Modal logic NL for common language

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ABSTRACT: Despite initial appearance, paradoxes in classical logic, when comprehension is unrestricted, do not go away even if the law of excluded middle is dropped, unless the law of noncontradiction is eliminated as well, which makes logic much less powerful. Is there an alternative way to preserve unrestricted comprehension of common language, while retaining power of classical logic? The answer is yes, when provability modal logic is utilized. Modal logic NL is constructed for this purpose. Unless a paradox is provable, usual rules of classical logic follow. The main point for modal logic NL is to tune the law of excluded middle so that we allow for ϕ and its negation $\neg\phi$ to be both false in case a paradox provably arises. Curry's paradox is resolved differently from other paradoxes but is also resolved in modal logic NL. The changes allow for unrestricted comprehension and naïve set theory, and allow us to justify use of common language in formal sense.

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1 Expressiveness of Language, No-no paradox

Ideally, logic underlies both language and mathematics. However, different paradoxes in logic, such as the liar paradox, Russell's paradox and Fitch's paradox[1], suggest expressiveness of common language must be restricted. But is this really a natural conclusion?
One may ask why paradoxes are not merely of curiosity. After all, one may simply say some statements have no meaning. Should we impose that all sentences be either true or false? Why not the third truth value for paradoxes? For that, consider the barber paradox. The barber A is a person who shaves all of those and only those who do not shave themselves. This definition should be clearly non-paradoxical in common language. Yet investigated logically, we realize that this definition contains a paradox, because if A shaves A, then the definition of the barber is wrong, and if A does not shave A, then A should shave A by the definition.
Also consider the following no-no paradox[2], originally by Jean Buridan, a medieval philosopher. In statement (A), Socrates simply writes that Plato is lying by statement (B). In statement (B), Plato simply writes that Socrates is lying by statement (A). There seems no reason to restrict expressiveness of language used to write these statements. After all, by common intuition, this is just Socrates questioning Plato's integrity, and vice versa. Yet logically investigated, we arrive at the paradox, because of asymmetrical nature of possible conclusions. That is, either (A) is true with (B) being false, or (B) is true with (A) being false. But how can this be, when sentence (A) and (B) are symmetrical? This asymmetric conclusion is not by itself a problem, but think of the case when there is nothing else happening, and we only have statement (A) and (B). Is the asymmetric conclusion then really valid?
It is possible to go around this by adding a third truth value - indeterminate - so that a

symmetric conclusion can be kept. Except, when we introduce the third truth value, we can always "absorb" this third value to one of the two classical truth values - the natural choice would be falsity. This would require a modification (but not an entire rejection) to one law of classical logic - the law of excluded middle. We will soon explore this. For now, the point is that there is nothing wrong with staying in bivalent logic.

We may avoid questions by limiting expressiveness of language, but the statements just formulated were not simply to be treated as nonsense, to allude somewhat to Ludwig Wittgenstein. At least they are different from obvious nonsense. Curtailment of language expressiveness amounts to treating some commonly sensible sentences as nonsense.

We thus seek for a resolution that does not really modify classical logic machineries but still keeps **unrestricted comprehension** that allows us to evaluate sentences in common language. That is what this writing provides by constructing modal logic **NL** - the abbreviation for "the new logic". The philosophy of **NL** is simple, even though actual construction and details are not: if you prove a paradox of form $p \leftrightarrow \neg p$, then we should set both p and $\neg p$ to be false. This requires referring to provability, necessitating provability modal logic, along with one modification and new additions to classical propositional logic. The goal is to preserve the set of sentences that are true in all models in classical logic, which is mapped to the set of sentences that are true in all models in **NL**, while allowing for unrestricted comprehension.

