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Gaussian Conditionally Markov Sequences: Theory with Application

Reza Rezaie

University of New Orleans, rrezaie@uno.edu

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Reza Rezaie

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M.S. Shiraz University, 2009

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To my parents

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Abbreviations

a.s. almost surely

CM conditionally Markov

NG nonsingular Gaussian

ZMG zero-mean Gaussian

ZMNG zero-mean nonsingular Gaussian

Abstract

Markov processes have been widely studied and used for modeling problems. A Markov process has two main components (i.e., an evolution law and an initial distribution). Markov processes are not suitable for modeling some problems, for example, the problem of predicting a trajectory with a known destination. Such a problem has three main components: an origin, an evolution law, and a destination. The conditionally Markov (CM) process is a powerful mathematical tool for generalizing the Markov process. One class of CM processes, called CM_L, fits the above components of trajectories with a destination. The CM process combines the Markov property and conditioning. The CM process has various classes that are more general and powerful than the Markov process, are useful for modeling various problems, and possess many Markov-like attractive properties.

Reciprocal processes were introduced in connection to a problem in quantum mechanics and have been studied for years. But the existing viewpoint for studying reciprocal processes is not revealing and may lead to complicated results which are not necessarily easy to apply.

We define and study various classes of Gaussian CM sequences, obtain their models and characterizations, study their relationships, demonstrate their applications, and provide general guidelines for applying Gaussian CM sequences. We develop various results about Gaussian CM sequences to provide a foundation and tools for general application of Gaussian CM sequences including trajectory modeling and prediction.

We initiate the CM viewpoint to study reciprocal processes, demonstrate its significance, obtain simple and easy to apply results for Gaussian reciprocal sequences, and recommend studying reciprocal processes from the CM viewpoint. For example, we present a relationship between CM and reciprocal processes that provides a foundation for studying reciprocal processes from the CM viewpoint. Then, we obtain a model for nonsingular Gaussian reciprocal sequences with white dynamic noise, which is easy to apply. Also, this model is extended to the case of singular sequences and its application is demonstrated. A model for singular sequences has not been possible for years based on the existing viewpoint for studying reciprocal processes. This demonstrates the significance of studying reciprocal processes from the CM viewpoint.

Keywords: Stochastic process, conditionally Markov process, reciprocal process, Markov process, dynamic model, trajectory modeling and prediction.

Chapter 1

Introduction

1.1 Importance of this Research

For modeling a problem in probability theory, usually the following order should be considered [1]. First, if the problem is time-invariant, a random variable might be good enough. Otherwise, a stochastic process seems necessary. An independent process can be considered first for its simplicity. If such a simple process is not good enough, the next choice is usually a Markov process. A Markov process has two elements (an evolution law and an initial distribution). However, even the Markov process is not good enough for some problems. Then, sometimes a higher order (e.g., second order) Markov process is used. But such a model does not fit some problems well, for example, a time-varying problem with some information available about its future (e.g., destination). More specifically, consider the problem of predicting a trajectory with known destination. Such a problem in, e.g., air traffic control (ATC), has three elements: an origin, an evolution law, and a destination, to which the Markov process does not fit since it can not account for the destination information. In fact, the destination distribution of a Markov process is completely determined by its initial distribution and evolution law. The conditionally Markov (CM) process is a powerful mathematical tool for generalizing the Markov process. One class of CM processes called CM_L has the following elements: an evolution law and a joint distribution of the two endpoints (i.e., an initial distribution and a destination distribution conditioned on the initial). The CM_L process can model destination information and has a Markov-like evolution law, which is powerful and simple. The CM_L process is more suitable than the Markov process for modeling problems with destination information. For example, it can be used in ATC for trajectory modeling, prediction, and conflict detection.

Conditioning is a very powerful tool in probability theory. The Bayes rule follows from the definition of conditional probability. The concept of posterior probability, which relies on the concept of conditioning, is essential in probability and statistical inference. Conditioning is the key idea in the total probability theorem, which is extremely useful for many problems. The Markov property, being very important and widely used, is based on conditioning. The CM process combines the Markov property and conditioning. Different ways of combining the two lead to different classes of CM processes, which are more general and powerful than the Markov process. The CM process has various classes that are more general and powerful than the Markov process, are useful for modeling various problems, and possess many Markov-like attractive properties. CM processes are important for problem modeling and should be studied in order to provide useful results for their application. We define and study various classes of CM processes, obtain their dynamic models and characterizations, study their relationships, demonstrate their applications, and provide general guidelines for using CM processes in application.

Reciprocal processes were introduced in [2] in connection to the problem posed by Schrödinger [3]–[4]. Later, reciprocal processes were studied more in [5]–[40] and others. However, the existing viewpoint for studying reciprocal processes is not revealing and may lead to complicated results which are not necessarily easy to apply. Reciprocal processes are special CM processes. We initiate the CM viewpoint to study reciprocal processes, demonstrate its significance, show its power, obtain simple and easy to apply results for reciprocal processes, and recommend studying reciprocal processes from the CM viewpoint.

1.2 Existing Results and Our Contributions

Consider stochastic sequences[1] defined over $[0, N] = \{0, 1, \ldots, N\}$. For convenience, let the index be time. A sequence is Markov if and only if (iff) conditioned on the state at any time k, the segment before k is independent of the segment after k. A sequence is reciprocal iff conditioned on the states at any two times k_1 and k_2, the segment inside the interval (k_1, k_2) is independent of the segments outside $[k_1, k_2]$. In other words, inside and outside are independent given the boundaries. A sequence is CM over $[k_1, k_2]$ [2] iff conditioned on the state at time k_1 (k_2), the sequence is Markov over $[k_1 + 1, k_2]$ ($[k_1, k_2 - 1]$). Therefore, there are several classes of CM sequences with different k_1, k_2, and the conditioning time (i.e., conditioning at the first or the last time of the CM interval). So, the set of CM sequences is very large and its two important special classes are the Markov sequence and the reciprocal sequence.

Markov processes have been widely studied and used for modeling problems. However, they are not general enough in some cases [36]–[51], and more general processes are needed. The reciprocal process is a generalization of the Markov process. The CM process is a powerful mathematical tool for generalizing the Markov process.

In this chapter, we review existing results and our contributions in each chapter of the dissertation. In Chapter 2 to Chapter 6, we present results about CM sequences. We also point out applications of different classes of CM sequences. In Chapter 7, an application of CM sequences in trajectory modeling is discussed in more detail. First, we present a general overview of CM processes in theory and application.

1.2.1 CM Processes in Theory and Application

The CM process is a very large class of stochastic processes with various classes defined based on the Markov property and the conditioning. Some classes of Gaussian CM processes were defined in [52] based on mean and covariance functions, and later studied further in [29]. CM processes are powerful in both theory and application. However, their power has not been appreciated in the literature, and their study is limited to the above two papers. We demonstrate the power of CM processes (the CM property) in theory and application.

Reciprocal processes have been widely studied and used in various fields/problems, e.g., applied mathematics, theoretical physics, stochastic mechanics, image processing, intent inference, and acausal systems [2]–[51]. In these papers, reciprocal processes were defined, their properties were studied, their dynamic models were presented, their estimation was addressed, their importance and usefulness were demonstrated, and their applications in various problems were discussed. Reciprocal processes include the Markov process as a special case. The properties, models, and estimators of reciprocal processes presented in the literature are much more complicated than those of Markov processes. In essence, the literature studies the reciprocal process from inside the set (of reciprocal processes) without paying attention to processes outside. As we show later, this viewpoint may lead to complicated results and difficulties. Also, it does not reveal some hidden properties of the reciprocal process. Fortunately, as we demonstrate later, CM processes (including the reciprocal process) can provide an alternative and in fact better viewpoint for studying reciprocal processes with many benefits. From the CM viewpoint we can study the reciprocal process from outside of the set as well. This viewpoint gives a clearer picture of the reciprocal process, is more revealing, and leads to simpler results. This demon-

[1] Our definitions and some of our results work for both discrete index and continuous index processes; however, we present them all for discrete index processes (i.e., sequences).

[2] This is called the CM interval.

strates the power of CM processes in theory. However, the literature on the reciprocal process has not appreciated its relationship to the CM process and has not recognized the significance of studying the reciprocal process from the CM viewpoint. Only very few papers implicitly benefited from the CM property [30]–[31]. For example, as we show later, studying reciprocal sequences from the CM viewpoint is very insightful and fruitful. But there is no paper in the literature on studying reciprocal sequences from the CM viewpoint.

CM processes are powerful and flexible for modeling complicated problems (systems/ phenomena), where Markov processes are not adequate. The CM property is based on the Markov property and the conditioning. Different ways of combining the two give different CM classes. As we illustrate later, by an appropriate combination of the Markov property and the conditioning we can define a suitable CM process for modeling a given problem. The power of CM processes for problem modeling has not been recognized in the literature. We develop a theoretical foundation of (Gaussian) CM sequences/processes, obtain results/tools (properties, models, characterizations, representations, etc.) for their application, present guidelines for their use in problem modeling, and demonstrate their application. For example, we demonstrate an application of CM_L sequences to trajectory modeling with destination information. Some papers used (finite state) reciprocal sequences, which are special CM_L sequences, for modeling such trajectories [41]–[47]. CM_L sequences and the structure of their dynamic model provide a natural, simpler, and more general framework for modeling trajectories with destination information. However, they have not been used in the literature.

1.2.2 Chapter 2

The notion of Gaussian CM processes was introduced in [52] based on mean and covariance functions of Gaussian processes. [52] studied and characterized continuous time stationary Gaussian CM processes that are nonsingular on the interior of the time interval. [29] extended the definition of Gaussian CM processes (in [52]) to the general (Gaussian/nonGaussian) case. Furthermore, [29] commented on some properties of Gaussian CM processes and Gaussian reciprocal processes. By conditioning on the state of the process at the first time of the CM interval, different Gaussian CM processes were defined in [52]. However, it is possible not only to extend the definitions to non-Gaussian processes, but also to other CM processes by conditioning on the state at the last time of the CM interval. Such processes are useful for both theory and application. Despite their power in theory and application, to our knowledge, (unlike reciprocal processes) CM processes have not received much attention and have not been studied well to gain understanding and to obtain tools for application. In addition, the literature on the reciprocal process has not appreciated its relationship to the CM process well and has not benefited from it except implicitly in very few cases [30]–[31]. In particular, we are not aware of any paper studying Gaussian reciprocal sequences from the CM viewpoint.

The main goal of Chapter 2 is two-fold: 1) to define and study various useful classes of CM sequences and provide useful and easy to apply results for their application, e.g., for motion trajectory modeling with destination information, and 2) to lay a foundation for studying an important special class of CM sequences, the reciprocal sequence, from the CM viewpoint.

The contributions of Chapter 2 are as follows. In [52], Gaussian CM processes were defined by conditioning only on the state at the first time of the CM interval. We extend the definitions by conditioning on the state at the last time of the CM interval. The usefulness of such processes is discussed for their application (e.g., trajectory modeling) and also for studying reciprocal processes. Definitions and derivations presented in [52] (and other papers following [52]) are restricted to the Gaussian case. Here, to build the foundation rigorously, all definitions are presented in the formal probability language for the general (Gaussian/non-Gaussian) case, and properties of CM sequences are studied. Then, in order to present results in a simple language for application, simple formulas equivalent to the formal definitions are obtained. Forward and backward dynamic models of (stationary/non-stationary) nonsingular Gaussian (NG) CM sequences in a recursive form are obtained. These models are complete descriptions of the corresponding classes of NG CM sequences. Based on the models, characterizations of NG CM sequences are obtained. As a by-product, new factorizations of two covariance matrices, characterizing two classes of NG CM sequences, are presented.

From system theory, it is well known that the state concept is equivalent to the Markov property, that is, conditioned on the state at a time, the states before and after are independent. That is why there exists a simple recursive model for the evolution of the Markov sequence. However, for the general Gaussian sequence there is no simple recursive model for the evolution. The CM sequence is more general than the Markov sequence. Consequently, a CM sequence does not necessarily have the above concept of state, in general. Instead, it has a similar concept if it is conditioned at two instead of one time. That is why a simple recursive model also exists for the evolution of Gaussian CM sequences.

Part of the results presented in Chapter 2 have appeared in [53].

1.2.3 Chapter 3

Reciprocal processes have been used in many different areas of science and engineering (e.g., [36]–[51]) where stochastic processes more general than Markov processes are needed. [36]–[39] discussed reciprocal processes in the context of stochastic mechanics. In [40], the behavior of acausal systems was described using reciprocal processes. More specifically, on the one hand, reciprocal processes are a generalization of Markov processes. On the other hand, acausal systems can be seen as a generalization of causal systems [40]. Then, the relationship between acausal systems and reciprocal processes was studied in [40]. Also, Based on quantized state space, [41]–[46] used finite state reciprocal sequences for trajectory modeling, detection of anomalous trajectory pattern, intent inference, tracking, and track-before-detect. The idea of the reciprocal process was implicitly utilized in [48]–[49] for intent inference in vehicle's intelligent interactive displays. Application of reciprocal processes in image processing was discussed in [50]–[51]. The behavior of particles in the problem posed in [3]–[4] by Schrödinger can be explained in the reciprocal process setting [2].

Reciprocal processes were introduced in [2] and studied further in [5]–[35] and others. A reciprocal process was considered in [5] related to a first-passage time problem. [6]–[8] characterized the stationary Gaussian reciprocal process and presented a functional form of the corresponding covariance function. In [9]–[13], reciprocal processes were studied in a general setting. A stochastic calculus study of reciprocal processes was presented in [13]. [29] commented on the relationship between Gaussian CM processes and Gaussian reciprocal processes. Following [52], [29] considered Gaussian processes being nonsingular on the interior of the time interval of the process. Inspired by [52], a Wiener process-based representation of Gaussian CM processes was also presented. State evolution models of Gaussian reciprocal processes were presented and studied in [14]–[18]. A stochastic differential equation of Gaussian reciprocal processes and their properties were studied in [14]–[15]. A dynamic model of NG reciprocal sequences was presented in [18]. [16] studied Gaussian Markov sequences with the same Gaussian reciprocal model of [18]. The continuous time version of that problem was addressed in [17]. [19] obtained a representation of the Gaussian reciprocal process in terms of the Gaussian Markov process and connected it to a two-point boundary value problem. A covariance extension problem for reciprocal sequences was discussed in [20]. Parameter estimation for a special case of the Gaussian reciprocal model of [18] was addressed in [21]. [22]–[23] studied characterization of stationary multivariate Gaussian reciprocal processes in terms of their covariance. [24]–[28] considered modeling and estimation of finite state reciprocal sequences. The optimal smoothing of finite state reciprocal sequences was studied in [27]. Also, [28] presented the maximum likelihood estimation of finite state reciprocal sequences and studied its performance.

Despite many papers on the theory of reciprocal processes (e.g., [2], [5]–[35]), there is still a lack of easy to apply results/tools for their application. To make this issue clear and demonstrate the significance of studying reciprocal processes from the CM viewpoint, as an example, consider a dynamic model of NG reciprocal sequences presented in [18], which is the most significant paper on Gaussian reciprocal sequences. It was shown that the evolution of a NG reciprocal sequence can be described by a second-order nearest-neighbor model driven by locally correlated dynamic noise [18]. That model describes the NG reciprocal sequence completely (i.e., necessarily and sufficiently), and can be considered a natural generalization of the Markov model. However, due to its nearest neighbor structure and its colored dynamic noise, it is not easy to apply. Also, recursive estimation of a reciprocal sequence based on the model of [18] is challenging. That is why several papers [18], [32]–[35] tried to find a recursive estimator. Clearly, a simpler and easier to apply model for NG reciprocal sequences is desired. But it is difficult to derive such a model from the viewpoint of the existing literature including [18]. So, a simpler yet complete description of the NG reciprocal sequence in an alternative viewpoint is desired. CM sequences provide such a good viewpoint, leading to many benefits. In other words, the literature studies reciprocal sequences from inside the set of reciprocal sequences without paying attention to sequences outside. This viewpoint may lead to complicated results and difficulties. From the CM viewpoint, however, we can also study the reciprocal sequence from outside. The CM viewpoint gives a clearer picture of the reciprocal sequence (from outside), is more revealing, and leads to simple results. For example, we obtain a dynamic model with white dynamic noise for the NG reciprocal sequence from the CM viewpoint, based on which recursive estimation is straightforward.

The main goal of Chapter 3 is three-fold: 1) to propose studying reciprocal sequences from the CM viewpoint and demonstrate its significance, insightfulness, and fruitfulness, 2) to study NG reciprocal sequences from the CM viewpoint, 3) to obtain easy to apply results and tools for NG reciprocal sequences.

The main contributions of Chapter 3 are as follows. The reciprocal sequence is studied explicitly from the CM viewpoint, which is a larger set of sequences. Studying, modeling, and characterizing the reciprocal sequence from this viewpoint are different from those of [18] and the literature. This fruitful angle has several advantages. It provides more insight into the reciprocal sequence via its relationship to other CM sequences. As a result, new properties of the Gaussian reciprocal sequence are revealed. In addition, the CM sequence and the reciprocal sequence can be treated in the same way. This is not only theoretically interesting, but also useful for application. We demonstrate that the relationship between the Gaussian CM process and the Gaussian reciprocal process stated in [29] is incomplete. More specifically we elaborate on the comment of [29], show that the said relationship is not sufficient even for Gaussian processes, and obtain a relationship between the general (Gaussian/non-Gaussian) reciprocal and CM processes. A characterization of the NG reciprocal sequence is obtained based on its relationship to the CM sequence. This characterization is the same as that of [18], but it is obtained by a different approach and from a different viewpoint. We show that a NG sequence is reciprocal iff it is both CM_L (i.e., conditioned on the state at the last time N is Markov over $[0, N-1]$) and CM_F (i.e., conditioned on the state at the first time 0 is Markov over $[1, N]$). In addition, we discuss how characterizations change from a NG CM sequence to the NG reciprocal sequence and then to the NG Markov sequence; that is, how different classes of NG CM sequences contribute to the construction of the NG reciprocal sequence, namely a spectrum of characterizations from a CM class to the reciprocal class. Moreover, we obtain new dynamic models for the NG reciprocal sequence based on the forward and backward models of CM_L and CM_F sequences. We call these models reciprocal CM_L and reciprocal CM_F models. They are driven by white (rather than colored) noise and are easy to apply. Also, we discuss under what conditions these models are for NG Markov sequences.

Part of the results presented in Chapter 3 have appeared in [54].

1.2.4 Chapter 4

Due to its simple structure and whiteness of the dynamic noise, our reciprocal CM_L model is easy to apply, e.g., for trajectory modeling with destination. For example, recursive estimation of a reciprocal sequence based on a reciprocal CM_L model is straightforward. However, it is not clear how parameters of a reciprocal CM_L model can be designed in a problem. More generally, a CM_L sequence and its dynamic model (obtained in Chapter 2) can model trajectories with destination. However, guidelines for parameter design of a CM_L model are required. Following [9], [43] used a transition probability function of a finite state reciprocal sequence from a transition probability function of a finite state Markov sequence in a quantized state space for a problem of intent inference. But [43] did not discuss if all reciprocal transition probability functions can be obtained from a Markov transition probability function, which is critical for application. Also, it is not always feasible or easy to quantize the state space in some applications. NG Markov sequences modeled by the same reciprocal model of [18] were studied in [16]. However, the results are based on the model of [18], which is not simple or easy to apply.

The main goal of Chapter 4 is three-fold: 1) to present some approaches/guidelines for parameter design of CM_L, CM_F, and reciprocal CM_L models for their application, 2) to obtain a representation of NG CM_L, CM_F , and reciprocal sequences, revealing a key fact about these sequences, and to emphasize the significance of studying reciprocal sequences from the CM viewpoint, and 3) to present a full spectrum of dynamic models from a CM_L model to a reciprocal CM_L model and show how models of various intersections of CM classes can be obtained.

The main contributions of Chapter 4 are as follows. From the CM viewpoint, we not only show how a Markov model induces a reciprocal CM_L model, but also prove that *every* reciprocal CM_L model can be induced by a Markov model. Then, we give formulas to obtain parameters of the reciprocal CM_L model from those of the Markov model. This approach is more intuitive than a direct parameter design of a reciprocal CM_L model, because one usually has an intuitive understanding of Markov models. A full spectrum of dynamic models from a CM_L model to a reciprocal CM_L model is presented. This spectrum helps to understand the gradual change from a CM_L model to a reciprocal CM_L model. For application of other CM classes, e.g. intersection of two CM classes defined in Chapter 2, we need their dynamic models. It is demonstrated how dynamic models for intersections of NG CM sequences can be obtained. In addition to their usefulness for application, these models are particularly useful to describe the behavior of a sequence (e.g., a reciprocal sequence) belonging to more than one CM class. Based on a valuable observation, [29] discussed representations of NG continuous time CM processes (including NG continuous time reciprocal processes) in terms of a Wiener process and an uncorrelated NG vector. First, we show that the representation presented in [29] is not sufficient for a Gaussian process to be reciprocal (although [29] stated that the representation was sufficient, which has not been corrected so far). Then, we obtain a simple (necessary and sufficient) representation for NG reciprocal sequences from the CM viewpoint. As a result, the significance of studying reciprocal sequences from the CM viewpoint is demonstrated. Second, inspired by [29], we show that a NG CM_L (CM_F) sequence can be represented by a sum of a NG Markov sequence and an uncorrelated NG vector. This (necessary and sufficient) representation makes a key fact of CM sequences clear and provides some insight for parameter design of CM_L and CM_F models based on those of a Markov model and an uncorrelated NG vector. Third, we study the obtained representations of NG CM_L, CM_F, and reciprocal sequences in detail and, as a by-product, obtain new representations of some matrices, which are characterizations of NG CM_L, CM_F, and reciprocal sequences.

Part of the results presented in Chapter 4 have appeared in [55].

1.2.5 Chapter 5

From the viewpoint of singularity, one can consider two extreme cases for Gaussian sequences. One extreme is a sequence being almost surely constant throughout the time interval. The other extreme is a nonsingular sequence, i.e., a sequence with a nonsingular covariance matrix. For example, a Gaussian sequence can be singular because it is almost surely constant over time or at a time (i.e., the state over time or at a time is almost surely constant), or because the states of the sequence at two or more times are almost surely linearly dependent. There are various such causes (corresponding to different times) leading to singular Gaussian sequences. As a result, we have various singularity. It is desired to model and characterize all singular and nonsingular Gaussian sequences in a unified way.

Characterizations of NG Markov, reciprocal, and CM sequences presented in [56], [18], Chapters 2, and Chapter 3 are based on the inverse of the covariance matrix of the whole sequence. So, they do not work for singular sequences. In [57] a characterization was presented for the scalar-valued (singular/nonsingular) Gaussian Markov process in terms of the covariance function. However, that characterization does not work for the general vector-valued case. In [58] a characterization was presented for a special kind of NG reciprocal processes (i.e., second-order NG processes, that is, Gaussian processes with covariance matrices corresponding to any two times of the process being nonsingular) in terms of the covariance function of the process. [19] presented a characterization of the Gaussian reciprocal process based on the Markov property. That characterization is actually a representation of the reciprocal process in terms of the Markov process and is specifically for continuous time processes. [30] presented a different characterization of the Gaussian reciprocal process based on the Markov property. Characterizations of [19] and [30] converted the question about a characterization of the Gaussian reciprocal process to the question about a characterization of the Gaussian Markov process, which was left unanswered for the general vector-valued Gaussian process. Later studies on the covariance of Gaussian processes were mainly under nonsingularity assumption [59]–[61]. Despite the above attempts, to our knowledge, there is no characterization in terms of the covariance function for the general (singular/nonsingular) Gaussian CM (including reciprocal and Markov) process in the literature.

The well-posedness of the reciprocal dynamic model presented in [18] (i.e., the uniqueness of the sequence obeying the model) is guaranteed by the nonsingularity assumption for the covariance of the whole sequence. It can be seen that unlike the model of [18], the nonsingularity assumption is not critical for the uniqueness of sequences obeying CM dynamic models presented in Chapter 2. Dynamic models of the NG reciprocal sequence obtained in Chapter 2 does not work for singular sequences, although the nonsingularity assumption is not critical for its well-posedness. To our knowledge, there is no dynamic model for the Gaussian reciprocal sequence[3] in the literature. For example, it is not clear how the model of [18] can be extended to the Gaussian reciprocal sequence. More generally, there is no dynamic model for Gaussian CM sequences in the literature.

Although they make the analysis and modeling easy, nonsingularity assumptions restrict application of Gaussian CM (including reciprocal and Markov) sequences. Without such assumptions, we have a larger and more powerful set of sequences for modeling problems. Some problems can be modeled by a singular sequence better than a nonsingular one. For example, a NG CM_L sequence is used in Chapter 7 for trajectory modeling between an origin and a destination. Now assume that the origin/destination is known, i.e., some components of the state of the sequence at the origin/destination are almost surely constant. Then, a singular CM_L sequence is better for modeling such trajectories.

[3]In this subsection and in Chapter 5, by the "Gaussian sequence" we mean the general singular/nonsingular Gaussian sequence. Otherwise, we make it explicit if we only mean the nonsingular Gaussian sequence (i.e., covariance of the whole sequence being nonsingular).

The main goal of Chapter 5 is threefold: 1) to obtain dynamic models and characterizations of the general Gaussian CM sequence to unify singular and nonsingular Gaussian CM sequences theoretically, 2) to provide tools for application of (singular/nonsingular) Gaussian CM sequences, e.g., in trajectory modeling with destination information, 3) to emphasize the significance of studying reciprocal sequences from the CM viewpoint, e.g., by obtaining two dynamic models for the general Gaussian reciprocal sequence from the CM viewpoint.

The main contributions of Chapter 5 are as follows. Dynamic models and characterizations of (singular/nonsingular) Gaussian CM, reciprocal, and Markov sequences are obtained. Two types of characterizations are presented for Gaussian CM and reciprocal sequences. The first type is in terms of the covariance function of the sequence. The second type, which has a similar spirit to (but different from) those of [19] and [30], is based on the state concept in system theory (i.e., the Markov property). Then, by deriving a characterization for the general vector-valued Gaussian Markov sequence in terms of the covariance function, we can check the Markov property. Then, the second type of characterization of Gaussian CM and reciprocal sequences becomes complete and makes a better sense. It is shown that dynamic models of Gaussian CM sequences have a structure similar to those of NG CM sequences presented in Chapter 2, and the difference is in the values of their parameters. Therefore, the presented models unify singular and nonsingular CM sequences. We obtain two dynamic models for the Gaussian reciprocal sequence from the CM viewpoint. As a result, the significance and the fruitfulness of studying reciprocal sequences from the CM viewpoint is demonstrated. A full spectrum of models (characterizations) ranging from a CM_L model (characterization) to a reciprocal CM_L model (characterization) for Gaussian sequences is presented. The obtained models and characterizations unify singular and nonsingular Gaussian CM sequences. The representation of NG CM_L/CM_F sequences presented in Chapter 4 is extended to the general singular/nonsingular Gaussian case.

1.2.6 Chapter 6

The evolution of a Markov sequence can be modeled by a Markov, reciprocal, CM_L, or CM_F model[4]. Similarly, the evolution of a reciprocal sequence can be modeled by a reciprocal model [18] or a CM_L (CM_F) model (Chapter 2). Therefore, a CM sequence can be modeled by more than one model. One model can be easier to apply than another in an application. For example, a reciprocal CM_L model (Chapter 3) is easier to apply than a reciprocal model of [18] for trajectory modeling with destination information (Chapter 7). The dynamic noise is white for the former but colored for the latter. Also, the reciprocal model of [18] can be useful for some other purposes since it is a natural generalization of a Markov model in the nearest-neighbor structure. In addition, a Markov model is simpler than a reciprocal, CM_L, or CM_F model. So, if we have a reciprocal, CM_L, or CM_F model whose sequence is Markov, a Markov model is desired. Moreover, sometimes only a forward (backward) model is available when a backward (forward) one is required. So, it is important to determine these models from each other.

Two models are said to be *probabilistically equivalent*[5] if their sequences have the same distribution. In some cases, this definition of equivalent models is not sufficient because it is only about the distribution, not individual sample path. The two-filter smoothing approach is an example, where to verify the conditions required for derivation, one needs the relationship in dynamic noise and boundary values[6] between forward and backward Markov models for having the same sample path of the sequence [62]–[64]. In other words, it is desired to find forward and backward Markov models whose stochastic sequences are path-wise identical. Two models are said to be *algebraically equivalent* if their stochastic sequences are path-wise identical. Despite

[4]By a "dynamic model" or "model", we may mean a model with or without its boundary condition, as is clear from the context.

[5]Later, by "equivalent" we mean probabilistically equivalent.

[6]For a forward (backward) Markov model, a boundary value means an initial (a final) value.

several attempts, to our knowledge, there is no general and unified approach for determination of algebraically equivalent Markov, reciprocal, or CM models in the literature.

Motivated by the two-filter smoothing approach, determination of a backward Markov model from a forward Markov model has been the topic of several papers [65]–[71]. [65] studied a backward model for a second order (or Gaussian) process equivalent to a forward model. To derive a smoother for a Markov process, [66] obtained a reverse-time model describing a process statistically equivalent to the original process up to second-order properties. In [67]–[68], a derivation of a backward Markov model was presented based on the scattering theory. [69] derived backward Markov models for second order processes equivalent to the forward models in the sense that they give the same state covariance. The forward and backward Markov models derived in [65]–[69] are equivalent, but not algebraically equivalent. The backward Markov model presented in [70] is algebraically equivalent only for forward models with nonsingular state transition matrices, not for other models. For models with a singular state transition matrix, [70] only provides an equivalent backward model. Later papers followed the approach of [70] and, to our knowledge, there is no backward Markov model algebraically equivalent to a forward one for a singular state transition matrix in the literature. As a result, we can not check the required conditions of a two-filter smoother for a Markov model with a singular state transition matrix.

Given a Markov model, [18] determined an algebraically equivalent reciprocal model. However, [18] did not present a unified approach for determination of other algebraically equivalent CM models.

An important question in the theory of reciprocal processes is regarding Markov processes governed by the same reciprocal evolution law [16], [17], [9]. Given a reciprocal model of [18], [16] discussed determination of Markov sequences sharing the same reciprocal model. The continuous-time counterpart of that problem was addressed in [17]. Also, given a reciprocal transition density, [9] determined the required conditions on the joint endpoint distribution so that the process is Markov. It is desired to have a simple approach for studying and determining Markov models whose sequences share the same reciprocal/CM_L model. This is not only useful for understanding the relationship between these models and between their sequences, but also helpful for application of these models. For example, CM_L models induced by Markov models are discussed in Chapter 4 for trajectory modeling with destination information. It is shown that inducing a CM_L model by a Markov model is useful for parameter design of a reciprocal CM_L model for trajectory modeling with destination information. Also, it is shown that a reciprocal CM_L model can be induced by any Markov model whose sequence obeys the given reciprocal CM_L model (and some boundary condition). So, it is desired to determine all such Markov models and to study their relationship. But a simple approach for this purpose is lacking in the literature.

The main goal of Chapter 6 is threefold: 1) to study the relationships between dynamic models of different classes of CM sequences including Markov, reciprocal, CM_L, and CM_F, 2) to define and distinguish the notions of probabilisticly equivalent and algebraically equivalent dynamic models, and 3) to present a unified approach for determination of algebraically equivalent models.

Chapter 6 makes the following main contributions. The relationship between CM_L, CM_F, reciprocal, and Markov dynamic models for NG sequences are studied. The notion of algebraically equivalent models is defined versus (probabilistically) equivalent ones. Then, a general and unified approach is presented, based on which given one of the above models, any algebraically equivalent model can be obtained. The presented approach is simple and not restricted to the above models. As a special case, a backward Markov model algebraically equivalent to a forward Markov model is obtained. Unlike [70], this approach works for both singular and nonsingular state transition matrices. So, the required conditions in the derivation of two-filter smoothing can be verified for all Markov models (with singular/nonsingular state transition matrices). The reciprocal model algebraically equivalent to a Markov model presented in [18] is obtained as a special case of our result. A simple approach is presented for studying and determining Markov models whose sequences share the same reciprocal/CM_L model.

Part of the results presented in Chapter 6 have appeared in [72].

1.2.7 Chapter 7

Modeling and predicting trajectories with an intent or a destination have been studied in the literature. This problem has two steps: (a) trajectory modeling, (b) trajectory processing (filtering and prediction). The corresponding papers can be divided into two groups. One group of papers focus on trajectory processing without explicitly modeling trajectories with intent/destination. In the modeling step, they consider Markov models developed for trajectories with no intent or destination information. Also, in the processing step they use estimation approaches developed for the case of no intent or destination. Then, in the processing step they heuristically utilize the intent/destination information to improve trajectory filtering and prediction performance. Such approaches for intent-based trajectory prediction can be found in [73]–[81]. [73]–[76] presented some trajectory predictors based on hybrid estimation aided by intent information for air traffic control (ATC). In [77], the interacting multiple model (IMM) approach was used for trajectory prediction, where a higher weight was assigned to the model with the closest heading towards the waypoint. Then, a pseudo measurement of destination was used to improve the prediction. To incorporate destination information, [78]–[79] also used a pseudo measurement to improve state estimation. [80] presented an approach for trajectory prediction using an inferred intent based on a correlation factor. [81] used the intent information (broadcast by ADS-B) in a tracking filter to improve state estimation in ATC. The trajectory model is not clear in the above approaches. However, to study, generate, and analyze trajectories, it is desired to model them. A rigorous mathematical model of trajectories is a basis for a systematic approach for handling them.

Another group of papers first consider the trajectory modeling step and explicitly model trajectories with intent/destination information. Then, in the processing step they use the obtained model for filtering and prediction. Such a filter and trajectory predictor are derived from one principle and are not based on a heuristic combination of different pieces. Therefore, one can systematically study and analyze them. Due to many sources of uncertainty, trajectories are mathematically modeled as some stochastic processes. An approach was presented in [82] to incorporate predictive information in trajectory modeling. After quantizing the state space, [41]–[46] used finite-state reciprocal sequences for intent inference and trajectory modeling with destination/waypoint information. [41] presented an approach to determine anomalous trajectory patterns using stochastic context-free grammar and finite state reciprocal sequences to assist the human operator in a surveillance system. The inadequacy of Markov models for modeling trajectory patterns with a destination was also discussed. In addition, the complexity of the corresponding estimation approaches was pointed out. [42] presented several trajectory patterns based on the context-free grammar and reciprocal sequences in a quantized state space. [43] used context-free grammar and finite state reciprocal sequences for trajectory modeling and intent inference in a quantized state space. The presented trajectory filter was based on combining a finite state reciprocal sequence filter and a context-free grammar filter. A track extraction approach, that is, confirming target existence in a set of observations, was presented in [44] using a finite state reciprocal sequence in a quantized state space. [45] presented a smoother for a generalized finite state reciprocal sequence used for trajectory modeling in a quantized state space. A track-before-detect approach was presented in [46] using maximum likelihood estimation and finite state reciprocal sequences in a quantized state space. Reciprocal sequences provide an interesting mathematical model for trajectories with destination information. However, it is not always feasible or easy to quantize the state space. So, it is desirable to use continuous state reciprocal sequences to model trajectories. Gaussian sequences have continuous-state space. A dynamic model of NG reciprocal sequences was presented in [18]. However, due to the nearest-neighbor structure and the colored dynamic noise, the model of [18] is not easy to apply for trajectory modeling and its generalization is not easy. For example, following [18], a generalized Gaussian reciprocal sequence was presented in [47] for trajectory modeling. The approach of [48]–[49] for intent inference (e.g., in selecting an icon on an in-vehicle interactive display) based on bridging distributions can also be seen in the reciprocal process setting, although reciprocal processes were not explicitly used or mentioned. To emphasize that trajectories end up at a specific destination, we call them *destination-directed trajectories*. A class of stochastic sequences capable of modeling the main components of destination-directed trajectories (i.e., an origin, a destination, and motion in between) with an appropriate and easy to apply dynamic model is desired.

Consider a trajectory modeling problem, where there is information available about the destination of a moving object. An example is an airliner flying from an origin to a destination. For modeling trajectories in such a problem there are three main components: an origin, a destination, and motion in between. The behavior of a Markov sequence can be described by an evolution law and an initial probability density function. So, the Markov sequence is not flexible enough to model destination-directed trajectories. Given an initial density and an evolution law, the future of a Markov sequence is determined probabilistically. CM_L sequences have the following main components: a joint endpoint density (i.e., an origin density and a destination density conditioned on the origin) and a Markov-like evolution law. CM_L sequences are suitable for modeling destination-directed trajectories. Also, they can be easily and systematically generalized if necessary.

In Chapter 7, we propose the use of CM_L sequences for destination-directed trajectory modeling. Considering the main components of destination-directed trajectories, we demonstrate how naturally one would use the CM_L sequence for modeling such trajectories. This class of CM sequences provides a general framework for modeling destination-directed trajectories. The CM_L sequence models the main components of destination-directed trajectories and the only assumption in its definition is the Markov-like (i.e., conditionally Markov) property of its evolution law. We show how parameters of a CM_L model can be designed for destination-directed trajectory modeling. The CM_L sequence enjoys several desirable properties for trajectory modeling (for example in ATC). The Gaussian CM_L sequence, its realization, its properties, and its dynamic model (the CM_L model) are studied for the purpose of trajectory modeling. Filtering and prediction formulations are derived based on the CM_L model. The behavior of the filter is studied. Trajectory predictors with and without destination information are compared based on their formulations and some simulations. Several simulations are presented to demonstrate the results.

Part of the results presented in Chapter 7 have appeared in [83].

1.2.8 Conventions and Notations

We give conventions used in multiple chapters of the dissertation.

We consider stochastic sequences defined over the interval $[0, N]$, which is a general discrete-index interval. For convenience this discrete-index is called time. The following conventions are used:

$$[i, j] \triangleq \{i, i+1, \ldots, j-1, j\}, i < j$$
$$[x_k]_i^j \triangleq \{x_k, k \in [i, j]\}$$
$$[x_k] \triangleq [x_k]_0^N$$

$$x \triangleq [x_0', x_1', \ldots, x_N']'$$
$$i, j, k_1, k_2, l_1, l_2 \in [0, N]$$
$$\sigma([x_k]_i^j) \triangleq \sigma\text{-field generated by } [x_k]_i^j$$

where k in $[x_k]_i^j$ is a dummy variable. $[x_k]$ is a stochastic sequence. $[x_k]_i^j$ is not defined for $i > j$. Also, $\sigma([x_k]_i^j)$, for $i > j$, and $\sigma([x_k]_c^c \setminus \{x_c\})$ are defined as the trivial σ-field (i.e., including only the empty set and the whole set Ω). The symbols " $'$ " and "\" are used for matrix transposition and set subtraction, respectively. In addition, 0 may denote a zero scalar, vector, or matrix, as is clear from the context. $P\{\cdot\}$ denotes probability and $F(\cdot|\cdot)$ denotes a conditional comulative distribution function (CDF). Also, $p(\cdot)$ and $p(\cdot|\cdot)$ are a probability density function (PDF) and a conditional PDF, respectively. $\mathbb{R}$ denotes the set of real numbers. $\mathcal{N}(\mu_k, C_k)$ denotes the Gaussian distribution with mean μ_k and covariance C_k. Also, $\mathcal{N}(x_k; \mu_k, C_k)$ denotes the corresponding Gaussian density with (dummy) variable x_k. $C_{i,j}$ is a covariance function, and $C_i \triangleq C_{i,i}$. C is the covariance matrix of the whole sequence $[x_k]$ ($C = \text{Cov}(x)$). A Gaussian sequence $[x_k]$ is nonsingular if its covariance matrix C is nonsingular. The abbreviations ZMNG and NG are used for "zero-mean nonsingular Gaussian" and "nonsingular Gaussian". For a matrix A, $A_{[r_1:r_2, c_1:c_2]}$ denotes its submatrix consisting of (block) rows r_1 to r_2 and (block) columns c_1 to c_2 of A. For square matrices M_k, we have

$$\text{diag}(M_0, M_1, \ldots, M_N) \triangleq \begin{bmatrix} M_0 & 0 & \cdots & 0 \\ 0 & M_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & M_N \end{bmatrix}$$

The evolution of a sequence can be modeled by a forward or a backward model. The forward direction is the default. For forward direction/models, we drop the term "forward", but for backward direction/models we make "backward" explicit. We have different dynamic models including a Markov model, a reciprocal model, a reciprocal CM_L/CM_F model, and a CM_L/CM_F model. For example, a Markov model has an initial condition and a CM_L model has a boundary condition. For unification, we may call an initial condition a boundary condition. Sometime we need to refer to a dynamic model including its boundary condition, but sometimes we need to refer to a dynamic model without its boundary condition. The term "dynamic model" or "model" is used to refer to both these cases when the meaning is clear from the context. But to avoid confusion in some cases we use the term "evolution model" to emphasize that we mean a dynamic model without considering its initial/boundary condition. For example, a "Markov evolution model" means a Markov model without considering its initial condition. Also, a "CM_L evolution model" means a CM_L model without considering its boundary condition.

Some equations and statements hold almost surely (and not strictly), as is clear from the context. For clarity, in some cases we mention it explicitly. The abbreviation "a.s." stands for "almost surely". Definitions and some of the results work for both discrete-time and continuous-time processes, but we present them all for discrete-time processes (i.e., sequences).

Chapter 2

Modeling and Characterizing Nonsingular Gaussian CM Sequences

In this chapter, we 1) provide useful and easy to apply results for application of CM sequences, e.g., for motion trajectory modeling with destination information, and 2) lay a foundation for studying an important special class of CM sequences, the reciprocal sequence, from the CM viewpoint.

2.1 Definitions and Preliminaries

To build a solid foundation, we start from definitions in the formal probability language. However, all the main results are presented in a simple language ready for application. We assume the stochastic sequences are defined with respect to an underlying probability triple $(\Omega, \mathcal{A}, P)$.

2.1.1 CM Definitions and Notations

A sequence $[x_k]$ is $[k_1, k_2]$-$CM_c, c \in \{k_1, k_2\}$, (i.e., CM over $[k_1, k_2]$) iff conditioned on the state at time k_1 (k_2), the sequence is Markov over $[k_1+1, k_2]$ ($[k_1, k_2-1]$). To build a solid foundation, we need a formal definition of CM sequences (Definition 2.1.1 below). To provide results for application, however, later we present Corollary 2.1.5, which is equivalent to Definition 2.1.1.

Definition 2.1.1. *$[x_k]$ is $[k_1, k_2]$-$CM_c, c \in \{k_1, k_2\}$, if for every $j \in [k_1, k_2]$*

$$P\{AB|x_j, x_c\} = P\{A|x_j, x_c\}P\{B|x_j, x_c\} \tag{2.1}$$

where $A \in \sigma([x_k]_{j+1}^{k_2} \setminus \{x_c\})$ and $B \in \sigma([x_k]_{k_1}^{j-1} \setminus \{x_c\})$.

The interval $[k_1, k_2]$ of a $[k_1, k_2]$-CM_c sequence is called the *CM interval* of the sequence. By Definition 2.1.1, the sequence is defined over the interval $[0, N]$ but the CM interval is $[k_1, k_2]$.

Remark 2.1.2. *We use the following notation*

$$[k_1, k_2]\text{-}CM_c = \begin{cases} [k_1, k_2]\text{-}CM_F & \text{if} \quad c = k_1 \\ [k_1, k_2]\text{-}CM_L & \text{if} \quad c = k_2 \end{cases}$$

where the subscript "F" or "L" is used because the conditioning is at the first or the last time of the CM interval.

Remark 2.1.3. *When the CM interval of a sequence is the whole time interval, it is dropped: a $[0, N]$-CM_c sequence is called CM_c.*

A CM_0 (CM_N) sequence is called a CM_F (CM_L) sequence. For the backward direction, a CM_0 (CM_N) sequence is a CM_L (CM_F) sequence. We consider mainly the forward direction. For the backward direction we present only results that are useful for some applications (e.g., smoothing).

We define that every sequence with a length smaller than 3 (i.e., $\{x_0, x_1\}$, $\{x_0\}$, and $\{\}$) is Markov. Similarly, every sequence is $[k_1, k_2]$-CM_c, $|k_2 - k_1| < 3$. So, CM_L and $CM_L \cap [k_1, N]$-CM_F, $k_1 \in [N-2, N]$ are the same.

Assuming $[x_k]$ is a $[k_1, k_2]$-CM_c sequence, then $[x_k]_{k_1}^{k_2}$ is a CM_c sequence.

Different values of k_1, k_2, and c define different classes of CM sequences. For example, CM_F and $[1, N]$-CM_L are two classes. By $CM_F \cap [1, N]$-CM_L we mean a sequence being both CM_F and $[1, N]$-CM_L. We use similar notations for intersections of other classes.

2.1.2 Preliminaries (for Gaussian CM Sequences)

In this subsection, some results are presented, to be used in proofs in later sections. The goal is to find simple necessary and sufficient conditions for Gaussian sequences to be CM.

Lemma 2.1.4. *$[x_k]$ is $[k_1, k_2]$-$CM_c, c \in \{k_1, k_2\}$, iff for every Borel measurable function f,*

$$E[f(x_k)|[x_i]_{k_1}^{j}, x_c] = E[f(x_k)|x_j, x_c] \tag{2.2}$$

for every $j, k \in [k_1, k_2], j < k$, or equivalently,

$$E[f(x_k)|[x_i]_{j}^{k_2}, x_c] = E[f(x_k)|x_j, x_c] \tag{2.3}$$

for every $k, j \in [k_1, k_2], k < j$.

Proof. We prove (2.2) first. It can be seen that Definition 2.1.1 of the $[k_1, k_2]$-CM_c sequence and (2.4) below are equivalent; that is, $[x_k]$ is $[k_1, k_2]$-CM_c iff

$$P\{A|[x_i]_{k_1}^{j}, x_c\} = P\{A|x_j, x_c] \tag{2.4}$$

for every $j \in [k_1, k_2 - 1]$, where $A \in \sigma([x_k]_{j+1}^{k_2} \setminus \{x_c\})$ [85], [6]. Also, (2.4) holds iff

$$E[h([x_i]_{j+1}^{k_2} \setminus \{x_c\})|[x_i]_{k_1}^{j}, x_c] = E[h([x_i]_{j+1}^{k_2} \setminus \{x_c\})|x_j, x_c] \tag{2.5}$$

for every $j \in [k_1, k_2 - 1]$ and every Borel measurable function h. Clearly (2.2) follows from (2.5). So, we need to show that if (2.2) holds, so does (2.5). We will do it by mathematical induction on j. For $j = k_2 - 1$, (2.5) follows from (2.2). Fix $l \in [k_1, k_2 - 2]$. Assume that (2.5) holds for $j = l + 1$, that is,

$$E[g([x_i]_{l+2}^{k_2} \setminus \{x_c\})|[x_i]_{k_1}^{l+1}, x_c] = E[g([x_i]_{l+2}^{k_2} \setminus \{x_c\})|x_{l+1}, x_c] \tag{2.6}$$

for every Borel measurable function g. Then, we prove that it holds for $j = l$,

$$\begin{aligned}
E[h([x_i]_{l+1}^{k_2} \setminus \{x_c\})|[x_i]_{k_1}^{l}, x_c] &= E\Big[E[h([x_i]_{l+1}^{k_2} \setminus \{x_c\})|[x_i]_{k_1}^{l}, x_{l+1}, x_c]\Big|[x_i]_{k_1}^{l}, x_c\Big] \\
&= E\Big[E[h([x_i]_{l+1}^{k_2} \setminus \{x_c\})|x_l, x_{l+1}, x_c]\Big|[x_i]_{k_1}^{l}, x_c\Big] \\
&= E\Big[E[h([x_i]_{l+1}^{k_2} \setminus \{x_c\})|x_l, x_{l+1}, x_c]\Big|x_l, x_c\Big] \\
&= E[h([x_i]_{l+1}^{k_2} \setminus \{x_c\})|x_l, x_c]
\end{aligned} \tag{2.7}$$

for every Borel measurable function h. Note that the second equality follows from (2.6), and the third equality is due to (2.2) (note that $E[h([x_i]_{l+1}^{k_2} \setminus \{x_c\})|x_l, x_{l+1}, x_c]$ is a function of x_l, x_{l+1}, and x_c). By mathematical induction, (2.5) is concluded. (Note that the required integrability condition for nested expectations [86] holds.)

The following has been used in the third equality of (2.7). By Corollary 2.1.5 below, (2.8) below follows from (2.2)

$$F(\xi_{l+1}|[x_i]_{k_1}^{l}, x_c) = F(\xi_{l+1}|x_l, x_c), \forall \xi_{l+1} \in \mathbb{R}^d \tag{2.8}$$

where $F(\cdot|\cdot)$ is the conditional CDF of x_{l+1} and d is the dimension of x_{l+1}. Then, (2.9) below follows from (2.8):

$$F(\xi_l, \xi_{l+1}, \xi_c|[x_i]_{k_1}^l, x_c) = F(\xi_l, \xi_{l+1}, \xi_c|x_l, x_c) \tag{2.9}$$

where $F(\cdot|\cdot)$ is the conditional CDF of x_l, x_{l+1}, and x_c. On the other hand, let $g_1(x_l, x_{l+1}, x_c) = E[h([x_i]_{l+1}^{k_2} \setminus \{x_c\})|x_l, x_{l+1}, x_c]$. By the definition of conditional expectation, g_1 is a Borel measurable function. Based on (2.9), it is concluded that (see the proof of Corollary 2.1.5 for more details)

$$E[g_1(x_l, x_{l+1}, x_c)|[x_i]_{k_1}^l, x_c) = E[g_1(x_l, x_{l+1}, x_c)|x_l, x_c) \tag{2.10}$$

which has been used in the third equality of (2.7). A similar fact has been used in the second equality of (2.7), too.

