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UNDERSTANDING PRESERVICE TEACHERS' SPATIAL REASONING AND HOW  
IT AFFECTS THEIR WORK WITH ELEMENTARY STUDENTS

by

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A DISSERTATION

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Major: Educational Studies

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Under the Supervision of Professor Lorraine M. Males

Lincoln, Nebraska

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# UNDERSTANDING PRESERVICE TEACHERS' SPATIAL REASONING AND HOW IT AFFECTS THEIR WORK WITH ELEMENTARY STUDENTS

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University of Nebraska, 2019

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Spatial reasoning involves those skills that allow one to mentally picture and manipulate objects which plays a unique role in learning and succeeding in science, technology, engineering, and mathematics fields (STEM). Despite the urgent need for strong spatial reasoning skills, our current education system spends little time fostering elementary students' visual and spatial reasoning skills. This is becoming increasingly problematic as the need to become literate in the STEM fields has never been greater.

The purpose of this qualitative study was to examine the spatial reasoning skills that preservice teachers demonstrated and how their spatial reasoning skills were used in the enactment of the tasks of teaching. Thirty-two preservice teachers completed a spatial reasoning task. Each preservice teacher then teamed with their practicum partner, created an adapted plan using the same spatial reasoning task, and enacted their plan with an elementary student in Grades K-5.

Finding from this study indicate that the spatial reasoning skills of preservice teachers are weak, which hinders flexible thinking when observing elementary students engaged in a spatial reasoning task. How learners represent and connect pieces of knowledge is a critical factor in whether they will understand it deeply and can use it in problem solving.

This dissertation is dedicated to:

God, for giving me this opportunity and the strength to accomplish it.

My parents, Marvin and Suzanne, for their example, support, and love.

My sister, Alise, my brothers, Brad and Jason, and each of their families, for always  
encouraging me.

To all my family and friends, who took an interest in this pursuit of a life-long dream of  
mine.

To all my former students, who graciously shared their mathematical thinking and  
inspired my teaching as well as this work.

To all preservice elementary teachers who begin this mathematical journey with  
reservations. Be assured, you can do this.

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## **CHAPTER 1: INTRODUCTION**

Science, technology, engineering, and mathematics (STEM) learning requires problem solving and mathematical skills. Spatial thinking, or mentally manipulating information about the structure of the shapes and spaces in one's environment, is crucial for developing skills that support STEM learning (Newcombe, 2010; Wolfgang, Stannard, & Jones, 2001). Complex mathematical problem solving rests on spatial skills and links between spatial and mathematical skills being established (Gunderson, Ramirez, Beilock, & Levine, 2012). Because spatial reasoning is vitally important for success in STEM careers and because spatial skills are foundational to mathematics learning, spatial reasoning has become a priority in education (National Council of Teachers of Mathematics [NCTM], 2007).

### **Statement of the Problem**

We are all familiar with the adage “give a man a fish, and you feed him for a day, teach a man to fish, and you feed him for a lifetime.” Too often, we are feeding our students instead of teaching them how to solve the problem of feeding themselves. Mathematically speaking, this resembles when students are struggling to work a problem (disequilibrium), and the teacher wants to help lead them to get the answer so their struggling and frustration will be over which is only helpful for the moment and not the next time they face the same challenge (Carter, 2008). Robbing students of the productive struggle when solving problems tends to steal their learning.

The focus on preparing students for STEM careers has increased the classroom emphasis on spatial reasoning, problem solving, and critical thinking. “Critical thinking is a skill that is impossible to teach directly but must be intertwined with content...some

argue that placing too strong an emphasis on students acquiring knowledge alone leaves them struggling when faced with more complex problems” (Christodoulou, 2014, p. 78). In essence, critical thinking requires the ability to reason and to use that reasoning to solve problems. Spatial reasoning is a component of critical thinking in math (Ontario Ministry of Education, 2014). It allows the learner to engage in visualizing, perspective taking, mental transformations, and composing and decomposing (shapes, numbers, measurements, data, and algebraic expressions) as he/she works to understand a problem. Spatial reasoning also relates to how well someone will be able to analyze new and potentially abstract information and then apply that information appropriately (Peirce, n.d.). In this way, it is related to problem solving and critical thinking, two primary components of cognitive aptitude which refers to a person’s ability to process information, learn new skills, and apply new information to solve problems.

Children are born with a natural curiosity (Galinsky, 2010). Give a child a toy and watch him or her play for hours. Listen to the questions a child asks. Children have a thirst to understand things (Deangelis, 2014). Instead of teaching through problem-based learning involving spatial reasoning and critical thinking, students are taught how to take tests and how to respond to basic questions. This needs to change if we want our children to succeed in a world that demands they use these skills together.

Andreas Schleicher, the Organization for Economic Cooperation and Development’s (OECD) director of education and skills, said during a recent webinar entitled *New PISA Results: Putting U.S. Achievement in Global Context* that the United States students’ levels of proficiency appear to decline as they advance to higher grades, contrary to the trend in many higher performing countries. He also highlighted another



troubling issue suggested by the PISA results. “Students are often good at answering the first layer of a problem in the United States. But as soon as students have to go deeper and answer the more complex part of a problem, they have difficulties” (Schleicher, 2016). This suggests that educationally, we place too strong of an emphasis on children acquiring knowledge alone which leaves them struggling when faced with more complex problems, and their learning is hindered when knowledge is front-loaded and all the thinking and critical questioning is left until later (Bonawitz, Shafto, Gweon, Goodman, Spelke & Shultz, 2011; Buchsbaum, Gopnik, Griffiths & Shafto, 2011).

By including spatial reasoning skills, students begin to improve visual memory, become independent thinkers, strengthen concentration efforts and perseverance, boost their capacity to predict and anticipate success and consequences, and advance their ability to use criteria to drive decision making and to evaluate alternatives. For example, structured block play is one way students begin to practice critical thinking skills. Through this type of play, young children imagine and manipulate spatial information in their heads visualizing three-dimensional objects as they try to recreate a construction by consulting a model or a blueprint (Casey & Bobb, 2003). More specifically, children must analyze what they see, perceive the parts that make up the whole and figure out how the parts relate to each other as they create these constructions (Dewar, 2017).

The OECD suggests that those countries where students do best at problem solving, are not only good at teaching the core subjects but are good at providing learning opportunities that prepare students well for complex, real-life problems (2016). Therefore, we must make core subjects like math and science relevant for students, and at the same time, foster creativity, curiosity, and a passion for problem solving (Deangelis,

2014). This is where STEM education should step in. STEM is about using math and science to solve real-world challenges and obstacles. This applied, project-based way of teaching and learning allows students to understand and appreciate the relevance of their work to their own lives and the world around them (Fortus, Krajcikb, Dersheimerb, Marx, & Mamlok-Naamand, 2005). Once they grasp core concepts, students can take complicated problems and use their creativity and curiosity to research, design, test, and improve a viable solution (Deangelis, 2014).

Take, for example, toys like LEGOs, unit blocks, and K'Nex, which nurture STEM skills by supporting open-ended experimentation. They require flexible thinking about cause and effect and are naturally suited to encourage the kind of reasoning and designing that allow children to build and fail and build again similar to the way scientists work to test theories (Barak & Levenberg, 2016). The language and social skills that are practiced in a classroom block corner exist alongside the math and geometry concepts that evolve from grouping, and adding and subtracting those halves and doubles. The design and spatial reasoning skills developing from the columns, ramps, curves, and buttresses exist alongside new understandings of gravity and balance (Gronlund & Stewart, 2011). And the perseverance and resilience that develop from tackling a challenging problem are valuable in all aspects of life (Logan, Lowrie & Bateup, 2017).

How learners represent and connect pieces of knowledge (i.e., critical thinking) is a critical factor in whether they will understand it deeply and can use it in problem solving (NRC, 2001). Thus, learning with understanding is more powerful than rote learning because the organization required improves retention, promotes fluency, and

facilitates learning related material. The central notion that deep understanding requires that learners connect pieces of knowledge and that connection, in turn, is a critical factor in whether they can use what they know productively in solving problems (NRC, 2001; Kolodner, Camp, Crismond, Fasse, Gray, Holbrook, Puntambekar, & Ryan, 2003). These skills, spatial reasoning, critical thinking, and problem solving, do not magically appear overnight. They take time and practice to develop.

Educating students in STEM subjects prepares students for life, regardless of the profession they choose to follow (Diaz & King, 2007). When integrated into problem solving activities, these subjects teach students how to think critically and how to solve problems which are skills that can be used throughout life. Content knowledge and the ability to solve problems must both be priorities for our students. While some ability to solve problems comes from knowledge of facts and persistence, other problem solving requires curiosity and critical thinking (Bertram, 2014). Without this broader skillset, our students will not be prepared for the jobs of the future or equipped to succeed in an ever-changing world. Our education system, then, must focus not only on building students' content knowledge, but also inspiring an entrepreneurial mindset encompassing spatial reasoning and problem solving (Satchwell & Loepp, 2002).

It all starts with those responsible for the instruction of mathematical thoughts and key ideas, the classroom teachers (Grossman, Wilson, & Shulman, 1989). Research indicates that teachers' mathematical skills and comfort level are significant indicators in the overall learning of their students (Putnam, Heaton, Prawat, & Remillard, 1992). A teacher's lack of experience with process-oriented teaching (i.e., an instructional model in which learners are taught to employ suitable learning and thinking activities to construct,

change, and utilize their knowledge of a particular subject) (Vermunt & Verschaffel, 2000) appears to send the message to students that not everyone is capable of understanding difficult math concepts. “More than one-third of elementary teachers admit they experience some kind of math anxiety and teachers who don’t understand math well tend to focus on teaching algorithms rather than underlying concepts (Brown, Westenskow, & Moyer-Packenham, 2011).

The math anxiety that teachers experience carries consequences beyond the individual as teachers who report high levels of math anxiety also report a reduction in the belief that they are capable of successfully carrying out their teaching responsibilities to their students (i.e., teaching self-efficacy and confidence) (Ramirez, Hooper, Kersting, Ferguson, & Yeager, 2018; Bursal & Paznokas, 2006; Gresham, 2008). Math anxious teachers have also been found to teach in a very inflexible manner that favors traditional and rigid forms of instruction (Trujillo & Hadfield, 1999; Vinson, 2001), and spend less time attending to students’ questions (Bush, 1989) a practice that has the potential to communicate to students that math is either something you know or do not know. Math anxious teachers primarily promote algorithmic teaching which makes the teacher the primary source of information rather than encouraging student-level reasoning (Karp, 1991) and typically only ask for a single solution to a problem (Bush, 1989). In summary, the literature suggests that math anxious teachers, through their teaching comments, behaviors, and teaching practices, may create an environment that devalues sense-making and effort in lieu of an emphasis on memorization and innate ability (Ramirez, et al., 2018).

