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# On Tetrahedralisations Containing Knotted and Linked Line Segments

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**Abstract**

This paper considers a set of twisted line segments in 3d such that they form a knot (a closed curve) or a link of two closed curves. Such line segments appear on the boundary of a family of 3d *indecomposable* polyhedra (like the Schönhardt polyhedron) whose interior cannot be tetrahedralised without additional vertices added. On the other hand, a 3d (non-convex) polyhedron whose boundary contains such line segments may still be decomposable as long as the twist is not too large. It is therefore interesting to consider the question: when there exists a tetrahedralisation contains a given set of knotted or linked line segments?

In this paper, we studied a simplified question with the assumption that all vertices of the line segments are in convex position. It is straightforward to show that no tetrahedralisation of 6 vertices (the three-line-segments case) can contain a trefoil knot. Things become interesting when the number of line segments increases. Since it is necessary to create new interior edges to form a tetrahedralisation. We provided a detailed analysis for the case of a set of 4 line segments. This leads to a crucial condition on the orientation of pairs of new interior edges which determines whether this set is decomposable or not. We then prove a new theorem about the decomposability for a set of $n$ ($n \geq 3$) knotted or linked line segments. This theorem implies that the family of polyhedra generalised from the Schönhardt polyhedron by Rambau [1] are all indecomposable.



## 1. Introduction

We say that a 3d polyhedron is *decomposable* if there exists a tetrahedralisation of this polyhedron which uses only its own vertices, i.e., no additional vertex is allowed. Otherwise, it is *indecomposable*. There are many published examples of indecomposable polyhedra, see [1–9]. Among these, the Schönhardt polyhedron [3], which is the twisted prism over a triangle, is the simplest one. Furthermore, Rambau proved that any twisted prisms over an $n$-gon ($n \geq 3$) is indecomposable [1]. It includes the Schönhardt polyhedron as a special case.

The existence of indecomposable polyhedra is a major difficulty in many problems. It is shown that to determine whether a given 3d polyhedron can be decomposed or not is NP-complete [10]. The Schönhardt polyhedron appears

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as an important example in the study of the flip-graph of all triangulations of a given point set [11]. Since additional points, so-called *Steiner points*, may be needed to form a tetrahedralisation, however it is shown that for a polyhedron with $n$ vertices, $\Omega(n^2)$ Steiner points is indeed necessary in the worst case [6]. In 3d tetrahedral mesh generation, the existence of indecomposable polyhedra has been repeatedly cited as the major obstacle in the boundary recovery step, see e.g. [12–15]. We just mention few (major) difficulties in 3d boundary recovery:

- There is no guarantee that any constraint (edges or faces) can be recovered by a sequence of local topological operations, so-called (edge and face) flips.
- There is no guarantee that any vertex from a given tetrahedral mesh can be removed by a sequence of flips.
- There is no rule to determine which location should Steiner points be placed.
- There is no a priori estimation on the necessary number of Steiner points to tetrahedralise a given 3d polyhedron.

On the other hand, several classes of 3d non-convex polyhedra which can be tetrahedralised without Steiner points are presented [16,17]. In particular, Shewchuk proved a condition which guarantees the existence of a constrained Delaunay tetrahedralisation [18].

The motivation of this work starts from the following observations: Figure 1 shows a 3d non-convex polyhedron with 6 vertices. It is combinatorially equivalent to the well-known Schönhardt polyhedron [3]. When we connect the three non-convex line segments (shown in red color) with the three convex line segments (shown in black color), we get a piecewise linear trefoil knot which has the minimum stick number 6.

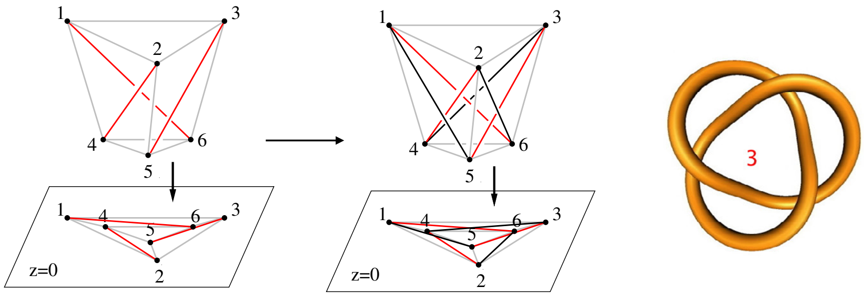


Fig. 1. The line segment trefoil knot obtained from a 3d non-convex polyhedron which is combinatorially equivalent to the Schönhardt polyhedron [3].

Figure 2 shows another 3d non-convex polyhedron with 8 vertices. It is combinatorially equivalent to the twisted prism over a 4-gon (a square). When we connect the four non-convex line segments with the 4 convex line segments, we get two closed piecewise linear curves which are linked together. It is a double link which has four crossings.

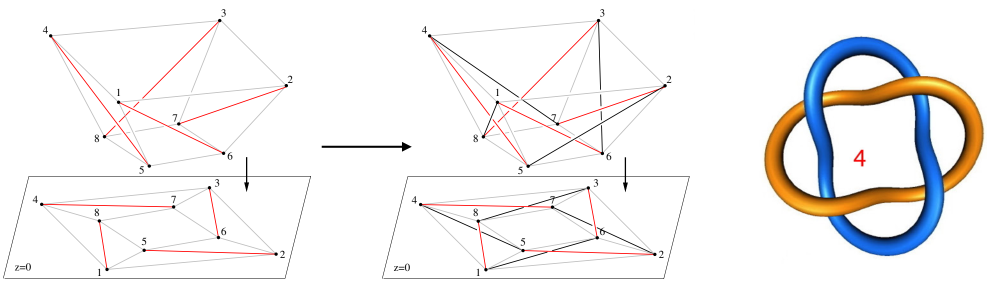


Fig. 2. The line segment link from the twisted 4-gon prism.

In general, one can obtain knots or links with arbitrary number of crossings from the boundary of a non-convex polyhedra which contains a set of twisted line segments. Figure 3 and Figure 5 show two more examples.

In this paper, we study the decomposability of 3d non-convex polyhedra whose boundary contains knotted or linked line segments. This family includes the Rambau's polyhedra [1]. It is not obvious whether such a polyhedron

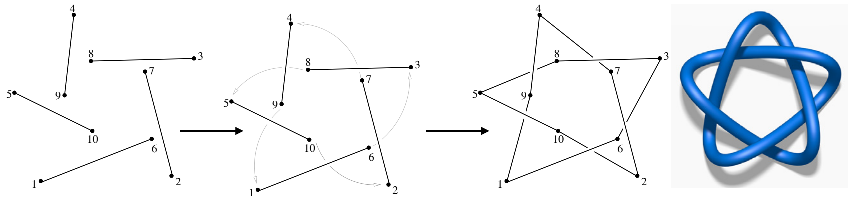

Fig. 3. The line segment knot from the twisted 5-gon prism. In fact, it is a set of pentafoil knot which is formed by 10 line segments.

is decomposable or not. It depends on many issues, like the shape of the top and bottom polygons, the relative orientation of them, and the amount of twist. Rambau's theorem [1] states that a (just slightly) twisted prism with congruent and parallel top and bottom faces is indecomposable. This condition is sufficient but not necessary. Our experience shows that even the top and bottom faces of a prism are not congruent and not parallel, if it is twisted enough, then the prism will be indecomposable.

