2007

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Published in Advances in Mathematics of Communications 1:1 (2007), pp 111-130.
doi: 10.3934/amc.2007.1.111

s-EXTREMAL ADDITIVE $\mathbb{F}_4$ CODES

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(Communicated by Patrick Sole)

ABSTRACT. Binary self-dual codes and additive self-dual codes over $\mathbb{F}_4$ have in common interesting properties, for example, Type I, Type II, shadows, etc. Recently Bachoc and Gaborit introduced the notion of s-extremality for binary self-dual codes, generalizing Elkies' study on the highest possible minimum weight of the shadows of binary self-dual codes. In this paper, we introduce a concept of s-extremality for additive self-dual codes over $\mathbb{F}_4$, give a bound on the length of these codes with even distance d, classify them up to minimum distance $d = 4$, give possible lengths and (shadow) weight enumerators for which there exist s-extremal codes with $5 \leq d \leq 11$ and give five s-extremal codes with $d = 7$. We construct four s-extremal codes of length $n = 13$ and minimum distance $d = 5$. We relate an s-extremal code of length $3d$ to another s-extremal code of that length, and produce extremal Type II codes from s-extremal codes.

1. INTRODUCTION

Conway and Sloane [5] introduced the shadow of a binary self-dual code in order to get additional constraints in the weight enumerator of a singly-even binary self-dual code. Let C be a singly-even (or Type I) binary self-dual code of length n and C_0 its doubly-even subcode. The shadow S of C is defined as

$$S := C_0^\perp \setminus C,$$

equivalently

$$S = \{\mathbf{w} \in \mathbb{F}_2^n \mid \mathbf{v} \cdot \mathbf{w} \equiv \frac{1}{2}\,\mathrm{wt}(\mathbf{v}) \pmod 2 \quad \text{for every } \mathbf{v} \in C\}.$$

Let d be the minimum distance of C and s the minimum weight of S. Bachoc and Gaborit [3] showed that $2d+s \leq \frac{n}{2}+4$, except in the case $n \equiv 22 \pmod{24}$ and

$d = 4[n/24] + 6$, where $2d + s = n/2 + 8$. Binary codes attaining these bounds are called *s-extremal* [3]. In fact, Elkies [7] already classified binary s-extremal codes for $d = 2$ and $d = 4$. Bachoc and Gaborit considered the case when $d = 6$ and showed that there exist binary s-extremal codes of length n with $d = 6$ if and only if $22 \leq n \leq 44$.

In a similar manner to that of Conway and Sloane [5], additional constraints of the weight enumerator of the shadow of an additive self-dual Type I code over $\mathbb{F}_4$ were used by Rains [13] to derive the best known upper bound on the highest possible minimum distance of these codes as follows. Let d_I (d_{II}, respectively) be the minimum weight of an additive self-dual Type I (Type II, respectively) code of length $n > 1$. Then

$$d_I \leq \begin{cases} 2\left\lfloor \frac{n}{6} \right\rfloor + 1 & \text{if } n \equiv 0 \pmod 6 \\ 2\left\lfloor \frac{n}{6} \right\rfloor + 3 & \text{if } n \equiv 5 \pmod 6 \\ 2\left\lfloor \frac{n}{6} \right\rfloor + 2 & \text{otherwise} \end{cases} \tag{1.1}$$

$$d_{II} \leq 2\left\lfloor \frac{n}{6} \right\rfloor + 2. \tag{1.2}$$

A code meeting the appropriate bound is called *extremal.*

After the introduction of s-extremal binary self-dual codes, it is natural to ask whether there exists a concept of s-extremal additive $\mathbb{F}_4$ codes. If so, can we classify them? In this paper, we introduce a concept of s-extremal codes for additive self-dual codes over $\mathbb{F}_4$, give a bound on the possible lengths of such codes related to their distances for even d, classify them up to minimum distance $d = 4$, and give possible lengths (only strongly conjectured for odd d) and (shadow) weight enumerators for which there exist s-extremal codes with $5 \leq d \leq 11$. We construct four s-extremal codes of length $n = 13$ and minimum distance $d = 5$ with the trivial automorphism group, which do not appear in any literature. We relate an s-extremal code of length $3d$ to another s-extremal code of that length, and produce extremal Type II codes from s-extremal codes.

2. s-Extremal Additive $\mathbb{F}_4$ Codes

We recall basic definitions on additive $\mathbb{F}_4$ codes from [4], [9].

Definition 2.1. An *additive* $\mathbb{F}_4$ *code* C *of length* n is a subset $C \subset \mathbb{F}_4^n$ which is a vector space over $\mathbb{F}_2$. We say that C is an $(n, 2^k)$ *code* if it has 2^k codewords. If $c \in C$, the weight of c, denoted by $wt(c)$, is the Hamming weight of c and the minimum distance (or minimum weight) d of C is the smallest weight among any non-zero codeword in C. We call C an $(n, 2^k, d)$ *code.*

Definition 2.2. Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{F}_4^n$. The *trace inner product* of $\mathbf{x}$ and $\mathbf{y}$ is given by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^{n} \mathrm{Tr}(x_i y_i^2)$$

where $\mathrm{Tr} : \mathbb{F}_4 \to \mathbb{F}_2$ is the trace map $\mathrm{Tr}(\alpha) = \alpha + \alpha^2$.

Definition 2.3. If C is an additive code, its *dual*, denoted $C^\perp$, is the additive code $\{\mathbf{x} \in \mathbb{F}_4^n \mid \langle \mathbf{x}, \mathbf{c} \rangle = 0 \text{ for all } \mathbf{c} \in C\}$. If C is an $(n, 2^k)$ code, then $C^\perp$ is an $(n, 2^{2n-k})$ code. C is *self-orthogonal* if $C \subseteq C^\perp$ and *self-dual* if $C = C^\perp$. If C is self-dual, it is an $(n, 2^n)$ code. For an additive self-dual code over $\mathbb{F}_4$, if all codewords have even weight, the code is *Type II*, otherwise it is *Type I*.

Definition 2.4. Let C be an additive $\mathbb{F}_4$ code of length n which is self-dual with respect to the trace inner product. The *shadow* $S = S(C)$ of C is given by

$$S = \{\mathbf{w} \in \mathbb{F}_4^n \mid \langle \mathbf{v}, \mathbf{w} \rangle \equiv \mathrm{wt}(\mathbf{v}) \pmod 2 \text{ for every } \mathbf{v} \in C\},$$

where $\mathrm{wt}(\mathbf{v})$ is the Hamming weight of $\mathbf{v}$. If C is Type II $S(C) = C$, while if C is Type I $S(C)$ is a coset of C.

The next theorem, which is the $\mathbb{F}_4$-analog of [3, Theorem 1], is the first main result of this paper. Its proof is given in Section 3 below.

Theorem 2.5. *Let C be a Type I additive $\mathbb{F}_4$ code of length n, self-dual with respect to the trace inner product, let $d = d_{\min}(C)$ be the minimum distance of C, let $S = S(C)$ be the shadow of C, and let $s = \mathrm{wt}_{\min}(S)$ be the minimum weight of S. Then $2d+s \leq n+2$ unless $n = 6m+5$ and $d = 2m+3$, in which case $2d+s = n+4$.*

Theorem 2.5 motivates the next definition.

Definition 2.6. Let C be a Type I additive $\mathbb{F}_4$ code of length n, self-dual with respect to the trace inner product, let $d = d_{\min}(C)$ be the minimum distance of C, let $S = S(C)$ be the shadow of C, and let $s = \mathrm{wt}_{\min}(S)$ be the minimum weight of S. We say C is *s-extremal* if the bound of Theorem 2.5 is met, i.e., if $2d+s = n+2$ except if $n = 6m+5$ and $d = 2m+3$ in which case $2d+s = n+4$.

Remark 2.7. It will follow from the proof of Theorem 2.5 that the weight enumerator of any s-extremal code is uniquely determined and can be explicitly computed from the values of n and d (or n and s).

3. Proof of Theorem 2.5

We will make integral use of Gleason's Theorem for additive $\mathbb{F}_4$-codes. The statement of this theorem is recalled below.

Theorem 3.1 ([11],[13]). *Let C be an additive $\mathbb{F}_4$ code of length n which is self-dual with respect to the trace inner product. Let $S = S(C)$ be the shadow of C, and let $C(x, y)$ and $S(x, y)$ be the homogeneous weight enumerators of C and S, respectively. Then*

$$S(x, y) = \frac{1}{|C|} C(x + 3y, y - x)$$

and there are polynomials

$$P(X, Y) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} u_i X^{n-2i} Y^i \text{ and } Q(X, Y) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} v_i X^{n-2i} Y^i$$

over $\mathbb{R}$ such that

$$C(x, y) = P(x + y, x^2 + 3y^2) = Q(x + y, y(x - y))$$

and

$$S(x, y) = P(2y, x^2 + 3y^2) = Q(2y, \frac{y^2 - x^2}{2}).$$

Lemma 3.2. *Let C be an additive $\mathbb{F}_4$ code of length n which is self-dual with respect to the trace inner product. Let $S = S(C)$ be the shadow of C. Every vector in S has weight congruent to n modulo 2. Moreover, if we let $s = \mathrm{wt}_{\min}(S)$ be the minimum weight of S and write $s = n - 2r$, then the coefficients u_i and v_i in the polynomials $P(X,Y)$ and $Q(X,Y)$ of Theorem 3.1 are 0 for $r+1 \le i \le \lfloor \frac{n}{2} \rfloor$.*

Proof. We have

$$\begin{aligned} S(x,y) = P(2y, x^2+3y^2) &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} u_i (2y)^{n-2i}(x^2+3y^2)^i \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} u_i 2^{n-2i} y^{n-2i} \sum_{j=0}^{i} \binom{i}{j} x^{2j} 3^{i-j} y^{2i-2j} \\ &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{i} u_i 2^{n-2i} 3^{i-j} \binom{i}{j} x^{2j} y^{n-2j}. \end{aligned}$$

The first statement of the lemma is now clear since the exponent on y corresponds to the weight of the vector.