Since teaching mathematics requires a more advanced level of mathematical understanding, it is imperative that teachers continue to hone their mathematical skills so that they are prepared to not only teach math, viewing the content from multiple perspectives (Moseley, 2000), but also understand the mathematical minds of their students who have real thoughts and ideas as well as address their misconceptions (Lee, Meadows, & Lee, 2003). A high proportion of preservice teachers enter teacher education with a limited range of mathematical experiences and show little evidence of connected thinking that recognizes the place and role of mathematics (Chen & Mu, 2010; Witt, Goode, & Ibbett, 2013). There are concerns worldwide about teacher content knowledge in mathematics as the capacity to deeply understand, identify, and use mathematical thinking across a range of contexts requires experience of its use beyond the mathematics classroom (ACER, 2009; Steen, 2001). In order to facilitate the complex reasoning and divergent thinking skills within challenging activities students need to be engaging in (National Governors Association Center for Best Practices, 2010), preservice teachers need to possess a thorough conceptual and procedural understanding of mathematics, as well as an understanding of research-based, developmentally appropriate pedagogical practices (Robichaux-Davis & Guarino, 2016; Ball, 2000; Ball, 1990; Ma, 1999).

Previously, researchers (Van der Sandt & Nieuwoudt, 2003; Fuys, Geddes, & Tischler, 1998; Mayberry, 1983; Cunningham & Roberts, 2010; Milsaps, 2013; Perry & Dockett, 2002; Pickreign, 2007; Reinke, 1997; Ward, 2004) have found that preservice elementary and middle-level teachers lack a level of geometric thinking necessary to future success as mathematics teachers. Preservice teachers typically demonstrate a

procedural understanding of geometry as evidenced by memorized definitions and properties of shapes, rather than a conceptual understanding based on a synthesis of properties of shapes and recognition of minimal properties that define shapes (Robichaux-Davis & Guarino, 2016; Cunningham & Roberts, 2010). Besides attaining the necessary content, preservice teachers must also possess adequate pedagogical content knowledge of geometry, as well as knowledge of particular materials appropriate for geometry instruction (Shulman, 1986). Thus, in order to teach geometric ideas and concepts, preservice teachers must have knowledge of various representations, appropriate examples and non-examples, and instructional materials that do not misrepresent the concepts being taught (Milsaps, 2013). Furthermore, preservice elementary teachers have been found to have significantly weaker spatial reasoning and spatial visualization skills when compared to other undergraduates, particularly those majoring in engineering fields, architecture, and mathematics (Robichaux, 2007).

Research indicates that students should be developing spatial reasoning from the beginning of their schooling, enabling them to successfully apply spatial strategies when faced with complex mathematical problems (Shumway, 2013). However, the absence of spatial reasoning in textbooks, state standards, and classroom instructions (Wheatley, 2002) suggests the need to develop lessons and activities to supplement instruction. Spatial reasoning must be recognized as a fundamental part of K-12 education, as it is a key ingredient for problem solving. While most mathematical classrooms focus on skills associated with numbers and operations, spatial reasoning along with problem solving are hidden at best but are usually not incorporated in the learning done by students (NRC, 2006). Because spatial reasoning is not a stand-alone discipline, standards for spatial

reasoning need to consist of general guidelines for what students need to know about spatial reasoning concepts so that they can learn to apply new approaches when solving challenging problems in a range of subject areas. Without a concentrated effort to structure learning in such a way that spatial reasoning becomes the focal point within daily lessons, we as a nation will not see improvement (Harris, Newcombe, & Hirsh-Pasek, 2013).

### **Purpose of the Study and Research Questions**

In this study, I examined the spatial reasoning skills that preservice elementary teachers demonstrated and how their spatial reasoning skills were used in the enactment of the tasks of teaching. Specifically, I focused on how the spatial reasoning skills of preservice elementary teachers may influence how they adapted and used the same task with an elementary student. The following sub-questions supported the central research question.

- (1) What do preservice elementary teachers do when engaging in a task that requires spatial reasoning?
- (2) What do preservice elementary teachers do when planning for and enacting the spatial reasoning task with elementary students?
- (3) What do preservice teachers write about in written reflections related to their spatial reasoning skills and their working with an elementary student on a spatial reasoning task?

## Definition of Terms

- **Accommodation** Any modification of an existing scheme of formation of a new cognitive structure when it is not possible to fit information into an existing structure.
- **Cognitive Development** When the child constructs new knowledge by being engaged in active self-discovery while interacting with objects in their environment.
- **Intersubjectivity** When a child attempts solving a difficult task alone, then a more knowledgeable person (peer or adult) begins working with the child discussing what they collectively know and what they are unsure of and finally through collaboration, a solution is generated.
- **Reflection** When a shape is flipped across a reflection line which produces a mirror image.
- **Rotation** When a shape is turned on a rotational point inside or outside the shape. Also, when the object is rotated  $360^\circ$ , it returns to its original position.
- **Scaffolding** Bridging the demands of the new task with existing knowledge and skills.
- **Spatial Orientation** The ability to look at a fixed figure from several different points of view.
- **Spatial Reasoning** Spatial reasoning involves the ability to think and reason by comparing, manipulating, and transforming mental shapes.
- **Spatial Visualization** The ability to visually compare shapes that have changed position on a plane or in space.



- **Transformations** The ways used to describe the changes in position of a shape which include a slide, reflection, and/or rotation. Many children have a difficult time understanding that the shape stays the same even when it is moved; they have not developed what Piaget has called conservation.
- **van Hiele Levels of Geometric Thought** A model used to understand and describe how children's geometric thinking develops over time: Level 0 – Visualization, Level 1 – Analysis, Level 2 – Informal Deduction, Level 3 – Formal Deduction, and Level 4 – Rigor.
- **Zone of Proximal Development** What a child can achieve independently and what they can achieve with guidance, encouragement, and collaboration from a skilled peer.

### **Importance of the Study**

This study showed the importance of STEM learning within an elementary classroom. It supported the benefits from problem-based learning that come from critical thinking, spatial reasoning, collaboration, and problem solving. It benefited preservice teachers by encouraging them to take risks in order to provide a rich learning experience for their students.

## **CHAPTER 2: LITERATURE REVIEW**

Although the acquisition of spatial reasoning is clearly interwoven within the NCTM recommendations (NCTM, 2000, 2006) and the Common Core State Standards (National Governors Association Center for Best Practices, 2010), much of the mathematics children are exposed to in the United States during their elementary years primarily addresses numbers sense (Griffin, 2004). Thus, the geometry content is often placed in a subsidiary position, and when geometry is taught systematically, the stress is on shape naming and shape attributes rather than spatial reasoning (Brownell, Chen, & Ginet, 2014; Clements, 2004). Many educators recognize the importance of including well-planned mathematics instruction throughout the day (Clements, 2001) and relish the opportunity to help children engage in mathematics problem solving and critical thinking (Butera, Friesen, Palmer, Lieber, Horn, Hanson, & Czaja, 2014), but finding instructional resources can be difficult (Bafie, 2002). The focus of mathematics instruction is frequently limited to teaching just numbers and operations (number recognition and counting) and geometry (shapes) (Butera, Palmer, Lieber, & Schneider, 2011).

### **Spatial Reasoning Skills**

“Spatial reasoning – or thinking about objects, their locations and shapes, their relations to one another, and the paths they take when they move – is key to helping children succeed in the STEM disciplines” (Newcombe, 2010, p. 29). It relates to how well someone will be able to analyze new and potentially abstract information and then apply that information appropriately. In this way, it is directly related to problem solving and critical thinking, two major components of cognitive aptitude which refers to a person’s ability to process information, learn new skills, and apply new information to

solve problems. Spatial reasoning is the basis of critical thinking in math. It allows the learner to engage in visualizing, perspective taking, mental transformations (sliding, rotating, and reflecting), orienting, dimension shifting, pathfinding, diagraming, modeling, symmetry, scaling, map-making, composing and decomposing (shapes, numbers, measurements, data, and algebraic expressions) and designing (Davis, Okamoto, & Whiteley, 2015), as he/she works to understand a problem. It is worth mentioning that every mathematical problem does not use all of these spatial reasoning ideas, but each problem that students work should allow them to engage in at least one or more of these critical thinking ideas. Spatial reasoning is often difficult and requires flexible thinking (National Research Council, 2006) as people in the real world frequently get lost or give directions that are difficult to follow and/or that contain mistakes. They get frustrated when attempting to put together “easy to assemble” furniture, and they become angry at each other when trying to pack a small car for a long trip (Scultheis & Carlson, 2013).

The NCTM recommends that at least half of mathematics teaching and learning should focus on spatial reasoning (NCTM, 2006, 2010). A focus on spatial reasoning allows mathematics to become a more visual endeavor and connect with what “real” mathematicians do when they are exploring patterns in the world and making discoveries. For example, long division is usually learned through the use of procedures in the standard algorithm, which leads to unconnected ideas and processes, causing students to struggle through in order to eventually learn the concept. What if we restructured the learning of division and connected it to total area being split into equal groups of the same size and modeled accordingly? What if we encouraged collaboration (like real

mathematicians) through students working in groups to discuss their collective solving of the problem? Through collaboration, students feel safe trying new ideas and learning from each other to visually solve a division problem similar to the constructivist work on intersubjectivity by Vygotsky. By exploring the spatial aspects of mathematics, we make it more accessible, more engaging, and more relevant (Evans, Kochalka, Ngoon, Wu, Qin, Battista, & Menon, 2015). Despite calls to bring geometry and spatial thinking to the forefront of early math curricula, local and international studies reveal that geometry and spatial sense receive the least amount of attention in the elementary years of mathematics (Moss, Bruce, Caswell, Flynn, & Hawes, 2016; Sarama & Clements, 2009a), making it an underserved area of mathematics instruction.

In its report, *Learning to Think Spatially*, the National Research Council (2006) has highlighted spatial reasoning as a “major blind spot” (p. 7) in education and calls on educators and researchers to pay attention. They stress that individuals are daily moving about in a spatially represented environment (e.g., the town they live in and/or the building they work in) where spatial reasoning has transformative and robust importance in helping to ease this task (Newcombe & Frick, 2010). Along with moving in a spatially represented environment, children and adults alike must communicate within this space.

Research in the psychology of intelligence and cognitive processes shows that verbal thinking is influenced by spatial thinking (Carroll, 1993; Bornstein, 2009). Spatial thinking also helps with reasoning in all domains (e.g., maps for directions and/or venn diagrams used to solve logical problems). Notice that some domains are not obviously spatial ones, but they still require the skill set learned through spatial reasoning to navigate this space. By focusing on spatial reasoning instead, students begin to improve

visual memory, become independent thinkers, strengthen concentration efforts and perseverance, boost their capacity to predict and anticipate success and consequences, and advance their ability to use criteria to drive decision making and to evaluate alternatives utilized within the STEM disciplines (Newcombe & Frick, 2010).

Reasoning about spatial compositions and their properties are linked to one's ability to progress and performance in various STEM fields. Children and adolescents who have higher spatial reasoning skills in middle school and high school are more likely to major in the STEM disciplines in college and to pursue STEM careers (Shea, Lubinski, & Benbow, 2001; Wai, Lubinski, & Benbow, 2009).