Instead of considering the whole boundary of such polyhedron, we focus only on the sets of twisted line segments, that is, those edges shown in red in Figure 1 and 2. We believe that they are one of the fundamental reasons that cause the indecomposability of these non-convex polyhedra. On the other hand, this question seems not being well studied in the literature of computational geometry and mesh generation. So far, very few work considered tetrahedralisations of a set of line segments. One interesting work is by Yang and Wang [22], who investigated several algorithmic problems in finding tetrahedralisations whose edge set is a subset of the edge set of a tetrahedralisation. They showed that it is NP-complete to determine whether a set of line segments is tetrahedralisable.

*Outline.* We first define such a set of twisted line segments in Section 2. We then study the decomposability question from the simplest case, which only has three line segments in Section 3. The result is not surprising (and maybe already done somewhere before). We show that no tetrahedralisation of 6 vertices can contain a trefoil knot. Things become interesting when the number of line segments increases. Importantly, we found there exist tetrahedralisations of 8 vertices (in convex position) whose edge graphs can contain a double link. This certainly makes the general decomposability question of an arbitrary set non-trivial. We first give an analysis of the decomposability for the case when there are only 4 line segments in Section 4. We gain insight to the crucial conditions for the decomposability. We then proof a theorem about the decomposability in general case with arbitrary number of line segments in Section 5. Finally, we discuss some possible applications of this study in 3d mesh generation in Section 6.

## 2. Definitions

Consider two non-coplanar and non-parallel vectors in $\mathbb{R}^3$. Their spatial relation is in one of the two orientation conventions, namely the Left-handed Rule and the Right-handed Rule, see Figure 4. Let $v - u$ denote the vector from point $u$ to point $v$. Assume that there are two parallel vectors, $v - u$ and $y - x$, if the points $u$, $x$ don't move and the vectors twisted by left hand such that they form a pair of non-coplanar and non-parallel vectors, we say they follow the Left-handed Rule, see Fig 4 left. Likewise, if the vectors twisted by right hand, they follow the Right-handed Rule, see Fig 4 right. We say that two vectors $v - u$ and $y - x$ form a *left-handed turn* if they follow the Left-handed Rule. Likewise, they form a *right-handed turn* if they follow the Right-handed Rule.

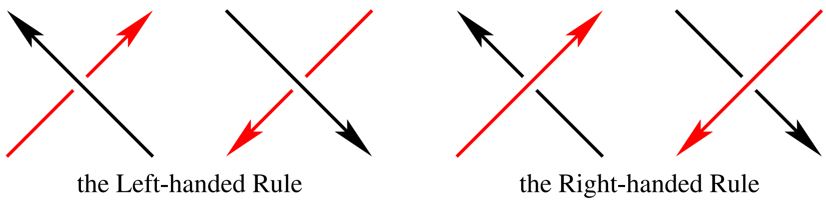


Fig. 4. The two orientations to distinguish two vectors in $\mathbb{R}^3$.

Let $A := \{a_0, a_1, \ldots, a_{m-1}\}$ and $B := \{b_0, b_1, \ldots, b_{m-1}\}$ be two cyclically ordered sets of $m$ ($m \geq 3$) distinct points along two closed curves in $\mathbb{R}^3$, respectively. We are interested in the following set of $m$ line segments which connects vertices from $A$ to $B$, see Figure 5 Left:

$$\mathcal{S}_m := \{a_0b_0, a_1b_1, \ldots, a_{m-1}b_{m-1}\}.$$

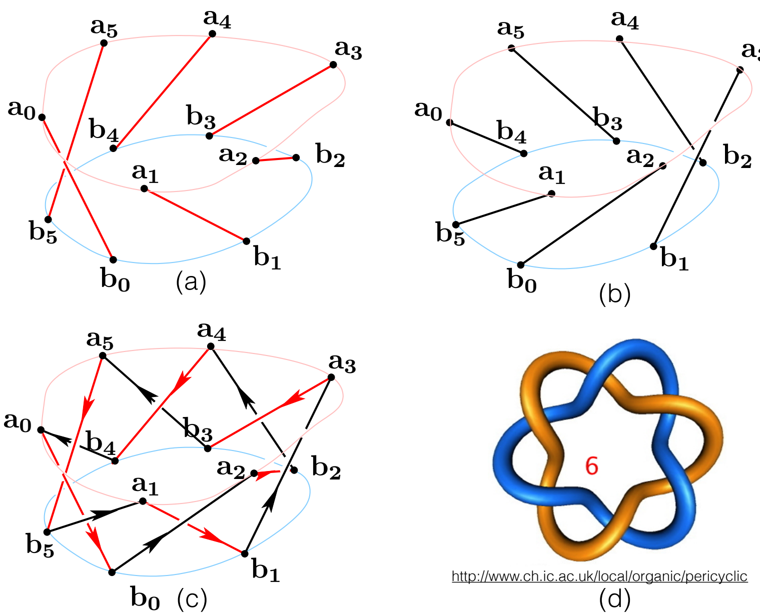


Fig. 5. A left-handed cyclic set of line segments, $\mathcal{S}_6$ (a) the dual line segments of it, in fact, it is a right-handed one (b). The link $k = \cup\mathcal{K}_m$ (c) and its corresponding topological version (d).

Throughout the rest of this paper, the indices of the points $a_i \in A$ and $b_j \in B$ are values with in the range $0, \ldots, m-1$. Additions and subtractions involving indices will be modulo $m$.

We say that $\mathcal{S}_m$ is a *cyclic set of line segments* if it has one of the following **properties**: for all $i = 0, \ldots, m-1$,

(1) $b_i - a_i$ and $a_{i+1} - b_{i-1}$ form a left-handed turn; or
(2) $b_i - a_i$ and $a_{i-1} - b_{i+1}$ form a right-handed turn.

We say that $\mathcal{S}_m$ is *left-handed* if it is in case *(1)*, otherwise, it is *right-handed*. Figure 5 shows an example of a left-handed $\mathcal{S}_m$.