Now let s be the minimal weight of any vector in S and write $s = n - 2r$. Then $n - 2r$ is the smallest exponent which appears on y with nonzero coefficient and so for $l > r$, we have

$$0 = \text{ coefficient of } x^{2l} y^{n-2l} = \sum_{i=l}^{\lfloor \frac{n}{2} \rfloor} u_i 2^{n-2i} 3^{i-l} \binom{i}{l}.$$

From this we see that u_i appears in the expression for the coefficient of $x^{2l}y^{n-2l}$ for $l \le i \le \lfloor \frac{n}{2} \rfloor$. Thus we recursively obtain the desired $\lfloor \frac{n}{2} \rfloor - r$ equations $u_{\lfloor \frac{n}{2} \rfloor} = 0$, $u_{\lfloor \frac{n}{2} \rfloor - 1} = 0, \ldots, u_{r+1} = u_{\lfloor \frac{n}{2} \rfloor - (\lfloor \frac{n}{2} \rfloor - (r+1))} = 0$.

To see that the v_i's are also 0 for $i \ge r+1$, notice that

$$(x+y)^2 - 2y(x-y) = x^2 + 3y^2$$

and so writing $\phi = x + y$, $\rho = x^2 + 3y^2$ and $\psi = y(x-y)$ we have

$$C(x,y) = Q(\phi, \psi) = P(\phi, \rho) = P(\phi, \phi^2 - 2\psi).$$

Using the second and last terms in the above equation, we get

$$\begin{aligned} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} v_i \phi^{n-2i} \psi^i &= \sum_{j=0}^{r} u_j \phi^{n-2j} (\phi^2 - 2\psi)^j \\ &= \sum_{j=0}^{r} u_j \sum_{k=0}^{j} (-2)^k \binom{j}{k} \phi^{n-2k} \psi^k. \end{aligned}$$

It is clear that no term ψ^k with $k \ge r+1$ can occur in this last sum, and so we must have $v_i = 0$ for $r+1 \le i \le \lfloor \frac{n}{2} \rfloor$ since ϕ and ψ are algebraically independent. $\square$

We are now ready to prove Theorem 2.5.

Proof of Theorem 2.5. If $2d + s < n + 2$, then we don't wish to say anything, so assume $2d + s \geq n + 2$. Set $t = n + 2 - 2d = n - 2(d - 1)$, which is nonnegative by the Singleton bound for nonlinear codes [12, p. 71]. So $0 \leq t \leq s$ and we have

$$C(x, y) = 1 + A_d x^{n-d} y^d + \dots$$
$$S(x, y) = B_t x^{n-t} y^t + B_{t+2} x^{n-t-2} y^{t+2} + \dots$$

where we are using the fact from Lemma 3.2 that all weights in S are congruent to n modulo 2 and we are not assuming that B_t is nonzero.

Note that if $B_t \neq 0$ then $t = s$, i.e., $2d + s = n + 2$. If $B_t = 0$ then $t < s$, i.e., $2d + s > n + 2$. We wish to show that $B_t = 0$ only in the case $n = 6m + 5$ and $d = 2m + 3$, and that in that case, $B_{t+2} \neq 0$, i.e., $s = t + 2$, i.e., $2d + s = n + 4$.

From the assumption $2d+s \geq n+2$ and substituting $s = n-2r$, we get $r \leq d-1$. If $r = 0$, then $s = n$. In this case, we have $d = 1$ by Lemma 3.2, and so $2d+s = n+2$ as desired. Hence we may assume $r > 0$.

We have that B_t is the coefficient of $x^{n-t}y^t$ in $S(x, y)$ and as $v_i = 0$ for $i \geq d$ by Lemma 3.2, we may write

$$\begin{aligned} S(x, y) &= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} v_i (2y)^{n-2i} \left(\frac{y^2 - x^2}{2} \right)^i \\ &= \sum_{i=0}^{d-1} v_i 2^{n-2i} y^{n-2i} 2^{-i} \sum_{j=0}^{i} (-1)^j \binom{i}{j} y^{2i-2j} x^{2j} \\ &= \sum_{i=0}^{d-1} \sum_{j=0}^{i} (-1)^j 2^{n-3i} \binom{i}{j} v_i x^{2j} y^{n-2j}. \end{aligned}$$

We see that B_t comes from the summand where $i = j = d - 1$, i.e.,

$$B_t = (-1)^{d-1} 2^{n-3(d-1)} \binom{d-1}{d-1} v_{d-1} = (-1)^{d-1} 2^{n-3d+3} v_{d-1}.$$

Hence $B_t = 0$ if and only if $v_{d-1} = 0$.

Next, start with the equation

$$1 + A_d y^d + \dots = C(1, y) = \sum_{i=0}^{d-1} v_i (1 + y)^{n-2i} (y(1 - y))^i.$$

Dividing both sides by $(1 + y)^n$, we get

$$\frac{1}{(1 + y)^n} (1 + A_d y^d + \dots) = \sum_{i=0}^{d-1} v_i \left(\frac{y(1 - y)}{(1 + y)^2} \right)^i.$$

Write $f(y) = \frac{1}{(1+y)^n}$ and $g(y) = \frac{y(1-y)}{(1+y)^2}$. Then we have

$$\begin{aligned} f(y) &= \sum_{i=0}^{d-1} v_i g(y)^i - f(y)(A_d y^d + \dots) \\ &= \sum_{i=0}^{d-1} v_i g(y)^i - A_d y^d + O(y^{d+1}). \end{aligned}$$

Since

$$\begin{aligned} g(y) &= \frac{y(1-y)}{(1+y)^2} \\ &= (y-y^2)\sum_{i\geq 0}(-1)^{i+1} i y^{i-1} \\ &= \sum_{i\geq 1}(-1)^{i+1}(2i-1)y^i, \end{aligned}$$

$g(y)^j = y^j + O(y^{j+1})$ for any $j \geq 1$. We see that the first few terms in the expansion of $f(y)$ as a power series in $g(y)$ are $v_0, \ldots, v_{d-1}, -A_d$. So by using the Bürman-Lagrange Theorem below, we see that the weight enumerator of an s-extremal code is uniquely determined.

We recall the Bürman-Lagrange Theorem (as stated in [13]): If $f(x)$ and $g(x)$ are formal power series with $g(0) = 0$ and $g'(0) \neq 0$ and the coefficients κ_i are defined by

$$f(x) = \sum_{i\geq 0} \kappa_i g(x)^i,$$

then

$$\kappa_i = \frac{1}{i}\left(\text{coefficient of } x^{i-1} \text{ in } f'(x)\left(\frac{x}{g(x)}\right)^i\right).$$

Our functions $f(y)$ and $g(y)$ satisfy these hypotheses, and we have

$$\begin{aligned} v_{d-1} &= \kappa_{d-1} \\ &= \frac{1}{d-1}\left(\text{coefficient of } y^{d-2} \text{ in } f'(y)\left(\frac{y}{g(y)}\right)^{d-1}\right) \\ &= \frac{1}{d-1}\left(\text{coefficient of } y^{d-2} \text{ in } \left(\frac{-n}{(1+y)^{n+1}}\right)\left(\frac{(1+y)^2}{1-y}\right)^{d-1}\right) \\ &= \frac{-n}{d-1}\left(\text{coefficient of } y^{d-2} \text{ in } \frac{1}{(1+y)^{n-2d+3}(1-y)^{d-1}}\right). \end{aligned}$$

We are now ready show that $B_t = 0$, i.e., $v_{d-1} = 0$, if and only if $n = 6m+5$ and $d = 2m+3$. We may rewrite v_{d-1} as

$$v_{d-1} = \frac{-n}{d-1}\left(\text{coefficient of } y^{d-2} \text{ in } \frac{1}{(1+y)^{n-3d+4}(1-y^2)^{d-1}}\right)$$

and we consider three cases.

If $n - 3d + 4 < 0$, then we are looking at the coefficient of y^{d-2} in the product of the polynomial $(1+y)^{3d-4-n}$ and the power series $(1+y^2+y^4+\ldots)^{d-1}$. That coefficient will certainly be nonzero (in fact, positive), which means $v_{d-1} \neq 0$ and $B_t \neq 0$.

If $n - 3d + 4 > 0$, then we have

$$
\begin{aligned}
v_{d-1} &= \frac{-n}{d-1} \sum_{\substack{j,\,k \geq 0 \\ j+2k=d-2}} (-1)^j \binom{n-3d+3+j}{j}\binom{d-2+k}{k} \\
&= \frac{n}{d-1}(-1)^{d+1} \sum_{\substack{j,\,k \geq 0 \\ j+2k=d-2}} \binom{n-3d+3+j}{j}\binom{d-2+k}{k} \\
&\neq 0
\end{aligned}
$$

and so again $B_t \neq 0$.