A proof of (2.3) is similar. To prove sufficiency, it suffices to show that (2.11) follows from (2.3) by mathematical induction. (Observe that for $j = k_1 + 1$, (2.11) follows from (2.3). Then, assume (2.11) holds for $j = l - 1$ (fix $l \in [k_1 + 2, k_2]$) and prove it for $j = l$.)

$$E[h([x_i]_{k_1}^{j-1} \setminus \{x_c\})|[x_i]_j^{k_2}, x_c] = E[h([x_i]_{k_1}^{j-1} \setminus \{x_c\})|x_j, x_c] \tag{2.11}$$

□

Corollary 2.1.5. *$[x_k]$ is $[k_1, k_2]$-$CM_c, c \in \{k_1, k_2\}$, iff its CDF satisfies*

$$F(\xi_k|[x_i]_{k_1}^j, x_c) = F(\xi_k|x_j, x_c), \forall \xi_k \in \mathbb{R}^d \tag{2.12}$$

for every $j, k \in [k_1, k_2], j < k$, or equivalently,

$$F(\xi_k|[x_i]_j^{k_2}, x_c) = F(\xi_k|x_j, x_c), \forall \xi_k \in \mathbb{R}^d \tag{2.13}$$

for every $k, j \in [k_1, k_2], k < j$, where d is the dimension of x_k.

Proof. It is enough to show that (2.2) is equivalent to (2.12) and (2.3) is equivalent to (2.13). We briefly address the former and skip the latter, since they are similar.

Assume (2.2) holds. Then, let $f(x_k) = 1_A(x_k)$ ($1_A(x_k) = 1$ for $x_k \in A$, and $1_A(x_k) = 0$ for $x_k \notin A$), where $A = \{x_k^1 \leq \xi_k^1\} \times \{x_k^2 \leq \xi_k^2\} \times \cdots \times \{x_k^d \leq \xi_k^d\}$, $x_k = [x_k^1, x_k^2, \ldots, x_k^d]'$, and $\xi_k = [\xi_k^1, \xi_k^2, \ldots, \xi_k^d]'$. Then, the RHS (LHS) of (2.2) is equal to the RHS (LHS) of (2.12).

Assume (2.12) holds. Then, $P\{B|[x_i]_{k_1}^j, x_c\} = P\{B|x_j, x_c\}$ for every $B \in \sigma(x_k)$ [87], and (2.2) is concluded. □

Remark 2.1.6. *Due to simplicity we recommend considering Corollary 2.1.5 as the definition of $[k_1, k_2]$-CM_c sequences in application.*

For Gaussian sequences, Lemma 2.1.4 is equivalent to the following.

Lemma 2.1.7. *A Gaussian sequence $[x_k]$ is $[k_1, k_2]$-$CM_c, c \in \{k_1, k_2\}$, iff*

$$E[x_k|[x_i]_{k_1}^j, x_c] = E[x_k|x_j, x_c] \tag{2.14}$$

for every $j, k \in [k_1, k_2], j < k$, or equivalently,

$$E[x_k|[x_i]_j^{k_2}, x_c] = E[x_k|x_j, x_c] \tag{2.15}$$

for every $j, k \in [k_1, k_2], k < j$.

Proof. We prove (2.14). Proof of (2.15) is similar and is skipped. Necessity: By Lemma 2.1.4, for a $[k_1, k_2]$-CM_c sequence $[x_k]$, (2.14) holds.

Sufficiency: Let $[x_k]$ be a Gaussian sequence for which (2.14) holds. The conditional covariance can be calculated as

$$\text{Cov}(x_k|[x_i]_{k_1}^j, x_c) = E\Big[\Big(x_k - E[x_k|[x_i]_{k_1}^j, x_c]\Big)\Big(\cdot\Big)'\Big|[x_i]_{k_1}^j, x_c\Big]$$

On the other hand, for conditional expectation we have $E[(x_k - E[x_k|[x_i]_{k_1}^j, x_c])g([x_i]_{k_1}^j, x_c)] = 0$ for every Borel measurable function g. Thus, $x_k - E[x_k|[x_i]_{k_1}^j, x_c]$ is orthogonal to (and due to Gaussianity independent of) $[x_i]_{k_1}^j$ and x_c. Therefore, noting (2.14), we have

$$\begin{aligned}\text{Cov}(x_k|[x_i]_{k_1}^j, x_c) &= E\Big[\Big(x_k - E[x_k|x_j, x_c]\Big)\Big(\cdot\Big)'\Big] \\ &= E\Big[\Big(x_k - E[x_k|x_j, x_c]\Big)\Big(\cdot\Big)'\Big|x_j, x_c\Big] = \text{Cov}(x_k|x_j, x_c) \end{aligned} \tag{2.16}$$

Due to Gaussianity, (2.14) and (2.16) lead to the equality of the corresponding conditional distributions. In other words, a Gaussian conditional distribution is completely determined by its conditional expectation [57]. Therefore, (2.2) holds and the sequence $[x_k]$ is $[k_1, k_2]$-CM_c. □

2.2 Dynamic Models of CM_c Sequences

The CM_c sequence is an important class of CM sequences. For example, a CM_L sequence can be used for motion trajectory modeling with destination information (Chapter 7). In addition, CM_L and CM_F sequences play a very important role in the study of the reciprocal sequence from the CM viewpoint.

A dynamic model for the zero-mean nonsingular Gaussian (ZMNG) reciprocal sequence was presented in [18]. Inspired by [18], we first present a model for evolution of the ZMNG CM_c sequence, called a CM_c model. Then, we discuss a model of the nonsingular Gaussian (NG) CM_c sequence. The following lemma demonstrates construction of a CM_c model for the ZMNG CM_c sequence.

Lemma 2.2.1. *Let $[x_k]$ be a ZMNG CM_c sequence with covariance function C_{l_1,l_2}. Then, its evolution obeys*

$$x_k = G_{k,k-1}x_{k-1} + G_{k,c}x_c + e_k, \quad k \in [1, N] \setminus \{c\} \tag{2.17}$$

where $[e_k]$ ($G_k = Cov(e_k)$) is a zero-mean white NG sequence, and boundary condition[1]

$$x_0 = e_0, \quad x_c = G_{c,0}x_0 + e_c \ (\textit{for } c = N) \tag{2.18}$$

or equivalently[2]

$$x_c = e_c, \quad x_0 = G_{0,c}x_c + e_0 \ (\textit{for } c = N) \tag{2.19}$$

Proof. We prove the lemma in three steps: (i) model construction, (ii) boundary conditions and the whiteness of $[e_k]$, and (iii) nonsingularity of covariance matrices $G_k, k \in [0, N]$.

(i) Model construction: Since $[x_k]$ is CM_c, by Lemma 2.1.7 for every $k \in [1, N] \setminus \{c\}$ we have

$$E[x_k|[x_i]_0^{k-1}, x_c] = E[x_k|x_{k-1}, x_c] \tag{2.20}$$

[1] Note that (2.18) means that for $c = N$ we have $x_0 = e_0$ and $x_N = G_{N,0}x_0 + e_N$. Also, for $c = 0$ we have $x_0 = e_0$. It is similar for (2.19).

[2] It should be clear that e_0 and e_N in (2.18) and in (2.19) are not necessarily the same. Just for simplicity we use the same notation.

Since $[x_k]$ is Gaussian, for $c=0$ and $k=1$ we have $E[x_k|x_{k-1},x_c] = C_{1,0}C_0^{-1}x_0$. Let $G_{1,0} \triangleq \frac{1}{2}C_{1,0}C_0^{-1}$. For other c and k values (i.e., $c=0$ and $k \in [2,N]$, and $c=N$ and $k \in [1,N-1]$),

$$E[x_k|x_{k-1},x_c] = [C_{k,k-1} \quad C_{k,c}] \begin{bmatrix} C_{k-1} & C_{k-1,c} \\ C_{c,k-1} & C_c \end{bmatrix}^{-1} \begin{bmatrix} x_{k-1} \\ x_c \end{bmatrix}$$

Let $[G_{k,k-1} \quad G_{k,c}] \triangleq [C_{k,k-1} \quad C_{k,c}] \begin{bmatrix} C_{k-1} & C_{k-1,c} \\ C_{c,k-1} & C_c \end{bmatrix}^{-1}$. So, for every $k \in [1,N] \setminus \{c\}$ and $c \in \{0,N\}$, we have $E[x_k|x_{k-1},x_c] = G_{k,k-1}x_{k-1} + G_{k,c}x_c$. Define e_k, $k \in [1,N] \setminus \{c\}$, as

$$\begin{aligned} e_k &= x_k - E[x_k|x_{k-1},x_c] \\ &= x_k - G_{k,k-1}x_{k-1} - G_{k,c}x_c \end{aligned} \tag{2.21}$$

Then, for $c=0$ and $k=1$, $G_1 \triangleq \text{Cov}(e_1) = C_1 - C_{1,0} \cdot C_0^{-1}C'_{1,0}$. For other c and k values,

$$G_k \triangleq \text{Cov}(e_k) = C_k - [C_{k,k-1} \quad C_{k,c}] \begin{bmatrix} C_{k-1} & C_{k-1,c} \\ C_{c,k-1} & C_c \end{bmatrix}^{-1} [C_{k,k-1} \quad C_{k,c}]'$$

$[e_k]_{[1,N]\setminus\{c\}}$ is a zero-mean white Gaussian sequence uncorrelated with x_0 and x_c. It can be verified as follows. By the definition of conditional expectation and based on (2.20) we have

$$E[(x_k - E[x_k|x_{k-1},x_c])g([x_j]_0^{k-1},x_c)] = E[(x_k - E[x_k|[x_i]_0^{k-1},x_c])g([x_j]_0^{k-1},x_c)] = 0 \tag{2.22}$$

for every Borel measurable function g. Thus, by (2.21) and (2.22), e_k is uncorrelated with $[x_i]_0^{k-1}$ and x_c. Then, for $k \geq j$,

$$E[e_k e_j'] = E[e_k(x_j - G_{j,j-1}x_{j-1} - G_{j,c}x_c)'] = \begin{cases} G_k & k=j \\ 0 & \text{otherwise} \end{cases} \tag{2.23}$$

Likewise for $j \geq k$. Therefore, $E[e_k e_k'] = G_k$ and $E[e_k e_j'] = 0, k \neq j$. So, $[e_k]_{[1,N]\setminus\{c\}}$ is white.

(ii) Boundary conditions: For $c=0$, we have $G_0 \triangleq C_0$. Let $c=N$ and consider (2.18). Since x_0 and x_N are jointly Gaussian, we have $E[x_N|x_0] = G_{N,0}x_0$, where $G_{N,0} = C_{N,0}C_0^{-1}$. Then, we define $e_N \triangleq x_N - G_{N,0}x_0$, where e_N is a ZMNG vector with covariance $G_N = C_N - C_{N,0}C_0^{-1}C'_{N,0}$. Also, by the definition of conditional expectation, e_N is uncorrelated with x_0 (i.e., $E[(x_N - E[x_N|x_0])g(x_0)] = 0$ for every Borel measurable function g). Also, for notational unification $e_0 \triangleq x_0$ with covariance $G_0 \triangleq C_0$.

Similarly for $c=N$ and (2.19), we have $x_0 = G_{0,N}\ x_N + e_0$, $G_{0,N} = C_{0,N}C_N^{-1}$, and $G_0 = C_0 - C_{0,N}C_N^{-1}C'_{0,N}$, where e_0 is a ZMNG vector with covariance G_0, uncorrelated with x_N. Also, set $e_N \triangleq x_N$ with covariance $G_N \triangleq C_N$.

By (2.22), $[e_k]_{[1,N]\setminus\{c\}}$ is uncorrelated with x_0 and x_c, and thus uncorrelated with e_0 and e_c. So, $[e_k]$ is white.

(iii) From (2.30) in the proof of Lemma 2.2.5 below, nonsingularity of the covariance matrices G_k, $k \in [0,N]$, follows from nonsingularity of the covariance matrix of $[x_k]$. □

Lemma 2.2.2. *For $c=N$, the boundary conditions* (2.18) *and* (2.19) *can be obtained from each other.*

Proof. For clarity, denote (2.18) and (2.19) as

$$x_0 = e_0^1, \quad x_N = G_{N,0}x_0 + e_N^1 \tag{2.24}$$

$$x_N = e_N, \quad x_0 = G_{0,N}x_N + e_0 \tag{2.25}$$

where $G_0^1 = E[e_0^1(e_0^1)']$ and $G_N^1 = E[e_N^1(e_N^1)']$. Now, we obtain (2.25) from (2.24). We will have (2.25) if e_0 and e_N are chosen such that $e_0^1 = G_{0,N}x_N + e_0$ and $e_N = G_{N,0}x_0 + e_N^1$, where it can be easily seen that e_0 and e_N are uncorrelated with $[e_k]_1^{N-1}$, because e_0^1, e_N^1, x_0, and x_N are uncorrelated with $[e_k]_1^{N-1}$. Also,

$$\begin{aligned}E[e_0e_N'] =&E[(e_0^1 - G_{0,N}x_N)(e_N^1 + G_{N,0}x_0)'] \\ =&E[e_0^1(e_N^1)'] + E[e_0^1x_0']G_{N,0}' - G_{0,N}E[x_N(e_N^1)'] - G_{0,N}E[x_Nx_0']G_{N,0}' \\ =&E[e_0^1(e_0^1)']G_{N,0}' - G_{0,N}E[(G_{N,0}e_0^1 + e_N^1)(e_N^1)'] - G_{0,N}E[(G_{N,0}e_0^1 + e_N^1)(e_0^1)']G_{N,0}' \\ =&G_0^1G_{N,0}' - G_{0,N}G_N^1 - G_{0,N}G_{N,0}G_0^1G_{N,0}' \\ =&C_0(C_{N,0}C_0^{-1})' - C_{0,N}C_N^{-1}(C_N - C_{N,0}C_0^{-1}C_{0,N}) - C_{0,N}C_N^{-1}C_{N,0}C_0^{-1}C_{0,N} = 0\end{aligned}$$

which means e_0 and e_N are uncorrelated. Similarly, one can obtain (2.24) from (2.25). □

So, for $c = N$, (2.18) and (2.19) have different forms, but are equivalent. Therefore, for brevity later we may refer to only one of them, although similar results hold for the other.

Remark 2.2.3. *Boundary condition* (2.19) *emphasizes the role and importance of x_N in a CM_L model (i.e., the evolution law from $k = 0$ to $k = N - 1$ depends on x_N).*

It is important that a dynamic model gives a unique covariance function of the corresponding sequence [18]. As the following lemma shows, this is the case for model (2.17).

Lemma 2.2.4. *Model* (2.17) *along with* (2.18) *or* (2.19) *for every parameter value admits a unique covariance function.*

Proof. Let $[x_k]$ obey (2.17) along with (2.18) or (2.19) with $c = N$. That is,

$$\mathcal{G}x = e, \quad e \triangleq [e_0', \dots, e_N']' \tag{2.26}$$

where $\mathcal{G}$ will be given below. Post-multiplying both sides by x' and taking expectation, we have $\mathcal{G}C = U$, where $C = \text{Cov}(x)$ and $U = \text{Cov}(e, x)$.

To show the uniqueness of the covariance function, it suffices to show that $\mathcal{G}$ is nonsingular. Consider (2.18) for which $\mathcal{G}$ is

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -G_{1,0} & I & 0 & \cdots & 0 & -G_{1,N} \\ 0 & -G_{2,1} & I & 0 & \cdots & -G_{2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -G_{N-1,N-2} & I & -G_{N-1,N} \\ -G_{N,0} & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{2.27}$$

The determinant of a partitioned matrix $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is $|A| = |A_{11}| \cdot |A_{22}|$ if $A_{12} = 0$ or $A_{21} = 0$ [88]. So, it can be seen that $|\mathcal{G}| \neq 0$ for every choice of the parameters, as follows. Since $\mathcal{G}_{[1:1,2:N+1]} = 0$, we have $|\mathcal{G}| = |\mathcal{G}_{[2:N+1,2:N+1]}|$. For a similar reason (i.e. $\mathcal{G}_{[N+1:N+1,2:N]} = 0$), we have $|\mathcal{G}_{[2:N+1,2:N+1]}| = |\mathcal{G}_{[2:N,2:N]}|$, where it is clear that $|\mathcal{G}_{[2:N,2:N]}| = 1$. Therefore, model (2.17)–(2.18) always admits a unique covariance function.

Since (2.18) and (2.19) are equivalent (Lemma 2.2.2), model (2.17) with (2.19) always admits a unique covariance function, too. It can be also verified based on the nonsingularity of $\mathcal{G}$ corresponding to (2.19), which is

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & -G_{0,N} \\ -G_{1,0} & I & 0 & \cdots & 0 & -G_{1,N} \\ 0 & -G_{2,0} & I & 0 & \cdots & -G_{2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -G_{N-1,N-2} & I & -G_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{2.28}$$

Let $[x_k]$ obey (2.17)–(2.18) with $c = 0$. That is, $\mathcal{G}x = e, e \triangleq [e'_0, \ldots, e'_N]'$, where $\mathcal{G}$ is

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -2G_{1,0} & I & 0 & \cdots & 0 & 0 \\ -G_{2,0} & -G_{2,1} & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -G_{N-1,0} & 0 & \cdots & -G_{N-1,N-2} & I & 0 \\ -G_{N,0} & 0 & 0 & \cdots & -G_{N,N-1} & I \end{bmatrix} \tag{2.29}$$

Since (2.29) is nonsingular, (2.17)–(2.18) always admits a unique covariance function. □

Lemma 2.2.5. $[x_k]$ *governed by* (2.17)–(2.18) *is always nonsingular (for every parameter value).*

Proof. Let $[x_k]$ obey (2.17)–(2.18), where the covariance matrices G_k, $k \in [0, N]$, are nonsingular. Based on (2.26), we have $\mathcal{G}x = e$, where $\mathcal{G}$ is given by (2.27) for $c = N$ and by (2.29) for $c = 0$. Then, the covariance matrix of $[x_k]$ can be obtained as

$$C = \mathcal{G}^{-1} G (\mathcal{G}')^{-1} \tag{2.30}$$

where $G = \text{Cov}(e) = \text{diag}(G_0, \ldots, G_N)$ and by the proof of Lemma 2.2.4, $\mathcal{G}$ is nonsingular. Since all G_k, $k \in [0, N]$, are nonsingular, G is nonsingular. Therefore, by (2.30), $[x_k]$ is nonsingular. □

By the previous lemmas, a model for the ZMNG CM_c sequence was constructed and some related properties were studied. Now, we can present the main result for the CM_c model as follows.

Theorem 2.2.6. *A ZMNG sequence* $[x_k]$ *with covariance function* C_{l_1,l_2} *is* CM_c *iff it obeys* (2.17) *along with* (2.18) *or* (2.19).

Proof. The necessity was proved as Lemma 2.2.1. So, we just need to prove the sufficiency. This amounts to proving that $[x_k]$ is (i) nonsingular and (ii) Gaussian CM_c. Lemma 2.2.5 has established (i). So, we just need to prove (ii).

Since $[x_k]$ is Gaussian, by Lemma 2.1.7, $[x_k]$ is CM_c if $E[x_k|[x_i]_0^j, x_c] = E[x_k|x_j, x_c]$, for every $j, k \in [0, N] \setminus \{c\}, j < k$. From (2.17) we have $x_k = G_{k,j}x_j + G_{k,c|j}x_c + e_{k|j}$, where the matrices $G_{k,j}$ and $G_{k,c|j}$ can be obtained from parameters of (2.17), and $e_{k|j}$ is a linear combination of $[e_l]_{j+1}^k$. Since $[e_k]$ is white, $[e_l]_{j+1}^k$ (and so $e_{k|j}$) is uncorrelated with $[x_k]_0^j$ and x_c. Thus, we have $E[x_k|[x_i]_0^j, x_c] = E[x_k|x_j, x_c]$, meaning that $[x_k]$ is CM_c. □

Let $z \sim \mathcal{N}(\mu_z, C_z)$ and $y \sim \mathcal{N}(\mu_y, C_y)$ be jointly Gaussian random vectors with cross-covariance $C_{z,y}$. Also, let $\check{z}$ and $\check{y}$ be zero-mean parts of z and y, respectively. We have $E[z|y] = \mu_z + C_{z,y}C_y^+(y - \mu_y)$, where '+' denotes the Moore-Penrose inverse. On the other hand, $E[\check{z}|\check{y}] = C_{z,y}C_y^+(\check{y})$. So, $E[z|y] - \mu_z = E[\check{z}|\check{y}]$. Then, by Lemma 2.1.7, a Gaussian sequence is CM_c iff its zero-mean part is CM_c. Therefore, a Gaussian sequence $[x_k]$ with mean function μ_k, $k \in [0, N]$, is CM_c iff its zero-mean part $[x_k - \mu_k]$ obeys (2.17)–(2.18). Thus, we only present models of zero-mean CM sequences.

Backward Markov/hybrid models have been developed and used, e.g., for smoothing [65]–[71], [89]. The evolution of the CM_c sequence can also be modeled by a backward CM_c model. A backward CM_c model may provide more insight and tools regarding the CM_c sequence. Also, it is useful for smoothing. The next proposition presents a backward CM_c model.

Proposition 2.2.7. *A ZMNG* $[x_k]$ *is* CM_c *iff*

$$x_k = G^B_{k,k+1}x_{k+1} + G^B_{k,c}x_c + e^B_k, k \in [0, N-1] \setminus \{c\} \tag{2.31}$$

$$x_c = e^B_c, \quad x_N = G^B_{N,c}x_c + e^B_c \ (\textit{for } c = 0) \tag{2.32}$$

and $[e_k^B]$ $(G_k^B = Cov(e_k^B))$ *is a zero-mean white NG sequence.*

Proof. A proof is paraller to that of the CM_c model (Theorem 2.2.6). The only difference is in time order. □

Similar to Theorem 2.2.6, we have a different form of the boundary condition equivalent to (2.32):

$$x_N = e_N^B, \quad x_c = G_{c,N}^B x_N + e_c^B \text{ (for } c = 0) \tag{2.33}$$

Similar to (2.30), the covariance matrix of $[x_k]$ can be obtained as

$$C = (\mathcal{G}^B)^{-1} G^B [(\mathcal{G}^B)']^{-1} \tag{2.34}$$

where $G^B = \text{diag}(G_0^B, \cdots, G_N^B)$ and $\mathcal{G}^B$ for $c = N$ is

$$\mathcal{G}^B = \begin{bmatrix} I & -G_{0,1}^B & 0 & \cdots & 0 & -G_{0,N}^B \\ 0 & I & -G_{1,2}^B & \cdots & 0 & -G_{1,N}^B \\ 0 & 0 & I & -G_{2,3}^B & \cdots & -G_{2,N}^B \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I & -2G_{N-1,N}^B \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{2.35}$$

(2.35) will be used in the next sections.

2.3 Characterization of CM_c Sequences

Definition 2.3.1. *A symmetric positive definite matrix is called* CM_L *if it has form* (2.36) *and* CM_F *if it has form* (2.37).

$$\begin{bmatrix} A_0 & B_0 & 0 & \cdots & 0 & 0 & D_0 \\ B_0' & A_1 & B_1 & 0 & \cdots & 0 & D_1 \\ 0 & B_1' & A_2 & B_2 & \cdots & 0 & D_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & B_{N-3}' & A_{N-2} & B_{N-2} & D_{N-2} \\ 0 & \cdots & 0 & 0 & B_{N-2}' & A_{N-1} & B_{N-1} \\ D_0' & D_1' & D_2' & \cdots & D_{N-2}' & B_{N-1}' & A_N \end{bmatrix} \tag{2.36}$$

$$\begin{bmatrix} A_0 & B_0 & D_2 & \cdots & D_{N-2} & D_{N-1} & D_N \\ B_0' & A_1 & B_1 & 0 & \cdots & 0 & 0 \\ D_2' & B_1' & A_2 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_{N-2}' & \cdots & 0 & B_{N-3}' & A_{N-2} & B_{N-2} & 0 \\ D_{N-1}' & \cdots & 0 & 0 & B_{N-2}' & A_{N-1} & B_{N-1} \\ D_N' & 0 & 0 & \cdots & 0 & B_{N-1}' & A_N \end{bmatrix} \tag{2.37}$$

Here A_k, B_k, and D_k are matrices in general.

Remark 2.3.2. *We use* CM_c *to mean both* CM_L *and* CM_F *matrices: A* CM_c *matrix for* $c = N$ *is* CM_L *and for* $c = 0$ *is* CM_F.

Remark 2.3.3. *A* CM_c *sequence is one defined in Subsection 2.1.1, but a* CM_c *matrix is one defined by Definition 2.3.1.*

First, several new factorizations of CM_c matrices are presented in the following lemma. Then, based on the lemma, characterizations of CM_c sequences are obtained.

Lemma 2.3.4. *A CM_c matrix A with $d \times d$ blocks can be uniquely factorized as $A = V'DV$, where D is block diagonal with $d \times d$ blocks, and V is a block matrix (with $d \times d$ blocks) with the same dimension as A: (i) for CM_L, V is in the form of* (2.38), (2.39), *or* (2.40)*; (ii) for CM_F, V is in the form of* (2.41), (2.42), *or* (2.43).

$$\begin{bmatrix} I & * & 0 & \cdots & 0 & * \\ 0 & I & * & \cdots & 0 & * \\ 0 & 0 & I & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I & * \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{2.38}$$

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ * & I & 0 & \cdots & 0 & * \\ 0 & * & I & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & * & I & * \\ * & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{2.39}$$

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & * \\ * & I & 0 & \cdots & 0 & * \\ 0 & * & I & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & * & I & * \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{2.40}$$

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ * & I & 0 & \cdots & 0 & 0 \\ * & * & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & 0 & \cdots & * & I & 0 \\ * & 0 & 0 & \cdots & * & I \end{bmatrix} \tag{2.41}$$

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & * \\ * & I & * & \cdots & 0 & 0 \\ * & 0 & I & * & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ * & 0 & \cdots & 0 & I & * \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{2.42}$$

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ * & I & * & \cdots & 0 & 0 \\ * & 0 & I & * & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ * & 0 & \cdots & 0 & I & * \\ * & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{2.43}$$

where $*$ is not necessarily zero.

Proof. See Appendix A. □

It is known that a positive definite matrix has a unique triangular factorization [88]. Corollary 2.3.4 shows that the unique triangular factorization of a CM_L matrix has a special form as (2.38). In addition, it shows that a CM_L matrix has non-triangular factorizations of forms (2.39) and (2.40), which are unique. Moreover, given a CM_L matrix, the proof of Corollary 2.3.4 shows how the matrices V and D of the factorizations can be easily calculated. The same is true for a CM_F matrix.

Theorem 2.3.5. *A NG sequence with covariance matrix C is CM_c iff C^{-1} is CM_c.*

Proof. We can prove the theorem based on results about the relationship between the conditional independence of some Gaussian variables and their covariance matrix (e.g., see [60]). However, here we present a proof based on the CM_c dynamic model and a factorization presented in Lemma 2.3.4. It suffices to consider the ZMNG sequence. Necessity: Consider $c = N$. Let $[x_k]$ be a ZMNG CM_L sequence. By Lemma 2.2.1, $[x_k]$ obeys (2.17)–(2.18). From (2.30), we have

$$C^{-1} = \mathcal{G}' G^{-1} \mathcal{G} \tag{2.44}$$

where $\mathcal{G}$ is given by (2.27) and $G = \text{diag}(G_0, \ldots, G_N)$. Substituting $\mathcal{G}$ in (2.27) into (2.44) leads to a C^{-1} that is CM_L. The same proof works for $c = 0$ (i.e., CM_F).

Sufficiency: We need to show that for every CM_c matrix C^{-1}, there exists a Gaussian CM_c sequence with covariance matrix C. This has been shown in the proof of Lemma 2.3.4 based on the CM_c matrix factorization. □

Markov and reciprocal sequences are special CM_c sequences (Chapter 3). That is why characterizations of NG Markov [56] and NG reciprocal sequences [18] are special cases of those of NG CM_c sequences.

Remark 2.3.6. *Given a CM_c matrix C^{-1}, parameters of the forward and backward CM_c models of a ZMNG CM_c sequence with covariance matrix C can be directly (and uniquely) determined in terms of the entries of C^{-1}.*

Remark 2.3.6 is verified in Lemma B.1 and Lemma B.2 (Appendix B). The uniqueness is clear either by Lemma B.1 and Lemma B.2 or the definition (uniqueness) of conditional expectation. Parameters of a CM_c model can be calculated based on the covariance function of the sequence. However, Remark 2.3.6 says that the parameters can be *directly* determined in terms of the entries of C^{-1} without calculating C. This is particularly useful for determination of parameters of a backward (forward) CM_c model in terms of those of a forward (backward) CM_c model for the same sequence (by equating C^{-1} calculated from the two models). In addition, it is useful for determination of Markov sequences governed by the same CM_c model. This is related to an important question in the theory of reciprocal processes regarding determination of Markov processes governed by the same reciprocal evolution law (Chapter 6).

2.4 Dynamic Models of $[k_1, k_2]$-CM_c Sequences

$[k_1, k_2]$-CM_c sequences are important for the study of the reciprocal sequence (Chapter 3). Also, an application of $[0, k_2]$-CM_L sequences is in trajectory modeling with waypoint or destination information (Chapter 4). Characterizations of NG $[k_1, k_2]$-CM_c sequences are obtained in Chapter 3. A dynamic model of $[0, k_2]$-CM_c sequences is given next.

Proposition 2.4.1. *A ZMNG $[x_k]$ with covariance function C_{l_1,l_2} is $[0, k_2]$-CM_c ($k_2 \in [1, N-1]$) iff*

$$x_k = G_{k,k-1} x_{k-1} + G_{k,c} x_c + e_k, \quad k \in [1, k_2] \setminus \{c\} \tag{2.45}$$

$$x_c = e_c, \quad x_0 = G_{0,c} x_c + e_0 \ (\textit{for } c = k_2) \tag{2.46}$$

$$x_k = \sum_{i=0}^{k-1} G_{k,i} x_i + e_k, \quad k \in [k_2+1, N] \tag{2.47}$$

and $[e_k]$ *(*$G_k = Cov(e_k)$*) is a zero-mean white NG sequence.*

Proof. Necessity: By the definition of the $[0, k_2]$-CM_c sequence, $[x_k]_0^{k_2}$ is CM_c. Therefore, by Theorem 2.2.6, $[x_k]_0^{k_2}$ obeys (2.45)–(2.46). Also, its parameters can be calculated from the covariance function (see the proof of Theorem 2.2.6). For the evolution over $[k_2+1, N]$ we have $E[x_k|[x_i]_0^{k-1}] = \sum_{i=0}^{k-1} G_{k,i} x_i, k \in [k_2+1, N]$, where $\begin{bmatrix} G_{k,0} & \cdots & G_{k,k-1} \end{bmatrix} = C_{[k+1:k+1,1:k]} C_{[1:k,1:k]}^{-1}$. We define $e_k = x_k - \sum_{i=0}^{k-1} G_{k,i} x_i, k \in [k_2+1, N]$, where $G_k \triangleq \text{Cov}(e_k) = C_k - C_{[k+1:k+1,1:k]} C_{[1:k,1:k]}^{-1} \cdot C'_{[k+1:k+1,1:k]}$. By the definition of conditional expectation we have $E[(x_k - E[x_k|[x_i]_0^{k-1}])g([x_j]_0^{k-1})] = 0$, $k \in [k_2+1, N]$, for every Borel measurable function g. Therefore, $[e_k]_{k_2+1}^N$ is a zero-mean white Gaussian sequence (see the proof of Lemma 2.2.1) with the nonsingular covariances G_k (nonsingularity of $G_k, k \in [0, N]$, follows from nonsingularity of $[x_k]$ and nonsingularity of T in (2.48) (see the proof of Lemma 2.2.1)). Also, $[e_k]_{k_2+1}^N$ is uncorrelated with $[e_k]_0^{k_2}$. So, $[e_k]$ is white.

Sufficiency: Consider $c = k_2$. Let $[x_k]$ obey (2.45), (2.46), and (2.47). Then, we can write

$$Tx = e \tag{2.48}$$

where $e \triangleq [e'_0, \ldots, e'_N]'$, $T = \begin{bmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{bmatrix}$,

$$T_{11} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & -G_{0,k_2} \\ -G_{1,0} & I & 0 & \cdots & 0 & -G_{1,k_2} \\ 0 & -G_{2,0} & I & 0 & \cdots & -G_{2,k_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -G_{k_2-1,k_2-2} & I & -G_{k_2-1,k_2} \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{2.49}$$

$$T_{21} = \begin{bmatrix} -G_{k_2+1,0} & -G_{k_2+1,1} & \cdots & -G_{k_2+1,k_2} \\ -G_{k_2+2,0} & -G_{k_2+2,1} & \cdots & -G_{k_2+2,k_2} \\ \vdots & \vdots & \cdots & \vdots \\ -G_{N,0} & -G_{N,1} & \cdots & -G_{N,k_2} \end{bmatrix} \tag{2.50}$$

and T_{22} is

$$\begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ -G_{k_2+2,k_2+1} & I & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ -G_{N-1,k_2+1} & \cdots & -G_{N-1,N-2} & I & 0 \\ -G_{N,k_2+1} & \cdots & -G_{N,N-2} & -G_{N,N-1} & I \end{bmatrix}$$

From (2.48) it is clear that $[x_k]_0^{k_2}$ is CM_L (see Theorem 2.2.6 and (2.28)).

Also, the covariances $G_k, k \in [0, N]$, and the matrix T are nonsingular. Thus, by (2.48) $[x_k]$ is nonsingular. Therefore, $[x_k]$ is a ZMNG $[0, k_2]$-CM_L sequence.

For $c = 0$ we have a similar proof with the following difference. For $c = k_2$, (2.49) is in the form of (2.40) (consider (2.46)). It can be seen that for $c = 0$, T_{11} is in the form of (2.41). So, $[x_k]_0^{k_2}$ is CM_F and $[x_k]$ is a ZMNG $[0, k_2]$-CM_F sequence.

Since T is always nonsingular, (2.45)–(2.47) admit a unique covariance function for every parameter value (see proof of Lemma 2.2.4). □

It is meaningful to compare the evolution model of the $[0, k_2]$-CM_c sequence over $[k_2 + 1, N]$ with a general Gaussian sequence. First, consider the following lemma.

Lemma 2.4.2. *$[x_k]$ is a ZMNG sequence with covariance function C_{l_1,l_2} iff it obeys*

$$x_k = \sum_{i=0}^{k-1} L_{k,i}x_i + d_k, k \in [1, N], \quad x_0 = d_0 \tag{2.51}$$

and $[d_k]$ ($L_k = Cov(d_k)$) is a zero-mean white NG sequence.

Comparing (2.47) and (2.51) indicates that given a sample path of the sequence over $[0, k_2]$, a ZMNG $[0, k_2]$-CM_c sequence has the same evolution over $[k_2 + 1, N]$ as that of a general ZMNG sequence.

Proofs of Proposition 2.4.3 and 2.4.4 are parallel to that of Proposition 2.4.1.

Proposition 2.4.3. *A ZMNG $[x_k]$ is $[k_1, N]$-CM_c ($k_1 \in [1, N-1]$) iff*

$$\begin{aligned}
x_k &= G^B_{k,k+1}x_{k+1} + G^B_{k,c}x_c + e^B_k, k \in [k_1, N-1] \setminus \{c\} \\
x_c &= e^B_c, \quad x_N = G^B_{N,c}x_c + e^B_N \ (\textit{for } c = k_1) \\
x_k &= \sum_{i=k+1}^{N} G^B_{k,i}x_i + e^B_k, \quad k \in [0, k_1 - 1]
\end{aligned}$$

and $[e^B_k]$ ($G^B_k = Cov(e^B_k)$) is a zero-mean white NG sequence.

Proposition 2.4.4. *A ZMNG $[x_k]$ is $[k_1, k_2]$-CM_c ($k_1, k_2 \in [1, N-1]$) iff*

$$\begin{aligned}
x_k &= G_{k,k-1}x_{k-1} + G_{k,c}x_c + e_k, k \in [k_1 + 1, k_2] \setminus \{c\} \\
x_c &= e_c, \quad x_{k_1} = G_{k_1,c}x_c + e_{k_1} \ (\textit{for } c = k_2) \\
x_k &= \sum_{i=k+1}^{N} G^B_{k,i}x_i + e_k, \quad k \in [0, k_1 - 1] \\
x_k &= \sum_{i=k_1}^{k-1} G_{k,i}x_i + e_k, \quad k \in [k_2 + 1, N]
\end{aligned}$$

and $[e_k]$ ($G_k = Cov(e_k)$) is a zero-mean white NG sequence.

Chapter 3

Reciprocal Sequences from the CM Viewpoint

In this chapter, we 1) propose studying the reciprocal sequence from the CM viewpoint and demonstrate its significance and fruitfulness, 2) study the NG reciprocal sequence from the CM viewpoint, and 3) obtain easy to apply results and tools for the NG reciprocal sequence.

3.1 Reciprocal Sequences

A sequence is reciprocal iff conditioned on the states at any two times j and l, the segment inside the interval (j, l) is independent of the segments outside $[j, l]$. A formal definition is as follows.

Definition 3.1.1. *$[x_k]$ is reciprocal if $\forall j, l \in [0, N]$, $j < l$,*

$$P\{AB|x_j, x_l\} = P\{A|x_j, x_l\}P\{B|x_j, x_l\} \tag{3.1}$$

where $A \in \sigma([x_k]_{j+1}^{l-1})$ and $B \in \sigma([x_k] \setminus [x_k]_j^l)$.

To provide results for application, later we present Corollary 3.1.7, which is equivalent to Definition 3.1.1.

A sequence is Markov iff conditioned on the state at any time j, the segment before j is independent of the segment after j. Formally, we have the following definition.

Definition 3.1.2. *$[x_k]$ is Markov if $\forall k_1 \in [0, N]$,*

$$P\{AB|x_j\} = P\{A|x_j)P(B|x_j\} \tag{3.2}$$

where $A \in \sigma([x_k]_0^{j-1})$ and $B \in \sigma([x_k]_{j+1}^N)$.

Lemma 3.1.3. *$[x_k]$ is Markov iff*

$$F(\xi_k|[x_i]_0^j) = F(\xi_k|x_j) \tag{3.3}$$

for every $j < k$, or equivalently,

$$F(\xi_k|[x_i]_j^N) = F(\xi_k|x_j) \tag{3.4}$$

for every $k < j$, where $\xi_k \in \mathbb{R}^d$ and d is the dimension of x_k.

Lemma 3.1.4. *A Gaussian $[x_k]$ is Markov iff*

$$E[x_k|[x_i]_0^j] = E[x_k|x_j] \tag{3.5}$$

for every $j < k$, or equivalently,

$$E[x_k|[x_i]_j^N] = E[x_k|x_j) \tag{3.6}$$

for every $k < j$.

Proofs of Lemmas 3.1.3 and 3.1.4 are similar to those of Corollary 2.1.5 and Lemma 2.1.7, respectively.

3.1.1 Reciprocal Characterization from CM Viewpoint

First, the relationship between the CM sequence and the reciprocal sequence is presented in Theorem 3.1.5 for the general Gaussian/non-Gaussian case. Then, according to this relationship, the reciprocal characterization of [18] is obtained based on the characterizations of CM sequences.

Theorem 3.1.5. $[x_k]$ *is reciprocal iff it is*

$$\text{(i) } [k_1, N]\text{-}CM_F,\ \forall k_1 \in [0, N], \text{ and } CM_L$$

or equivalently

$$\text{(ii) } [0, k_2]\text{-}CM_L,\ \forall k_2 \in [0, N], \text{ and } CM_F$$

Proof. Necessity: Let $[x_k]$ be reciprocal. Comparing Definition 2.1.1 and Definition 3.1.1, we can see that Definition 2.1.1 with $[k_1, c] = [0, k_2]$ and Definition 2.1.1 with $[c, k_2] = [k_1, N]$ are both special cases of Definition 3.1.1. Therefore, $[x_k]$ is both $[0, k_2]$-CM_L, $\forall k_2 \in [0, N]$ and $[k_1, N]$-CM_F, $\forall k_1 \in [0, N]$.

Sufficiency: We prove the sufficiency for (i). Proof of the sufficiency of (ii) is similar. It can be seen that (3.1) and (3.7) below are equivalent; that is, $[x_k]$ is reciprocal iff

$$P\{B|[x_i]_{k_1}^{k_2}] = P\{B|x_{k_1}, x_{k_2}] \tag{3.7}$$

for every $k_1, k_2 \in [0, N]$ $(k_1 < k_2)$, where $B \in \sigma([x_k] \setminus [x_k]_{k_1}^{k_2})$ [85]. On the other hand, (3.7) and (3.8) below are equivalent, meaning that $[x_k]$ is reciprocal iff

$$E[g([x_k]_0^{k_1-1})h([x_k]_{k_2+1}^N)|[x_i]_{k_1}^{k_2}] = E[g([x_k]_0^{k_1-1})h([x_k]_{k_2+1}^N)|x_{k_1}, x_{k_2}] \tag{3.8}$$

for every $k_1, k_2 \in [0, N]$ $(k_1 < k_2)$ and every Borel measurable function g and h.

Similarly, it can be seen that the definition of $[k_1, N]$-CM_F (Definition 2.1.1) and (3.9) below are equivalent; that is, $[x_k]$ is $[k_1, N]$-CM_F iff

$$P\{B|[x_i]_{k_1}^{k_2}\} = P\{B|x_{k_1}, x_{k_2}\} \tag{3.9}$$

for every $k_1, k_2 \in [0, N]$ $(k_1 < k_2)$, where $B \in \sigma([x_k]_{k_2+1}^N)$ [85]. Also, (3.9) and (3.10) below are equivalent, meaning that $[x_k]$ is $[k_1, N]$-CM_F iff

$$E[h([x_k]_{k_2+1}^N)|[x_i]_{k_1}^{k_2}] = E[h([x_k]_{k_2+1}^N)|x_{k_1}, x_{k_2}] \tag{3.10}$$

for every $k_2 \in [k_1 + 1, N - 1]$ and every Borel measurable function h.

By Definition 2.1.1, $[x_k]$ is CM_L iff

$$E[g([x_k]_0^{k_1-1})|[x_i]_{k_1}^N] = E[g([x_k]_0^{k_1-1})|x_{k_1}, x_N] \tag{3.11}$$

for every $k_1 \in [1, N - 1]$ and every Borel measurable function g. Now, let $[x_k]$ be $[k_1, N]$-CM_F, $\forall k_1 \in [0, N]$, and CM_L. We show that (3.8) holds. We have

$$\begin{aligned}
E[g([x_k]_0^{k_1-1})h([x_k]_{k_2+1}^N)|[x_i]_{k_1}^{k_2}] &= E\Big[E[g([x_k]_0^{k_1-1})h([x_k]_{k_2+1}^N)|[x_i]_{k_1}^{k_2}, [x_i]_{k_2+1}^N]\Big|[x_j]_{k_1}^{k_2}\Big] \\
&= E\Big[h([x_k]_{k_2+1}^N)E[g([x_k]_0^{k_1-1})|[x_i]_{k_1}^{k_2}, [x_i]_{k_2+1}^N]\Big|[x_i]_{k_1}^{k_2}\Big] \\
&= E\Big[h([x_k]_{k_2+1}^N)E[g([x_k]_0^{k_1-1})|x_{k_1}, [x_i]_{k_2}^N]\Big|[x_i]_{k_1}^{k_2}\Big] \\
&= E\Big[h([x_k]_{k_2+1}^N)E[g([x_k]_0^{k_1-1})|x_{k_1}, [x_i]_{k_2}^N]\Big|x_{k_1}, x_{k_2}\Big] \\
&= E\Big[E[g([x_k]_0^{k_1-1})h([x_k]_{k_2+1}^N)|x_{k_1}, [x_i]_{k_2}^N]\Big|x_{k_1}, x_{k_2}\Big] \\
&= E[g([x_k]_0^{k_1-1})h([x_k]_{k_2+1}^N)|x_{k_1}, x_{k_2}]
\end{aligned}$$

where the third equality holds because $[x_k]$ is CM_L and thus by (3.11) we have

$$E[g([x_k]_0^{k_1-1})|[x_i]_{k_1}^N] = E[g([x_k]_0^{k_1-1})|x_{k_1}, [x_i]_{k_2}^N]$$

The fourth equality holds because $[x_k]$ is $[k_1, N]$-CM_F for every $k_1 \in [0, N]$ and so by (3.10) we have

$$E[q([x_k]_{k_2}^N, x_{k_1})|[x_i]_{k_1}^{k_2}] = E[q([x_k]_{k_2}^N, x_{k_1})|x_{k_1}, x_{k_2}]$$

for every Borel measurable function q. Therefore, $[x_k]$ is reciprocal. (Note that the required integrability condition for nested expectations [86] holds.)

Due to the symmetry ($[k_1, N]$-CM_F, $\forall k_1 \in [0, N]$, and CM_L vs. $[0, k_2]$-CM_L, $\forall k_2 \in [0, N]$, and CM_F), sufficiency of (ii) is proved using (3.8), (3.12) and (3.13) below. Similar to (3.11), $[x_k]$ is $[0, k_2]$-CM_L iff

$$E[g([x_k]_0^{k_1-1})|[x_i]_{k_1}^{k_2}] = E[g([x_k]_0^{k_1-1})|x_{k_1}, x_{k_2}] \tag{3.12}$$

for every $k_1 \in [1, k_2 - 1]$ and every Borel measurable function g. Also, considering $k_1 = 0$ in (3.10), $[x_k]$ is CM_F iff

$$E[h([x_k]_{k_2+1}^N)|[x_i]_0^{k_2}] = E[h([x_k]_{k_2+1}^N)|x_0, x_{k_2}] \tag{3.13}$$

for every $k_2 \in [1, N - 1]$ and every Borel measurable function h. □

Note that Theorem 3.1.5, Lemma 3.1.6, and Corollary 3.1.7 below are for the general (Gaussian/non-Gaussian) case.

[29] commented on the relationship between the Gaussian CM process and the Gaussian reciprocal process, where a part of the condition, i.e., $[k_1, N]$-CM_F ($\forall k_1 \in [0, N]$) was mentioned, but the other part, i.e., CM_L was overlooked. We will show in Section 3.2 that the condition presented in [29] is not sufficient for a Gaussian process to be reciprocal.

From the proof of Theorem 3.1.5, a sequence that is $[k_1, N]$-CM_F ($\forall k_1 \in [0, N]$) and CM_L or equivalently $[0, k_2]$-CM_L ($\forall k_2 \in [0, N]$) and CM_F is actually $[k_1, N]$-CM_F and $[0, k_2]$-CM_L ($\forall k_1, k_2 \in [0, N]$). It means that a sequence is reciprocal iff it is $[k_1, N]$-CM_F and $[0, k_2]$-CM_L ($\forall k_1, k_2 \in [0, N]$). This was pointed out for the Gaussian case in [30]. However, [30] did not discuss if the condition presented in [29] is sufficient even for the Gaussian case. By the relationship between the CM sequence and the reciprocal sequence it can be seen that the set of CM sequences is much larger than that of reciprocal sequences.

The following lemma presents an equation which is equivalent to the definition of the reciprocal sequence. Lemma 3.1.6 follows from [6]. However, our proof is based on the relationship between the reciprocal sequence and the CM sequence (Theorem 3.1.5), which is simple and different from that of [6]. This proof demonstrates the advantage of the CM viewpoint for studying reciprocal sequences.

Lemma 3.1.6. *$[x_k]$ is reciprocal iff*

$$E[f(x_k)|[x_i]_0^j, [x_i]_l^N] = E[f(x_k)|x_j, x_l] \tag{3.14}$$

for every $j, k, l \in [0, N]$ ($j < k < l$) and every Borel measurable function f.

Proof. Necessity: It can be seen that (3.1) is equivalent to

$$P\{A|[x_i]_0^j, [x_i]_l^N\} = P\{A|x_j, x_l\} \tag{3.15}$$

where $A \in \sigma([x_k]_{j+1}^{l-1})$ [85]. Let $[x_k]$ be a reciprocal sequence. Then, (3.14) follows from (3.15).

Sufficiency: It is based on Theorem 3.1.5. Assume that (3.14) holds for $[x_k]$. Then,

$$\begin{aligned} E[f(x_k)|[x_i]_0^j, x_l] &= E\Big[E[f(x_k)|[x_i]_0^j, [x_i]_l^N]\Big|[x_i]_0^j, x_l\Big] \\ &= E\Big[E[f(x_k)|x_j, x_l]\Big|[x_i]_0^j, x_l\Big] \\ &= E[f(x_k)|x_j, x_l] \end{aligned}$$

where the second equality holds due to (3.14). So, by Corollary 2.1.5, $[x_k]$ is $[0, l]$-CM_L. Similarly, we have

$$E[f(x_k)|x_j, [x_i]_l^N] = E[f(x_k)|x_j, x_l]$$

meaning that, by Corollary 2.1.5, $[x_k]$ is $[j, N]$-CM_F. Then, by Theorem 3.1.5, $[x_k]$ is reciprocal. □

Corollary 3.1.7. *$[x_k]$ is reciprocal iff*

$$F(\xi_k|[x_i]_0^j, [x_i]_l^N) = F(\xi_k|x_j, x_l) \tag{3.16}$$

for every $j, k, l \in [0, N]$ $(j < k < l)$, where $F(\cdot|\cdot)$ is the conditional cumulative distribution function (CDF) of x_k, $\xi_k \in \mathbb{R}^d$, and d is the dimension of x_k.

Proof. See the proof of Corollary 2.1.5. □

Corollary 3.1.7 (in a simple language) is equivalent to Definition 3.1.1.

Remark 3.1.8. *Due to its simplicity, we recommend Corollary 3.1.7 as the definition of reciprocal sequences in application.*

Lemma 3.1.6 reduces to the following lemma for the Gaussian case.

Lemma 3.1.9. *A Gaussian sequence $[x_k]$ is reciprocal iff*

$$E[x_k|[x_i]_0^j, [x_i]_l^N] = E[x_k|x_j, x_l] \tag{3.17}$$

for every $j, k, l \in [0, N]$ $(j < k < l)$.