The National Research Council (2006) warns if careful focus is not taken seriously, spatial reasoning “will remain locked in a curious educational twilight zone: extensively relied on across the K-12 curriculum but not explicitly and systematically instructed in any part of the curriculum” (p. 7). Geometry and spatial reasoning in the early years typically focus on having children label and sort shapes (Clements, 2004) as well as spatial orientations and terms such as relative position (Levine, Ratliff, Huttenlocher, & Cannon, 2012), yet cognitive science (Casey, Andrews, Schindler, Kersh, Samper, & Copley, 2008) and educational research (Kersh, Casey, & Young, 2008; Cheng & Mix, 2014) shows that young children are capable of – and interested in – more dynamic and complex spatial thinking (Moss, et al., 2016).

The various research studies below demonstrate the relationship between spatial reasoning and mathematical ability. For example, one research study found that the quality of block play at four years of age was a predictor of high school mathematics achievement (Wolfgang, Stannard, & Jones, 2001). Another study found a relationship

between young children's construction skills (such as playing with jigsaw puzzles and blocks) and strong number sense as well as success in solving mathematical word problems (Nath & Szűcs, 2014). Researchers have also underlined that the link between spatial reasoning and math is so strong that it is "almost as if they are one and the same thing" (Dehaene, 1997, p. 125). Wai, et al., (2009) in their research, used a stratified random sample of high school students and analyzed their spatial reasoning skills against predicted STEM education and career choices. They found a significant connection between spatial reasoning skills in high school and STEM outcomes. Despite the research suggesting its need, spatial opportunities are largely absent from elementary classrooms (NRC, 2006). It should also be noted that having more advanced spatial reasoning skills is linked to better mathematics achievement (Casey, Nuttall, & Pezaris, 1997; Geary, Saults, Lui, & Hoard, 2000), and in a recent longitudinal study, it was found that spatial reasoning skills in kindergarten were stronger predictors of ninth grade math school achievement than fourth grade math achievement scores (Krajewski & Ennemoser, 2009).

Reflecting on the strength of this relationship, others have noted that "spatial instruction will have a two-for-one effect" that yields benefits in mathematics as well as spatial reasoning (Verdine, Golinkoff, Hirsh-Pasek, & Newcombe, 2014, p. 13). Of course, the practices of mathematicians also benefit from spatial reasoning; many mathematicians stress that their work involving motions and spatial reasoning in 3-D space relies strongly on visual and spatial representations and forms of understanding (e.g., computational geometry and robotics in computer science, design and analysis of linkages in mechanical engineering, control of formations of moving robots and

localization of sensor networks in electrical and computer engineering, and modeling protein structures and their motions, with its impact on protein function and drug designs) (Whiteley, Sinclair, & Davis, 2015). Such work has included developing physical models, running simulations over time, and preparing animations of how things unfold in time and space – all in support of spatial reasoning and communication with interdisciplinary collaboration.

### **Spatial Reasoning Incorporated into Various Mathematical Strands**

Various strands of mathematics are inherently spatial. Think about what happens when we compare the area of two polygons, such as a rhombus and a rectangle (Moss, et al., 2016). To be successful, we can draw on spatial strategies such as composition and decomposition of two-dimensional shapes, mental rotation, and visualization. In fact, research shows that spatial reasoning is linked to performance within many strands of mathematics including: basic magnitude and counting skills (Thompson, Nuerk, Moeller, & Cohen Kadosh, 2013), mental arithmetic (Kyttälä & Lehto, 2008), word problems (Hegarty & Kozhevnikov, 1999), algebra (Tolar, Lederberg, & Fletcher, 2009), calculus (Sorby, Casey, Veurink, & Dulaney, 2013), and advanced mathematics (Wei, Yuan, Chen, & Zhou, 2012).

Basic magnitude and counting skills incorporate mental rotation and numerical skills which are linked together. In a recent study by Mix and Cheng (2012), they showed that training children on a mental rotation task (i.e., a task that involves mentally imagining what an object would look like if it would be rotated) improved later performance in mathematical tasks such as arithmetic. Various research in number representation indicates that spatial training can also predict and/or improve other types

of number skills (Thompson, et al., 2013). The development of mental rotation in young children is one of the predictors for numerical competence (Marmor, 1975). If individual differences in mental rotation abilities at early developmental stages can partially predict later numerical abilities, this may be able to help validate other early warning signs of number difficulties (Thompson, et al., 2013).

Visuospatial working memory is responsible for processing and the short-term storage of visual and spatial information, which is infused into mental arithmetic. “To understand the value of a digit within a number and its relation to other digits, it is important to perceive its visuospatial location” (Kyttälä & Lehto, 2008, p. 79). Geometric figures, diagrams, and curves are examples of visual spatial materials, and visual contour is detected in verbal material (e.g., digits or other mathematical symbols). Concrete visuospatial support (e.g., explaining things visually, encouraging students to use visuospatial help: fingers, drawings, etc.) benefits students with poor visuospatial abilities and leads to better success in math.

The construction of schematic spatial representations in solving mathematical problems might also be thought of as the construction of high quality spatial representations, in that they represent the essential information relevant to solving the problem and omit superfluous details. There is a distinction between the two types of visual-spatial representations used in mathematical problem solving: schematic representations that primarily encodes the spatial relations described in a problem and pictorial representations that primarily encodes the visual appearance of the objects or persons described (Csíkos, Szitányi, & Kelemen, 2012) and each representation is related differently to problem solving success (Hegarty & Kozhevnikov, 1999). Successful



problem solvers typically create a complete pictorial representation, either mental or sketched, of the problem schema, which in turn, facilitates the encoding and retrieval of information needed to solve problems (Xin, Jitendra, & Deatline-Buchman, 2005; Didierjean & Cauzinille-Marmèche, 1998; Fuson & Willis, 1989; Marshall, 1995; Mayer, 1982). Problem schema acquisition allows the learner to use the representation to solve a range of different (i.e., containing varying surface features) but structurally similar problems (Sweller, Chandler, Tierney, & Cooper, 1990).

The strong effect of computational fluency on algebra achievement is seen through the emphasis on procedural problems in algebra assessments (Tolar, Lederberg, & Fletcher, 2009). The combined direct and indirect effects of computational fluency on algebra achievements are as strong as the effect of the level of algebraic education. Problems used in the Tolar, Lederberg, and Fletcher (2009) study were typical of the type of problems found in many high school and college algebra curricula and involved solving algebraic equations, simplifying algebraic expressions, and translating between symbolic and graphical representations of functional relations. These problems may be solved entirely by retrieving procedures and algorithms from long-term memory and implementing them. This emphasis on procedure, both at the algebraic and numerical levels, may be why computational fluency was highly related to algebra achievement.

A lot of calculus problems involve visualizing slopes and areas under curves which are included in spatial reasoning. Students must also understand inflection points and how they relate to maximum and minimum slopes. Frequently, students will be shown a 2-D graph and asked to visualize it as a 3-D function. “Just having extra practice visualizing and sketching objects in 3-D space may potentially increase a student’s

capacity to use visualization when solving calculus problems” (Sorby, Casey, Veurink, & Dulaney, 2013, p. 28).

Taken together, research suggests (Moss, et al., 2016) that spatial reasoning skills offer a potentially powerful means of supporting children’s mathematical thinking and learning. Teachers’ performance of procedures help students develop their spatial reasoning: (1) encourage students to mentally manipulate the objects, (2) support the use of geometric terms by intentionally making an effort to connect the students’ everyday language with geometric terminology, and (3) cultivate students’ ownership of ideas by describing both how they moved the shapes and their struggles to do so.

### **Mathematical Knowledge for Teaching**

Teachers need to know more and different mathematics than what is needed by other adults in other professions (Hill, 2010; Baumert, Kunter, Blum, Brunner, Voss, Jordan, Klusmann, Krauss, Neubrand, & Tsai, 2010; An, Kulm, & Wu, 2004; Ball, Thames, & Phelps 2008). According to Ball, Hill, and Bass (2005), the quality of mathematics teaching depends on teachers’ mathematical content knowledge, and alarmingly, many United States teachers lack firm mathematical understanding and skill. Hill and Ball (2009) and Charalambous (2010) have established that overall mathematical ability does not fully account for the knowledge and skills needed for effective mathematics teaching. They have uncovered a select type of knowledge, called specialized content knowledge, that is needed by teachers that is specifically mathematical, separate from pedagogy and knowledge of students, which is not needed in other professional settings (Ball, et al., 2008; Hill, Schilling, & Ball, 2004). Since the daily task of teaching requires knowledge beyond that which is needed to reliably carry

out a mathematical algorithm, it encompasses interpreting someone else's work, representing and fabricating links between ideas in multiple forms, developing alternative explanations, choosing usable definitions, decompressed or unpacked mathematical reasoning as well as pedagogical thinking (Ball, 2003; Ball, et al., 2005; Hill & Ball, 2009).

For teachers to be prepared to teach quality mathematics, teacher educators must ensure that preservice teachers have opportunities to develop the mathematical knowledge that is specific to their needs (Lee, Meadows, & Lee, 2003; Davis & Krajcik, 2005). "Improving the mathematics learning of every child depends on making central the learning opportunities of our teachers (Ball, 2003, p. 9)." To understand the knowledge needed to teach mathematics, one must grasp an overall picture of what teaching math looks like and sounds like in a school classroom setting. Since the goal of teaching mathematics is to improve the opportunities students' have for learning mathematics, let us begin by analyzing the mathematical demands of the work of teaching which is extremely different in comparison to the knowledge and skill required of other kinds of work like engineering or nursing or construction.

First, knowing mathematics for teaching requires knowing, in detail, the topics and ideas that are fundamental to the school curriculum and beyond (Ball, 2003; Harris & Sass, 2007). Teaching mathematics involves using tools and skills for reasoning about mathematical ideas, representations, and solutions, as well as what constitutes coursework. It demands the understanding of the insides of students' ideas, their roots and connections, their reasons and the ways in which they are being represented both with the connection of the current mathematical topics and how particular ideas anticipate

later ones (Fennema, Franke, Carpenter, & Carey, 1993). The work of helping others learn mathematics often entails unpacking or decompressing ideas as they surface.

Second, the knowledge for teaching mathematics is different from the mathematical knowledge needed for other mathematically intensive occupations and professions (Ball, 2003). Interpreting someone else's error, representing ideas in multiple forms, developing alternative explanations, and choosing a usable definition are all examples of the problems that teachers must solve and central to the work of teaching (Harris & Sass, 2007).