Finally, consider the case where $n-3d+4=0$. Write $n = 6m+l$ where $0 \leq l \leq 5$. Then $0 = n - 3d + 4 = 6m + l - 3d + 4$, and so $3d = 6m + l + 4$, i.e.,

$$d = 2m + \frac{l+4}{3}.$$

Since d must be an integer, this means $l = 2$ or $l = 5$. If $l = 2$, we have $n = 6m+2$, $d = 2m+2$, and

$$v_{d-1} = v_{2m+1} = \frac{-(6m+2)}{2m+1}\left(\text{coefficient of } y^{2m} \text{ in } \frac{1}{(1-y^2)^{2m+1}}\right) \neq 0$$

and so $B_t \neq 0$. If $l = 5$, we have $n = 6m+5$, $d = 2m+3$ and then

$$v_{d-1} = v_{2m+2} = \frac{-(6m+5)}{2m+2}\left(\text{coefficient of } y^{2m+1} \text{ in } \frac{1}{(1-y^2)^{2m+2}}\right) = 0$$

and hence $B_t = 0$ in this case. Thus we have completed the proof of the fact that $B_t = 0$ if and only if $n = 6m+5$ and $d = 2m+3$.

We now need to show that if $n = 6m+5$ and $d = 2m+3$, then $B_{t+2} \neq 0$. We have

$$B_{t+2}x^{n-t-2}y^{t+2} + \cdots = S(x,y) = \sum_{i=0}^{d-2}\sum_{j=0}^{i}(-1)^j 2^{n-3i}\binom{i}{j} v_i x^{2j} y^{n-2j}.$$

Thus B_{t+2} is obtained by taking j to satisfy $n-2j = t+2$ in the summand on the right. As $n = 6m+5$ and $t = n+2-2d = 6m+5+2-4m-6 = 2m+1$, we need $j = 2m+1 = d-2$. Thus we must have $i = d-2$ as well and we get

$$B_{t+2} = (-1)^{d-2}2^{n-3d+6}\binom{d-2}{d-2}v_{d-2} = -4v_{d-2}.$$

Using Bürman-Lagrange again, we get

$$\begin{aligned}
v_{d-2} &= \kappa_{d-2} \\
&= \frac{1}{d-2}\left(\text{coefficient of } y^{d-3} \text{ in } \left(\frac{-n}{(1+y)^{n+1}}\right)\left(\frac{(1+y)^2}{1-y}\right)^{d-2}\right) \\
&= \frac{-n}{d-2}\left(\text{coefficient of } y^{d-3} \text{ in } \frac{1}{(1+y)^{n-2d+5}(1-y)^{d-2}}\right) \\
&= \frac{-n}{d-2}\left(\text{coefficient of } y^{d-3} \text{ in } \frac{1}{(1+y)^{n-3d+7}(1-y^2)^{d-2}}\right) \\
&= \frac{-n}{d-2} \sum_{\substack{j,k\geq 0 \\ j+2k=d-3}} (-1)^j \binom{n-3d+j+6}{j}\binom{d+k-3}{k} \\
&= \frac{n}{d-2}(-1)^d \sum_{\substack{j,k\geq 0 \\ j+2k=d-3}} \binom{n-3d+j+6}{j}\binom{d+k-3}{k}
\end{aligned}$$

which is certainly nonzero. Hence B_{t+2} is nonzero and we have $s = t+2 = n-2d+4$ as desired. This completes the proof. □

4. A bound for the length of an s-extremal code

In this section, we give an upper bound for the length of an s-extremal code with even minimum distance d. This bound generalizes additive $\mathbb{F}_4$ codes the case of binary s-extremal codes of [8].

Theorem 4.1. *An s-extremal code with length n and even minimum distance d must satisfy $n < 3d$.*

Proof. Gleason's theorem gives us

$$C(1,y) = 1 + A_d y^d + \cdots = \sum_{0\leq i\leq \lfloor \frac{n}{2}\rfloor} v_i (1+y)^{n-2i}(y(1-y))^i.$$

If C is s-extremal and has even minimum distance, then it follows from Definition 2.6 that $s = n-2(d-1)$. We have shown in the proof of Lemma 3.2 that $v_i = 0$ if $i \geq d$ and we may once again divide both sides of the above equation by $(1+y)^n$ to get

$$\sum_{i=0}^{d-1} v_i \left(\frac{y(1-y)}{(1+y)^2}\right)^i = \frac{1}{(1+y)^n} + \frac{1}{(1+y)^n}\left(A_d y^d + A_{d+1}y^{d+1} + \cdots\right).$$

Let $g(y) = \dfrac{y(1-y)}{(1+y)^2}$ and from the above expression, we see that $v_0, \ldots, v_{d-1}, -A_d$ are the first coefficients of the development of $\dfrac{1}{(1+y)^n}$ as a series in $g(y)$. Applying the Bürman-Lagrange Theorem, we then have

$$\begin{aligned}
A_d &= \frac{n}{d}\left(\text{coefficient of } y^{d-1} \text{ in } \frac{1}{(1+y)^{n-2d+1}(1-y)^d}\right) \\
&= \frac{n}{d}\left(\text{coefficient of } y^{d-1} \text{ in } \frac{1}{(1+y)^{n-3d+1}(1-y^2)^d}\right)
\end{aligned}$$

If $n = 3d + \alpha$ where $\alpha \geq 0$, then

$$\begin{aligned} A_d &= \frac{n}{d}\left(\text{coefficient of } y^{d-1} \text{ in } \frac{1}{(1+y)^{1+\alpha}(1-y^2)^d}\right) \\ &= \frac{n}{d} \sum_{\substack{j,k\geq 0 \\ j+2k=d-1}} (-1)^j \binom{\alpha+j}{j}\binom{d+k-1}{k} \end{aligned}$$

Now, if d is even, then $d-1 = j+2k$ must be odd which shows that j must be odd. This implies that all the terms in the summation must be negative, showing that $A_d < 0$ and leading to a contradiction. Therefore $n < 3d$. □

Remark 4.2. For the odd distance case it also seems that there is always a negative coefficient in the possible s-extremal weight enumerators at some point, but we could not find any clear pattern to use to prove this as for the even case.

Next, we consider a lower bound for the length of an s-extremal code of length n and minimum distance d. First notice that it follows from Eq. (1.1) that any Type I additive self-dual code of length n and minimum distance d satisfies $n \geq 3d - 5$. In the case that d is even and C is s-extremal, we have:

Lemma 4.3. *If d is even, then any s-extremal code with length n and minimum distance d satisfies $n \geq 3d - 4$.*

Proof. We see from the proof of Theorem 2.5 that if $n - 3d + 4 < 0$, then v_{d-1} is positive. This implies $B_t = (-1)^{d-1}2^{n-3d+3}v_{d-1}$ is negative if d is even. This is impossible. Hence $n - 3d + 4 \geq 0$. □

Putting this all together, we have:

Corollary 4.4. *If d is even, then any s-extremal code of length n with minimum distance d satisfies $d - 2 \leq s < d + 2$ and $3d - 4 \leq n \leq 3d - 1$.*

5. A construction of s-extremal additive $\mathbb{F}_4$ codes

We recall the shortening of additive $\mathbb{F}_4$ codes from [9]. Let C be a self-dual additive $(n, 2^n, d)$ code with its generator matrix G. Choose any column of G, say the i^{th} one. The entries in column i can be any of 0, 1, ω, or $\overline{\omega}$. By row reducing G (to obtain another generator matrix we call G again), we can make all the entries in column i equal to 0 except for one or two entries; if two they would be two of the three values 1, ω, or $\overline{\omega}$. The *shortened code of C' on coordinate i*, denoted C' is the code with generator matrix G' obtained from G by eliminating one row of G with a nonzero entry in column i and then eliminating column i. [If there is only one nonzero entry in column i of G, then C' is C shortened in the usual sense.] Clearly C' is a self-dual additive $(n-1, 2^{n-1}, d')$ code with $d' \geq d - 1$.

Proposition 5.1. *Suppose C is an extremal Type II additive $\mathbb{F}_4$ code of length n. If $n \equiv 0$ or $2 \pmod 6$, then any shortening of C is s-extremal.*

Proof. Suppose first that $n \equiv 0 \pmod 6$. Writing $n = 6k$, we have that C has $M := 2^{3k}$ codewords and minimum distance $d = 2k+2$. We may choose a generator matrix G of C which has the form:

$$G = \left(\begin{array}{c|c} 0 & \\ \vdots & G_0 \\ 0 & \\ \hline \omega & \mathbf{y} \\ \hline 1 & \mathbf{x} \end{array}\right)$$

where G_0 is a matrix of size $(3k-2) \times (n-1)$ over $\mathbb{F}_4$ and $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{F}_4^{n-1}$. Without loss of generality, we shorten with respect to the first column and last row of this generator matrix, and so our code C' has generator matrix

$$G' = \begin{pmatrix} G_0 \\ \mathbf{y} \end{pmatrix}.$$

Letting C_0' be the code generated by G_0, $C_2' = \mathbf{y} + C_0'$, $C_1' = \mathbf{x} + C_0'$ and $C_3' = (\mathbf{x}+\mathbf{y}) + C_0'$, we see that

$$C = \{0|C_0'\} \cup \{\omega|C_2'\} \cup \{1|C_1'\} \cup \{\bar{\omega}|C_3'\}.$$

Further, the shortened code C' is simply $C_0' \cup C_2'$ and its shadow S' is $C_1' \cup C_3'$. Letting n', d' and s' denote the length, minimum distance and minimum shadow weight of C', we have $n' = 6k-1$, $d' \geq 2k+1$ and $s' \geq 2k+1$. But then

$$2d' + s' \geq 2(2k+1) + (2k+1) = 6k+3 = (6k-1)+4.$$

Since $6k-1 \equiv 5 \pmod 6$, we see that C' is s-extremal.

