

On Solving the Problem of Identifying Unreliable Sensors Without a Knowledge of the Ground Truth: The Case of Stochastic Environments*

Anis Yazidi[†] and B. John Oommen[‡] Morten Goodwin[§]

Abstract

In many applications, data from different sensors are aggregated in order to obtain more reliable information about the process that the sensors are monitoring. However, the quality of the aggregated information is intricately dependent on the reliability of the individual sensors. In fact, unreliable sensors will tend to report erroneous values of the ground truth, and thus degrade the quality of the fused information. Finding strategies to identify unreliable sensors can assist in having a counter-effect on their respective detrimental influences on the fusion process, and this has been a focal concern in the literature. The purpose of this paper is to propose a solution to an extremely pertinent problem, namely, that of identifying which sensors are unreliable *without any knowledge of the ground truth*. This fascinating paradox can be formulated in simple terms as trying to identify *stochastic liars* without any additional information about the truth. Though apparently impossible, we will show that it is feasible to solve the problem, a claim that *is counter-intuitive in and of itself*. To the best of our knowledge, this is the first reported solution to the aforementioned paradox. Legacy work and the reported literature have merely addressed assessing the reliability of a sensor by comparing its reading to the ground truth either in an online or an offline manner. The informed reader will observe that the so-called Weighted Majority Algorithm is a representative example of a large class of such legacy algorithms. Unfortunately, the fundamental assumption of revealing the ground truth cannot be always guaranteed (or even expected) in many real life scenarios. Indeed, accessing the ground truth might be impossible, and since only noisy reports from the sensors are available, rendering the task of identifying the unreliable sensors is apparently impossible. While some extensions of the Condorcet Jury theorem [9] can lead to a probabilistic guarantee on the quality of the fused process, they do not provide a solution to the Unreliable Sensor Identification (USI) problem. The essence of our approach involves studying the agreement of each sensor with the rest of the sensors, and not comparing the reading of the individual sensors with the ground truth – as advocated in the literature. Under some mild conditions on the reliability of the sensors, we can prove that we can, indeed, filter out the unreliable ones. Our approach leverages the power of the theory of Learning Automata (LA) so as to gradually learn the identity of the reliable and unreliable sensors. To achieve this, we resort

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[†]This author can be contacted at: Dept. of Computer Science, University College of Oslo and Akershus, Oslo. A part from this work was done by this author when he was earlier with Teknova AS, Sorlandet Kunnskapsark, Kristiansand 4630. E-mail: anis.yazidi@uia.no.

[‡]Author's status: *Chancellor's Professor, Fellow: IEEE and Fellow: IAPR*. This author can be contacted at: School of Computer Science, Carleton University, Ottawa, Canada : K1S 5B6. The author is also an Adjunct Professor with University of Agder, Grimstad, Norway. E-mail: oommen@scs.carleton.ca.

[§]This author can be contacted at: Department of ICT, University of Agder, Grimstad 4630, Norway. He also holds a position with Teknova AS, Sorlandet Kunnskapsark, Kristiansand 4630. E-mail: morten.goodwin@uia.no.

to a team of *LA*, where a distinct automaton is associated with each sensor. The solution provided here has been subjected to rigorous experimental tests, and the results presented are, in our opinion, both novel and conclusive.

Keywords: *Sensor Fusion, Learning Automata*

1 Introduction

In many applications, data from different sources is received, processed and then fused, to obtain more reliable information about the process being monitored. This is often the case in industrial applications where multiple redundant sensors are used to measure the same quantities [20,21], and for example, in nuclear or space applications, where human intervention is not possible. Sensors usually provide imprecise and uncertain observations. The field of sensor fusion involves a set of redundant sensors measuring the same physical quantity. This redundancy permits the operators to obtain a robustness of sorts, whenever some sensors are prone to error.

Furthermore, fused data will reduce or eliminate the effects due to failures of a few sensors operating in the system. Most of the research on fusing multiple sensor data merely assume that the confidence levels in the measurements are known. The accuracy of an observation can be computed by comparing the current observation with the reference data set and/or by performing physical investigation. However, performing a physical investigation or having a reference data set is not practical in many monitoring scenarios, although it is possible to adopt such measures during training or within a limited scope. To the best of our knowledge, trying to assess the reliability of a sensor without any additional information about the ground truth is still an open research question that has not been addressed before, and our strategy for resolving this will be discussed in the body of this paper.

The first question to be addressed is whether the problem of detecting an unreliable sensor without knowing the ground truth is even a solvable problem. Our position is that if there is no other information, it is a futile venture. But if we consider the fact that there is a set of sensors, all of which are measuring the same quantity, the information provided by the *other* sensors can provide invaluable metrics about how good any specific sensor is. This, indeed, is the philosophy that we advocate. The question of how the information from the other sensors is to be gleaned and processed is really, in and of itself, unsolved. Suffice it to state that we emphasize that our solution to the problem lies in investigating the level of agreement between the various data sources/sensors, which, in turn, constitutes valuable information to fuse them in an efficient manner. In simple words, we assert the rather fascinating claim that given a group of sensors, we can find the sub-group of unreliable sensors without any knowledge of the ground truth, if we also permit each sensor to be compared to the others!

In order to position our work in relation with the existing work, we shall present a brief review of the state-of-the-art related to data fusion. The legacy research has focused on fusing sensor information under either known or estimated confidence levels. Most current data fusion methods employ probabilistic descriptions of observations and processes, and use Bayesian principles to combine this information. Other approaches rely on principles derived from evidential reasoning including Dempster-Shafer inference theory [8] and subjective logic [23]. Elmenreich [17] presented a novel algorithm that uses the estimated variance of each sensor measurement in order to find the optimal averaging weights. Another theme akin to multi-sensor fusions involves “prediction using expert advice” [28], where the performance is always nearly as good as the best forecasting strategy. The fault-tolerant averaging algorithm was first introduced by Marzullo [29] in the context of time synchronization in distributed systems. Afterwards, it was used in the domain of information fusion to fuse a set of abstract sensors into a single

reliable abstract sensor that is correct even when some of the original sensors are incorrect or faulty. Consensus algorithms, such as majority voting, are suitable for fusing binary measurements.

Most approaches rely on accessing the ground truth to compute the accuracy of a sensor. The work by Hossain [22] is representative of such approaches: It computes the accuracy by comparing the observations provided by the sensor with the ground truth in the training data. This can be experimentally computed by comparing the outcome of the online sensor observations with the ground truth, and by repeating this process multiple times. The approach of Hossain and his colleagues considers the opinions of the sensors in performing a common observation, and proceeds to group the opinions into two subgroups, namely those which support the occurrence of the event and those which oppose it. The scheme then determines the winning group and increases the confidence of the sensors in that group (by considering this event as a “reward”), while, at the same time, it decreases the confidence of the sensors of the other group.

The research presented here also relates to the field of “Softsensing”, which is an emerging technology that can be perceived as a software alternative or complement to traditional hardware sensors, where several measurements are processed simultaneously. With the increasingly wide deployment of sensor technology in the industry, obtaining robust indirect measurements has become unquestionably recognized as a central topic. While the problem is particularly interesting to the industry, it is also extremely appealing from a research perspective.

A myriad of pieces of literature can be cited that concentrate on using majority voting to faulty sensor fusion. The premise for invoking majority voting is that the decision of the group is better than the decision of the individual sensor.

The Condorcet Jury Theorem demonstrates that the Majority group is always better at selecting superior alternatives than any single individual member [9]. There are some limitations to the hypotheses governing the theorem. In fact, it requires that each individual makes the right decision with a probability $p > 0.5$, and that all individuals are homogenous in p . Probably the most notable extension of this is the scenario when the population is not homogenous. Boland [9] assumes that the voters can be divided into two groups. The first group consists of individuals whose “true” interest lie in one direction, while the other group consists of those whose “true” interests lie in the other. When mapped to the case of sensor aggregation, we again have two groups, where the first group consists of reliable sensors that possess the “true” interest of reporting the ground truth, while the alternate group of unreliable sensors possess a “true” interest in misreporting it.

It is worth mentioning that the theory of sensor fusion has found wide deployment in the field of “reputation systems” where users who want to promote a particular product or service can flood the domain (i.e., the social network) with sympathetic votes, while those who want to get a competitive edge over a specific product or service can “badmouth” it unfairly. Thus, although these systems can offer generic recommendations by aggregating user-provided opinions, unfair ratings may degrade the trustworthiness of such systems. This problem, of separating “fair” and “unfair” agents for a specific service, is called the Agent-Type Partitioning Problem (*ATPP*). Determining ways to solve the (*ATPP*) [55] and thus counter the detrimental influence of unreliable agents on a Reputation System, has been a focal concern of a number of very interesting studies [11, 16, 33, 46, 56, 57].

It is worth noting that the task of combining reports from different witnesses is akin to the problem of fusing possibly conflicting sources of information [2, 18, 27]. Buchegger and Le Boudec [11] tackled the latter issue as follows: They proposed a Bayesian reputation mechanism in which each node isolates malicious nodes by applying a so-called *deviation test* methodology. Their approach requires each agent to have enough *direct* experience with

the services so that he can evaluate the trustworthiness of the reports of the witnesses. While this is a desirable option, unfortunately, in real life, such an assumption does not always hold, specially when the number of possible services is large. In [12], Chen and Singh evaluated the quality of feedback responses assuming that a feedback is credible if it is consistent with the majority of feedback responses for a given user. Their approach, though promising, unfortunately, suffers from a deterioration in the performance when the ratio of deceptive agents is high. In [56], Yu and Singh devised a modified weighted majority algorithm to combine reports from several witnesses to determine the ratings of another agent. The main shortcoming of the work reported in [56] is its relatively slow rate of convergence. In contrast, in [54], Witby and Jøsang presented a Bayesian approach to filter out dishonest feedback based on an iterated filtering approach. In their approach, the authors extended the so-called “Beta” reputation system earlier presented by Jøsang and Ismail [24].