2 Refinements to classical logic

What is classical propositional logic? It is defined in terms of axiom schema:

- **Modus Ponens**. This is not an axiom and just an inference rule.
- **Then-1**: $p \to (q \to p)$
- **Then-2**: $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
- **And-1, And-2**: $p \wedge q \to p$, $p \wedge q \to q$
- **And-3**: $p \to (q \to (p \wedge q))$
- **Or-1, Or-2**: $p \to p \vee q$, $q \to p \vee q$
- **Or-3**: $(p \to q) \to ((r \to q) \to (p \vee r \to q))$
- **Principle of explosion**: $\bot \to p$
- **Not-1'**: $(p \to \bot) \to \neg p$ (Refined in the new logic)
- **Not-2'**: $\neg p \to (p \to \bot)$
- **Iff-1, Iff-2**: $(p \leftrightarrow q) \to ((p \to q) \wedge (q \to p))$
- **Iff-3**: $(p \to q) \to ((q \to p) \to (p \leftrightarrow q))$

- **Double negation elimination and introduction**: $\neg\neg p \leftrightarrow p$

The axiom schema that are **dropped** initially from the new logic is **Not-1'**. Other axioms remain intact in the new logic. In intuitionist logic the axiom schema dropped is double negation elimination (and introduction), so this is an atypical choice. Refining **Not-1'** reflects the idea that falsity of p does not mean truth of $\neg p$, and both p and its negation $\neg p$ may turn out to be false, as required for resolution of Russell's paradox without restricting comprehension.
Furthermore, since Curry's paradox that we wish to avoid without restricting comprehension requires invoking **Then-2**, there is a need to refine it as well. We keep **Then-2** but add an additional axiom (**Then-4**) to address Curry's paradox.
The goal is to preserve classical proofs that invoke the law of excluded middle, **Not-1'** or **Then-2**, so **Refined Not-1'** and **Then-4** are to be introduced as to preserve classical proofs as valid proofs in the new logic but without having to restrict comprehension. **Refined Not-1'** and **Then-4** go, along with additional axioms, though there would have to be more, added to the new logic as:

- **Refined Not-1'**:

$$\neg\Box((p \to \neg p) \land (\neg p \to p)) \to ((p \to \bot) \to \neg p) \tag{2.1}$$

- **Then-4**:

$$(p \leftrightarrow (p \to r)) \to ((p \to \bot) \land ((p \to r) \to \bot)) \tag{2.2}$$

- **Law of noncontradiction**: $(p \land \neg p) \to \bot$
- **De Morgan's laws**: $\neg(p \lor q) \leftrightarrow \neg p \land \neg q$, $\neg(p \land q) \leftrightarrow \neg p \lor \neg q$
- **Not-4'**:

$$\Box((p \to \neg p) \land (\neg p \to p)) \to \neg\Box p \land \neg\Box\neg p \tag{2.3}$$

Then-4 has precedence over **Then-2**. Thus, in case of a conflict between **Then-4** and **Then-2**, **Then-4** must be used instead.
Refined Not-1' and **Not-4'** are where modal operator $\Box$ enters, as part of provability logic. The logical system utilized for the modal operator is modal logic **GL** (also referred to as **KW**). For now all that matters is that $\Box$ means provable, with $\Diamond \equiv \neg\Box\neg$. Note that **GL** itself does not include axioms of classical propositional logic. Thus, the modifications to classical propositional logic are safe to use along with **GL**.
(However, in existing literature, consequences of **GL** or **S4** are often proved with some of the axioms in classical proposition logic without explicit mentions - most of time, the principle of explosion. Thus some cautions are required.) Also, **contrapositive relation** of $(p \to q) \to (\neg q \to \neg p)$ does follow from the principle of explosion, despite the modification to classical propositional logic.
The axiom set of **Then-1**, **Then-2**, **Then-4**, **And-1**, **And-2**, **And-3**, **Or-1**, **Or-2**, **Or-3**, **principle of explosion**, **Refined Not-1'**, **Not-2'**, **Iff-1**, **Iff-2**, **Iff-3**, **double negation**

elimination and introduction, **law of noncontradiction**, **De Morgan's laws**, **Not-4'** and modal logic **GL** "partially" define the new logic. Let us call this logic **NL-lite**.