The above property means that each line segment $a_ib_i \in \mathcal{S}_m$ is "tangled" with both of its neighbours, i.e., $a_{i-1}b_{i-1}$ and $a_{i+1}b_{i+1}$. In particular, there is another set $\mathcal{S}'_m$ of line segments which goes from $B$ to $A$, such that $\mathcal{S}_m$ and $\mathcal{S}'_m$ are in a bijective relation. The elements of $\mathcal{S}'_m$ are:

(i) if $\mathcal{S}_m$ is left-handed, then the corresponding line segment of $a_ib_i \in \mathcal{S}_m$ is $b_{i-1}a_{i+1} \in \mathcal{S}'_m$, i.e.,

$$\mathcal{S}'_m := \{b_0a_2, b_1a_3, \ldots, b_{m-2}a_0, b_{m-1}a_1\};$$

(ii) if $\mathcal{S}_m$ is right-handed, then the corresponding line segment of $a_ib_i \in \mathcal{S}_m$ is $b_{i+1}a_{i-1} \in \mathcal{S}'_m$, i.e.,

$$\mathcal{S}'_m := \{b_1a_{m-1}, b_2a_0, \ldots, b_{m-1}a_{m-3}, b_0a_{m-2}\}.$$

Each line segment in $\mathcal{S}'_m$ will pass the same "outside" of its correspond line segment in $\mathcal{S}_m$.

We show that this property makes the two sets of line segments knotted or linked together. Let $\mathcal{K}_m := \mathcal{S}_m \cup \mathcal{S}'_m$ be the collection of $2m$ line segments, see Figure 5 (c).

**Proposition 1.** *$\mathcal{K}_m$ is either an alternating knot with crossing number m, or an alternating link of two closed curves with linking number $\pm m/2$.*

*Proof.* Given a $\mathcal{K}_m$, we show that it corresponds to a (knot or link) diagram in the plane. The construction of the diagram is as follows: first duplicate two vertices: either $a_i$ and $b_{i-1}$ (if $\mathcal{S}_m$ is left-handed), or $a_i$ and $b_{i+1}$ (if $\mathcal{S}_m$ is right-handed), then spread all line segments in $\mathcal{K}_m$ in the plane in such a way that the crossings are at the edge pairs $a_ib_i$ and $b_{i-1}a_{i+1}$ (if $\mathcal{S}_m$ is left-handed) or $a_ib_i$ and $b_{i+1}a_{i-1}$ (if $\mathcal{S}_m$ is right-handed). We then obtain a diagram which has a one-to-one correspondence with $\mathcal{K}_m$ with respect to their vertices, edges, and crossings, respectively.

It remains to determine the type of each crossing, i.e., whether it is an under-cross or an over-cross. This can be determined by the property of $\mathcal{S}_m$ as follows: if $\mathcal{S}_m$ is left-handed, we make all $a_ib_i$'s under-cross their corresponding edges $b_{i-1}a_{i+1}$'s. If $\mathcal{S}_m$ is right-handed, we make all $a_ib_i$'s over-cross their corresponding edges $b_{i+1}a_{i-1}$'s. Figure 6 shows two obtained diagrams.

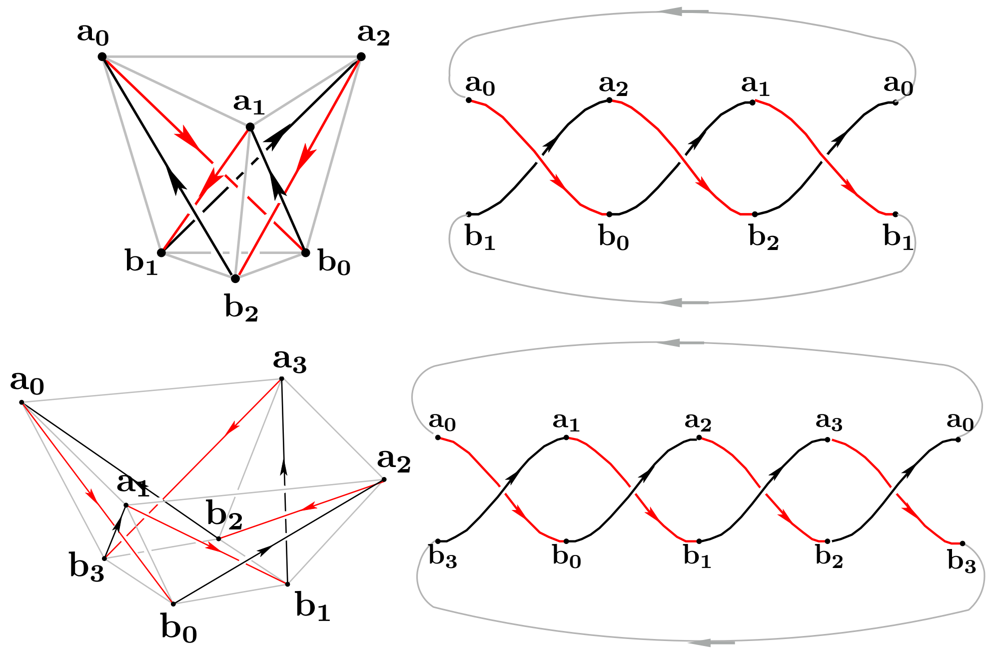


Fig. 6. Top: the knot diagram of a right-handed trefoil knot $\mathcal{K}_3$. Bottom: the link diagram of a left-handed double link $\mathcal{K}_4$.

The obtained diagram has $m$ crossings. We then check that whether $k$ is an alternating knots (or links) as follows. We orient all line segments in $\mathcal{S}_m$ from $a_i$'s to $b_i$'s, and all line segments in $\mathcal{S}'_m$ from $b_j$'s to $a_{j+2}$'s (if $\mathcal{S}_m$ is left-handed) or $b_j$'s to $a_{j-2}$'s (if $\mathcal{S}_m$ is right-handed). We then trace the corresponding oriented edges in the diagram. We see that the crossings are alternating as we followed the directions of the oriented line segments of $\mathcal{K}_m$. Hence $k$ is a knot if $m$ is even or a link if $m$ is odd. □

We say that $\mathcal{K}_m$ is the knot or (the link) of $\mathcal{S}_m$ when $m$ is odd (or $m$ is even). We use $\mathcal{K}_0$ to denote an unknot (a circle).

Let $V$ be a set of points in $\mathbb{R}^3$. A *tetrahedralisation of V* is defined as a 3d simplicial complex $\mathcal{T}$ whose vertex set is a subset of $V$ and the underlying space (the union of all simplices) of $\mathcal{T}$ is the convex hull of $V$. Note in this definition, we do not require that $\mathcal{T}$ uses all vertices of $V$. While $\mathcal{T}$ must contain no additional vertex of $V$.

It is obvious that a tetrahedralisation of $V$ exists, like the famous Delaunay tetrahedralisation of $V$. Indeed, there are many possible tetrahedralisations of $V$. However, it is not obvious if we want a tetrahedralisation of $V$ which also contains some specific edges which are given.

In the following sections, we show some results about tetrahedralisations of knotted or linked line segments.

In the rest of this paper, it is convenient to use the terms "decomposable" and "indecomposable" directly to a set $\mathcal{S}_m$ and its related knot $k$. We say that a set $\mathcal{S}_m$ (or a knot $k$) is *decomposable* if there is a tetrahedralisation of the vertices of $\mathcal{S}_m$ which contains all line segments of $\mathcal{S}_m$ (or the knot $k$). Otherwise, we say that the set $\mathcal{S}_m$ (or the knot $k$) is indecomposable.