This problem, of separating reliable and unreliable sensors, is called the Sensor-Type Partitioning Problem (*STPP*). Put in a nutshell, in this paper, we propose to solve the above-mentioned paradoxical *STPP* using tools provided by Learning Automata (LA), which have proven powerful potential in efficiently and quickly learning the optimal action when operating in unknown stochastic environments. It adaptively, and in an on-line manner, gradually learns the identity and characteristics of the sensors that are reliable and those that are unreliable. In addition, we will provide two approaches for fusing the sensor readings which leverage the convergence result of our LA-based partitioning. Rigorous theoretical results and a host of empirical results will be presented in this paper. Our work differs from the aforementioned research since we aim to infer the confidence of the measurements based on their level of agreements *in the absence of knowledge of the ground truth*.

1.1 Paper Organization

Earlier, in Section 1 we introduced the research problem and presented a brief survey of the available solutions for dealing with reliable and unreliable sensors. The rest of the paper is organized as follows. First of all, in Section 2, we submit a formal statement of the problem. Then, in Section 3 we present a brief overview of the field of LA. Thereafter, in Section 4 we present our solution, which is the LA-based scheme for identifying unreliable sensors in a stochastic environment in the absence of knowledge of the ground truth. Experimental results obtained by rigorously testing our solution for a variety of scenarios and for agents with different characteristics, are presented in Section 5. Section 6 concludes the paper and discusses open avenues for future work.

2 Modeling the Problem

We consider a population of N sensors, $\mathcal{S} = \{s_1, s_2, \dots, s_N\}$. Let the real situation of the environment at the time instant t be modeled by a binary variable $T(t)$, which can take one of two possible values, 0 and 1. The value of T is unknown and can only be inferred through measurements from sensors. The output from the sensor s_i is referred to as x_i . Let π be the probability of the state of the ground truth, i.e., $T = 0$ with probability π .

To formalize the scenario, we record four possibilities:

- $x_i = T$ (where $x_i = 0$ or 1): This is the case when the sensor correctly reports the ground truth.
- $x_i \neq T$ (where $x_i = 0$ or 1): This is the case when the sensor faultily reports the ground truth.

In our discussions, we make one simplifying assumption: The probability of the sensor reporting a value erroneously is symmetric. In other words, in terms of the binary detection problem, we assume that the probability of a false alarm and the so-called miss probability are both equal. Formally, we assume that:

$$Prob(x_i = 0|T = 1) = Prob(x_i = 1|T = 0). \quad (1)$$

Further, let q_i denote the Fault Probability (FP) of sensor s_i , where:

$$q_i = Prob(x_i = 0|T = 1) = Prob(x_i = 1|T = 0).$$

Similarly, we define the Correctness Probability (CP) of sensor s_i as $p_i = 1 - q_i$.

It is easy to prove that the total probability $Prob(x_i = T)$ is, indeed, p_i , since, in fact:

$$\begin{aligned} Prob(x_i = T) &= Prob(T = 0)Prob(x_i = 0|T = 0) + Prob(T = 1)Prob(x_i = 1|T = 1) \\ &= \pi p_i + (1 - \pi)p_i \\ &= p_i. \end{aligned} \quad (2)$$

Thus, the quantity $p_i = Prob(x_i = T)$ can be re-written as $p_i = Prob(I\{x_i = T\} = 1)$, where $I\{\cdot\}$ is the Indicator function.

We refer to a sensor as being reliable when it has a FP $q_i < 0.5$. Conversely, the sensor is unreliable when it has a FP $q_i > 0.5$. Equivalently, we can define a reliable sensor to be one that has a CP $p_i > 0.5$ and an unreliable sensor as one that has a CP of $p_i < 0.5$.

Observe that as a result of this model, a reliable sensor will probabilistically tend to report 0 when the ground truth is 0, and 1 when the ground truth is 1. Otherwise, it is clearly, unreliable. Our aim, then, is to partition the sensors as being reliable or unreliable. Furthermore, once partitioned, our aim is to use the partitioning as a basis for better fusion.

To simplify the analysis¹, we assume that every p_i can assume one of two possible values from the set $\{p_R, p_U\}$, where $p_R > 0.5$ and $p_U < 0.5$. Then, a sensor s_i is said to be reliable if $p_i = p_R$, and is said to be unreliable if $p_i = p_U$. To render the problem non-trivial and interesting, we assume that p_R and p_U are unknown to the algorithm.

Based on the above, the set of reliable sensors is $\mathcal{S}_R = \{s_i | p_i = p_R\}$, and the set of unreliable sensors is $\mathcal{S}_U = \{s_i | p_i = p_U\}$.

We now formalize the Sensor-Type Partitioning Problem (*STPP*). The *STPP* involves a set of N sensors², $\mathcal{S} = \{s_1, s_2, \dots, s_N\}$, where each sensor s_i is characterized by a fixed but unknown probability p_i of it sensing the ground truth correctly. The *STPP* involves partitioning \mathcal{S} into 2 mutually exclusive and exhaustive groups so as to obtain a 2-partition $\mathbb{G} = \{G_U, G_R\}$, such that each group, G_R , of size, N_R , and G_U , of size N_U , exclusively contains only the sensors of its own type, i.e., which are either reliable or unreliable respectively.

We define $P_{(N_R-1, N_U)}$ as the probability of a deterministic majority voting scheme, which involves the opinions of $N_R - 1$ reliable sensors and N_U unreliable ones, to yield the correct decision using the majority rule. In other words, this is the probability that a majority of more than $(N_R - 1 + N_U)/2$ of the sensors will advocate the ground truth. Similarly, we define $P_{(N_R, N_U-1)}$ as the probability of a deterministic majority voting scheme, which involves

¹This assumption, however, does not simplify the problem. Indeed, p_R can be assigned to be the smallest value of all the values of p_i for the reliable sensors, and p_U can be assigned to be the largest value of all the values of p_i for the unreliable ones.

²Throughout this paper, since we will be invoking majority-like decisions, we assume that $N = N_R + N_U$ is an even number.

the opinions of N_R reliable sensors and $N_U - 1$ unreliable ones, to yield the correct decision using the majority rule. As one can see, this quantity is the same: It too is the probability that a majority of more than $(N_R + N_U - 1)/2$ of the sensors will, in turn, advocate the ground truth.

To render the problem meaningful and solvable³, we shall assume that:

$$(N_R - 1)p_R + N_U p_U > (N_R + N_U)/2.$$

This mild condition that we require in this paper, is founded on a fundamental premise that has to hold in any sustainable society, where telling the “truth” is considered a virtue, while “lying” is considered detrimental and harmful to the society. The rationale for invoking this is the following: A reliable sensor will tend to agree with the averaged/aggregated opinion of the rest of other sensors, and thus by comparing the reading of any specific sensor with the rest of other sensors, we hypothesize that we will be able to detect sensors that deviate from the accepted norm even without knowing the ground truth. Of course, in the interest of completeness, we will also investigate the opposite case in which the phenomenon of “lying” is more prevalent than that of saying the “truth”. Indeed, we will also present some theoretical results for the case where:

$$(N_R - 1)p_R + N_U p_U < (N_R + N_U)/2 - 1.$$

3 Stochastic Learning Automata

Learning Automata(LA) have been used in systems that have incomplete knowledge about the Environment in which they operate [1,35,43,50]. The learning mechanism attempts to learn from a *stochastic Teacher* which models the Environment. In his pioneering work, Tsetlin [51] attempted to use LA to model biological learning. In general, a random action is selected based on a probability vector, and these action probabilities are updated based on the observation of the Environment’s response, after which the procedure is repeated.

The term “Learning Automata” was first publicized by Narendra and Thathachar [35]. The goal of LA is to “determine the optimal action out of a set of allowable actions” [1]. The distinguishing characteristic of automata-based learning is that the search for the optimizing parameter vector is conducted in the space of probability *distributions* defined over the parameter space, rather than in the parameter space itself [49].

In the first LA designs, the transition and the output functions were time invariant, and for this reason these LA were considered “Fixed Structure Stochastic Automata” (FSSA). Tsetlin, Krylov, and Krinsky [51] presented notable examples of this type of automata.