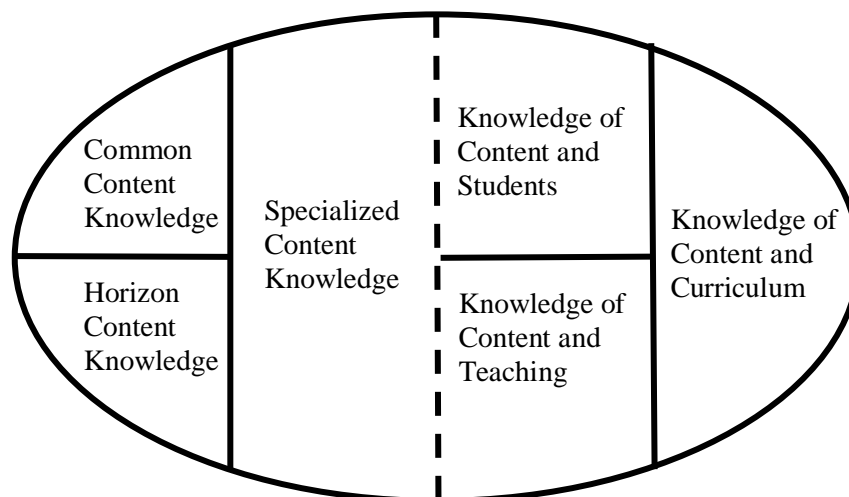
Third, mathematical knowledge for teaching must be assistive for the work that teaching entails: from offering clear explanations, to posing good problems, to mapping across alternative models, to examining instructional material with a keen and critical mathematical eye, to modifying or correcting inaccurate or incorrect analyses. Teaching mathematics is a serious and demanding field of mathematical work which is not, even at the elementary level, a watered-down version of "real" mathematics (Ball, 2003; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992).

The improvement of the teaching of mathematics depends on the furtherance of our understanding of its mathematical nature and demands which begins with opportunities for teachers to acquire the appropriate mathematical knowledge and skills to do this work well (Greenberg & Walsh, 2008). For example, teachers can attend carefully designed courses and workshops geared to learning more mathematics, but seldom do they learn mathematics in ways that they will need to use it in their work. Knowing the importance of different representations and being able to make strategic use of them in teaching is essential as well as being interested in alternative methods and

having the skills to inspect them, consider their potential for generalization, and having the tools to do so is another facet of the mathematical knowledge needed for teaching (Harris & Sass, 2007).

Mathematical Knowledge for Teaching (see Figure 2.1) encompasses two distinct areas: Subject Matter Knowledge and Pedagogical Content Knowledge.

### **Subject Matter Knowledge**



### **Pedagogical Content Knowledge**

*Figure 2.1.* Domains of Mathematical Knowledge for Teaching (reproduced from Ball, Thames, & Phelps, 2008, p. 403)

Within Subject Matter Knowledge, Ball, et al., (2008) included Common Content Knowledge, Specialized Content Knowledge, and Horizon Content Knowledge. Common Content Knowledge is “mathematical knowledge and skill used in settings other than teaching” (p. 399). For example, teachers and accountants alike must be able to recognize incorrect arithmetic solutions and use correct terminology and notation, so this type of mathematical knowledge is not unique to teaching. Specialized Content Knowledge is “mathematical knowledge not typically needed for purposes other than teaching” (p. 400), such as knowing mathematical explanations for common rules or procedures;

constructing and/or linking non-symbolic representations of mathematical subject matter; interpreting, understanding, and responding to non-standard mathematical methods and solutions; deploying mathematical definitions or proofs in accurate yet also grade-level-appropriate ways; or being able to analyze common mathematical errors in student work (Hill, 2010). Horizon Content Knowledge is “an awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (p. 403), such as the need for a first-grade teacher to know the mathematics students will learn later in order to help them learn first-grade content in a way that will enable more connections to be made in third grade.

In this model, Pedagogical Content Knowledge is the knowledge of how students learn content or of ways to teach specific topics (Shulman, 1986; Wilson, Shulman, & Richert, 1987; Hill, 2010) and it is also divided into three domains: Knowledge of Content and Students, Knowledge of Content and Teaching, and Knowledge of Content and Curriculum. Knowledge of Content and Curriculum looks at what kind of instructional materials are available for teaching and learning a mathematical concept, what approaches these materials take, and how effective they are in helping students learning the given mathematical concept (Shulman, 1987). The other two domains emerged from the research of Ball, Thames, and Phelps (2008). Knowledge of Content and Students joins knowledge of the subject with knowledge of students, e.g., what examples students will find interesting, how students will approach a task, what difficulties students might have with a concept, or how to interpret a student’s incomplete knowledge.

To clarify the differences between Common Content Knowledge, Specialized Content Knowledge, and Knowledge of Content and Students, Ball, Thames, and Phelps (2008) explained,

recognizing a wrong answer is Common Content Knowledge, whereas sizing up the nature of the error, especially an unfamiliar error, typically requires nimbleness in thinking about meaning in ways that are distinctive of Specialized Content Knowledge. In contrast, familiarity with common errors and deciding which of several errors students are most likely to make are examples of Knowledge of Content and Students. (p. 401)

Knowledge of Content and Teaching involves the knowledge used to make instructional decisions, understanding the advantages and disadvantages of various representations of a concept, and deciding when to pursue a student's question or to move on to a new task to develop the concept. Baumert and colleagues (2010), as well as Fennema and Franke (1992), found that teachers' pedagogical content knowledge was more predictive of student learning gains than content knowledge.

### **Mathematical Knowledge for Teaching Geometry**

The National Council of Teachers of Mathematics (NCTM) (2000) stressed the recent elevation of geometry by stating that "geometry offers an aspect of mathematical thinking that is different from, but connected to, the world of numbers" (p. 97). Clements and Battista (1992) pointed out that geometry can be considered as a tool to describe, analyze, and understand the world in which we live as well as a tool that can be applied to other areas of mathematics. Several mathematics educators have maintained that geometry promotes students' knowledge relating to space and the relationship of objects

within it, skills of deductive reasoning, and the ability to solve real-life problems in which geometrical vocabulary and properties present themselves (French, 2004; Presmeg, 2006; Marchis, 2012).

Geometry lends itself well to making “rich connections with the rest of mathematics, including topics and themes in discrete and continuous mathematics as algorithmic thinking, geometric series, optimization, functions, limits, trigonometry and more” (Goldenberg, Cuoco, & Mark, 1998, p. 23). Geometry is one of the focus areas for the NCTM (2000) Content Standards and NCTM (2006) Curriculum Focal Points and as such, prospective elementary and middle grade teachers must be prepared to teach this subject effectively. More recently, geometry is also the only content strand that is in every grade, K-12, in the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices, 2010). The National Assessment of Educational Progress identified weaknesses in the performance of U.S. students on mathematical concepts, in particular geometry concepts, as compared with students in other countries (Gonzales, Calsyn, Jocelyn, Kastberg, Arafeh, Williams, & Tsen, 2000). More specifically, students lacked the ability to perform well on spatial visualization and problem solving, which are factors that affect success in geometry and geometric problem solving (Battista, 1999). Success in math and geometry is closely related to the strength of the individual’s spatial reasoning and visualization ability (Holzinger & Swineford, 1946). Hegarty & Waller (2005) found that spatial reasoning skills, together with intelligence and visual perception, are required to develop mathematical thinking.

A contributing factor to U.S. students’ weak performance on geometric concepts could be attributed to the mathematical knowledge for teaching geometric concepts held



by teachers (Unal, Jakubowski, & Corey, 2009). Two national studies of the preparation of middle school teachers (*Breaking the Cycle* report and *Mathematics Teaching in the 21<sup>st</sup> Century*) found that prospective middle grades teachers' mathematics knowledge in the areas of algebra and geometry to be weak in comparison with potential middle grades teachers in other countries (Center for Research in Math and Science Education, 2010; Schmidt, Tatto, Bankov, Blomeke, Cedillo, Cogan, Han, Houang, Hsieh, Paine, Santillan, & Schwille, 2007). Therefore, prospective middle grades teachers may not possess the subject matter knowledge and pedagogical content knowledge needed to effectively teach geometric concepts (Grover & Conner, 2000; Swafford, Jones, & Thorton, 1997). Furthermore, preservice elementary teachers have been found to have significantly weaker spatial reasoning and spatial visualization skills when compared to other undergraduates, particularly those majoring in engineering fields, architecture, and mathematics (Robichaux, 2007).

### **Teaching and Beliefs**

Mathematical content knowledge is just one part that plays a significant role in teacher preparedness to teach mathematics. The way in which a teacher approaches mathematical situations also has to do with their beliefs (Stodolsky & Grossman, 1995). For example, since there is no procedure to follow to visualize spatial relations; one must rely on connections made through experiences and the ability to make mental images. If the teacher believes spatial relations are not mathematically important, they will choose other math concepts to focus on and not provide opportunities for students to improve their spatial reasoning. Certain beliefs that teachers hold seem to mediate the effects of teachers' knowledge on their teaching practices (Grossman, Wilson, & Shulman, 1989).

If they believe that math is a subject of rules and routines which have to be remembered, then their approach to problems that do not fit this mold will be uneasy and impact their teaching (Ozarka, 2014). These beliefs are prominent in the development of conceptual knowledge, where hypothesizing, finding evidence, and seeking explanations is very different from the procedural approach of applying rules and routines in recognizable contexts.

If teachers lack confidence in their subject matter knowledge, they may avoid taking a risk in the classroom (e.g., if the teacher is not able to solve or understand a solution to a mathematical problem, they will avoid assigning that problem to their students) and be guarded in responding to a student's unexpected questions and ideas about solving problems (Putnam, Heaton, Prawat, & Remillard, 1992). They are also fearful that their own mathematical skills will be challenged by teaching students in the older grades (Wolfram, 2016). This lack of confidence can result in teachers being more careful and incorporating a range of resource materials when lesson planning (e.g., attempting to find problems they themselves can answer and not challenging their students with a variety of complex problems to solve) or it might appear as a lack of interest by the teacher which may carry over into negative feelings about mathematics and/or deficient planning and preparation (Beswick, Watson, & Brown, 2006). All in all, the flexibility and deep interconnectedness of mathematics suffers when the teacher has weak mathematical subject knowledge.

### **The Malleability of Spatial Reasoning Skills**

Research shows that spatial reasoning skills are malleable (Uttal, Meadow, Tipton, Hand, Alden, Warren, & Newcombe, 2013) and that experiences like block

building (Casey, et al., 2008) and puzzle play (Levine, Ratliff, Huttenlocher, & Cannon, 2012) can alter spatial reasoning. Harris, Newcombe, & Hirsh-Pasek (2013) suggest that during the elementary school years is an ideal time to introduce spatial reasoning, especially if spatial reasoning is considered a fundamental cognitive process underlying STEM success. Visualizing, mentally transforming, and composing shapes (putting two or more together to form a composite) and decomposing shapes (separating a composite shape into smaller individual shapes) are important skills that young students can develop over time if engaged in appropriate, sequenced, mathematical experiences with intentional actions taken by the teacher (Brown, 2009; Clements, Wilson, & Sarama, 2004). Sekiyama, Kinoshita, & Soshi, (2014) suggest that children aged around seven to eight years fall in a transition period for spatial reasoning, wherein more mental processing emerges from the earlier physical and illustrated approaches.

