightarrow \infty} \|f_n\| = M \tag{2}$$

We claim that (1) and (2) holds with  $m = \frac{1}{4}$  and  $M = 1$ . First observe that we have  $\lim_{n \rightarrow \infty} \|f_n\| = 1$  immediately from the requirements, so (2) holds for  $M = 1$ . In order to prove the two inequalities in (1), let  $(t_n)_{n=1}^\infty$  be an arbitrary sequence with finitely many non zero terms. First we will prove that  $1/4 \sup_n |t_n| \leq \left\| \sum_n t_n f_n \right\|$ . We can assume by scaling that  $\sup |t_n| = 1$ . Since  $(t_n)_{n=1}^\infty$  has finitely many non zero terms, its norm is attained at, say,  $n = N$ , i.e.  $|t_N| = 1$ . Put  $x_N = x_{k_N}$  where  $x_{k_N}$  is the point where  $p_{k_N}$  and thus  $f_N$  attains its norm. Then

$$\begin{aligned} \left\| \sum_{n \in \mathbb{N}} t_n f_n \right\| &\geq |t_N f_N(x_N)| - \left| \sum_{n \neq N} t_n f_n(x_N) \right| \\ &\geq 1 - \frac{1}{2^N} - \sum_{n \neq N} |f_n(x_N)| \\ &> 1 - \frac{1}{2^N} - \frac{1}{4} \geq \frac{1}{4}. \end{aligned}$$

We conclude that the left hand side of the inequality (1) holds. Now we will show the right hand side of this inequality holds, i.e. we want to prove that  $\left| \sum_n t_n f_n(x) \right| \leq 1$  for all  $x \in [0, 1]$ . Since  $f_n \geq 0$  for all  $n = 1, 2, \dots$ , we may assume that every  $t_n$  is positive. Now, if  $x \notin \cup_n (a_n, b_n)$ , we have

$$\sum_n t_n f_n(x) \leq \sum_n f_n(x) \leq \sum_n \frac{1}{2^{2n}} \leq \frac{1}{3}.$$

If, on the other hand  $x \in (a_{n'}, b_{n'})$  for some  $n' \in \mathbb{N}$ , then

$$\begin{aligned} \sum_n t_n f_n(x) &\leq f_{n'}(x) + \sum_{n < n'} f_n(x) + \sum_{n > n'} f_n(x) \\ &\leq 1 - \frac{1}{2^{n'}} + \frac{n' - 1}{2^{2n'}} + \frac{1}{2^{2n'}} \\ &\leq 1 + \frac{n' - 2^{n'}}{2^{2n'}} \leq 1 - \frac{1}{2^{2n'}} < 1. \end{aligned}$$

These combined yields the right hand side of the inequality (1), so the proof is complete. □

A Banach space  $X$  contains an *asymptotically isometric copy* of  $\ell_1$  if it contains a sequence  $(x_n)_{n=1}^\infty$  for which there exists a sequence  $(\delta_n)_{n=1}^\infty$  in  $(0, 1)$ , decreasing to 0, and such that

$$\sum_{n=1}^m (1 - \delta_n) |a_n| \leq \left\| \sum_{n=1}^m a_n x_n \right\| \leq \sum_{n=1}^m |a_n|$$

for each finite sequence  $(a_n)_{n=1}^m$  in  $\mathbb{R}$ .

Merging ([11, Theorem 2]) and [12, Lemma 2.3] gives us that if either the Banach space  $X$  contains an asymptotically isometric copy of  $c_0$  or if  $X^*$  is octahedral, then  $X^*$  contains an asymptotically isometric copy of  $\ell_1$ . So, we have two ways of proving

**Corollary 2.7.**  $M(\Lambda)^*$  contains an asymptotically isometric copy of  $\ell_1$ .

Moreover, we have

**Corollary 2.8.**  $M(\Lambda)^{**}$  contains an isometrically isomorphic copy of  $L_1[0, 1]$ .

*Proof.* This follows from Corollary 2.7 and [13, Theorem 2]. □

*Remark 2.9* (Added in proof).

- (a) One of the anonymous referees invited the authors to consider the question whether Müntz spaces also could be octahedral (as  $C[0, 1]$  is). Here is a preliminary answer: Combine the so called Clarkson-Erdős-Schwartz theorem (see [9, Theorem 6.2.3]) in tandem with a result of P. Wojtaszczyk (see [14, Theorem 1]). Then we see that when  $\Lambda$  consists of natural numbers,  $M(\Lambda)$  is isomorphic to a subspace of  $c_0$ . Since an octahedral space contains a copy of  $\ell_1$ , we have a negative answer for a big class of Müntz spaces.
- (b) We have mentioned [12, Lemma 2.3] as reference for the fact that an octahedral space contains an asymptotically isometric copy of  $\ell_1$ . It has come to our knowledge that this result, even with the same proof, was published earlier by Yamina Yagoub-Zidi, see [15, Proposition 3.3].

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