The proof in the case $n \equiv 2 \pmod 6$ is exactly the same, except that we have $n' = 6k+1$ and conclude

$$2d' + s' \geq 2(2k+1) + (2k+1) = 6k+3 = (6k+1)+2,$$

which gives the result. □

Remark 5.2. The analogous result for the case $n = 6k+4$ does not hold. For example, consider the extremal Type II code QC_{10a} [9] of length 10 and minimum distance 4. If we shorten this code with respect to the first column and the fifth row, we get a Type I code of length 9 and minimum distance 3 with weight enumerator $C(x,y) = x^9 + 3x^6y^3 + 18x^5y^4 + 63x^4y^5 + 120x^3y^6 + 153x^2y^7 + 117xy^8 + 37y^9$. This cannot be the weight enumerator of an s-extremal code with $d = 3$ and $n = 9$ by the below classification.

6. Classification of s-extremal codes

In this section, we classify all s-extremal codes of minimum distance at most 4 and give partial results for s-extremal codes of higher minimum distances.

Suppose C is an s-extremal code of minimum distance d. Then we have the d equations $A_0 = 1$, $A_i = 0$ for $1 \leq i \leq d-1$ in the unknown coefficients u_i in the Gleason polynomial $P(X,Y)$. If d is not of the form $d = 2m+3$ for some nonnegative m, i.e., if $d = 1$ or d is even, then the length n of the code must not be congruent to 5 modulo 6, and d and the minimum shadow weight s must satisfy $2d + s = n + 2$. Thus we have $s = n - 2(d-1)$ and so, by Lemma 3.2, only the d coefficients $u_0, \ldots, u_{d-1}$ can be nonzero. Thus there is a unique solution to the system of linear equations, and hence there is a unique possible weight enumerator for an s-extremal code of length n and minimum distance d.

If, on the other hand, d is odd and at least 3, then there are two cases to consider. If n is not congruent to 5 modulo 6, then the argument above shows that there is a unique possible weight enumerator. If n is congruent to 5 modulo 6, then in order for C to be s-extremal, we must have $s = n - 2(d-2)$. By Lemma 3.2, only the $d-1$ coefficients $u_0, \ldots, u_{d-2}$ can be nonzero. Hence we have d equations in $d-1$ unknowns and there need not be a solution.

For each value of d, we first compute the possible values of n such that there is an s-extremal code of length n and minimum distance d. We then explicitly compute the putative weight enumerator for each possible pair (d, n). Since the shadow enumerator, and hence the minimum weight of the shadow, is determined by the weight enumerator, any code with this weight enumerator is necessarily s-extremal. So we only need to find the codes with these weight enumerators. Some of the simpler cases were treated by Höhn [11]; many of the others were treated by Gaborit, et al., in [9] or by Danielsen and Parker in [6].

We now begin our classification.

$d = 1$: In this case we have $s = n$. Thus $P(X,Y) = u_0 X^n$ and so $C(x,y) = u_0(x+y)^n$. Since $A_0 = 1$, we have $u_0 = 1$, i.e., $C(x,y) = (x+y)^n$. If $C \not\sim \mathbb{F}_2^n$, then C has at least two (and hence three) distinct words of weight 1 supported on the i^{th} coordinate for some i. Let $\mathbf{w}$ be the unique word of weight n in C. Then, for some i, there is a word $\mathbf{c} \in C$ of weight 1 supported on the i^{th} coordinate with $c_i \neq w_i$. But then $\mathbf{w} + \mathbf{c} \neq \mathbf{w}$ is a word of weight n in C, a contradiction. Hence $C \sim \mathbb{F}_2^n$. (This is also shown in [11].)

$d = 2$: In this case we have $0 < s = n - 2$. Höhn [11] proves that there are $A_2 = \frac{n}{2}(5-n)$ words of weight 2, and so we must have $n \leq 4$. The weight enumerators of C for the possible values of n are as follows:

n	$C(x,y)$
3	$x^3 + 3xy^2 + 4y^3$
4	$x^4 + 2x^2y^2 + 8xy^3 + 5y^4$

Note that the shadow enumerators can be computed from the weight enumerators using Theorem 3.1.

There is a unique code up to equivalence in each case. For $n = 3$, it is generated (over $\mathbb{F}_2$) by the vectors $(1,1,0)$, $(1,0,1)$ and (ω,ω,ω). For $n = 4$, it is generated by $(1,1,0,0)$, $(0,0,1,1)$, $(1,0,\omega,\omega)$, $(\omega,\omega,1,0)$.

$d = 3$: Since $3 = 2(0) + 3$, a code of minimal distance 3 can be s-extremal if either it has length $n = 5$ and minimum shadow weight $s = 3$, or if has length $n \not\equiv 5 \pmod 6$ and minimum shadow weight $s = n - 4$. In the former case, the weight enumerator of the code is $C(x,y) = x^5 + 10x^2y^3 + 15xy^4 + 6y^5$ and there is a unique code, called the *shorter hexacode* by Höhn [11]. In the latter case, one finds $6 \leq n \leq 10$ as follows: Since $s = n - 4$, we have $n > 4$. The coefficient of x^2y^{n-2} in the shadow enumerator is $2^{n-6}n(13-n)$, which is negative for $n \geq 14$. Finally, one can check that the coefficient of $x^{n-6}y^n$ in the weight enumerator of the code is negative if $n = 12$ or 13.

$n = 6$: The weight enumerator of the code is $C(x,y) = x^6 + 8x^3y^3 + 21x^2y^4 + 24xy^5 + 10y^6$. There is a unique code with this weight enumerator, called the *odd hexacode* by Höhn [11].

$n = 7$: We have $C(x,y) = x^7 + 7x^4y^3 + 21x^3y^4 + 42x^2y^5 + 42xy^6 + 15y^7$. By [6], there are three codes with this weight enumerator out of four codes with $n = 7$ and $d = 3$. Incidentally, we may obtain them by shortening the three extremal even self-dual codes of length $n = 8$ and minimum distance $d = 4$ [9] and applying Proposition 5.1.

$n = 8$: We have $C(x, y) = x^8 + 8x^5y^3 + 18x^4y^4 + 48x^3y^5 + 88x^2y^6 + 72xy^7 + 21y^8$. Again appealing to the classification done by [6], we see that there are exactly three s-extremal codes of length 8 and minimum distance 3.

$n = 9$: We have $C(x, y) = x^9 + 12x^6y^3 + 18x^5y^4 + 36x^4y^5 + 120x^3y^6 + 180x^2y^7 + 117xy^8 + 28y^9$. There is only one s-extremal code of this length and minimum distance by [6].

$n = 10$: We have $C(x, y) = x^{10} + 20x^7y^3 + 30x^6y^4 + 12x^5y^5 + 100x^4y^6 + 300x^3y^7 + 345x^2y^8 + 180xy^9 + 36y^{10} = (x^5 + 10x^2y^3 + 15xy^4 + 6y^5)^2$. There exists only one s-extremal code of this length by [6]; it is interesting to note that this code is decomposable. This must be a direct sum of two shorter hexacodes because $n = 5$ is the shortest length for additive self-dual codes of length n with $d = 3$, i.e., the shorter hexacode.

$d = 4$: Using a similar argument to that used above to find the possible lengths of s-extremal codes of minimum distance 3, we see that any s-extremal code of minimum distance 4 must have length n with $8 \leq n \leq 10$.

$n = 8$: We have $C(x, y) = x^8 + 26x^4y^4 + 64x^3y^5 + 72x^2y^6 + 64xy^7 + 29y^8$. There are exactly two codes with this weight enumerator. They are denoted by f_{100} and f_{101} using the notation of [6], or by QC_{8a} and QC_{8b} in the notation of [9].

$n = 9$: We get $C(x, y) = x^9 + 18x^5y^4 + 72x^4y^5 + 120x^3y^6 + 144x^2y^7 + 117xy^8 + 40y^9$. Of the eight Type I codes of length 9 and minimum distance 4 found in [9], precisely five are s-extremal: QC_{9a}, QC_{9b}, QC_{9d}, QC_{9f}, and QC_{9h}; this information can be verified via [6] also.

$n = 10$: We get $C(x, y) = x^{10} + 10x^6y^4 + 72x^5y^5 + 160x^4y^6 + 240x^3y^7 + 285x^2y^8 + 200xy^9 + 56y^{10}$. Of the 120 Type I codes of length 10 and minimum distance 4, exactly 15 are s-extremal [6].