Proof. Necessity: Let $[x_k]$ be a Gaussian reciprocal sequence. Clearly (3.17) follows from (3.14).

Sufficiency: We present a proof based on Theorem 3.1.5. Let $[x_k]$ be a Gaussian sequence for which (3.17) holds for every $j, k, l \in [0, N]$ $(j < k < l)$. Then, as in the proof of Lemma 3.1.6, we have

$$\begin{aligned} E[x_k|[x_i]_0^j, x_l] &= E[x_k|x_j, x_l] \\ E[x_k|x_j, [x_i]_l^N] &= E[x_k|x_j, x_l] \end{aligned}$$

meaning that $[x_k]$ is $[0, l]$-CM_L and $[j, N]$-CM_F (Lemma 2.1.7). Then, by Theorem 3.1.5, $[x_k]$ is reciprocal. □

Note that Lemma 3.1.9 works for both singular and nonsingular Gaussian sequences.

In order to characterize the NG reciprocal sequence based on Theorem 3.1.5, we obtain characterizations of NG $[k_1, k_2]$-CM_c sequences. We first consider the general case, and then address some important special cases.

Proposition 3.1.10. *Let $A = C^{-1}$ be the inverse of the covariance matrix of a NG sequence. The sequence is $[k_1, k_2]$-CM_c ($k_1, k_2 \in [1, N-1]$) iff $\Delta_{B_{11}}$ has the CM_c form, where*

$$\Delta_{B_{11}} = B_{22} - B'_{12} B_{11}^{-1} B_{12} \tag{3.18}$$

$$B = \Delta_{A_{bb}} = A_{aa} - A_{ab} A_{bb}^{-1} A'_{ab} \tag{3.19}$$

and for a $(k_2+1)d \times (k_2+1)d$ matrix X we have $X_{11} = X_{[1:k_1,1:k_1]}$, $X_{22} = X_{[k_1+1:k_2+1,k_1+1:k_2+1]}$, $X_{12} = X_{[1:k_1,k_1+1:k_2+1]}$, and for an $(N+1)d \times (N+1)d$ matrix Y we have $Y_{aa} = Y_{[1:k_2+1,1:k_2+1]}$, $Y_{bb} = Y_{[k_2+2:N+1,k_2+2:N+1]}$, and $Y_{ab} = Y_{[1:k_2+1,k_2+2:N+1]}$ ($d \times d$ is the dimension of each block entry of these matrices).

Proof. We have

$$A^{-1} = \begin{bmatrix} \Delta_{A_{bb}}^{-1} & -\Delta_{A_{bb}}^{-1} A_{ab} A_{bb}^{-1} \\ -A_{bb}^{-1} A'_{ab} \Delta_{A_{bb}}^{-1} & A_{bb}^{-1} + A_{bb}^{-1} A'_{ab} \Delta_{A_{bb}}^{-1} A_{ab} A_{bb}^{-1} \end{bmatrix}$$

Also,

$$C = \begin{bmatrix} C_{aa} & C_{ab} \\ C'_{ab} & C_{bb} \end{bmatrix} = A^{-1}$$

Clearly $C_{aa} = \Delta_{A_{bb}}^{-1}$. Then, define $B \triangleq \Delta_{A_{bb}}$. We have

$$B^{-1} = \begin{bmatrix} B_{11}^{-1} + B_{11}^{-1} B_{12} \Delta_{B_{11}}^{-1} B'_{12} B_{11}^{-1} & -B_{11}^{-1} B_{12} \Delta_{B_{11}}^{-1} \\ -\Delta_{B_{11}}^{-1} B'_{12} B_{11}^{-1} & \Delta_{B_{11}}^{-1} \end{bmatrix}$$

Also, let

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D'_{12} & D_{22} \end{bmatrix} \triangleq C_{aa}$$

Since $B^{-1} = D$, we have $D_{22}^{-1} = \Delta_{B_{11}}$. Therefore, the sequence $[x_k]$ is $[k_1, k_2]$-CM_c iff $\Delta_{B_{11}}$ has the CM_c form, where $\Delta_{B_{11}}$ is given by (3.18). □

Definition 3.1.11. *A positive definite matrix A is called a $[k_1, k_2]$-CM_c matrix (or A is said to have the $[k_1, k_2]$-CM_c form) if $\Delta_{B_{11}}$ in* (3.18) *has the CM_c form.*

The characterization of the $[k_1, k_2]$-CM_c sequence can be presented in a different formulation (Proposition 3.1.12). Actually, Proposition 3.1.10 and Proposition 3.1.12 below give different formulations of the same characterization. However, in some cases one formulation is more convenient to use than the other (Section 3.2).

Proposition 3.1.12. *Let $A = C^{-1}$ be the inverse of the covariance matrix of a NG sequence. The sequence is $[k_1, k_2]$-CM_c ($k_1, k_2 \in [1, N-1]$) iff $\Delta_{B_{22}}$ has the CM_c form, where*

$$\Delta_{B_{22}} = B_{11} - B_{12} B_{22}^{-1} B'_{12} \tag{3.20}$$

$$B = \Delta_{A_{11}} = A_{22} - A'_{12} A_{11}^{-1} A_{12} \tag{3.21}$$

and $B_{11} = B_{[1:k_2-k_1+1,1:k_2-k_1+1]}$, $B_{22} = B_{[k_2-k_1+2:N-k_1+1,k_2-k_1+2:N-k_1+1]}$, $B_{12} = B_{[1:k_2-k_1+1,k_2-k_1+2:N-k_1+1]}$, $A_{11} = A_{[1:k_1,1:k_1]}$, $A_{22} = A_{[k_1+1:N+1,k_1+1:N+1]}$, and $A_{12} = A_{[1:k_1,k_1+1:N+1]}$.

Proof. Similar to the proof of Proposition 3.1.10. □

Corollary 3.1.13. *Let $A = C^{-1}$ be the inverse of the covariance matrix of a NG sequence. (i) The sequence is $[0, k_2]$-CM_c ($k_2 \in [1, N-1]$) iff $\Delta_{A_{bb}}$ in* (3.19) *has the CM_c form. (ii) The sequence is $[k_1, N]$-CM_c ($k_1 \in [1, N-1]$) iff $\Delta_{A_{11}}$ in* (3.21) *has the CM_c form.*

Proof. Proofs of (i) and (ii) are special cases of those of Proposition 3.1.10 and Proposition 3.1.12, respectively. □

In order to make clear the properties of the $[0, k_2]$-CM_c sequence[1], we study the sequence over $[k_2 + 1, N]$, too. In the following, we discuss the distribution of $[x_k]_{k_2+1}^N$. First, relevant properties of the positive definite matrices are reviewed. Consider a symmetric matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M'_{12} & M_{22} \end{bmatrix}$$

Then, $M > 0$ iff $M_{11} > 0$ and $M_{22} - M'_{12}M_{11}^{-1}M_{12} > 0$; $M > 0$ iff $M_{22} > 0$ and $M_{11} - M_{12}M_{22}^{-1}M'_{12} > 0$.

Now, let $[x_k]$ be a NG sequence with the covariance matrix (following the notation of Proposition 3.1.10)

$$C = \begin{bmatrix} C_{aa} & C_{ab} \\ C'_{ab} & C_{bb} \end{bmatrix} \tag{3.22}$$

$[x_k]$ is $[0, k_2]$-CM_c iff $(C_{aa})^{-1}$ has the CM_c form. Then, given C_{aa} of a NG $[0, k_2]$-CM_c sequence, the only restriction on C_{bb} and C_{ab} is that $C_{bb} - (C_{ab})'(C_{aa})^{-1}C_{ab} > 0$. As an example, for $C_{ab} = (C_{aa})^{\frac{1}{2}}B$, where B is any nonsingular matrix, we have $C_{bb} - (C_{ab})'(C_{aa})^{-1}C_{ab} = C_{bb} - B'B > 0$. Now, let $C_{ab} = 0$. Then, there is no more restriction on C_{bb} (other than positive definiteness). It can be also seen from the covariance matrix C with $C_{ab} = 0$. Therefore, there exist $[0, k_2]$-CM_c sequences for which the covariance of $[x_k]_{k_2+1}^N$ can be any positive definite matrix without extra restriction.

Marginal distributions of the NG $[k_1, k_2]$-CM_c sequence over $[0, k_1 - 1]$ and $[k_2 + 1, N]$ can be similarly studied. Let its covariance matrix be

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C'_{12} & C_{22} & C_{23} \\ C'_{13} & C'_{23} & C_{33} \end{bmatrix}$$

where $C_{11} = C_{[1:k_1,1:k_1]}$, $C_{12} = C_{[1:k_1,k_1+1:k_2+1]}$, $C_{13} = C_{[1:k_1,k_2+2:N+1]}$, $C_{22} = C_{[k_1+1:k_2+1,k_1+1:k_2+1]}$, $C_{23} = C_{[k_1+\ 1:k_2+1,k_2+2:N+1]}$, and $C_{33} = C_{[k_2+2:N+1,k_2+2:N+1]}$. Given C_{22} (where C_{22}^{-1} has the CM_c form), we have

$$\begin{aligned} &C_{11} - C_{12}C_{22}^{-1}C'_{12} > 0 \\ &C_{33} - \begin{bmatrix} C'_{13} & C'_{23} \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} \\ C'_{12} & C_{22} \end{bmatrix}^{-1} \begin{bmatrix} C_{13} \\ C_{23} \end{bmatrix} > 0 \end{aligned}$$

or

$$\begin{aligned} &C_{33} - C'_{23}C_{22}^{-1}C_{23} > 0 \\ &C_{11} - \begin{bmatrix} C_{12} & C_{13} \end{bmatrix} \begin{bmatrix} C_{22} & C_{23} \\ C'_{23} & C_{33} \end{bmatrix}^{-1} \begin{bmatrix} C'_{12} \\ C'_{13} \end{bmatrix} > 0 \end{aligned}$$

A characterization of the NG reciprocal sequence was presented in [18]. Based on the above characterizations of $[k_1, k_2]$-CM_c sequences, that characterization of the NG reciprocal sequence can be obtained from the CM viewpoint (Theorem 3.1.14 below). The corresponding proof is simple and different from the one presented in [18].

[1] One can similarly study $[k_1, N]$-CM_c.

Theorem 3.1.14. *A NG sequence with the covariance matrix C is reciprocal iff C^{-1} is cyclic (block) tri-diagonal as*

$$\begin{bmatrix} A_0 & B_0 & 0 & \cdots & 0 & 0 & D_0 \\ B'_0 & A_1 & B_1 & 0 & \cdots & 0 & 0 \\ 0 & B'_1 & A_2 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & B'_{N-3} & A_{N-2} & B_{N-2} & 0 \\ 0 & \cdots & 0 & 0 & B'_{N-2} & A_{N-1} & B_{N-1} \\ D'_0 & 0 & 0 & \cdots & 0 & B'_{N-1} & A_N \end{bmatrix} \tag{3.23}$$

Proof. Necessity: By Theorem 3.1.5, characterization of the NG reciprocal sequence is the same as that of the NG sequence being CM_F and $[0,k_2]$-CM_L, $\forall k_2 \in [0,N]$. Let $[x_k]$ be a NG sequence (with the covariance matrix C), which is CM_F and $[0,k_2]$-CM_L, $\forall k_2 \in [0,N]$. By Theorem 2.3.5, C^{-1} is (block) cyclic tri-diagonal, because a matrix being both CM_L and CM_F is cyclic tri-diagonal.

Sufficiency: Assume that the inverse of the covariance matrix (C^{-1}) of a NG $[x_k]$ is cyclic (block) tri-diagonal. A cyclic tri-diagonal matrix has the CM_F and the $[0,k_2]$-CM_L forms $\forall k_2 \in [0,N]$. Thus, by Theorem 2.3.5 and Corollary 3.1.13, $[x_k]$ is CM_F and $[0,k_2]$-CM_L, $\forall k_2 \in [0,N]$. Therefore, by Theorem 3.1.5, $[x_k]$ is reciprocal. □

The following corollary of Theorem 3.1.14 has an important implication about the relationship between the CM sequence and the reciprocal sequence. Once more, it demonstrates the significance of the CM viewpoint for studying the reciprocal sequence.

Corollary 3.1.15. *A NG sequence is reciprocal iff it is both CM_L and CM_F.*

Proof. Clearly, a NG sequence with the covariance matrix C is both CM_L and CM_F iff C^{-1} has the form of (3.23). So, by Theorem 3.1.14, it is reciprocal iff it is both CM_L and CM_F. □

By Corollary 3.1.15, a NG sequence being both CM_L and CM_F is $[k_1,k_2]$-CM_L and $[k_1,k_2]$-CM_F, $\forall k_1,k_2 \in [0,N]$.

A characterization of the NG Markov sequence is as follows [56].

Remark 3.1.16. *A NG sequence with the covariance matrix C is Markov iff C^{-1} is (block) tri-diagonal as* (3.23) *with* $D_0 = 0$.

Theorem 2.3.5, Theorem 3.1.14, Corollary 3.1.15, and Remark 3.1.16 show how CM, reciprocal, and Markov sequences are connected.

3.1.2 Reciprocal CM_c Dynamic Models

A dynamic model of the ZMNG reciprocal sequence was presented in [18], where the evolution of reciprocal sequences is described by a second-order nearest-neighbor model driven by locally correlated dynamic noise [18]. That model can be considered a generalization of the Markov model. However, due to the autocorrelation of the dynamic noise and the nearest-neighbor structure, it is not necessarily easy to apply (see Subsection 3.1.3). In the following, two models of the ZMNG reciprocal sequence are presented from the CM viewpoint. These models have white dynamic noise, and their corresponding recursive estimators are easily obtained.

Dynamic models of CM_c sequences were presented in Chapter 2. On the other hand, the reciprocal sequence is a special CM_c sequence. The following theorem gives the condition under which a CM_c model is for a reciprocal sequence. Also, it is shown that every ZMNG reciprocal sequence has such a dynamic model.

Theorem 3.1.17. *A ZMNG $[x_k]$ with the covariance function C_{l_1,l_2} is reciprocal iff it obeys* (2.17) *along with* (2.18) *or* (2.19)*, and*

$$G_k^{-1}G_{k,c} = G'_{k+1,k}G_{k+1}^{-1}G_{k+1,c} \tag{3.24}$$

$\forall k \in [1, N-2]$ *for $c = N$, or $\forall k \in [2, N-1]$ for $c = 0$. Moreover, for $c = N$, $[x_k]$ is Markov iff in addition to* (3.24)*, we have*

$$G_N^{-1}G_{N,0} = G'_{1,N}G_1^{-1}G_{1,0} \tag{3.25}$$

for (2.18)*, or equivalently*

$$G_0^{-1}G_{0,N} = G'_{1,0}G_1^{-1}G_{1,N} \tag{3.26}$$

for (2.19)*. Also, for $c = 0$, the reciprocal sequence is Markov iff in addition to* (3.24)*, we have*

$$G_{N,0} = 0 \tag{3.27}$$

Proof. Let $c = N$. A ZMNG sequence is CM_L iff it obeys (2.17) along with (2.18) or (2.19). Also, by Theorem 2.3.5, a NG sequence is CM_L iff its C^{-1} is CM_L given by (2.36). Entries of C^{-1} of a ZMNG CM_L sequence can be calculated in terms of the parameters of (2.17) along with (2.18) or (2.19) (see (3.29) below). On the other hand, by Theorem 3.1.14, a NG sequence is reciprocal iff its C^{-1} is cyclic (block) tri-diagonal given by (3.23). Therefore, a ZMNG sequence is reciprocal iff it obeys (2.17) along with (2.18) or (2.19) and its C^{-1} is given by (2.36) with $D_1 = \cdots = D_{N-2} = 0$. By $D_1 = \cdots = D_{N-2} = 0$, we obtain (3.24). In addition, by Remark 3.1.16, a NG sequence is Markov iff in addition to $D_1 = \cdots = D_{N-2} = 0$, we have $D_0 = 0$, which leads to (3.25) for (2.18), and (3.26) for (2.19) (see (B.5),(B.8),(B.11) in Appendix B.1 for the explicit relationship between $D_i, i \in [0, N-2]$ and parameters of the CM_L dynamic model).

Proof of the theorem for $c = 0$ is similar to that of $c = N$. By $D_2 = D_3 = \cdots = D_{N-1} = 0$, we get (3.24) and by $D_N = 0$ we obtain (3.27) (see (B.41)–(B.42) in Appendix B.2 for the explicit relationship between $D_i, i \in [2, N]$ and parameters of the CM_F dynamic model). □

The Markov sequence is a subset of the reciprocal sequence, and the reciprocal sequence is a subset of the CM_c sequence. A CM_c model is a complete (i.e., necessary and sufficient) description of the CM_c sequence. Theorem 3.1.17 shows under what conditions a CM_c model is a complete description of the reciprocal sequence, and under what conditions a CM_c model is a complete description of the Markov sequence. In other words, Theorem 3.1.17 shows simple and explicit iff conditions for the CM (model) to reduce to the reciprocal, and for the reciprocal to reduce to the Markov.

Theorem 3.1.17 can be presented in a different way.

Corollary 3.1.18. *Model* (2.17) *along with* (2.18) *or* (2.19) *is one for a ZMNG reciprocal sequence iff the matrix*

$$A = \mathcal{G}'G^{-1}\mathcal{G} \tag{3.28}$$

is (block) cyclic tri-diagonal, where $G = diag(G_0, G_1, \ldots, G_N)$, and for $c = N$ the matrix $\mathcal{G}$ is (2.27) *for* (2.18)*,* (2.28) *for* (2.19)*, and for $c = 0$, $\mathcal{G}$ is* (2.29)*. In addition, the sequence is Markov iff A in* (3.28) *is (block) tri-diagonal.*

Proof. Let $[x_k]$ be a ZMNG CM_c sequence that obeys (2.17) along with (2.18) or (2.19). Then,

$$\mathcal{G}x = e \triangleq [e'_0, e'_1, \ldots, e'_N]'$$

where for $c = 0$, $\mathcal{G}$ is given by (2.29); for $c = N$, $\mathcal{G}$ is given by (2.27) for (2.18) and by (2.28) for (2.19). C^{-1} of $[x_k]$ is calculated as

$$C^{-1} = \mathcal{G}' G^{-1} \mathcal{G} \tag{3.29}$$

which is a CM_c matrix (Theorem 2.3.5). By Theorem 3.1.14, $[x_k]$ is reciprocal iff its C^{-1} (i.e., $\mathcal{G}'G^{-1}\mathcal{G}$) is (block) cyclic tri-diagonal. In addition, by Remark 3.1.16, $[x_k]$ is Markov iff $\mathcal{G}'G^{-1}\mathcal{G}$ is (block) tri-diagonal. □

The ZMNG reciprocal sequence can be modeled by either the reciprocal CM_c model of Theorem 3.1.17 or the reciprocal model of [18]. We use the term "reciprocal CM_c model" for our model (Theorem 3.1.17) and the term "reciprocal model" for the model of [18].

Remark 3.1.19. *A CM_c model of a reciprocal sequence is called a reciprocal CM_c model. In this way, we distinguish between the reciprocal model of [18] and our reciprocal CM_c model.*

Similarly, the Markov sequence is a special CM_c sequence. Theorem 3.1.17, gives the necessary and sufficient condition under which a CM_c model is actually a dynamic model of the ZMNG Markov sequence. A CM_c model describing a Markov sequence is called a *Markov CM_c model.*

The following proposition presents a backward model of the reciprocal sequence.

Proposition 3.1.20. *A ZMNG sequence $[x_k]$ with the covariance function C_{l_1,l_2} is reciprocal iff it obeys* (2.31) *along with* (2.32) *or* (2.33) *and*

$$(G^B_{k+1})^{-1} G^B_{k+1,c} = (G^B_{k,k+1})' (G^B_k)^{-1} G^B_{k,c} \tag{3.30}$$

$\forall k \in [1, N-2]$ *for $c = 0$, or $\forall k \in [0, N-3]$ for $c = N$. Moreover, for $c = 0$, $[x_k]$ is Markov iff in addition to* (3.30)*, we have*

$$(G^B_N)^{-1} G^B_{N,0} = (G^B_{N-1,N})' (G^B_{N-1})^{-1} G^B_{N-1,0} \tag{3.31}$$

for (2.32)*, or equivalently*

$$(G^B_0)^{-1} G^B_{0,N} = (G^B_{N-1,0})' (G^B_{N-1})^{-1} G^B_{N-1,N} \tag{3.32}$$

for (2.33)*. Also, for $c = N$, $[x_k]$ is Markov iff in addition to* (3.30)*, we have*

$$G^B_{0,N} = 0 \tag{3.33}$$

Proof. It is similar to that of Theorem 3.1.17 and is omitted. □

The dynamic model presented in Proposition 3.1.20 is called a backward reciprocal CM_c model.

3.1.3 Recursive Estimation of Reciprocal Sequences

A dynamic model was presented in [18] for the NG reciprocal sequence. That dynamic model with the second-order nearest neighbor structure is a complete (i.e., necessary and sufficient) description of the NG reciprocal sequence. It was shown in [18] that the well-posedness of the model[2] is guaranteed by a condition on all parameters of the model (i.e., parameters should be determined in a way leading to a nonsingular sequence). That condition on all the parameters of the model is not easy to check. On the contrary, our CM_c dynamic models (including reciprocal

[2]A dynamic model is well-posed iff it admits a unique solution. For the Gaussian case, a model is well-posed iff it admits a unique covariance function [18].

CM_c models) are always well-posed for every value of their parameters (Chapter 2). In addition, the condition on parameters of a CM_c model of a reciprocal sequence is much simpler than the well-posedness condition for the model of [18].

Due to the nearest neighbor structure and the colored dynamic noise, recursive estimation of a reciprocal sequence based on the model of [18] is not straightforward. That is why several papers were presented for recursive estimation of NG reciprocal sequences based on that dynamic model. For example, [32]–[33] presented some recursive estimators for NG reciprocal sequences based on the model of [18] with different boundary conditions using a higher-order state whose dimension was 3 times of that of the state of the sequence. Later, a different recursive estimator was presented in [34] using a higher-order state whose dimension was 2 times of that of the state of the sequence. After lengthy and complicated manipulation of the second-order model of [18], [34] obtained a first-order forward/backward dynamic model with white dynamic noise. Then, derivation of a recursive estimator based on the obtained first-order forward/backward model was straightforward based on existing results from linear system theory. This straightforwardness is why the approach of [34] is highly desired. The difficulty in the approach of [34] is to obtain the forward/backward first-order model with white dynamic noise from the second-order model of [18]. However, it can be easily seen that the first-order forward/backward model of [18] is actually a forward/backward CM_L model (Theorem 2.2.6 ($c = N$) and Proposition 2.2.7 ($c = 0$)), which is available without any effort from the CM viewpoint. Therefore, it demonstrates the significance of the CM viewpoint for studying reciprocal sequences.

Note that the first-order forward/backward model in [34] (which turned out to be a forward/backward CM_L model) was obtained after complicated manipulation of the dynamic model of [18]. That is why the relationship between parameters of the obtained first-order forward/backward model and those of the dynamic model of [18] is complicated. Consequently, the relationship between different parameters of the obtained first-order forward/backward model is complicated and not straightforward. However, we obtained the forward/backward reciprocal CM_L model directly from the CM viewpoint. Thus, we do not need to connect parameters of our forward/backward reciprocal CM_L model to those of the model of [18]. Also, the relationship between different parameters of a forward/backward reciprocal CM_L model is clear. Moreover, a reciprocal CM_c model clearly shows how a CM_c model of the CM_c sequence reduces to its special case (i.e., reciprocal CM_c model) for the reciprocal sequence (Theorem 3.1.17). This is desired for unifying treatment of CM and reciprocal sequences leading to more insight into reciprocal sequences.

Recursive estimation (filtering/smoothing/prediction) based on our reciprocal CM_c model is straightforward based on the results from linear system theory (Chapter 7).

3.2 Characterizations: Other CM Classes vs. Reciprocal

In order to reveal the relationship between various CM sequences (including reciprocal sequences), their intersections are studied. As a result, we show how the characterizations change from a CM sequence to the reciprocal sequence.

We do not consider trivial special cases. For example, it is obvious by definition that every sequence is $[k_1, k_2]$-CM_c, $0 < k_2 - k_1 \leq 2$. We are interested in the general cases with arbitrary N, k_1, and k_2. The only difference between some classes of CM sequences is time direction. For example, the $CM_L \cap [0, k]$-CM_L sequence and the $CM_F \cap [k, N]$-CM_F sequence (i.e., the backward $CM_L \cap [k, N]$-CM_L sequence) differ only in time direction. Due to the similarity, we only consider one of such cases.

3.2.1 $CM_L \cap [k_1, N]$-CM_F

By Theorem 2.3.5 and Corollary 3.1.13 we have

- Special case: A sequence is $CM_L \cap [N-3, N]$-CM_F iff its C^{-1} is given by (2.36) with $D_{N-2} = 0$.

- Special case: A sequence is $CM_L \cap [N-4, N]$-CM_F iff its C^{-1} is given by (2.36) with $D_{N-3} = D_{N-2} = 0$.

- General case: A sequence is $CM_L \cap [k_1, N]$-CM_F iff its C^{-1} is given by (2.36) with

$$D_{k_1+1} = D_{k_1+2} = \cdots = D_{N-3} = D_{N-2} = 0$$

- Important special case: A sequence is $CM_L \cap CM_F$ iff its C^{-1} is given by (2.36) with

$$D_1 = D_2 = \cdots = D_{N-2} = 0$$

  which is actually the reciprocal characterization (Corollary 3.1.15).

It is thus seen how the characterizations (i.e., C^{-1}) gradually change from CM_L to reciprocal and then Markov. In addition, based on the results above, the $CM_L \cap [k_1, N]$-CM_F sequence has been characterized for every $k_1 \in [0, N-1]$.

Similarly, a sequence is $CM_F \cap [0, k_2]$-CM_L iff its C^{-1} is given by (2.37) with $D_2 = D_3 = \cdots = D_{k_2-1} = 0$.

3.2.2 $CM_L \cap [0, k_2]$-CM_L ($CM_F \cap [k_1, N]$-CM_F)

In this subsection, we study characterizations of $CM_L \cap [0, k_2]$-CM_L sequences to see their relationship with the reciprocal sequence. As a result, the condition presented in [29] is addressed.

By Theorem 2.3.5 and Corollary 3.1.13 we have

- Special case: A sequence is $CM_L \cap [0, 3]$-CM_L iff its C^{-1} is given by (2.36) with $D_0 U_{0,3} D_2' = 0$, where $U_{0,3} = R_{[N-3:N-3, N-3:N-3]}$ and $R = (A_{[5:N+1, 5:N+1]})^{-1}$. Clearly, a trivial solution is $D_2 = 0$.

- Special case: A sequence is $CM_L \cap [0, 4]$-CM_L iff its C^{-1} is given by (2.36) with $D_0 U_{0,4} D_2' = 0$, $D_0 U_{0,4} D_3' = 0$, and $D_1 U_{0,4} D_3' = 0$, where $U_{0,4} = R_{[N-4:N-4, N-4:N-4]}$ and $R = (A_{[6:N+1, 6:N+1]})^{-1}$. A trivial solution is $D_2 = D_3 = 0$.

- General case: A sequence is $CM_L \cap [0, k_2]$-CM_L iff its C^{-1} is given by (2.36) with

$$\begin{aligned}
&D_0 U_{0,k_2} D_i' = 0, \quad i = 2, \ldots, k_2 - 1 \\
&D_1 U_{0,k_2} D_i' = 0, \quad i = 3, \ldots, k_2 - 1 \\
&\quad \vdots \\
&D_{k_2-3} U_{0,k_2} D_{k_2-1}' = 0
\end{aligned}$$

  where $U_{0,k_2} = R_{[N-k_2:N-k_2, N-k_2:N-k_2]}$ and $R = (A_{[k_2 +2:N+1, k_2+2:N+1]})^{-1}$. A trivial solution is $D_2 = D_3 = \cdots = D_{k_2-1} = 0$.

- Important special case: A sequence is $CM_L \cap [0, N-1]$-CM_L iff its C^{-1} is given by (2.36) with

$$\begin{aligned}
&D_0 U_{0,N-1} D_i' = 0, \quad i = 2, \ldots, N-2 \\
&D_1 U_{0,N-1} D_i' = 0, \quad i = 3, \ldots, N-2 \\
&\quad \vdots \\
&D_{N-4} U_{0,N-1} D_{N-2}' = 0
\end{aligned}$$

  where $U_{0,N-1} = (A_N)^{-1}$.

A trivial solution for the last case above is $D_2 = D_3 = \cdots = D_{N-2} = 0$ and D_1 can be non-zero. Note that this trivial solution is actually a trivial solution of all the above equations for $CM_L \cap [0,3]$-CM_L through $CM_L \cap [0, N-1]$-CM_L. On the other hand, a sequence is $\cap_{k_2=1}^{N}[0,k_2]$-CM_L iff its C^{-1} is given by (2.36) and all the above equations (for $CM_L \cap [0,3]$-CM_L through $CM_L \cap [0, N-1]$-CM_L) hold. As a result, a sequence with C^{-1} given by (2.36) with $D_2 = D_3 = \cdots = D_{N-2} = 0$ is $\cap_{k_2=1}^{N}[0,k_2]$-CM_L. From the CM_L sequence to the $\cap_{k_2=1}^{N}[0,k_2]$-CM_L sequence, as we consider intersections of more classes of sequences, the set of solutions for $D_1, \ldots, D_{N-2}$ shrinks in general, while a trivial solution for $D_2, \ldots, D_{N-2}$ (i.e., $D_2 = \cdots = D_{N-2} = 0$) always exists. However, it can be seen that for the $\cap_{k_2=1}^{N}[0,k_2]$-CM_L sequence, there is no guarantee that $D_1, D_2, \ldots, D_{N-2}$ are all equal to zero. Therefore, $\cap_{k_2=1}^{N}[0,k_2]$-CM_L is not necessarily reciprocal. Similarly, $\cap_{k_1=0}^{N-1}[k_1,N]$-$CM_F$ is not necessarily reciprocal. This means that the condition of [29] is not sufficient because equation (4) in [29] (which is the foundation of the argument of [29]) is necessary but not sufficient for a Gaussian process to be reciprocal.

Note that we need to check if a solution of the above equations is valid, that is, there exists a sequence with such a C^{-1}. This is because the above equations do not guarantee the positive definiteness of C^{-1}. In order to show the existence of a sequence corresponding to a solution of the above equations (for $CM_L \cap [0,3]$-CM_L through $CM_L \cap [0, N-1]$-CM_L), we can find parameters of a CM_L model (Theorem 2.2.6) of a sequence with the corresponding C^{-1} (i.e., satisfying the above equations).

The trivial solution ($D_2 = D_3 = \cdots = D_{N-2} = 0$ and $D_1 \neq 0$) really exists for all the above equations (for $CM_L \cap [0,3]$-CM_L through $CM_L \cap [0, N-1]$-CM_L), that is, there exists a sequence with C^{-1} given by (2.36) with $D_2 = D_3 = \cdots = D_{N-2} = 0$ and $D_1 \neq 0$. Based on the relation of (block) entries of a CM_L matrix and the parameters of the CM_L model and its boundary condition (Appendix B.1), we can determine parameters of a CM_L model and its boundary condition so that $D_2 = D_3 = \cdots = D_{N-2} = 0$ and $D_1 \neq 0$. Then, by Theorem 2.2.6, there exists a CM_L sequence corresponding to the above trivial solution. For example, the set of parameters of a CM_L model $(G_{k,k-1}, G_{k,N}, G_k)$, $k \in [1, N-1]$, with $G_{k,N} = 0$, $k \in [2, N-1]$, and $G_{1,N} \neq 0$ leads to $D_2 = D_3 = \cdots = D_{N-2} = 0$ and $D_1 \neq 0$.

From the above equations (for the characterization of the $CM_L \cap [0,k_2]$-CM_L sequence), it can be seen that for scalar sequences, a $CM_L \cap [0,l_2]$-CM_L sequence is $CM_L \cap [0,k_2]$-CM_L if $k_2 < l_2$. However, this is not true for vector-valued sequences in general. Here is a counterexample.

Example 3.2.1. Consider CM sequences defined over $[0,N]$ with $N = 5$. According to the results above, the characterization of $CM_L \cap [0,3]$-CM_L is given by (2.36) with $D_0 U_{0,3} D_2' = 0$, where $U_{0,3} = R_{[2:2,2:2]} = (A_5 - B_4' A_4^{-1} B_4)^{-1}$ and $R = (A_{[5:6,5:6]})^{-1}$. Also, the characterization of $CM_L \cap [0,4]$-CM_L is given by (2.36) with

$$D_0 U_{0,4} D_2' = 0, D_0 U_{0,4} D_3' = 0, D_1 U_{0,4} D_3' = 0 \tag{3.34}$$

where $U_{0,4} = (A_5)^{-1}$.

Based on the parametric relation of a CM_L model and C^{-1} of its sequence, we have (Appendix B.1)

$$\begin{aligned}
D_0 &= -G_{5,0} G_5^{-1} + G_{1,0}' G_1^{-1} G_{1,5} \\
D_i &= -G_i^{-1} G_{i,5} + G_{i+1,i}' G_{i+1}^{-1} G_{i+1,5}, \quad i = 1,2,3 \\
A_5 &= G_5^{-1} + \sum_{i=0}^{4} G_{i,5}' G_i^{-1} G_{i,5} \\
B_4 &= -G_4^{-1} G_{4,5} \\
A_4 &= G_4^{-1}
\end{aligned}$$

Let $G_i = I, i = 0, \ldots, 5$, $G_{4,3} = G_{3,5} = G_{0,5} = 0$, $G_{1,5} = I$. Also,

$$G_{4,5} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad G_{2,5} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad G_{0,5} = \begin{bmatrix} \frac{5}{6} & \frac{5}{6} & \frac{5}{6} \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Then, we can see that $D_3 = 0$, $D_0 U_{0,4} D_2' = 0$, but $D_0 U_{0,3} D_2' \neq 0$. Thus, for the above choice of parameters, (3.34) holds but $D_0 U_{0,3} D_2' \neq 0$. Thus, for vector-valued sequences, $CM_L \cap [0, 4]$-CM_L is not $CM_L \cap [0, 3]$-CM_L in general.

3.2.3 More About Intersections of CM Classes Relative to Reciprocal

In this subsection, intersections of some other interesting CM classes are studied relative to the reciprocal sequence.

For the Gaussian sequence the only restrictions on C^{-1} are the symmetry and the positive definiteness. The restrictions on C^{-1} gradually increase from the Gaussian sequence to the Gaussian CM_c sequence to the Gaussian reciprocal sequence, and then to the Gaussian Markov sequence. A $CM_L \cap [k_1, N]$-CM_F sequence is reciprocal over $[k_1, N]$ (Corollary 3.1.15).

It can be seen that a $\cap_{k_2=1}^{N} [0, k_2]$-$CM_L \cap_{k_1=1}^{N-1} [k_1, N]$-$CM_F$ sequence is not necessarily reciprocal because $D_1, D_2, \ldots, D_{N-2}$ in (2.36) are not necessarily zero for such a sequence. However, the $CM_L \cap CM_F$ sequence is reciprocal. It means that to be reciprocal the sequence must be both CM_L and CM_F. In addition, a $CM_L \cap [0, N-1]$-CM_F sequence is not necessarily reciprocal. This can be seen as follows. A sequence is $CM_L \cap [0, N-1]$-CM_F iff its C^{-1} is given by (2.36) with

$$\begin{aligned}
&D_1 (A_N)^{-1} D_i' = 0, \quad i = 3, \ldots, N-2 \\
&D_1 (A_N)^{-1} B_{N-1}' = 0 \\
&D_2 (A_N)^{-1} D_i' = 0, \quad i = 4, \ldots, N-2 \\
&D_2 (A_N)^{-1} B_{N-1}' = 0 \\
&\vdots \\
&D_{N-4} (A_N)^{-1} D_{N-2}' = 0 \\
&D_{N-4} (A_N)^{-1} B_{N-1}' = 0 \\
&D_{N-3} (A_N)^{-1} B_{N-1}' = 0
\end{aligned}$$

The above equations have a trivial solution $D_1 = D_2 = \cdots = D_{N-2} = 0$, which corresponds to the reciprocal sequence. Another trivial solution is

$$D_3 = D_4 = \cdots = D_{N-2} = B_{N-1} = 0 \tag{3.35}$$

where D_1 and D_2 are non-zero. So, a $CM_L \cap [0, N-1]$-CM_F sequence is not necessarily reciprocal. Consider the following choice of parameters of the CM_L model: $G_{i,N} = 0$, $i = 3, \ldots, N-1$, and other parameters equal to the identity matrix I. This set of parameters satisfies (3.35) (see Appendix B.1 for the relation of the CM_L model parameters and the (block) entries of C^{-1}).

Using Theorem 2.3.5, Proposition 3.1.10 or 3.1.12, and Corollary 3.1.13, we can study the relationship between some other CM classes. We skip the details and only present some results.

A $\cap_{k_2=1}^{N-1} [0, k_2]$-$CM_L \cap_{k_1=0}^{N-1} [k_1, N]$-$CM_F$ sequence is not necessarily CM_L. If it were, it would be reciprocal (Corollary 3.1.15), but it is not reciprocal. So, it is not CM_L. This again shows the role of the CM_L sequence and the CM_F sequence in the construction of the reciprocal sequence.

An interesting class of CM sequences is $[0,l]$-$CM_L \cap [l,N]$-CM_F ($l \in [1, N-1]$). Conditioning on x_l, the sequence is Markov over $[0, l-1]$ and Markov over $[l+1, N]$. A sequence is $[0,l]$-$CM_L \cap [l,N]$-CM_F iff its C^{-1} has both $[0,l]$-CM_L and $[l,N]$-CM_F forms.

A scalar $CM_L \cap [k_1, k_2]$-CM_L sequence is $CM_L \cap [l_1, l_2]$-CM_L, $k_1 \leq l_1 < l_2 < k_2$. However, this is not necessarily true for vector-valued sequences. A scalar $CM_L \cap [k_1, N-1]$-CM_F sequence is $CM_L \cap [l_1, l_2]$-$CM_F, k_1 < l_1 < l_2 \leq N-1$. In general, a $CM_L \cap [k_1, k_2]$-CM_F sequence is not necessarily $CM_L \cap [l_1, l_2]$-$CM_F, k_1 < l_1 < l_2 \leq k_2 \leq N-1$.

Chapter 4

Models and Representations of Gaussian Reciprocal and Other Gaussian CM Sequences

In this chapter, we 1) present some approaches/guidelines for parameter design of CM_L, CM_F, and reciprocal CM_L models for their application, 2) present a full spectrum of dynamic models ranging from a CM_L model to a reciprocal CM_L model, 3) show how models of various intersections of CM classes can be obtained, and 4) obtain a representation of NG CM_L, CM_F, and reciprocal sequences, revealing a key fact behind these sequences, and demonstrate the significance of studying reciprocal sequences from the CM viewpoint.

4.1 Dynamic Models of Reciprocal and Intersections of CM Classes

4.1.1 Reciprocal Sequences

By Theorem 3.1.17, one can determine whether a CM_L evolution model is for a reciprocal sequence or not. In other words, it gives the required conditions on the parameters of a CM_L evolution model to design a reciprocal CM_L evolution model. However, Theorem 3.1.17 does not provide an approach for designing the parameters. Theorem 4.1.3 below provides such an approach. First, we have a lemma.

Lemma 4.1.1. *The set of reciprocal sequences modeled by a reciprocal CM_L evolution model (2.17) with parameters $(G_{k,k-1}, G_{k,N}, G_k)$, $k \in [1, N-1]$ includes Markov sequences.*

Proof. By Theorem 3.1.17, (2.17) (for $c = N$) satisfying (3.24) with (2.19) models a reciprocal sequence. By Theorem 2.3.5, C^{-1} of such a sequence is cyclic (block) tri-diagonal given by (2.36) with $D_1 = \cdots = D_{N-2} = 0$ and

$$D_0 = G'_{1,0} G_1^{-1} G_{1,N} - G_0^{-1} G_{0,N} \tag{4.1}$$

(see (B.11) in Appendix B.1).

Now, consider a reciprocal sequence modeled by (2.17) satisfying (3.24) with the parameters $(G_{k,k-1}, G_{k,N}, G_k), k \in [1, N-1]$, and boundary condition (2.19) with the parameters $G_{0,N}$, G_0, and G_N, where

$$G_{0,N} = G_0 G'_{1,0} G_1^{-1} G_{1,N} \tag{4.2}$$

meaning that $D_0 = 0$. This reciprocal sequence is Markov (Theorem 2.3.5). Note that since for every possible value of the parameters of the boundary condition the sequence is nonsingular reciprocal modeled by the same reciprocal CM_L evolution model, choice (4.2) is valid. Thus, there exist Markov sequences belonging to the set of reciprocal sequences modeled by a reciprocal CM_L evolution model (2.17) with the parameters $(G_{k,k-1}, G_{k,N}, G_k)$, $k \in [1, N-1]$. □

Lemma 4.1.2. *A ZMNG $[y_k]$ is Markov iff it obeys*

$$y_k = M_{k,k-1} y_{k-1} + e_k, \quad k \in [1, N] \tag{4.3}$$

where $y_0 = e_0$ and $[e_k]$ ($Cov(e_k) = M_k$) is a zero-mean white NG sequence.

Theorem 4.1.3. *(Markov-induced CM_L (evolution) model) A ZMNG $[x_k]$ is reciprocal iff it can be modeled by a CM_L model (2.17) and (2.19) (for $c = N$) induced by a Markov evolution model (4.3), that is, iff the parameters $(G_{k,k-1}, G_{k,N}, G_k)$, $k \in [1, N-1]$, of the CM_L evolution model (2.17) of $[x_k]$ can be determined by the parameters $(M_{k,k-1}, M_k)$, $k \in [1, N]$, of a Markov evolution model (4.3), as*

$$G_{k,k-1} = M_{k,k-1} - G_{k,N} M_{N|k} M_{k,k-1} \tag{4.4}$$

$$G_{k,N} = G_k M'_{N|k} C^{-1}_{N|k} \tag{4.5}$$

$$G_k = (M_k^{-1} + M'_{N|k} C^{-1}_{N|k} M_{N|k})^{-1} \tag{4.6}$$

where $M_{N|k} = M_{N,N-1} \cdots M_{k+1,k}$, $C_{N|k} = \sum_{n=k}^{N-1} M_{N|n+1} M_{n+1} M'_{N|n+1}$, $k \in [1, N-1]$, and $M_{N|N} = I$, where $M_{k,k-1}$, $k \in [1, N]$, are square matrices, and M_k, $k \in [1, N]$, are positive definite with the dimension of x_k.

Proof. First, we show how (4.4)–(4.6) are obtained and prepare the setting for our proof.

Given the square matrices $M_{k,k-1}, k \in [1, N]$, and the positive definite matrices $M_k, k \in [1, N]$, there exists a ZMNG Markov sequence $[y_k]$ (Lemma 4.1.2):

$$y_k = M_{k,k-1} y_{k-1} + e^M_k, \quad k \in [1, N], \quad y_0 = e^M_0 \tag{4.7}$$

where $[e^M_k]$ is a zero-mean white NG sequence with covariances $M_k, k \in [0, N]$.

Since every Markov sequence is CM_L, we can obtain a CM_L model of $[y_k]$ as

$$y_k = G_{k,k-1} y_{k-1} + G_{k,N} y_N + e^y_k, \quad k \in [1, N-1] \tag{4.8}$$

where $[e^y_k]$ is a zero-mean white NG sequence with covariances $G_k, k \in [1, N-1], G^y_0, G^y_N$, and boundary condition

$$y_N = e^y_N, \quad y_0 = G^y_{0,N} y_N + e^y_0 \tag{4.9}$$

Parameters of (4.8) can be obtained as follows. By (4.7), we have $p(y_k|y_{k-1}) = \mathcal{N}(y_k; M_{k,k-1} y_{k-1}, M_k)$. Since $[y_k]$ is Markov, we have, for $\forall k \in [1, N-1]$,

$$\begin{aligned} p(y_k|y_{k-1}, y_N) &= \frac{p(y_k|y_{k-1}) p(y_N|y_k, y_{k-1})}{p(y_N|y_{k-1})} \\ &= \frac{p(y_k|y_{k-1}) p(y_N|y_k)}{p(y_N|y_{k-1})} \\ &= \mathcal{N}(y_k; G_{k,k-1} y_{k-1} + G_{k,N} y_N, G_k) \end{aligned} \tag{4.10}$$

and it turns out that $G_{k,k-1}$, $G_{k,N}$, and G_k are given by (4.4)–(4.6) [90], where we have $p(y_k|y_{k-1}) = \mathcal{N}(y_k; M_{k,k-1} y_{k-1}, M_k)$.

Now, we construct a sequence $[x_k]$ modeled by the same evolution model (4.8) as

$$x_k = G_{k,k-1} x_{k-1} + G_{k,N} x_N + e_k, \quad k \in [1, N-1] \tag{4.11}$$

where $[e_k]$ is a zero-mean white Gaussian sequence with nonsingular covariances G_k, and boundary condition

$$x_N = e_N, \quad x_0 = G_{0,N} x_N + e_0 \tag{4.12}$$

but with different parameters of the boundary condition (i.e., $(G_N, G_{0,N}, G_0) \neq (G^y_N, G^y_{0,N}, G^y_0)$). By Theorem 2.2.6, $[x_k]$ is a ZMNG CM_L sequence. (Note that parameters of (4.8) and (4.11) are the same ($G_{k,k-1}, G_{k,N}, G_k, k \in [1, N-1]$), but parameters of (4.9) ($G^y_{0,N}, G^y_0, G^y_N$) and (4.12) ($G_{0,N}, G_0, G_N$) are different.)

Sufficiency: we prove sufficiency; that is, a CM_L model with the parameters (4.4)–(4.6) is a reciprocal CM_L model. It suffices to show that the parameters (4.4)–(4.6) satisfy (3.24), and consequently $[x_k]$ is reciprocal. Substituting (4.4)–(4.6) in (3.24), for the right hand side of (3.24), we have

$$G'_{k+1,k}G^{-1}_{k+1}G_{k+1,N} = \\ M'_{N|k}C^{-1}_{N|k+1} - M'_{N|k}C^{-1}_{N|k+1}M_{N|k+1}(M^{-1}_{k+1} + M'_{N|k+1}C^{-1}_{N|k+1}M_{N|k+1})^{-1}M'_{N|k+1}C^{-1}_{N|k+1}$$

and for the left hand side of (3.24), we have $G^{-1}_k G_{k,N} = M'_{N|k}C^{-1}_{N|k} = M'_{N|k}\ (C_{N|k+1} + M_{N|k+1}M_{k+1}M'_{N|k+1})^{-1}$, where from the matrix inversion lemma it follows that (3.24) holds. Therefore, $[x_k]$ is reciprocal. So, equations (2.17) and (2.19) with (4.4)–(4.6) model a ZMNG reciprocal sequence.

Necessity: Let $[x_k]$ be ZMNG reciprocal. By Theorem 3.1.17 $[x_k]$ obeys (2.17) and (2.19) with (3.24). By Lemma 4.1.1, the set of reciprocal sequences modeled by a reciprocal CM_L evolution model contains Markov and non-Markov sequences (depending on the parameters of the boundary condition). So, a sequence modeled by a reciprocal CM_L evolution model and a boundary condition determined as in the proof of Lemma 4.1.1 (i.e., satisfying (4.2)) is actually a Markov sequence whose C^{-1} is (block) tri-diagonal (i.e., (2.36) with $D_0 = \cdots = D_{N-2} = 0$). Given this C^{-1}, we can obtain parameters of a Markov model (4.7) ($M_{k,k-1}, k \in [1, N]$, $M_k, k \in [0, N]$) of a Markov sequence with the given C^{-1} as follows. C^{-1} of a Markov sequence can be calculated in terms of parameters of a Markov CM_L model or in terms of parameters of a Markov model. Equating these two formulations of C^{-1}, parameters of the Markov model are obtained in terms of parameters of the Markov CM_L. Thus, for $k = N-2, N-3, \ldots, 0$,

$$M^{-1}_N = A_N \tag{4.13}$$

$$M_{N,N-1} = -M_N B'_{N-1} \tag{4.14}$$

$$M^{-1}_{k+1} = A_{k+1} - M'_{k+2,k+1}M^{-1}_{k+2}M_{k+2,k+1} \tag{4.15}$$

$$M_{k+1,k} = -M_{k+1}B'_k \tag{4.16}$$

$$M^{-1}_0 = A_0 - M'_{1,0}M^{-1}_1 M_{1,0} \tag{4.17}$$

where

$$A_0 = G^{-1}_0 + G'_{1,0}G^{-1}_1 G_{1,0} \tag{4.18}$$

$$A_k = G^{-1}_k + G'_{k+1,k}G^{-1}_{k+1}G_{k+1,k}, k \in [1, N-2] \tag{4.19}$$

$$A_{N-1} = G^{-1}_{N-1} \tag{4.20}$$

$$A_N = G^{-1}_N + \sum_{k=0}^{N-1} G'_{k,N}G^{-1}_k G_{k,N} \tag{4.21}$$

$$B_k = -G'_{k+1,k}G^{-1}_{k+1}, k \in [0, N-2] \tag{4.22}$$

$$B_{N-1} = -G^{-1}_{N-1}G_{N-1,N} \tag{4.23}$$

Following (4.10) to get a reciprocal CM_L model from this Markov model, we have (4.4)–(4.6).