## 3. Three Line Segments

In this section, we consider the simplest case which $m = 3$, i.e., a cyclic set of three line segments. This case is relatively simple. It is shown in Figure 1 that a $\mathcal{S}_3$ appears in the boundary of the Schönhardt polyhedron, and it results the simplest knot – the trefoil knot – with the minimum stick number 6. The following theorem is certainly not surprising.

**Theorem 2.** *An $\mathcal{S}_3$ is indecomposable.*

There are indeed many ways to proof the above theorem. Since we're not aware of a published proof for this, we provide our own proof in below. Our proof uses only basic geometry in which we see that the shape of the Schönhardt polyhedron naturally appears.

*Proof.* We prove the case when it is a left-handed $\mathcal{S}_3$. The proof for the right-handed case is same.

Let $\mathcal{L} = \{a_0b_0, a_1b_1, a_2b_2\}$ be a left-handed $\mathcal{S}_3$, see Figure 7 Left. Let $V$ be the vertex set of $\mathcal{L}$. $V$ must in convex position. Otherwise, at least one pair of edge vectors can not form a left-handed turn, which violates the property of $\mathcal{S}_3$. The convexity of $V$ is useful. It implies that the three edges of $\mathcal{S}'_3$:

$$b_2a_1, b_0a_2, b_1a_0$$

must lie on the convex hull of $V$. Then they must appear in any tetrahedralisation of $V$. Assume there exists a tetrahedralisation of $V$ which contains all three line segments of $\mathcal{L}$, it must also contain the three tetrahedra which are formed by corresponding line segments in $\mathcal{S}_3$ and $\mathcal{S}'_3$, see Figure 7 (b):

$$a_0b_0b_2a_1, a_1b_1b_0a_2, a_2b_2b_1a_0.$$

The remaining volume after removing the three tetrahedra from the convex hull of $V$ is a non-convex polyhedron which is combinatorially equivalent to the Schönhardt polyhedron, see Figure 7 Right. Then it is easy to check that no four vertices of $V$ are able to form a tetrahedron whose interior lies inside of this polyhedron. □

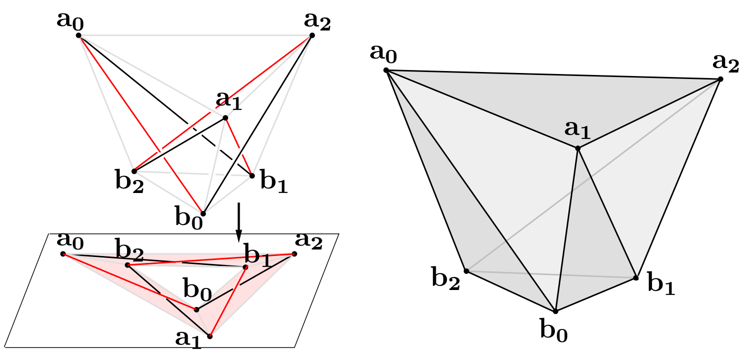


Fig. 7. (Letf) A left-handed $\mathcal{S}_3$ and the three tetrahedra formed by corresponding pairs in $\mathcal{S}_3$ and $\mathcal{S}'_3$. (Right) The remaining non-convex polyhedron which cannot be triangulated.

The following corollary is straightforward.

**Corollary 3.** *There is no tetrahedralisation of six vertices that contains a trefoil knot in its edge graph.*

We comment that for point set with just one more vertex, the above corollary is false. It is possible that a tetrahedralisation of 7 vertices contains a trefoil knot in its edge graph. The simplest example (with 7 vertices) is constructed in the book [11, Chap 3, Example 3.6.15, page 139] which is a modified example in [19]. This is also the simplest non-regular tetrahedralisation which will cause the failure of 3d monotone flip algorithms in generating Delaunay tetrahedralisations.

## 4. Four Line Segments

In this section, we consider the case of a cyclic set of four line segments, a $\mathcal{S}_4$. Although it contains just one more line segments than a $\mathcal{S}_3$, it can form much more configurations. Unlike a $\mathcal{S}_3$, there indeed exist tetrahedralisations of 8 vertices (in convex position) whose edge graphs contain a $\mathcal{S}_4$. This makes the decomposability question of a $\mathcal{S}_4$ more interesting.

In general, the 8 vertices of a $\mathcal{S}_4$ are not necessarily in convex position. To make the analysis as simple as possible, in this paper, we only consider the case which all vertices of a $\mathcal{S}_4$ are in convex position.

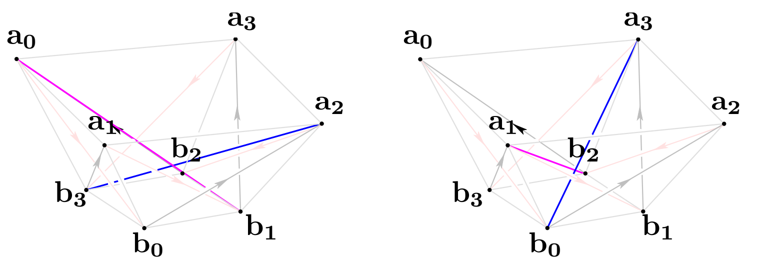


Fig. 8. The four possible interior edges of a left-handed $\mathcal{S}_4$. Left: $a_0b_1$ and $a_2b_3$. Right: $a_1b_2$ and $a_3b_0$.

Let $\mathcal{L} = \{a_0b_0, a_1b_1, a_2b_2, a_3b_3\}$ be a left-handed $\mathcal{S}_4$, see Figure 6 Bottom-Left for an example. Note that any decomposition of a $\mathcal{S}_4$ (if it exists) must contain new edges that lie in the interior of the convex hull of $\mathcal{L}$. Due to the convex position assumption, it is easy to show that: Any new interior edge must connect a vertex $a_i$ and a vertex $b_j \notin \{b_i, b_{i-1}, b_{i-2}\}$. Hence there are four possible new interior edges (refer to Figure 8), which are:

$$a_0b_1, a_1b_2, a_2b_3, a_3b_0.$$

The following description will explain that if there is only one added edge, $S_4$ is decomposed to a $S_3$ and a decomposable part. According to the description above, the $S_3$ part is indecomposable. But, is $S_4$ must indecomposable? The answer is no. $S_4$ may become decomposable by adding two of the new interior edges, which can be added without intersecting the line segments in $\mathcal{L}$. So whether $S_4$ is decomposable depends on the position of the two new added interior edges. A further observation is that the two new added interior edges must satisfy a condition, that is, there is a face of the convex hull of $S_4$, such that the projections of the two edges in the face have no intersection. In fact, there are only two pairs of edges that satisfies the condition, they are

$$\{a_0b_1, a_2b_3\}, \{a_1b_2, a_3b_0\},$$

see Figure 8. Each pair of edges are line segments in different planes, so the end points of them can form a tetrahedra. It is possible that none of these two tetrahedra can be inserted without intersecting the line segments in $\mathcal{L}$. This happens when the line segments in $\mathcal{L}$ are twisted too much. If this is the case, then $\mathcal{L}$ is indecomposable.