$d = 5$: As $5 = 2(1) + 3$, it is possible to have an s-extremal code of minimum distance 5 and length $n \equiv 5 \pmod 6$. In that case, the length must be $6(1)+5 = 11$ and the weight enumerator is $C(x, y) = x^{11} + 66x^6y^5 + 198x^5y^6 + 330x^4y^7 + 495x^3y^8 + 550x^2y^9 + 330xy^{10} + 78y^{11}$. By [6] or [9], we see that there is a unique code with this weight enumerator and hence a unique s-extremal code of minimum distance 5 and length 11.

In the case $n \not\equiv 5 \pmod 6$, we find that any s-extremal code of minimum distance 5 must have length n with $12 \leq n \leq 15$.

$n = 12$: We get $C(x, y) = x^{12} + 48x^7y^5 + 188x^6y^6 + 432x^5y^7 + 765x^4y^8 + 1040x^3y^9 + 972x^2y^{10} + 528xy^{11} + 122y^{12}$. Of the 63 Type I codes of length 12 and minimum distance 5, exactly 59 are s-extremal [6].

$n = 13$: We get $C(x, y) = x^{13} + 39x^8y^5 + 156x^7y^6 + 468x^6y^7 + 1053x^5y^8 + 1690x^4y^9 + 2028x^3y^{10} + 1716x^2y^{11} + 858xy^{12} + 183y^{13}$. Five codes with this weight enumerator can be found in [10]. They all have nontrivial automorphism groups. We further construct the first examples of s-extremal codes of length $n = 13$ and $d = 5$ with the *trivial* automorphism group. The generator matrices of these codes are given by $A_{13} + \omega I_{13}$, where A_{13} is a symmetric $(0,1)$-matrix of size 13×13 with diagonal zero and I_{13} is the identity matrix of that size. It is easy to check that such a matrix gives an additive self-dual code over $\mathbb{F}_4$. In fact, the converse holds too. (See [6] for a proof.) In order to save space we only give a lower triangular part of A_{13} row by row. For example, the vector $(1; 00; 110)$ refers to the following matrix:

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

The four vectors producing s-extremal codes of length $n = 13$ and $d = 5$ are the following.

$$\begin{aligned}
&(1; 10; 001; 0000; 00101; 111110; 0001100; 01011010;\\
&\ 010001011; 1110000101; 10111111110; 011100000110),\\
&(0; 10; 110; 1000; 01011; 010000; 1000001; 10100111;\\
&\ 101000010; 1100110101; 11010000111; 111111101110),\\
&(1; 11; 100; 0000; 10100; 101101; 1100101; 11001000;\\
&\ 110101010; 1111110111; 01001110011; 000011010010),\\
&(0; 11; 111; 1001; 01111; 110000; 0110001; 11101111;\\
&\ 011100010; 1000110001; 01001010001; 100110001000)
\end{aligned}$$

$n = 14$: We get $C(x,y) = x^{14} + 42x^9y^5 + 119x^8y^6 + 408x^7y^7 + 1281x^6y^8 + 2492x^5y^9 + 3486x^4y^{10} + 3864x^3y^{11} + 3038x^2y^{12} + 1386xy^{13} + 267y^{14}$. There is at least one s-extremal code [10]; it is one of the five Type I 4-circulant codes.

$n = 15$: It is unknown whether there exists an s-extremal code of $n = 15$ and $d = 5$, but the putative weight enumerator of such a code would be $C(x,y) = x^{15} + 63\,x^{10}\,y^5 + 105\,x^9y^6 + 225\,x^8\,y^7 + 1305\,x^7\,y^8 + 3430\,x^6\,y^9 + 5418\,x^5\,y^{10} + 6930\,x^4\,y^{11} + 7350\,x^3\,y^{12} + 5355\,x^2\,y^{13} + 2205\,x\,y^{14} + 381\,y^{15}$.

We summarize the above results in Table 1.

TABLE 1. Summary of s-extremal codes for $1 \le d \le 5$

d	n	# of Codes	Weight Enumerators $C(x,y)$; $S(x,y)$
1	≥ 2	1(i.e., $\mathbb{F}_2^n$)	$(x+y)^n$; $2^n y^n$
2	3	1	$x^3 + 3xy^2 + 4y^3$; $3x^2y + 5y^3$
	4	1	$x^4 + 2x^2y^2 + 8xy^3 + 5y^4$; $8x^2y^2 + 8y^4$
	5	1	$x^5 + 10x^2y^3 + 15xy^4 + 6y^5$; $20x^2y^3 + 12y^5$
	6	1	$x^6 + 8x^3y^3 + 21x^2y^4 + 24xy^5 + 10y^6$; $3x^4y^2 + 42x^2y^4 + 19y^6$
3	7	3	$x^7 + 7x^4y^3 + 21x^3y^4 + 42x^2y^5 + 42xy^6 + 15y^7$; $14x^4y^3 + 84x^2y^5 + 30y^7$
	8	3	$x^8 + 8x^5y^3 + 18x^4y^4 + 48x^3y^5 + 88x^2y^6 + 72xy^7 + 21y^8$; $48x^4y^4 + 160x^2y^6 + 48y^8$
	9	1	$x^9 + 12x^6y^3 + 18x^5y^4 + 36x^4y^5 + 120x^3y^6 + 180x^2y^7 + \cdots$; $144x^4y^5 + 288x^2y^7 + 80y^9$
	10	1	$(x^5 + 10x^2y^3 + 15xy^4 + 6y^5)^2$; $400x^4y^6 + 480x^2y^8 + 144y^{10}$
	8	2	$x^8 + 26x^4y^4 + 64x^3y^5 + 72x^2y^6 + 64xy^7 + 29y^8$; $4x^6y^2 + 36x^4y^4 + 172x^2y^6 + 44y^8$
4	9	5	$x^9 + 18x^5y^4 + 72x^4y^5 + 120x^3y^6 + 144x^2y^7 + 117xy^8 + 40y^9$; $12x^6y^3 + 108x^4y^5 + 324x^2y^7 + 68y^9$
	10	15	$x^{10} + 10x^6y^4 + 72x^5y^5 + 160x^4y^6 + 240x^3y^7 + \cdots$; $40x^6y^4 + 280x^4y^6 + 600x^2y^8 + 104y^{10}$
	11	1	$x^{11} + 66x^6y^5 + 198x^5y^6 + 330x^4y^7 + 495x^3y^8 + 550x^2y^9 + \cdots$; $132x^6y^5 + 660x^4y^7 + 1100x^2y^9 + 156y^{11}$
	12	59	$x^{12} + 48x^7y^5 + 188x^6y^6 + 432x^5y^7 + 765x^4y^8 + \cdots$; $15x^8y^4 + 356x^6y^6 + 1530x^4y^8 + 1956x^2y^{10} + 239y^{12}$
5	13	at least 9	$x^{13} + 39x^8y^5 + 156x^7y^6 + 468x^6y^7 + 1053x^5y^8 + 1690x^4y^9 + \cdots$; $78x^8y^5 + 936x^6y^7 + 3380x^4y^9 + 3432x^2y^{11} + 366y^{13}$
	14	at least 1	$x^{14} + 42x^9y^5 + 119x^8y^6 + 408x^7y^7 + 1281x^6y^8 + 2492x^5y^9 + \cdots$; $308x^8y^6 + 2352x^6y^8 + 7224x^4y^{10} + 5936x^2y^{12} + 564y^{14}$
	15	unknown	$x^{15} + 63x^{10}y^5 + 105x^9y^6 + 225x^8y^7 + 1305x^7y^8 + 3430x^6y^9 + \cdots$; $1080x^8y^7 + 5600x^6y^9 + 15120x^4y^{11} + 10080x^2y^{13} + 888y^{15}$

$d = 6$: There is no known s-extremal code with $d = 6$, but the putative weight and shadow enumerators are as follows:

$n = 14$:

$C(x, y) = x^{14} + 161x^8y^6 + 576x^7y^7 + 1113x^6y^8 + 2240x^5y^9 + 3738x^4y^{10} + 4032x^3y^{11} + 2870x^2y^{12} + 1344xy^{13} + 309y^{14}$;
$S(x, y) = 21x^{10}y^4 + 203x^8y^6 + 2562x^6y^8 + 7014x^4y^{10} + 6041x^2y^{12} + 543y^{14}$.

$n = 15$:

$C(x, y) = x^{15} + 105x^9y^6 + 540x^8y^7 + 1305x^7y^8 + 2800x^6y^9 + 5418x^5y^{10} + 7560x^4y^{11} + 7350x^3y^{12} + 5040x^2y^{13} + 2205xy^{14} + 444y^{15}$, and $S(x, y) = 63x^{10}y^5 + 765x^8y^7 + 6230x^6y^9 + 14490x^4y^{11} + 10395x^2y^{13} + 825y^{15}$.

$n = 16$:

$C(x, y) = x^{16} + 56x^{10}y^6 + 480x^9y^7 + 1410x^8y^8 + 3200x^7y^9 + 7056x^6y^{10} + 12096x^5y^{11} + 14840x^4y^{12} + 13440x^3y^{13} + 8760x^2y^{14} + 3552xy^{15} + 645y^{16}$, and $S(x, y) =$
$224x^{10}y^6 + 2400x^8y^8 + 14784x^6y^{10} + 29120x^4y^{12} + 17760x^2y^{14} + 1248y^{16}$.