Spatial reasoning skills can be enhanced through practice. By practicing spatial reasoning skills, students' mathematical performances are significantly increased. For example, Deno (1995) found positive correlations between non-academic activities (e.g., model building, sketching, and assembly of parts) and spatial reasoning. Balke-Aurell's (1982) study found that students educated in schools using a verbally oriented curriculum had more growth in verbal abilities, whereas those taught in schools using a technical curriculum showed more growth in spatial reasoning. More recently, Bairaktarova, Reyes, Nassr, & Carlton (2015) found that spatial reasoning skills can be developed through playing video games, musical experience, art, and childhood toys (Legos, Lincoln Logs, etc.), hand-eye coordination sports, and technical education since these skills are learned and not inherent.

## **Manipulatives Enhance Learning**

Children construct much of their knowledge through active manipulation of the environment (Beaty, 1984; Bredekamp & Copple, 1997; Lee, 1992; McInerney & McInerney, 2002). Children aged between five and eight benefit from active experiences like hands-on discovery learning while working with tangible objects known as manipulatives that support a developing association between the concrete representation and symbolic representation (Clements & McMillen, 1996). Children in this age bracket rely on active manipulation of real materials to connect abstract materials, such as ideas and statements, to something observable and imaginable which begins to shape their reasoning processes (Kaplan, 2000). Children at this age can reason logically as long as principles are applied to concrete examples (Santrock, 1998). Learning, through Vygotsky's ideas on intersubjectivity and Piaget's work with cognitive development, occurs when constructive play enables children to combine their repetitive sensorimotor ideas with the symbolic representation of ideas (Fenson & Schell, 1985; Santrock, 1998).

The idea that the manipulation of physical objects plays an important role in the learning process of all children was first advocated by Pestalozzi, who argues the importance of things before words, concrete before abstract (Nathan & von Gunten, 1989). In more recent times, both Froebel and Montessori supported this philosophy. Froebel created a set of 20 "gifts," objects such as balls, blocks, and sticks designed to help children recognize and appreciate common patterns and forms found in nature (Resnick, 1998). Montessori developed educational objects or "materials" to enhance learning for children aged between three and twelve years (Lillard, 2005). The use of Cuisenaire Rods and Pattern Blocks in early elementary classrooms is a testament to the

importance of concrete manipulative materials to support children as they develop abstract concepts (Resnick, 1998).

The use of tools, for example, is a highly spatial activity. Newcomb (2013) points out that the development and use of tools relies on spatial thinking: “to create a successful tool, one must first imagine a shape that is relevant to a particular function, such as cutting or digging, and then fashion that shape out of larger forms” (p. 102). We can see how the use of manipulatives in the mathematics classroom can help to consolidate understanding and concept development as visualization and problem solving are inherent in their use.

Although it is important that manipulatives are made available to students, an even more critical consideration is how to ensure the use of manipulatives in meaningful ways – as integral to the thinking and the problem solving. In other words, the learning task is designed so that manipulatives are not just used to communicate or show representations of thinking after the cognitive work of problem solving is done; they are the tools with which the problem is solved (Ontario Ministry of Education, 2104). They are integral to the task, not an add-on or an option for students if they choose. Manipulatives help build understanding through visual and kinesthetic means by providing a hands-on exploration of numeric quantities and algebraic expressions. The power of manipulatives is in helping students move between concrete representations and abstract ideas, as well as visually understanding and internalizing abstract concepts.

### **The Development of Problem Solving and Reasoning Skills**

Problem solving and reasoning become more complex as young children gain new abilities to ask questions and gather information. Their inclination to be curious,

explore, experiment, ask questions, and develop their own theories about their surroundings enhances their overall learning (Early Childhood Learning & Knowledge Center, 2018). Since mathematics includes generalizations and abstractions, math skills help young children connect ideas, develop logical and abstract thinking, analyze, and question, as well as understand the world around them (Diezmann & Yelland, 2000). Children develop math concepts and skills through active exploration and discovery in the context of stimulating learning opportunities and intentional teaching strategies (Derman-Sparks, Edwards, & National Association for the Education of Young Children, 2010). Providing children with opportunities to play, explore, and encounter mathematical patterns and structures (e.g., identifying shapes and patterns, counting, addition, subtraction, and spatial reasoning) as they engage with mathematics in various ways will help them to develop science, technology, engineering and mathematics (STEM) concepts (Moomaw & Davis, 2010; Ginsburg, Inoue, & Seo, 1999; Sarama & Clements, 2009).

Considering children's play naturally employs skills of observation and experimentation, it can also lead to the development of specific process models for how things should be constructed and how things work, thus signaling important elements of engineering thinking (Cunningham, Lachapelle, & Davis, 2018). Mitchell and Burton (1984) argue that the use of construction toys (e.g., blocks, Legos, K'nex) provide opportunities for children to value and exploit their spatial skills. Children's ability to think, reason, and use information allows them to acquire knowledge, understand the world around them, and make appropriate decisions (Cunningham, 2017). They tend to think with an open-minded curiosity and possess the motivation to solve problems that

require math, spatial reasoning, and creativity (Chouinard, Harris, & Maratsos, 2007) as well as build knowledge through active questioning and information gathering. This process of active learning and acquisition of knowledge occurs during play with materials, play with ideas, and play with others. Vygotsky stressed the importance of play in the learning process of young children (Bodrova & Leong, 2004).

The National Association of Early Childhood Specialists in the State Departments of Education's (NAEYC) guidelines for early childhood curriculum accept the Piagetian notion that children construct knowledge through interaction with materials and communication with people as stressed by Vygotsky's ideas of intersubjectivity (NAEYC, 1991). What is "concrete" to the child may have more to do with what is meaningful and manipulable than with physical characteristics. Benefits are not automatic, of course. Thoughtful use, including carefully selected activities, is the key (Clements & Nastasi, 1992). Piaget demonstrated that young children learn about geometric shapes, not from taking mental pictures of objects, but from actions they perform on objects; for example, children can walk a rectangular path to help them understand the main characteristics of a rectangle. Our world calls for reflective mathematical thinkers, which emphasizes discussing and solving problems in geometry, number sense, and patterns with the help of manipulatives and/or technology (Clements, 1999).

Research indicates that students should be developing spatial reasoning from the beginning of their schooling, enabling them to successfully apply spatial reasoning skills when faced with complex mathematical problems. However, the absence of spatial reasoning in textbooks and classroom instructions (Wheatley, 2002) suggests the need to

develop lessons and activities to supplement instruction. Kersh, Casey, and Young (2008) recommend that instruction includes lessons with block-building activities structured around specific problems that are designed to encourage spatial thinking, including lessons for building bridges, stairs, and towers (Shumway, 2013).

As the childhood years provide the foundation for later learning, specific opportunities for the development of spatial reasoning should be implemented in educational settings. Spatial and algebraic reasoning are important for learning advanced mathematical concepts. Therefore, it is beneficial for students in the elementary grades to engage in activities that support mathematical reasoning. Current practice in preschool and elementary school settings appear to provide contrasting opportunities for the development of spatial reasoning. Whereas preschool settings appear to provide multiple and varied opportunities for the development of spatial reasoning, opportunities in more formal school settings (e.g., elementary school) are minimal at best (Dunphy, Dooley, Shiel, Butler, Corcoran, Ryan, & Travers, 2014). It is from these lacking practices of exposure to spatial reasoning tasks within the school setting and the minimal preparation and experiences of preservice elementary teachers with spatial reasoning tasks that this study was created.

### **Theoretical Foundation**

In this study, I used the van Hiele Levels of Geometric Thought framework to better understand the geometric ideas involving spatial reasoning. The van Hiele Levels are situated within the cognitive development theories of Lee Vygotsky and Jean Piaget. These theorists both focused their constructivist work on how a child learns and develops



even though they have separate ideas on how this is accomplished. In the next section, I describe these theories.

**Vygotsky and Social Constructivism.** Vygotsky's theory of sociocultural learning emphasizes the importance of culture and context in understanding what occurs in society and constructing knowledge based on this understanding (Derry, 1999). This knowledge is co-constructed, and engaged individuals learn from one another. "Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first, between people (interpsychological) and then inside the child (intrapsychological)" (Vygotsky, 1978, p. 57). Vygotsky strongly believed that community plays a central role in the process of "making meaning" and social learning tends to precede development (Vygotsky, 1978). The environment in which children grow up will influence how they think and what they think about (Vygotsky, 1987).

A fundamental aspect of Vygotsky's theory is the Zone of Proximal Development. This is the difference between what a child can achieve independently and what a child can achieve with guidance, encouragement, and collaboration from a skilled peer (Vygotsky, 1978). Vygotsky believes learning occurs in this zone since the elements of the task that are initially beyond the student's capacity begin to reveal themselves as the student concentrates and completes parts of the task within his/her range of competence intermixed with scaffolded concepts helping to bridge the demands of the new task with existing knowledge and skills within the given task. This scaffolding not only produces immediate results but also instills the skills necessary for independent problem solving in the future. Vygotsky focused on the connections between people and the sociocultural context in which they act and interact in shared experiences (Crawford,

1996). According to Vygotsky, people use tools that develop from a culture, such as speech and writing, to mediate their social environments (Vygotsky, 1987). Initially, children develop these tools to serve solely as social functions and ways to communicate needs. Vygotsky believed that the internalization of these tools led to higher thinking skills (David, 2014).

The idea of intersubjectivity is also part of the work of Vygotsky and social constructivism. Intersubjectivity is evident within the solving of a given task by two or more individuals. First, each individual works on the task alone, attempting to solve the problem. Once each individual has wrestled with the task, they now work with another discussing what they know and what they are still unsure of as they are attempting to solve the task. This interaction can be expressed in many forms (e.g., written, verbal, or gestures) as a way to communicate what one knows about the problem while the other listens intently. After each person has shared their individual knowledge, they collectively through collaboration, generate a solution to the given task. These individuals benefit through shared discussion, clarification of ideas, and evaluation of others' ideas when attempting to solve a given task. This process displays intersubjectivity between two or more individuals working together to solve a problem. Collaborative problem solving and interactive decision making enrich the overall learning of students in constructivist learning situations (Satchwell & Loepp, 2002).

**Piaget and Cognitive Development.** Piaget's concept of cognitive development affirms that children go through specific stages according to their intellect and ability to perceive mature relationships. These childhood stages occur in the same order in all children, across all cultures and backgrounds (Weisz, 2018). However, the age at which

the stage materializes may vary slightly from child to child. The knowledge the child is able to construct from their experiences is central to their current level of cognitive function. A child constructs new knowledge when they engage in active self-discovery, as they interact with objects in their environment. These independent explorations come from organized patterns of physical action known as schemes, and they are the foundation for the child's further development.

Piaget's thoughts on social interaction being necessary for development is in agreement with Vygotsky. The exchanging of ideas and cooperation with others should occur between peers since they are more likely to cooperate as equals instead of adults and children (Karpov, 2006; Piaget, 1976). Within this exchange of ideas, students are able to see multiple perspectives and have the opportunity to change their way of thinking (Brown & Palincsar, 1989). These social interactions by themselves are not acceptable for scholarly development; they must incorporate highly cognitive demanding tasks for students to show intellectual growth (Lourenco & Machado, 1996; Piaget, 1950/1995).