What remains to be proven is that the parameters of the model obtained by (4.4)–(4.6) are the same as those of the CM_L model calculated directly based on the covariance function of $[x_k]$. By Theorem 2.2.6, the model constructed from (4.4)–(4.6) is a valid CM_L model. In addition, given a CM_L matrix (a positive definite cyclic (block) tri-diagonal matrix is a special CM_L matrix) as the C^{-1} of a sequence, the set of parameters of the CM_L evolution model and boundary condition of the sequence is unique (it can be seen by the almost sure uniqueness of a conditional expectation (Chapter 2)). Thus, the parameters (4.4)–(4.6) must be the same as those obtained directly from the covariance function of $[x_k]$. Thus, a ZMNG reciprocal sequence $[x_k]$ obeys (2.17) and (2.19) with (4.4)–(4.6). □

Note that by matrix inversion lemma, (4.6) is equivalent to $G_k = M_k - M_k M'_{N|k}(C_{N|k} + M_{N|k}M_k M'_{N|k})^{-1}M_{N|k}M_k$.

Note that Theorem 4.1.3 holds true for every combination of the parameters, i.e., square matrices $M_{k,k-1}$ and positive definite matrices $M_k, k \in [1,N]$. By (4.4)–(4.6), parameters of every reciprocal CM_L model are obtained from $M_{k,k-1}, M_k, k \in [1,N]$, which are parameters of a Markov evolution model (4.3). This is particularly useful for parameter design of a reciprocal CM_L model. In Chapter 7 we use Theorem 4.1.3 for parameter design of a CM_L model for motion trajectory modeling with destination information.

Markov sequences modeled by the same reciprocal evolution model of [18] were studied in [16]. This is an important topic in the theory of reciprocal processes [9]. In the following, Markov sequences modeled by the same CM_L evolution model (2.17) are studied and determined. Following the notion of a reciprocal transition density derived from a Markov transition density [9], a CM_L evolution model *induced* by a Markov model is defined as follows. A Markov sequence can be modeled by either a Markov model (4.3) or a CM_L model (2.17). Such a CM_L evolution model is called the CM_L evolution model *induced* by the Markov evolution model since parameters of the former can be obtained from those of the latter (see (4.10) or (4.13)–(4.23)). Definition 4.1.4 is for the Gaussian case.

Definition 4.1.4. *Consider a Markov evolution model* (4.3) *with parameters* $M_{k,k-1}, k \in [1,N]$, M_k, $k \in [1,N]$. *The* CM_L *(evolution) model* (2.17) *with parameters* $(G_{k,k-1}, G_{k,N}, G_k)$, $k \in [1,N-1]$, *given by* (4.4)–(4.6) *is called the* CM_L *(evolution) model induced by the Markov (evolution) model or simply the Markov-induced* CM_L *(evolution) model.*

Corollary 4.1.5. *A* CM_L *model* (2.17) *is for a reciprocal sequence iff it can be so induced by a Markov model* (4.3)*.*

Proof. See our proof of Theorem 4.1.3. □

By the proof of Theorem 4.1.3, given a reciprocal CM_L evolution model (2.17) (satisfying (3.24)), we can choose a boundary condition satisfying (4.2) and then obtain a Markov model (4.3) for a Markov sequence that obeys the given reciprocal CM_L evolution model (see (4.13)–(4.23)). Since parameters of the boundary condition (i.e., $G_{0,N}$, G_0, and G_N) satisfying (4.2) can take many values, there are many such Markov models and their parameters are given by (4.13)–(4.17).

The idea of obtaining a reciprocal evolution law from a Markov evolution law was used in [3], [9], and later for finite-state reciprocal sequences in [41], [27]. Also, [16] studied Markov sequences with the same reciprocal evolution model of [18]. Our contributions are different. First, our reciprocal CM_L model above is from the CM viewpoint. Second, Theorem 4.1.3 not only induces a reciprocal CM_L evolution model by a Markov evolution model, but also shows that *every* reciprocal CM_L evolution model can be induced by a Markov evolution model (by necessity and sufficiency of Theorem 4.1.3). This is important for application of reciprocal sequences (i.e., parameter design of a reciprocal CM_L model) because one usually has an intuitive understanding of Markov models (Chapter 7). Third, our proof of Theorem 4.1.3 is constructive and shows how a given reciprocal CM_L evolution model can be induced by a Markov evolution model. Fourth, our constructive proof of Theorem 4.1.3 gives all possible Markov evolution models by which a given reciprocal CM_L evolution model can be induced. Note that only one CM_L evolution model can be induced by a given Markov evolution model (it can be verified by (4.13)–(4.23)). However, a given reciprocal CM_L evolution model can be induced by many different Markov evolution models. This is because (4.2) holds for many different choices of parameters of the boundary condition (i.e., $G_{0,N}$, G_0, and G_N) each of which leads to a Markov model with parameters given by (4.13)–(4.17) (see the proof of necessity of Theorem 4.1.3).

4.1.2 Intersections of CM Classes

In some applications sequences with more than one CM property (i.e., belonging to more than one CM class) are desired. An example is trajectories with waypoint and destination information. A CM_L sequence can be used for modeling trajectories with destination information (Chapter 7). Assume not only the destination density (at time N) but also the density of the state at a time $k_2(< N)$ is known (i.e., waypoint information). First, consider only the waypoint information at k_2 (without destination information). In other words, we know the state density at k_2 but not after. With a CM evolution law between 0 and k_2, such trajectories can be modeled as a $[0, k_2]$-CM_L sequence. Now, consider only the destination information (density) without waypoint information. Such trajectories can be modeled as a CM_L sequence. Then, trajectories with waypoint and destination information can be modeled as a sequence being both $[0, k_2]$-CM_L and CM_L, denoted as $CM_L \cap [0, k_2]$-CM_L. In other words, the sequence has both the CM_L property and the $[0, k_2]$-CM_L property. Studying the evolution of other sequences belonging to more than one CM class, for example $CM_L \cap [k_1, N]$-CM_F, is also useful for studying reciprocal sequences. The NG reciprocal sequence is equivalent to $CM_L \cap CM_F$ (Chapter 3). Proposition 4.1.6 below presents a dynamic model of $CM_L \cap [k_1, N]$-CM_F sequences, based on which one can see a full spectrum of models from a CM_L sequence to a reciprocal sequence.

Proposition 4.1.6. *A ZMNG $[x_k]$ is $CM_L \cap [k_1, N]$-CM_F iff it obeys* (2.17) *and* (2.19) *with* ($\forall k \in [k_1 + 1, N - 2]$)

$$G_k^{-1} G_{k,N} = G'_{k+1,k} G_{k+1}^{-1} G_{k+1,N} \tag{4.24}$$

Proof. A ZMNG CM_L sequence has a CM_L model (2.17) and (2.19) (Theorem 2.2.6). Also, a NG sequence is $[k_1, N]$-CM_F iff its C^{-1} has the $[k_1, N]$-CM_F form (Corollary 3.1.13). Then, a sequence is $CM_L \cap [k_1, N]$-CM_F iff it obeys (2.17) and (2.19), where C^{-1} of the sequence has the $[k_1, N]$-CM_F form, which is equivalent to (4.24) (see Appendix B.1 for calculation of C^{-1} in terms of parameters of a CM_L model). □

Proposition 4.1.6 shows how models change from a CM_L model to a reciprocal CM_L model for $k_1 = 0$ (compare (4.24) and (3.24) (for $c = N$)). Note that CM_L and $CM_L \cap [k_1, N]$-CM_F, $k_1 \in [N - 2, N]$ are equivalent (Subsection 2.1.1).

Following the idea of the proof of Proposition 4.1.6, we can obtain models for intersections of different CM classes, for example $CM_c \cap [k_1, k_2]$-$CM_c \cap [m_1, m_2]$-CM_c sequences. However, the above approach does not lead to simple results in some cases, e.g., $CM_L \cap [0, k_2]$-CM_L sequences. A different way of obtaining a model for $CM_L \cap [0, k_2]$-CM_L sequences is presented in Proposition 4.1.7.

Proposition 4.1.7. *A ZMNG $[x_k]$ is $CM_L \cap [0, k_2]$-CM_L iff*

$$x_k = G_{k,k-1} x_{k-1} + G_{k,k_2} x_{k_2} + e_k, k \in [1, k_2 - 1] \tag{4.25}$$

$$x_{k_2} = e_{k_2}, \quad x_0 = G_{0,k_2} x_{k_2} + e_0 \tag{4.26}$$

$$x_N = \sum_{i=0}^{k_2} G_{N,i} x_i + e_N \tag{4.27}$$

$$x_k = G_{k,k-1} x_{k-1} + G_{k,N} x_N + e_k, k \in [k_2 + 1, N - 1] \tag{4.28}$$

where $[e_k]$ ($Cov(e_k) = G_k$) is a zero-mean white NG sequence,

$$G'_{N,j} G_N^{-1} G_{N,i} = 0 \tag{4.29}$$

$$G_l^{-1} G_{l,k_2} = G'_{l+1,l} G_{l+1}^{-1} G_{l+1,k_2} + G'_{N,l} G_N^{-1} G_{N,k_2} \tag{4.30}$$

$j = 0, \ldots, k_2 - 3$, $i = j + 2, \ldots, k_2 - 1$, *and* $l = 0, \ldots, k_2 - 2$.

Proof. Necessity: Let $[x_k]$ be a ZMNG $CM_L \cap [0,k_2]$-CM_L sequence. Let $p(\cdot)$ and $p(\cdot|\cdot)$ be its density and conditional density, respectively. Then,

$$x_{k_2} \sim p(x_{k_2}) \tag{4.31}$$
$$x_0 \sim p(x_0|x_{k_2}) \tag{4.32}$$

Since $[x_k]$ is $CM_L \cap [0,k_2]$-CM_L, it is $[0,k_2]$-CM_L. Thus, for $k \in [1, k_2-1]$,

$$x_k \sim p(x_k|x_0,\ldots,x_{k-1},x_{k_2}) = p(x_k|x_{k-1},x_{k_2}) \tag{4.33}$$

Also, since $[x_k]$ is CM_L, for $k \in [k_2+1, N]$,

$$x_N \sim p(x_N|x_0,\ldots,x_{k_2}) \tag{4.34}$$
$$x_k \sim p(x_k|x_0,\ldots,x_{k-1},x_N) = p(x_k|x_{k-1},x_N) \tag{4.35}$$

According to (4.31)–(4.32), we have $x_{k_2} = e_{k_2}$ and $x_0 = G_{0,k_2}x_{k_2}+e_0$, where e_0 and e_{k_2} are uncorrelated ZMNG with nonsingular covariances G_0 and G_{k_2}, $G_{0,k_2} = C_{0,k_2}C_{k_2}^{-1}$, $G_{k_2} = C_{k_2}$, $G_0 = C_0 - C_{0,k_2}C_{k_2}^{-1}C'_{0,k_2}$, and C_{l_1,l_2} is the covariance function of $[x_k]$. For $k \in [1,k_2-1]$, by (4.33), we have $x_k = G_{k,k-1}x_{k-1} + G_{k,k_2}x_{k_2} + e_k$, $G_k = \text{Cov}(e_k)$ (Theorem 2.2.6), $[G_{k,k-1}, G_{k,k_2}] = [C_{k,k-1}, C_{k,k_2}] \begin{bmatrix} C_{k-1} & C_{k-1,k_2} \\ C_{k_2,k-1} & C_{k_2} \end{bmatrix}^{-1}$, and $G_k = C_k - [C_{k,k-1}, C_{k,k_2}] \begin{bmatrix} C_{k-1} & C_{k-1,k_2} \\ C_{k_2,k-1} & C_{k_2} \end{bmatrix}^{-1}$ $\cdot [C_{k,k-1}, C_{k,k_2}]'$

For $k \in [k_2+1, N]$, by (4.34), we have $x_N = \sum_{i=0}^{k_2} G_{N,i}x_i + e_N$, $G_N = \text{Cov}(e_N)$, and

$$[G_{N,0}, G_{N,1}, \ldots, G_{N,k_2}] = C_{[N+1:N+1,1:k_2+1]}(C_{[1:k_2+1,1:k_2+1]})^{-1}$$
$$G_N = C_N - C_{[N+1:N+1,1:k_2+1]}(C_{[1:k_2+1,1:k_2+1]})^{-1}C'_{[N+1:N+1,1:k_2+1]}$$

Here, $C_{[r_1:r_2,c_1:c_2]}$ denotes the submatrix of the covariance matrix C of $[x_k]$ including the block rows r_1 to r_2 and the block columns c_1 to c_2. [1]

By (4.35), we have $x_k = G_{k,k-1}x_{k-1} + G_{k,N}x_N + e_k, k \in [k_2+1, N-1]$, $G_k = \text{Cov}(e_k)$, and

$$[G_{k,k-1}, G_{k,N}] = [C_{k,k-1}, C_{k,N}] \begin{bmatrix} C_{k-1} & C_{k-1,N} \\ C_{N,k-1} & C_N \end{bmatrix}^{-1}$$
$$G_k = C_k - [C_{k,k-1}, C_{k,N}] \begin{bmatrix} C_{k-1} & C_{k-1,N} \\ C_{N,k-1} & C_N \end{bmatrix}^{-1} [C_{k,k-1}, C_{k,N}]'$$

In the above, $[e_k]$ is a zero-mean white NG sequence with covariances G_k.

Now we show that (4.29)–(4.30) hold. We construct C^{-1} of the whole sequence $[x_k]$ and obtain (4.29)–(4.30) from the fact that C^{-1} is both CM_L and $[0,k_2]$-CM_L. $[x_k]_0^{k_2}$ obeys (4.25)–(4.26). So, by Theorem 2.2.6, $[x_k]_0^{k_2}$ is CM_L. Then, by Theorem 2.3.5, $(C_{[1:k_2+1,1:k_2+1]})^{-1}$ is CM_L for every value of parameters of (4.25)–(4.26) (i.e., C^{-1} is $[0,k_2]$-CM_L). C^{-1} of $[x_k]$ is calculated by stacking (4.25)–(4.28) as follows. We have

$$\mathcal{G}x = e \tag{4.36}$$

where $x \triangleq [x'_0, x'_1, \ldots, x'_N]'$, $e \triangleq [e'_0, e'_1, \ldots, e'_N]'$, $\mathcal{G} = \begin{bmatrix} \mathcal{G}_{11} & 0 \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{bmatrix}$,

[1]Note that C is an $(N+1)\times(N+1)$ matrix for a scalar sequence.

$$\mathcal{G}_{21} = \begin{bmatrix} 0 & \cdots & 0 & -G_{k_2+1,k_2} \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -G_{N,0} & \cdots & -G_{N,k_2-1} & -G_{N,k_2} \end{bmatrix}$$

$$\mathcal{G}_{11} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & -G_{0,k_2} \\ -G_{1,0} & I & 0 & \cdots & 0 & -G_{1,k_2} \\ 0 & -G_{2,0} & I & 0 & \cdots & -G_{2,k_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -G_{k_2-1,k_2-2} & I & -G_{k_2-1,k_2} \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix}$$

$$\mathcal{G}_{22} = \begin{bmatrix} I & 0 & \cdots & 0 & -G_{k_2+1,N} \\ -G_{k_2+2,k_2+1} & I & 0 & \cdots & -G_{k_2+2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & -G_{N-1,N-2} & I & -G_{N-1,N} \\ 0 & \cdots & 0 & 0 & I \end{bmatrix}$$

Then,

$$C^{-1} = \mathcal{G}' G^{-1} \mathcal{G} \tag{4.37}$$

where $G = \text{diag}(G_0, G_1, \ldots, G_N)$. Since $[x_k]$ is CM_L, C^{-1} has the CM_L form, which is equivalent to (4.29)–(4.30).

Sufficiency: We need to show that a sequence modeled by (4.25)-(4.30) is $CM_L \cap [0, k_2]$-CM_L, that is, its C^{-1} has both CM_L and $[0, k_2]$-CM_L forms. Since $[x_k]_0^{k_2}$ obeys (4.25)–(4.26), $(C_{[1:k_2+1,1:k_2+1]})^{-1}$ has the CM_L form for every choice of parameters of (4.25)–(4.26) (Theorem 2.2.6 and Theorem 2.3.5). So, $[x_k]$ governed by (4.25)-(4.30) is $[0, k_2]$-CM_L. Also, C^{-1} can be calculated by (4.37). It can be seen that (4.29)–(4.30) is equivalent to C^{-1} having the CM_L form. Thus, a sequence modeled by (4.25)-(4.30) is $CM_L \cap [0, k_2]$-CM_L. The Gaussianity of $[x_k]$ follows clearly from linearity of (4.25)-(4.28). Also, $[x_k]$ is nonsingular due to (4.37), the nonsingularity of $\mathcal{G}$, and the positive definiteness of G. □

4.2 Representations of CM and Reciprocal Sequences

A representation of NG continuous-time CM processes in terms of a Wiener process and an uncorrelated NG vector was presented in [29]. Inspired by [29], we show that a NG CM_c sequence can be represented by a sum of a NG Markov sequence and an uncorrelated NG vector. We also show how to use a NG Markov sequence and an uncorrelated NG vector to construct a NG CM_c sequence. This is useful for construction of a CM_L/CM_F model in application.

Proposition 4.2.1. *A ZMNG $[x_k]$ is CM_c iff it can be represented as*

$$x_k = y_k + \Gamma_k x_c, \quad k \in [0, N] \setminus \{c\} \tag{4.38}$$

where $[y_k] \setminus \{y_c\}$ [2] is a ZMNG Markov sequence, x_c is a ZMNG vector uncorrelated with $[y_k] \setminus \{y_c\}$, and Γ_k are some matrices.

[2] For $c = N$, $[y_k] \setminus \{y_c\} = [y_k]_0^{N-1}$, and for $c = 0$, $[y_k] \setminus \{y_c\} = [y_k]_1^N$.

Proof. Let $c = N$. Necessity: It is shown that a ZMNG CM_L $[x_k]$ can be represented as (4.38). $[x_k]$ obeys

$$x_k = G_{k,k-1}x_{k-1} + G_{k,N}x_N + e_k, \quad k \in [1, N-1] \tag{4.39}$$
$$x_0 = G_{0,N}x_N + e_0 \tag{4.40}$$
$$x_N = e_N \tag{4.41}$$

where $[e_k]$ ($G_k = \text{Cov}(e_k)$) is zero-mean white NG.

According to (4.40), we consider $y_0 = e_0$ and $\Gamma_0 = G_{0,N}$. So, $x_0 = y_0 + \Gamma_0 x_N$. For $k \in [1, N-1]$, we have

$$\begin{aligned} x_k &= G_{k,k-1}x_{k-1} + G_{k,N}x_N + e_k \\ &= G_{k,k-1}(y_{k-1} + \Gamma_{k-1}x_N) + G_{k,N}x_N + e_k \\ &= G_{k,k-1}y_{k-1} + e_k + (G_{k,k-1}\Gamma_{k-1} + G_{k,N})x_N \end{aligned}$$

By induction, $[x_k]$ can be represented as $x_k = y_k + \Gamma_k x_N$, $k \in [0, N-1]$, where for $k \in [1, N-1]$, $y_k = U_{k,k-1}y_{k-1} + e_k$, $U_{k,k-1} = G_{k,k-1}$, $\Gamma_k = G_{k,k-1}\Gamma_{k-1} + G_{k,N}$, $y_0 = e_0$, $\Gamma_0 = G_{0,N}$, and x_N is uncorrelated with the Markov sequence $[y_k]_0^{N-1}$, because x_N is uncorrelated with $[e_k]_0^{N-1}$.

What remains is to show the nonsingularity of $[y_k]_0^{N-1}$ and the random vector x_N. Since the sequence $[x_k]$ is nonsingular, x_N is nonsingular. Also, we have $y_0 = e_0$. In addition, the covariances $G_k, k \in [0, N]$, are nonsingular. Thus, $U_k = \text{Cov}(e_k)$, $k \in [0, N-1]$, are all nonsingular. Similar to (4.37), we have $C^{\mathsf{y}} = \text{Cov}(\mathsf{y}) = W^{-1}UW'^{-1}$, where $\mathsf{y} = [y'_0, y'_1, \ldots, y'_{N-1}]'$, $U = \text{diag}(U_0, U_1, \ldots, U_{N-1})$ and W is a nonsingular matrix. Therefore, $[y_k]_0^{N-1}$ is nonsingular because U and W are nonsingular.

Sufficiency: We show that given a ZMNG Markov sequence $[y_k]_0^{N-1}$ uncorrelated with a ZMNG vector x_N, $[x_k]$ constructed as $x_k = y_k + \Gamma_k x_N, k \in [0, N-1]$ is a ZMNG CM_L sequence, where Γ_k are some matrices. Therefore, it suffices to show that $[x_k]$ obeys (2.17) and (2.19). Since $[y_k]_0^{N-1}$ is a ZMNG Markov sequence, it obeys (Lemma 4.1.2) $y_k = U_{k,k-1}y_{k-1} + e_k$, $k \in [1, N-1]$, $y_0 = e_0$, where $[e_k]_0^{N-1}$ is a zero-mean white NG sequence with covariances U_k.

We have $x_0 = y_0 + \Gamma_0 x_N$. So, consider $G_{0,N} = \Gamma_0$. Then, for $k \in [1, N-1]$, we have

$$\begin{aligned} x_k &= y_k + \Gamma_k x_N = U_{k,k-1}y_{k-1} + e_k + \Gamma_k x_N \\ &= U_{k,k-1}x_{k-1} + (\Gamma_k - U_{k,k-1}\Gamma_{k-1})x_N + e_k \end{aligned} \tag{4.42}$$

We consider $G_{k,k-1} = U_{k,k-1}$ and $G_{k,N} = \Gamma_k - U_{k,k-1}\Gamma_{k-1}$. Covariances U_k, $k \in [0, N-1]$ and $\text{Cov}(x_N)$ are nonsingular. So, covariances $G_k = \text{Cov}(e_k), k \in [0, N]$ (let $e_N = x_N$), are all nonsingular. So, $[x_k]$ is nonsingular (it can be shown similar to (4.37)). Thus, by (4.42), it can be seen that $[x_k]$ obeys (2.17) (note that $[e_k]$ is white). So, $[x_k]$ is a ZMNG CM_L sequence.

For $c = 0$ we have a parallel proof. So, we skip the details and only present some results required later. Necessity: Let $c = 0$. The proof is based on the CM_F model. Let $[x_k]$ be a ZMNG CM_F sequence governed by (2.17)–(2.18) (for $c = 0$). It is possible to represent $[x_k]$ as (4.38) with the Markov sequence $[y_k]_1^N$ governed by $y_k = U_{k,k-1}y_{k-1} + e_k$, $k \in [2, N]$, where for $k \in [2, N]$, $U_{k,k-1} = G_{k,k-1}$, $\Gamma_1 = 2G_{1,0}$, $\Gamma_k = G_{k,k-1}\Gamma_{k-1} + G_{k,0}$.

Sufficiency: Let $[y_k]_1^N$ be a ZMNG Markov sequence governed by $y_k = U_{k,k-1}y_{k-1} + e_k$, $k \in [2, N]$, where $[e_k]_1^N$ (let $y_1 = e_1$) is a zero-mean white NG sequence with covariances U_k. Also, let x_0 be a ZMNG vector uncorrelated with the sequence $[y_k]_1^N$. It can be shown that the sequence $[x_k]$ constructed by (4.38) obeys (2.17)–(2.18) (for $c = 0$), where for $k \in [2, N]$, $G_{k,k-1} = U_{k,k-1}$, $G_{1,0} = \frac{1}{2}\Gamma_1$, and $G_{k,0} = \Gamma_k - U_{k,k-1}\Gamma_{k-1}$. □

Proposition 4.2.1 makes a key fact behind the NG CM_c sequence clear, that is, every NG CM_c sequence can be represented as a sum of two components: a NG Markov sequence and an uncorrelated NG vector. As a result, it provides some insight and guideline for design of CM_c models in application. Below we explain the idea for designing a CM_L model for motion trajectory modeling with destination information. A CM_L model is more general than a reciprocal CM_L model. Consequently, the following guideline for CM_L model design includes the approach of Theorem 4.1.3 as a special case. The guideline is as follows. First, a Markov model (e.g., a nearly constant velocity model) with the given origin distribution (without considering other information) is considered. The sequence modeled by this model is $[y_k]_0^{N-1}$ in (4.38). Assume the destination (distribution of x_N) is known. Then, based on Γ_k, the Markov sequence $[y_k]_0^{N-1}$ is modified to satisfy the available information in the problem (e.g., about the general form of trajectories) leading to the desired trajectories $[x_k]$ which end up at the destination. A direct attempt to design parameters of a CM_L model for this problem is hard. However, the above guideline makes parameter design easier and intuitive. In addition, one can learn Γ_k (which shows the impact of the destination) from a set of trajectories. In the following, the representation of Proposition 4.2.1 is studied further to provide insight and tools for its application.

The following representation of CM_c matrices is a by-product of Proposition 4.2.1.

Corollary 4.2.2. *Let C be an $(N+1)d \times (N+1)d$ positive definite block matrix (with $(N+1)$ blocks in each row/column and each block with dimension $d \times d$). C^{-1} is CM_c iff*

$$C = B + \Gamma D \Gamma' \tag{4.43}$$

where D is a $d \times d$ positive definite matrix and (i) for $c = N$, $B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\Gamma = \begin{bmatrix} S \\ I \end{bmatrix}$, (ii) for $c = 0$, $B = \begin{bmatrix} 0 & 0 \\ 0 & B_1 \end{bmatrix}$, $\Gamma = \begin{bmatrix} I \\ S \end{bmatrix}$, where $(B_1)^{-1}$ is $Nd \times Nd$ block tri-diagonal, S is $Nd \times d$, and I is the $d \times d$ identity matrix.

Proof. Let $c = N$. Necessity: By Theorem 2.3.5, for every CM_L matrix C^{-1}, there exists a ZMNG CM_L sequence $[x_k]$ with the covariance C. By Proposition 4.2.1, we have

$$x = y + \Gamma x_N \tag{4.44}$$

where $x \triangleq [x_0', x_1', \ldots, x_{N-1}', x_N']$, $\mathsf{y} \triangleq [y_0', y_1', \ldots, y_{N-1}']'$, $y \triangleq [\mathsf{y}', 0]'$, $S \triangleq [\Gamma_0', \Gamma_1', \ldots, \Gamma_{N-1}']'$, $\Gamma \triangleq [S', I]'$, and $[y_k]_0^{N-1}$ is a ZMNG Markov sequence uncorrelated with the ZMNG vector x_N. Then, by (4.44), we have

$$\text{Cov}(x) = \text{Cov}(y) + \Gamma \text{Cov}(x_N) \Gamma' \tag{4.45}$$

because y and x_N are uncorrelated. Then, (4.45) leads to (4.43), where $C \triangleq \text{Cov}(x)$, $B \triangleq \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} = \text{Cov}(y)$, $B_1 \triangleq \text{Cov}(\mathsf{y})$, $D \triangleq \text{Cov}(x_N)$, and by Remark 3.1.16, $(B_1)^{-1}$ is block tri-diagonal. Therefore, for every CM_L matrix C^{-1} we have (4.43).

Sufficiency: Let $(B_1)^{-1}$ be an $Nd \times Nd$ block tri-diagonal matrix, D be a $d \times d$ positive definite matrix, and S be an $Nd \times d$ matrix. By Theorem 2.3.5, for every $Nd \times Nd$ block tri-diagonal matrix $(B_1)^{-1}$, there exists a Gaussian Markov sequence $[y_k]_0^{N-1}$ with $(C^{\mathsf{y}})^{-1} = (B_1)^{-1}$, where $C^{\mathsf{y}} = \text{Cov}(\mathsf{y})$ and $\mathsf{y} = [y_0', y_1', \ldots, y_{N-1}']'$. Also, given a $d \times d$ positive definite matrix D, there exists a Gaussian vector x_N with $\text{Cov}(x_N) = D$. Let x_N and $[y_k]_0^{N-1}$ be uncorrelated. By Proposition 4.2.1, $[x_k]$ constructed by (4.44) is a CM_L sequence. Also, by Theorem 2.3.5, C^{-1} of $[x_k]$ is a CM_L matrix. With $C \triangleq \text{Cov}(x)$, (4.43) follows from (4.45). Thus, for every block tri-diagonal matrix $(B_1)^{-1}$, every positive definite matrix D, and every matrix S, C^{-1} is a CM_L matrix. The proof for $c = 0$ is similar. □

Corollary 4.2.3. *For every CM_c sequence, the representation* (4.38) *is unique.*

Proof. Let $c = N$, and $[x_k]$ be a CM_L sequence governed by (2.17) with parameters $(G_{k,k-1}, G_{k,N}, G_k)$, $k \in [1, N-1]$, and (2.19) with the parameters $(G_{0,N}, G_0, G_N)$. By Proposition 4.2.1, $[x_k]$ can be represented as (4.38). Parameters (denoted by $U_{k,k-1}, k \in [1, N-1]$, $U_k, k \in [0, N-1]$) of the Markov model (4.3) of $[y_k]_0^{N-1}$, covariance of x_N denoted by D, and the matrices Γ_k, $k \in [0, N-1]$, can be calculated in terms of the parameters of the CM_L model as follows (see the proof of Proposition 4.2.1):

$$D = G_N, \quad \Gamma_0 = G_{0,N} \tag{4.46}$$

$$U_k = G_k, \quad k \in [0, N-1] \tag{4.47}$$

$$U_{k,k-1} = G_{k,k-1}, \quad k \in [1, N-1] \tag{4.48}$$

$$\Gamma_k = G_{k,k-1}\Gamma_{k-1} + G_{k,N}, \quad k \in [1, N-1] \tag{4.49}$$

Now, assume that there exists a different representation of the form (4.38) for $[x_k]$. Denote parameters of the corresponding Markov model by $\tilde{U}_{k,k-1}, k \in [1, N-1]$, $\tilde{U}_k, k \in [0, N-1]$, and the weight matrices by $\tilde{\Gamma}_k$, $k \in [0, N-1]$ (covariance of x_N is D). By the proof of Proposition 4.2.1, parameters of the corresponding CM_L model are

$$G_{0,N} = \tilde{\Gamma}_0, \quad G_N = D \tag{4.50}$$

$$G_{k,k-1} = \tilde{U}_{k,k-1}, \quad k \in [1, N-1] \tag{4.51}$$

$$G_{k,N} = \tilde{\Gamma}_k - \tilde{U}_{k,k-1}\tilde{\Gamma}_{k-1}, \quad k \in [1, N-1] \tag{4.52}$$

$$G_k = \tilde{U}_k, \quad k \in [0, N-1] \tag{4.53}$$

Parameters of a CM_L model of a CM_L sequence are unique (Appendix B.1). Comparing (4.46)-(4.49) and (4.50)-(4.53), it can be seen that the parameters $\tilde{U}_{k,k-1}, k \in [1, N-1]$, $\tilde{U}_k, k \in [0, N-1]$, and $\tilde{\Gamma}_k$, $k \in [0, N-1]$, are the same as $U_{k,k-1}, k \in [1, N-1]$, $U_k, k \in [0, N-1]$, and Γ_k, $k \in [0, N-1]$. In other words, parameters of the representation (4.38) are unique. Uniqueness of (4.38) for $c = 0$ can be proven similarly. □

Based on a valuable observation, [29] discussed the relationship between Gaussian CM and Gaussian reciprocal processes. Then, based on the obtained relationship, [29] presented a representation of NG reciprocal processes. It was shown in Chapter 3 that the relationship between Gaussian CM and Gaussian reciprocal processes presented in [29] was incomplete, that is, the presented condition was not sufficient for a Gaussian process to be reciprocal (although [29] stated that it was sufficient, which has not been corrected so far). Then, the relationship between CM and reciprocal processes for the general (Gaussian/non-Gaussian) case was presented (Theorem 3.1.5). In addition, it was shown that CM_L in Theorem 3.1.5 was the missing part in the results of [29]. Consequently, it can be seen that the representation presented in [29] is not sufficient for a NG process to be reciprocal and its missing part is the representation of CM_L processes.

In the following, we present a simple necessary and sufficient representation of NG reciprocal sequences from the CM viewpoint. It demonstrates the significance of studying reciprocal sequences from the CM viewpoint.

Proposition 4.2.4. *A ZMNG $[x_k]$ is reciprocal iff it can be represented as both*

$$x_k = y_k^L + \Gamma_k^L x_N, \quad k \in [0, N-1] \tag{4.54}$$

$$x_k = y_k^F + \Gamma_k^F x_0, \quad k \in [1, N] \tag{4.55}$$

where $[y_k^L]_0^{N-1}$ and $[y_k^F]_1^N$ are ZMNG Markov sequences, x_N and x_0 are ZMNG vectors uncorrelated with $[y_k^L]_0^{N-1}$ and $[y_k^F]_1^N$, respectively, and Γ_k^L and Γ_k^F are some matrices.

Proof. A NG $[x_k]$ is reciprocal iff it is both CM_L and CM_F (Theorem 2.3.5). On the other hand, $[x_k]$ is CM_L (CM_F) iff it can be represented as (4.54) ((4.55)) (Proposition 4.2.1). So, $[x_k]$ is reciprocal iff it can be represented as both (4.54) and (4.55). □

By (4.54)–(4.55) the relation between sample paths of the two Markov sequences is $y_k^L + \Gamma_k^L x_N = y_k^F + \Gamma_k^F x_0$, $k \in [1, N-1]$, $y_0^L + \Gamma_0^L x_N = x_0$, $x_N = y_N^F + \Gamma_N^F x_0$.

The following representation of cyclic block tri-diagonal matrices is a by-product of Proposition 4.2.4.

Corollary 4.2.5. *Let C be an $(N+1)d \times (N+1)d$ positive definite block matrix (with $(N+1)$ blocks in each row/column and each block with dimension $d \times d$). Then, C^{-1} is cyclic block tri-diagonal iff*

$$C = B^L + \Gamma^L D^L (\Gamma^L)' = B^F + \Gamma^F D^F (\Gamma^F)' \tag{4.56}$$

where D^L and D^F are $d \times d$ positive definite matrices, $B^L = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\Gamma^L = \begin{bmatrix} S_1 \\ I \end{bmatrix}$, $B^F = \begin{bmatrix} 0 & 0 \\ 0 & B_2 \end{bmatrix}$, $\Gamma^F = \begin{bmatrix} I \\ S_2 \end{bmatrix}$, $(B_1)^{-1}$ and $(B_2)^{-1}$ are $Nd \times Nd$ block tri-diagonal, S_1 and S_2 are $Nd \times d$, and I is the $d \times d$ identity matrix.

Proof. Necessity: Let C^{-1} be a positive definite cyclic block tri-diagonal matrix. So, C^{-1} is CM_L and CM_F. Then, by Corollary 4.2.2 we have (4.56). Sufficiency: Let a positive definite matrix C be written as (4.56). By Corollary 4.2.2, C^{-1} is CM_L and CM_F and consequently cyclic block tri-diagonal. □

The reciprocal sequence is an important special CM_L (CM_F) sequence. So, it is important to know under what conditions the representation (4.38) is for a reciprocal sequence.

Proposition 4.2.6. *Let $[y_k] \backslash \{y_c\}$, $c \in \{0, N\}$, be a ZMNG Markov sequence, $y_k = U_{k,k-1} y_{k-1} + e_k, k \in [1, N] \setminus \{a\}$,*

$$a = \begin{cases} 1 & \text{if } c = 0 \\ N & \text{if } c = N \end{cases}, \quad r = \begin{cases} 1 & \text{if } c = 0 \\ N-1 & \text{if } c = N \end{cases}$$

where $[e_k] \setminus \{e_c\}$ is a zero-mean white NG sequence with covariances U_k (for $c = 0$ we have $e_1 = y_1$; for $c = N$ we have $e_0 = y_0$). Also, let x_c be a ZMNG vector with a covariance C_c uncorrelated with the Markov sequence $[y_k] \setminus \{y_c\}$. Let $[x_k]$ be constructed as

$$x_k = y_k + \Gamma_k x_c, \quad k \in [0, N] \setminus \{c\} \tag{4.57}$$

where Γ_k are some matrices. Then, $[x_k]$ is reciprocal iff $\forall k \in [1, N-1] \setminus \{r\}$,

$$U_k^{-1}(\Gamma_k - U_{k,k-1}\Gamma_{k-1}) = U'_{k+1,k} U_{k+1}^{-1}(\Gamma_{k+1} - U_{k+1,k}\Gamma_k) \tag{4.58}$$

Moreover, $[x_k]$ is Markov iff in addition to (4.58), we have

$$(U_0)^{-1}\Gamma_0 = U'_{1,0} U_1^{-1}(\Gamma_1 - U_{1,0}\Gamma_0), \ (\text{for } c = N) \tag{4.59}$$

$$\Gamma_N - U_{N,N-1}\Gamma_{N-1} = 0, \ (\text{for } c = 0) \tag{4.60}$$

Proof. By Proposition 4.2.1, $[x_k]$ constructed by (4.57) is a CM_c sequence. Parameters of the CM_L model (i.e., $c = N$) are calculated by (4.50)–(4.53) (($\tilde{U}_{k,k-1}, k \in [1, N-1]$, $\tilde{U}_k, k \in [0, N-1]$ and $\tilde{\Gamma}_k, k \in [0, N-1]$ are replaced by $U_{k,k-1}, k \in [1, N-1]$, $U_k, k \in [0, N-1]$, and $\Gamma_k, k \in [0, N-1]$). Parameters of the CM_F model (i.e., $c = 0$) are calculated as $G_{k,k-1} = U_{k,k-1}$, $k \in [2, N]$, $G_k = U_k$, $k \in [1, N]$, $G_{1,0} = \frac{1}{2}\Gamma_1$, $G_0 = D$, $G_{k,0} = \Gamma_k - U_{k,k-1}\Gamma_{k-1}$, $k \in [2, N]$. Then, by Proposition 3.1.17, the CM_c sequence $[x_k]$ is reciprocal iff (4.58) holds. Also, $[x_k]$ is Markov iff in addition to (4.58), (4.59) holds for $c = N$ and (4.60) for $c = 0$. □

Due to their importance in design of CM_c dynamic models, the main elements of representation (4.38) are formally defined.

Definition 4.2.7. *In* (4.38)*,* $[y_k] \setminus \{y_c\}$ *is called an underlying Markov sequence and its Markov evolution model (i.e., its Markov model without considering the initial condition) is called an underlying Markov (evolution) model. Also,* $[x_k]$ *is called a* CM_c *sequence constructed from the underlying Markov sequence and its* CM_c *evolution model (i.e., the* CM_c *model of* $[x_k]$ *without considering its boundary condition) is called a* CM_c *(evolution) model constructed from the underlying Markov (evolution) model.*

Corollary 4.2.8. *For* CM_c *models, having the same underlying Markov evolution model is equivalent to having the same* $G_{k,k-1}, G_k$*,* $\forall k \in [1, N] \setminus \{a\}$ *(*$a = N$ *for* $c = N$*, and* $a = 1$ *for* $c = 0$*).*

Proof. Given a Markov evolution model with parameters $U_{k,k-1}, U_k, k \in [1, N] \setminus \{a\}$, by our proof of Proposition 4.2.1, parameters of a CM_c evolution model constructed from the Markov evolution model are $G_{k,k-1} = U_{k,k-1}$, $G_{k,c} = \Gamma_k - U_{k,k-1}\Gamma_{k-1}$, $G_k = U_k$, $k \in [1, N] \setminus \{a\}$. Clearly all CM_c models so constructed have the same $G_{k,k-1}, G_k, k \in [1, N] \setminus \{a\}$.

For a CM_c evolution model with the parameters $G_{k,k-1}, G_{k,c}, G_k$, $\forall k \in [1, N] \setminus \{a\}$, parameters of its underlying Markov evolution model are uniquely determined as (see the proof of Proposition 4.2.1)

$$U_{k,k-1} = G_{k,k-1}, \quad U_k = G_k, \quad k \in [1, N] \setminus \{a\} \tag{4.61}$$

So, CM_c evolution models with the same $G_{k,k-1}, G_k$, $\forall k \in [1, N] \setminus \{a\}$, are constructed from the same underlying Markov evolution model. □

In the following, we try to distinguish between two concepts which are both useful in the application of CM_L and reciprocal sequences: 1) a CM_L evolution model *induced* by a Markov evolution model (Definition 4.1.4) and 2) a CM_L evolution model *constructed* from its underlying Markov evolution model (Definition 4.2.7).

By Theorem 4.1.3, a CM_L evolution model induced by a Markov evolution model is actually a reciprocal CM_L evolution model. In other words, non-reciprocal CM_L evolution models can not be so induced (with (4.4)–(4.6)) by any Markov evolution model. By Corollary 4.1.5, every reciprocal CM_L evolution model can be induced by a Markov evolution model. However, the corresponding Markov evolution model is not unique. In addition, every Markov sequence modeled by a Markov evolution model is also modeled by the CM_L evolution model induced by the Markov evolution model.

Every CM_L evolution model can be constructed from its underlying Markov evolution model, which is unique (Corollary 4.2.3). So, an underlying Markov evolution model plays a fundamental role in constructing a CM_L evolution model. However, an underlying Markov sequence is not modeled by the constructed CM_L evolution model.

The underlying Markov evolution model of a reciprocal CM_L evolution model induced by a Markov evolution model is determined as follows. Let $M_{k,k-1}, M_k$, $\forall k \in [1, N]$, be the parameters of a Markov evolution model (4.3). Parameters of the reciprocal CM_L evolution model induced by this Markov evolution model are calculated by (4.4)–(4.6). Then, by (4.61), parameters of the underlying Markov evolution model denoted by $(U_{k,k-1}, U_k)$, $\forall k \in [1, N-1]$, are

$$U_{k,k-1} = M_{k,k-1} - (U_k M'_{N|k} C^{-1}_{N|k}) M_{N|k-1} \tag{4.62}$$

$$U_k = (M_k^{-1} + M'_{N|k} C^{-1}_{N|k} M_{N|k})^{-1} \tag{4.63}$$

where $M_{N|k} = M_{N,N-1} \cdots M_{k+1,k}$ for $k \in [0, N-1]$, $C_{N|k} = \sum_{n=k}^{N-1} M_{N|n+1} M_{n+1} M'_{N|n+1}$ for $k \in [1, N-1]$, and $M_{N|N} = I$.

Chapter 5

Singular/Nonsingular Gaussian CM Sequences

In this chapter, we 1) obtain dynamic models and characterizations of the general Gaussian CM (including reciprocal and Markov) sequence to unify singular and nonsingular Gaussian CM sequences theoretically, 2) provide tools for application of (singular/nonsingular) Gaussian CM sequences, e.g., in trajectory modeling with destination information, 3) emphasize the significance of studying reciprocal sequences from the CM viewpoint, e.g., by obtaining two dynamic models for the general (singular/nonsingular) Gaussian reciprocal sequence from the CM viewpoint.

For a matrix P, $P_{i,j}$ denotes the (block) entry at (block) row $i+1$ and (block) column $j+1$ of P. Also, $P_i \triangleq P_{i,i}$. For example, C is the covariance matrix of the whole sequence $[x_k]$, $C_{i,j}$ is the covariance function[1], and $C_i \triangleq C_{i,i}$. By the "Gaussian sequence" we mean the general singular/nonsingular Gaussian sequence. Otherwise, we make it explicit if we only mean the NG sequence. The abbreviation ZMG is used for "zero-mean Gaussian".

5.1 Dynamic Model and Characterization of CM_c Sequences

5.1.1 Dynamic Model

The following theorem presents a model of ZMG CM_c sequences called a CM_c model. A Gaussian sequence is CM_c iff its zero-mean part is CM_c (Chapter 2). So, based on Theorem 5.1.1, a model of nonzero-mean Gaussian CM_c sequences can be easily obtained.

Theorem 5.1.1. *A ZMG $[x_k]$ is $CM_c, c \in \{0, N\}$, iff it obeys*

$$x_k = G_{k,k-1}x_{k-1} + G_{k,c}x_c + e_k, \quad k \in [1, N] \setminus \{c\} \tag{5.1}$$

where $[e_k]$ is a zero-mean white Gaussian sequence with covariances G_k, and boundary condition[2]

$$x_c = e_c, \quad x_0 = G_{0,c}x_c + e_0 \text{ (for } c = N) \tag{5.2}$$

or equivalently[3]

$$x_0 = e_0, \quad x_c = G_{c,0}x_0 + e_c \text{ (for } c = N) \tag{5.3}$$

Proof. Necessity: We first prove it for $c = N$ (i.e., CM_L). Let $[x_k]$ be a ZMG CM_L sequence with covariance function C_{l_1,l_2}. It is shown that $[x_k]$ is modeled by (5.1) along with (5.2) or (5.3). First, we obtain boundary condition (5.3). Let $x_0 = e_0$, where e_0, a ZMG vector with

[1] Note that $i, j \in [0, N]$, but matrix C has (block) rows (columns) 1 to $N + 1$.

[2] Note that (5.2) means that for $c = N$ we have $x_N = e_N$ and $x_0 = G_{0,N}x_N + e_0$; for $c = 0$ we have $x_0 = e_0$. Likewise for (5.3).

[3] e_0 and e_N in (5.2) are not necessarily the same as e_0 and e_N in (5.3). Just for simplicity we use the same notation.

covariance C_0, is defined for notational unification. The conditional expectation $E[x_N|x_0]$ is the a.s. unique Borel measurable function of x_0 for which

$$E[(x_N - E[x_N|x_0])g(x_0)] = 0 \tag{5.4}$$

for every Borel measurable function g.

We show now that there exists B for which $E[(x_N - Bx_0])g(x_0)] = 0$ for every Borel measurable function g. Then, by the uniqueness of the conditional expectation in (5.4), we conclude $E[x_N|x_0] = Bx_0$ [57], [91].

Let B satisfy the following normal equation

$$BC_0 = C_{N,0} \tag{5.5}$$

which always has a solution $B = C_{N,0}(C_0)^+ + S(I - C_0(C_0)^+)$ for any matrix S, where the superscript "+" means the Moore-Penrose inverse (MP-inverse) [1]. Since $[x_k]$ is zero-mean, (5.5) can be rewritten as

$$E[(x_N - Bx_0)x_0'] = 0 \tag{5.6}$$

which means $x_N - Bx_0$ is uncorrelated with (and orthogonal to, because $[x_k]$ is zero-mean) x_0. Due to the Gaussianity of $[x_k]$, $x_N - Bx_0$ and x_0 are independent and we have

$$E[(x_N - Bx_0)g(x_0)] = 0 \tag{5.7}$$

for every Borel measurable function g. Comparing (5.4) and (5.7), and by the uniqueness of the conditional expectation, we have $E[x_N|x_0] = Bx_0$ for B given above (i.e., solution of (5.5)). Also, since $\text{Cov}((I - C_0(C_0)^+)x_0) = 0$, we have $(I - C_0(C_0)^+)x_0 \overset{a.s.}{=} E[(I - C_0(C_0)^+)x_0] = 0$. Therefore, $E[x_N|x_0] = C_{N,0}(C_0)^+x_0$. We define e_N as $e_N = x_N - C_{N,0}(C_0)^+x_0$. By (5.6), e_N and e_0 are uncorrelated. Also, the covariance of e_N is $C_N - C_{N,0}(C_0)^+C_{N,0}'$.

We can obtain (5.2) as $x_N = e_N$ and $x_0 = C_{0,N}(C_N)^+x_N + e_0$, where e_N and e_0 are independent ZMG vectors with covariances C_N and $C_0 - C_{0,N}(C_N)^+(C_{0,N})'$, respectively.

Following a similar argument as above, based on the definition of the conditional expectation $E[x_k|y_{k-1}]$, $y_k = [x_k', x_N']'$, we obtain $E[x_k|y_{k-1}] = A_k y_{k-1}$, where $A_k = C^{xy}_{k,k-1}(C^y_{k-1})^+ + S(I - C^y_{k-1}(C^y_{k-1})^+)$, $C^y_{k-1} = \text{Cov}(y_{k-1})$, and $C^{xy}_{k,k-1} = \text{Cov}(x_k, y_{k-1})$. In addition, we have $(I - C^y_{k-1}(C^y_{k-1})^+)y_{k-1} = 0$, a.s., because $\text{Cov}((I - C^y_{k-1}(C^y_{k-1})^+)y_{k-1}) = 0$ and $E[(I - C^y_{k-1}(C^y_{k-1})^+)y_{k-1}] = 0$. Thus, we have a.s.

$$E[x_k|y_{k-1}] = C^{xy}_{k,k-1}(C^y_{k-1})^+y_{k-1} \tag{5.8}$$

We define e_k, $\forall k \in [1, N-1]$, as

$$e_k = x_k - E[x_k|x_{k-1}, x_N] \tag{5.9}$$

where $[e_k]$ is a zero-mean white Gaussian sequence (with covariances $C_k - C^{xy}_{k,k-1}(C^y_{k-1})^+ \cdot(C^{xy}_{k,k-1})', k \in [1, N-1]$), which can be verified as follows. By the definition of the conditional expectation $E[x_k|[x_i]_0^{k-1}, x_N]$, we have

$$E[(x_k - E[x_k|[x_i]_0^{k-1}, x_N])g([x_i]_0^{k-1}, x_N)] = 0 \tag{5.10}$$

for every bounded Borel measurable function g. Then, by Lemma 2.1.7, (5.10) leads to

$$E[(x_k - E[x_k|x_{k-1}, x_N])g([x_i]_0^{k-1}, x_N)] = 0 \tag{5.11}$$

Since $x_k - E[x_k|x_{k-1}, x_N]$ is uncorrelated with $g([x_i]_0^{k-1}, x_N)$, it can be seen from (5.9) that $[e_k]$ is white ($E[e_k e_j'] = 0$, $k \neq j$). Thus, given any ZMG CM_L sequence, its evolution obeys (5.1) along with (5.2) or (5.3).

Proof of necessity for $c = 0$ (i.e., CM_F) is similar. We have $x_0 = e_0$, $x_1 = C_{1,0}(C_0)^+x_0 + e_1$, and $x_k = C^{xy}_{k,k-1}(C^y_{k-1})^+y_{k-1} + e_k, k \in [2, N]$, where $G_0 = C_0$, $G_1 = C_1 - C_{1,0}C_0^+C'_{1,0}$, and $G_k = C_k - C^{xy}_{k,k-1}(C^y_{k-1})^+(C^{xy}_{k,k-1})', k \in [2, N]$.