$d = 7$: Gulliver and Kim [10] give one circulant s-extremal code of length 17, $C_{17,1}$ with weight enumerator
$C(x, y) = x^{17} + 408x^{10}y^7 + 1530x^9y^8 + 3400x^8y^9 + 8160x^7y^{10} + 17136x^6y^{11} + 25704x^5y^{12} + 28560x^4y^{13} + 24480x^3y^{14} + 15096x^2y^{15} + 5661xy^{16} + 936y^{17}$
and $S(x, y) =$
$816x^{10}y^7 + 6800x^8y^9 + 34272x^6y^{11} + 57120x^4y^{13} + 30192x^2y^{15} + 1872y^{17}$.
They also give four circulant s-extremal codes of length 19, $C_{19,1}$, $C_{19,2}$, $C_{19,3}$, $C_{19,4}$, each of which has weight enumerator
$C(x, y) = x^{19} + 228x^{12}y^7 + 1026x^{11}y^8 + 3496x^{10}y^9 + 10488x^9y^{10} + 25308x^8y^{11} + 50616x^7y^{12} + 82992x^6y^{13} + 106704x^5y^{14} + 105564x^4y^{15} + 79173x^3y^{16} + 42408x^2y^{17} + 14136xy^{18} + 2148y^{19}$ and
$S(x, y) = 456x^{12}y^7 + 6992x^{10}y^9 + 50616x^8y^{11} + 165984x^6y^{13} + 211128x^4y^{15} + 84816x^2y^{17} + 4296y^{19}$.

The other possible lengths are $n = 18$, 20, or 21. There are no known examples for these lengths, but the putative weight and shadow enumerators are as follows:

$n = 18$:

$C(x, y) = x^{18} + 288x^{11}y^7 + 1314x^{10}y^8 + 3680x^9y^9 + 9432x^8y^{10} + 21312x^7y^{11} + 38136x^6y^{12} + 52416x^5y^{13} + 55440x^4y^{14} + 44448x^3y^{15} + 25317x^2y^{16} + 8928xy^{17} + 1432y^{18}$, and $S(x, y) = 84x^{12}y^6 + 2376x^{10}y^8 + 19116x^8y^{10} + 76272x^6y^{12} + 110700x^4y^{14} + 50760x^2y^{16} + 2836y^{18}$.

$n = 20$:

$C(x, y) = x^{20} + 240x^{13}y^7 + 750x^{12}y^8 + 2720x^{11}y^9 + 10992x^{10}y^{10} + 29520x^9y^{11} + 62220x^8y^{12} + 116160x^7y^{13} + 179040x^6y^{14} + 213648x^5y^{15} + 197145x^4y^{16} + 139680x^3y^{17} + 70960x^2y^{18} + 22320xy^{19} + 3180y^{20}$, and
$S(x, y) = 1920x^{12}y^8 + 19968x^{10}y^{10} + 128640x^8y^{12} + 353280x^6y^{14} + 397440x^4y^{16} + 140800x^2y^{18} + 6528y^{20}$.

$n = 21$:

$C(x,y) =$
$x^{21} + 360x^{14}y^7 + 630x^{13}y^8 + 1120x^{12}y^9 + 10080x^{11}y^{10} + 34776x^{10}y^{11} + 74340x^9y^{12} + 146160x^8y^{13} + 264960x^7y^{14} + 380184x^6y^{15} + 418257x^5y^{16} + 362880x^4y^{17} + 245280x^3y^{18} + 118440x^2y^{19} + 35028xy^{20} + 4656y^{21}$, and
$S(x,y) = 7280x^{12}y^9 + 54432x^{10}y^{11} + 317520x^8y^{13} + 735168x^6y^{15} + 740880x^4y^{17} + 231840x^2y^{19} + 10032y^{21}$.

Table 2 summarizes the possible lengths of s-extremal codes for $6 \le d \le 11$. For even d, the bound comes from Section 4. For the odd case the '*' in the table means that, for this d, the possible lengths are only conjectured in the sense that for greater n there always seems to be a negative coefficient in the weight enumerator. See the Appendix for the detailed weight enumerators of the s-extremal codes considered in this table.

TABLE 2. Possible range of n for $6 \le d \le 11$

d	6	7*	8	9*	10	11*
n	14..16	17..21	20..22	23..27	26..28	29..33

7. Codes related to s-extremal codes

In this section we describe how to produce two Type I or Type II codes over $\mathbb{F}_4$ from the shadow of an s-extremal code. As in the case of the shadow of a binary Type I code, it follows from the definition of the shadow of a Type I code C over $\mathbb{F}_4$ with the even subcode C_0 of C that $C_0^\perp = C_0 \cup C_2 \cup C_1 \cup C_3$, where $C = C_0 \cup C_2$ and the shadow $S = C_1 \cup C_3$ with a correction of $S(C) = C_0^\perp \backslash C_0$ in [14, p. 203] as $S(C) = C_0^\perp \backslash C$. Note that C_i $(i = 1,2,3)$ are nonzero cosets of C_0 in $C_0^\perp$. We give an orthogonality among C_i $(i = 0, \cdots, 3)$ in Table 3, where $\perp$ means that two cosets are orthogonal and / means that they are not. Hence $C_0 \cup C_1$ and $C_0 \cup C_3$ are self-dual.

TABLE 3. Orthogonality for the cosets of C_0 in $C_0^\perp$

	C_0	C_2	C_1	C_3
C_0	$\perp$	$\perp$	$\perp$	$\perp$
C_2	$\perp$	$\perp$	/	/
C_1	$\perp$	/	$\perp$	/
C_3	$\perp$	/	/	$\perp$

Proposition 7.1. *Suppose C is an s-extremal code of length n and minimum distance d satisfying $2d + s = n + 2$. If d is odd and $s = d + 2$, then $C_0 \cup C_1$ or $C_0 \cup C_3$ is an s-extremal code with minimum distance $d' = d + 1$ and the minimum shadow weight $s' = s - 2 = d$.*

Proof. Let $C^{(1)} := C_0 \cup C_1$ and $C^{(3)} := C_0 \cup C_3$. Since $n = 3d$ is odd, all weights in S of C are odd by Lemma 3.2. Hence both $C^{(1)}$ and $C^{(3)}$ are of Type I. We may assume that C_1 contains a vector $\mathbf{x}$ of minimum weight s. Then the minimum distance of $C^{(1)}$ is $\min\{d(C_0), d(C_1)\} = \min\{d+1, s\} = d+1$. The shadow of $C^{(1)}$ is $C_2 \cup C_3$ and its minimum weight is $\min\{d(C_2), d(C_3)\} = \min\{d, \geq s\} = d$. As $2(d+1)+d = 3d+2 = n+2$, $C^{(1)}$ is an s-extremal code. ☐

Example 7.2. The s-extremal code of length $n = 9$ with $d = 3$ in Table 1 produces an s-extremal code of the same length with minimum distance 4. Similarly if there is an s-extremal code of length 15 with $d = 5$, then there must exist an s-extremal code of $n = 15$ and $d = 6$. Further, if there is an s-extremal code of length 21 with $d = 7$, then there is an s-extremal code of that length with $d = 8$. Existence of these codes of lengths 15 and 21 are unknown.

Proposition 7.3. *Suppose C is an s-extremal code of length $n = 6m + 2$ and minimum distance d. If d is odd and $s = d + 1$, then both $C_0 \cup C_1$ and $C_0 \cup C_3$ are extremal Type II codes with minimum distance $d + 1$. Moreover the weight enumerators of C_1 and C_3 are the same and are explicitly determined.*

Proof. Let $C^{(1)} := C_0 \cup C_1$ and $C^{(3)} := C_0 \cup C_3$. By Lemma 3.2 all weights in S of C are even. So $C^{(1)}$ is of Type II. The minimum distance of $C^{(1)}$ is $\min\{d(C_0), d(C_1)\} = d + 1$. Similarly $d(C^{(3)})$ is $d + 1$. Since $s = d + 1$ and $2d+s = n+2 = 6m+4$, we get $d+1 = 2m+2 = 2 * \lfloor \frac{n}{6} \rfloor + 2$. Hence the two codes are extremal. Finally we recall that the weight enumerator of an extremal Type II code is uniquely determined by its length and that the weight enumerator of C_0 is explicitly determined by that of C. Therefore the weight enumerator of C_1 and C_3 are the same and are explicitly determined. ☐

Example 7.4. The three s-extremal codes of length 8 and $d = 3$ produce the three extremal Type II codes of length 8 and $d = 4$. The weight enumerator of an extremal Type II code of length 8 and $d = 4$ is known as $x^8 + 42x^4y^4 + 168x^2y^6 + 45y^8$ (for example in [10]). Since the weight enumerator of the even subcode of the three s-extremal codes of length 8 and $d = 3$ is $x^8 + 18x^4y^4 + 88x^2y^6 + 21y^8$ from Table 1, we get $W_{C_1} = 24x^4y^4 + 80x^2y^6 + 24y^4 = W_{C_3}$. We have checked that $W_{C_1} + W_{C_3} = S(x, y)$. From Table 1 we know that there exists an s-extremal code of length 14 with $d = 5$. By Proposition 7.3 we can decompose $S(x, y)$ as $W_{C_1} = W_{C_3} = 154x^8y^6 + 1176x^6y^8 + 3612x^4y^{10} + 2968x^2y^{12} + 282y^{14}$ and there should exists a Type II code of $n = 14$ and $d = 6$. Similarly if there exists an s-extremal code of $n = 26$ and $d = 9$, then there must exist an extremal Type II code of length 26 with minimum distance 10 whose existence is unknown [10].