Later, Vorontsova and Varshavskii [35] introduced a class of stochastic automata known in the literature as Variable Structure Stochastic Automata (VSSA). The solution we present here, essentially falls within this family and so we shall explain this family in greater detail in Section 4. In the definition of a VSSA, the LA is completely defined by a set of actions (one of which is the output of the automaton), a set of inputs (which is usually the response of the Environment) and a learning algorithm, T . The learning algorithm [35] operates on a vector (called *the Action Probability vector*)

$$P(t) = [p_1(t), \dots, p_R(t)]^T,$$

where $p_i(t)$ ($i = 1, \dots, R$) is the probability that the automaton will select the action α_i at time ‘ t ’,

$$p_i(t) = \Pr[\alpha(t) = \alpha_i], i = 1, \dots, R, \text{ and it satisfies}$$

³If this condition is not satisfied, it means that we are dealing with a system from which no meaningful measurements can be inferred.

$$\sum_{i=1}^R p_i(t) = 1 \forall t.$$

Note that the algorithm $T : [0,1]^R \times A \times B \rightarrow [0,1]^R$ is an updating scheme where $A = \{\alpha_1, \alpha_2, \dots, \alpha_R\}$, $2 \leq R < \infty$, is the set of output actions of the automaton, and B is the set of responses from the Environment. Thus, the updating is such that

$$P(t+1) = T(P(t), \alpha(t), \beta(t)),$$

where $P(t)$ is the action probability vector, $\alpha(t)$ is the action chosen at time t , and $\beta(t)$ is the response it has obtained.

If the mapping T is chosen in such a manner that the Markov process has absorbing states, the algorithm is referred to as an absorbing algorithm. Many families of VSSA that possess absorbing barriers have been reported [35]. Ergodic VSSA have also been investigated [35,38]. Further, in order to increase their speed of convergence, the concept of discretizing the probability space was introduced [38,48]. This concept is implemented by restricting the probability of choosing an action to a finite number of values in the interval $[0,1]$. Following the discretization concept, many of the continuous VSSA have been discretized; indeed, discrete versions of almost all continuous automata have been reported [38]. Finally, Pursuit and Estimator-based LA were introduced to be faster schemes, characterized by the fact that they pursue what can be reckoned to be the *current* optimal action or the set of current optimal actions [38]. The updating algorithm improves its convergence results by using the history to maintain an estimate of the probability of each action being rewarded, in what is called the *reward-estimate* vector. Families of Pursuit and Estimator-based LA have been shown to be faster than VSSA [49]. Indeed, even faster discretized versions of these schemes have been reported [1,38].

With regard to applications, the entire field of LA and stochastic learning has had a myriad of applications [26,34,35,43,50], which (apart from the many applications listed in these books) include solutions for problems in network and communications [32,37,40,42], network call admission, traffic control, quality of service routing [3,4,53], distributed scheduling [47], training hidden Markov models [25], neural network adaptation [30], intelligent vehicle control [52], and even fairly theoretical problems such as graph partitioning [39]. Besides these fairly generic applications, with a little insight, LA can be used to assist in solving (by, indeed, learning the associated parameters) the stochastic resonance problem [13], the stochastic sampling problem in computer graphics [14], the problem of determining roads in aerial images by using geometric-stochastic models [6], and various location problems [10]. Similar learning solutions can also be used to analyze the stochastic properties of the random waypoint mobility model in wireless communication networks [7], to achieve spatial point pattern analysis codes for GISs [44], to digitally simulate wind field velocities [41], to interrogate the experimental measurements of global dynamics in magneto-mechanical oscillators [15], and to analyze spatial point patterns [5]. LA-based schemes have already been utilized to learn the best parameters for neural networks [30], optimizing QoS routing [53], and bus arbitration [37] – to mention a few other applications.

4 The Solution

4.1 Overview of Our Solution

In this paper, we provide a novel solution to the *STTP*, based on the field of LA that was briefly surveyed above. We intend to take advantage of the fact that LA combine rapid and accurate convergence with low computational complexity. In addition to its computational simplicity, unlike most reported approaches, as mentioned earlier,

our scheme does not require prior knowledge of the ground truth. Rather, it adaptively, and in an on-line manner, gradually learns the identity and characteristics of the sensors which tend to provide reliable readings, and of those which tend to provide unreliable ones.

Our solution involves a team of LA where each LA is uniquely attached to (or rather, associated with) a specific sensor, on a one-to-one basis. Each automaton \mathcal{A}^i attached to sensor s_i , has two actions.

By suitably modeling the agreement or disagreement of the opinions about the sensed ground truth between each sensor and the rest of the other sensors, we can appropriately model these as responses from the corresponding “Environment”. Using these synthesized responses, our scheme will intelligently group the sensors according to the readings that they report about the ground truth. Since a sensor is reliable if it reports the ground truth correctly with a probability $p_i > 0.5$ (and unreliable otherwise), we will design our scheme so that it can infer the similar sensors and collect them into their respective groups. In other words, we will infer the crucial sensor identities from the random stream of sensor reports.

The fusion part of our scheme will be based on the result of a prior partitioning phase. Ultimately, the aim behind identifying the set of unreliable sensors, \mathcal{S}_U , is to improve the performance of the fusion process for inferring the ground truth. The result of the convergence of the team of LA , which results in a partitioning that infers the identity of the sensor, will serve as an input to the fusion process. In this vein, we shall present two approaches for fusing the results, and study their performances in the section that describes the experimental result. The first fusion approach only considers the measurements from the reliable sensors as being informative, and simultaneously discards measurements from the unreliable sensors. As opposed to this, the second approach attempts to intelligently combine (or fuse) the measurements from both the reliable and the unreliable sensors to yield an accurate value of the ground truth. In this approach, the reading from an unreliable sensor is modified so that it can be considered informative.

The first formal result concerning the performance of the LA is given below.

Theorem 1. *Consider the scenario when $(N_R - 1)p_R + N_U p_U > (N_R + N_U)/2$ and when $N_R + N_U - 1 \geq 3$. Let $s_i \in \mathcal{S}_R$. Consider now the agreement between the opinion of a reliable sensor s_i and the opinion of the majority formed by all the rest of the sensors $S \setminus \{s_i\} = (\mathcal{S}_R \setminus \{s_i\}) \cup \mathcal{S}_U$. Let $y_{(N_R-1, N_U)}$ be the decision of a majority voting scheme $S \setminus \{s_i\}$, based on the responses of $N_R - 1$ reliable and N_U unreliable sensors. Then, if x_i is the output of s_i : $Prob(x_i = y_{(N_R-1, N_U)}) > 0.5$.*

Proof: Since $N_R + N_U - 1$ is an odd number, $(N_R + N_U)/2$ is the smallest integer that suffices to yield a majority vote among $N_R + N_U - 1$ votes.

Let $P_{(N_R-1, N_U)}$ be the probability that the majority of the votes of the sensors $S \setminus \{s_i\}$ adheres to the ground truth, where s_i belongs to \mathcal{S}_R . Formally $P_{(N_R-1, N_U)}$ can be written as:

$$P_{(N_R-1, N_U)} = Prob(y_{(N_R-1, N_U)} = T), \quad (3)$$

where $y_{(N_R-1, N_U)}$ denotes the decision of the majority voting mechanism as a result of the sensors $S \setminus \{s_i\}$. We see that $y_{(N_R-1, N_U)}$ is a random variable defined as below:

$$y_{(N_R-1, N_U)} = \begin{cases} 1 & \text{If } \sum_{\substack{k=1 \\ k \neq i}}^{N_R+N_U} I\{x_k = 1\} \geq (N_R + N_U)/2 \\ 0 & \text{Otherwise.} \end{cases}$$

A simple observation of the complementary case under which $y_{(N_R-1, N_U)} = 0$, permits us to re-write the above expression as:

$$y_{(N_R-1, N_U)} = \begin{cases} 1 & \text{If } \sum_{\substack{k=1 \\ k \neq i}}^{N_R+N_U} I\{x_k = 1\} \geq (N_R + N_U)/2 \\ 0 & \text{If } \sum_{\substack{k=1 \\ k \neq i}}^{N_R+N_U} I\{x_k = 0\} \geq (N_R + N_U)/2. \end{cases} \quad (4)$$

Based on the above expressions, we can now compute the total probability $P_{(N_R-1, N_U)}$ as:

$$\begin{aligned} P_{(N_R-1, N_U)} &= \text{Prob}(y_{(N_R-1, N_U)} = T) \\ &= \text{Prob}(y_{(N_R-1, N_U)} = 0|T = 0)\text{Prob}(T = 0) + \text{Prob}(y_{(N_R-1, N_U)} = 1|T = 1)\text{Prob}(T = 1) \\ &= \pi \text{Prob}\left(\sum_{\substack{k=1 \\ k \neq i}}^{N_R+N_U} I\{x_k = 0\} \geq (N_R + N_U)/2|T = 0\right) \\ &\quad + (1 - \pi) \text{Prob}\left(\sum_{\substack{k=1 \\ k \neq i}}^{N_R+N_U} I\{x_k = 1\} \geq (N_R + N_U)/2|T = 1\right). \end{aligned} \quad (5)$$

By virtue of the symmetric fault model that we have invoked, expressed in Eq. (1), it is easy to prove that:

$$\text{Prob}\left(\sum_{\substack{k=1 \\ k \neq i}}^{N_R+N_U} I\{x_k = 0\} \geq (N_R + N_U)/2|T = 0\right) = \text{Prob}\left(\sum_{\substack{k=1 \\ k \neq i}}^{N_R+N_U} I\{x_k = 1\} \geq (N_R + N_U)/2|T = 1\right). \quad (6)$$

Replacing Eq. (6) in Eq. (5) yields:

$$\begin{aligned} P_{(N_R-1, N_U)} &= \pi \text{Prob}(y_{(N_R-1, N_U)} = 0|T = 0) + (1 - \pi) \text{Prob}(y_{(N_R-1, N_U)} = 0|T = 0) \\ &= \text{Prob}(y_{(N_R-1, N_U)} = 0|T = 0) = \text{Prob}(y_{(N_R-1, N_U)} = 1|T = 1). \end{aligned} \quad (7)$$

Since $s_i \in \mathcal{S}_R$, we see that $S \setminus \{s_i\} = (\mathcal{S}_R \setminus \{s_i\}) \cup \mathcal{S}_U$ contains $N_R - 1$ reliable sensors and N_U unreliable ones. Indeed, as a consequence of this, by considering the various possible combinations by which the voting can take place, we see that the exact expression of $\text{Prob}(y_{(N_R-1, N_U)} = 0|T = 0)$ is given by [9]:

$$\text{Prob}(y_{(N_R-1, N_U)} = 0|T = 0) = \sum_{x_1=(N_R+N_U-1)/2}^{N_R+N_U-1} \sum_{x_2=k}^{N_U} \binom{N_R-1}{x_1} \binom{N_U}{x_2} p_R^{x_1} (1-p_R)^{N_R-1-x_1} p_U^{x_2} (1-p_U)^{N_U-x_2}$$

where $k = x_1 - (N_R - 1)$ if $x_1 - (N_R - 1) > 0$, and $k = 0$ otherwise.