Sufficiency: Our proof of sufficiency is similar to that of the ZMNG CM_c model in Chapter 2. From (5.1), we have $x_k = G_{k,j}x_j + G_{k,c|j}x_c + e_{k|j}$, where $G_{k,j}$ and $G_{k,c|j}$ can be obtained from parameters of (5.1), and $e_{k|j}$ is a linear combination of $[e_l]_{j+1}^k$. Since $[e_k]$ is white, $[e_l]_{j+1}^k$ (and so $e_{k|j}$) is uncorrelated with $[x_k]_0^j$ and x_c. So, we have $E[x_k|[x_i]_0^j, x_c] = E[x_k|x_j, x_c]$. Then, by Lemma 2.1.7, $[x_k]$ is CM_c.

(5.13) and (5.15) (below) are always nonsingular. Then, by (5.12), (5.1)–(5.2) (for every parameter value) admit a unique covariance function (i.e., a unique sequence). Similarly, (5.1) and (5.3) admit a unique covariance function for every parameter value (see Lemma 2.2.4). □

The boundary conditions (5.2) and (5.3) are equivalent. So, later we only consider one of them.

Consider (5.1)–(5.2) for $c = N$. We have

$$\mathcal{G}x = e \tag{5.12}$$

where $e \triangleq [e'_0, e'_1, \ldots, e'_N]'$, $x \triangleq [x'_0, x'_1, \ldots, x'_N]'$, and $\mathcal{G}$ is

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & -G_{0,N} \\ -G_{1,0} & I & 0 & \cdots & 0 & -G_{1,N} \\ 0 & -G_{2,0} & I & 0 & \cdots & -G_{2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -G_{N-1,N-2} & I & -G_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{5.13}$$

From (5.12), the covariance matrix of x (i.e., C) is calculated as

$$C = \mathcal{G}^{-1}G(\mathcal{G}')^{-1} \tag{5.14}$$

where $G = \text{diag}(G_0, \ldots, G_N)$. Similarly, for $c = 0$, the covariance is given by (5.14), where $G = \text{diag}(G_0, \ldots, G_N)$ and $\mathcal{G}$ is

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -2G_{1,0} & I & 0 & \cdots & 0 & 0 \\ -G_{2,0} & -G_{2,1} & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -G_{N-1,0} & 0 & \cdots & -G_{N-1,N-2} & I & 0 \\ -G_{N,0} & 0 & 0 & \cdots & -G_{N,N-1} & I \end{bmatrix} \tag{5.15}$$

By (5.14), we can determine the imposed condition on the parameters of (5.1)–(5.2) due to a specific singularity. An example is as follows.

Corollary 5.1.2. *A ZMG $[x_k]$ with covariance function C_{l_1,l_2} is CM_L with the matrices*

$$\begin{bmatrix} C_k & C_{k,N} \\ C_{N,k} & C_N \end{bmatrix}, k \in [0, N-2] \tag{5.16}$$

being nonsingular iff

$$x_k = G_{k,k-1}x_{k-1} + G_{k,N}x_N + e_k, k \in [1, N-1] \tag{5.17}$$

$$x_N = e_N, \quad x_0 = G_{0,N}x_N + e_0 \tag{5.18}$$

where $[e_k]$ *is a white Gaussian sequence with covariances* G_k*, and the matrices*

$$\begin{bmatrix} P_k & P_{k,N} \\ P_{N,k} & P_N \end{bmatrix}, k \in [0, N-2] \tag{5.19}$$

are nonsingular (positive definite[4]*), with* $P = \mathcal{G}^{-1}G\mathcal{G}'^{-1}$*,* $G = diag(G_0, \ldots, G_N)$*, and* $\mathcal{G}$ *being given by* (5.13)*.*

Proof. A ZMG $[x_k]$ is CM_L iff we have (5.17)–(5.18). Also, P is the covariance of $[x_k]$ (see (5.14)). So, (5.16) and (5.19) are equal. □

By having different values of the parameters, (5.1)–(5.2) can model all Gaussian CM_c sequences ranging from a nonsingular CM_c sequence to a singular CM_c sequence a.s. zero throughout the time interval. For example, let $|G_k| = 0, \forall k \in [0, N]$ ($| \cdot |$ denotes the determinant operator), and all other parameters of (5.1)–(5.2) be zero. By (5.14), such a CM_c model is for a white sequence with $|C_k| = 0, \forall k \in [0, N]$ (for a scalar-valued sequence, it is actually an a.s. zero sequence). Another extreme is when all the matrices G_k are nonsingular leading to a nonsingular Gaussian CM_c sequence.

Let $[x_k]$ be a ZMG CM_L sequence. x_n and $y_{n-1} = [x'_{n-1}, x'_N]'$ are a.s. linearly dependent iff e_n is a.s. zero (i.e., $\text{Cov}(e_n) = 0$). It can be verified by (5.9).

Let $[x_k]$ be a ZMG CM_L sequence. x_n is a.s. zero (i.e., $\text{Cov}(x_n) = 0$) iff both e_n and $C^{xy}_{n,n-1}(C^y_{n-1})^+ y_{n-1}$ are a.s. zero. It is verified as follows. By (5.9), x_n is a.s. zero iff we have a.s.

$$e_n + C^{xy}_{n,n-1}(C^y_{n-1})^+ y_{n-1} = 0 \tag{5.20}$$

Post-multiplying both sides of (5.20) by e'_n and taking expectation, it is concluded that $\text{Cov}(e_n) = 0$, where the fact that e_n is orthogonal to x_{n-1} and x_N, has been used (see (5.11)). Then, by (5.20), we have a.s. $C^{xy}_{n,n-1}(C^y_{n-1})^+ y_{n-1} = 0$. Therefore, x_n is a.s. zero iff both terms of (5.20) are a.s. zero.

5.1.2 Characterization

Two characterizations are presented for Gaussian CM_c sequences with any kind of singularity. The first characterization is as follows.

Theorem 5.1.3. *A Gaussian* $[x_k]$ *with covariance function* C_{l_1,l_2} *is* $CM_c, c \in \{0, N\}$*, iff*

$$C_{k,i} = \begin{bmatrix} C_{k,j} & C_{k,c} \end{bmatrix} \begin{bmatrix} C_j & C_{j,c} \\ C_{c,j} & C_c \end{bmatrix}^+ \begin{bmatrix} C_{j,i} \\ C_{c,i} \end{bmatrix} \tag{5.21}$$

$\forall i, j, k \in [0, N] \setminus \{c\}$*,* $i < j < k$*, where the superscript "+" means the MP-inverse.*

Proof. A Gaussian sequence is CM_c iff its zero-mean part is CM_c. Also, a sequence and its zero-mean part have the same covariance function. So, it suffices to consider zero-mean sequences.

Necessity: Let $[x_k]$ be a ZMG CM_c sequence with covariance function C_{l_1,l_2}. Define

$$r(k, j) = x_k - E[x_k|y_j] \tag{5.22}$$

$\forall j, k \in [0, N] \setminus \{c\}$, $j < k$, and $y_j \triangleq [x'_j, x'_c]'$. Then, since $[x_k]$ is Gaussian, (5.22) leads to (see (5.8))

$$r(k, j) = x_k - \begin{bmatrix} C_{k,j} & C_{k,c} \end{bmatrix} \begin{bmatrix} C_j & C_{j,c} \\ C_{c,j} & C_c \end{bmatrix}^+ \begin{bmatrix} x_j \\ x_c \end{bmatrix} \tag{5.23}$$

[4] P is always positive (semi)definite.

On the other hand, by the definition of the conditional expectation $E[x_k|[x_i]_0^j, x_c]$, we have

$$E[(x_k - E[x_k|[x_i]_0^j, x_c])g([x_i]_0^j, x_c)] = 0 \tag{5.24}$$

for every Borel measurable function g. Then, by Lemma 2.1.7, we have

$$E[(x_k - E[x_k|x_j, x_c])g([x_i]_0^j, x_c)] = 0 \tag{5.25}$$

By (5.25), $r(k,j)$ is uncorrelated with $[x_i]_0^j$ and x_c. So, post-multiplying both sides of (5.23) by x_i', $\forall i \in [0, j-1] \setminus \{c\}$, and taking expectation, we obtain (5.21), where $i, j, k \in [0, N] \setminus \{c\}$, $i < j < k$.

Sufficiency: Let $[x_k]$ be a ZMG sequence with covariance function C_{l_1,l_2} satisfying (5.21), $\forall i, j, k \in [0, N] \setminus \{c\}$, $i < j < k$. Since $[x_k]$ is Gaussian, we have

$$E[x_k|x_j, x_c] = \begin{bmatrix} C_{k,j} & C_{k,c} \end{bmatrix} \begin{bmatrix} C_j & C_{j,c} \\ C_{c,j} & C_c \end{bmatrix}^+ \begin{bmatrix} x_j \\ x_c \end{bmatrix} \tag{5.26}$$

Define

$$r(k,j) = x_k - \begin{bmatrix} C_{k,j} & C_{k,c} \end{bmatrix} \begin{bmatrix} C_j & C_{j,c} \\ C_{c,j} & C_c \end{bmatrix}^+ \begin{bmatrix} x_j \\ x_c \end{bmatrix} \tag{5.27}$$

where based on (5.21), it is concluded that $r(k,j)$ is uncorrelated with (and since $[x_k]$ is zero-mean, orthogonal to) $[x_i]_0^{j-1} \setminus \{c\}$ (it is seen by post-multiplying both sides of (5.27) by x_i', $\forall i \in [0, j-1] \setminus \{c\}$ and taking expectation). In addition, $r(k,j)$ is orthogonal to x_j and x_c. It can be verified based on (5.26) and the definition of the conditional expectation $E[x_k|x_j, x_c]$, where $E[(x_k - E[x_k|x_j, x_c])g(x_j, x_c)] = 0$ for every Borel measurable function g. Then, due to the Gaussianity, $r(k,j)$ is independent of $[x_i]_0^j$ and x_c, and consequently $r(k,j)$ is uncorrelated with $g([x_i]_0^j, x_c)$ for every Borel measurable function g. Thus, by the a.s. uniqueness of the conditional expectation in (5.24),

$$E[x_k|[x_i]_0^j, x_c] = \begin{bmatrix} C_{k,j} & C_{k,c} \end{bmatrix} \begin{bmatrix} C_j & C_{j,c} \\ C_{c,j} & C_c \end{bmatrix}^+ \begin{bmatrix} x_j \\ x_c \end{bmatrix} \tag{5.28}$$

So, by (5.26) and (5.28), $\forall j, k \in [0, N] \setminus \{c\}$, $j < k$, we have $E[x_k|[x_i]_0^j, x_c] = E[x_k|x_j, x_c]$. Then, by Lemma 2.1.7, $[x_k]$ is CM_c. □

The following characterization of the Gaussian CM_c sequence is based on the concept of state in system theory (i.e., Markov property).

Corollary 5.1.4. *A Gaussian $[x_k]$ is CM_c iff $[y_k] \setminus \{y_c\}$ [5] is Markov, where $y_k \triangleq [x_k', x_c']', \forall k \in [0, N] \setminus \{c\}$.*

Proof. It can be verified by Lemma 2.1.7 or Theorem 5.1.1. □

5.2 Characterization and Dynamic Model of Reciprocal Sequences

5.2.1 Characterization

In [58] a characterization was presented for the Gaussian reciprocal process with a special kind of nonsingularity called the second-order nonsingularity. $[x_k]$ is second-order nonsingular if the covariance of $y = [x_m', x_n']'$ for every $n, m \in [0, N]$ is nonsingular. Inspired by [58], in Theorem 5.2.2 below, a characterization of the Gaussian reciprocal sequence is presented. First, we need a corollary of Theorem 5.1.3. By definition, $[x_k]$ is $[k_1, k_2]$-CM_c iff $[x_k]_{k_1}^{k_2}$ is CM_c. So, we have the following corollary.

[5] For $c = N$, $[y_k] \setminus \{y_c\} \triangleq [y_k]_0^{N-1}$, and for $c = 0$, $[y_k] \setminus \{y_c\} \triangleq [y_k]_1^N$.

Corollary 5.2.1. *A Gaussian* $[x_k]$ *with covariance function* C_{l_1,l_2} *is* $[k_1,k_2]$*-*$CM_c, c \in \{k_1,k_2\}$*, iff*

$$C_{k,i} = \begin{bmatrix} C_{k,j} & C_{k,c} \end{bmatrix} \begin{bmatrix} C_j & C_{j,c} \\ C_{c,j} & C_c \end{bmatrix}^+ \begin{bmatrix} C_{j,i} \\ C_{c,i} \end{bmatrix} \tag{5.29}$$

$\forall i,j,k \in [k_1,k_2] \setminus \{c\}$*,* $i < j < k$*.*

Theorem 5.2.2. *A Gaussian* $[x_k]$ *with covariance function* C_{l_1,l_2} *is reciprocal iff*

$$C_{k,i} = \begin{bmatrix} C_{k,j} & C_{k,l} \end{bmatrix} \begin{bmatrix} C_j & C_{j,l} \\ C_{l,j} & C_l \end{bmatrix}^+ \begin{bmatrix} C_{j,i} \\ C_{l,i} \end{bmatrix} \tag{5.30}$$

(a) $\forall i,j,k,l \in [0,N]$ *with* $l < i < j < k$*, and (b)* $\forall i,j,k \in [0,N-1]$ *with* $i < j < k < l = N$ *(or equivalently (a)* $\forall i,j,k,l \in [0,N]$ *with* $i < j < k < l$*, and (b)* $\forall i,j,k \in [1,N]$ *with* $0 = l < i < j < k$*).*

Proof. A proof is based on Theorem 3.1.5 and Corollary 5.2.1. □

First, the characterization presented in [58] only works for second-order nonsingular Gaussian reciprocal sequences. The characterization of Theorem 5.2.2 works for all Gaussian reciprocal sequences. Second, Theorem 3.1.5 implies the equality of two sets of sequences, i.e., $\cap_{k_1=0}^N [k_1,N]$-$CM_F \cap_{k_2=0}^N [0,k_2]$-$CM_L = \cap_{k_1=0}^N [k_1,N]$-$CM_F \cap CM_L$. Accordingly, and by Corollary 5.2.1, for a Gaussian sequence (5.30) holds for (a) $\forall i,j,k,l \in [0,N]$ with $l < i < j < k$, and (b) $\forall i,j,k,l \in [0,N]$ with $i < j < k < l$ iff (5.30) holds for (a) $\forall i,j,k,l \in [0,N]$ with $l < i < j < k$, and (b) $\forall i,j,k \in [0,N-1]$ with $i < j < k < l = N$. Although the two conditions are equivalent, the latter is simpler (and more revealing) than the former. It seems [58] was not aware of the simpler condition. We obtained the simpler condition based on studying reciprocal sequences from the CM viewpoint, which is different from that of [58]. It shows how insightful the CM viewpoint is for studying reciprocal sequences.

Another characterization of the Gaussian reciprocal sequence is based on the concept of state in system theory (i.e. Markov property).

Corollary 5.2.3. *i) A Gaussian* $[x_k]$ *is reciprocal iff* $[y_k]_{k_1+1}^N$ *with* $y_k \triangleq [x_k', x_{k_1}']', \forall k \in [k_1+1,N]$*,* $\forall k_1 \in [0,N]$*, and* $[y_k]_0^{N-1}$ *with* $y_k \triangleq [x_k', x_N']', \forall k \in [0,N-1]$*, are Markov. ii) A Gaussian* $[x_k]$ *is reciprocal iff* $[y_k]_0^{k_2-1}$ *with* $y_k \triangleq [x_k', x_{k_2}']', \forall k \in [0,k_2-1]$*,* $\forall k_2 \in [0,N]$*, and* $[y_k]_1^N$ *with* $y_k \triangleq [x_k', x_0']', \forall k \in [1,N]$*, are Markov.*

Proof. A proof is based on Theorem 3.1.5, Corollary 5.1.4, and the fact that $[x_k]$ is $[k_1,k_2]$-CM_c iff $[x_k]_{k_1}^{k_2}$ is CM_c. □

5.2.2 Dynamic Model

To our knowledge, the only dynamic model for Gaussian reciprocal sequences is the one presented in [18], which is for the NG reciprocal sequence. The nonsingularity assumption is critical for that model, because its well-posedness (i.e., the uniqueness of the sequence obeying the model) is guaranteed by the nonsingularity of the whole sequence. There is not any model for the general (singular/nonsingular) Gaussian reciprocal sequence in the literature, and it is not clear how to obtain such a model. For example, it is not clear how the model of [18] can be extended to the general (singular/nonsingular) case. The CM viewpoint is very fruitful for studying reciprocal sequences. The following theorem presents two models for the general (singular/nonsingular) Gaussian reciprocal sequence from the CM viewpoint. They are called reciprocal CM_c models.

Theorem 5.2.4. *A ZMG $[x_k]$ is reciprocal iff it obeys* (5.1)–(5.2) *and*

$$P_{k,i} = \begin{bmatrix} P_{k,j} & P_{k,l} \end{bmatrix} \begin{bmatrix} P_j & P_{j,l} \\ P_{l,j} & P_l \end{bmatrix}^+ \begin{bmatrix} P_{j,i} \\ P_{l,i} \end{bmatrix} \tag{5.31}$$

(i) for $c = N$ and $\forall i, j, k, l \in [0, N]$, $l < i < j < k$, and $\mathcal{G}$ given by (5.13)*, or equivalently (ii) for $c = 0$ and $\forall i, j, k, l \in [0, N]$, $i < j < k < l$, and $\mathcal{G}$ given by* (5.15)*, where $P = (\mathcal{G})^{-1}G(\mathcal{G}')^{-1}$ and $G = diag(G_0, \ldots, G_N)$.*

Proof. A reciprocal sequence is CM_c. A ZMG sequence is CM_c iff it obeys (5.1)–(5.2). The covariance matrix of a sequence modeled by a CM_c model can be calculated in terms of the parameters of the model and its boundary condition (the calculated covariance matrix is denoted by P above). A Gaussian sequence is reciprocal iff its covariance function satisfies (5.30). Since model (5.1)–(5.2) is for a CM_c sequence, P already satisfies condition (b) of Theorem 5.2.2 (note that condition (b) of Theorem 5.2.2 is a CM_c characterization for $c = N$ or $c = 0$). So, a Gaussian sequence is reciprocal iff it obeys (5.1)–(5.2) (for $c = N$ or $c = 0$) and P satisfies (5.31). □

The results of this section support the idea of studying reciprocal sequences from the CM viewpoint.

5.3 Characterizations and Dynamic Models of Other CM Sequences

It is useful for both application and theory to study sequences belonging to more than one CM class. For example, an application of $CM_L \cap [0, k_2]$-CM_L sequences in trajectory modeling with a waypoint and a destination was discussed in Chapter 4. Also, by Theorem 3.1.5, a reciprocal sequence belongs to several CM classes. This is particularly useful for studying reciprocal sequences from the CM viewpoint (e.g., Theorem 5.2.2 and Theorem 5.2.4). In addition, a dynamic model of $CM_L \cap [k_1, N]$-CM_F sequences is useful for obtaining a full spectrum of models ranging from a CM_L model to a reciprocal CM_L model.

Corollary 5.3.1. *A Gaussian $[x_k]$ is $CM_L \cap [k_1, N]$-CM_F iff it obeys* (5.1)–(5.2) *(for $c = N$), and*

$$P_{k,i} = \begin{bmatrix} P_{k,j} & P_{k,k_1} \end{bmatrix} \begin{bmatrix} P_j & P_{j,k_1} \\ P_{k_1,j} & P_{k_1} \end{bmatrix}^+ \begin{bmatrix} P_{j,i} \\ P_{k_1,i} \end{bmatrix} \tag{5.32}$$

$\forall i, j, k \in [k_1 + 1, N]$, $i < j < k$, where

$$P = \mathcal{G}^{-1}G(\mathcal{G}')^{-1} \tag{5.33}$$

$G = diag(G_0, \ldots, G_N)$, and $\mathcal{G}$ is given by (5.13)*.*

Proof. A sequence is $CM_L \cap [k_1, N]$-CM_F iff it is CM_L and $[k_1, N]$-CM_F. By Theorem 5.1.1, a Gaussian sequence is CM_L iff it obeys (5.1)–(5.2) (for $c = N$). Also, the covariance matrix of a CM_L sequence can be calculated as (5.33). On the other hand, by Corollary 5.2.1, a Gaussian sequence is $[k_1, N]$-CM_F iff its covariance function satisfies (5.29) (let $k_2 = N$ and $c = k_1$ in Corollary 5.2.1). Therefore, a Gaussian sequence is $CM_L \cap [k_1, N]$-CM_F iff it obeys (5.1)–(5.2) and (5.32) holds. □

Following the idea of Corollary 5.3.1, one can obtain models of other CM sequences belonging to more than one CM class, e.g., $CM_L \cap [k_1, N]$-$CM_F \cap [l_1, N]$-CM_F. As a result, by Theorem 3.1.5, Corollary 5.3.1, and Theorem 5.2.4, one can see a full spectrum of models ranging from a CM_L model (Theorem 5.1.1) to a reciprocal CM_L model (Theorem 5.2.4).

Characterizations presented in Corollary 5.1.4 and Corollary 5.2.3 are based on the Markov property. To complete those characterizations, we need a characterization of the Gaussian Markov sequence based on covariance functions. A characterization was presented in [57] for the scalar-valued Gaussian Markov process, but its generalization to the vector-valued case is not trivial. The following corollary presents a characterization of the vector-valued general (singular/nonsingular) Gaussian Markov sequence. To our knowledge, there is no such a characterization in the literature.

Corollary 5.3.2. *A Gaussian $[x_k]$ with covariance function C_{l_1,l_2} is Markov iff $C_{k,i} = C_{k,j}C_j^+C_{j,i}$, $\forall i,j,k \in [0,N]$, $i<j<k$.*

Proof. Our proof is parallel to that of Theorem 5.1.3. The main differences are as follows. For the proof of necessity, instead of $r(k,j)$ in (5.22), we need to define $r(k,j) = x_k - E[x_k|x_j]$. Also, instead of Lemma 2.1.7, we should use Lemma 3.1.4. For the proof of sufficiency, instead of $r(k,j)$ in (5.27), we need to define $r(k,j) = x_k - C_{k,j}C_j^+x_j$. □

Inspired by [29], a representation of the ZMNG CM_c sequence as a sum of a ZMNG Markov sequence and an uncorrelated ZMNG vector was presented in Proposition 4.2.1 (Chapter 4). We now extend it to the ZMG CM_c sequence. Proposition 5.3.3 can be proved based on Theorem 5.1.1. We omit the proof.

Proposition 5.3.3. *A ZMG $[x_k]$ is $CM_c, c \in \{0,N\}$, iff it can be represented as $x_k = y_k + \Gamma_k x_c, k \in [0,N] \setminus \{c\}$, where $[y_k] \setminus \{y_c\}$ is a ZMG Markov sequence, x_c is a ZMG vector uncorrelated with $[y_k] \setminus \{y_c\}$, and Γ_k are some matrices.*

A corollary of Proposition 5.3.3 is as follows.

Corollary 5.3.4. *An $(N+1)d \times (N+1)d$ matrix (with $(N+1)$ blocks in each row/column and each block with dimension $d \times d$) is the covariance matrix of a d-dimensional vector-valued Gaussian CM_c sequence iff $C = B + \Gamma D \Gamma'$, where D is a $d \times d$ positive semi-definite matrix and (i) for $c=N$, $B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$, $\Gamma = \begin{bmatrix} S \\ I \end{bmatrix}$, (ii) for $c=0$, $B = \begin{bmatrix} 0 & 0 \\ 0 & B_1 \end{bmatrix}$, $\Gamma = \begin{bmatrix} I \\ S \end{bmatrix}$, where B_1 is an $Nd \times Nd$ covariance matrix of a d-dimensional vector-valued Gaussian Markov sequence, S is an arbitrary $Nd \times d$ matrix, and I is the $d \times d$ identity matrix.*

Chapter 6

Algebraically Equivalent Dynamic Models of Gaussian CM Sequences

In this chapter, we 1) study the relationships between dynamic models of different classes of CM sequences including Markov, reciprocal, CM_L, and CM_F, 2) define and distinguish the notions of probabilistically equivalent and algebraically equivalent dynamic models, 3) present a unified approach for determination of algebraically equivalent models, and 4) present a simple approach for studying/determining Markov sequences sharing the same reciprocal/CM_L model.

The term "boundary value" is used for random vectors in equations as "boundary condition". A boundary condition (value) for a forward (backward) Markov model means an initial (a final) condition (value).

6.1 Preliminaries: Dynamic Models

Forward and backward CM_L, CM_F, reciprocal, and Markov models are reviewed Chapter 2, Chapter 3, [18].

Let $[x_k]$ be a zero-mean NG sequence.

Markov Model

$[x_k]$ is Markov iff

$$x_k = M_{k,k-1}x_{k-1} + e^M_k, k \in [1, N] \tag{6.1}$$

where $x_0 = e^M_0$ and $[e^M_k]$ ($\text{Cov}(e^M_k) = M_k$) is a zero-mean white NG sequence. We have

$$\mathcal{M}x = e^M, \quad e^M = [(e^M_0)', (e^M_1)', \ldots, (e^M_N)']' \tag{6.2}$$

where $\mathcal{M}$ is the nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -M_{1,0} & I & 0 & \cdots & 0 & 0 \\ 0 & -M_{2,1} & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -M_{N-1,N-2} & I & 0 \\ 0 & 0 & 0 & \cdots & -M_{N,N-1} & I \end{bmatrix} \tag{6.3}$$

From (6.2), the inverse of the covariance matrix of $[x_k]$ is

$$C^{-1} = \mathcal{M}'M^{-1}\mathcal{M} \tag{6.4}$$

where $M = \text{Cov}(e^M) = \text{diag}(M_0, M_1, \ldots, M_N)$. C^{-1} is (block) tri-diagonal (Remark 3.1.16).

Backward Markov Model

$[x_k]$ is Markov iff

$$x_k = M^B_{k,k+1}x_{k+1} + e^{BM}_k, k \in [0, N-1] \tag{6.5}$$

where $x_N = e^{BM}_N$ and $[e^{BM}_k]$ ($\text{Cov}(e^{BM}_k) = M^B_k$) is a zero-mean white NG sequence.

We have

$$\mathcal{M}^B x = e^{BM}, \quad e^{BM} = [(e^{BM}_0)', \ldots, (e^{BM}_N)']' \tag{6.6}$$

$$C^{-1} = (\mathcal{M}^B)'(M^B)^{-1}\mathcal{M}^B \tag{6.7}$$

where $M^B = \text{Cov}(e^{BM}) = \text{diag}(M^B_0, \ldots, M^B_N)$, C^{-1} is (block) tri-diagonal, and $\mathcal{M}^B$ is the nonsingular matrix

$$\begin{bmatrix} I & -M^B_{0,1} & 0 & \cdots & 0 & 0 \\ 0 & I & -M^B_{1,2} & 0 & \cdots & 0 \\ 0 & 0 & I & -M^B_{2,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & I & -M^B_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{6.8}$$

Reciprocal Model

$[x_k]$ is reciprocal iff

$$R^0_k x_k - R^-_k x_{k-1} - R^+_k x_{k+1} = e^R_k, \quad k \in [1, N-1] \tag{6.9}$$

where $[e^R_k]^{N-1}_1$ is a zero-mean colored Gaussian sequence with $E[e^R_k(e^R_k)'] = R^0_k$, $k \in [1, N-1]$, $E[e^R_k(e^R_{k+1})'] = -R^+_k$, $k \in [1, N-2]$, $E[e^R_k(e^R_j)'] = 0, |k-j| > 1$, $R^+_k = (R^-_{k+1})'$, $k \in [1, N-2]$ and boundary condition (i) or (ii) below, with parameters of (6.9) and either boundary condition leading to a nonsingular sequence.

(i) The first type:

$$R^0_0 x_0 - R^-_0 x_N - R^+_0 x_1 = e^R_0 \tag{6.10}$$

$$R^0_N x_N - R^-_N x_{N-1} - R^+_N x_0 = e^R_N \tag{6.11}$$

where $E[e^R_0(e^R_1)'] = -R^+_0$, $E[e^R_N(e^R_0)'] = -R^+_N$, $E[e^R_0(e^R_0)'] = R^0_0$, $E[e^R_0(e^R_k)'] = 0$, $k \in [2, N-1]$, $E[e^R_N(e^R_k)'] = 0$, $k \in [1, N-2]$, $E[e^R_N(e^R_N)'] = R^0_N$, $E[e^R_{N-1}(e^R_N)'] = -R^+_{N-1}$, $(R^-_0)' = R^+_N$, $(R^-_N)' = R^+_{N-1}$, $(R^-_1)' = R^+_0$.

(ii) The second type: $[x'_0, x'_N]' \sim \mathcal{N}(0, C_{\{0,N\}})$, which can be written as

$$x_0 = e^R_0, \quad x_N = R_{N,0}x_0 + e^R_N \tag{6.12}$$

or equivalently

$$x_N = e^R_N, \quad x_0 = R_{0,N}x_N + e^R_0 \tag{6.13}$$

where e^R_0 and e^R_N are uncorrelated zero-mean NG vectors[1] with covariances R^0_0 and R^0_N, and uncorrelated with $[e^R_k]^{N-1}_1$.

[1] e^R_0 and e^R_N (and their covariances) in (6.12) are not necessarily the same as those in (6.13) or in the first boundary condition. Just for simplicity we use the same notation.

Consider (6.9) and boundary condition[2] (6.10)–(6.11) with appropriate parameters leading to a nonsingular sequence. Then,

$$\mathfrak{R}x = e^R, \quad e^R = [(e_0^R)', \ldots, (e_N^R)']' \tag{6.14}$$
$$C^{-1} = \mathfrak{R}' R^{-1} \mathfrak{R} = R \tag{6.15}$$

where $R = \text{Cov}(e^R) = \mathfrak{R}$ and $\mathfrak{R}$ is

$$\begin{bmatrix} R_0^0 & -R_0^+ & 0 & \cdots & 0 & -R_0^- \\ -R_1^- & R_1^0 & -R_1^+ & 0 & \cdots & 0 \\ 0 & -R_2^- & R_2^0 & -R_2^+ & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -R_{N-1}^- & R_{N-1}^0 & -R_{N-1}^+ \\ -R_N^+ & 0 & 0 & \cdots & -R_N^- & R_N^0 \end{bmatrix} \tag{6.16}$$

Since the sequence is nonsingular, so is (6.16) [18]. Then, $C^{-1} = R$ is (block) cyclic tri-diagonal (3.23).

Model (6.9) and its boundary condition (either the first or the second type) are well-posed (i.e., they admit a unique sequence) if their parameters lead to a nonsingular sequence [18]. Not all choices of the parameters lead to a nonsingular covariance matrix.

A reciprocal model is symmetric. So, its forward and backward are the same.

Remark 6.1.1. *Model* (6.9) *(with either boundary condition) is called a reciprocal model, to be distinguished from our reciprocal CM_L and CM_F models.*

CM_c Models

$[x_k]$ is CM_c, $c \in \{0, N\}$, iff (2.17) along with (2.18) or (2.19).

For $c = 0$, we have a CM_F model. Then,

$$\mathcal{G}^F x = e^F, \quad e^F \triangleq [e_0', \ldots, e_N']' \tag{6.17}$$
$$C^{-1} = (\mathcal{G}^F)'(G^F)^{-1}\mathcal{G}^F \tag{6.18}$$

where $G^F = \text{Cov}(e^F) = \text{diag}(G_0, \ldots, G_N)$ and $\mathcal{G}^F$ is the nonsingular matrix (2.29). C^{-1} is a CM_F matrix (2.37).

For $c = N$, we have a CM_L model. Then,

$$\mathcal{G}^L x = e^L, \quad e^L \triangleq [e_0', \ldots, e_N']' \tag{6.19}$$
$$C^{-1} = (\mathcal{G}^L)'(G^L)^{-1}\mathcal{G}^L \tag{6.20}$$

where $G^L = \text{Cov}(e^L) = \text{diag}(G_0, \ldots, G_N)$, $\mathcal{G}^L$ is the nonsingular matrix (2.27) for (2.18) and (2.28) for (2.19). C^{-1} is a CM_L matrix (2.36).

Theorem 3.1.17 gives the reciprocal/Markov CM_c model.

Backward CM_c Models

$[x_k]$ is CM_c, $c \in \{0, N\}$, iff it obeys (2.31) along with (2.32) or (2.33).

For $c = 0$, we have a backward CM_L model. Then,

$$\mathcal{G}^{BL} x = e^{BL}, \quad e^{BL} = [(e_0^B)', \ldots, (e_N^B)']' \tag{6.21}$$
$$C^{-1} = (\mathcal{G}^{BL})'(G^{BL})^{-1}\mathcal{G}^{BL} \tag{6.22}$$

[2] Boundary condition (ii) is discussed only in Section 6.4. In all other sections, we consider boundary condition (i).

where C^{-1} is a CM_F matrix, $G^{BL} = \text{Cov}(e^{BL}) = \text{diag}(G_0^B, \ldots, G_N^B)$, $\mathcal{G}^{BL}$ is the nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & -G_{0,N}^B \\ -G_{1,0}^B & I & -G_{1,2}^B & \cdots & 0 & 0 \\ -G_{2,0}^B & 0 & I & -G_{2,3}^B & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -G_{N-1,0}^B & 0 & \cdots & 0 & I & -G_{N-1,N}^B \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{6.23}$$

for (2.33), and $\mathcal{G}^{BL}$ for (2.32) is the nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -G_{1,0}^B & I & -G_{1,2}^B & \cdots & 0 & 0 \\ -G_{2,0}^B & 0 & I & -G_{2,3}^B & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -G_{N-1,0}^B & 0 & \cdots & 0 & I & -G_{N-1,N}^B \\ -G_{N,0}^B & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{6.24}$$

For $c = N$, we have a backward CM_F model. Then,

$$\mathcal{G}^{BF} x = e^{BF}, \quad e^{BF} = [(e_0^B)', \ldots, (e_N^B)']' \tag{6.25}$$

$$C^{-1} = (\mathcal{G}^{BF})'(G^{BF})^{-1}\mathcal{G}^{BF} \tag{6.26}$$

where C^{-1} is a CM_L matrix, $G^{BF} = \text{Cov}(e^{BF}) = \text{diag}(G_0, \ldots, G_N)$, and $\mathcal{G}^{BF}$ is the nonsingular matrix (2.35).

Theorem 3.1.20 gives a backward reciprocal/Markov CM_c model.

Forward and backward CM_L (CM_F) models have similar structures. They differ only in the time direction.

For a Markov model, $[e_k^M]_1^N$ is the dynamic noise and e_0^M is the initial value. For a CM_L model, $[e_k]_1^{N-1}$ is the dynamic noise and e_0 and e_N are the boundary values. For a CM_F model, $[e_k]_1^N$ is the dynamic noise and e_0 is the boundary value. For a reciprocal model, $[e_k^R]_1^{N-1}$ is the dynamic noise and e_0^R and e_N^R are the boundary values. Likewise for backward models.

Let $[x_k]$ be a CM sequence modeled by any of the above models. Then,

$$Tx = v, \quad v = [v_0', \ldots, v_N']' \tag{6.27}$$

where the vector v includes the dynamic noise and the boundary values. The matrix T is determined by parameters of the corresponding model. T is nonsingular for the forward and backward CM_L, CM_F, and Markov models. Also, since $[x_k]$ is assumed nonsingular, T is also nonsingular for the reciprocal model.

Definition 6.1.2. *Two models $T_1 x = v$ and $T_2 y = w$ are (probabilistically) equivalent if x and y have the same distribution.*

Definition 6.1.3. *Two models $T_1 x = v$ and $T_2 y = w$ are algebraically equivalent if $x = y$.*

6.2 Determination of Algebraically Equivalent Models: A Unified Approach

By Definitions 6.1.2 and 6.1.3, (algebraically) equivalence is mutual, i.e., if model 2 is (algebraically) equivalent to model 1, then so is model 1 to model 2.

To determine an equivalent model, we need to fix its parameters. Thus, we have the following proposition.

Proposition 6.2.1. *Any two forward/backward CM_L, CM_F, reciprocal, or Markov models*

$$T_1 x = v \tag{6.28}$$
$$T_2 y = w \tag{6.29}$$

are equivalent iff

$$T_2' P_2^{-1} T_2 = T_1' P_1^{-1} T_1 \tag{6.30}$$

where $v = [v_0', \ldots, v_N']'$ and $w = [w_0', \ldots, w_N']'$ are the vectors of the dynamic noise and boundary values with covariances $Cov(v) = P_1$ and $Cov(w) = P_2$.

Proof. The inverse of the covariance matrix of the sequence obeying model (6.28) is $C^{-1} = T_1'(P_1)^{-1}T_1$ because $E[(T_1x)(T_1x)'] = E[vv']$. Similarly, for the sequence obeying (6.29), we have $C^{-1} = T_2'(P_2)^{-1}T_2$. Two models are equivalent iff their sequences have the same covariance matrix; thus we have (6.30). □

Due to the special structures of T_1, P_1, T_2, and P_2, parameters of model 2 can be easily obtained from parameters of model 1 using (6.30) (see Appendix B for more details). Then, P_2 and T_2 are known. Note that parameters of model 2 so calculated are unique. This can be easily verified based on (6.30) for all models (see Appendix B). This uniqueness also follows from the definition of conditional expectation.

Clearly, algebraically equivalent models are equivalent. The next proposition gives a relationship of dynamic noise and boundary values for two equivalent models to be algebraically equivalent.

Proposition 6.2.2. *Two equivalent models* (6.28) *and* (6.29) *are algebraically equivalent if*

$$T_2'(P_2)^{-1}w = T_1'(P_1)^{-1}v \tag{6.31}$$

Proof. Let P_2, T_2, P_1, and T_1 be given (Proposition 6.2.1). Given model (6.28), we show how (6.31) leads to an algebraically equivalent model (6.29). First, we show that w has the desired covariance P_2. By (6.31), we have

$$T_2'(P_2)^{-1}\text{Cov}(w)(P_2)^{-1}T_2 = T_1'(P_1)^{-1}\text{Cov}(v)(P_1)^{-1}T_1$$

From $\text{Cov}(v) = P_1$ and (6.30) it follows that

$$\text{Cov}(w) = P_2(T_2')^{-1}T_2'(P_2)^{-1}T_2(T_2)^{-1}P_2 = P_2$$

Thus, w is the required vector.

Now we show that (6.31) implies that models (6.28) and (6.29) generate the same sample path of the sequence. We have

$$\begin{aligned} T_1'(P_1)^{-1}T_1 y &\overset{(6.30)}{=} T_2'(P_2)^{-1}T_2 y \overset{(6.29)}{=} T_2'(P_2)^{-1}w \overset{(6.31)}{=} T_1(P_1)^{-1}v \overset{(6.28)}{=} T_1(P_1)^{-1}T_1 x \\ &\implies y = x \end{aligned}$$

So, (6.29) and (6.28) are algebraically equivalent. □

By Propositions 6.2.1 and 6.2.2, given a model, one can construct an algebraically equivalent model. For two algebraically equivalent models, how are the sample paths of their dynamic noise and boundary values related? The next proposition answers this question.

Proposition 6.2.3. *For two algebraically equivalent forward /backward CM_L, CM_F, reciprocal, or Markov models*

$$T_1 x = v \tag{6.32}$$
$$T_2 y = w \tag{6.33}$$

the sample paths of v and w are related by (6.31), where $v = [v_0', \dots, v_N']'$ and $w = [w_0', \dots, w_N']'$ are vectors of the dynamic noise and boundary values with covariances $Cov(v) = P_1$ and $Cov(w) = P_2$, and the nonsingular matrices T_1 and T_2 are determined by the model parameters.

Proof. Algebraic equivalence (i.e., $x = y$) of (6.32) and (6.33) yields

$$T_2^{-1} w = T_1^{-1} v \tag{6.34}$$

It follows from the equivalence of (6.32) and (6.33) that

$$C^{-1} = T_1' P_1^{-1} T_1 = T_2' P_2^{-1} T_2 \tag{6.35}$$

Then, using (6.34) and (6.35), we have $(T_2' P_2^{-1} T_2) T_2^{-1} w = (T_1' P_1^{-1} T_1) T_1^{-1} v$, which leads to (6.31). □

Remark 6.2.4. (6.31) *is equivalent to* (6.34).

Although (6.34) looks simpler, for the construction of algebraically equivalent models, (6.31) is preferred for the following reasons. The matrices P_1 and P_2 in (6.31) for the forward/backward CM_L, CM_F, and Markov models are block diagonal, and their inverses can be easily calculated. Also, for the reciprocal model, no calculation is needed since $P = T$ in (6.27) (see Subsection 6.1). However, calculation of the inverses of T_1 and T_2 in (6.34) is not straightforward in general.

6.3 Algebraically Equivalent Models: Examples

Following Propositions 6.2.1 and 6.2.2, algebraically equivalent forward/backward CM_L, CM_F, reciprocal, or Markov models can be obtained. Two such examples are presented in this section, and more in appendices. Appendix B shows how parameters of equivalent models can be uniquely determined from each other (Proposition 6.2.1). Appendix C shows how the dynamic noise and boundary values of algebraically equivalent models are related (Proposition 6.2.2).

6.3.1 Forward and Backward Markov Models

By (6.30), parameters of a backward Markov model (6.5) are obtained from those of a forward one (6.1). For $k = 2, 3, \dots, N$,

$$(M_0^B)^{-1} = M_0^{-1} + M_{1,0}' M_1^{-1} M_{1,0} \tag{6.36}$$
$$M_{0,1}^B = M_0^B M_{1,0}' M_1^{-1} \tag{6.37}$$
$$(M_{k-1}^B)^{-1} = M_{k-1}^{-1} + M_{k,k-1}' M_k^{-1} M_{k,k-1} - (M_{k-2,k-1}^B)' (M_{k-2}^B)^{-1} M_{k-2,k-1}^B \tag{6.38}$$
$$M_{k-1,k}^B = M_{k-1}^B M_{k,k-1}' M_k^{-1} \tag{6.39}$$
$$(M_N^B)^{-1} = M_N^{-1} - (M_{N-1,N}^B)' (M_{N-1}^B)^{-1} M_{N-1,N}^B \tag{6.40}$$

By (6.31), the dynamic noise and boundary values of the two models are related by

$$(M_0^B)^{-1} e_0^{BM} = M_0^{-1} e_0^M - M_{1,0}' M_1^{-1} e_1^M \tag{6.41}$$
$$(M_k^B)^{-1} e_k^{BM} = (M_{k-1,k}^B)' (M_{k-1}^B)^{-1} e_{k-1}^{BM} + M_k^{-1} e_k^M - M_{k+1,k}' M_{k+1}^{-1} e_{k+1}^M, k \in [1, N-1] \tag{6.42}$$
$$(M_N^B)^{-1} e_N^{BM} = (M_{N-1,N}^B)' (M_{N-1}^B)^{-1} e_{N-1}^{BM} + M_N^{-1} e_N^M \tag{6.43}$$

By these equations, given a backward model, one can obtain its algebraically equivalent forward model.

For a forward Markov model with a nonsingular state transition matrix, [70] determined the relationship of the dynamic noise and boundary values between algebraically equivalent forward and backward models. But in the case of singular state transition matrices, forward and backward models of [70] are not algebraically equivalent, but only (probabilistically) equivalent. Our (6.36)–(6.40) and (6.41)–(6.43) give algebraically equivalent forward and backward models whether the state transition matrix is singular or nonsingular. Based on (6.41)–(6.43), we can verify the required condition for the two-filter smoother [62]–[64] for Markov models with singular/nonsingular state transition matrices.

6.3.2 Reciprocal CM_L and Reciprocal Models

By (6.30), parameters of a reciprocal model are obtained from those of a reciprocal CM_L model. For (2.17)–(2.18), parameters of the reciprocal model are

$$R_0^0 = G_0^{-1} + G'_{1,0}G_1^{-1}G_{1,0} + G'_{N,0}G_N^{-1}G_{N,0} \tag{6.44}$$

$$R_k^0 = G_k^{-1} + G'_{k+1,k}G_{k+1}^{-1}G_{k+1,k}, k \in [1, N-2] \tag{6.45}$$

$$R_{N-1}^0 = G_{N-1}^{-1} \tag{6.46}$$

$$R_N^0 = G_N^{-1} + \sum_{k=1}^{N-1} G'_{k,N}G_k^{-1}G_{k,N} \tag{6.47}$$

$$R_k^+ = G'_{k+1,k}G_{k+1}^{-1}, k \in [0, N-2] \tag{6.48}$$

$$R_{N-1}^+ = G_{N-1}^{-1}G_{N-1,N} \tag{6.49}$$

$$R_0^- = -G'_{1,0}G_1^{-1}G_{1,N} + G'_{N,0}G_N^{-1} \tag{6.50}$$

and for (2.17) and (2.19) we have (6.45)–(6.46), (6.48)–(6.49), and

$$R_0^0 = G_0^{-1} + G'_{1,0}G_1^{-1}G_{1,0} \tag{6.51}$$

$$R_N^0 = G_N^{-1} + \sum_{k=1}^{N-1} G'_{k,N}G_k^{-1}G_{k,N} + G'_{0,N}G_0^{-1}G_{0,N} \tag{6.52}$$

$$R_0^- = G_0^{-1}G_{0,N} - G'_{1,0}G_1^{-1}G_{1,N} \tag{6.53}$$

By (6.31), the dynamic noise and boundary values of the two models are related by: for (2.17)–(2.18),

$$e_0^R = G_0^{-1}e_0 - G'_{1,0}G_1^{-1}e_1 - G'_{N,0}G_N^{-1}e_N \tag{6.54}$$

$$e_k^R = G_k^{-1}e_k - G'_{k+1,k}G_{k+1}^{-1}e_{k+1}, k \in [1, N-2] \tag{6.55}$$

$$e_{N-1}^R = G_{N-1}^{-1}e_{N-1} \tag{6.56}$$

$$e_N^R = -\sum_{k=1}^{N-1} G'_{k,N}G_k^{-1}e_k + G_N^{-1}e_N \tag{6.57}$$

and for (2.17) and (2.19), (6.54) and (6.57) are replaced by

$$e_0^R = G_0^{-1}e_0 - G'_{1,0}G_1^{-1}e_1 \tag{6.58}$$

$$e_N^R = -\sum_{k=1}^{N-1} G'_{k,N}G_k^{-1}e_k + G_N^{-1}e_N - G'_{0,N}G_0^{-1}e_0 \tag{6.59}$$

By these equations, one can obtain an algebraically equivalent reciprocal CM_L model from a reciprocal model. This is important because a reciprocal CM_L model is easier to apply than a reciprocal model (Chapter 3, Chapter 7).

6.4 More About Algebraically Equivalent Models

6.4.1 Models Algebraically Equivalent to a Reciprocal Model

This section presents two approaches for determination of models algebraically equivalent to a reciprocal model (6.9) along with (6.12) or (6.13), or the other way round. We consider only boundary condition (6.13). The same approach works for boundary condition (6.12).

We first show how to determine parameters of a reciprocal model (6.9) and (6.13) equivalent to other models. For example, from the parameters of a reciprocal CM_L model (2.17) and (2.19), those of its equivalent reciprocal model (6.9) and (6.13) are obtained as follows. Regardless of its boundary condition, model (6.9) is obtained based on conditional expectations [18], so its parameters are as given in Subsection 6.3.2 for a NG reciprocal sequence (i.e., with a given covariance matrix). (2.19) and (6.13) are the same since they are both obtained from the joint density of x_0 and x_N, which is the same for both reciprocal and reciprocal CM_L models.

Similarly, from parameters of a reciprocal model (6.9) and (6.13), we can uniquely determine parameters of its equivalent reciprocal CM_L model (2.17) and (2.19). Also, by (6.30), parameters of other equivalent models can be determined.

Algebraically equivalent models are discussed next.

The First Approach

We show that the unified approach of Section 6.2 (i.e., (6.31)) works for models algebraically equivalent to a reciprocal model (6.9) and (6.13).

First, we determine the structure of T, P, and ξ in (6.27) for model (6.9) and (6.13). We have

$$\mathfrak{R}_r x = e^r \tag{6.60}$$

where $e^r \triangleq [(e_0^R)', \ldots, (e_N^R)']'$ and

$$\mathfrak{R}_r = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & -R_{0,N} \\ -R_1^- & R_1^0 & -R_1^+ & \cdots & 0 & 0 \\ 0 & -R_2^- & R_2^0 & -R_2^+ & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -R_{N-1}^- & R_{N-1}^0 & -R_{N-1}^+ \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{6.61}$$

It is nonsingular because its submatrix of the block rows and columns 2 to N is nonsingular since (6.16) is nonsingular. Its nonsingularity can be verified based on the determinant of a partitioned matrix [92]. Also, the covariance of e^r is

$$R_r = \begin{bmatrix} R_0^0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & R_1^0 & -R_1^+ & \cdots & 0 & 0 \\ 0 & -R_2^- & R_2^0 & -R_2^+ & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -R_{N-1}^- & R_{N-1}^0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & R_N^0 \end{bmatrix} \tag{6.62}$$

which is likewise nonsingular since its submatrix of block rows and columns 2 to N is same as that of (6.16) because model (6.9) is independent of boundary condition [18].

With (6.61) and (6.62), models algebraically equivalent to (6.9) and (6.13) can be obtained by (6.31).

The Second Approach

In the first approach, $(R_r)^{-1}$ is required in (6.31), which is not desirable since R_r is not block diagonal. In the following, we present a simple relationship in dynamic noise and boundary values between a reciprocal model and an algebraically equivalent reciprocal CM_L model.

It suffices to construct a reciprocal CM_L model algebraically equivalent to a reciprocal model. Then, by Proposition 6.2.2 other algebraically equivalent models can be obtained.