Intellectual development involves continual adaptation, whereby individuals construct new and more sophisticated cognitive structures (schemes) as they engage in new experiences. Equilibration is a process of maintaining a cognitive balance between our existing knowledge and new experiences (Durwin & Reese-Weber, 2017). When individuals are confronted with new experiences, they have a sense of disequilibrium, a discrepancy between their existing way of knowing and the new experiences they are engaging in. After wrestling with this new exploration, they finally reach a conclusion that not only builds their cognitive intelligence but restores balance in their cognitive system (Piaget, 1985).

Cognitive adaptation can be achieved through assimilation and accommodation, which work together to help the individual maintain equilibration (Piaget, 1970; Sternberg, 2003). Assimilation involves integrating new information or a new experience into an existing cognitive structure and accommodation consists of any modification of an existing scheme of formation of a new cognitive structure when it is not possible to fit information into an existing structure (Durwin & Reese-Weber, 2017). For example, Coon and Mitterer (2012) description of a child seeing a zebra for the first time and calls it a horse. The child assimilates this information into her schema for a horse. When the child accommodates information, she takes into consideration the different properties of a zebra compared to a horse, perhaps calling a zebra a horse with stripes. When she eventually learns the name of zebra, she has accommodated this information.

**van Hiele Levels of Geometric Thought.** Children intuitively use geometric motions when solving puzzles which is foundational to spatial reasoning as they turn the pieces, flip them over, and slide them into place working towards all the pieces fitting together (van Hiele, 1999) which are tasks accomplished at Level 1 or Level 2 of van Hiele's Levels of Geometric Thought. Built on the Piagetian idea that the child is an active learner (Piaget, 1926/1959), he begins solving the puzzles and is able to put some pieces together. He soon becomes stumped and unable to complete the rest of the puzzle on his own. Suppose a more skilled person, such as an adult, sibling, or peer who has more experience with puzzles, provides a little bit of assistance which Vygotsky calls scaffolding (Vygotsky, 1987). The more experienced person might point to an empty space on the puzzle and encourage the child to find a piece that fits that spot. If the child remains stumped, the helper might point out a piece or rotate it to help the child see the

relationship. The helper acts to motivate the child and provide support to help the child finish the puzzle, emphasizing that they worked together to accomplish this. Effective scaffolding works within what Vygotsky calls the Zone of Proximal Development which is the gap between the child's competence level, what he can do alone, and what he can do with assistance (Vygotsky, 1987). Throughout this process of putting the puzzle together, the helper adjusts their responses to meet the needs of the child in the moment as they work to accomplish the goal of finishing the puzzle. With time, the child internalizes the hints and is able to accomplish building the puzzle on his own (FERNYHOUGH, 2008).

The van Hiele Levels of Geometric Thought is foundational for understanding the growth of children's geometry knowledge. Pierre van Hiele and Dina van Hiele-Geldof developed this model as they sought to understand and describe how children's geometric thinking develops over time (van Hiele, 1959/1985). This big picture model is very useful for describing categories of thinking you may see in any elementary or secondary classroom, and for planning future instruction. The van Hiele model has five levels, numbered 0-4 (see Table 2.1).

Table 2.1

*van Hiele Levels of Geometric Thought with Descriptor and Examples*

<b>Level</b>	<b>Descriptor</b>	<b>Example (van Hiele, 1959/1985)</b>
Level 0 – Visualization	Children pay attention to what a shape looks like.	A rectangle looks like a door.
Level 1 – Analysis	Children notice the properties of shapes and begin to develop vocabulary for these properties.	A rectangle had four sides and all right angles.

Table 2.1 – Continued

Level 2 – Informal Deduction	Children begin to build arguments about relationships between the properties of classes of shapes.	Like a rectangle, a square also has four sides and all right angles, so a square is a special kind of rectangle.
Level 3 – Formal Deduction	Students support their claims systematically with chains of logical reasoning. Mathematicians call these chains of reasoning proofs.	A proof of the claim that the sum of the measures of the interior angles of any triangle is $180^\circ$ .
Level 4 – Rigor	Students look beyond the constraints of the familiar and consider alternative geometries.	The kinds of questions people investigate at this level are: How is geometry different on the spherical surface of the Earth than it is on an infinite flat plane? and How is geometry different if – like a taxicab navigating a city – there are no diagonal lines but only right angles?

Each level describes a category of student thinking about geometry, and the levels are hierarchical. This means that students thinking at a higher level of the model must have had experiences thinking at the lower levels too. In this sense, lower levels are contained within the higher ones.

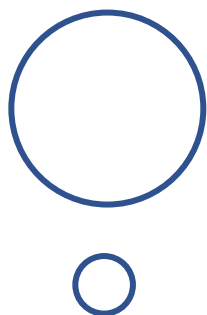
The school curriculum in the elementary grades has students sorting shapes based on the number of sides and learning many vocabulary terms (National Governors Association Center for Best Practices, 2010) which are opportunities to learn at Level 0 and Level 1. Upper elementary and middle school curricula have incorporated a lot of meaningful work around the properties of shapes (Danielson, 2016). For example, students begin to think about the minimum information necessary to define figures and observations go beyond the properties into mathematical arguments about the properties as students start to engage in an intuitive level of “proof” at Level 2 which should ease the transition of these students into high school geometry or Level 3 work (Mason, 2002).

For instance, students can explore relationships, produce conjectures, and start to decide if the conjectures are true at Level 3. Along these same lines, the structure of axioms, definitions, theorems, etc., begin to develop at Level 3 and students are able to work with abstract statements as well as draw conclusions based more on logic than intuition (Way, 2011). Instead of a natural evolution of their prior geometric work, students are struggling to understand the purpose and structure of mathematical proofs causing an abrupt change. van Hiele Level 4 has been reserved for undergraduate math majors and beyond.

The van Hiele model provides a structure for understanding this geometric confusion which many students experience. In order to build formal and logical arguments at Level 3, you need to have practice making informal arguments at Level 2 (Shenk, 1985). When students do not have that practice prior to high school geometry, proof writing becomes more of an exercise in trying to guess what the teacher wants, rather than exploring the forms and constraints of logic. The van Hiele model is based on the idea that you need experience at each level before you can move to the next, and that instruction is a necessary ingredient at each level (Way, 2011). Therefore, it is impossible to go from Level 1 thinking directly to high school geometry. If students do not have experience and instruction building informal geometry arguments, they will not learn to write proofs. They may learn to imitate the form of a two-column proof, but they will not build understanding through mathematical arguments as they do (Shenk, 1985). They will also be less likely to transfer their understanding of geometry to other problem solving situations.

## **Describing Student Learning with the van Hiele Model**

The Common Core State Standards, and those of many non-Common Core states, are structured to provide students with instructional experiences that progress through the van Hiele levels (Danielson, 2016). In kindergarten, students identify and describe shapes (National Governors Association Center for Best Practices, 2010). They describe relationships of shapes to each other and in space. For example, a kindergarten student is shown the two circles in Figure 2.2 and asked to describe what they see.



*Figure 2.2. Kindergarten Students Asked to Describe What They See*

A typical response might be stating that they see two circles. A more advanced response by a kindergarten student would state that the bigger circle is above the smaller one, but this sort of relationship many kindergartners need to practice noticing and describing. Much of this is van Hiele Level 0 work (van Hiele, 1959/1985). This shows that very young children are capable of noticing more than what they can articulate concerning complex relationships among geometric figures (Mashburn, Pianta, Hamre, Downer, Barbarin, Bryant, Burchinal, Early, & Howes, 2008). Opportunities to learn at Levels 0 and 1 are important for developing the language to discuss and critique the beginnings of these mathematically mature ideas (Mason, 2002).



Across the primary grades, children extend this work by looking for similarities and differences among groups of shapes (Van de Walle, 2001). They consider the meaning of such words as right, square, and angle. Often this learning requires a construction of knowledge that is always being refined based on the new experiences students encounter (Sinclair, Moss, Hawes, & Stephenson, 2018). For example, students describe a rectangle as a shape with four corners and four straight sides. Their “shape thinking” is challenged when they have to describe a square. They know it has four corners and four sides, but the sides of the square all look the same. Eventually, students describe in more detail the four corners of a rectangle and square as L-shaped, which ultimately leads students to call these angles right angles (Van de Walle, 2001). As students continue their work of classifying shapes, a parallelogram will resemble a rectangle with four straight sides and four angles, but more specifically, the opposite angles will be congruent and not right angles (Mason, 2002).

Writing a definition that is both true and complete is challenging Level 1 work since students often rely on visualized examples (van Hiele, 1959/1985; Sinclair, Moss, Hawes, & Stephenson, 2018). For example, it is true that a rectangle is a four-sided shape but this is not a complete definition of a rectangle. A student who can look at a collection of quadrilaterals and accurately select all the rectangles may struggle when explaining (using correct mathematical language) why the selected shapes are rectangles. Similarly, a student who has developed a complete definition of a rectangle “a four-sided polygon with all right angles” may not recognize that this definition allows squares (Mason, 2002). This is normal and important Level 1 work.

Even while students continue to wrestle with these Level 1 ideas, they also begin to work on relationships among the properties they work with (Van de Walle, 2001). For example, a group discussion about a collection of triangles and quadrilaterals could result in a student noticing that whenever triangles have sides of the same length (equilateral), their angles are the same too (equiangular); but this is not true for quadrilaterals (Mason, 2002). This is a Level 2 observation since students look beyond the particulars of the shapes they are looking at, and they look across different categories of shapes to make and defend claims about geometric relationships (Crowley, 1987).

Argumentation at Level 2 makes up some of the richest, most interesting work of elementary and middle school geometry (Vojkuvkova, 2012). We see that definitions themselves are not fixed for all time but change according to need, context, and aesthetics as students wrestle with new geometric ideas concerning properties of shapes, speculating about relationships, and building informal arguments (Van de Walle, 2001). Grappling with these ideas gives students opportunities to accept, reject, and modify claims shared by their peers, a process Vygotsky calls intersubjectivity. This ongoing Level 2 work is essential as students learn to reason in geometry. They cannot be expected to prove geometric theorems at Level 3 until they have built up an extensive understanding of the various properties of geometric figures, the relationships between the properties, and how these properties are ordered (Crowley, 1987).

### **Purpose**

In this study, I examined the spatial reasoning skills that preservice elementary teachers demonstrated and how their spatial reasoning skills were used in the enactment of the tasks of teaching. Specifically, I focused on how the spatial reasoning skills of

preservice elementary teachers may influence how they adapted and used the same task with an elementary student.

### **CHAPTER 3: METHODOLOGY**

In this chapter, I describe the research design and plan that guided this study, the participants, the timeline, data collection, data analysis, establishing credibility, research permission, and ethical considerations.