We can avoid evaluating the above complicated expression by a subtle result due to Miller [31]. In fact, if we define $\bar{p}_{(N_R-1, N_U)}$, the mean competence of individual s_i in a heterogeneous group $S \setminus \{s_i\} = \mathcal{S}_R \setminus \{s_i\} \cup \mathcal{S}_U$ as:

$$\bar{p}_{(N_R-1, N_U)} = \frac{(N_R - 1)p_R + N_U p_U}{N_R + N_U - 1}, \quad (8)$$

we can apply Theorem 4 due to Boland [9], which is an extension of the Condorcet Jury theorem for heterogenous groups, to demonstrate that:

If $\bar{p}_{(N_R-1, N_U)} > \frac{1}{2} + \frac{1}{2(N_R+N_U-1)}$ and $|S \setminus \{s_i\}| = N_R - 1 + N_U \geq 3$,

then the following equation holds:

$$Prob(y_{(N_R-1, N_U)} = 0 | T = 0) > \bar{p}_{(N_R-1, N_U)}.$$

Now let us prove that Eq. (9) below holds true within the settings of our problem space:

$$Prob(y_{(N_R-1, N_U)} = 0 | T = 0) > \bar{p}_{(N_R-1, N_U)}. \quad (9)$$

We achieve this by proving that the condition $\bar{p}_{(N_R-1, N_U)} > \frac{1}{2} + \frac{1}{2(N_R+N_U-1)}$ is met. Indeed, since we assumed that $(N_R - 1)p_R + N_U p_U > (N_R + N_U)/2$ is true:

$$\begin{aligned} (N_R - 1)p_R + N_U p_U > (N_R + N_U)/2 &\Rightarrow \frac{(N_R - 1)p_R + N_U p_U}{N_R + N_U - 1} > \frac{N_R + N_U}{2(N_R + N_U - 1)} \\ &\Rightarrow \bar{p}_{(N_R-1, N_U)} > \frac{(N_R + N_U - 1) + 1}{2(N_R + N_U - 1)} \\ &\Rightarrow \bar{p}_{(N_R-1, N_U)} > \frac{1}{2} + \frac{1}{2(N_R + N_U - 1)}. \end{aligned} \quad (10)$$

By virtue of having proved Eq. (10), we can deduce that the fundamental result of Eq. (9) is true.

Since from Eq. (10), $\bar{p}_{(N_R-1, N_U)} > \frac{1}{2} + \frac{1}{2(N_R+N_U-1)}$, we see that, trivially, $\bar{p}_{(N_R-1, N_U)} > 1/2$. Utilizing the latter inequality in conjunction with Eq. (9) yields:

$$Prob(y_{(N_R-1, N_U)} = 0 | T = 0) > 1/2. \quad (11)$$

Using Eq. (7) together with the above equation, we obtain:

$$P_{(N_R-1, N_U)} > 1/2. \quad (12)$$

Eq. (12) confirms the elegant result that the aggregated opinion formed by a majority voting scheme from among the $S \setminus \{s_i\} = S_R \setminus \{s_i\} \cup S_U$ sensors, namely $y_{(N_R-1, N_U)}$, will tend to inform us of the ground truth with a probability larger than 1/2.

Now let us compute the mutual agreement probability $Prob(x_i = y_{(N_R-1, N_U)})$ between the reading x_i of the sensor $s_i \in S_R$, and the aggregated opinion $y_{(N_R-1, N_U)}$ of the rest of the sensors $S \setminus \{s_i\} = S_R \setminus \{s_i\} \cup S_U$. Indeed,

$$\begin{aligned} Prob(x_i = y_{(N_R-1, N_U)}) &= Prob[(x_i = T \wedge y_{(N_R-1, N_U)} = T) \vee (x_i = 1 - T \wedge y_{(N_R-1, N_U)} = 1 - T)] \\ &= Prob[x_i = T \wedge y_{(N_R-1, N_U)} = T] + Prob[x_i = 1 - T \wedge y_{(N_R-1, N_U)} = 1 - T] \\ &= Prob(x_i = T) \cdot Prob(y_{(N_R-1, N_U)} = T) + Prob(x_i = 1 - T) \cdot Prob(y_{(N_R-1, N_U)} = 1 - T) \\ &= Prob(x_i = T) \cdot Prob(y_{(N_R-1, N_U)} = T) + (1 - Prob(x_i = T)) \cdot (1 - Prob(y_{(N_R-1, N_U)} = T)) \\ &= p_R \cdot P_{(N_R-1, N_U)} + (1 - p_R) \cdot (1 - P_{(N_R-1, N_U)}), \end{aligned} \quad (13)$$

where the second line in the above set of equations is because of the mutually exclusive nature of the events, and the third line is because of the independence of x_i and $y_{(N_R-1, N_U)}$.

We will now prove that $Prob(x_i = y_{(N_R-1, N_U)}) > 1/2$. In order to prove this inequality, let us consider the function $g(\cdot)$ defined as the convex combination:

$$g(\rho) = p_R \cdot \rho + (1 - p_R) \cdot (1 - \rho),$$

whence, it is easy to see that:

$$g(P_{(N_R-1, N_U)}) = Prob(x_i = y_{(N_R-1, N_U)}). \quad (14)$$

Let us investigate the dynamics of $g(\rho)$ by studying its derivative function, $g'(\rho)$, which specifically, has the form $g'(\rho) = 2p_R - 1$. Since, by definition, $p_R > 1/2$, we can confirm that $2p_R - 1 > 0$ which is equivalent to stating that $g'(\rho) > 0$. $g(\rho)$ is thus a *strictly increasing* function.

We further know that $g(1/2) = 1/2p_R + 1/2(1 - p_R) = 1/2$. Thus, by virtue of the strictly increasing property of the function $g(\cdot)$:

$$\text{if } \rho > 1/2 \Rightarrow g(\rho) > g(1/2) = 1/2. \quad (15)$$

Observe that, in particular, we can apply the inequality (15) for the particular case when $\rho = P_{(N_R-1, N_U)}$. Since we have previously demonstrated in Eq. (12) that $P_{(N_R-1, N_U)} > 1/2$, if we replace ρ by $P_{(N_R-1, N_U)}$ in the inequality (15), we get:

$$g(P_{(N_R-1, N_U)}) > 1/2.$$

As per Eq. (14), this is equivalent to

$$Prob(x_i = y_{(N_R-1, N_U)}) > 1/2, \quad (16)$$

which concludes the proof! □

We shall now consider the converse case of omitting an unreliable sensor, and prove the analogous result.

Theorem 2. Consider the scenario when $(N_R - 1)p_R + N_U p_U > (N_R + N_U)/2$ and when $N_R + N_U - 1 \geq 3$. Let $s_i \in \mathcal{S}_U$. Consider now the agreement between the opinion of an unreliable sensor s_i and the opinion of the majority formed by all the rest of the sensors $S \setminus \{s_i\} = \mathcal{S}_R \cup \mathcal{S}_U \setminus \{s_i\}$. Let $y_{(N_R, N_U-1)}$ be the decision of a majority voting scheme formed of $S \setminus \{s_i\}$, based on the responses of N_R reliable and $N_U - 1$ unreliable sensors. Then, if x_i is the output of s_i : $Prob(x_i = y_{(N_R, N_U-1)}) > 0.5$.

Proof: As in the previous case, since $N_R + N_U - 1$ is an odd number, $(N_R + N_U)/2$ is the smallest integer that suffices to yield a majority vote among $N_R + N_U - 1$ votes. Further, since we are now dealing with the exclusion of a single unreliable sensor, this excluded one belongs to the group \mathcal{S}_U , implying that the entire set that we are taking the voting from consists of N_R reliable sensors and $N_U - 1$ unreliable sensors, i.e., $S \setminus \{s_i\} = \mathcal{S}_R \cup \mathcal{S}_U \setminus \{s_i\}$.