What is **Refined Not-1'** stating? It say that if p is not in a verifiably (provably) paradoxical situation in the classical logic, then p resulting in contradiction (which means p is false both in classical logic and the new logic) must mean $\neg p$. Thus, in usual non-paradoxical situations of classical logic, the new logic and classical logic should agree. **Not-4'** says that in case a paradox involving p and $\neg p$ is provable, then both p and $\neg p$ are unprovable. In addition, in such a paradoxical case, p and $\neg p$ cannot be true by the law of noncontradiction, and thus they are both false.

Refined Not-1' results in a peculiar feature of **NL-lite**: ϕ that can only be proved through classical "proof by contradiction" now is considered unprovable in **NL-lite**, and this will not change in the full modal logic **NL** (the new logic). However, since we allow for the law of excluded middle when no paradox can be proven for ϕ, it is still true that for non-paradoxical classical non-modal propositions, all existing proofs of p are valid. It is just that $\Box p$ no longer holds if proving p requires a proof by contradiction. All these proofs of p that do not have $\Box p$ are considered "post-proofs" which do not satisfy $\vdash p$, since one cannot prove p within finite deductions. It requires infinite deductions to prove p, with all models demonstrated to satisfy p. In this sense, completeness is broken in **NL**.

It is known by Solovay's arithmetical completeness theorem that the notion of provability definable in modal logic **GL** is equivalent to the notion of provability definable in Peano arithmetic (PA), in sense that $GL \vdash A \leftrightarrow PA \vdash f(A)$ for all possible f, where $f(\bot) = \bot$, $f(p \to q) = (f(p) \to f(q))$ and $f(\Box p) = Prov(\ulcorner f(p) \urcorner)$. $Prov$ refers to a provability predicate in PA, and f is about translating a sentence into an arithmetical sentence in PA. Thus, the question is whether the notion of provability definable in PA really is enough for our purpose. The answer is no. We do need more axioms attached to provability modal logic to get the right power.

Why is this the case? This is because the intention of the new logic is to resolve paradoxes in classical logic when comprehension is not restricted. Thus it is natural that PA is not powerful enough to prove sentences that we intend to prove in the new logic. The complications in particular arise from the non-exclusion of the full law of excluded middle, and the next axioms mostly are about new required details on the refined law of excluded middle. Thus what appear next are additional axioms that would have to be assumed for the new logic.

- **NL1**: $\Box(p \to \bot) \to \Box\neg\Box p$. This is refinement of **Not-1'** for $\neg\Box$ in place of negation ($\neg$). Converse does not hold always. The reason why converse does not hold always is that unprovability of a sentence does not always mean that a sentence derives a contradiction.

- **NL2**: $\neg\Box p \vee \Box p$. This is the law of excluded middle for the provability operator. Again, the law of excluded middle does not generally hold.

- **NL3**: $\Box(\neg\Box p \to \bot) \to \Box p$, $\Box(\Box p \to \bot) \to \Box\neg\Box p$. Again, the general **Not-1'** axiom schema not involving provability predicates does not hold, and recourse to **Refined**

Not-1' and **Not-4'** is required.

- **NL4**: $\Box p \lor \Box q \to \Box(p \lor q)$. Quasi-De Morgan's law for $\Box$.
- **NL5**: $\Box p \land \Box q \leftrightarrow \Box(p \land q)$. Quasi-De Morgan's law for $\Box$.
- **NL6**: $\neg\Box(p \lor q) \to \neg\Box p \land \neg\Box q$. De Morgan's law for $\neg\Box$. Equivalent to **NL4** via the principle of contraposition.
- **NL7**: $\neg\Box p \lor \neg\Box q \leftrightarrow \neg\Box(p \land q)$. De Morgan's law for $\neg\Box$. Equivalent to **NL5** via the principle of contraposition.
- **Law of noncontradiction non-equivalence**: This is neither a refinement nor a new law of the new logic, but a consequence of the new logic. The law of noncontradiction is not equivalent to $\neg(p \land \neg p)$, though when the law of excluded middle is granted, equivalence does hold. Note also that since De Morgan's laws are valid in the new logic as well, $\neg(p \land \neg p)$ essentially is the re-statement of the law of excluded middle, which is not granted fully, restricted by the refined law of excluded middle.
- **Implication non-equivalence**: Again, this is neither a refinement nor a new law of the new logic. Implication $p \to q$ no longer simply is equivalent to $\neg p \lor q$. As will be with other cases, if the law of excluded middle is granted, then equivalence does hold.
- **Refined proof by contradiction**: Again, this is a consequence of the new logic, rather than a law. While proving $\neg p$ by deriving contradiction from p does not hold, it does hold trivially that falsity of p and $\neg\Box p$ follow from p resulting in contradiction. In addition, if the law of excluded middle is granted, then one can indeed prove $\neg p$ from p resulting in contradiction.