8. Conclusion

We have introduced a concept of s-extremal codes for additive self-dual codes over $\mathbb{F}_4$. More precisely, for an additive self-dual $\mathbb{F}_4$ code C of length n with minimum distance d, it satisfies $2d + s \leq n + 2$ unless $n = 6m + 5$ and $d = 2m + 3$, in which case $2d + s = n + 4$, where s is the minimum weight of the shadow of C. Then we have given a bound on the length of s-extremal codes with even length, and classified them up to minimum distance $d = 4$. We have shown that any shortening of extremal Type II codes of length $n \equiv 0$ or $2 \pmod 6$ produces an s-extremal code of length $n - 1$. Furthermore, we have given possible lengths and (shadow) weight enumerators for which there exist s-extremal codes with $5 \leq d \leq 11$ and five s-extremal codes with $d = 7$. We have given four new s-extremal codes of length

$n = 13$ and minimum distance $d = 5$. We also have described a way to relate an s-extremal code of length $3d$ to another s-extremal code of that length, and produced extremal Type II codes from s-extremal codes.

In Theorem 3.1 of [3], Bachoc and Gaborit showed that the set of words of weight i in a binary s-extremal code holds 1-designs (and even 2-designs in some cases) using the idea of harmonic weight enumerators of binary codes as introduced by Bachoc in [1]. As a future work, it would be interesting to find t-designs (possibly with repeated blocks) in s-extremal codes over $\mathbb{F}_4$. Bachoc's idea [2] of using harmonic weight enumerators is not directly applicable to s-extremal codes over $\mathbb{F}_4$ due to the requirement of linearity.

Appendix

In this appendix, we give detailed weight enumerators of the s-extremal codes treated in Table 2.

$d = 8$: There is no known s-extremal code with $d = 8$. The putative weight enumerators and shadow enumerators are as follows.

$n = 19$:

If $n = 19$, then the coefficients of $C(x, y)$ and $S(x, y)$ are nonnegative. But the highest possible minimum weight of a Type I code of length 19 is 7 [14]. So we exclude $n = 19$.

$n = 20$:

$C(x,y) = x^{20} + 990x^{12}y^8 + 4160x^{11}y^9 + 9552x^{10}y^{10} + 25920x^9y^{11} + 65820x^8y^{12} + 120960x^7y^{13} + 174240x^6y^{14} + 210048x^5y^{15} + 200745x^4y^{16} + 141120x^3y^{17} + 69520x^2y^{18} + 22080xy^{19} + 3420y^{20}$, and
$S(x,y) = 120x^{14}y^6 + 1080x^{12}y^8 + 22488x^{10}y^{10} + 124440x^8y^{12} + 357480x^6y^{14} + 394920x^4y^{16} + 141640x^2y^{18} + 6408y^{20}$.

$n = 21$:

$C(x,y) = x^{21} + 630x^{13}y^8 + 3640x^{12}y^9 + 10080x^{11}y^{10} + 27216x^{10}y^{11} + 74340x^9y^{12} + 158760x^8y^{13} + 264960x^7y^{14} + 367584x^6y^{15} + 418257x^5y^{16} + 370440x^4y^{17} + 245280x^3y^{18} + 115920x^2y^{19} + 35028xy^{20} + 5016y^{21}$, and
$S(x,y) = 360x^{14}y^7 + 4760x^{12}y^9 + 61992x^{10}y^{11} + 304920x^8y^{13} + 747768x^6y^{15} + 733320x^4y^{17} + 234360x^2y^{19} + 9672y^{21}$.

$n = 22$:

$C(x,y) =$
$x^{22} + 330x^{14}y^8 + 3080x^{13}y^9 + 10164x^{12}y^{10} + 27216x^{11}y^{11} + 79464x^{10}y^{12} + 194040x^9y^{13} + 367620x^8y^{14} + 577632x^7y^{15} + 763917x^6y^{16} + 814968x^5y^{17} + 676060x^4y^{18} + 425040x^3y^{19} + 192192x^2y^{20} + 55176xy^{21} + 7404y^{22}$, and
$S(x,y) = 1320x^{14}y^8 + 16632x^{12}y^{10} + 168168x^{10}y^{12} + 722040x^8y^{14} + 1539384x^6y^{16} + 1345960x^4y^{18} + 386232x^2y^{20} + 14568y^{22}$.

$d = 9$: There is no known s-extremal code with $d = 9$. The putative weight enumerators and shadow enumerators are as follows.

$n = 23$:

$C(x,y) = x^{23} + 2530x^{14}y^9 + 10626x^{13}y^{10} + 26082x^{12}y^{11} + 78246x^{11}y^{12} + 223146x^{10}y^{13} + 478170x^9y^{14} + 830346x^8y^{15} + 1245519x^7y^{16} + 1562022x^6y^{17} + 1562022x^5y^{18} + 1221990x^4y^{19} + 733194x^3y^{20} + 317262x^2y^{21} + 86526xy^{22} + 10926y^{23}$, and
$S(x,y) = 5060x^{14}y^9 + 52164x^{12}y^{11} + 446292x^{10}y^{13} + 1660692x^8y^{15} + +3124044x^6y^{17} + 2443980x^4y^{19} + 634524x^2y^{21} + 21852y^{23}$.

$n = 24$:

$C(x,y) = x^{24} + 1760x^{15}y^9 + 8712x^{14}y^{10} + 26208x^{13}y^{11} + 80556x^{12}y^{12} + 235872x^{11}y^{13} + 567864x^{10}y^{14} + 1122528x^9y^{15} + 1876941x^8y^{16} + 2657952x^7y^{17} + 3116344x^6y^{18} + 2949408x^5y^{19} + 2203740x^4y^{20} + 1259808x^3y^{21} + 517896x^2y^{22} + 135072xy^{23} + 16554y^{24}$, and
$S(x,y) = 495x^{16}y^8 + 15048x^{14}y^{10} + 165732x^{12}y^{12} + 1131768x^{10}y^{14} + 3753882x^8y^{16} + 6235768x^6y^{18} + 4404708x^4y^{20} + 1036872x^2y^{22} + 32943y^{24}$.

$n = 25$:

$C(x,y) = x^{25} + 1375x^{16}y^9 + 6600x^{15}y^{10} + 23400x^{14}y^{11} + 81900x^{13}y^{12} + 245700x^{12}y^{13} + 631800x^{11}y^{14} + 1403160x^{10}y^{15} + 2630925x^9y^{16} + 4153050x^8y^{17} + 5537400x^7y^{18} + 6144600x^6y^{19} + 5530140x^5y^{20} + 3936900x^4y^{21} + 2147400x^3y^{22} + 844200x^2y^{23} + 211050xy^{24} + 24831y^{25}$,
and
$S(x,y) = 2750x^{16}y^9 + 46800x^{14}y^{11} + 491400x^{12}y^{13} + 2806320x^{10}y^{15} + 8306100x^8y^{17} + 12289200x^6y^{19} + 7873800x^4y^{21} + 1688400x^2y^{23} + 49662y^{25}$.

$n = 26$:

$C(x,y) = x^{26} + 1430x^{17}y^9 + 4719x^{16}y^{10} + 16848x^{15}y^{11} + 80340x^{14}y^{12} + 259560x^{13}y^{13} + 671580x^{12}y^{14} + 1625520x^{11}y^{15} + 3442725x^{10}y^{16} + 6048900x^9y^{17} + 8973250x^8y^{18} + 11359920x^7y^{19} + 11999988x^6y^{20} + 10271976x^5y^{21} + 6970860x^4y^{22} + 3641040x^3y^{23} + 1373970x^2y^{24} + 329238xy^{25} + 36999y^{26}$,
and $S(x,y) = 12012x^{16}y^{10} + 143520x^{14}y^{12} + 1394640x^{12}y^{14} + 6795360x^{10}y^{16} + 18046600x^8y^{18} + 23927904x^6y^{20} + 13974480x^4y^{22} + 2739360x^2y^{24} + 74988y^{26}$.

$n = 27$:

$C(x,y) = x^{27} + 2145x^{18}y^9 + 3861x^{17}y^{10} + 4563x^{16}y^{11} + 69732x^{15}y^{12} + 288900x^{14}y^{13} + 712260x^{13}y^{14} + 1738620x^{12}y^{15} + 4182165x^{11}y^{16} + 8301150x^{10}y^{17} + 13534950x^9y^{18} + 19034730x^8y^{19} + 23057892x^7y^{20} + 23202036x^6y^{21} + 18885204x^5y^{22} + 12249900x^4y^{23} + 6150690x^3y^{24} + 2232009x^2y^{25} + 512109xy^{26} + 54811y^{27}$, and $S(x,y) = 47736x^{16}y^{11} + 423360x^{14}y^{13} + 3837600x^{12}y^{15} + 16061760x^{10}y^{17} + 38610000x^8y^{19} + 46043712x^6y^{21} + 24654240x^4y^{23} + 4425408x^2y^{25} + 113912y^{27}$.

$d = 10$: There is no known s-extremal code with $d = 10$. The putative weight enumerators and shadow enumerators are as follows.