Again, as we stated, we are dealing with a society where “truth prevails over lying”, and so:

$$(N_R - 1)p_R + N_U p_U > (N_R + N_U)/2.$$

Since we know that $p_R > p_U$, $p_R - p_U > 0$, whence we can obtain the interesting conclusion:

$$\begin{aligned}
(N_R - 1)p_R + N_U p_U > (N_R + N_U)/2 &\Rightarrow (N_R - 1)p_R + N_U p_U + (p_R - p_U) > (N_R + N_U)/2 + (p_R - p_U) \\
&\Rightarrow N_R p_R + (N_U - 1)p_U > (N_R + N_U)/2 + (p_R - p_U) > (N_R + N_U)/2 \\
&\Rightarrow N_R p_R + (N_U - 1)p_U > (N_R + N_U)/2.
\end{aligned} \tag{17}$$

Let $P_{(N_R, N_U-1)}$ be the probability that the majority of the votes of the sensors $S \setminus \{s_i\}$ adheres to the ground truth, where s_i belongs to S_U . Formally $P_{(N_R, N_U-1)}$ can be written as:

$$P_{(N_R, N_U-1)} = \text{Prob}(y_{(N_R, N_U-1)} = T),$$

where $y_{(N_R, N_U-1)}$ is the decision of the majority voting scheme decided on by $S \setminus \{s_i\}$.

Following the same line of arguments as in the proof of Theorem 1 (the details are omitted to avoid repetition), we can conclude that:

$$\text{Prob}(x_j = y_{(N_R, N_U-1)}) = p_U \cdot P_{(N_R, N_U-1)} + (1 - p_U) \cdot (1 - P_{(N_R, N_U-1)}). \tag{18}$$

Let $\bar{p}_{(N_R, N_U-1)} = \frac{N_R p_R + (N_U - 1)p_U}{N_R + N_U - 1}$ be the mean value of individual p_i 's in the heterogeneous group. We now apply the result by Boland (identified as Theorem 4 in [9]), which is, indeed, an extension of the Condorcet Jury theorem for heterogeneous groups. This leads to the result that:

If $\bar{p}_{(N_R, N_U-1)} > \frac{1}{2} + \frac{1}{2(N_R + N_U - 1)}$ and $|S \setminus \{s_i\}| = N_R - 1 + N_U \geq 3$, then the following equation holds:

$$\text{Prob}(y_{(N_R, N_U-1)} = 0 | T = 0) > \bar{p}_{(N_R, N_U-1)}. \tag{19}$$

Starting from the inequality (17) and following the same arguments of the proof as in Theorem 1 we obtain:

$$\begin{aligned}
N_R p_R + (N_U - 1)p_U > (N_R + N_U)/2 &\Rightarrow \frac{N_R p_R + (N_U - 1)p_U}{N_R + N_U - 1} > \frac{N_R + N_U}{2(N_R + N_U - 1)} \\
&\Rightarrow \bar{p}_{(N_R, N_U-1)} > \frac{1}{2} + \frac{1}{2(N_R + N_U - 1)}.
\end{aligned} \tag{20}$$

Using the above result and by following steps analogous to those in Theorem 1, we can prove that:

$$P_{(N_R, N_U-1)} > 1/2. \tag{21}$$

We will now prove that $\text{Prob}(x_i = y_{(N_R, N_U-1)}) < 1/2$. To achieve this, consider the function $h(\cdot)$ defined by:

$$h(\rho) = p_U \cdot \rho + (1 - p_U) \cdot (1 - \rho) \tag{22}$$

whence, it is easy to see that: $h(P_{(N_R, N_U-1)}) = \text{Prob}(x_i = y_{(N_R, N_U-1)})$.

Let us investigate the dynamics of $h(\rho)$ by studying its derivative, $h'(\rho)$. Since $h'(\rho) = 2p_U - 1$, and $p_U < 1/2$, we see that $2p_U - 1 < 0$ which is equivalent to the conclusion that $h'(\rho) < 0$. Therefore $h(x)$ is a strictly decreasing function.

As a boundary condition, we see that $h(1/2) = 1/2 p_U + 1/2(1 - p_U) = 1/2$. Indeed, by virtue of the fact that

the function $h(\cdot)$ is strictly decreasing we obtain:

$$\text{If } \rho > 1/2 \Rightarrow h(\rho) < h(1/2) = 1/2. \quad (23)$$

In particular, we now apply the inequality (23) for the particular case when $\rho = P_{(N_R, N_U-1)}$. We have previously demonstrated in Eq. (21) that $P_{(N_R, N_U-1)} > 1/2$. Consequently, we obtain:

$$h(P_{(N_R, N_U-1)}) < 1/2$$

which is equivalent to:

$$\text{Prob}(x_i = y_{(N_R, N_U-1)}) < 1/2,$$

proving the theorem. □

4.2 Construction of the Learning Automata

The results that we presented in the previous section form the basis of our LA-based solution. We explain this below, including the strategy by which the majority vote is invoked.

In the partitioning strategy, with each sensor s_i we associate a 2-action L_{RI} automaton \mathcal{A}^i , $(\Sigma^i, \Pi^i, \Gamma^i, \Upsilon^i, \Omega^i)$, where Σ^i is the set of actions, Π^i is the set of action probabilities, Γ^i is the set of feedback inputs from the Environment, and Υ^i is the set of action probability updating rules.

1. *The set of actions of the automaton:* (Σ^i)

The two actions of the automaton are α_k^i , for $k \in \{0, 1\}$, i.e, α_0^i and α_1^i

2. *The action probabilities:* (Π^i)

$P_k^i(n)$ represent the probabilities of selecting the action α_k^i , for $k \in \{0, 1\}$, at step n . Initially, $P_k^i(0) = 0.5$, for $k = 0, 1$.

3. *The feedback inputs from the Environment to each automaton:* (Γ^i)

Let the automaton select either the the action α_0^i or α_1^i . Then, the responses from the Environment and the corresponding probabilities are tabulated below. For a chosen action, the Environment will respond by a “Reward”, or a “Penalty”. The conditional probabilities of the “Reward”, and “Penalty” are also specified in the tables.

ACTION	ASSOCIATED PROBABILITY	
	REWARD	PENALTY
α_0^i	$\text{Prob}(x_i = y_{(N_R-1, N_U)})$	$1 - \text{Prob}(x_i = y_{(N_R-1, N_U)})$
α_1^i	$1 - \text{Prob}(x_i = y_{(N_R-1, N_U)})$	$\text{Prob}(x_i = y_{(N_R-1, N_U)})$

Table 1: Reward and Penalty probabilities for sensor $s_i \in \mathcal{S}_R$

A brief explanation about the equations in these tables could be beneficial.

ACTION	ASSOCIATED PROBABILITY	
	REWARD	PENALTY
α_0^i	$Prob(x_i = y_{(N_R, N_U-1)})$	$1 - Prob(x_i = y_{(N_R, N_U-1)})$
α_1^i	$1 - Prob(x_i = y_{(N_R, N_U-1)})$	$Prob(x_i = y_{(N_R, N_U-1)})$

Table 2: Reward and Penalty probabilities for sensor $s_i \in \mathcal{S}_U$

- (a) The LA system is rewarded if it chooses action α_0^i , in which case the reading of the sensor s_i agrees with the opinion of the majority voting scheme associated with $S \setminus \{s_i\}$. This occurs with probability $Prob(x_i = y_{(N_R-1, N_U)})$ whenever $s_i \in \mathcal{S}_R$ and with probability $Prob(x_i = y_{(N_R, N_U-1)})$ whenever $s_i \in \mathcal{S}_U$.
- (b) Alternatively, the system is rewarded if it chooses action α_1^i , in which case the reading of the sensor s_i disagrees with the opinion of the majority voting scheme associated with $S \setminus \{s_i\}$. This occurs with probability $1 - Prob(x_i = y_{(N_R-1, N_U)})$ whenever $s_i \in \mathcal{S}_R$ and with probability $1 - Prob(x_i = y_{(N_R, N_U-1)})$ whenever $s_i \in \mathcal{S}_U$.
- (c) The penalty scenarios are the reversed ones.

4. *The action probability updating rules:* (Υ^i)

First of all, since we are using the L_{RI} scheme, we ignore all the penalty responses. Upon reward, we obey the following updating rule :

If α_k^i for $k \in \{0, 1\}$ was rewarded then,

$$\begin{aligned} P_{1-k}^i(n+1) &\leftarrow \theta \times P_{1-k}^i(n) \\ P_k^i(n+1) &\leftarrow 1 - \theta \times P_{1-k}^i(n), \end{aligned}$$

where $0 \ll \theta < 1$ is the L_{RI} reward parameter.

Before we prove the properties of the overall system, we first state a fundamental result of the L_{RI} learning schemes which we will repeatedly allude to in the rest of the paper.

Lemma 1. *An L_{RI} learning scheme with parameter $0 \ll \theta < 1$ is ϵ -optimal, whenever an optimal action exists. In other words, $\lim_{\theta \rightarrow 1} \lim_{n \rightarrow \infty} P_k^i(n) \rightarrow 1$.*

The above result is well known [26, 35, 45]. By virtue of this property, we are guaranteed that for any L_{RI} scheme with the two actions $\{\alpha_0, \alpha_1\}$, if $\exists k \in \{0, 1\}$ such that $c_k^i < c_{1-k}^i$, then the action α_k^i is optimal, and for this action $P_k^i(n) \rightarrow 1$ as $n \rightarrow \infty$ and $\theta \rightarrow 1$, where the $\{c_k^i\}$, are the penalty probabilities for the two actions of the automaton \mathcal{A}^i .

By invoking the property of the L_{RI} learning scheme, we state and prove the convergence property of the overall system.