We show that (6.63) below makes an equivalent reciprocal model algebraically equivalent to a reciprocal CM_L model (2.17) and (2.19):

$$e^r = T_{R|CM_L} e \tag{6.63}$$

where $T_{R|CM_L}$ is the nonsingular matrix

$$\begin{bmatrix} I & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & G_1^{-1} & -G'_{2,1}G_2^{-1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & G_2^{-1} & -G'_{3,2}G_3^{-1} & \cdots & 0 & 0 \\ 0 & 0 & 0 & G_3^{-1} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & G_{N-1}^{-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \tag{6.64}$$

$e^r \triangleq [(e_0^R)', \ldots, (e_N^R)']'$ is the vector of dynamic noise and boundary values of the reciprocal model, and $e \triangleq [e'_0, \ldots, e'_N]'$ is of the reciprocal CM_L model.

Let $[e_k]$ be white (since it is for a reciprocal CM_L model). We show that $[e_k^R]$ has the properties of reciprocal dynamic noise and boundary values. By (6.64), the covariance of $[e_k^R]_1^{N-1}$ is cyclic tridiagonal. So, $[e_k^R]_1^{N-1}$ can serve as dynamic noise of a reciprocal model (6.9). It is a function of $[e_k]_1^{N-1}$ with $e_0^R = e_0$ and $e_N^R = e_N$. Then, since $[e_k]$ is white, $[e_k^R]_1^{N-1}$ is uncorrelated with e_0^R and e_N^R and consequently with x_0 and x_N. Therefore, $[e_k^R]_1^{N-1}$ can serve as reciprocal dynamic noise, and e_0^R and e_N^R as boundary values.

Now, we show that (6.63) leads to the same sample path of the sequence obeying the reciprocal CM_L model and the reciprocal model. From (6.63), we have

$$e_k^R = G_k^{-1}e_k - G'_{k+1,k}G_{k+1}^{-1}e_{k+1}, k \in [1, N-2] \tag{6.65}$$

Substituting e_k and e_{k+1} of the CM_L model (2.17) into (6.65), after some manipulation, we get

$$\begin{aligned} e_k^R = {} & (G_k^{-1} + G'_{k+1,k}G_{k+1}^{-1}G_{k+1,k})x_k - G_k^{-1}G_{k,k-1}x_{k-1} - G'_{k+1,k}G_{k+1}^{-1}x_{k+1} + \\ & (-G_k^{-1}G_{k,N} + G'_{k+1,k}G_{k+1}^{-1}G_{k+1,N})x_N \end{aligned} \tag{6.66}$$

Using (3.24), (6.66) becomes

$$e_k^R = (G_k^{-1} + G'_{k+1,k}G_{k+1}^{-1}G_{k+1,k})x_k - G_k^{-1}G_{k,k-1}x_{k-1} - G'_{k+1,k}G_{k+1}^{-1}x_{k+1} \tag{6.67}$$

(6.67) has the properties (of the structure and parameters) of (6.9) and thus can serve as a reciprocal model for $k \in [1, N-2]$. In addition, for $k = N-1$, based on (6.63) we have

$$e_{N-1}^R = G_{N-1}^{-1}e_{N-1} \tag{6.68}$$

Substituting e_{N-1} of (2.17), we have

$$e_{N-1}^R = G_{N-1}^{-1}x_{N-1} - G_{N-1}^{-1}G_{N-1,N-2}x_{N-2} - G_{N-1}^{-1}G_{N-1,N}x_N \tag{6.69}$$

(6.69) can serve as a reciprocal model for $k = N-1$. So, by (6.67), (6.69) and since (2.19) and (6.13) are identical, (6.63) leads to the same sample path of the sequence obeying the two models (and their boundary conditions). In other words, the two models are algebraically equivalent.

Next, from a reciprocal model (6.9) and (6.13), we construct its algebraically equivalent reciprocal CM_L model. Calculation of the parameters of (2.17) and (2.19) from those of (6.9) and (6.13) was discussed above. So, $T_{R|CM_L}$ is known. First, we show that e in (6.63) has a (block) diagonal covariance matrix, i.e, $[e_k]$ is white (which is the case for a reciprocal CM_L model). According to (6.9) and (6.13), e_0^R and e_N^R are uncorrelated, and uncorrelated with $[e_k^R]_1^{N-1}$. By (6.63), we have $e_0 = e_0^R$ and $e_N = e_N^R$. Also, $[e_k]_1^{N-1}$ are linear combinations of $[e_k^R]_1^{N-1}$. So, e_0 and e_N are mutually uncorrelated and uncorrelated with $[e_k]_1^{N-1}$. Therefore, we only need to show that $[e_k]_1^{N-1}$ is white. The covariance of $[(e_1^R)', \ldots, (e_{N-1}^R)']'$ is $(R_r)_{[2:N,2:N]}$, i.e., matrix (6.62) without the first and the last block rows and columns. By (6.63), we have

$$(R_r)_{[2:N,2:N]} = (T_{R|CM_L})_{[2:N,2:N]}(\text{Cov}(e))_{[2:N,2:N]}(T_{R|CM_L})'_{[2:N,2:N]} \tag{6.70}$$

Let C be the covariance matrix of the reciprocal sequence. Now calculate C^{-1} based on the reciprocal CM_L model (2.17) and (2.19) (Appendix B). The tridiagonal matrix $(C^{-1})_{[2:N,2:N]}$ can be decomposed as

$$(C^{-1})_{[2:N,2:N]} = (T_{R|CM_L})_{[2:N,2:N]}G_{[2:N,2:N]}(T_{R|CM_L})'_{[2:N,2:N]} \tag{6.71}$$

where $G_{[2:N,2:N]} = \text{diag}(G_1, \ldots, G_{N-1})$. Comparing $(R_r)_{[2:N,2:N]}$ with (6.16), it can be seen that $(R_r)_{[2:N,2:N]} = (C^{-1})_{[2:N,2:N]}$. Comparing (6.71) and (6.70), we have

$$(\text{Cov}(e))_{[2:N,2:N]} = G_{[2:N,2:N]}$$

meaning that $[e_k]_1^{N-1}$ is white. So, $[e_k]$ is white.

Next, we show that (6.63) leads to algebraic equivalence of the reciprocal model and the reciprocal CM_L model. (6.63) for $k = N-1$ is

$$e_{N-1}^R = G_{N-1}^{-1}e_{N-1} \tag{6.72}$$

Using e_{N-1}^R from the reciprocal model (6.9), we obtain

$$R_{N-1}^0 x_{N-1} - R_{N-1}^- x_{N-2} - R_{N-1}^+ x_N = G_{N-1}^{-1}e_{N-1}$$

Expressing R_{N-1}^0, R_{N-1}^-, and R_{N-1}^+ of the reciprocal model in terms of parameters of the reciprocal CM_L model (specifically (6.46), (6.48), (6.49)) yields

$$G_{N-1}^{-1}x_{N-1} - (G_{N-1}^{-1}G_{N-1,N-2})x_{N-2} - (G_{N-1}^{-1}G_{N-1,N})x_N = G_{N-1}^{-1}e_{N-1}$$

which leads to

$$x_{N-1} - G_{N-1,N-2}x_{N-2} - G_{N-1,N}x_N = e_{N-1} \tag{6.73}$$

Clearly (6.73) is a CM_L model (2.17) for $k = N-1$ with an e_{N-1} that is related to e_{N-1}^R by (6.72). Then, By (6.63), for $k \in [1, N-2]$, we have

$$e_k^R = G_k^{-1}e_k - G'_{k+1,k}G_{k+1}^{-1}e_{k+1} \tag{6.74}$$

Substituting e_k^R of the reciprocal model (6.9) into (6.74) yields

$$R_k^0 x_k - R_k^- x_{k-1} - R_k^+ x_{k+1} = G_k^{-1}e_k - G'_{k+1,k}G_{k+1}^{-1}e_{k+1} \tag{6.75}$$

Substituting e_{k+1} from the reciprocal CM_L model (2.17) into (6.75), we obtain

$$(G_k^{-1} + G'_{k+1,k}G_{k+1}^{-1}G_{k+1,k})x_k - G_k^{-1}G_{k,k-1}x_{k-1} - G'_{k+1,k}G_{k+1}^{-1}x_{k+1} = \\ G_k^{-1}e_k - G'_{k+1,k}G_{k+1}^{-1}(x_{k+1} - G_{k+1,k}x_k - G_{k+1,N}x_N) \tag{6.76}$$

After manipulation of (6.76), we obtain

$$G_k^{-1}x_k - G_k^{-1}G_{k,k-1}x_{k-1} - G'_{k+1,k}G_{k+1}^{-1}G_{k+1,N}x_N = G_k^{-1}e_k \tag{6.77}$$

Using (3.24) for the coefficient of x_N in (6.77), (6.77) leads to

$$x_k - G_{k,k-1}x_{k-1} - G_{k,N}x_N = e_k \tag{6.78}$$

This is a CM_L model (2.17) for $k \in [1, N-2]$ with an $[e_k]_1^{N-2}$ that is related to $[e_k^R]_1^{N-2}$ by (6.63). Also, the two models have identical boundary conditions. So, (6.63) connects the two models by having the same sample paths of the reciprocal sequence. In other words, using (6.63), the reciprocal model and the reciprocal CM_L model are algebraically equivalent.

6.4.2 Parameters of Equivalent Markov and Reciprocal Models

By (6.30), parameters of equivalent models can be uniquely determined (Appendix B). In some cases given parameters of a model, one can calculate parameters of an equivalent model in a different way. Due to the uniqueness, the apparently different results must be the same. For example, in the following we consider an approach (different from (6.30)) for calculating parameters of a reciprocal model equivalent to a Markov model. Then, we show that the results are actually the same as those of Appendix B.

Given a Markov model (6.1) of $[x_k]$, by (6.30), parameters of an equivalent reciprocal model (6.9) are (Appendices B.4 and B.3), for $k \in [1, N-1]$,

$$R_k^0 = M_k^{-1} + M'_{k+1,k}M_{k+1}^{-1}M_{k+1,k} \tag{6.79}$$

$$R_k^+ = M'_{k+1,k}M_{k+1}^{-1} \tag{6.80}$$

$$R_k^- = M_k^{-1}M_{k,k-1} \tag{6.81}$$

Parameters of the reciprocal model (6.9) can be also obtained as follows. The transition density of $[x_k]$ is

$$p(x_k|x_{k-1}) = \mathcal{N}(x_k; M_{k,k-1}x_{k-1}, M_k) \tag{6.82}$$

Given (6.82), by the Markov property, we have

$$p(x_k|x_{k-1}, x_{k+1}) = \frac{p(x_k|x_{k-1})p(x_{k+1}|x_k)}{p(x_{k+1}|x_{k-1})} \\ = \mathcal{N}(x_k; R_{k,k-1}x_{k-1} + R_{k,k+1}x_{k+1}, R_k)$$

Then, we define r_k as

$$r_k = x_k - R_{k,k-1}x_{k-1} - R_{k,k+1}x_{k+1} \tag{6.83}$$

where the covariance of r_k is R_k and

$$R_{k,k-1} = M_{k,k-1} - (M_k^{-1} + M'_{k+1,k}M_{k+1}^{-1}M_{k+1,k})^{-1}M'_{k+1,k}M_{k+1}^{-1}M_{k+1,k}M_{k,k-1}$$

$$R_{k,k+1} = (M_k^{-1} + M'_{k+1,k}M_{k+1}^{-1}M_{k+1,k})^{-1}M'_{k+1,k}M_{k+1}^{-1}$$

$$R_k = (M_k^{-1} + M'_{k+1,k}M_{k+1}^{-1}M_{k+1,k})^{-1}$$

Pre-multiplying both sides of (6.83) by R_k^0 (which is nonsingular), we obtain

$$R_k^0 x_k = R_k^0 R_{k,k-1} x_{k-1} + R_k^0 R_{k,k+1} x_{k+1} + R_k^0 r_k \tag{6.84}$$

By the uniqueness of parameters, we must have $R_k^0 R_{k,k-1} = R_k^-$, $R_k^0 R_{k,k+1} = R_k^+$, and $\text{Cov}(R_k^0 r_k) = R_k^0$. Comparing the parameters of (6.84) with (6.79), (6.80), and (6.81), it is not clear that $R_k^0 R_{k,k-1} = R_k^-$, which, however, can be verified using

$$(M_k^{-1} + M'_{k+1,k} M_{k+1}^{-1} M_{k+1,k})^{-1} (M_k^{-1} + M'_{k+1,k} M_{k+1}^{-1} M_{k+1,k}) M_{k,k-1} = M_{k,k-1}$$

6.5 Markov Models and Reciprocal/CM_L Models

An important question in the theory of reciprocal processes is about Markov processes governed by the same reciprocal evolution law [16]–[17], [9]. It is desired to determine Markov evolution models (i.e., without the initial condition) of Markov sequences, which obey a reciprocal CM_L evolution model (and an arbitrary boundary condition). Also, given two Markov evolution models, whether their sequences share the same CM_L evolution model? Studying such issues will gain a better understanding of the models and their sequences, and is useful for their application. For example, in Chapter 7 we discuss CM_L evolution models *induced* by Markov evolution models (presented in Chapter 4) for trajectory modeling with destination information, and show that inducing a CM_L evolution model by a Markov evolution model is useful for parameter design of a reciprocal CM_L evolution model. Also, in Chapter 4 we showed that a reciprocal CM_L evolution model can be induced by any Markov evolution model whose sequences obey the given reciprocal CM_L evolution model (and some boundary condition). So, it is desired to determine all such Markov evolution models and their relationships. In the following, a simple approach is presented for studying and determining different Markov models whose sequences share the same reciprocal/CM_L evolution model.

Relationships between different models (and their boundary conditions) can be studied based on the entries of C^{-1} calculated from the models and their boundary conditions. Some entries of C^{-1} depend on evolution model parameters only and others depend also on boundary condition (Appendix B). Proofs of the following results are based on Appendix B.

The next proposition gives conditions for Markov models of Markov sequences to share the same reciprocal model.

Proposition 6.5.1. *Two Markov sequences modeled by Markov models* (6.1) *with parameters* $M_{k,k-1}^{(i)}$, $M_k^{(i)}$, $k \in [1, N]$, $i = 1, 2$, *share the same reciprocal evolution model* (6.9) *iff*

$$\begin{aligned}&(M_k^{(1)})^{-1} + (M_{k+1,k}^{(1)})'(M_{k+1}^{(1)})^{-1} M_{k+1,k}^{(1)} = \\ &(M_k^{(2)})^{-1} + (M_{k+1,k}^{(2)})'(M_{k+1}^{(2)})^{-1} M_{k+1,k}^{(2)}, k \in [1, N-1]\end{aligned} \tag{6.85}$$

$$(M_{k+1,k}^{(1)})'(M_{k+1}^{(1)})^{-1} = (M_{k+1,k}^{(2)})'(M_{k+1}^{(2)})^{-1}, k \in [0, N-1] \tag{6.86}$$

Proof. Two sequences share the same reciprocal evolution model iff their C^{-1} (3.23) have the same entries $A_1, A_2, \ldots, A_{N-1}, B_0, B_1, \ldots, B_{N-1}$. So, two Markov sequences having Markov models with parameters $M_{k,k-1}^{(i)}$, $M_k^{(i)}$, $k \in [1, N]$, $i = 1, 2$, share the same reciprocal model iff (6.85)–(6.86) hold. □

Sequences modeled by any Markov model (6.1) satisfying

$$R_k^0 = M_k^{-1} + M'_{k+1,k} M_{k+1}^{-1} M_{k+1,k}, k \in [1, N-1] \tag{6.87}$$

$$R_k^+ = M'_{k+1,k} M_{k+1}^{-1}, k \in [0, N-1] \tag{6.88}$$

share a given reciprocal evolution model with parameters R_k^0, $k \in [1, N-1]$, and R_k^+, $k \in [0, N-1]$ (with some boundary condition) (see Proposition 6.5.1). Therefore, all Markov models whose sequences share a reciprocal model are determined.

Proposition 6.5.2. *Two sequences share the same reciprocal evolution model* (6.9) *iff they share the same reciprocal CM_L evolution model* (2.17) *($c = N$).*

Proof. Two sequences share the same reciprocal evolution model (6.9) (reciprocal CM_L evolution model (2.17) ($c = N$)) iff their C^{-1} (3.23) have the same entries $A_1, \ldots, A_{N-1}, B_0, \ldots, B_{N-1}$. So, two sequences share the same reciprocal evolution model (6.9) iff they share the same reciprocal CM_L evolution model (2.17) ($c = N$). □

By Proposition 6.5.2 and (6.87)–(6.88) we can determine all Markov models whose sequences share a reciprocal CM_L evolution model (2.17). All we need to do is to replace the model parameters in (6.87)–(6.88) (i.e., R^0_k and R^+_k) with the corresponding (block) entries of the C^{-1} calculated from the parameters of (2.17) (see Subsection 6.3.2 or Appendix B).

The following proposition determines conditions for two Markov sequences sharing the same reciprocal evolution model to share the same Markov evolution model.

Proposition 6.5.3. *Two Markov sequences sharing the same reciprocal evolution model* (6.9) *share the same Markov evolution model* (6.1) *iff for the parameters of* (6.11) *we have*

$$(R^0_N)^{(1)} = (R^0_N)^{(2)} \tag{6.89}$$

or equivalently $M^{(1)}_N = M^{(2)}_N$, where the superscripts (1) *and* (2) *correspond to the first and the second sequence.*

Proof. Two sequences share the same reciprocal evolution model iff their C^{-1} (3.23) have the same entries $A_1, \ldots, A_{N-1}, B_0, \ldots, B_{N-1}$. Two Markov sequences share the same Markov evolution model iff their C^{-1} ((3.23) with $D_0 = 0$) have the same entries $A_1, \ldots, A_N, B_0, \ldots, B_{N-1}$. So, two Markov sequences sharing the same reciprocal evolution model share the same Markov evolution model iff they have the same A_N, i.e., (6.89) holds (see (B.72)). □

More general relationships between different forward/backward CM_L, CM_F, reciprocal, Markov models can be studied based on the entries of C^{-1} calculated from the models and the boundary conditions. In general, we can obtain conditions for two sequences sharing the same evolution model to share the same evolution model of different type.

Chapter 7

Trajectory Modeling, Filtering, and Prediction Using CM Sequences

In this chapter, we discuss an application of CM sequences in modeling trajectories with destination information. To emphasize that the trajectory ends up at a specific destination, we call it *destination-directed trajectory (DDT)*.

7.1 DDT Modeling

To model the trajectory of a moving object without the notion of destination there are two main components: the evolution (motion) law and the origin. On the other hand, the Markov sequence is determined by two components: an evolution law and an initial density. Sample paths of a Markov sequence can be used for modeling such trajectories. For example, a nearly constant velocity/acceleration/turn (with white noise) model describes a Markov sequence. Markov property is simple and effective and this is the reason for its widespread use in application and theory.

In trajectory modeling problems there might be some information available about the destination. A case in point is in air traffic control (ATC), where destination of flight is available. The main components of destination-directed trajectories are an origin, a destination, and motion in between. The Markov sequence is not flexible enough for DDT modeling because its final density is determined by its initial density and evolution law. Therefore, a more general class of stochastic sequences with an initial density, evolution law, and final density as main components is desired.

In the following, some properties desirable for a DDT model and the corresponding inference are discussed. Such a model should take the three main components of DDT into account. It should be able to model any origin and destination. Also, the evolution law (as the most important part of the model) should be able to describe trajectories corresponding to any origin and destination. In other words, the model should be general enough to describe trajectories in different scenarios according to available information. In addition, the evolution law should be simple and easy to apply, yet has the potential to be generalized to more powerful ones if necessary. Moreover, it is desired to model the relationship between the trajectories at the origin and the destination. In some applications (e.g., ATC) an accurate prior density of the destination state might be available. In some other applications, based on the available information about the destination, an approximate prior density might be available. An automatic update of the prior density (to the posterior, a more concentrated density) is desired as more measurements are received. The impact of destination is the key to DDT modeling. However, the state estimate over time (especially far from the destination) should not be sensitive to (the mismatch of) the prior destination density. Also, it is useful to have guidelines for a suitable design of an approximate prior destination density to decrease the mismatch impact on state estimate near the destination.

CM_L sequences provide a general framework for DDT modeling that enjoys the above desirable properties. Some of these properties are about modeling and some others regarding filtering/prediction. Therefore, some of them are addressed in this section and others are discussed in Subsection 7.2.3, after presenting filtering of CM_L sequences in Subsections 7.2.1 and 7.2.2.

Some desirable properties of CM_L sequences for DDT modeling are as follows: 1) they fit well the need to model the main DDT components, 2) they have a Markov-like evolution law, which is simple and well understood, 3) they include reciprocal sequences as a special case (Chapter 3), 4) the CM_L dynamic model ((7.6) below) has an appropriate structure for describing DDT, 5) the CM_L model can systematically model the impact of destination on the evolution of trajectories (see (7.6) below), 6) the CM_L model has white dynamic noise which is desirable for simplicity, 7) state estimation based on the CM_L model is straightforward, and 8) CM_L sequences (and their dynamic models) can be simply and systematically generalized, if necessary. Later, we elaborate these and some other properties of CM_L sequences for DDT modeling and prediction.

Here we only briefly compare the structure of our CM_L model and the reciprocal model of [18] for DDT modeling. The model of [18] has a nearest-neighbor structure (i.e., the current state depends on the previous state and the next state). As a result, for estimation of the current state, prior information (density) of the next state is required. However, such information is not available. Based on our CM_L model, for estimation of the current state, information about the last state (destination) is required. For our problem (i.e., trajectory modeling with destination information) such information is available. Also, dynamic noise of the reciprocal model of [18] is colored. As a result, state estimation based on that model is not straightforward. However, dynamic noise of our CM_L model is white and its state estimation is straightforward.

7.1.1 CM_L Sequences for DDT Modeling

Let the trajectory be modeled as a sequence $[x_k]$. In probability theory, one can interpret the main elements of a DDT (i.e., an origin, a destination, and motion in between) as follows. The origin (destination) is modeled by a density function of x_0 (x_N). The relationship between the origin and the destination is modeled by their joint density, i.e., joint density of x_0 and x_N. Since the destination (i.e., density of x_N) is (assumed) known, the evolution law can be modeled as a conditional density (over the space of sample paths) given the state at destination x_N. Different choices of this conditional density correspond to different evolution laws. The simplest choice is that conditioned on x_N the density is equal to the product of its marginals: $p([x_k]_0^{N-1}|x_N) = \prod_{k=0}^{N-1} p(x_k|x_N)$. However, this choice is often inadequate. Then, the next choice is a conditional density corresponding to the Markov sequence: $p([x_k]_0^{N-1}|x_N) = p(x_0|x_N) \prod_{k=1}^{N-1} p(x_k|x_{k-1}, x_N)$. This is the evolution law of the CM_L sequence (Chapter 2). The main elements of a CM_L sequence $[x_k]$ are: a joint density of x_0 and x_N—in other words, an initial density and a final density conditioned on the initial, or equivalently, the other way round—in addition an evolution law, where the evolution law is conditionally Markov (conditioned on x_N). The above argument naturally leads to CM_L sequences for DDT modeling. Following the same argument, we can consider more general and complicated evolution laws, if necessary. For example, the conditional law (conditioned on x_N) can be higher-order Markov instead of first-order Markov. Therefore, by choosing conditional laws, all DDT can be modeled.

The CM_L sequence is studied in more detail below to demonstrate its use for DDT modeling. In the following, sample path generation of the Markov sequence and the CM_L sequence is discussed.

There are many different ways for sample path generation of a stochastic sequence. Let $p(\cdot)$ and $p(\cdot|\cdot)$ denote any joint and conditional density function, respectively. The *causal* approach for sample path generation is based on the following representation

$$p([x_k]) = p(x_N|[x_i]_0^{N-1}) \cdots p(x_2|x_1, x_0)p(x_1|x_0)p(x_0) \tag{7.1}$$

meaning that first x_0 is generated, and then conditioned on the realization of x_0, x_1 is generated, and so on. For generation of x_k, realizations of all the previous states are required. This approach is causal because the realization of x_k does not depend on the realizations of any future state. Depending on the properties of a sequence, there might be simpler ways for sample path generation. In the following, Markov sequence sample path generation is discussed. Then, a simple approach for CM_L sample path generation is presented.

Let $[x_k]$ be a Markov sequence. Then, $p(x_k|[x_i]_0^{k-1}) = p(x_k|x_{k-1})$. Therefore, the causal approach of (7.1) leads to a simple way for sample path generation, which can be seen in two steps: first the initial state is generated from $p(x_0)$, then the subsequent states are generated from the transition density $p(x_k|x_{k-1})$ step by step as follows:

$$p([x_k]) = p([x_k]_1^N|x_0)p(x_0) = \Big(\prod_{i=1}^{N} p(x_i|x_{i-1})\Big)p(x_0) \tag{7.2}$$

Corresponding to (7.2), we have model (6.1).

Unlike for the Markov sequence, the causal sample path generation (7.1) does not lead to a simple way for the CM_L sequence sample path generation. Following (7.1), it can be seen that the state of a ZMNG CM_L sequence $[x_k]$ generally obeys

$$x_k = \sum_{i=0}^{k-1} F_{k,i}x_i + d_k, \quad k \in [1, N] \tag{7.3}$$

where x_0 is uncorrelated with $[d_k]_1^N$, which is a zero-mean white NG sequence. But this model is not simple for application. By definition, for a CM_L sequence $[x_k]$, we have $p(x_k|[x_i]_0^{k-1}, x_N) = p(x_k|x_{k-1}, x_N)$. A simple way for the CM_L sample path generation is as follows: first generate the endpoint states from their joint density $p(x_0, x_N)$, and then generate other states based on the transition density $p(x_k|x_{k-1}, x_N)$. For example, we can first generate x_N from $p(x_N)$ and then x_0 from $p(x_0|x_N)$. So, we have

$$\begin{aligned} p([x_k]) &= p([x_k]_0^{N-1}|x_N)p(x_N) && (7.4)\\ &= p([x_k]_1^{N-1}|x_0, x_N)p(x_0|x_N)p(x_N) \\ &= \Big(\prod_{i=1}^{N-1} p(x_i|x_{i-1}, x_N)\Big)p(x_0|x_N)p(x_N) && (7.5) \end{aligned}$$

It should be noticed that given a joint density of a CM_L sequence $[x_k]$, (7.1) and (7.5) give the same set of paths.

Corresponding to (7.5), we have CM_L model (2.17) and (2.19) ($c = N$).

For trajectory modeling we need non-zero-mean sequences. A non-zero-mean NG sequence is CM_L iff its zero-mean part follows a CM_L model (Chapter 2). Similarly, a non-zero-mean NG sequence is Markov iff its zero-mean part follows a Markov model. The CM_L model (and its boundary condition) of the non-zero-mean Gaussian CM_L sequences considered in the simulations (for DDT modeling) is as follows. Let μ_0 (μ_N) and C_0 (C_N) be the mean and covariance of the origin (destination) state distribution. Also, let $C_{0,N}$ be the cross-covariance of x_0 and x_N. We have

$$x_k = G_{k,k-1}x_{k-1} + G_{k,N}x_N + e_k, \quad k \in [1, N-1] \tag{7.6}$$

$$x_N = \mu_N + e_N, \quad x_0 = \mu_0 + G_{0,N}(x_N - \mu_N) + e_0 \tag{7.7}$$

where $G_{0,N} = C_{0,N}C_N^{-1}$, $G_0 = \text{Cov}(e_0) = C_0 - C_{0,N}C_N^{-1}(C_{0,N})'$, and $G_N = \text{Cov}(e_N) = C_N$. Parameter design for (7.6) is discussed in Subsection 7.1.2.

In (7.5) or (7.6), x_N is generated before other states. In other words, the last state is generated first and realizations of other states depend on the realization of the last one. Therefore, the model is not causal. Is this non-causal model applicable for DDT modeling, filtering, and prediction in reality? The answer to this question is based on the filter derived for the CM_L sequence in Section 7.2. Therefore, the applicability of model (7.6) is discussed in Subsection 7.2.3. Here we only mention that the non-causal model (7.6) requires *information* about x_N (i.e., $p(x_N)$), which is available. Therefore, this model is totally applicable.

7.1.2 CM_L Model Parameter Design for DDT Modeling

To use the CM_L model for DDT modeling, we need an approach for its parameter design. Next we present such an approach.

We show how Theorem 4.1.3 can be used to design parameters of a CM_L model for DDT modeling. DDT can be modeled based on two key assumptions: (i) the moving object follows a Markov model (7.8) below (e.g., a nearly constant velocity model) without the destination information (destination density), and (ii) the joint origin and destination density is known (which can be different from that of the Markov model in (i)). In reality, if the joint density is not known, an approximate density can be used (the density mismatch impact is studied in Section 7.4). Now, (by (i)) let $[y_k]$ be Markov modeled by

$$y_k = M_{k,k-1}y_{k-1} + e_k^M, \quad k \in [1, N], \quad y_0 = e_0^M \tag{7.8}$$

where $[e_k^M]$ is a zero-mean white NG sequence with covariances $M_k, k \in [0, N]$. Every Markov sequence is CM_L. So, $[y_k]$ can be modeled by a CM_L model as

$$y_k = G_{k,k-1}y_{k-1} + G_{k,N}y_N + e_k^y, \quad k \in [1, N-1] \tag{7.9}$$

where $[e_k^y]$ is a zero-mean white NG sequence with covariances $G_k, k \in [1, N-1]$, G_0^y, G_N^y, and boundary condition

$$y_N = e_N^y, \quad y_0 = G_{0,N}^y y_N + e_0^y \tag{7.10}$$

We now obtain parameters of (7.9). Based on the Markov property of $[y_k]$, we have

$$\begin{aligned} p(y_k|y_{k-1}, y_N) &= \frac{p(y_k, y_{k-1}, y_N)}{p(y_{k-1}, y_N)} \\ &= \frac{p(y_k|y_{k-1})p(y_N|y_k, y_{k-1})}{p(y_N|y_{k-1})} \\ &= \frac{p(y_k|y_{k-1})p(y_N|y_k)}{p(y_N|y_{k-1})} \\ &= \mathcal{N}(y_k; G_{k,k-1}y_{k-1} + G_{k,N}y_N; G_k), \quad k \in [1, N-1] \end{aligned} \tag{7.11}$$

and $G_{k,k-1}$, $G_{k,N}$, and G_k are obtained as

$$G_{k,k-1} = M_{k,k-1} - G_{k,N}M_{N|k}M_{k,k-1} \tag{7.12}$$

$$G_{k,N} = G_k M'_{N|k} C_{N|k}^{-1} \tag{7.13}$$

$$G_k = (M_k^{-1} + M'_{N|k} C_{N|k}^{-1} M_{N|k})^{-1} \tag{7.14}$$

where $M_{N|N} = I$,

$$\begin{aligned} M_{N|k} &= M_{N,N-1} \cdots M_{k+1,k}, \quad k \in [1, N-1] \\ C_{N|k} &= \sum_{n=k}^{N-1} M_{N|n+1}M_{n+1}M'_{N|n+1}, \quad k \in [1, N-1] \\ p(y_k|y_{k-1}) &= \mathcal{N}(y_k; M_{k,k-1}y_{k-1}, M_k) \end{aligned}$$

and $M_{k,k-1}, M_k, k \in [1, N]$, are parameters of (7.8).

Now, we construct a different sequence $[x_k]$ modeled also by (7.9) as

$$x_k = G_{k,k-1}x_{k-1} + G_{k,N}x_N + e_k, \quad k \in [1, N-1] \tag{7.15}$$

where $[e_k]$ is a zero-mean white NG sequence with covariances $G_k, k \in [0, N]$, and boundary condition

$$x_N = e_N, \quad x_0 = G_{0,N}x_N + e_0 \tag{7.16}$$

but with different parameters of the boundary condition (i.e., $(G_N, G_{0,N}, G_0) \neq (G^y_N, G^y_{0,N}, G^y_0)$). Note that parameters of (7.9) and (7.15) are the same ($G_{k,k-1}, G_{k,N}, G_k, k \in [1, N-1]$), but parameters of (7.10) ($G^y_{0,N}, G^y_0, G^y_N$) and (7.16) ($G_{0,N}, G_0, G_N$) are different. So, $[y_k]$ and $[x_k]$ are two different sequences. By Theorem 2.2.6, $[x_k]$ is a ZMNG CM_L sequence.

The sequences $[y_k]$ and $[x_k]$ have the same CM_L model (i.e., (7.15) and (7.9) have the same parameters $G_{k,k-1}, G_{k,N}, G_k, k \in [1, N-1]$) or equivalently the same transition density (7.11) or the same evolution law. But since parameters of the boundary condition (7.16) (i.e., $(G_N, G_{0,N}, G_0)$) are arbitrary, $[x_k]$ can have any joint endpoint density. The two assumptions ((i) and (ii)) above naturally leads to the CM_L sequence $[x_k]$ whose CM_L model is the same as that of $[y_k]$ while the former can model any origin and destination. Model (7.15) with (7.12)–(7.14) is desired for DDT modeling based on (i) and (ii) above.

The CM_L model (7.15) with parameters (7.12)–(7.14) is called the CM_L model *induced* by the Markov model (7.8) (or simply the Markov-induced CM_L model) since parameters of the former are obtained from parameters of the latter (Chapter 4). Such a CM_L model is used in our simulations presented in Subsection 7.4.

7.2 DDT Filtering

Consider CM_L model (7.6)–(7.7) and the measurement model

$$z_k = H_k x_k + v_k, \quad k \in [1, N] \tag{7.17}$$

where $[v_k]_1^N$ is a zero-mean white Gaussian noise with $\text{Cov}(v_k) = R_k$ and uncorrelated with $[e_k]$ in (7.6)–(7.7).

The goal is to obtain the minimum mean square error (MMSE) estimate $\hat{x}_k = E[x_k|z^k]$ and its mean square error (MSE) matrix given all measurements from the beginning to time k denoted as $z^k = \{z_1, z_2, \ldots, z_k\}$, where z^0 means no measurement.

We present two formulations of the filter. The first one is simpler, but the second one provides a better intuitive understanding of the behavior of the DDT filter and its main components.

7.2.1 First Formulation

Let $s_k = [x'_k, x'_N]'$. Then, (7.6) can be written as

$$s_k = G^s_{k,k-1}s_{k-1} + e^s_{k-1}, \quad k \in [1, N-1] \tag{7.18}$$

where

$$G^s_{k,k-1} = \begin{bmatrix} G_{k,k-1} & G_{k,N} \\ 0 & I \end{bmatrix}, \quad e^s_k = \begin{bmatrix} e_{k+1} \\ 0 \end{bmatrix}, \quad G^s_k = \text{Cov}(e^s_k) = \begin{bmatrix} G_{k+1} & 0 \\ 0 & 0 \end{bmatrix}$$

Also, (7.17) is written as

$$z_k = H^s_k s_k + v_k, \quad k \in [1, N-1] \tag{7.19}$$

where $H_k^s = [H_k, 0]$. Given $\hat{s}_0 = E[s_0]$ and $\Sigma_0 = \mathrm{Cov}(s_0)$, based on (7.18) and (7.19), the MMSE estimator and its MSE matrix are

$$\hat{s}_k = E[s_k|z^k] = \hat{s}_{k|k-1} + C_{s_k,z_k} C_{z_k}^{-1}(z_k - H_k^s \hat{s}_{k|k-1}) \tag{7.20}$$

$$\Sigma_k = E[(s_k - \hat{s}_k)(s_k - \hat{s}_k)'] = \Sigma_{k|k-1} - C_{s_k,z_k} C_{z_k}^{-1}(C_{s_k,z_k})' \tag{7.21}$$

where $\hat{s}_{k|k-1} = G_{k,k-1}^s \hat{s}_{k-1}$, $\Sigma_{k|k-1} = G_{k,k-1}^s \Sigma_{k-1} (G_{k,k-1}^s)' + G_{k-1}^s$, $C_{s_k,z_k} = \Sigma_{k|k-1}(H_k^s)'$, $C_{z_k} = H_k^s \Sigma_{k|k-1}(H_k^s)' + R_k$. The estimate of x_k and its MSE are

$$\hat{x}_k = [I, 0]\hat{s}_k \tag{7.22}$$

$$P_k = [I, 0]\Sigma_k [I, 0]' \tag{7.23}$$

Given $\hat{s}_{N-1}$ and Σ_{N-1}, we have

$$\hat{x}_{N|N-1} = [0, I]\hat{s}_{N-1} \tag{7.24}$$

$$P_{N|N-1} = [0, I]\Sigma_{N-1}[0, I]' \tag{7.25}$$

where $\hat{x}_{N|N-1}$ is the estimate of x_N given all the measurements up to time $N-1$ and $P_{N|N-1}$ is the corresponding MSE matrix. Given z_N, we have the update

$$\hat{x}_N = \hat{x}_{N|N-1} + C_{x_N,z_N} C_{z_N}^{-1}(z_N - H_N \hat{x}_{N|N-1}) \tag{7.26}$$

$$P_N = P_{N|N-1} - C_{x_N,z_N} C_{z_N}^{-1}(C_{x_N,z_N})' \tag{7.27}$$

where $C_{x_N,z_N} = P_{N|N-1}(H_N)'$ and $C_{z_N} = H_N P_{N|N-1}(H_N)' + R_N$.

The filter is as follows.

- **Initialization**

$$\hat{s}_0 = \begin{bmatrix} \mu_0 \\ \mu_N \end{bmatrix}, \quad \Sigma_0 = \begin{bmatrix} C_0 & C_{0,N} \\ (C_{0,N})' & C_N \end{bmatrix}$$

- **For $k \in [1, N-1]$:**

$$\begin{aligned}
\hat{s}_{k|k-1} &= G_{k,k-1}^s \hat{s}_{k-1} \\
\Sigma_{k|k-1} &= G_{k,k-1}^s \Sigma_{k-1}(G_{k,k-1}^s)' + G_{k-1}^s \\
C_{s,z} &= \Sigma_{k|k-1}(H_k^s)' \\
C_z &= H_k^s \Sigma_{k|k-1}(H_k^s)' + R_k \\
\hat{s}_k &= \hat{s}_{k|k-1} + C_{s,z} C_z^{-1}(z_k - H_k^s \hat{s}_{k|k-1}) \\
\Sigma_k &= \Sigma_{k|k-1} - C_{s,z} C_z^{-1}(C_{s,z})' \\
\hat{x}_k &= [I, 0]\hat{s}_k \\
P_k &= [I, 0]\Sigma_k [I, 0]'
\end{aligned}$$

- **For $k = N$:**

$$\begin{aligned}
\hat{x}_{N|N-1} &= [0, I]\hat{s}_{N-1} \\
P_{N|N-1} &= [0, I]\Sigma_{N-1}[0, I]' \\
C_{x,z} &= P_{N|N-1}(H_N)' \\
C_z &= H_N P_{N|N-1}(H_N)' + R_N \\
\hat{x}_N &= \hat{x}_{N|N-1} + C_{x,z} C_z^{-1}(z_N - H_N \hat{x}_{N|N-1}) \\
P_N &= P_{N|N-1} - C_{x,z} C_z^{-1}(C_{x,z})'
\end{aligned}$$

7.2.2 Second Formulation

The filter is derived based on propagation of posterior density $p(x_k|z^k)$ over time. For $k \in [1, N-1]$ we can write

$$p(x_k|z^k) = \int p(x_k|x_N, z^k)p(x_N|z^k)dx_N \tag{7.28}$$

For calculation of $p(x_k|z^k)$ based on (7.28), the propagation of $p(x_k|x_N, z^k)$ and $p(x_N|z^k)$ (the key terms of the filter) over time is required.

From the boundary condition of the CM_L model, a prior jointly Gaussian endpoint density $p(x_0, x_N)$ with the following mean and covariance is available:

$$\begin{bmatrix} C_0 & C_{0,N} \\ (C_{0,N})' & C_N \end{bmatrix}, \quad \begin{bmatrix} \mu_0 \\ \mu_N \end{bmatrix}$$

Recursive calculation of $p(x_k|x_N, z^k)$ can be done as follows. We have $p(x_0|x_N) = \mathcal{N}(x_0; \mu_{0|N}, \Sigma_{0|N})$, where $\mu_{0|N} = \mu_0 + C_{0,N}C_N^{-1}(x_N - \mu_N)$ and $\Sigma_{0|N} = C_0 - C_{0,N}C_N^{-1}(C_{0,N})'$. For the recursive calculation it is useful to define the following terms $\mu_{0|N} = b_0 + B_0x_N$, $b_0 = \mu_0 - C_{0,N}C_N^{-1}\mu_N$, $B_0 = C_{0,N}C_N^{-1}$, $\mathfrak{B}_0 = \Sigma_{0|N}$. Then,

$$p(x_0|x_N) = \mathcal{N}(x_0; \mu_{0|N}, \Sigma_{0|N}) = \mathcal{N}(x_0; b_0 + B_0x_N, \mathfrak{B}_0) \tag{7.29}$$

Let the conditional density from time $k-1$, and the CM_L transition density based on model (7.6) be given as

$$p(x_{k-1}|x_N, z^{k-1}) = \mathcal{N}(x_{k-1}; b_{k-1} + B_{k-1}x_N, \mathfrak{B}_{k-1}) \tag{7.30}$$

$$p(x_k|x_{k-1}, x_N) = \mathcal{N}(x_k; G_{k,k-1}x_{k-1} + G_{k,N}x_N, G_k) \tag{7.31}$$

Then, for $k \in [1, N-1]$,

$$\begin{aligned} p(x_k|x_N, z^{k-1}) &= \int p(x_k|x_{k-1}, x_N)p(x_{k-1}|x_N, z^{k-1})dx_{k-1} \\ &= \mathcal{N}(x_k; G_{k,k-1}b_{k-1} + D_kx_N, S_k) \end{aligned} \tag{7.32}$$

where $S_k = G_k + G_{k,k-1}\mathfrak{B}_{k-1}(G_{k,k-1})'$ and $D_k = G_{k,k-1}B_{k-1} + G_{k,N}$. By (7.17), we have

$$p(z_k|x_k) = \mathcal{N}(z_k; H_kx_k, R_k) \tag{7.33}$$

For $k \in [1, N-1]$,

$$p(x_k|x_N, z^k) = \frac{p(z_k|x_k)p(x_k|x_N, z^{k-1})}{p(z_k|x_N, z^{k-1})} = \mathcal{N}(x_k; b_k + B_kx_N, \mathfrak{B}_k)$$

where

$$\begin{aligned} b_k &= G_{k,k-1}b_{k-1} + \mathfrak{B}_k(H_k)'R_k^{-1}(z_k - H_kG_{k,k-1}b_{k-1}) \\ B_k &= G_{k,k-1}B_{k-1} + G_{k,N} - \mathfrak{B}_k(H_k)'R_k^{-1}H_k(G_{k,k-1}B_{k-1} + G_{k,N}) \\ \mathfrak{B}_k &= S_k - S_k(H_k)'(R_k + H_kS_k(H_k)')^{-1}H_kS_k \end{aligned}$$

So, the propagation of $p(x_k|x_N, z^k)$ is complete.

The second density $p(x_N|z^k)$ can be recursively calculated as follows. For the purpose of recursive calculation it is useful to write $p(x_N) = \mathcal{N}(x_N; a_0, A_0)$, where $a_0 = \mu_N$ and $A_0 = C_N$. For $k \in [1, N-1]$, we have

$$p(x_N|z^k) = \frac{p(z_k|x_N, z^{k-1})p(x_N|z^{k-1})}{p(z_k|z^{k-1})} \tag{7.34}$$

where from $k-1$ we have

$$p(x_N|z^{k-1}) = \mathcal{N}(x_N; a_{k-1}, A_{k-1}) \tag{7.35}$$

Also,

$$\begin{aligned} p(z_k|x_N, z^{k-1}) &= \int p(z_k|x_k)p(x_k|x_N, z^{k-1})dx_k \\ &= \mathcal{N}\Big(z_k; H_k(G_{k,k-1}b_{k-1} + D_k x_N), R_k + H_k S_k (H_k)'\Big) \end{aligned} \tag{7.36}$$

where $p(x_k|x_N, z^{k-1})$ and $p(z_k|x_k)$ are available by (7.32) and (7.33), respectively. Then, substituting (7.35) and (7.36) into (7.34), we get

$$p(x_N|z^k) = \mathcal{N}(x_N; a_k, A_k) \tag{7.37}$$

where

$$\begin{aligned} a_k =& a_{k-1} + A_k(D_k)'(H_k)'(R_k + H_k S_k(H_k)')^{-1}(z_k - H_k G_{k,k-1}b_{k-1} - H_k D_k a_{k-1}) \\ A_k =& A_{k-1} - A_{k-1}(D_k)'(H_k)'(R_k + H_k S_k(H_k)' + H_k D_k A_{k-1}(D_k)'(H_k)')^{-1} H_k D_k A_{k-1} \end{aligned}$$

Thus, the propagation of $p(x_N|z^k)$ for $k \in [1, N-1]$ is complete.

Given the key terms $p(x_k|x_N, z^k)$ and $p(x_N|z^k)$, the posterior density $p(x_k|z^k)$ for $k \in [1, N-1]$ can be calculated by (7.28), which results in

$$\begin{aligned} p(x_k|z^k) &= \int \mathcal{N}(x_k; b_k + B_k x_N, \mathfrak{B}_k)\mathcal{N}(x_N; a_k, A_k)dx_N \\ &= \mathcal{N}(x_k; B_k a_k + b_k, \mathfrak{B}_k + B_k A_k (B_k)') \end{aligned}$$

Then, the MMSE estimate and its MSE matrix are

$$\hat{x}_k = B_k a_k + b_k \tag{7.38}$$

$$P_k = \mathfrak{B}_k + B_k A_k (B_k)' \tag{7.39}$$

For $k = N$, the posterior density is

$$p(x_N|z^N) = \frac{p(z_N|x_N)p(x_N|z^{N-1})}{p(z_N|z^{N-1})}$$

where $p(x_N|z^{N-1}) = \mathcal{N}(x_N; a_{N-1}, A_{N-1})$ is available from time $N-1$, and $p(z_N|x_N)$ is given by (7.33). Then,

$$\begin{aligned} \hat{x}_N &= a_{N-1} + P_N(H_N)'R_N^{-1}(z_N - H_N a_{N-1}) \\ P_N &= A_{N-1} - A_{N-1}(H_N)'(R_N + H_N A_{N-1}(H_N)')^{-1} H_N A_{N-1} \end{aligned}$$

The filter is as follows.

- **Initialization:**

$$\begin{aligned} b_0 =& \mu_0 - C_{0,N} C_N^{-1} \mu_N \\ B_0 =& C_{0,N} C_N^{-1} \\ \mathfrak{B}_0 =& C_0 - C_{0,N} C_N^{-1} (C_{0,N})' \\ \hat{x}_0 =& a_0 = \mu_N \\ P_0 =& A_0 = C_N \end{aligned}$$

- **For** $k \in [1, N-1]$**:**

$$
\begin{aligned}
S_k =& G_k + G_{k,k-1}\mathfrak{B}_{k-1}(G_{k,k-1})' \\
\mathfrak{B}_k =& S_k - S_k(H_k)'(R_k + H_k S_k (H_k)')^{-1} H_k S_k \\
b_k =& G_{k,k-1}b_{k-1} + \mathfrak{B}_k (H_k)' R_k^{-1}(z_k - H_k G_{k,k-1} b_{k-1}) \\
B_k =& G_{k,k-1}B_{k-1} + G_{k,N} - \mathfrak{B}_k (H_k)' R_k^{-1} H_k (G_{k,k-1}B_{k-1} + G_{k,N}) \\
D_k =& G_{k,k-1}B_{k-1} + G_{k,N} \\
A_k =& A_{k-1} - A_{k-1}(D_k)'(H_k)'(R_k + H_k S_k (H_k)' + H_k D_k A_{k-1}(D_k)'(H_k)')^{-1} H_k D_k A_{k-1} \\
a_k =& a_{k-1} + A_k (D_k)'(H_k)'(R_k + H_k S_k (H_k)')^{-1}(z_k - H_k G_{k,k-1} b_{k-1} - H_k D_k a_{k-1}) \\
P_k =& \mathfrak{B}_k + B_k A_k (B_k)' \\
\hat{x}_k =& B_k a_k + b_k
\end{aligned}
$$

- **For** $k = N$**:**

$$
\begin{aligned}
P_N =& A_{N-1} - A_{N-1}(H_N)'\Big(R_N + H_N A_{N-1}(H_N)'\Big)^{-1} H_N A_{N-1} \\
\hat{x}_N =& a_{N-1} + P_N (H_N)' R_N^{-1}\Big(z_N - H_N a_{N-1}\Big)
\end{aligned}
$$

7.2.3 Discussion

The CM_L Sequence For DDT Modeling

Consider a flight from an origin to a destination. Let the trajectories of the flight be modeled by sample paths of a CM_L sequence $[x_k]$. In other words, it is assumed that the flight follows the CM_L sequence $[x_k]$. Although we don't know which CM_L sample path the flight is following, at every time a measurement of the state of the flight is available. The goal is to obtain an estimate of the state (and then obtain a predicted state) by processing the measurements. (7.6) is non-causal, but our filter (7.38)–(7.39) still works in a causal way because it uses only causal (statistical) information. Therefore, there is no problem regarding the applicability of the CM_L model due to its non-casuality. If the exact density is not available, an approximate one can be used. The mismatch impact is studied in Subsection 7.4.

By (7.4), first, x_N is generated from $p(x_N)$. Then, conditioned on the realization of x_N, other states are realized. Given x_N, one can intuitively interpret CM_L sample path generation as the realization of one of the sample paths going through the given x_N. This approach of path generation helps to understand the behavior of the filter based on (7.28). In (7.28), $p(x_k|z^k)$ is a weighted sum of $p(x_k|x_N, z^k)$, where the weights are proportional to the posterior destination density $p(x_N|z^k)$. As measurements are received over time, the posterior destination density is updated. In other words, the uncertainty about the state x_N reduces. Also, for every value of x_N, the conditional density $p(x_k|x_N, z^k)$ is propagated over time. Thus, as $p(x_N|z^k)$ gets more concentrated, higher weights are given to conditional densities $p(x_k|x_N, z^k)$ with more likely x_N (according to $p(x_N|z^k)$). It means the conditional densities $p(x_k|x_N, z^k)$ with more likely x_N play more important roles in determination of $p(x_k|z^k)$. The above explanation, based on the second formulation of the filter (Subsection 7.2.2), shows that the behavior of a DDT filter is quite intuitive.