Now assume we could add at least one of the above two tetrahedra. Without loss of generality, we consider the case by adding the tetrahedron: $a_1b_2a_3b_0$. This introduces two new edges, $a_1b_2$ and $a_3b_0$ which form a new crossing, into the link diagram of $\mathcal{L}$, see Figure 9. We can apply the following topological modifications on the link diagram:

(1) take apart the four edges from the original link diagram: $a_1b_1$, $b_0a_2$, $a_2b_2$, and $b_1a_3$, and replace them by the two new edges: $a_1b_2$ and $a_3b_0$,
(2) form a new knot using the four removed edges in step *(1)* with the two new edges.

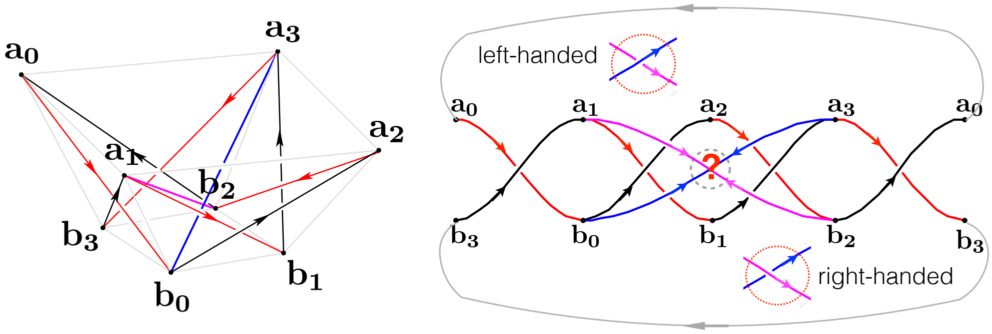


Fig. 9. The double link $\mathcal{K}_4$ of a $\mathcal{S}_4$ (left) and its link diagram (right) with two new edges $a_1b_2$ and $a_3b_0$ added. The type of the highlighted new crossing can be either left-handed or right-handed.

In step *(1)*, the crossings in the original diagram is reduced by 1 (we removed two crossings, added one). This way, a double link $\mathcal{K}_4$ is decomposed into two knots $k_1$ and $k_2$, whose edges are respectively:

$$k_1 : a_0b_0, \underline{b_0a_3}, a_3b_3, b_3a_1, \underline{a_1b_2}, b_2a_0,$$
$$k_2 : a_1b_1, b_1a_3, \underline{a_3b_0}, b_0a_2, a_2b_2, \underline{b_2a_1}.$$

These two knots are glued together at the new edges $a_1b_2$ and $a_3b_0$ (underlined). Each of them contains two line segments of $\mathcal{L}$.

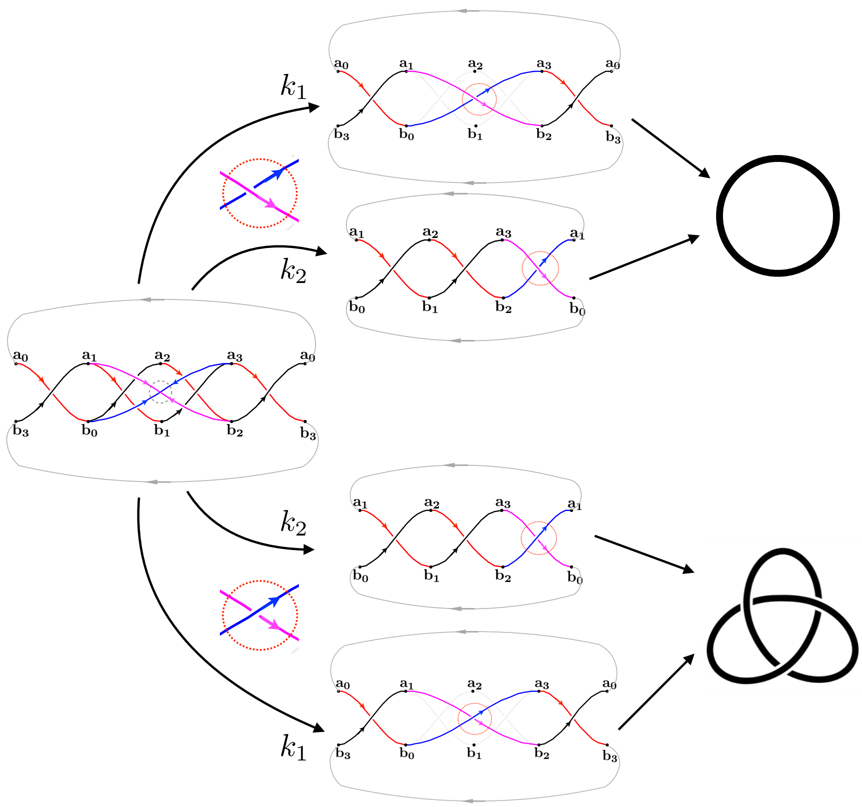


Fig. 10. An illustration of the two possible decompositions of a (left-handed) double link by inserting a pair of crossing edges in its link diagram. One leads to two circles (top) when it is an over-cross (a right-handed trun), and one leads to two trefoil knots (bottom) when it is an under-cross (a left-handed trun).

The type of the knots depends on how the new edges $a_1b_2$ and $a_3b_0$ cross each other. The link diagram (Figure 9) tells us which edge vectors we need to check. In this example, since $\mathcal{L}$ is left-handed, we should check the two edge vectors: $b_2 - a_1$ and $a_3 - b_0$. There are two cases, see Figure 10:

(1) These two vectors are right-handed (which corresponds to an over-cross), by applying the Reidermeister moves on $k_1$ and $k_2$, we see that they are just circles (unknots), $\mathcal{K}_0$'s, see Figure 10 top. It can be shown that one can construct a tetrahedralisation of $k_1 \cup k_2$ which contains all edges of $k_1$ and $k_2$. Hence $\mathcal{L}$ is decomposable.
(2) These two vectors are left-handed (which corresponds to an under-cross) then both $k_1$ and $k_2$ are trefoil knots, $\mathcal{K}_3$'s, see Figure 10 Bottom. Note that both $k_1$ and $k_2$ only contain two line segments of $\mathcal{L}$, which means, they both have one free edge which could be ignored. This means that $k_1 \setminus \{a_1b_2\}$ and $k_2 \setminus \{a_3b_0\}$ can be tetrahedralised individually. However, one could not find a common tetrahedralisation of $k_1$ and $k_2$. For example, any tetrahedralisation of $k_1 \setminus \{a_1b_2\}$ must use the edge $a_3b_0$, but with this edge it is not possible to decompose $k_2$. Therefore, at least one of the two new edges, i.e, $a_1b_2$ and $a_3b_0$, must be omitted in any tetrahedralisation of $k_1 \setminus \{a_1b_2\}$ and $k_2 \setminus \{a_3b_0\}$. In this case, the decomposability of $\mathcal{L}$ cannot be determined.