Theorem 3. *Consider the scenario when $(N_R - 1)p_R + N_U p_U > (N_R + N_U)/2$ and when $N_R + N_U - 1 \geq 3$. If each of the LA in the system uses the L_{RI} scheme with a parameter θ which is arbitrarily close to unity, the following is true:*

$$\begin{aligned} \text{If } s_i \in \mathcal{S}_R, \quad &\text{then } \lim_{\theta \rightarrow 1} \lim_{n \rightarrow \infty} P_1^i(n) \rightarrow 1; \\ \text{If } s_i \in \mathcal{S}_U, \quad &\text{then } \lim_{\theta \rightarrow 1} \lim_{n \rightarrow \infty} P_0^i(n) \rightarrow 1. \end{aligned}$$

Proof: To prove the theorem, we shall treat the two cases separately.

Case 1: $s_i \in \mathcal{S}_R$: Based on the result of Theorem 1, we know that the following inequality holds:

$$Prob(x_i = y_{(N_R-1, N_U)}) > 0.5.$$

Therefore, we can deduce that

$$Prob(x_i = y_{(N_R-1, N_U)}) > 1 - Prob(x_i = y_{(N_R-1, N_U)}). \quad (24)$$

If we now consider the entries of Table 1 that specify the penalty probabilities $s_i \in \mathcal{S}_R$, we see that:

$$c_1^i = Prob(x_i = y_{(N_R-1, N_U)}) < c_0^i = 1 - Prob(x_i = y_{(N_R-1, N_U)}),$$

implying thus that the action α_1^i is the optimal one. Consequently, by virtue of Lemma 1, for this action:

$$P_1^i(n) \rightarrow 1 \text{ as } n \rightarrow \infty \text{ and } \theta \rightarrow 1,$$

proving the result for this case!

Case 2: $s_i \in \mathcal{S}_U$: If we now consider the result of Theorem 2, we see that the following inequality holds:

$$Prob(x_i = y_{(N_R, N_U-1)}) < 0.5.$$

Consequently, we observe that

$$Prob(x_i = y_{(N_R, N_U-1)}) < 1 - Prob(x_i = y_{(N_R, N_U-1)}). \quad (25)$$

The analogous actions and associated probabilities from Table 2 specify the penalty probabilities $s_i \in \mathcal{S}_U$, as:

$$c_0^i = 1 - Prob(x_i = y_{(N_R, N_U-1)}) < c_1^i = Prob(x_i = y_{(N_R, N_U-1)}),$$

implying that the action α_0^i is optimal. Consequently, for this action:

$$P_0^i(n) \rightarrow 1 \text{ as } n \rightarrow \infty \text{ and } \theta \rightarrow 1.$$

The theorem is thus proven. □

4.3 Theoretical Results for the Case Where: “Lying Prevails over Truth”

In the previous section, we had investigated the theoretical results for the case when the Environment is characterized by the condition: $(N_R - 1)p_R + N_U p_U > (N_R + N_U)/2$, which we, informally, expressed as the scenario in which “Truth Prevails over Lying”. In the application domain, this, of course, corresponds to the scenario when it is more likely for the sensors to be reliable than unreliable. However, there is also the extremely interesting and fascinating case when “Lying Prevails over Truth-Telling”, i.e., when it is more likely for the sensors to be unreliable than reliable. In this section, we will analyze this scenario, and provide the theoretical results for the case where $(N_R - 1)p_R + N_U p_U < (N_R + N_U)/2 - 1$.

Theorem 4. Consider the scenario when $N_R p_R + (N_U - 1)p_U < (N_R + N_U)/2 - 1$ and when $N_R + N_U - 1 \geq 3$. Let $s_i \in \mathcal{S}_R$. Consider now the agreement between the opinion of a reliable sensor s_i and the opinion of the majority formed by all the rest of the sensors $S \setminus \{s_i\} = (\mathcal{S}_R \setminus \{s_i\}) \cup \mathcal{S}_U$. Let $y_{(N_R-1, N_U)}$ be the decision of a majority voting scheme $S \setminus \{s_i\}$, based on the responses of $N_R - 1$ reliable and N_U unreliable sensors. Then, if x_i is the output of s_i : $Prob(x_i = y_{(N_R-1, N_U)}) < 0.5$.

Proof: The proof relies on a symmetry property which we shall record presently. As mentioned in the theorem's statement, we work with the premise that $N_R p_R + (N_U - 1) p_U < (N_R + N_U)/2 - 1$. To begin with, if $q_R = 1 - p_R$ and $q_U = 1 - p_U$, we shall prove that under these conditions,

$$(N_R - 1)q_R + N_U q_U > (N_R + N_U)/2.$$

Working through the premise of the theorem, we see that:

$$\begin{aligned} (N_R + N_U)/2 - 1 > (N_R - 1)p_R + N_U p_U &\Rightarrow (N_R + (N_U - 1)) - (N_R p_R + (N_U - 1) p_U) > (N_R + N_U)/2 \\ &\Rightarrow N_R(1 - p_R) + (N_U - 1)(1 - p_U) > (N_R + N_U)/2 \\ &\Rightarrow N_R q_R + (N_U - 1)q_U > (N_R + N_U)/2. \end{aligned} \quad (26)$$

However, we can affirm that $q_U - q_R > 0$ since $p_R > p_U$. By utilizing the fact that $q_U - q_R > 0$:

$$\begin{aligned} N_R q_R + (N_U - 1)q_U + (q_U - q_R) > (N_R + N_U)/2 + (q_U - q_R) &\Rightarrow (N_R - 1)q_R + N_U q_U > (N_R + N_U)/2 + (q_U - q_R) \\ &\Rightarrow (N_R - 1)q_R + N_U q_U > (N_R + N_U)/2 \\ &\Rightarrow (N_R - 1)q_R + N_U q_U > (N_R + N_U)/2. \end{aligned} \quad (27)$$

The last assertion of the above equation, i.e., that $(N_R - 1)q_R + N_U q_U > (N_R + N_U)/2$, specifies a rather straightforward, but non-obvious fact. Indeed, by a simple replacement of the variables, (i.e., by replacing p_R by q_R , and p_U by q_U), and by observing that $q_U > 0.5$ and $q_R < 0.5$, we can invoke arguments similar to those used in Theorem 1 to prove our intended result.

To do this, let $\bar{q}'_{(N_R, N_U-1)} = \frac{N_R q_R + (N_U - 1)q_U}{N_R + N_U - 1}$ be the mean of individual q_i 's in a heterogeneous group, where, by definition, q_i is defined as $q_i = \text{Prob}(x_i = 1|T = 0) = \text{Prob}(x_i = 0|T = 1)$. We now again apply the result by Boland (identified as Theorem 4 in [9]), which is, indeed, an extension of the Condorcet Jury theorem for these heterogonous groups. This leads to the result that:

If $\bar{q}'_{(N_R, N_U-1)} > \frac{1}{2} + \frac{1}{2(N_R + N_U - 1)}$ and $|S \setminus \{s_i\}| = N_R - 1 + N_U \geq 3$, then the following equation holds:

$$\text{Prob}(y_{(N_R, N_U-1)} = 1|T = 0) = \text{Prob}(y_{(N_R, N_U-1)} = 0|T = 1) > \bar{q}'_{(N_R, N_U-1)}. \quad (28)$$

By invoking arguments similar to those used in the other proofs, we can deduce:

$$(N_R - 1)q_R + N_U q_U > (N_R + N_U)/2 \implies q_{(N_R, N_U-1)} > 0.5.$$

Consequently,

$$\text{Prob}(y_{(N_R, N_U-1)} = 1|T = 0) = \text{Prob}(y_{(N_R, N_U-1)} = 0|T = 1) > 1/2, \quad (29)$$

which is equivalent to stating:

$$\text{Prob}(y_{(N_R, N_U-1)} = 0|T = 0) = \text{Prob}(y_{(N_R, N_U-1)} = 0|T = 0) < 1/2, \quad (30)$$

where the latter quantity is, indeed, $P_{(N_R, N_U-1)}$.

Consider now the following equation that we obtained in the process of proving Theorem 1:

$$\text{Prob}(x_j = y_{(N_R, N_U-1)}) = p_U \cdot P_{(N_R, N_U-1)} + (1 - p_U) \cdot (1 - P_{(N_R, N_U-1)}), \quad (31)$$

using which we shall prove that $\text{Prob}(x_j = y_{(N_R, N_U-1)}) < 1/2$. To do this, we re-consider the same function $g(\cdot)$ that we defined previously in the proof of Theorem 1. The only difference is that the quantity involved in the inequality obeys $P_{(N_R, N_U-1)} < 1/2$. Utilizing the strictly increasing property of $g(\cdot)$, we obtain for

$$1/2 > x, g(1/2) = 1/2 > g(x).$$

Indeed, in particular for $x = P_{(N_R, N_U-1)}$, we have:

$$g(P_{(N_R, N_U-1)}) < 1/2 \implies \text{Prob}(x_i = y_{(N_R-1, N_U)}) < 0.5,$$

proving the theorem. \square

The next theorem, which deals with the analogous case of excluding an unreliable sensor, follows.