An essential part of the filter is the update of destination density $p(x_N|z^k)$ (Subsection 7.2.2). As measurements are received over time, the posterior destination density becomes more concentrated. The (assumed) known prior destination density $p(x_N)$ is not necessarily accurate. If not known, an approximate (mismatched) prior destination density can be used. Given the CM_L model, the destination density is updated as measurements are received. It can be seen in the simulations (Section 7.4) that the impact of the destination density mismatch on state estimates far from the destination is negligible. Also, by an appropriate design of the approximate prior destination density, this mismatch impact can be reduced for estimates of the states close to the destination (Section 7.4).

Reciprocal CM_L Model vs. Reciprocal Model of [18] for Estimation

Recursive estimation of a reciprocal sequence based on the reciprocal CM_L model and the reciprocal model of [18] was discussed in Subsection 3.1.3. It was shown that the reciprocal CM_L model gives a much simpler recursive estimator. In addition, we emphasize that the structure of the reciprocal CM_L model fits DDT much better than the reciprocal model of [18]. Because the former can directly model/incorporate destination density (due to the term x_N in (2.17)), but the latter is difficult to incorporate such information (due to the nearest neighbor structure of (6.9)).

Markov Model vs. Reciprocal CM_L Model for Estimation

By Theorem 3.1.17, given a reciprocal CM_L model, there exist boundary conditions that lead to Markov sequences. So, such a Markov sequence can be modeled by a Markov model (7.8) or a reciprocal CM_L model (Chapter 3, Chapter 6). The CM_L filter (Subsection 7.2.1 or Subsection 7.2.2) is MMSE optimal. For a Markov sequence, one can also derive the MMSE optimal filter based on the Markov model. Therefore, for a Markov sequence both these filters calculate the conditional mean $E[x_k|z^k]$ and are actually the same.

7.3 DDT Prediction

Given a CM_L model and measurements up to time k, the trajectory can be predicted. Let $[x_k]$ be a CM_L sequence modeled by (7.18). Assume that the output of the filter $p(s_k|z^k) = \mathcal{N}(s_k; \hat{s}_k, \Sigma_k)$ at time k is available (Subsection 7.2.1). The predicted density at time $k+n \in [k+1, N-1]$ is ($s_k = [x'_k, x'_N]'$)

$$p(s_{k+n}|z^k) = \int p(s_{k+n}|s_k)p(s_k|z^k)ds_k \tag{7.40}$$

where the second term of the integrand is the output of the filter (7.20)–(7.21) at time k, and the first term is determined by (7.18). For $k+n \in [k+1, N-1]$, the predicted state and its MSE matrix are obtained as

$$\hat{s}_{k+n|k} = G^s_{k+n|k}\hat{s}_k \tag{7.41}$$

$$\Sigma_{k+n|k} = K_{k+n|k} + G^s_{k+n|k}\Sigma_k(G^s_{k+n|k})' \tag{7.42}$$

where

$$G^s_{k+n|k} = G^s_{k+n,k+n-1}G^s_{k+n-1,k+n-2}\cdots G^s_{k+1,k}, \quad G^s_{k|k} = I, \forall k \tag{7.43}$$

$$K_{k+n|k} = \sum_{i=k}^{k+n-1} G^s_{k+n|i+1}G^s_i(G^s_{k+n|i+1})' \tag{7.44}$$

Then, the predicted x_{k+n} and its MSE matrix are

$$\hat{x}_{k+n|k} = [I, 0]\hat{s}_{k+n|k} \tag{7.45}$$

$$P_{k+n|k} = [I, 0]\Sigma_{k+n|k}[I, 0]' \tag{7.46}$$

Also,

$$\hat{x}_{N|k} = [0, I]\hat{s}_k \tag{7.47}$$
$$P_{N|k} = [0, I]\Sigma_k[0, I]' \tag{7.48}$$

The predictor is as follows:

- $\hat{s}_k$ and Σ_k are available by the filter.
- **For** $k + n \in [k + 1, N - 1]$**:**

$$\begin{aligned}
\hat{s}_{k+n|k} &= G^s_{k+n|k}\hat{s}_k \\
\Sigma_{k+n|k} &= K_{k+n|k} + G^s_{k+n|k}\Sigma_k(G^s_{k+n|k})' \\
\hat{x}_{k+n|k} &= [I, 0]\hat{s}_{k+n|k} \\
P_{k+n|k} &= [I, 0]\Sigma_{k+n|k}[I, 0]'
\end{aligned}$$

- **For** $k + n = N$**:**

$$\begin{aligned}
\hat{x}_{N|k} &= [0, I]\hat{s}_k \\
P_{N|k} &= [0, I]\Sigma_k[0, I]'
\end{aligned}$$

It is desirable to compare trajectory prediction formulations obtained with and without incorporating destination information. To do so, we compare trajectory predictors obtained based on a Markov model and on the Markov-induced CM_L model (Theorem 4.1.3). In addition to the above formulation, we present an alternative formulation for DDT prediction for the Markov-induced CM_L model. This formulation is particularly useful for comparing trajectory predictors with and without destination information. For simplicity, we assume a time-invariant Markov model (7.8) (i.e., $M_{k,k-1} = F$ and $M_k = Q$) of $[y_k]$. We have

$$\begin{aligned}
p(y_{k+n}|y_k) &= \mathcal{N}(y_{k+n}; F^n y_k, C_{k+n|k}) \\
p(y_N|y_{k+n}) &= \mathcal{N}(y_N; F^{N-(k+n)} y_{k+n}, C_{N|k+n})
\end{aligned}$$

where for $k + n \in [k + 1, N - 1]$, $C_{k+n|k} = \sum_{i=0}^{n-1} F^i Q(F^i)'$, $C_{N|k+n} = \sum_{i=0}^{N-k-n-1} F^i Q(F^i)'$. By the Markov property, for the transition density we have

$$\begin{aligned}
p(y_{k+n}|y_k, y_N) &= \frac{p(y_{k+n}|y_k)p(y_N|y_{k+n}, y_k)}{p(y_N|y_k)} \\
&= \frac{p(y_{k+n}|y_k)p(y_N|y_{k+n})}{p(y_N|y_k)} && (7.49) \\
&= \mathcal{N}(y_{k+n}; W_{k+n,k}y_k + U_{k+n,k}y_N, \mathcal{W}_{k+n,k}) && (7.50)
\end{aligned}$$

where

$$\begin{aligned}
W_{k+n,n} =& F^n - U_{k+n,n}F^{N-k} && (7.51) \\
U_{k+n,n} =& \mathcal{W}_{k+n|k}(F^{N-(k+n)})'C^{-1}_{N|k+n} && (7.52) \\
\mathcal{W}_{k+n|k} =& C_{k+n|k} - C_{k+n|k}(F^{N-(k+n)})'(C_{N|k+n}+ \\
& F^{N-(k+n)}C_{k+n|k}(F^{N-(k+n)})')^{-1}F^{N-(k+n)}C_{k+n|k} && (7.53) \\
E_{k+n|k} =& [W_{k+n,n}, U_{k+n,n}] && (7.54)
\end{aligned}$$

Let $h(y_{k+n}, y_k, y_N) = p(y_{k+n}|y_k, y_N)$. For the transition density of the Markov-induced CM_L model of $[x_k]$, we have[1] (Appendix D)

$$p(x_{k+n}|x_k, x_N) = h(x_{k+n}, x_k, x_N) \tag{7.55}$$

For trajectory prediction, we can write ($s_k = [x_k', x_N']'$)

$$p(x_{k+n}|z^k) = \int p(x_{k+n}|s_k)p(s_k|z^k)ds_k \tag{7.56}$$

Using (7.55) in (7.56), the trajectory predictor based on the Markov-induced CM_L model is, for $k+n \in [k+1, N-1]$,

$$\hat{x}_{k+n|k} = E_{k+n|k}\hat{s}_k \tag{7.57}$$

$$P_{k+n|k} = \mathcal{W}_{k+n|k} + E_{k+n|k}\Sigma_k(E_{k+n|k})' \tag{7.58}$$

with (7.51)–(7.54), and for $k+n = N$ we have (7.47)–(7.48).

The predictor is as follows:

- $\hat{s}_k$ and Σ_k are available by the filter.
- **For** $k+n \in [k+1, N-1]$**:**

$$\hat{x}_{k+n|k} = E_{k+n|k}\hat{s}_k$$
$$P_{k+n|k} = \mathcal{W}_{k+n|k} + E_{k+n|k}\Sigma_k(E_{k+n|k})'$$

- **For** $k+n = N$**:**

$$\hat{x}_{N|k} = [0, I]\hat{s}_k$$
$$P_{N|k} = [0, I]\Sigma_k[0, I]'$$

The trajectory predictor based on Markov model (7.8) of a Markov sequence $[y_k]$ is obtained by

$$p(y_{k+n}|z^k) = \int p(y_{k+n}|y_k)p(y_k|z^k)dy_k$$

where the second term of the integrand is available from the filter and the first term is determined by the Markov model. Then, for $M_{k,k-1} = F$ and $M_k = Q$, the predicted y_{k+n} and its MSE matrix are

$$\hat{y}_{k+n|k} = F^n\hat{y}_k \tag{7.59}$$

$$P_{k+n|k} = C_{k+n|k} + F^nP_k(F^n)' \tag{7.60}$$

where $\hat{y}_k$ and P_k are provided by the corresponding filter (the filter derived based on the Markov model).

It is useful to compare the DDT predictor (7.57)–(7.58) with the trajectory predictor (7.59)–(7.60).

[1] Note that with an abuse of notation, $p(y_{k+n}|y_k, y_N)$ means transition density of $[y_k]$ and $p(x_{k+n}|x_k, x_N)$ transition density of $[x_k]$.

7.4 Simulations

Performance of the CM_L sequence in DDT modeling was evaluated via simulations. Four examples were considered to study the following topics: trajectories in different scenarios, filtering, destination density update, and trajectory prediction. Then, an example is presented to demonstrate an application of a singular CM_L sequence in trajectory modeling.

Consider a two-dimensional scenario, where the state of a moving object at time k is $x_k = [\mathsf{x}, \dot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}]'_k$ with position $[\mathsf{x}, \mathsf{y}]'$ and velocity $[\dot{\mathsf{x}}, \dot{\mathsf{y}}]'$. Mean and covariance of the origin (destination) are denoted by μ_0 and C_0 (μ_N and C_N). The cross-covariance between them is denoted by $C_{0,N}$. To compare performance of the CM_L modeling with that of the Markov modeling for trajectories, we considered a Markov-induced CM_L model (Theorem 4.1.3). For the corresponding Markov model (7.8), for every $k \in [1, N]$, we have

$$M_{k,k-1} = F = \text{diag}(F_1, F_1), \quad F_1 = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \tag{7.61}$$

$$M_k = Q = \text{diag}(Q_1, Q_1), \quad Q_1 = q \begin{bmatrix} T^3/3 & T^2/2 \\ T^2/2 & T \end{bmatrix} \tag{7.62}$$

where $T = 15$ second (sampling interval between $k-1$ and k), $q = 0.01$, and $N = 100$.

In simulations we considered the Markov model $y_k = M_{k,k-1} y_{k-1} + e_k^M$, $e_k^M \sim \mathcal{N}(0, M_k)$, $k \in [1, N]$, $y_0 = e_0^M$, with the above parameters, where $e_0^M \sim \mathcal{N}(\mu_0, C_0)$. Also, we considered the Markov-induced CM_L model

$$x_k = G_{k,k-1} x_{k-1} + G_{k,N} x_N + e_k \tag{7.63}$$

$$x_0 = \mu_0 + C_{0,N} C_N^{-1} (x_N - \mu_N) + e_0 \tag{7.64}$$

$$x_N = \mu_N + e_N \tag{7.65}$$

where $e_k \sim \mathcal{N}(0, G_k)$, $k \in [1, N-1]$, $e_N \sim \mathcal{N}(0, C_N)$, $e_0 \sim \mathcal{N}(0, C_0 - C_{0,N} C_N^{-1} C'_{0,N})$, and parameters of (7.63) are given by (7.12)–(7.14) as

$$G_{k,k-1} = F - G_{k,N} F^{N-k+1} \tag{7.66}$$

$$G_{k,N} = G_k (F^{N-k})' C_{N|k}^{-1} \tag{7.67}$$

$$\begin{aligned} G_k &= (Q^{-1} + (F^{N-k})' C_{N|k}^{-1} F^{N-k})^{-1} \\ &= Q - Q(F^{N-k})'(C_{N|k} + F^{N-k} Q (F^{N-k})')^{-1} F^{N-k} Q \end{aligned} \tag{7.68}$$

where $C_{N|k} = \sum_{n=k}^{N-1} F^{N-n-1} Q (F^{N-n-1})'$. The time duration is the same $[0, N]$ in all scenarios.

Example 7.4.1. In this example, trajectories generated by the above Markov-induced CM_L model were studied. Different scenarios were considered.

- Scenario 1: Let the means and the covariances of the origin and the destination densities be given by (7.69)–(7.72). Fig. 7.1 shows some CM_L trajectories from the origin to the destination, generated by the Markov-induced CM_L model. To compare the two models (the Markov model and the Markov-induced CM_L model), we plot the trajectories of Fig. 7.1 (solid lines) and those of the Markov sequence (dash lines) in Fig.7.2. Both sequences model the origin well. Also, near the origin their difference is small. However, later their difference grows. This is due to the poor performance of the Markov model in incorporating the destination information. Also, Figs. 7.3 and 7.4 show the x and y components of the velocity for Markov (50 dash lines) and CM_L (50 solid lines) sequences. For clarity, Fig. 7.5 also shows y-velocity for the CM_L sequence separately. Variations of velocity components are intuitive by comparing Markov and CM_L trajectories in Figs. 7.3 and 7.4. The x-position mean of the destination is 130000 while the x-position at

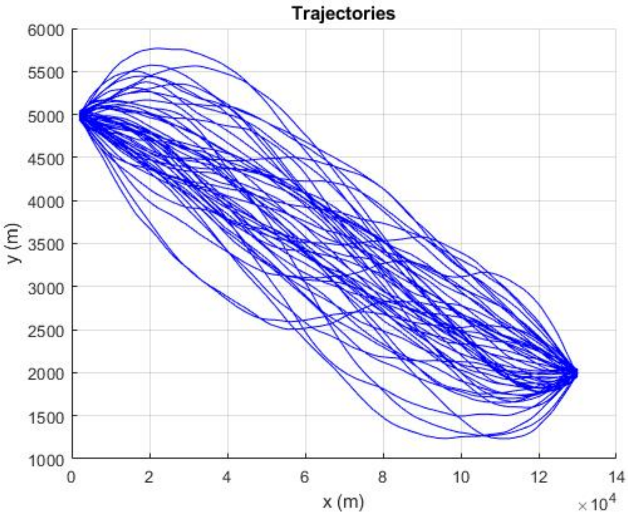

Figure 7.1: CM_L trajectories from an origin to a destination (Example 1, Scenario 1).

the end of Markov trajectories is around 110000. The x-velocity means at the origin and the destination are the same. So, the x-velocity for the CM_L sequence should be greater than that of the Markov sequence on the way (Fig. 7.3) to satisfy the x-position at the destination. Note that the x-velocity for the Markov sequence does not change much overall. Also, note that both sequences have the same time duration $[0, N]$. The y-velocity means of the origin and destination densities are the same. But the y-position means of the origin (5000) is larger than that of the destination (2000). So, the y-velocity of the CM_L sequence slightly decreases on the way (Fig. 7.4).

$$\mu_0 = [2000, 70, 5000, 0]' \tag{7.69}$$
$$C_0 = C_N = \text{diag}(A, A) \tag{7.70}$$
$$\mu_N = [130000, 70, 2000, 0]' \tag{7.71}$$
$$C_{0,N} = \text{diag}(B, B) \tag{7.72}$$
$$A = \begin{bmatrix} 1000 & 40 \\ 40 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} 800 & 20 \\ 20 & 7 \end{bmatrix}$$

- Scenario 2: Let the means and the covariances be given by (7.69)–(7.72), except $\mu_N = [80000, 70, 2000, 0]$. Fig. 7.6 shows some trajectories of the CM_L and Markov sequences. Similar to the scenario 1, variations of velocity components are intuitive by comparing Markov and CM_L trajectories in Figs. 7.7 and 7.8. The x-position mean of the destination is 80000 while the x-position at the end of Markov trajectories is around 110000. The x-velocity means at the origin and the destination are the same. So, the x-velocity for the CM_L sequence should be smaller than that of the Markov sequence on the way (Fig. 7.7) to satisfy the x-position at the destination. Note that the x-velocity for the Markov sequence does not change much overall; also, the time duration for both sequences is the same $[0, N]$. It is meaningful to compare the x-velocity in Figs. 7.3 and 7.7. The variations of y-velocities are similar in Figs. 7.4 and 7.8.

- Scenario 3: Let the means and the covariances be given by (7.69)–(7.72), except $\mu_0 = [2000, 70, 5000, 10]$ and $\mu_N = [130000, 70, 2000, -10]$. Trajectories of the corresponding CM_L sequence are shown in Fig. 7.9. Fig. 7.10 shows trajectories of the Markov and the

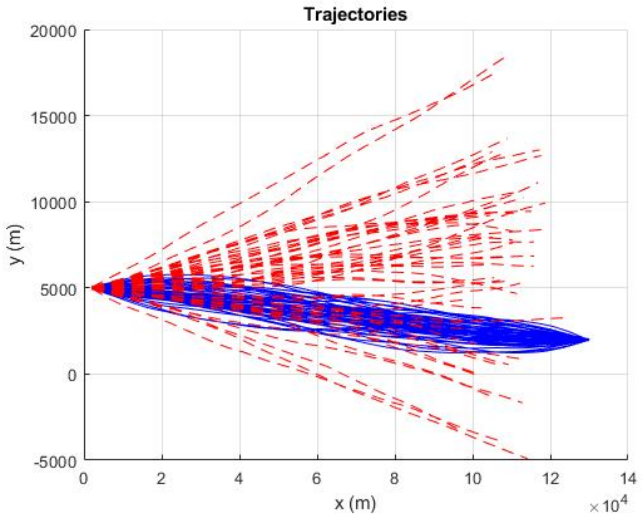

Figure 7.2: CM_L (solid lines) and Markov (dash lines) trajectories (Example 1, Scenario 1).

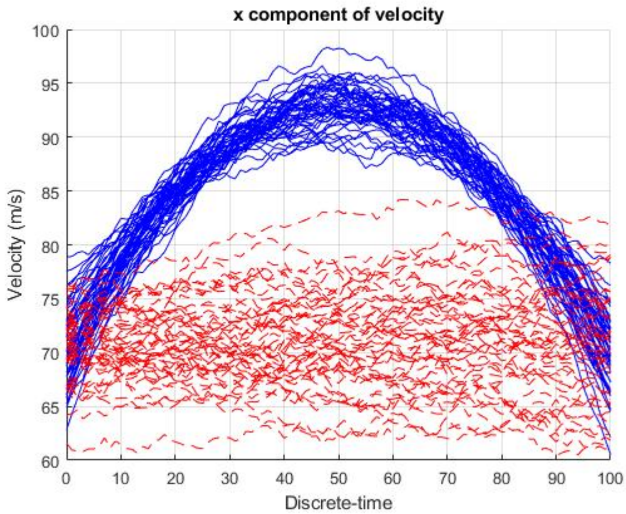

Figure 7.3: x-velocity for CM_L and Markov trajectories (Example 1, Scenario 1).

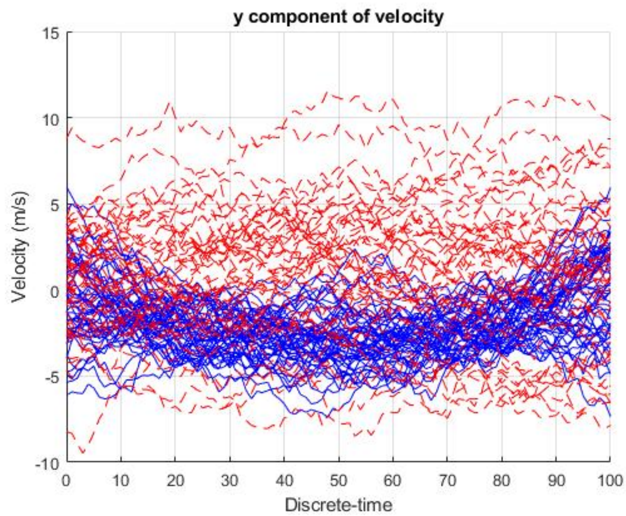

Figure 7.4: y-velocity for CM_L and Markov trajectories (Example 1, Scenario 1).

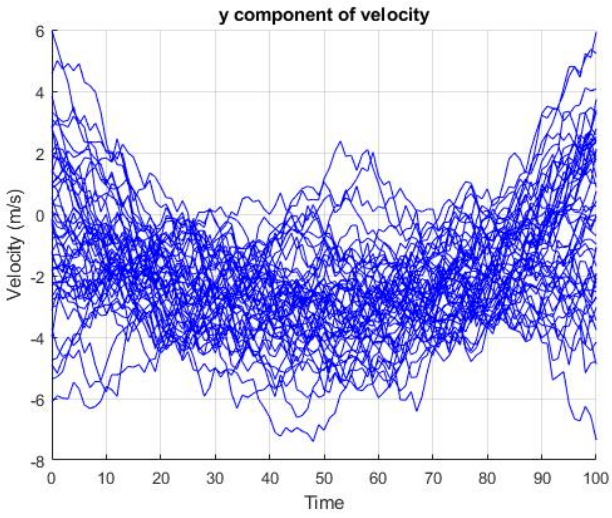

Figure 7.5: y-velocity for CM_L trajectories (Example 1, Scenario 1).

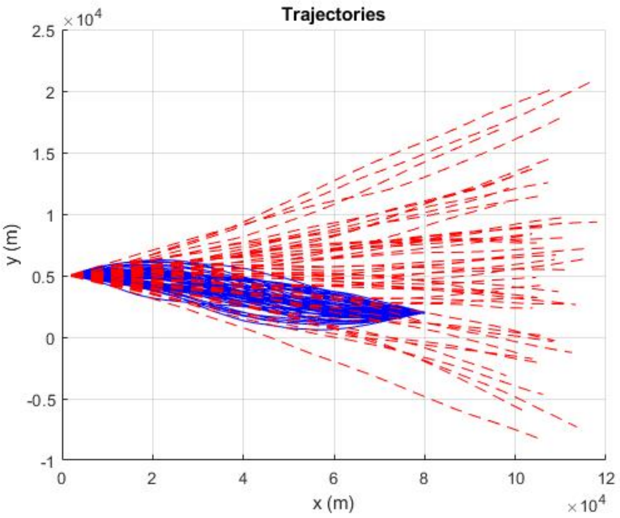

Figure 7.6: CM_L and Markov trajectories (Example 1, Scenario 2).

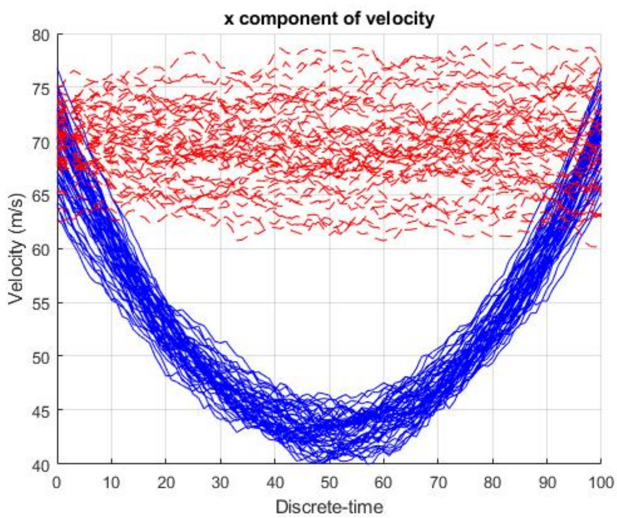

Figure 7.7: x-velocity for CM_L and Markov trajectories (Example 1, Scenario 2).

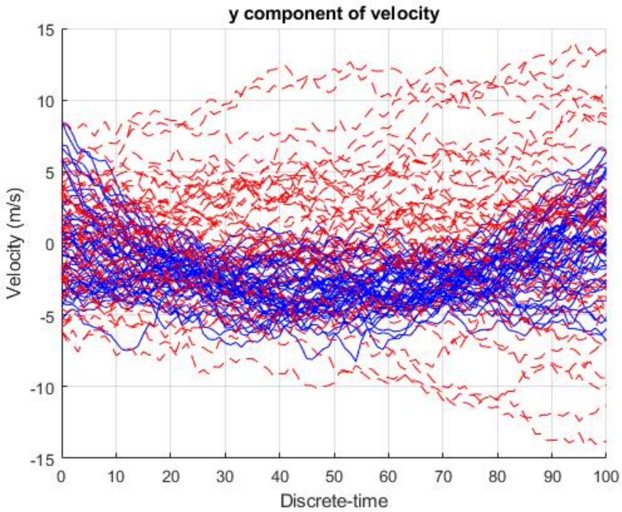

Figure 7.8: y-velocity for CM_L and Markov trajectories (Example 1, Scenario 2).

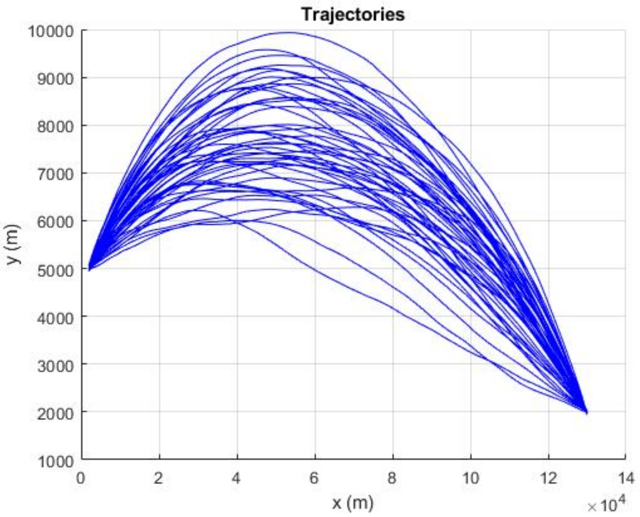

Figure 7.9: CM_L trajectories from an origin to a destination (Example 1, Scenario 3).

CM_L sequence. The variations of the x and y components of the velocity in Figs. 7.11 and 7.12 are intuitive based on the origin and destination means of position and velocity.

- Scenario 4: Let the means and the covariances be given by (7.69)–(7.72), except $\mu_0 = [2000, 70, 5000, 10]$ and $\mu_N = [130000, 70, 2000, 10]$. Trajectories of the CM_L sequence are shown in Fig 7.13.

- Scenario 5: Let the means and the covariances be given by (7.69)–(7.72), except $\mu_0 = [2000, 70, 5000, 10]$ and $\mu_N = [130000, 70, 2000, 0]$. Trajectories of the CM_L sequence are shown in Fig 7.14.

Example 7.4.1 shows how the CM_L sequence can model trajectories taking the origin and the destination information into account.

In the ATC application, the origin and the destination of a flight are two airports. So, the origin and the destination densities are often available. However, in other applications the exact origin and destination densities are not necessarily available. Thus, in the following, some mismatched cases are considered. The matched case (i.e., the true μ_0, C_0, μ_N, C_N, and $C_{0,N}$

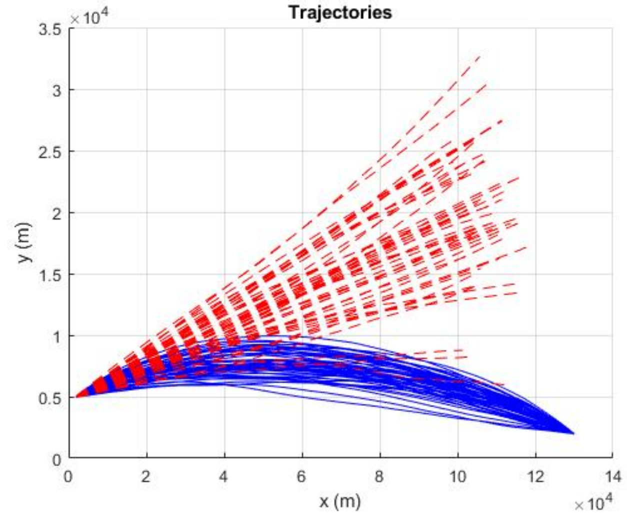

Figure 7.10: CM_L and Markov trajectories (Example 1, Scenario 3).

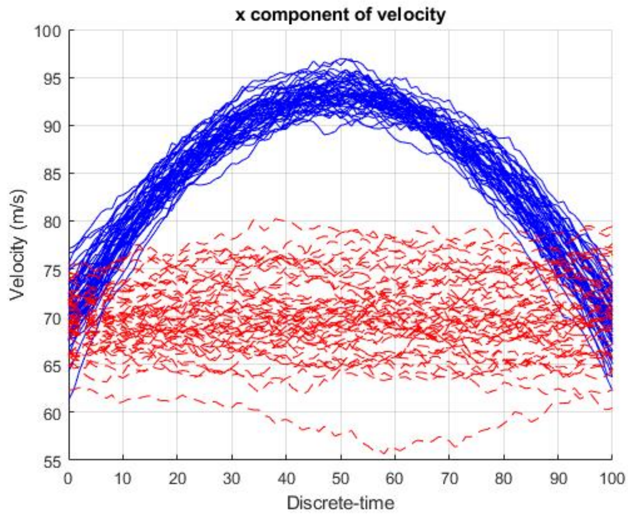

Figure 7.11: x-velocity for CM_L and Markov trajectories (Example 1, Scenario 3).

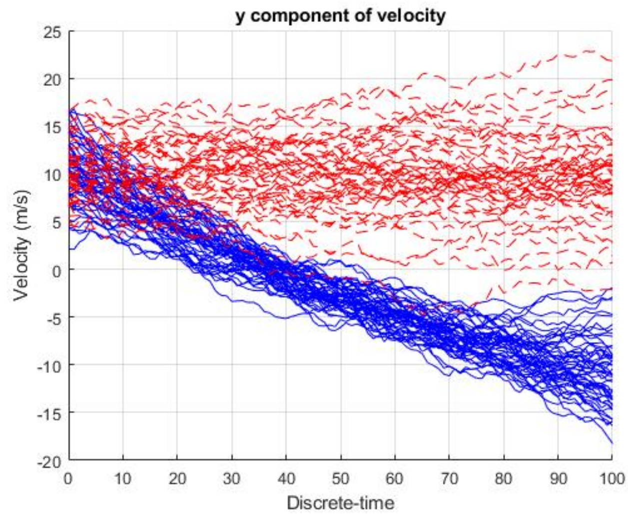

Figure 7.12: y-velocity for CM_L and Markov trajectories (Example 1, Scenario 3).

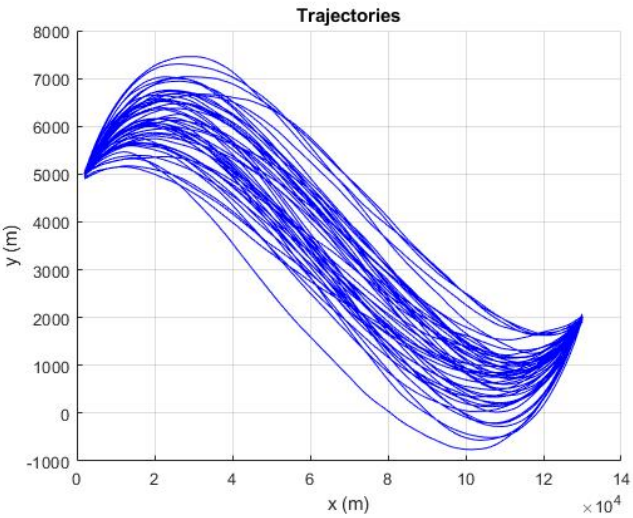

Figure 7.13: CM_L trajectories from an origin to a destination (Example 1, Scenario 4).

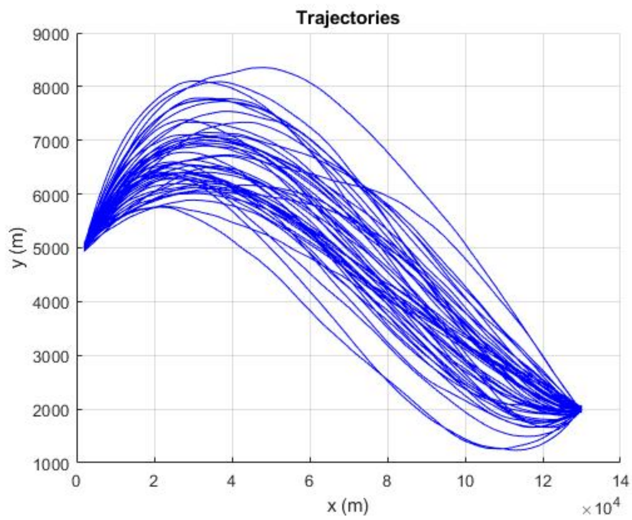

Figure 7.14: CM_L trajectories from an origin to a destination (Example 1, Scenario 5).

are given by (7.69)–(7.72)) is considered as case (i). The mismatched cases are:

- Case (ii): (7.70), (7.72), and

$$\mu_0 = [2300, 60, 5300, 10]' \tag{7.73}$$
$$\mu_N = [130300, 60, 2300, 10]' \tag{7.74}$$

- Case (iii):

$$\mu_0 = [2300, 60, 5300, 10]' \tag{7.75}$$
$$C_0 = C_N = \text{diag}(10000, 100, 10000, 100) \tag{7.76}$$
$$\mu_N = [130300, 60, 2300, 10]' \tag{7.77}$$
$$C_{0,N} = \text{diag}(7000, 60, 7000, 60) \tag{7.78}$$

- Case (iv): (7.75), (7.77), and

$$C_0 = C_N = \text{diag}(100, 1, 100, 1) \tag{7.79}$$
$$C_{0,N} = \text{diag}(90, 0.8, 90, 0.8) \tag{7.80}$$

- Case (v): (7.69), (7.71), and

$$C_0 = C_N = \text{diag}(10000, 100, 10000, 100) \tag{7.81}$$
$$C_{0,N} = \text{diag}(7000, 60, 7000, 60) \tag{7.82}$$

Example 7.4.2. Filtering performance is studied. The true trajectories were generated by the Markov-induced CM_L model (case (i)). Since the Markov sequence is a special CM_L sequence, this approach for generation of true trajectories is totally fair for both CM_L and Markov models (see Subsection 7.2.3 about a Markov model vs. a reciprocal CM_L model for estimation). The measurement model is given by (7.17), where

$$H_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
$$R_k = \text{diag}(100, 100)$$

Figs. 7.15 and 7.16 show the logarithm of the average Euclidean error (AEE) [93] of the position ($\text{AEE}^p_{k|k}$) and the velocity ($\text{AEE}^v_{k|k}$) estimates based on the CM_L model and the Markov model using measurements up to time k. The AEE of the position and velocity estimates are given by

$$\text{AEE}^p_{k|k} = \frac{1}{M}\sum_{i=1}^{M}\sqrt{(\mathsf{x}^{(i)}_k - \hat{\mathsf{x}}^{(i)}_{k|k})^2 + (\mathsf{y}^{(i)}_k - \hat{\mathsf{y}}^{(i)}_{k|k})^2}$$
$$\text{AEE}^v_{k|k} = \frac{1}{M}\sum_{i=1}^{M}\sqrt{(\dot{\mathsf{x}}^{(i)}_k - \hat{\dot{\mathsf{x}}}^{(i)}_{k|k})^2 + (\dot{\mathsf{y}}^{(i)}_k - \hat{\dot{\mathsf{y}}}^{(i)}_{k|k})^2}$$

where $[\mathsf{x}^{(i)}_k, \mathsf{y}^{(i)}_k]'$ and $[\dot{\mathsf{x}}^{(i)}_k, \dot{\mathsf{y}}^{(i)}_k]'$ are the true position and velocity at time k on the ith Monte Carlo run, $[\hat{\mathsf{x}}^{(i)}_{k|k}, \hat{\mathsf{x}}^{(i)}_{k|k}]'$ and $[\hat{\dot{\mathsf{x}}}^{(i)}_{k|k}, \hat{\dot{\mathsf{y}}}^{(i)}_{k|k}]'$ their estimates using measurements up to time k, and $M = 1000$ is the number of Monte Carlo runs. The results of the CM_L model for all different mismatched endpoints are shown (Figs. 7.15 and 7.16). However, for the Markov model only the result of the matched case (i.e., case (i)) is presented. In case (ii), the mismatched means of the origin and destination densities lead to some bias in the CM_L model. This is the reason for

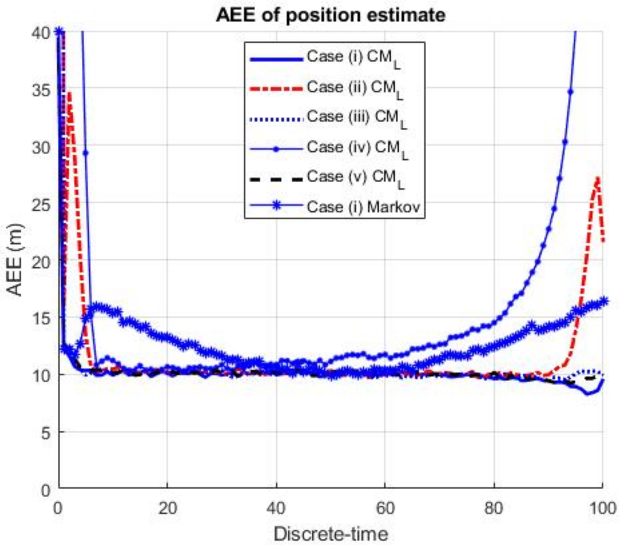

Figure 7.15: AEE of position estimate ($\mathrm{AEE}^p_{k|k}$) (Example 2).

estimation performance degradation near the origin and especially near the destination in case (ii). However, the mismatch impact is not significant far from the origin and the destination, which is intuitive. By an appropriate (large enough) choice of the origin and the destination covariances, the impact of mismatched means can be compensated as it is seen in case (iii). An inappropriate (too small) choice of the origin and the destination covariances can make the impact of mismatched means even worse (case (iv)). On the other hand, the impact of large covariances of the origin and the destination is not that serious (case (v)). The differences in estimation performance in case (i), case (iii), and case (v) are not significant. So, if the origin or the destination mean mismatch is likely in a scenario, one should design the covariances accordingly to compensate the model bias. Note that estimation performance based on the Markov model can be much worse than that of Figs. 7.15 and 7.16 for other scenarios presented in Figs. 7.9, 7.13, 7.14.

Example 7.4.3. Destination density update is an important part of the filter for the CM_L sequence (Section 7.2.2). In other words, estimation of x_N plays an important role in filtering and prediction. Dynamic and measurement models are the same as the above. The AEE of the (prediction) estimates of the position and velocity components of x_N given measurement up to time k are given by

$$\mathrm{AEE}^p_{N|k} = \frac{1}{M}\sum_{i=1}^{M}\sqrt{(\mathsf{x}^{(i)}_N - \hat{\mathsf{x}}^{(i)}_{N|k})^2 + (\mathsf{y}^{(i)}_N - \hat{\mathsf{y}}^{(i)}_{N|k})^2}$$
$$\mathrm{AEE}^v_{N|k} = \frac{1}{M}\sum_{i=1}^{M}\sqrt{(\dot{\mathsf{x}}^{(i)}_N - \hat{\dot{\mathsf{x}}}^{(i)}_{N|k})^2 + (\dot{\mathsf{y}}^{(i)}_N - \hat{\dot{\mathsf{y}}}^{(i)}_{N|k})^2}$$

where $[\mathsf{x}^{(i)}_N, \mathsf{y}^{(i)}_N]'$ and $[\dot{\mathsf{x}}^{(i)}_N, \dot{\mathsf{y}}^{(i)}_N]'$ are the true position and velocity at time N on the ith Monte Carlo run, $[\hat{\mathsf{x}}^{(i)}_{N|k}, \hat{\mathsf{y}}^{(i)}_{N|k}]'$ and $[\hat{\dot{\mathsf{x}}}^{(i)}_{N|k}, \hat{\dot{\mathsf{y}}}^{(i)}_{N|k}]'$ their estimates using measurements up to time k, and $M = 1000$ is the number of Monte Carlo runs.

Figs. 7.17 and 7.18 show how the predicted x_N in case (i) gets better as more measurements are received especially near the destination. The mismatched means (model bias) in case (ii) degrade the estimation performance. Appropriate choices of the covariances in case (iii) enhance the performance while inappropriate choices of the covariances in case (iv) make the bias impact worse. It demonstrates the importance of appropriate choices of the origin and destination covariances in the presence of mismatched means.

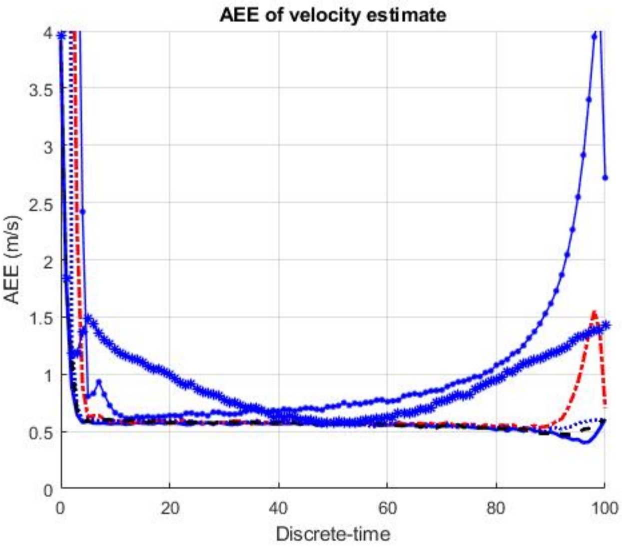

Figure 7.16: AEE of velocity estimate ($\text{AEE}^v_{k|k}$) (Example 2).

Due to the mismatched endpoint densities, the predicted x_N can deteriorate over time. It can be verified as follows. Based on a CM_L model (7.6) we can write $x_k = A_k x_0 + B_k x_N + r_k$, $k \in [1, N-1]$, where A_k and B_k are some matrices, and r_k is a linear function of the CM_L dynamic noise. Also, from (7.7) we have $x_0 = \mu_0 + G_{0,N}(x_N - \mu_N) + e_0$. For example, assume the means of the origin (μ_0) and the destination (μ_N) are mismatched. We have $\mu_0 = \mu_0^{true} + \tilde{\mu}_0$ and $\mu_N = \mu_N^{true} + \tilde{\mu}_N$, where $\tilde{\mu}_0$ and $\tilde{\mu}_0$ are mismatch terms. Using the above formulas for x_k and x_0, we can write the measurement at time k (i.e., (7.17)) in terms of x_N as $z_k = L_k x_N + d_k + b_k + w_k$, where L_k is a matrix, d_k is a linear function of μ_0^{true} and μ_N^{true}, w_k is a linear function of the measurement noise and the CM_L dynamic noise, and b_k is a bias term due to mismatched means (i.e., b_k is a function of the mismatch terms $\tilde{\mu}_0$ and $\tilde{\mu}_N$). It can be seen that depending on the bias at different times, the predicted x_N can deteriorate over time occasionally (Fig. 7.18).

Example 7.4.4. Trajectory prediction is studied in this example. Dynamic and measurement models are the same as in the above. CM_L trajectory prediction is possible based on (7.45)–(7.46) or (7.57)–(7.58) for $k+n \in [k+1, N-1]$, and (7.47)–(7.48) for $k+n = N$. It is assumed that the measurements are available up to time $k = 9$, based on which the filter's output is available. Fig. 7.19 shows the logarithm of the AEE of the position prediction obtained based on the CM_L model and the Markov model. The AEE of the position prediction is

$$\text{AEE}_{k+n|k} = \frac{1}{M} \sum_{i=1}^{M} \sqrt{(\mathsf{x}^{(i)}_{k+n} - \hat{\mathsf{x}}^{(i)}_{k+n|k})^2 + (\mathsf{y}^{(i)}_{k+n} - \hat{\mathsf{y}}^{(i)}_{k+n|k})^2}$$

where $[\mathsf{x}^{(i)}_{k+n}, \mathsf{y}^{(i)}_{k+n}]'$ is the true position at $k+n$ ($k+n = 10, \ldots, 100$) on the ith Monte Carlo run, $[\hat{\mathsf{x}}^{(i)}_{k+n|k}, \hat{\mathsf{y}}^{(i)}_{k+n|k}]'$ is its prediction using measurements up to time $k = 9$, and $M = 1000$ is the number of Monte Carlo runs. Results of the CM_L model in all different mismatched endpoints are shown. However, for the Markov model only the result of case (i) is shown. The ratio of $\text{AEE}^p_{100|9}$ of the Markov model to $\text{AEE}^p_{100|9}$ of the CM_L model $\frac{\text{AEE}^p_{100|9}(\text{Markov})}{\text{AEE}^p_{100|9}(CM_L)}$ is 545.75, which is huge. Performance of the Markov model in other cases is close to case (i) or worse. Fig. 7.19 shows that the origin and destination mismatched means degrade the prediction performance in case (ii). Prediction performance in case (ii) and case (iii) are close. However, an appropriate (large enough) covariance of the destination can compensate a large bias due to a highly mismatched destination mean. An inappropriate (small) covariance of the destination

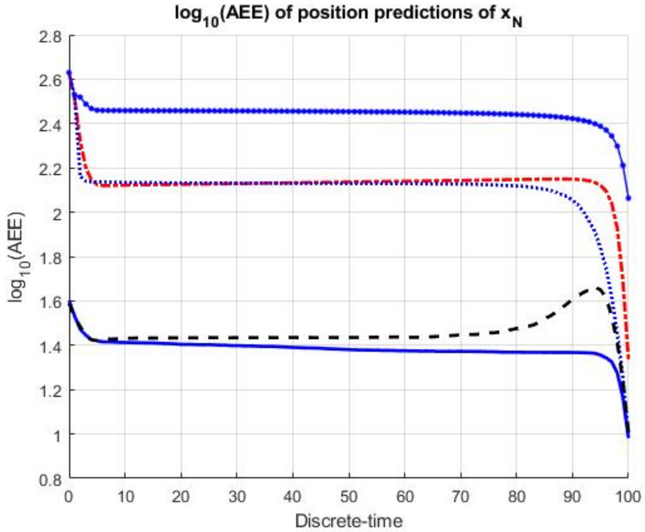

Figure 7.17: Log of AEE of position predictions of x_N ($\mathrm{AEE}^p_{N|k}$) (Example 3).

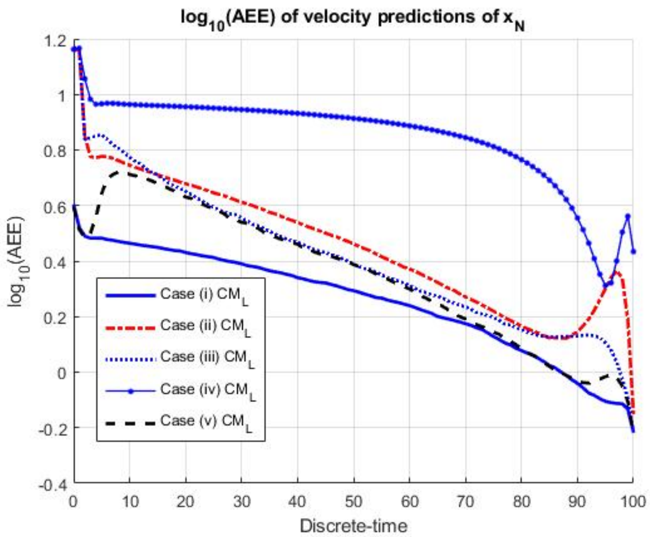

Figure 7.18: Log of AEE of velocity predictions of x_N ($\mathrm{AEE}^v_{N|k}$) (Example 3).

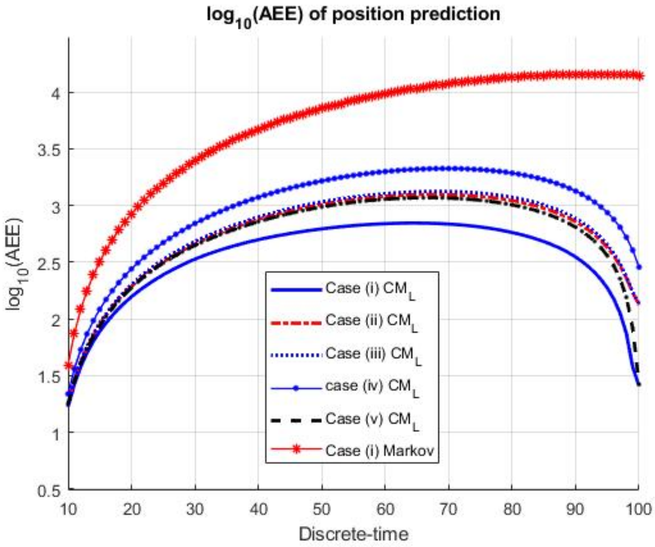

Figure 7.19: Log of AEE of position prediction ($\log_{10}(\text{AEE}_{9+n|9})$) (Example 4).

can make the bias impact (case (ii)) even worse (case (iv)). Prediction performance in case (v) is better than in case (ii) especially near the destination.