Let us review the axioms of the new logic - or one can call it modal logic **NL** in the following list. (Note that there is no **Not-3'** axiom in **NL**. There is no **Then-3** as well. Also, **NL7** is excluded, as it is equivalent to **NL5**, and **NL6** is excluded, as it is equivalent to **NL4**.)

- **K**: $\Box(p \to q) \to (\Box p \to \Box q)$
- **W**: $\Box(\Box p \to p) \to \Box p$
- **Then-1**: $p \to (q \to p)$
- **Then-2**: $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
- **Then-4**: $(p \leftrightarrow (p \to r)) \to ((p \to \bot) \land ((p \to r) \to \bot))$, with precedence over **Then-2**.
- **And-1, And-2**: $p \land q \to p$, $p \land q \to q$
- **And-3**: $p \to (q \to (p \land q))$
- **Or-1, Or-2**: $p \to p \lor q$, $q \to p \lor q$

- **Or-3**: $(p \to q) \to ((r \to q) \to (p \lor r \to q))$
- **Principle of explosion**: $\bot \to p$
- **Refined Not-1'** $\neg\Box((p \to \neg p) \land (\neg p \to p)) \to ((p \to \bot) \to \neg p)$
- **Not-2'**: $\neg p \to (p \to \bot)$
- **Not-4'**: $\Box((p \to \neg p) \land (\neg p \to p)) \to \neg\Box p \land \neg\Box\neg p$
- **Iff-1, Iff-2**: $(p \leftrightarrow q) \to ((p \to q) \land (q \to p))$
- **Iff-3**: $(p \to q) \to ((q \to p) \to (p \leftrightarrow q))$
- **Double negation elimination and introduction**: $\neg\neg p \leftrightarrow p$
- **Law of noncontradiction**: $(p \land \neg p) \to \bot$
- **De Morgan's laws**: $\neg(p \lor q) \leftrightarrow \neg p \land \neg q$, $\neg(p \land q) \leftrightarrow \neg p \lor \neg q$
- **NL1**: $\Box(p \to \bot) \to \Box\neg\Box p$
- **NL2**: $\neg\Box p \lor \Box p$
- **NL3**: $\Box(\neg\Box p \to \bot) \to \Box p$, $\Box(\Box p \to \bot) \to \Box\neg\Box p$
- **NL4**: $\Box p \lor \Box q \to \Box(p \lor q)$
- **NL5**: $\Box p \land \Box q \leftrightarrow \Box(p \land q)$

with inference rules being modus ponens, uniform substitution and necessitation ($\vdash p \to \vdash \Box p$).

2.1 Role of $\neg\Box$ in Refined Not-1'

Why does **Refined Not-1'** have $\neg\Box$ instead of simple negation? The reason is that we would like to eliminate a model where despite unprovability of a paradox, a paradox is considered to hold and thus the law of excluded middle is disallowed. We want to keep **Not-1'** for all models when classical logic should be considered equivalent to the new logic, allowing us to continue using proofs based on **Not-1'** for non-paradoxical circumstances safely.