Theorem 5. *Consider the scenario when $N_R p_R + (N_U - 1)p_U < (N_R + N_U)/2 - 1$ and when $N_R + N_U - 1 \geq 3$. Let $s_i \in \mathcal{S}_U$. Consider now the agreement between the opinion of an unreliable sensor s_i and the opinion of the majority formed by all the rest of the sensors, $S \setminus \{s_i\} = \mathcal{S}_R \cup \mathcal{S}_U \setminus \{s_i\}$. Let $y_{(N_R, N_U-1)}$ be the decision of a majority voting scheme based on the responses of $S \setminus \{s_i\}$, consisting of N_R reliable and $N_U - 1$ unreliable sensors. Then, if x_i is the output of s_i : $\text{Prob}(x_i = y_{(N_R, N_U-1)}) > 0.5$*

Proof: The proof again relies on a symmetry property akin to the one seen in the previous theorem. We, of course, assume that the society of sensors in which “lying prevails over truth”, and thus, as stated in the theorem statement, $N_R p_R + (N_U - 1)p_U < (N_R + N_U)/2 - 1$. Also, without repeating the notation and the arguments invoked in Theorem 4, we utilize the result that:

$$N_R q_R + (N_U - 1)q_U > (N_R + N_U)/2. \quad (32)$$

The proof now follows the same parallel as the previous one. The primary principle involves defining

$$\bar{q}_{(N_R-1, N_U)} = \frac{(N_R-1)q_R + N_U q_U}{N_R + N_U - 1}.$$

Then we continue to prove that $P_{(N_R, N_U-1)} < 0.5$.

As in the case of Theorem 2, we use⁴ the function $h(\cdot)$ and its decreasing property to show that:

$$h(P_{(N_R, N_U-1)}) > 1/2 \implies \text{Prob}(x_i = y_{(N_R-1, N_U)}) > 0.5,$$

thus proving the theorem. \square

The final theorem which shows the power of the team of LA in such a scenario, follows.

Theorem 6. *Consider the scenario when $(N_R - 1)p_R + N_U p_U < (N_R + N_U)/2 - 1$ and that $N_R + N_U - 1 \geq 3$. Given the L_{RI} scheme with a parameter θ which is arbitrarily close to unity, the following is true:*

$$\begin{aligned} \text{If } s_i \in \mathcal{S}_R, \quad \text{then } \lim_{\theta \rightarrow 1} \lim_{n \rightarrow \infty} P_0^i(n) &\rightarrow 1; \\ \text{If } s_i \in \mathcal{S}_U, \quad \text{then } \lim_{\theta \rightarrow 1} \lim_{n \rightarrow \infty} P_1^i(n) &\rightarrow 1. \end{aligned}$$

⁴The algebraic details of the proof are omitted to avoid repetition.

Proof: To prove the theorem, we again treat the two cases separately.

Case 1: $s_i \in \mathcal{S}_R$: Based on the result of Theorem 4, we can see that the inequality $Prob(x_i = y_{(N_R-1, N_U)}) < 0.5$ holds. We can thus deduce that:

$$Prob(x_i = y_{(N_R-1, N_U)}) < 1 - Prob(x_i = y_{(N_R-1, N_U)}). \quad (33)$$

If we now consider the entries of Table 1 that specify the penalty probabilities $s_i \in \mathcal{S}_R$, we see that:

$$c_1^i = Prob(x_i = y_{(N_R-1, N_U)}) > c_0^i = 1 - Prob(x_i = y_{(N_R-1, N_U)}),$$

implying that for this case, the action α_0^i is the optimal one. Consequently, by virtue of Lemma 1, for this action:

$$P_0^i(n) \rightarrow 1 \text{ as } n \rightarrow \infty \text{ and } \theta \rightarrow 1,$$

proving the result for this case.

Case 2: $s_i \in \mathcal{S}_U$: In this case, based on the result of Theorem 5, we see that the following inequality holds: $Prob(x_i = y_{(N_R, N_U-1)}) > 0.5$.

Therefore we can confirm that

$$Prob(x_i = y_{(N_R, N_U-1)}) > 1 - Prob(x_i = y_{(N_R, N_U-1)}). \quad (34)$$

From the entries of Table 2 that specify the penalty probabilities $s_i \in \mathcal{S}_U$, we obtain:

$$c_0^i = 1 - Prob(x_i = y_{(N_R, N_U-1)}) > c_1^i = Prob(x_i = y_{(N_R, N_U-1)}).$$

This implies that the action α_1^i is the optimal one, and for this action:

$$P_1^i(n) \rightarrow 1 \text{ as } n \rightarrow \infty \text{ and } \theta \rightarrow 1.$$

The theorem is thus proven. \square

4.3.1 Remarks and some Additional Notation

For the case when $N_R p_R + (N_U - 1) p_U > (N_R + N_U)/2$, once the partitioning has taken place, all the LA will have converged to their appropriate partitions. From the results of Theorem 3, we see that the reliable sensors will have converged to action α_1^i , while the unreliable ones will have converged to action α_0^i – both with an arbitrarily large probability. Analogously, from Theorem 6, we see the similar result for the case when $N_R p_R + (N_U - 1) p_U < (N_R + N_U)/2 - 1$. In this case, the actions will be inverted when compared to Theorem 3. In fact, when $N_R p_R + (N_U - 1) p_U < (N_R + N_U)/2 - 1$, the reliable sensors will converge to action α_0^i , while the unreliable ones to action α_1^i with an arbitrarily large probability. We can summarize these results as below:

- Partitioning when $N_R p_R + (N_U - 1) p_U > (N_R + N_U)/2$

- $G_R = \{s_i \in S \text{ such that } \lim_{n \rightarrow \infty} P_1^i(n) = 1\}$

- $G_U = \{s_i \in S \text{ such that } \lim_{n \rightarrow \infty} P_0^i(n) = 1\}$.

- Partitioning when $N_R p_R + (N_U - 1) p_U < (N_R + N_U)/2 - 1$

- $G_R = \{s_i \in S \text{ such that } \lim_{n \rightarrow \infty} P_0^i(n) = 1\}$

- $G_U = \{s_i \in S \text{ such that } \lim_{n \rightarrow \infty} P_1^i(n) = 1\}$.

Indeed, since the results are ϵ -optimal results, if θ is not arbitrarily close to unity, some of the *LA* might fail to converge to the optimal action and thus the set S_R may not necessary be equivalent to G_R . However, as θ is arbitrarily close to unity, G_R will converge exactly to S_R .

4.4 Fusion approaches

We now present two simple fusion schemes that make use of the partitionings in order to improve the quality of the aggregated opinion from the different sensors for guessing the ground truth.

4.4.1 Fusion Scheme with Exclusion: Discarding the opinions of the unreliable sensors

A possible strategy to increase the accuracy of the fusion process is to employ a simple majority voting strategy that excludes all the sensors whose *LA* converged to the action G_U during the partitioning phase. This means that the prediction of the ground truth will be exclusively based on the sensors whose *LA* converged to the action G_R .

4.4.2 Fusion Scheme with Inversion: Inverting the opinions of the unreliable sensors

In this subsection, instead of excluding the readings of the unreliable sensors, we propose intelligently combining the readings from both the reliable and unreliable sensors when evaluating the ground truth. In fact, we opt to invert the decision of the unreliable sensors as inferred by the *LA* algorithm, rendering them to be informative. Thus, for every reading x_i from a sensor s_i whose *LA* has converged to the action G_U , we record the reverse of the reading. Indeed, the majority voting scheme will be equivalent to one that aggregates the votes from a group of sensors consisting of:

- N_R reliable sensors, each possessing a correctness probability p_R , and
- N_U unreliable that have been rendered reliable and that possess a correctness probability $p'_U = 1 - p_U$ (where $p'_U = 1 - p_U > 0.5$). By the phrase, rendered reliable, we mean that we are inverting the respective readings of the sensors in G_U .

We now report the experimental results that we have obtained by testing the strategies explained in the previous sections.

5 Experimental results

The performance of the *LA*-based partitioning as well as the two fusion schemes that make use of the partitioning, have been rigorously tested by simulation in a variety of parameter settings, and the results that we have obtained are truly conclusive. In the interest of brevity, we merely report a few representative (and typical) experimental results, so that the power of our proposed methodology can be justified. In the experiments, the settings were chosen so that the condition $N_R p_R + (N_U - 1) p_U > (N_R + N_U)/2$ was met, reflecting the phenomenon where “the truth prevails over lying”.

5.1 Performance of the Partitioning

We first examine the convergence speed of the *LA* algorithm. Since a *LA* is associated with every sensor (whether it is reliable or unreliable), where each possesses its own distinct reward probabilities for its respective actions, they will, clearly, have different convergence speeds, as is well-known in the theory of *LA*. Observe that the convergence of the individual *LA* is defined in terms of its ϵ -convergence, where the *LA* were deemed to have converged if one of its action probabilities attained the value $1 - \epsilon^5$. Formally:

- If $P_0^i(n) \geq 1 - \epsilon$, then the *LA* has converged to the action α_0^i ;
- If $P_1^i(n) \geq 1 - \epsilon$, then the *LA* has converged to the action α_1^i .

We also initialized all the *LA* at time instant $t = 0$, to have the values: $P_0^i(t) = P_1^i(t) = 0.5$.

To render the results meaningful, we took an ensemble average of 1,000 experiments, and computed the average convergence times for the *LA* associated with the sensors in \mathcal{S}_R and for those in \mathcal{S}_U . Although the experiments related to the convergence speeds were performed for different settings, we only report some representative results in which we fixed N_R to 20, N_U to 10 and $\theta = 0.8$, and where we also simultaneously varied p_R and p_U . In fact, it turns out that these parameters will influence the agreement probability (reward probability), and consequently the speed of convergence as per the theoretical results reported earlier. The results obtained are given in Table 3.