The above analysis shows that the decomposability of a $\mathcal{S}_4$ can be determined. For a left-handed $\mathcal{S}_4$, if there exists a pair of interior edges, i.e., $a_0b_1, a_2b_3$ or $a_1b_2, a_3b_0$, such that they form a right-handed turn, then this $\mathcal{S}_4$ is decomposable. If none of these two pairs of interior edge exists or they form left-handed turns, then it is indecomposable. The decomposability of a right-handed $\mathcal{S}_4$ can be determined in the same way. We summarise the above result in the following theorem.

**Theorem 4.** *Let $\mathcal{L} = \{a_0b_0, a_1b_1, a_2b_2, a_3b_3\}$ be a $\mathcal{S}_4$. Assume the vertices of $\mathcal{L}$ are in convex position. Then*

*(1) If $\mathcal{L}$ is left-handed, then $\mathcal{L}$ is decomposable, either*
  *(1.1) if the interior of the tetrahedron $a_0b_1a_2b_3$ does not intersect any line segments of $\mathcal{L}$, and $b_1 - a_0$ and $a_2 - b_3$ form a right-handed turn; or*
  *(1.2) if the interior of the tetrahedron $a_1b_2a_3b_0$ does not intersect any line segments of $\mathcal{L}$, and $b_2 - a_1$ and $a_3 - b_0$ form a right-handed turn.*
*(2) If $\mathcal{L}$ is right-handed, then $\mathcal{L}$ is decomposable, either*
  *(2.1) if the interior of the tetrahedron $a_0b_3a_2b_1$ does not intersect any line segments of $\mathcal{L}$, and $b_3 - a_0$ and $a_2 - b_1$ form a left-handed turn; or*
  *(2.2) if the interior of the tetrahedron $a_1b_0a_3b_2$ does not intersect any line segments of $\mathcal{L}$, and $b_0 - a_1$ and $a_3 - b_2$ form a left-handed turn.*
*(3) $\mathcal{L}$ is indecomposable if it satisfies neither (1) nor (2).*

## 5. *n* Line Segments

In this section, we consider the case of a set of arbitrary number of cyclical line segments, $\mathcal{S}_n$, where $n \geq 3$. For simplicity, we consider the case which all vertices of a $\mathcal{S}_n$ are in convex position.

Let $\mathcal{L} = \{a_0b_0, a_1b_1, \ldots, a_{n-1}b_{n-1}\}$ be a left-handed $\mathcal{S}_n$. Let $\mathcal{K}_n$ be the knot (or link) of $\mathcal{L}$. Any decomposition of $\mathcal{L}$ (if it exists) must contain new interior edges that lie in the interior of the convex hull of $\mathcal{L}$. Due to the convex position assumption, it is easy to show that: Any new interior edge must connect a vertex $a_i$ and a vertex $b_j \notin \{b_{[i]}, b_{[i-1]}, b_{[i-2]}\}$ where $[i] = \{p | p \equiv i \bmod n\}$. Hence the set of possible new interior edges are

$$\{a_{[i]}b_{[j]} \mid i, j \text{ satisfies that } [i - j] > [1]\}.$$

When $n = 4$, there are only two pairs of interior edges that can be added without intersecting the line segments in $\mathcal{L}$, while $n > 4$, there are much more choices of the added interior edges. Like the case of a $\mathcal{S}_4$, we consider adding the following pairs of edges,

$$a_{[i]}b_{[j]}, a_{[j+1]}b_{[i-1]}.$$

If a tetrahedron $a_{[i]}b_{[j]}a_{[j+1]}b_{[i-1]}$ does not intersect any line segment in $\mathcal{L}$, it can decompose $\mathcal{K}_n$ into a pair of two simpler knots (or links), $\mathcal{K}_p$ and $\mathcal{K}_q$, with less crossing numbers than $\mathcal{K}_n$. The type of these knots (or links) depends on how the new edges cross each other. The decomposability of $\mathcal{L}$ depends on whether the two knots (or links) can be decomposed or not. By repeatedly applying this on the $\mathcal{K}_p$ and $\mathcal{K}_q$, respectively, either we find that we could not decompose a $\mathcal{K}_p$ or $\mathcal{K}_q$ anymore, or we will obtain a collection of the simplest knots, i.e., $\mathcal{K}_3$'s (trefoils) and $\mathcal{K}_0$'s (unknots). This allows us to check the decomposability of $\mathcal{L}$ from the types from the final knots.

Our main contribution of this section is a new theorem to determine the decomposability of a $\mathcal{S}_n$, which is stated in the following theorem.

**Theorem 5.** *Let $\mathcal{L}$ be a $\mathcal{S}_n$. Assume the vertices of $\mathcal{L}$ are in convex position. Then*

*(1) If $\mathcal{L}$ is left-handed, and the knot (or link) of $\mathcal{L}$ can be decomposed into a set of simplest knots by adding a set of tetrahedra: $a_{[i]}b_{[j]}a_{[j+1]}b_{[i-1]}$, then $\mathcal{L}$ is decomposable if at least one pair of edges: $a_{[i]}b_{[j]}$ and $a_{[j+1]}b_{[i-1]}$ form a right-handed turn.*

*(2) If $\mathcal{L}$ is right-handed, and the knot (or link) of $\mathcal{L}$ can be decomposed into a set of simplest knots by adding a set of tetrahedra: $a_{[i]}b_{[j]}a_{[j-1]}b_{[i+1]}$, then $\mathcal{L}$ is decomposable if at least one pair of edges: $a_{[i]}b_{[j]}$ and $a_{[j-1]}b_{[i+1]}$ form a left-handed turn.*

*(3) $\mathcal{L}$ is indecomposable if it satisfies neither (1) nor (2).*

Instead of directly proving the above theorem, we show that the above problem is closely related to the Cutting Pattern Problem [20,21]. The basic idea is: given a knot (or link) $\mathcal{K}_n$ in $\mathbb{R}^3$, we associate it to a convex $n$-gon in the plane so that the decomposability of $\mathcal{K}_n$ is equivalent to find a triangulation of this $n$-gon with an appropriate 3-tuple $\pm1$ assignments for each triangle of it.

Let $\mathcal{L} := \{a_0b_0, a_1b_1, \ldots, a_{n-1}b_{n-1}\}$ be a $\mathcal{S}_n$. Let $\mathcal{K}_n$ be the knot (or link) of $\mathcal{S}_n$. Let $D$ be a convex $n$-gon in the plane, whose vertices are one-to-one mapped to the vertices $b_0, b_1, \ldots, b_{n-1}$ of $\mathcal{S}_n$. The boundary edges of $D$ are: $b_0b_1, b_1b_2, \ldots, b_{n-1}b_0$. Each boundary edge $b_ib_{i+1}$ represents a crossing in the original knot (or link) $\mathcal{K}_n$, it is assigned a +1 (−1) sign if it is a left-handed (right-handed) turn. Figure 11 shows an example.