3 Paradoxes of classical logic examined in NL

So far, we have stuck with use of truth and falsity as in bivalent classical logic or usual usages. But it may be beneficial to separate the notion of truth in common language from the notion of truth in formal logic. After all, when people say, "this sentence is false," do they really use the word "false" in conventional formal logic understanding? Furthermore, there really is no suitable truth predicate in formal logic by Tarski's impossibility results[4] and thus we cannot really say S: $\neg T(S)$, where T is truth predicate. We can only say S:

$\neg S$. Thus some separation may seem desirable.
We thus explore both notions of truth - formal and common language, without changing what true and false mean in formal logic. Modal logic provides a possible way that truth in common language may be interpreted, if we translate $\Box p$ as p being "true" in common language, while $\neg\Box p$ as p being "false" in common language.
But more importantly, how **NL** resolves paradoxes in classical logic are explored.

3.1 Liar, Russell's, Barber paradox

Sentence S says, "S is false." In modal logic **NL**, S is $\neg\Box S$, if we make conversion of the word "false" to "unprovable", with "true" to "provable"($\Box$). If S is provable, then S must be unprovable. ($\Box S \rightarrow \Box\neg\Box S$. By $\Box p \rightarrow p$ of modal logic **GL**, $\neg\Box S$ and thus $\Box S \rightarrow \neg\Box S$, contradiction.) But if S is unprovable, there is no provable contradiction. Thus, S is determined unprovable ($\neg\Box S$) by axiom **Refined Not-1'**, and furthermore, we get $\neg\Box\neg\Box S$ as well. Also, while S is unprovable, S is "true" as in conventional formal logic. Suppose $\neg S$. Then, $\neg S \rightarrow \Box S$ from contraposition of $\neg\Box S \rightarrow S$ by definition of S. By $\Box p \rightarrow p$, $\neg S \rightarrow S$, contradiction. However, assuming S exhibits no provable contradiction. Thus, S is unprovably true in terms of formal logic.
Suppose we re-define S as $\neg S$, the usual formulation of liar paradox in classical logic but examined in **NL**. Then we basically are examining case of Russell's paradox (with S substituted with $x \in x$). Since this clearly is a provable paradox, both S and $\neg S$ must be false in terms of formal logic, since this is the only way axioms remain consistent due to the law of noncontradiction. Both are unprovable (or false, in terms of common language) as well by **Not-4'**.
Barber paradox is identical, as the problem is about $S(b) \leftrightarrow \neg S(b)$ where S refers to shaving and b refers to the barber. Thus, the paradox is resolved in the same way as Russell's paradox - $S(b)$ and $\neg S(b)$ are false in terms of formal logic. So should the barber shave herself? The answer is no.
Let us examine Russell's paradox again, with details filled in. Construct set R allowed by unrestricted comprehension as $R = \{x|x \notin x\}$. Thus, $R \in R \leftrightarrow R \notin R$. This is a paradox, so R and $\neg R$ are both false and unprovable, in terms of formal logic. So should R be an element of R? The answer is no, because $R \notin R$ is false in terms of formal logic.
Barber b is defined as the one who shaves all those, and only those who do not shave themselves. x not shaving herself is defined as $\neg S(x)$. The question is, given this definition, $\neg S(b) \leftrightarrow S(b)$. But since both $S(b)$ and $\neg S(b)$ are false, in terms of formal logic, b is not included in the set of those who do not shave themselves. Thus, while the barber b should not shave herself in reality, $\neg S(b)$ is false as well, so b now goes onto shave others who do not shave themselves peacefully.

3.2 Curry's paradox, Then-4

For this paradox, we will not talk of the common language notion of truth and falsity. Curry's paradox is that if a sentence C that says $C \rightarrow F$ exists, then its mere existence (called curry sentence) without evaluation of its truth or falsity would mean that any claim can be proven. The idea goes as follows. $C \leftrightarrow (C \rightarrow F)$ by definition. Thus, $C \rightarrow F$ must