(p_R, p_U)	Average Convergence time for $s_i \in \mathcal{S}_R$	Average Convergence time for $s_i \in \mathcal{S}_U$
(0.8, 0.1)	40.91	43.37
(0.8, 0.2)	36.41	44.11
(0.85, 0.1)	31.16	30.84
(0.85, 0.2)	29.53	35.82
(0.9, 0.1)	26.14	26.71
(0.9, 0.2)	25.90	32.81
(0.95, 0.1)	23.51	25.88
(0.95, 0.2)	23.47	32.27

Table 3: Average convergence time for the case when $(N_R, N_U) = (20, 10)$.

By examining this table, we observe:

1. Remarkably, the *LA* converge very rapidly. In fact, on the average, the *LA* were able to determine the optimal partition in less than 44.11 time instances, which, incidentally, was the largest value in the table.
2. Earlier, we proved that the probability of a reward is a decreasing function of p_U whenever we deal with an unreliable sensor. As we fix p_R and vary p_U , we observe that the convergence speed decreases, which, in this case, translates into a decreased reward probability.
3. In addition, we p_R is increased towards unity and as p_U is decreased closer to 0, the convergence speed increases for both the individual *LA* and for those included in \mathcal{S}_R . This reflects the concept that the environment becomes “easier” when the sensor is less noisy (i.e., (p_R, p_U) approaches $(1, 0)$) and consequently, the *LA* converge faster to the optimal actions. By “easier”, we mean that the difference between the reward probabilities of the actions of the *LA* becomes larger, and thus, the *LA* will converge both faster and with a higher probability to the optimal action. This is consistent with the well-known results in the field of *LA*.

⁵The value of ϵ was set to be 0.01.

4. Consider the case when $(p_R, p_U) = (0.95, 0.1)$ as reported in the table. The respective convergence speeds for the LA associated with the reliable and unreliable sensors are 23.51 and 25.88 respectively. However, as the sensors became more noisy by decreasing p_R to 0.8, the task of differentiating between the partitions became more difficult. Indeed, the convergence speed for LA associated with a reliable sensor dropped down to 36.41, and the speed of the LA associated with unreliable sensor became 44.11.

5.2 Fusion Scheme with Exclusion

We now compare the “Fusion Scheme with Exclusion” explained in Section 4.4.1 with the deterministic Majority Voting (MV) strategy that incorporates all the sensors in S . As detailed earlier, the latter scheme relies exclusively on the decision of the vote of the majority of the sensors that converged to the G_R partition. Let $P(C_c)$ denote the probability of the consensus being correct, i.e, that the probability that the vote of the majority coincides with the ground truth. Table 4 reports the result of the comparison for the case when N_R and N_U are both equal to 10.

(p_R, p_U)	$P(C_c)$ for Fusion Scheme with Exclusion	$P(C_c)$ for MV for all sensors
(0.75, 0.45)	0.921	0.766
(0.75, 0.4)	0.921	0.87
(0.75, 0.35)	0.921	0.599
(0.75, 0.3)	0.921	0.5
(0.8, 0.45)	0.972	0.84
(0.8, 0.4)	0.972	0.775
(0.8, 0.35)	0.972	0.574
(0.8, 0.3)	0.9672	0.604

Table 4: Comparisons of the value of $P(C_c)$, the probabilities of the consensus being correct for different values of (p_R, p_U) , and for the different approaches for $N_R = 10$ and $N_U = 10$.

We observe from the table:

1. The distribution of T does not play a role in determining the value of $P(C_c)$ for the Fusion Scheme with Exclusion because of the symmetry property of the fault. As one can see, the results we report are conclusive. In fact, we were able to increase the value of $P(C_c)$ quite remarkably. For example, for the case when $(p_R, p_U) = (0.75, 0.3)$, our scheme yielded a value of 0.921 for $P(C_c)$, while the scheme which operated with the majority voting involving all the sensors yielded the value of only 0.5.
2. The value of $P(C_c)$ for our Fusion Scheme with Exclusion is immune to the variation of p_U . For example, for the entries corresponding to $p_R = 0.75$, we see that $P(C_c)$ is equal to 0.921 even if p_U changes, for example, by taking the values 0.45, 0.35 and 0.3.

Consider now the case when we double the value N_R from 10 to 20 while the value of N_U is equal to 10. As expected, we see from Table 5, the value of $P(C_c)$ for our scheme increases and approaches unity.

5.3 Fusion Scheme with Inversion

In Table 6, we report the results when we fix N_R to 20 and N_U to 10 and compare the result of a simple MV scheme involving all sensors with the Fusion Scheme with Inversion presented in Section 4.4.2. We can make the following observations:

(p_R, p_U)	$P(C_C)$ for Fusion Scheme with Exclusion	$P(C_C)$ for MV for all sensors
(0.75, 0.45)	0.9986	0.943
(0.75, 0.4)	0.9986	0.919
(0.75, 0.35)	0.9986	0.888
(0.75, 0.3)	0.9986	0.85
(0.8, 0.45)	0.997	0.98
(0.8, 0.4)	0.997	0.97
(0.8, 0.35)	0.997	0.954
(0.8, 0.3)	0.997	0.934

Table 5: Comparisons of $P(C_C)$, the probabilities of the consensus being correct for different values of (p_R, p_U) , and for the different approaches for $N_R = 20$ and $N_U = 10$.

1. Under a fixed value of p_R , a smaller value of p_U yields a higher value for $P(C_C)$ for the Fusion Scheme with Inversion. For example, for a fixed value of $p_R = 0.8$, $P(C_C)$ increases from 0.929 to 0.986 as we decrease p_U from 0.45 to 0.3. This is due to the fact that a smaller value for p_U actually implies a higher value for $1 - p_U$. Thus, a sensor which is highly unreliable, can be transformed into one that is highly reliable – thanks to the operation of inverting its reading!
2. The results for the case where we increase N_U to 20 is reported in Table 7. Indeed, in general we can affirm from Tables 6 and 7 that the Fusion Scheme with Inversion outperforms the simple majority voting involving all sensors in all the settings.
3. However, by comparing both Tables 6 and 7, we remark that $P(C_C)$ for the scheme with inversion does not necessarily increase as we increase N_R , the number of reliable sensors.

(p_R, p_U)	$P(C_C)$ for Fusion Scheme with Inversion	$P(C_C)$ $P(C_C)$ for MV for all sensors
(0.75, 0.45)	0.974	0.943
(0.75, 0.4)	0.984	0.919
(0.75, 0.35)	0.99	0.888
(0.75, 0.3)	0.994	0.85
(0.8, 0.45)	0.992	0.98
(0.8, 0.4)	0.995	0.97
(0.8, 0.35)	0.997	0.954
(0.8, 0.3)	0.998	0.934

Table 6: Comparisons of $P(C_C)$ the probabilities of the consensus being correct for different values of (p_R, p_U) , and for the different approaches for $N_R = 20$ and $N_U = 10$.

6 Conclusion

Sensor Fusion has become a prevalent research topic due to the wide deployment of sensor technology in the industry and in our daily life. In this paper, we have considered an extremely pertinent problem in the area of Sensor Fusion, namely the one of identifying unreliable sensors without knowing the ground truth. Although paradigms like the one that involves majority voting offer a generic prediction strategy for the ground truth by aggregating sensor-provided readings, they are prone to error caused by unreliable sensors. Clearly, such unreliable sensors may degrade the quality of the aggregated information.

(p_R, p_U)	$P(C_C)$ for Fusion Scheme with Inversion	$P(C_C)$ for MV for all sensors
(0.75, 0.45)	0.974	0.8821
(0.75, 0.4)	0.985	0.804
(0.75, 0.35)	0.994	0.699
(0.75, 0.3)	0.997	0.571
(0.8, 0.45)	0.987	0.941
(0.8, 0.4)	0.994	0.892
(0.8, 0.35)	0.998	0.816
(0.8, 0.3)	0.998	0.71

Table 7: Comparisons of $P(C_C)$ the probabilities of the consensus being correct for different values of (p_R, p_U) , and for the different approaches for $N_R = 20$ and $N_U = 20$.

A large body of the research in sensor fusion deduces the reliability of the sensors either online or offline by assuming that one can access the ground truth. While this is a desirable option, unfortunately, in real life, such an assumption does not always hold. The question of whether a solution to the problem even exists in this scenario is open. In this paper, we have presented a novel solution for the problem using tools provided by the family of Learning Automata (LA). Unlike most reported approaches, our scheme does not require prior knowledge of the ground truth. Instead, our solution gradually learns the identity and characteristics of the sensors which provide reliable readings, and of those who provide unreliable measurements.

In addition to presenting rigorous theoretical results for the unsolved problem, we have also included comprehensive empirical results that demonstrate that our LA-based scheme achieves optimal partitioning with a high convergence speed.

A possible extension of this research, which we are currently working on, is to develop the analogous methodology for continuous sensor readings instead of boolean ones. In addition, we advocate that it is possible to render the two phases of partitioning and fusion to be interleaving by using the information contained in the all N vectors $P^i(t) = [P_0^i(t), P_1^i(t)]$ at time n . Thus, the fusion can take place at each time instant t , instead of delaying or postponing the execution of the proposed fusion (with/without Inversion/Exclusion) until all the LA have converged.

The entire issue of whether we can use the field of Random Races [36] to achieve a comparison between the various sensors also holds a great potential. Finally, the question of investigating the effect of adding more voters on $P(C_C)$ has not been considered here. Some details about this scenario can be found in [19].

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