$n = 26$:

$C(x,y) = x^{26} + 6149x^{16}y^{10} + 28288x^{15}y^{11} + 68900x^{14}y^{12} + 219520x^{13}y^{13} + 711620x^{12}y^{14} + 1705600x^{11}y^{15} + 3362645x^{10}y^{16} + 5948800x^9y^{17} + 9073350x^8y^{18} + 11440000x^7y^{19} + 11919908x^6y^{20} + 10231936x^5y^{21} + 7010900x^4y^{22} + 3652480x^3y^{23} + 1362530x^2y^{24} + 327808xy^{25} + 38429y^{26}$,
and $S(x,y) = 715x^{18}y^8 + 5577x^{16}y^{10} + 169260x^{14}y^{12} + 1334580x^{12}y^{14} + 6885450x^{10}y^{16} + 17956510x^8y^{18} + 23987964x^6y^{20} + 13948740x^4y^{22} + 2745795x^2y^{24} + 74273y^{26}$.

$n = 27$:

$C(x,y) = x^{27} + 3861x^{17}y^{10} + 23868x^{16}y^{11} + 69732x^{15}y^12 + 211680x^{14}y^{13} + 712260x^{13}y^{14} + 1918800x^{12}y^{15} + 4182165x^{11}y^{16} + 8030880x^{10}y^{17} + 13534950x^9y^{18} + 19305000x^8y^{19} + 23057892x^7y^{20} + 23021856x^6y^{21} + 18885204x^5y^{22} + 12327120x^4y^{23} + 6150690x^3y^{24} + 2212704x^2y^{25} + 512109xy^{26} + 56956y^{27}$, and
$S(x,y) = 2145x^{18}y^9 + 28431x^{16}y^{11} + 500580x^{14}y^{13} + 3657420x^{12}y^{15} + 16332030x^{10}y^{17} + 38339730x^8y^{19} + 46223892x^6y^{21} + 24577020x^4y^{23} + 4444713x^2y^{25} + 111767y^{27}$.

$n = 28$:

$C(x,y) = x^{28} + 2002x^{18}y^{10} + 19656x^{17}y^{11} + 68523x^{16}y^{12} + 197568x^{15}y^{13} + 686520x^{14}y^{14} + 2066400x^{13}y^{15} + 4931745x^{12}y^{16} + 10221120x^{11}y^{17} + 18878860x^{10}y^{18} + 30030000x^{9}y^{19} + 40414374x^{8}y^{20} + 46043712x^{7}y^{21} + 44027256x^{6}y^{22} + 34515936x^{5}y^{23} + 21542430x^{4}y^{24} + 10325952x^{3}y^{25} + 3581298x^{2}y^{26} + 797384xy^{27} + 84719y^{28}$, and

$S(x,y) = 8008x^{18}y^{10} + 107016x^{16}y^{12} + 1476000x^{14}y^{14} + 9653280x^{12}y^{16} + 38038000x^{10}y^{18} + 80576496x^{8}y^{20} + 88207392x^{6}y^{22} + 43024800x^{4}y^{24} + 7176456x^{2}y^{26} + 168008y^{28}$.

$d = 11$: There is no known s-extremal code with $d = 11$. The putative weight enumerators and shadow enumerators are as follows.

$n = 29$:

$C(x,y) = x^{29} + 15834x^{18}y^{11} + 71253x^{17}y^{12} + 179046x^{16}y^{13} + 613872x^{15}y^{14} + 2140200x^{14}y^{15} + 5618025x^{13}y^{16} + 12350520x^{12}y^{17} + 24701040x^{11}y^{18} + 43543500x^{10}y^{19} + 65315250x^{9}y^{20} + 83454228x^{8}y^{21} + 91040976x^{7}y^{22} + 83413512x^{6}y^{23} + 62560134x^{5}y^{24} + 37431576x^{4}y^{25} + 17276112x^{3}y^{26} + 5781034x^{2}y^{27} + 1238793xy^{28} + 126006y^{29}$, and

$S(x,y) = 31668x^{18}y^{11} + 358092x^{16}y^{13} + 4280400x^{14}y^{15} + 24701040x^{12}y^{17} + 87087000x^{10}y^{19} + 166908456x^{8}y^{21} + 166827024x^{6}y^{23} + 74863152x^{4}y^{25} + 11562068x^{2}y^{27} + 252012y^{29}$.

$n = 30$:

$C(x,y) = x^{30} + 10920x^{19}y^{11} + 56875x^{18}y^{12} + 173880x^{17}y^{13} + 594810x^{16}y^{14} + 2068128x^{15}y^{15} + 5951745x^{14}y^{16} + 14444640x^{13}y^{17} + 31016440x^{12}y^{18} + 59033520x^{11}y^{19} + 97783686x^{10}y^{20} + 139510800x^{9}y^{21} + 170873820x^{8}y^{22} + 178382880x^{7}y^{23} + 156295230x^{6}y^{24} + 112510944x^{5}y^{25} + 64827000x^{4}y^{26} + 28815080x^{3}y^{27} + 9281295x^{2}y^{28} + 1920120xy^{29} + 190010y^{30}$, and

$S(x,y) = 3003x^{20}y^{10} + 93730x^{18}y^{12} + 1247535x^{16}y^{14} + 11813400x^{14}y^{16} + 62102950x^{12}y^{18} + 195567372x^{10}y^{20} + 341690310x^{8}y^{22} + 312650520x^{6}y^{24} + 129622815x^{4}y^{26} + 18571170x^{2}y^{28} + 379019y^{30}$.

$n = 31$:

$C(x,y) = x^{31} + 8463x^{20}y^{11} + 42315x^{19}y^{12} + 149730x^{18}y^{13} + 577530x^{17}y^{14} + 2003499x^{16}y^{15} + 6010497x^{15}y^{16} + 15992280x^{14}y^{17} + 37315320x^{13}y^{18} + 76251630x^{12}y^{19} + 137252934x^{11}y^{20} + 216241740x^{10}y^{21} + 294875100x^{9}y^{22} + 345616830x^{8}y^{23} + 345616830x^{7}y^{24} + 290653272x^{6}y^{25} + 201221496x^{5}y^{26} + 111658435x^{4}y^{27} + 47853615x^{3}y^{28} + 14880930x^{2}y^{29} + 2976186xy^{30} + 285015y^{31}$, and

$S(x,y) = 16926x^{20}y^{11} + 299460x^{18}y^{13} + 4006998x^{16}y^{15} + 31984560x^{14}y^{17} + 152503260x^{12}y^{19} + 432483480x^{10}y^{21} + 691233660x^{8}y^{23} + 581306544x^{6}y^{25} + 223316870x^{4}y^{27} + 29761860x^{2}y^{29} + 570030y^{31}$.

$n = 32$:

$C(x,y) = x^{32} + 8736x^{21}y^{11} + 29848x^{20}y^{12} + 100800x^{19}y^{13} + 547680x^{18}y^{14} + 2012832x^{17}y^{15} + 5875362x^{16}y^{16} + 16674048x^{15}y^{17} + 42966400x^{14}y^{18} + 94624320x^{13}y^{19} + 182499408x^{12}y^{20} + 313432704x^{11}y^{21} + 472350528x^{10}y^{22} + 615551040x^{9}y^{23} + 690813240x^{8}y^{24} + 663526656x^{7}y^{25} + 536812416x^{6}y^{26} + 357739424x^{5}y^{27} + 191337240x^{4}y^{28} + 79204800x^{3}y^{29} + 23825504x^{2}y^{30} + 4608288xy^{31} + 426021y^{32}$, and

$S(x,y) = 75712x^{20}y^{12} + 958080x^{18}y^{14} + 12291264x^{16}y^{16} + 84651520x^{14}y^{18} + 367016832x^{12}y^{20} + 942499584x^{10}y^{22} + 1383308160x^{8}y^{24} + 1072737792x^{6}y^{26} + 382983360x^{4}y^{28} + 47586944x^{2}y^{30} + 858048y^{32}$.

$n = 33$:
$C(x,y) = x^{33} + 13104x^{22}y^{11} + 24024x^{21}y^{12} + 11088x^{20}y^{13} + 459360x^{19}y^{14} + 2205456x^{18}y^{15} + 5821794x^{17}y^{16} + 16114032x^{16}y^{17} + 46814592x^{15}y^{18} + 113683680x^{14}y^{19} + 232681680x^{13}y^{20} + 427942944x^{12}y^{21} + 706874688x^{11}y^{22} + 1018686240x^{10}y^{23} + 1268312760x^{9}y^{24} + 1366361568x^{8}y^{25} + 1263917952x^{7}y^{26} + 984864496x^{6}y^{27} + 632200536x^{5}y^{28} + 326359440x^{4}y^{29} + 130735968x^{3}y^{30} + 38090448x^{2}y^{31} + 7123941xy^{32} + 634800y^{33}$, and $S(x,y) = 310464x^{20}y^{13} + 2969472x^{18}y^{15} + 36552384x^{16}y^{17} + 218718720x^{14}y^{19} + 867993984x^{12}y^{21} + 2025264384x^{10}y^{23} + 2741371776x^{8}y^{25} + 1965404672x^{6}y^{27} + 654160320x^{4}y^{29} + 75892608x^{2}y^{31} + 1295808y^{33}$.

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