be true by **Then-2**. But $(C \to F) \to C$. Thus C is true, so F is true.
The paradox is avoided in **NL** by **Then-4** that augments **Then-2**. It is in place to exactly avoid this circumstance, without restricting comprehension. Thus, $C \leftrightarrow (C \to F)$ would still hold - just that one cannot derive the chain of "C and thus $C \to F$ therefore F", because both C and $C \to F$ will be false.
Then-4 simply says that any curry sentence that allows us to prove a target proposition is false. But this sounds too easy, so why was this resolution not tried?
Consider classical logic. $p \to (p \to q)$ means $\neg p \vee (p \to q)$. Let $\neg p$ be taken, following the idea in **Then-4**. Then $(p \to q) \to p$ means $\neg(p \to q) \vee p$. $\neg p$ means that $\neg(p \to q)$ must be taken. This is equivalent to $p \wedge \neg q$. Thus we get a contradiction. There is no easy escape in classical logic.
Modal logic **NL** is different, in that falsity of p does not necessarily imply $\neg p$, and that $p \to q$ does not necessarily translate to $\neg p \vee q$. These changes are quite significant - this allows us a mean to bypass the above problem in classical logic and falsify a curry sentence, evaluated true in classical logic, in **NL**.

3.3 No-no paradox

In the above, we have discussed the no-no paradox[2]. Let us express it in terms of modal logic **NL**, with translation of truth in common language to be $\Box$ and falsity in common language as $\neg\Box$. (A) (A) says $\neg\Box B$. (B) (B) says $\neg\Box A$. Suppose A is true. Then, $A \to \neg\Box B$, thus $A \to \neg\Box\neg\Box A$. Generally, any multiple of $\neg\Box\neg\Box$ applied to A holds from A. In fact, to generalize further $A \leftrightarrow (\neg\Box\neg\Box)^k A$ holds.
Suppose instead that $\neg A$ holds. Then, $\neg A \leftrightarrow \Box(\neg\Box\neg\Box)^k \neg\Box A$.
Suppose that $\Box A$ holds. Then, $\Box A \leftrightarrow \Box(\neg\Box\neg\Box)^k A$.
Suppose that $\neg\Box A$ holds. Then, $\neg\Box A \leftrightarrow \neg\Box(\neg\Box\neg\Box)^k A$.
This essentially is the hell of unprovability that we cannot really do much of evaluation unless more is provided. Thus, $\neg\Box A$, with both A and B turning out to be true, in terms of formal logic. In common language, we can say that both statements are false. We get the hell of unprovability of $\neg\Box(\neg\Box)^k A$ as well.
In fact, this should be expected. We know that if Plato and Socrates are only discussing A and B that only refer to each other, then they really are discussing truth or falsity that says nothing much. So the statements must be both false in terms of common language. But in terms of formal logic, they must be trivially true symmetrically as well, by the fact that both are acknowledging vacuousness of each other's statements.

- **Importance of no-no paradox**: The no-no paradox demonstrates that it is heavily beneficial to translate "truth" in common language as "provable"($\Box$), and "falsity" as "unprovable"($\neg\Box$). We get to keep seemingly required symmetry there. Even for liar paradox, resulting analysis is far smoother (S or $\neg S$ is assigned true), if we translate "false" in common language as unprovable. In a way, we really do not have a good definition of what truth and falsity really are. Common language is very silent about this, and philosophy is filled with debates about this exact topic. One way of capturing what truth is would be recourse to how we think of truth in conventional understanding of formal logic. Here, we may take a different route and rely on a more established concept - provability. This choice alone guarantees that we avoid many of paradoxes.

- While we have separated truth in common language and truth in formal logic, one may attempt to equate provability with truth directly in formal logic. This alternative path is achievable by adding $\phi \to \Box\phi$ to modal logic, but this requires heavy modifications of **NL**. While this is doable and sometimes results in simpler analysis, this requires heavy deviations from classical logic. As matter of balance goes, it is beneficial to keep **NL**, and separate notion of truth in common language from that in formal logic.