#### **Research Design**

The research design chosen for this qualitative study was the descriptive case study, as outlined by Yin (2018).

#### **Case Study Design**

Since case study research involves studying a case within a real-life context (Yin, 2009), it fits the needs of this inquiry, which is how the spatial reasoning skills of preservice teachers (the case of the study) may have influenced how they adapted and used the same task with an elementary student (a real-life context). Stake (1995) states that the “first obligation” (p. 4) in case study research is to fully develop and understand the case at hand, specifically the spatial reasoning skills of preservice teachers. Case studies are chosen because the research requires the “close examination of people, topics, issues, or programs” (Hays, 2004, p. 218). Each of these items could be a case or what is termed a “bounded system,” which is defined as the focal point of the study (Stake, 1995). With preservice teachers being the case or focus of this research project, they were bounded by both the time frame of two months of data collection (see Figure 3.1) as well as the place where the data collection occurred which was the local school district and the university. When considering the size of this case, one should also mention that this study was a collective case study which involved one issue or concern (preservice teachers spatial reasoning) with multiple cases (preservice teachers) being selected

because the situations had a common characteristic (spatial reasoning) as a way to create a rich, thick description.

Yin (2018) suggests that the system can be studied, depending on purpose, with one of three types of case studies: exploratory case studies, explanatory case studies, and descriptive case studies. The descriptive case study was best suited for this research project as it was used to develop a document that fully illuminated the intricacies of an experience through a rich, thick description of the phenomenon (Merriam, 2009; Stake, 1995) involving the spatial reasoning skills that preservice elementary teachers demonstrated and how their spatial reasoning skills were used in the enactment of the tasks of teaching influenced the ways in which preservice teachers adapted and used the same task with an elementary student. I explored and explained the case using information gathering questions such as who, what, where, when, why, and how based on theoretical constructs (Yin, 2003b).

The case study's unique strength is "its ability to deal with a full variety of evidence – documents, artifacts, interviews, and observations" (Yin, 2003a, p. 8). Since "all evidence is of some use to the case study researcher, nothing is turned away" (Gillham, 2000, p. 20) indicating that the use of multiple sources of evidence is a key characteristic of case study research. According to Hartley (1994, 2004) data collection and analysis are "developed together in an iterative process," which can be a strength as it allows for theory development which is grounded in empirical evidence (p. 220; p. 329). Essential steps in the process of analyzing the data involve the development of categories as well as providing a careful description of the data. The data may then be organized

around certain topics, key themes or central questions, and finally, the data needs to be examined to see how they fit or fail to fit the expected categories (Kohlbacher, 2006).

Yin (2003a) maintains that data analysis consists of “examining, categorizing, tabulating, testing, or otherwise recombining both quantitative and qualitative evidence to address the initial propositions of a study” (p. 109). In general, “data analysis means a search for patterns in data” (Neuman, 1997, p. 426). These patterns can become the start of an analytic path, leading one further into the data and possibly suggesting additional relationships (Yin, 2018). This inductive strategy of working the data from the ground up, can yield noticeable benefits that have been demonstrated in grounded theory research (Corbin & Strauss, 2015; Glaser & Strauss, 1967). “The procedures assign various kinds of codes to the data, each code representing a concept or abstraction of potential interest. You can apply such procedures to all case studies, not just those trying to emulate grounded theory” (Yin, 2018, p. 169). In the end, “the ultimate goal of the case study is to uncover patterns, determine meanings, construct conclusions, and build theory” (Patton & Appelbaum, 2003, p. 67).

I analyzed my data using a holistic analysis approach of the entire case (preservice teachers’ spatial reasoning) in order to create a detailed description in which patterns and themes were developed through categorizing and analyzing coded data as it was gathered concerning: (1) What do preservice elementary teachers do when engaging in a task that requires spatial reasoning?, (2) What do preservice elementary teachers do when planning for and enacting the spatial reasoning task with elementary students?, and (3) What do preservice teachers write about in written reflections related to their spatial reasoning skills and their working with an elementary student on a spatial reasoning

task?. These themes were ones that helped generate an understanding of the complexity of the case and not necessarily for generalizations beyond the case (Yin, 2009). In the initial “open-coding” stages, the data was categorized into broad categories based on recurring themes. As more data was collected and analyzed, these categories were condensed and re-formed into new patterns and themes through axial coding, specifically looking for commonalities and discrepancies in the data. Finally, through selective coding, core categories were matched with other categories and compared to the existing database as a way to “fill in gaps” and enhance the description of the case being studied.

What follows are the research questions, the theoretical view, the rationale for the target population, the sample population, the research plan, the timeline, the methods used in collecting the data, how the data was analyzed, and how the data exposed a credible representation of preservice teachers experiences with a spatial reasoning task and how this related to their work with elementary students.

### **Research Questions**

In this study, I examined the spatial reasoning skills that preservice elementary teachers demonstrated and how their spatial reasoning skills were used in the enactment of the tasks of teaching. Specifically, I focused on how the spatial reasoning skills of preservice elementary teachers may influence how they adapted and used the same task with an elementary student. The following sub-questions supported the central research question.

- (1) What do preservice elementary teachers do when engaging in a task that requires spatial reasoning?

- (2) What do preservice elementary teachers do when planning for and enacting the spatial reasoning task with elementary students?
- (3) What do preservice teachers write about in written reflections related to their spatial reasoning skills and their working with an elementary student on a spatial reasoning task?

### Theoretical View

In this study, I used the van Hiele Levels of Geometric Thought framework to better understand the geometric ideas involving spatial reasoning. The van Hiele Levels are situated within the cognitive development theories of Lee Vygotsky and Jean Piaget. These theorists both focused their constructivist work on how a child learns and develops even though they have separate ideas on how this is accomplished. Table 3.1 provides a summary of the major ideas that provided structure for this study. The work of these theorists builds the connecting pieces of my work which helped answer my research

Table 3.1

#### *Theories and Frameworks Used in Analysis*

<b>Theorist or Framework</b>	<b>Major Ideas Used</b>	<b>Analysis Points Considered</b>
Vygotsky	Zone of Proximal Development	What the child can achieve independently and what they can achieve with guidance, encouragement, and collaboration from a skilled peer
	Scaffolding	Bridging the demands of the new task with existing knowledge and skills
	Co-constructed knowledge	Community plays a part in the “making meaning” The environment they grow up in influences how they think and what they think about
	Intersubjectivity	Working on task alone, working with another discussing what they know and what they are unsure of, then collectively (through collaboration) generate a solution



Table 3.1 – Continued

Piaget	Cognitive Development	Constructs new knowledge when the child is engaged in active self-discovery while interacting with objects in their environment Schemes – independent explorations which come from organized patterns of physical action
	Assimilation	Integrating new information or a new experience into an existing cognitive structure
	Accommodation	Any modification of an existing scheme or formation of a new cognitive structure when it is not possible to fit information into an existing structure
van Hiele	Levels of Geometric Thought	A model used to understand and describe how children’s geometric thinking develops over time: Level 0 – Visualization Level 1 – Analysis Level 2 – Informal Deduction Level 3 – Formal Deduction Level 4 – Rigor
Boaler & Brodie	Teacher Question Types	Types of questions teachers ask when working with students can be categorized into the following nine areas: 1. Linking and applying 2. Extending thinking 3. Probing – getting students to explain their thinking 4. Exploring mathematical meanings and/or relationships 5. Gathering information, leading students through a method 6. Generating discussion 7. Orienting and focusing 8. Inserting terminology 9. Establishing context

questions as preservice teachers worked with a spatial reasoning task. I used the van Hiele Levels of Geometric Thought framework for analysis when the preservice teachers individually solved the spatial reasoning task as well as when the preservice teachers were summarizing their experience with their elementary student’s interaction with the adapted task. I used Boaler and Brodie’s framework involving teacher questions types

when the preservice teachers created their adapted task, prepared their probing questions, and the incorporation of these questions during the enactment of the task. The cognitive development work of theorists Lee Vygotsky and Jean Piaget was referenced throughout the entire spatial reasoning task starting with the preservice teachers individually solving the spatial reasoning task, their work adapting and enacting the task with an elementary student and the preservice teachers' reflections concerning the complete project.

### **Rationale for Target Population**

Preservice teachers were an important group to study since “they will be future instructional leaders in the classroom making instructional decisions regarding the learning opportunities for their students” (Unal, Jakubowski, & Corey, 2009). Henderson (1988) found that preservice teachers' geometric thinking levels were reflected in their instruction and, as a result, the level of understanding of preservice teachers influenced students' difficulty or insight. In other words, how students learn geometric concepts greatly depends on the teacher and how they make instructional decisions at critical moments in the classroom (Unal, Jakubowski, & Corey, 2009). Gaining insight into preservice teachers' thinking is vital as this helps one understand and describe their spatial reasoning when they approach a mathematical task.

Within the First International Mathematics Study (1964), a good predictor of differences found in student performance concerning geometric tasks centered around this idea of “opportunity to learn.” While the abundance of learning opportunities may abound for students, the teachers whose geometric knowledge and/or spatial reasoning is limited may not have the capacity to make adjustments to the curriculum to address the varying learning needs of their students (Unal, Jakubowski, & Corey, 2009). For

teachers to be prepared to teach quality mathematics, teacher educators must ensure that preservice teachers have opportunities to develop the mathematical knowledge that is specific to their needs (Lee, Meadows, & Lee, 2003). “Improving the mathematics learning of every child depends on making central the learning opportunities of our teachers” (Ball, 2003, p. 9). As a descriptive study into this area, this study sought to further explore the relationship between preservice teachers’ spatial reasoning and their ability to adapt the same task to use with an elementary student.

### **Sample Population**

The participants in this study were preservice teachers in an Elementary Math Methods course at a university in the Midwest part of the United States. At this university, preservice teachers not only experience learning within a college classroom but are able to extend this learning by spending time in a local elementary classroom with real students who have real thoughts and ideas, real challenges, and real successes. Two different colleagues taught the two participating college sections. Pseudonyms were used within this study in order to conceal the true identities of the individual preservice teachers as well as the names of the elementary students. Table 3.2 provides an overview summarizing how 48 potential preservice teachers were reduced to 24 who participated in this study.