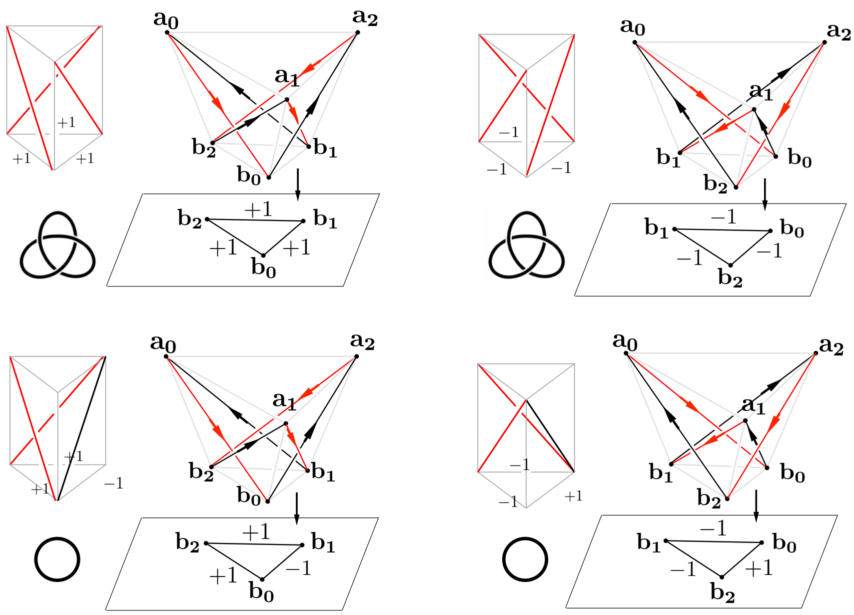


Fig. 11. A triangle whose edges all have the same signs represents a trefoil knot, hence it is an indecomposable case. A triangle with mixed signs corresponds to an unknot (circle), which is decomposable.

Let $\mathcal{T}$ be a triangulation of $D$ with no new vertex. Each interior edge $b_ib_j \in \mathcal{T}$, where $[i-j] > [1]$, represents a crossing formed by a pair of new edges which are added into $\mathcal{K}_n$. Let $t_1, t_2$ be two triangles in $\mathcal{T}$ which share at this edge. Both $t_1$ and $t_2$ get a sign for this edge as follows:

(1) If $b_i b_j$ represents a crossing different from the crossing of the boundary edges, both signs are the same and are opposite values to the one on boundary edges.
(2) If $b_i b_j$ represents a crossing which is the same as the crossing of the boundary edges, then the signs for $t_1$ and $t_2$ are opposite, i.e., if $t_1$ gets a $+1$ sign then $t_2$ gets a $-1$ sign, or vice versa.

Figure 12 shows a square which corresponds to a left-handed $\mathcal{S}_4$. All boundary edges of this square have a $+1$ sign. In the middle, the interior edge represents a right-handed turn, hence it have both $-1$ signs in its two triangles, the two triangles represent two unknots. Hence this $\mathcal{S}_4$ is decomposable as show in Section 4. In the right, then interior edge represents a left-handed turn, then one triangle gets mixed signs, which implies that its knot can be decomposed. However, it forces the other triangle to have all edges have the same signs, which implies this $\mathcal{S}_4$ is indecomposable.

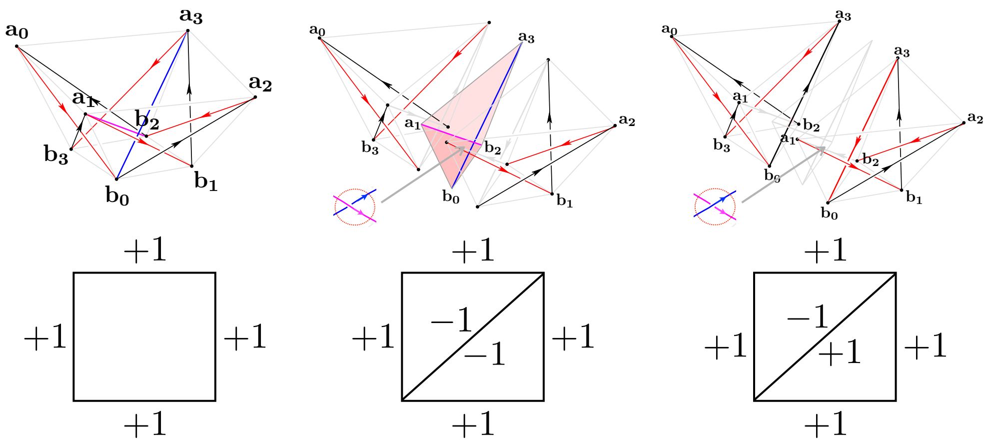


Fig. 12. The equivalent polygon (a square) of the double link $\mathcal{K}_4$ of a left-handed $\mathcal{S}_4$ and its two possible decompositions. In the right, it is possible to switch the $+1$ and $-1$ signs at the interior edge $b_0 b_2$. This corresponds to choose the (diagonal) edge $a_1 b_2$ instead of $a_3 b_0$.

By this way, our original decomposition problem of a $\mathcal{S}_n$ has been transformed into the Cutting Pattern Problem in [20]. If there exists an assignment of edges of a triangulation of $\mathcal{K}_n$ such that every triangle gets mixed signs, then this $\mathcal{S}_n$ is decomposable. If none such assignment exists, it is indecomposable.

*Proof.* We first prove (1), so it is same for (2). Consider $\mathcal{L}$ is left-handed. Assume there exist a decomposition of $\mathcal{K}_n$ by a set of tetrahedra: $a_{[i]} b_{[j]} a_{[j+1]} b_{[i-1]}$, and at least one pair of new edges $a_{[i]} b_{[j]}$ and $a_{[j+1]} b_{[i-1]}$ form a right-handed turn, we show that $\mathcal{L}$ is decomposable.

By the above transformation, this is equivalent to the case which there exists a triangulation $\mathcal{T}$ of the disk $D$, all boundary edges of $\mathcal{T}$ have a $+1$ sign (a left-handed turn), and at least one interior edge has a $-1$ sign (a right-handed turn) in both of its adjacent triangles, see Figure 13 Left. We need to show that it is possible to find an assignment of other interior edges such that all triangles of $\mathcal{T}$ have mixed signs. For $n = 4$, this is already the case. However, for an arbitrary $\mathcal{K}_n$, $n > 4$, this question is not obvious. Note that although there exist an edge with different signs, there may still exist many triangles have the same signs. The task is to show that we are always able to make an assignment such that every triangle has a mixed signs on its edges. In the following, we prove this by borrowing a result from the Cutting Pattern Problem [20].

Since we at least have an interior edge with both $-1$ signs. We could split the triangulation $\mathcal{T}$ along this edge to get two triangulations: $\mathcal{T}_1$ and $\mathcal{T}_2$, such that $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. Now both $\mathcal{T}_1$ and $\mathcal{T}_2$ have the property such that the boundary edges have at least one sign is different from others, see Figure 13 Right. This is a special case of the Cutting Pattern Problem with General Boundary Conditions [20, Definition 2]: every boundary edge of $D$ has a prescribed assignment that can take an arbitrary value of $\pm 1$. It has been proven in [20, Section 3.2] that it is always possible to find a mixed assignment on edges of triangles when there exist at least one different sign on the boundary.