Example 7.4.5. In the previous examples, we used a NG CM_L sequence to model trajectories between an origin and a destination. Now assume the destination (the position) is completely known (i.e., position components of the state of the sequence at destination are almost surely constant, which means the sequence is singular). The means and the covariances of the origin and the destination are

$$\mu_0 = [2000, 5, 2000, 20]' \tag{7.83}$$

$$C_0 = \text{diag}(A, A) \tag{7.84}$$

$$A = \begin{bmatrix} 100000 & 40 \\ 40 & 10 \end{bmatrix} \tag{7.85}$$

$$\mu_N = [15000, 5, 2000, -20]' \tag{7.86}$$

$$C_N = \text{diag}(0, 1, 0, 1) \tag{7.87}$$

$$C_{0,N} = \text{diag}(0, 2, 0, 2) \tag{7.88}$$

The Markov model used in this example is the same as in the above examples with μ_0 and C_0 given by (7.83)–(7.84). The Markov-induced CM_L model is the same as in the above examples, where the boundary condition is $x_0 = \mu_0 + C_{0,N}(C_N)^+(x_N - \mu_N) + e_0$, $e_0 \sim \mathcal{N}(0, C_0 - C_{0,N}(C_N)^+ C'_{0,N})$, and $x_N = \mu_N + e_N$, $e_N = [0, \alpha, 0, \beta]'$, $\alpha \sim \mathcal{N}(0,1)$, $\beta \sim \mathcal{N}(0,1)$, where α and β are independent. Also, μ_0, C_0, μ_N, C_N, and $C_{0,N}$ are given by (7.83)–(7.88).

Fig. 7.20 shows trajectories generated by the CM_L model. To demonstrate the behavior of the CM_L model induced by the Markov model, Fig. 7.21 shows trajectories generated by the CM_L model (50 solid lines) and the Markov model (50 dash lines). Also, Figs. 7.22 and 7.23 show the x and y components of the velocity for trajectories of both models. It can be seen how the velocity components for the CM_L sequence variations to satisfy the origin and the destination densities. This example demonstrates an application of a singular CM_L sequence for DDT modeling.

[84] presented a CM sequence for modeling trajectories with waypoints and destination information.

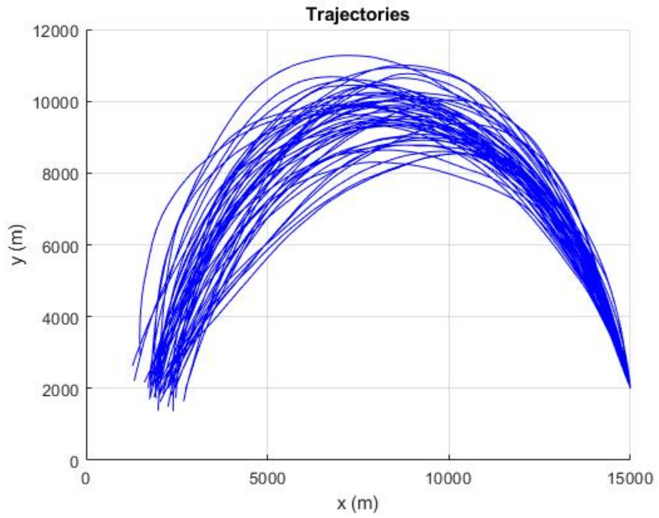

Figure 7.20: CM_L trajectories from an origin to a destination (Example 5).

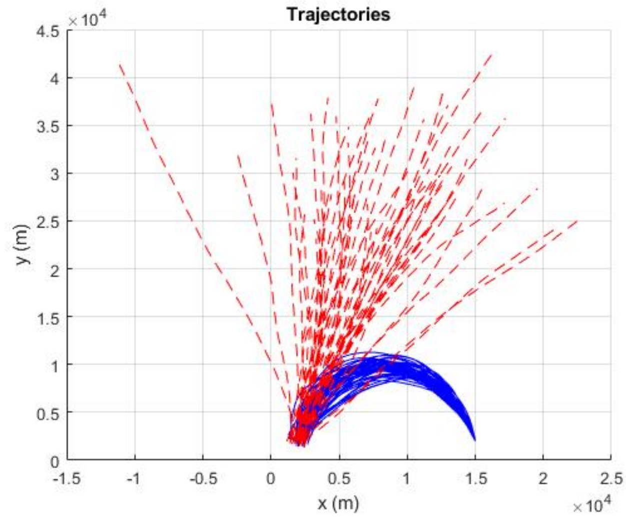

Figure 7.21: CM_L and Markov trajectories (Example 5).

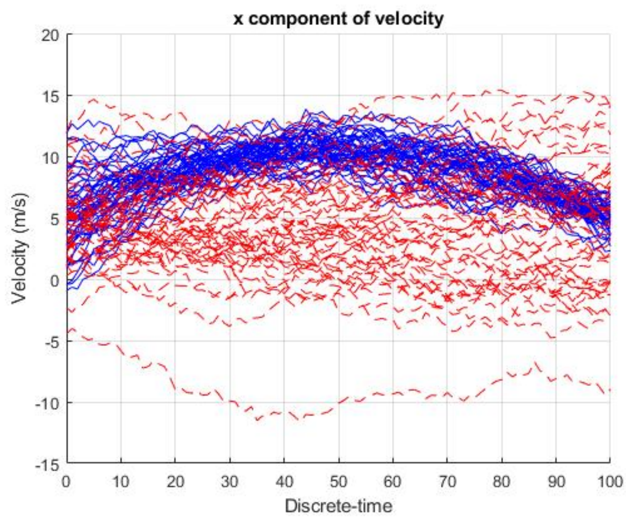

Figure 7.22: x-velocity for CM_L and Markov trajectories (Example 5).

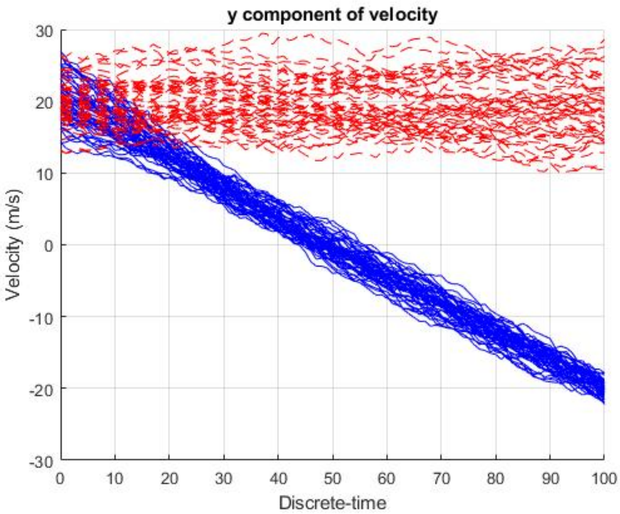

Figure 7.23: y-velocity for CM_L and Markov trajectories (Example 5).

Chapter 8

Conclusions and Future Work

We have developed a large class of stochastic sequences, called conditionally Markov (CM) sequences, and have demonstrated its power in theory and application. There are a wide variety of CM sequences useful for problem modeling. We have studied various Gaussian CM classes, obtained their dynamic models and characterizations, studied the relationships between their models, and pointed out their applications (Chapters 2–6). Chapters 2–6 have provided required models and tools for application of CM sequences. We highlight some of the obtained results.

- Nonsingular Gaussian (NG) CM_c sequences have a simple Markov-like recursive dynamic model in the state space with white dynamic noise ((2.17) along with (2.18) or (2.19)). The two boundary conditions (2.18) and (2.19) are equivalent. One can be more suitable than the other for a problem. (5.1) along with (5.2) or (5.3) is the extension of the above CM_c model to the general singular/nonsingular Gaussian sequences, where there is no nonsingularity condition on the covariances of the dynamic noise and the boundary values (i.e., $[e_k]$). There is no condition on the parameters of the CM_c model ((5.1) along with (5.2) or (5.3)) and the model is well-posed for any value of its parameters.

- Inverse of the covariance matrix of the NG CM_c sequence has a special structure (which differs from that of the NG Markov sequence only in the first/last row and column) that characterizes the sequence (Chapter 2). These characterizations clearly reveal the relationship between Markov, reciprocal, and CM_c sequences, i.e., a Markov sequence is a special reciprocal sequence, and a reciprocal sequence is a special CM_c sequence. These characterizations are also useful to obtain a reciprocal dynamic model from the CM viewpoint (Chapter 3). A more general characterization is in terms of the covariance function of the CM_c sequence that characterizes the general singular/nonsingular Gaussian CM_c sequence (Chapter 5).

- We initiated the CM viewpoint to study reciprocal processes. CM processes provide an insightful and fruitful viewpoint for studying reciprocal processes. For example, a NG sequence is reciprocal if and only if (iff) it is both CM_L and CM_F, i.e., the NG reciprocal sequence is equivalent to the intersection of the NG CM_L sequence and the NG CM_F sequence. This relationship simplifies studying the NG reciprocal sequence by studying the NG CM_L sequence and the NG CM_F sequence. For example, this idea leads to a reciprocal CM_L/CM_F dynamic model with white dynamic noise being easy to apply (Chapter 3). A full spectrum of characterizations and dynamic models from a NG CM class to the NG reciprocal class provides more insight into these classes (Chapter 3, Chapter 4).

- The evolution of a Markov sequence can be modeled by a CM_L model. Correspondingly, a Markov model can induce a CM_L model that is actually a reciprocal CM_L model. Also, every reciprocal CM_L model can be induced by a Markov model. This is particularly useful for parameter design of a reciprocal CM_L model based on those of a Markov model since one usually has an intuitive understanding of the Markov model (Chapter 4).

- By definition, a CM_c sequence is obtained based on combining the Markov property and conditioning. Every Gaussian CM_c sequence can be decomposed to a Gauss-Markov sequence and an independent Gaussian vector as the conditioning state (i.e., sum of a Gauss-Markov sequence and an independent Gaussian vector). Also, a sum of a Gauss-Markov sequence and an independent Gaussian vector gives a CM_c sequence, where the independent vector is the conditioning state. This is particularly useful for design of a CM_c sequence/model in application based on a Gauss-Markov sequence/model and an independent Gaussian vector (Chapter 4). Also, it makes the key fact about the CM_c sequence clear (i.e., the Markov property and the conditioning state).

- Singular CM (including reciprocal) sequences are desired for modeling some problems. For example, a singular CM_L sequence is desired for destination-directed trajectory modeling, where some components of the state at the destination are known (e.g., the destination position is (almost surely) constant). Our CM_c dynamic model works for both singular and nonsingular Gaussian sequences (Chapter 2, Chapter 5). The well-posedness of the reciprocal model presented in [18] is guaranteed by the nonsingularity of its sequence. This is why it has not been possible to generalize the model of [18] to the general singular/nonsingular case even after decades. However, from the CM viewpoint we have obtained a reciprocal CM_c model for the general singular/nonsingular Gaussian case. This demonstrates the significance of studying reciprocal sequences from the CM viewpoint.

- A CM (including Markov and reciprocal) sequence can be described by different models. For example, a reciprocal sequence can be modeled by a reciprocal model of [18], a (forward/backward) CM_L model, or a (forward/backward) CM_F model. These models are equivalent, but one can be more suitable than the other for a given problem. We defined two notions of equivalency for models: algebraic and probabilistic. Then, we presented a unified approach based on which given a model of a NG CM_c sequence, other (algebraically) equivalent models can be obtained. As a special case, given a forward Markov model, the presented approach gives an (algebraically) equivalent backward Markov model regardless of the singularity/nonsingularity of the Markov transition matrix. This makes it possible to check the required condition for two-filter smoothing for a Markov model with a singular transition matrix, which has not been possible before.

By definition, a process is Markov iff given the state at any time, the states before and after that are independent. A process is reciprocal iff given the states at any two times, the states between two times are independent of the states outside. The reciprocal process is a generalization of the Markov process. However, according to the definition and the properties of the reciprocal process, it can be seen that it is a complicated generalization of the Markov process. The CM process is a simpler and more flexible generalization of the Markov process based on conditioning. It has several classes and includes the reciprocal process as a special case. The CM process is a more powerful generalization of the Markov process for problem modeling.

In Chapters 2–6, we have provided required tools for application of CM sequences and have pointed out such applications. Then, as an example, we have elaborated an application of one CM class (i.e., CM_L) to *destination-directed trajectory* modeling in Chapter 7. In the following, we discuss some directions and ideas for future research in application of CM sequences.

As it can be seen from the results of Chapter 7, the impact of destination is significant on the behavior of trajectories when they are close to the destination, which is intuitive. Depending on different factors (e.g., sampling interval) the impact of destination can be small on the behavior of trajectories when they are far from it. Destination impact on the local (small scale) behavior of trajectories when they are far from the destination can be tiny, but on the global (large scale) behavior can be significant. Accordingly, we can consider different modeling scales. One for the local scale and the other for the global scale (this is somewhat similar to the idea of "meta-level tracking" [42]). In the former we can use the existing dynamic models/filters (without the destination notion) and in the latter we can use a CM_L dynamic model and the corresponding filter, where the two scales are connected. A good design of the two scales is based on the impact of the destination. The impact is negligible in one and significant in the other. Also, the two scales can change over time and can be even merged when trajectories are close to the destination.

Reciprocal CM_L models can be induced by a Markov model (Theorem 4.1.3). Their parameters can be designed based on those of the Markov model (Chapter 4). However, not all CM_L models can be induced by a Markov model. Therefore, Theorem 4.1.3 can not be used for parameter design of *all* CM_L models. The Markov-based representation of CM_L sequences (Proposition 4.2.1 and Proposition 5.3.3) makes key components of the Gaussian CM_L sequence clear: a Gauss-Markov sequence and an independent Gaussian vector. The result of that proposition is necessary and sufficient for a Gaussian sequence to be CM_L. It means that every Gaussian CM_L sequence can be constructed based on (4.38). Also, the superimposition of every Gauss-Markov sequence and an independent Gaussian vector (based on (4.38)) gives a CM_L sequence. On the other hand, in Chapter 7 we showed that CM_L sequences naturally model destination-directed trajectories. So, Proposition 4.2.1 is particularly useful for constructing a CM_L model for destination-directed trajectories. Superimposition of every Gauss-Markov sequence and an independent Gaussian vector models some trajectories from an origin to a destination. But it is desired to choose a Gauss-Markov sequence (called the underlying Markov sequence of a CM_L sequence (Definition 4.2.7)), an independent Gaussian vector, and appropriate coefficients (of the independent Gaussian vector in (4.38)) that lead to desired trajectories from the origin to the destination.

In Chapter 7, we showed that destination-directed trajectories can be naturally modeled by CM_L sequences (Section 7.1.1). Modeling the evolution law by a Markov conditional density (conditioned on the last state), the resultant sequence is a CM_L sequence that naturally models destination-directed trajectories. Instead of a Markov conditional density, we can consider more general and complicated conditional densities, for example, a higher-order Markov conditional density, to model the evolution law. Then, the resultant sequence is a higher-order CM_L sequence, i.e., a CM_L sequence with a higher-order Markov property. Application of such CM_L sequences in destination-directed trajectory modeling can be further studied.

CM sequences belonging to more than one CM class are useful in application. For example, reciprocal sequences, which have been used in various applications (Chapter 3–4), belong to several CM classes. Another example of CM sequences belonging to more than one CM class is $CM_L \cap [0, k_2]$-CM_L sequences. An application of such sequences in trajectory modeling with a waypoint and a destination was pointed out in Chapter 4, where a dynamic model of $CM_L \cap [0, k_2]$-CM_L sequences was also obtained. Details of an application of $CM_L \cap [0, k_2]$-CM_L sequences in trajectory modeling with a waypoint and a destination can be further studied.

In Chapter 7, we modeled the destination-directed trajectory of a single target by a CM_L sequence. We can also study a multi-target scenario, where information about destinations of targets is available. A CM_L dynamic model can be used for trajectory modeling of each target. By incorporating destination information, a CM_L model can improve data association. Also, a CM_L model can be used in the PHD filter framework for multi-target tracking with destination information. In addition, trajectory prediction based on a CM_L model can be used for the purpose of conflict detection in air traffic control. So, there are several directions for application of CM models in multi-target problems.

The idea of using CM_L sequences for trajectory modeling with destination information can be generalized to other problems using CM sequences. Note the critical role of a destination in destination-directed trajectory modeling. The trajectory of a flight depends on its destination. It is the destination that makes destination-directed trajectory modeling more complicated than trajectory modeling without a destination. By conditioning on the state at the destination, the remaining problem is a simple one of no ambiguity about the destination. Then, the conditional sequence is modeled as a Markov sequence, which is simple (Chapter 7). Thus, by conditioning, a complicated problem is reduced to a simple one. This idea can be used to handle many problems in which there are some "hubs" ("critical parts") affecting the problem as the source of complexity of the problem. In order to use conditioning effectively, we first should understand the problem well to distinguish such "hubs" in the problem (e.g., destination in the above problem). Then, by conditioning on the "hubs" the complicated problem is reduced to a simpler one easy to handle.

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Appendix

A Proof of Lemma 2.3.4

We first prove the following lemma about a factorization of a tridiagonal matrix. Such a factorization was used in [56] without proof.

Lemma A.1. *A positive definite block*[1] *tridiagonal matrix T can be factorized as $T = U'DU$, where U is upper block bidiagonal with unit diagonal*[2]*, and D is block diagonal.*

Proof. Let T be an $(N+1)d \times (N+1)d$ positive definite block tridiagonal matrix. T can be triangularly factorized as [88],

$$T = \begin{bmatrix} T_1 & T_{12} \\ T'_{12} & T_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ T'_{12}T_1^{-1} & I \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & \Delta_{T_1} \end{bmatrix} \begin{bmatrix} I & T_1^{-1}T_{12} \\ 0 & I \end{bmatrix} \tag{A.1}$$

where $\Delta_{T_1} = T_2 - T'_{12}T_1^{-1}T_{12}$. T_1 is $Nd \times Nd$, and T_2 is $d \times d$. Since T_1 is also a nonsingular block tridiagonal matrix we can factorize it (similar to (A.1)) as $T_1 = U'_1 D_{T_1} U_1$, where U_1 is upper triangular with unit diagonal and D_{T_1} is block diagonal whose first block $[D_{T_1}]_1$ is $(N-1)d \times (N-1)d$ and the second block $[D_{T_1}]_2$ is $d \times d$. Using the factorization of T_1 and (A.1), T can be factorized as follows

$$\begin{bmatrix} I & 0 \\ T'_{12}T_1^{-1} & I \end{bmatrix} \begin{bmatrix} U'_1 D_{T_1} U_1 & 0 \\ 0 & \Delta_{T_1} \end{bmatrix} \begin{bmatrix} I & T_1^{-1}T_{12} \\ 0 & I \end{bmatrix} =$$
$$\underbrace{\begin{bmatrix} U'_1 & 0 \\ T'_{12}T_1^{-1}U'_1 & I \end{bmatrix}}_{W'} \begin{bmatrix} D_{T_1} & 0 \\ 0 & \Delta_{T_1} \end{bmatrix} \underbrace{\begin{bmatrix} U_1 & U_1 T_1^{-1}T_{12} \\ 0 & I \end{bmatrix}}_{W}$$

Then, using $T_1 = U'_1 D_{T_1} U_1$, we have $T'_{12}T_1^{-1}U'_1 = T'_{12}U_1^{-1}D_{T_1}^{-1}$, where U_1^{-1} is upper triangular. Then, from the forms of T_{12}, U_1^{-1}, and $D_{T_1}^{-1}$, it can be seen that $T'_{12}T_1^{-1}U'_1$ is a $d \times dN$ block row matrix of the form $[0_{d\times d(N-1)} \quad *_{d\times d}]$—the first $N-1$ blocks being zero and the last block (denoted by $*$) not necessarily zero. Therefore, the structure of the last block column of W is the same as that of an upper block bidiagonal matrix. Then, we can continue the same procedure for the matrix $[D_{T_1}]_1$ and so on, to obtain $T = U'DU$ with U being upper block bidiagonal with unit diagonal and D block diagonal. □

In the following, we prove Lemma 2.3.4 using Lemma A.1. We prove (i) (in Lemma 2.3.4) and skip (ii) (they are similar).

First, triangular factorization of a CM_L matrix is obtained. Let $A_{(N+1)d\times(N+1)d}$ be a CM_L matrix. Since it is positive definite, it can be factorized as $A = V'DV$, where V is upper triangular with unit diagonal and D is block diagonal:

$$\begin{bmatrix} A_1 & A_{12} \\ A'_{12} & A_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ A'_{12}A_1^{-1} & I \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & \Delta_{A_1} \end{bmatrix} \begin{bmatrix} I & A_1^{-1}A_{12} \\ 0 & I \end{bmatrix} \tag{A.2}$$

[1] In this appendix, we consider block matrices (block tridiagonal or CM_c matrices) with $d \times d$ blocks.

[2] An $(N+1)d \times (N+1)d$ upper block bidiagonal matrix with unit diagonal is: identity matrices $I_{d\times d}$ as block diagonal elements, the first upper minor $d \times d$ block diagonal elements not necessarily zero, and all other elements zero.

where $\Delta_{A_1} = A_2 - A_{12}' A_1^{-1} A_{12}$, A_1 is $Nd \times Nd$, and A_2 is $d \times d$. Since A is CM_L, A_1 is positive definite block tridiagonal. So, A_1 can be factorized as $A_1 = V_1' D_{A_1} V_1$, where V_1 is upper block bidiagonal with unit diagonal and D_{A_1} is block diagonal. Then by (A.2), A can be factorized as

$$\begin{bmatrix} I & 0 \\ A_{12}' A_1^{-1} & I \end{bmatrix} \begin{bmatrix} V_1' D_{A_1} V_1 & 0 \\ 0 & \Delta_{A_1} \end{bmatrix} \begin{bmatrix} I & A_1^{-1} A_{12} \\ 0 & I \end{bmatrix} =$$
$$\begin{bmatrix} V_1' & 0 \\ A_{12}' A_1^{-1} V_1' & I \end{bmatrix} \begin{bmatrix} D_{A_1} & 0 \\ 0 & \Delta_{A_1} \end{bmatrix} \begin{bmatrix} V_1 & V_1 A_1^{-1} A_{12} \\ 0 & I \end{bmatrix} = V' D V$$

It can be seen that $A_{12}' A_1^{-1} V_1'$ is a $d \times dN$ block row matrix with not necessarily zero blocks[3]. In addition, V_1 is upper block bidiagonal with unit diagonal. Therefore, V in the factorization $A = V' D V$ has form (2.38). Also, D is block diagonal. The uniqueness of this factorization is discussed below.

Next, factorizations of a CM_L matrix in the forms of (2.39) and (2.40) are discussed.

Consider a ZMNG sequence with covariance matrix C governed by backward model (2.31)–(2.32). By (2.34), we have $C^{-1} = (\mathcal{G}^B)'(G^B)^{-1}\mathcal{G}^B$, where $G^B = \text{diag}(G_0^B, G_1^B, \ldots, G_N^B)$, G_k^B ($k \in [0, N]$) are nonsingular, and $\mathcal{G}^B$ is given by (2.35). It can be seen that matrices (2.38) and (2.35) have the same form. Thus, from $\mathcal{G}^B = U$ and $(G^B)^{-1} = D_A$, we can construct a backward model (2.31)–(2.32) for a ZMNG CM_L sequence with $C^{-1} = A$. Therefore, for every CM_L matrix A, there exists a unique ZMNG CM_L sequence $[x_k]$ with its C^{-1} equal to A (the uniqueness of such a sequence is obvious because the covariance matrix C determines the Gaussian sequence). On the other hand, by Lemma 2.2.1, given a ZMNG CM_L sequence, one can construct its model (2.17) along with (2.18) or (2.19). Also, the inverse of the covariance matrix of the sequence can be calculated by (2.30), where $\mathcal{G}$ is given by (2.27) for (2.18), and by (2.28) for (2.19). It can be seen that (2.27) is actually in the form of (2.39), and (2.28) is in the form of (2.40). In addition, given a CM_L matrix C^{-1}, parameters of the forward/backward model of a ZMNG CM_L sequence with the covariance matrix C are unique (see Remark 2.3.6). Therefore, a CM_L matrix can be uniquely factorized in the forms of (2.38), (2.39), and (2.40).

Also, given a CM_L matrix A, parameters of forward/backward models of a sequence with $C^{-1} = A$ can be easily obtained (Lemma B.1 in Appendix B). Then, (2.35), (2.27), and (2.28) give V, and G (G^B) gives D for factorization of the CM_L matrix. So, not only their structure, but also the values of the matrices V and D in the factorizations of Lemma 2.3.4 are also determined.

B (Probabilistically) Equivalent Models

Parameters of equivalent models can be calculated based on (6.30). Since there are several different models, in order to save space, it suffices to show i) how the entries of the inverse of the covariance matrix of a sequence (C^{-1}) can be written in terms of the parameters of the model and the boundary condition of the sequence, ii) how parameters of a model and its boundary condition can be calculated from the entries of C^{-1}. Then, based on (i) and (ii), given parameters of a model and its boundary condition, parameters of any equivalent model and its boundary condition can be uniquely determined.

[3]Note that in (A.1), T_{12}' has the form $[0_{d\times d(N-1)} \quad *_{d\times d}]$, and so $T_{12}' T_1^{-1} U_1'$ has the same form, but in (A.2), A_{12}' is a general $d \times dN$ block matrix and so $A_{12}' A_1^{-1} U_1'$ is also a general $d \times dN$ matrix.

B.1 CM_L Sequences

Forward CM_L Model ($c = N$)

For (2.17):

$$A_k = G_k^{-1} + G'_{k+1,k}G_{k+1}^{-1}G_{k+1,k}, k \in [1, N-2] \tag{B.1}$$

$$A_{N-1} = G_{N-1}^{-1} \tag{B.2}$$

$$B_k = -G'_{k+1,k}G_{k+1}^{-1}, k \in [0, N-2] \tag{B.3}$$

$$B_{N-1} = -G_{N-1}^{-1}G_{N-1,N} \tag{B.4}$$

$$D_k = -G_k^{-1}G_{k,N} + G'_{k+1,k}G_{k+1}^{-1}G_{k+1,N}, k \in [1, N-2] \tag{B.5}$$

for boundary condition (2.18):

$$A_0 = G_0^{-1} + G'_{1,0}G_1^{-1}G_{1,0} + G'_{N,0}G_N^{-1}G_{N,0} \tag{B.6}$$

$$A_N = G_N^{-1} + \sum_{k=1}^{N-1} G'_{k,N}G_k^{-1}G_{k,N} \tag{B.7}$$

$$D_0 = G'_{1,0}G_1^{-1}G_{1,N} - G'_{N,0}G_N^{-1} \tag{B.8}$$

and for (2.19):

$$A_0 = G_0^{-1} + G'_{1,0}G_1^{-1}G_{1,0} \tag{B.9}$$

$$A_N = G_N^{-1} + \sum_{k=1}^{N-1} G'_{k,N}G_k^{-1}G_{k,N} + G'_{0,N}G_0^{-1}G_{0,N} \tag{B.10}$$

$$D_0 = -G_0^{-1}G_{0,N} + G'_{1,0}G_1^{-1}G_{1,N} \tag{B.11}$$

Backward CM_F Model ($c = N$)

For (2.31)–(2.32):

$$A_0 = (G_0^B)^{-1} \tag{B.12}$$

$$A_{k+1} = (G_{k,k+1}^B)'(G_k^B)^{-1}G_{k,k+1}^B + (G_{k+1}^B)^{-1}, k \in [0, N-2] \tag{B.13}$$

$$A_N = \sum_{k=0}^{N-2}(G_{k,N}^B)'(G_k^B)^{-1}G_{k,N}^B + 4(G_{N-1,N}^B)'(G_{N-1}^B)^{-1}G_{N-1,N}^B + (G_N^B)^{-1} \tag{B.14}$$

$$B_k = -(G_k^B)^{-1}G_{k,k+1}^B, k \in [0, N-2] \tag{B.15}$$

$$B_{N-1} = (G_{N-2,N-1}^B)'(G_{N-2}^B)^{-1}G_{N-2,N}^B - 2(G_{N-1}^B)^{-1}G_{N-1,N}^B \tag{B.16}$$

$$D_0 = -(G_0^B)^{-1}G_{0,N}^B \tag{B.17}$$

$$D_k = (G_{k-1,k}^B)'(G_{k-1}^B)^{-1}G_{k-1,N}^B - (G_k^B)^{-1}G_{k,N}^B, k \in [1, N-2] \tag{B.18}$$

Lemma B.1. *Parameters of CM_L model (2.17) along with (2.18) or (2.19) and backward CM_F model (2.31)–(2.32) of a ZMNG CM_L sequence with the inverse of its covariance matrix equal to any given CM_L matrix (2.36) can be uniquely determined as follows.*

(i) CM_L model (2.17) ($c = N$):

$$G_{N-1}^{-1} = A_{N-1} \tag{B.19}$$

$$\begin{cases} k = N-1, \ldots, 2: \\ G_{k,k-1} = -G_k B'_{k-1} \\ G_{k-1}^{-1} = A_{k-1} - G'_{k,k-1}(G_k)^{-1}G_{k,k-1} \end{cases} \tag{B.20}$$

$$G_{1,0} = -G_1 B_0' \tag{B.21}$$
$$G_{N-1,N} = -G_{N-1} B_{N-1} \tag{B.22}$$
$$\begin{cases} k = N-1, \dots, 2: \\ G_{k-1,N} = G_{k-1} G'_{k,k-1} G_k^{-1} G_{k,N} - G_{k-1} D_{k-1} \end{cases} \tag{B.23}$$

Parameters of the boundary condition are: for (2.18)

$$G_N^{-1} = A_N - \sum_{k=1}^{N-1} G'_{k,N} G_k^{-1} G_{k,N} \tag{B.24}$$
$$G_{N,0} = G_N G'_{1,N} G_1^{-1} G_{1,0} - G_N D_0' \tag{B.25}$$
$$G_0^{-1} = A_0 - G'_{1,0} G_1^{-1} G_{1,0} - G'_{N,0} G_N^{-1} G_{N,0} \tag{B.26}$$

and for (2.19)*:*

$$G_0^{-1} = A_0 - G'_{1,0} G_1^{-1} G_{1,0} \tag{B.27}$$
$$G_{0,N} = G_0 G'_{1,0} G_1^{-1} G_{1,N} - G_0 D_0 \tag{B.28}$$
$$G_N^{-1} = A_N - \sum_{k=1}^{N-1} G'_{k,N} G_k^{-1} G_{k,N} - G'_{0,N} G_0^{-1} G_{0,N} \tag{B.29}$$

(ii) Backward CM_F model (2.31)–(2.32) *($c = N$):*

$$(G_0^B)^{-1} = A_0 \tag{B.30}$$
$$\begin{cases} k = 0, 1, \dots, N-2: \\ G^B_{k,k+1} = -G^B_k B_k \\ (G^B_{k+1})^{-1} = A_{k+1} - (G^B_{k,k+1})'(G^B_k)^{-1} G^B_{k,k+1} \end{cases} \tag{B.31}$$
$$G^B_{0,N} = -G^B_0 D_0 \tag{B.32}$$
$$\begin{cases} k = 1, 2, \dots, N-2: \\ G^B_{k,N} = G^B_k (G^B_{k-1,k})'(G^B_{k-1})^{-1} G^B_{k-1,N} - G^B_k D_k \end{cases} \tag{B.33}$$
$$2G^B_{N-1,N} = G^B_{N-1}(G^B_{N-2,N-1})'(G^B_{N-2})^{-1} G^B_{N-2,N} - G^B_{N-1} B_{N-1} \tag{B.34}$$
$$(G^B_N)^{-1} = A_N - \sum_{i=0}^{N-2} (G^B_{i,N})'(G^B_i)^{-1} G^B_{i,N} - 4(G^B_{N-1,N})'(G^B_{N-1})^{-1} G^B_{N-1,N} \tag{B.35}$$

B.2 CM_F Sequences

CM_F Model ($c = 0$)

For (2.17)–(2.18):

$$A_0 = G_0^{-1} + \sum_{k=2}^{N} G'_{k,0}(G_k)^{-1} G_{k,0} + 4G'_{1,0} G_1^{-1} G_{1,0} \tag{B.36}$$
$$A_k = G'_{k+1,k}(G_{k+1})^{-1} G_{k+1,k} + G_k^{-1}, k \in [1, N-1] \tag{B.37}$$
$$A_N = G_N^{-1} \tag{B.38}$$
$$B_0 = G'_{2,0} G_2^{-1} G_{2,1} - 2G'_{1,0} G_1^{-1} \tag{B.39}$$
$$B_k = -G'_{k+1,k}(G_{k+1})^{-1}, k \in [1, N-1] \tag{B.40}$$
$$D_k = G'_{k+1,0} G_{k+1}^{-1} G_{k+1,k} - G'_{k,0} G_k^{-1}, k \in [2, N-1] \tag{B.41}$$
$$D_N = -G'_{N,0} G_N^{-1} \tag{B.42}$$

Backward CM_L Model ($c = 0$)

For backward CM_L model (2.31):

$$A_1 =(G_1^B)^{-1} \tag{B.43}$$
$$A_k =(G_{k-1,k}^B)'(G_{k-1}^B)^{-1}G_{k-1,k}^B + (G_k^B)^{-1}, k \in [2, N-1] \tag{B.44}$$
$$B_0 =-(G_{1,0}^B)'(G_1^B)^{-1} \tag{B.45}$$
$$B_k =-(G_k^B)^{-1}G_{k,k+1}^B, k \in [1, N-1] \tag{B.46}$$
$$E_k =(G_{k-1,0}^B)'(G_{k-1}^B)^{-1}G_{k-1,k}^B - (G_{k,0}^B)'(G_k^B)^{-1}, k \in [2, N-1] \tag{B.47}$$
$$\tag{B.48}$$

for boundary condition (2.32):

$$A_0 =(G_0^B)^{-1} + \sum_{k=1}^{N-1}(G_{k,0}^B)'(G_k^B)^{-1}G_{k,0}^B + (G_{N,0}^B)'(G_N^B)^{-1}G_{N,0}^B \tag{B.49}$$
$$A_N =(G_{N-1,N}^B)'(G_{N-1}^B)^{-1}G_{N-1,N}^B + (G_N^B)^{-1} \tag{B.50}$$
$$E_N =(G_{N-1,0}^B)'(G_{N-1}^B)^{-1}G_{N-1,N}^B - (G_{N,0}^B)'(G_N^B)^{-1} \tag{B.51}$$

and for (2.33):

$$A_0 =(G_0^B)^{-1} + \sum_{k=1}^{N-1}(G_{k,0}^B)'(G_k^B)^{-1}G_{k,0}^B \tag{B.52}$$
$$A_N =(G_{N-1,N}^B)'(G_{N-1}^B)^{-1}G_{N-1,N}^B + (G_N^B)^{-1} + (G_{0,N}^B)'(G_0^B)^{-1}G_{0,N}^B \tag{B.53}$$
$$E_N =(G_{N-1,0}^B)'(G_{N-1}^B)^{-1}G_{N-1,N}^B - (G_0^B)^{-1}G_{0,N}^B \tag{B.54}$$

Lemma B.2. *Parameters of CM_F model* (2.17)–(2.18) *and backward CM_L model* (2.31) *along with* (2.32) *or* (2.33) *of a ZMNG CM_F sequence with the inverse of its covariance matrix equal to any given CM_F matrix* (2.37) *can be uniquely determined as follows.*

(i) CM_F model (2.17)–(2.18)*:*

$$G_N^{-1} = A_N \tag{B.55}$$
$$\begin{cases} k = N, N-1, \ldots, 2 : \\ G_{k,k-1} = -G_k B'_{k-1} \\ G_{k-1}^{-1} = A_{k-1} - G'_{k,k-1}(G_k)^{-1}G_{k,k-1} \end{cases} \tag{B.56}$$
$$G_{N,0} = -G_N E'_N \tag{B.57}$$
$$\begin{cases} k = N-1, N-2, \ldots, 2 : \\ G_{k,0} = G_k G'_{k+1,k} G_{k+1}^{-1} G_{k+1,0} - G_k E'_k \end{cases} \tag{B.58}$$
$$2G_{1,0} = G_1 G'_{2,1} G_2^{-1} G_{2,0} - G_1 B'_0 \tag{B.59}$$
$$G_0^{-1} = A_0 - \sum_{k=2}^{N} G'_{k,0} G_k^{-1} G_{k,0} - 4G'_{1,0} G_1^{-1} G_{1,0} \tag{B.60}$$

(ii) Backward CM_L model (2.31) *($c = 0$):*

$$(G_1^B)^{-1} = A_1 \tag{B.61}$$

$$\begin{cases} k = 1, 2, \ldots, N-2 : \\ G^B_{k,k+1} = -G^B_k B_k \\ (G^B_{k+1})^{-1} = A_{k+1} - (G^B_{k,k+1})'(G^B_k)^{-1} G^B_{k,k+1} \end{cases} \tag{B.62}$$

$$G^B_{N-1,N} = -G^B_{N-1} B_{N-1} \tag{B.63}$$

$$G^B_{1,0} = -G^B_1 B'_0 \tag{B.64}$$

$$\begin{cases} k = 2, \ldots, N-1 : \\ G^B_{k,0} = G^B_k (G^B_{k-1,k})'(G^B_{k-1})^{-1} G^B_{k-1,0} - G^B_k E'_k \end{cases} \tag{B.65}$$

Parameters of the boundary condition are: for (2.32)

$$(G^B_N)^{-1} = A_N - (G^B_{N-1,N})'(G^B_{N-1})^{-1} G^B_{N-1,N} \tag{B.66}$$

$$(G^B_0)^{-1} = A_0 - \sum_{k=1}^{N-1} (G^B_{k,0})'(G^B_k)^{-1} G^B_{k,0} - (G^B_{N,0})'(G^B_N)^{-1} G^B_{N,0} \tag{B.67}$$

$$G^B_{N,0} = G^B_N (G^B_{N-1,N})'(G^B_{N-1})^{-1} G^B_{N-1,0} - G^B_N E'_N \tag{B.68}$$

and for (2.33)*:*

$$(G^B_0)^{-1} = A_0 - \sum_{k=1}^{N-1} (G^B_{k,0})'(G^B_k)^{-1} G^B_{k,0} \tag{B.69}$$

$$G^B_{0,N} = G^B_0 (F^B_{N-1,0})'(G^B_{N-1})^{-1} G^B_{N-1,N} - G^B_0 E_N \tag{B.70}$$

$$(G^B_N)^{-1} = A_N - (G^B_{N-1,N})'(G^B_{N-1})^{-1} G^B_{N-1,N} - (G^B_{0,N})'(G^B_0)^{-1} G^B_{0,N} \tag{B.71}$$

B.3 Reciprocal Sequences

For reciprocal model (6.9) along with (6.10)–(6.11):

$$R^0_k = A_k, k \in [0, N] \tag{B.72}$$

$$R^+_k = (R^-_{k+1})' = -B_k, k \in [0, N-1] \tag{B.73}$$

$$R^-_0 = (R^+_N)' = -D_0 \tag{B.74}$$

Model (6.9) along with (6.12) or (6.13) was discussed in Section 6.4.

B.4 Markov Sequences

Markov Model (6.1)

$$A_0 = M_0^{-1} + M'_{1,0} M_1^{-1} M_{1,0} \tag{B.75}$$

$$A_k = M_k^{-1} + M'_{k+1,k} M_{k+1}^{-1} M_{k+1,k}, k \in [1, N-1] \tag{B.76}$$

$$A_N = M_N^{-1} \tag{B.77}$$

$$B_k = -M'_{k+1,k} M_{k+1}^{-1}, k \in [0, N-1] \tag{B.78}$$

Backward Markov Model (6.5)

$$A_0 = (M^B_0)^{-1} \tag{B.79}$$

$$A_k = (M^B_k)^{-1} + (M^B_{k-1,k})'(M^B_{k-1})^{-1} M^B_{k-1,k}, k \in [1, N-1] \tag{B.80}$$

$$A_N = (M^B_N)^{-1} + (M^B_{N-1,N})'(M^B_{N-1})^{-1} M^B_{N-1,N} \tag{B.81}$$

$$B_k = -(M^B_k)^{-1} M^B_{k,k+1}, k \in [0, N-1] \tag{B.82}$$

Lemma B.3. *Parameters of Markov model* (6.1) *and backward Markov model* (6.5) *of a ZMNG Markov sequence with the inverse of its covariance matrix equal to any given symmetric positive definite (block) tri-diagonal matrix can be uniquely determined as follows:*

(i) Markov model (6.1)*:*

$$M_N^{-1} = A_N \tag{B.83}$$

$$M_{N,N-1} = -M_N B'_{N-1} \tag{B.84}$$

$$\begin{cases} k = N-2, N-3, \ldots, 0: \\ M_{k+1}^{-1} = A_{k+1} - M'_{k+2,k+1} M_{k+2}^{-1} M_{k+2,k+1} \\ M_{k+1,k} = -M_{k+1} B'_k \end{cases} \tag{B.85}$$

$$M_0^{-1} = A_0 - M'_{1,0} M_1^{-1} M_{1,0} \tag{B.86}$$

(ii) Backward Markov model (6.5)*:*

$$(M_0^B)^{-1} = A_0 \tag{B.87}$$

$$M_{0,1}^B = -M_0^B B_0 \tag{B.88}$$

$$\begin{cases} k = 2, 3, \ldots, N: \\ (M_{k-1}^B)^{-1} = A_{k-1} - (M_{k-2,k-1}^B)'(M_{k-2}^B)^{-1} M_{k-2,k-1}^B \\ M_{k-1,k}^B = -M_{k-1}^B B_{k-1} \end{cases} \tag{B.89}$$

$$(M_N^B)^{-1} = A_N - (M_{N-1,N}^B)'(M_{N-1}^B)^{-1} M_{N-1,N}^B \tag{B.90}$$

C Algebraically Equivalent Models

Following (6.31), relationships of dynamic noise and boundary values between some algebraically equivalent models are presented.

C.1 Reciprocal Model and Markov Model

$$e_0^R = M_0^{-1} e_0^M - M'_{1,0} M_1^{-1} e_1^M \tag{C.1}$$

$$e_k^R = M_k^{-1} e_k^M - M'_{k+1,k} M_{k+1}^{-1} e_{k+1}^M, k \in [1, N-1] \tag{C.2}$$

$$e_N^R = M_N^{-1} e_N^M \tag{C.3}$$

These equations are the same as those obtained in [18] by a different approach.

C.2 CM_L Model and Markov Model

(i) CM_L model (2.17)–(2.18) ($c = N$):

$$G_0^{-1} e_0 - G'_{1,0} G_1^{-1} e_1 - G'_{N,0} G_N^{-1} e_N = M_0^{-1} e_0^M - M'_{1,0} M_1^{-1} e_1^M \tag{C.4}$$

$$G_k^{-1} e_k - G'_{k+1,k} G_{k+1}^{-1} e_{k+1} = M_k^{-1} e_k^M - M'_{k+1,k} M_{k+1}^{-1} e_{k+1}^M, k \in [1, N-2] \tag{C.5}$$

$$G_{N-1}^{-1} e_{N-1} = M_{N-1}^{-1} e_{N-1}^M - M'_{N,N-1} M_N^{-1} e_N^M \tag{C.6}$$

$$-\sum_{k=1}^{N-1} G'_{k,N} G_k^{-1} e_k + G_N^{-1} e_N = M_N^{-1} e_N^M \tag{C.7}$$

(ii) CM_L model (2.17) and (2.19) ($c = N$): we have (C.5)–(C.6), and

$$G_0^{-1} e_0 - G'_{1,0} G_1^{-1} e_1 = M_0^{-1} e_0^M - M'_{1,0} M_1^{-1} e_1^M \tag{C.8}$$

$$-\sum_{k=1}^{N-1} G'_{k,N} G_k^{-1} e_k + G_N^{-1} e_N - G'_{0,N} G_0^{-1} e_0 = M_N^{-1} e_N^M \tag{C.9}$$

C.3 CM_F Model and Reciprocal Model

$$e_0^R = G_0^{-1}e_0 - 2G'_{1,0}G_1^{-1}e_1 - \sum_{k=2}^{N} G'_{k,0}G_k^{-1}e_k \tag{C.10}$$
$$e_1^R = G_1^{-1}e_1 - G'_{2,1}G_2^{-1}e_2 \tag{C.11}$$
$$e_k^R = G_k^{-1}e_k - G'_{k+1,k}G_{k+1}^{-1}e_{k+1}, k \in [2, N-1] \tag{C.12}$$
$$e_N^R = G_N^{-1}e_N \tag{C.13}$$

C.4 CM_L Model and Backward CM_F Model

(i) CM_L model (2.17)–(2.18): we have

$$(G_0^B)^{-1}e_0^B = G_0^{-1}e_0 - G'_{1,0}G_1^{-1}e_1 - G'_{N,0}G_N^{-1}e_N \tag{C.14}$$
$$-(G_{k-1,k}^B)'(G_{k-1}^B)^{-1}e_{k-1}^B + (G_k^B)^{-1}e_k^B = G_k^{-1}e_k - G'_{k+1,k}G_{k+1}^{-1}e_{k+1}, k \in [1, N-2] \tag{C.15}$$
$$-(G_{N-2,N-1}^B)'(G_{N-2}^B)^{-1}e_{N-2}^B + (G_{N-1}^B)^{-1}e_{N-1}^B = G_{N-1}^{-1}e_{N-1} \tag{C.16}$$
$$\sum_{k=0}^{N-2}(G_{k,N}^B)'(G_k^B)^{-1}e_k^B + 2(G_{N-1,N}^B)'(G_{N-1}^B)^{-1}e_{N-1}^B - (G_N^B)^{-1}e_N^B = \sum_{k=1}^{N-1} G'_{k,N}G_k^{-1}e_k + G_N^{-1}e_N \tag{C.17}$$

(ii) CM_L model (2.17) and (2.19): we have (C.15)–(C.16), and

$$(G_0^B)^{-1}e_0^B = G_0^{-1}e_0 - G'_{1,0}G_1^{-1}e_1 \tag{C.18}$$
$$(G_N^B)^{-1}e_N^B - 2(G_{N-1,N}^B)'(G_{N-1}^B)^{-1}e_{N-1}^B - \sum_{k=0}^{N-2}(G_{k,N}^B)'(G_k^B)^{-1}e_k^B = -\sum_{k=1}^{N-1} G'_{k,N}G_k^{-1}e_k + G_N^{-1}e_N - G'_{0,N}G_0^{-1}e_0 \tag{C.19}$$

D Transition Density of a Markov-Induced CM_L Model

We show that the transition density $p_{CM_L}(x_{k+n}|x_k, x_N)$ ($k+n \in [k+1, N-1]$) of a CM_L sequence $[x_k]$ described by a Markov-induced CM_L model (Definition 4.1.4) is the same as the transition density $p_M(y_{k+n}|y_k, y_N)$ of the Markov sequence $[y_k]$ given by (7.49). Note that by Definition 4.1.4 (and (4.10)) we only know that $p_{CM_L}(x_{m+1}|x_m, x_N)$ and $p_M(y_{m+1}|y_m, y_N)$ are the same (for every $m \in [0, N-2]$).

First, note that by recursive use of the Markov-induced CM_L model we obtain

$$x_{k+n} = L_{k,n}x_k + L_{k,n,N}x_N + e_{k+n|k} \tag{D.1}$$

where $L_{k,n}$ and $L_{k,n,N}$ are some matrices, and $e_{k+n|k}$ (with $\mathcal{L}_{k,n} = \text{Cov}(e_{k+n|k})$) is a linear combination of $[e_k]_{k+1}^{k+n}$. Then, by (D.1) we have $p_{CM_L}(x_{k+n}|x_k, x_N) = \mathcal{N}(x_{k+n}; L_{k,n}x_k + L_{k,n,N}x_N, \mathcal{L}_{k,n})$. However, it is not obvious whether this transition is the same as $p_M(y_{k+n}|y_k, y_N)$ given by (7.49). To show that they are actually the same, define functions h and g based on transition densities of the Markov sequence $[y_k]$ as follows: $h(y_{k+n}, y_k, y_N) = p_M(y_{k+n}|y_k, y_N), k+n \in [k+1, N-1]$ and $g(y_j, y_i) = p_M(y_j|y_i), i, j \in [0, N], i < j$. By the definition of a Markov-induced CM_L model (see (4.10)), for the transition density of $[x_k]$, for every $m \in [0, N-2]$, we have

$$p_{CM_L}(x_{m+1}|x_m, x_N) = h(x_{m+1}, x_m, x_N) = \frac{g(x_{m+1}, x_m)g(x_N, x_{m+1})}{g(x_N, x_m)} \tag{D.2}$$

Then, since $[x_k]$ is CM_L, for $k+n \in [k+2, N-1]$, we have

$$p_{CM_L}(x_{k+n}|x_k, x_N) = \int p_{CM_L}(x_{k+n}|x_{k+n-1}, x_N) p_{CM_L}(x_{k+n-1}|x_{k+n-2}, x_N) \cdot$$

$$\cdots p_{CM_L}(x_{k+1}|x_k, x_N) dx_{k+n-1} \cdots dx_{k+1} = \frac{g(x_{k+n}, x_k) g(x_N, x_{k+n})}{g(x_N, x_k)} = h(x_{k+n}, x_k, x_N)$$

which is obtained by substituting all the terms of the integrand based on (D.2) and using (D.3) below.

$$g(x_{k+n}, x_k) = \int g(x_{k+n}, x_{k+n-1}) g(x_{k+n-1}, x_{k+n-2}) \cdots$$

$$\cdot\, g(x_{k+1}, x_k) dx_{k+n-1} dx_{x_{k+n-2}} \cdots dx_{k+1} \tag{D.3}$$

Thus, the two transitions $p_{CM_L}(x_{k+n}|x_k, x_N)$ and $p_M(y_{k+n}|y_k, y_N)$ (given by (7.49)) are the same.

Vita

Reza Rezaie received his Master from Shiraz University and his Bachelor from Kerman University both in Electrical Engineering. His current research interests are stochastic processes and systems, dynamical systems, and machine learning.