3.4 Knower and Fitch's Paradox

D: $\neg D$ is known.
E: E is unknown.
Let us write sentence D as $K(\neg D)$, with E written as $\neg K(E)$, where $K(P)$ refers to "P is known". Let us list some common epistemic assumptions used:

- **KF**: $K(P) \to P$
- **PK**: $\Box P \to K(P)$
- **CK**: $K(KF)$, $K(KF) \equiv K(K(P) \to P)$
- **IM**: $I(KF, P) \wedge CK \to K(P)$. $I(KF, P)$ means P is derivable from KF.

First, consider E first.

$$K(E) \to E \to \neg K(E)$$

Since we proved a contradiction, in classical logic, we would say that $\neg K(E)$ is proven. Thus,

$$\Box\neg K(E) \to K(\neg K(E)) \to K(E)$$

A paradox, since $K(E) \leftrightarrow \neg K(E)$. Modal logic **NL** disallows such a paradox. We can prove that assuming a paradox of $K(E) \leftrightarrow \neg K(E)$ cannot be proven, paradox does arise by the above argument, and thus the paradox must be provable. Thus the law of excluded middle is prohibited, and both $K(E)$ and $\neg K(E)$ must not be true, along with falsity of E and $\neg E$. But all assumptions above have been kept.
D is same in this regard that $D \leftrightarrow \neg D$ is proved in classical logic. This means that $K(\neg D) \leftrightarrow \neg K(\neg D)$. Again, the same argument applies so D and $\neg D$ must be false, along with $K(\neg D)$ and $\neg K(\neg D)$.

- **Common language truth**: While we can be fine with a sentence and its negation both being false, those uncomfortable with allowing this circumstance would better be served if we define truth in common language (as opposed to formal logic) as provable.

Fitch's paradox[1] works similarly - essentially, the paradox proves that all truth must already be known. The proof involves proving $\neg K(p \wedge \neg K(p))$ from $K(p \wedge \neg K(p))$, and thereby proving $\neg K(p \wedge \neg K(p))$ (which is disallowed in **NL** unless a paradox is unprovable), which then is used to prove that $p \to K(p)$. If we take it true, by an axiom, that there exists p such that $\neg(p \to K(p))$ holds and there exists no paradox, then by contraposition, we reach $U \equiv K(p \wedge \neg K(p))$. Thus we form a paradox of $U \leftrightarrow \neg U$, and thus both U and $\neg U$ must be false. But note that this line of thought assumes that an instance of $p \wedge \neg K(p)$ exists.

4 Conclusion

Modal logic **NL** was constructed as to resolve paradoxes in classical logic, while maintaining unrestricted comprehension. This construction allows consistent formation of naive set theory. The major strong point of **NL** is that expressiveness of common language is kept, such that we have a good mean of unifying analysis of formal and common language. Admittedly, while the general philosophy behind **NL** is simple enough ("eliminate paradoxes as we see"), actual construction is not, and along the way we sacrificed completeness - that if p must be semantically entailed, then p is syntactically entailed (in finite deductions) - which is not a minor sacrifice. It was shown that paradoxes in common language are more naturally resolved, if truth in common language refer to provability in formal logic.

References

[1] F. B. Fitch, *A logical analysis of some value concepts*, *The Journal of Symbolic Logic* **28** (1963) 135.

[2] R. Sorensen, *Vagueness and Contradiction*. Clarendon Press, 2001.

[3] S. Read, *Thomas Bradwardine, Insolubilia*. Peeters, 2010.

[4] A. Tarski, *Der wahrheitsbegriff in den formalisierten sprachen*, *Studia Philosophica* **1** (1936) 261.

[5] K. Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I*, *Monatshefte für Mathematik und Physik* **38** (1931) 173.

[6] G. Restall, *Truth-tellers in bradwardine's theory of truth*, in *Modern Views of Medieval Logic*, Peeters, 2018.

[7] S. Read, *Symmetry and paradoxes*, *History and Philosophy of Logic* **27** (2006) 307.

[8] D. Kaplan and R. Montague, *A paradox regained.*, *Notre Dame J. Formal Logic* **1** (1960) 79.

[9] M. Resnik, *On skolem's paradox*, *The Journal of Philosophy* **63** (1966) 425.