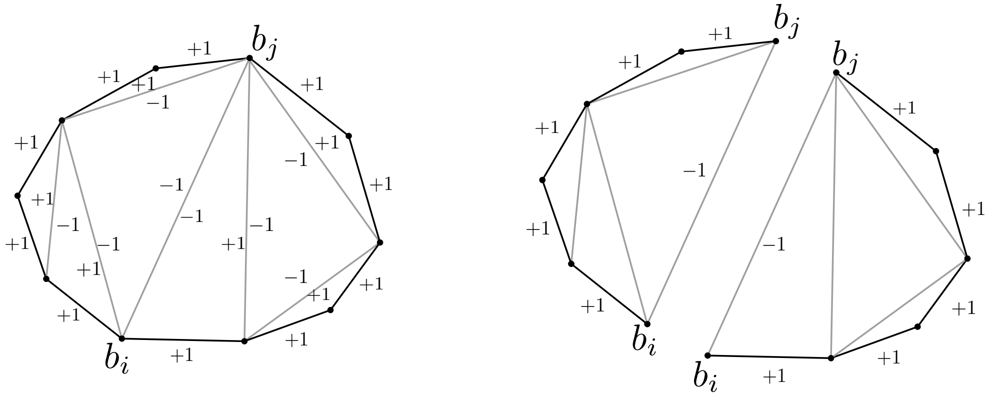


Fig. 13. Left: An initial ±1 assignment on the polygon. In particular, the edge $b_ib_j$ gets both −1 signs. Right: Split the polygon along the edge $b_ib_j$ leads to a moderate boundary condition for the Cutting Pattern Problem, which is solvable.

The proof of (3) needs more elaborations on the geometric properties of a $\mathcal{S}_n$, which is not possible to fit in the scope of this paper. In the following, the proof idea is highlighted. The full proof will be provided in a full preprint of this paper.

We only show the case for a left-handed $\mathcal{L}$, that is, if all pairs of interior edges: $a_{[i]}b_{[j]}$ and $a_{[j+1]}b_{[i-1]}$ (where $[i-j] > [1]$) form left-handed turns, then $\mathcal{L}$ is indecomposable.

First of all, this is obvious for the case of $n = 4$, where there are only two such possible tetrahedra: namely $a_0b_1a_2b_3$ and $a_1b_2a_3b_0$, which are the pair of diagonal edges. For $n > 4$, we have considered tetrahedra with the combination of the interior edges: $a_{[i]}b_{[j]}a_{[j+1]}b_{[i-1]}$ (where $[i-j] > [1]$).

For $n > 4$, there are more combinations of interior edges: $\{a_{[i]}b_{[j]} \mid i, j \text{ satisfies that } [i-j] > [1]\}$ which could form tetrahedra in the interior of the convex hull of $\mathcal{L}$. When such a tetrahedron does not intersect any line segment in $\mathcal{L}$, it can also be added to split the knot (or link) of $\mathcal{L}$. Fortunately, we still could use the 2d graph triangulation to show that by adding these tetrahedra will always lead to the same indecomposable cases as adding the pairs of edges in above.

Let $a_{[i]}b_{[j]}a_{[k]}b_{[l]}$ be a new tetrahedron which does not intersect any line segment of $\mathcal{L}$. By assumption, it must be $[i-j] > [1]$, $[k-l] > [1]$, $k \neq j+1$, and $l \neq i-1$. These conditions ensure that $a_{[i]}b_{[j]}$ and $a_{[k]}b_{[l]}$ are new interior edges, and this tetrahedron is not the one that we have considered in above.

The first key issue is to show is that at least one of the edges $a_{[i]}b_{[l]}$ and $a_{[k]}b_{[j]}$ is a new interior edge. This means, adding this tetrahedron will result a new interior edge to be added.

The second key issue is that, the adding of this new edge is equivalently to split the convex hull of $D$ into two parts along the edge $b_{[i-1]}b_l$ or $b_{[k-1]}b_{[j]}$. By our assumption, this interior edge represents a left-handed turn. The adding of this tetrahedron has fixed the edge $a_{[i]}b_{[l]}$ or $a_{[k]}b_{[j]}$ as part of the decomposition of the knots. Hence the boundary edge of the subpolygon gets a +1 sign. This makes the whole subgraph have all +1 signs on its boundary. The further decomposition this subpolygon will result the same situation, which shows that $\mathcal{L}$ is indecomposable. □

As a consequence of the above Theorem, we show that it implies the result of Rambau [1] which proved the indecomposability of a family of generalised Schönhardt polyhedra.

**Corollary 6 (Rambau[1]).** *The twisted non-convex prism over an n-gon cannot be triangulated as long as the top and bottom n-gons are congruent and the twist is not too large.*

*Proof.* Let $\mathcal{L}$ be the set of diagonal line segments on the boundary of such a (untwisted) prism. First of all, all the vertices of such prism are in convex position. By a slight twist, $\mathcal{L}$ becomes either a left-handed or a right-handed $\mathcal{S}_n$. Without loss of generality, we assume $\mathcal{L}$ is left-handed. Since the top and bottom $n$-gons are congruent and parallel, all the interior pairs of edges: $a_{[i]}b_{[j]}a_{[j+1]}b_{[i-1]}$ (where $[i-j] > [1]$), are coplanar before the twist. But they all form the same left-handed turn after just a slightly twist. By our Theorem, there is no tetrahedralisation whose edge graph contains $\mathcal{L}$, in other words, the prism is indecomposable. □

## 6. Discussion

In this paper, we study in the decomposability of 3d non-convex polyhedra whose boundary contains knotted or linked line segments. This family includes the Rambau's polyhedra. Instead of considering the whole boundary of such polyhedron, we focus only on the sets of twisted line segments. They are more fundamental in causing the indecomposability of these non-convex polyhedra. Our main result is a new theorem which provides the sufficient conditions to determine whether there exists or does not exist a tetrahedralisation for a set of $n$ ($n \geq 3$) line segments whose vertex set is in convex position.

If $n$ is not too large, then our theorem could be used to quickly determine whether an $n$-sided prismatic polyhedron can be tetrahedralised or not. One can first list all possible edge pairs in advance which are candidates to form a decomposition of the polyhedron, then check the orientations of these edge pairs, and compare them to the orientation of the given polyhedron. This algorithm may be applied in 3d mesh generation problems. For example, in 3d boundary recovery in which a set of constraints (edges and faces) need to be entirely preserved in a tetrahedral mesh. A common way to recover a missing constraint is to form a cavity inside the tetrahedralisation, and re-triangulate this cavity to include the missing constraint. Usually, such a cavity is a non-convex polyhedron which includes a small number of vertices, for example, $n$ is between 3 and 7. Our result may then be applied to quickly determine whether it can be tetrahedralised or not.

In the future, it would be very interesting to see how to extend our result to more general case.

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