Table 3.2

#### *Detailed Overview of Preservice Teacher Participant Selection*

<b>Total Number of Preservice Teacher Participants</b>	<b>Number of Preservice Teachers Considered for Removal</b>	<b>Criteria for Removal</b>
48	10	Declined to participate
38	4	Previously completed a geometry course for preservice teachers

Table 3.2 – Continued

34	2	Submitted the same work on the spatial reasoning task when asked to submit this work individually
32 – Analyzed individual work	7	Preservice teacher teams had one member agree to participate but not the other member
25	1	An odd number for a class section total meant one person did not have a practicum partner
24 – Analyzed, adapted, and enacted plan with elementary student		

Of the 48 potential preservice teachers in these sections, ten declined to participate in this study and four were omitted from further study because they had previously completed a geometry course which is designed to help preservice teachers develop an understanding of geometry as taught in the elementary school. By completing a geometry course, the four preservice teachers had previous knowledge related to the spatial reasoning task due to the recent review of transformations (reflecting, sliding, and rotating) in this geometry course. Therefore, I decided to remove these participants so that the remaining participants would have had similar university experiences related to transformations. Data from two students was disregarded because they submitted the same written work on the tile assignment instead of submitting their individual work which prevents a comparison between the van Hiele Levels of Geometric Thought for both the preservice teachers and the elementary student who completes the adapted task. This number narrowed slightly to 24 preservice teachers when participants completed the adaption of the task with their practicum partner as one partner agreed to participate in the study and the other person did not. Also, there was an odd number of preservice

teachers who agreed to participate in this study from one class section, which resulted in one person not having a partner. This meant that this participant did not have the benefit of collaborating on the adaption and enactment of the task. Thus, I chose to eliminate this participant from the study. Therefore, 24 preservice teachers participated in this study.

### Research Plan

Table 3.3 provides a summary of the research plan I used to study the research questions above.

Table 3.3

#### *Summary of Research Steps and Their Connection to the Research Questions*

<b>Research Steps</b>	<b>Research Purpose</b>	<b>Connection to Research Question</b>
1. Preservice teachers complete the Math Matters Tile Assignment	To determine individual spatial reasoning levels using van Hiele Levels of Geometric Thought	The spatial reasoning level of a preservice teacher will influence what they do when engaging in a task that requires spatial reasoning.
2. Preservice teacher teams adapted the Math Matters Tile Assignment for their teaching experience	To observe the variety of adaptations generated	This activity will provide insight into what preservice teachers do when adapting the task.
3. Preservice teacher teams enact plan with an elementary student	To observe the interaction of elementary students with the task as recorded in their journal and shared in interviews	This activity will provide insight into the enactment of the preservice teachers adapted spatial reasoning task with an elementary student.
4. Preservice teachers record written reflections of teaching experience	To observe examples as recorded in their journal of the thought processes of the preservice teacher teams with an elementary student	This activity will provide insight into what preservice teachers write about in written reflections related to their spatial reasoning skills and their working with an elementary student on a spatial reasoning task.

Table 3.3 – Continued

5. Phone interview	This is an opportunity for the researcher to clarify and validate data	This activity will provide insight into what preservice teachers write about in written reflections related to their spatial reasoning skills and their working with an elementary student on a spatial reasoning task.
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In their methods course, preservice teachers were given a spatial reasoning task to complete called Math Matters Tile Assignment (see Appendix A) which was adapted from the problem entitled, Shapes from Four Triangles (Sowder, Sowder, & Nickerson, 2008). I chose this spatial reasoning task because the solving of the task promotes triangle transformations (rotating, sliding, and reflecting) in order to generate all the solutions to the task. The completion of this task gave me the opportunity to gain insight into the spatial reasoning skills of individual preservice teachers which is particularly important since spatial reasoning relates to how well someone will be able to analyze new and potentially abstract information and then apply that information appropriately (Peirce, n.d.). In this way, it is directly related to problem solving and critical thinking, two major components of cognitive aptitude which refers to a person's ability to process information, learn new skills, and apply new information to solve problems. It holds individuals accountable for performing under "challenging conditions that demand strategic reasoning, insightfulness, perseverance, creativity, and craftsmanship to resolve a complex problem" (Costa & Kallick, 2000, p. 1).

Once each preservice teacher completed the Math Matters Tile Assignment, he/she teamed with their practicum partner and created a plan to adapt the task for use with an elementary student (see Appendix A, part 2). This plan involved adaptations to

the original task as well as generating questions to ask the elementary student in order to enhance their overall experience and provide insight into their spatial thinking as they worked to create solutions to the task. The intention of this part of the study was for the preservice teacher team to enact their adapted plan in order to gain the understanding of real students who have real ideas and how to adapt in the moment while working with a student in Grades K-5.

Following the enactment of the plan, the preservice teacher team journaled about their experience with the elementary student using the questions provided (see Appendix A, part 3) concerning: the prior knowledge of the elementary student, the mathematical understanding of the preservice teacher, strategies used by the elementary student when solving the task, questions asked by the preservice teacher during the enactment of the task, and summarizing the demonstrated spatial reasoning of the elementary student.

After coding the initial data (preservice teachers individual work on the problem, the adapted plan created by the preservice teacher team, the enactment of the plan with the elementary student, and the preservice teachers' reflections), seven preservice teachers were selected and interviewed about their entire experience as a way to clarify and validate generated codes. The preservice teachers were selected based on their individual work with the spatial reasoning task (i.e., did they have the correct number of solutions or were some repeated followed by how they generated their solutions using a base shape, guess and check, or rotating triangles which helped clarify their thinking when attempting to solve the task), the assessment of their work using the van Hiele levels (i.e., Level 1 or Level 2), their adaptations to the original task (i.e., was their practicum partner selected or not as I did not want to interview both members of the same

team), the grade level of their elementary student as I wanted various grades represented, and what types of probing questions they generated to ask their elementary student (i.e., those whose questions were mainly in three categories: linking and applying, extending thinking, and probing – getting students to explain their thinking).

The following interview questions (see Appendix C) were asked of the selected preservice teachers during their phone interview:

- Identify a rewarding time and a challenging time when working with your elementary student,
- Explain a time when you had to adapt in the moment,
- Rate your own comfort level with spatial reasoning, and
- Spatial reasoning as an important aspect of being a good elementary teacher.

These questions were asked in order to have the preservice teachers explain two main experiences (a rewarding time and a challenging time) they encountered when working with their elementary student which lead into an explanation by the preservice teacher when they adapted in the moment as their elementary student worked to solve the adapted spatial reasoning task, therefore, helping to answer the research question concerning planning and enacting the spatial reasoning task with an elementary student. I also wanted to know the preservice teachers' thoughts about their own comfort level regarding spatial reasoning as I was trying to determine their level of anxiety about the spatial reasoning task, therefore, helping to answer the research question about what preservice teachers do when engaging with a spatial reasoning task. Thinking a bit more globally, I asked the question about spatial reasoning and being a good elementary teacher to determine if this skill set would be beneficial for elementary teachers to possess therefore



helping to answer the research question about their working with an elementary student on a spatial reasoning task and their reflection about their own spatial reasoning skills.

### The Timeline

The following is the timeline (see Table 3.4) I followed when collecting data on this research project.

Table 3.4

#### *Timeline of Data Collection*

Month	Data Collected
November	Completed Math Matters Tile Assignment by individual preservice teachers
	Adaptations of the Math Matters Tile Assignment generated by preservice teacher teams
	Enacted adapted plan of the Math Matters Tile Assignment by preservice teacher teams with an elementary student
December	Written reflections by preservice teacher teams
	Phone interviews from selected individual preservice teachers

### Data Collection and Analysis

Table 3.5 indicates the research steps for this study, their connection to a research question, the data I collected, and the analysis I performed for each.

Table 3.5

#### *Summary of Research Steps with Data Collected*

Research Steps	Connection to Research Question	Data Collected	Analysis
1. Preservice teachers complete the Math Matters Tile Assignment	The spatial reasoning level of a preservice teacher will influence what they do when engaging in a task that requires spatial reasoning.	Solutions generated	<ul style="list-style-type: none"> <li>Counted total number of solutions and any repeated designs</li> <li>Open coded the data looking for common solution strategies and assumptions</li> <li>van Hiele Levels of Geometric Thought of the work done by the preservice teachers</li> </ul>
		Assumptions about the task	
		Solution approaches	
		Reasoning related to finding all solutions	

Table 3.5 – Continued

2. Preservice teacher teams adapted the Math Matters Tile Assignment for their teaching experience	This activity will provide insight into what preservice teachers do when adapting the task.	Written task adaptations	<ul style="list-style-type: none"> <li>• Open coded the data looking for common adaptations</li> <li>• Boaler and Brodie (2004) questioning framework</li> </ul>
		Probing questions generated by the preservice teacher teams and asked while the elementary student was working the adapted task	
3. Preservice teacher teams enact plan with an elementary student	This activity will provide insight into the enactment of the preservice teachers adapted spatial reasoning task with an elementary student.	Written documents – notes, questions asked with generated responses, journal entries	<ul style="list-style-type: none"> <li>• Open coded the data looking for common themes within the documents</li> </ul>
4. Preservice teachers record written reflections of teaching experience	This activity will provide insight into what preservice teachers write about in written reflections related to their spatial reasoning skills and their working with an elementary student on a spatial reasoning task.	Written reflections within their journals	<ul style="list-style-type: none"> <li>• Open coded the data looking for common themes within the written reflections</li> <li>• van Hiele Levels of Geometric Thought assessed by the preservice teachers from the work of the elementary student</li> </ul>
5. Phone interview	This activity will provide insight into what preservice teachers write about in written reflections related to their spatial reasoning skills and their working with an elementary student on a spatial reasoning task.	Transcripts of audio recordings	<ul style="list-style-type: none"> <li>• Open coded the data looking for supporting data of the previous generated themes</li> </ul>

Table 3.5 provides a summary of the research steps, their connection to a research question, the data I collected, and the analysis I performed. In the next section, I describe

the data that I collected. Each of the research steps where data was collected will be looked at more closely in the sections that follow.

**Data Collection.** In this section, I describe each of the pieces of data I collected.

*Math Matters Tile Assignment.* In order to begin to understand the experiences of the preservice teacher participants around a spatial reasoning task, they were asked to individually complete the Math Matters Tile Assignment (see Appendix A). They uploaded their completed work to Canvas, and I retrieved it from there. They had to think critically about how to find all the possible ways four triangles can be arranged in order to have a complete solution. By generating their solutions, preservice teachers were problem solving about a spatial reasoning task. Within the explanations of their solutions they submitted to Canvas, preservice teachers used various approaches to solving the task as well as expressed written assumptions concerning how they generated their solutions which alluded to their critical thinking concerning the task.

Along with their approaches and assumptions, preservice teachers presented reasons as to why they had found all the solutions possible to the spatial reasoning task. The above data generated from the preservice teachers' solutions are indicators considered when using the van Hiele Levels of Geometric Thought (van Hiele, 1984a). The work of a preservice teacher at Level 0 used combinations of triangles as solutions instead of a base shape to build from. Their work would also show that they did not find all 14 solutions to the task and have repeated designs as solutions. The work of a preservice teacher at Level 1 used guess and check as their main way to find all solutions to the spatial reasoning task and potentially not have repeated solutions. Their work also assumed visually what kind of triangle they were working with (e.g., isosceles triangles).

The work of a preservice teacher at Level 2 used a base shape (square, parallelogram, and a larger triangle) in helping them find all the solutions to the task. Their work included finding all 14 solutions and not have repeated solutions.

It is assumed that the work of these preservice teachers participating in the study is not at Level 3 or Level 4 based on these levels being reserved for formal proofs as seen in high school geometry and the advanced study of foundational geometry at the college level. This framework provided foundational geometric understanding which is vital to assessing ones grasp of spatial reasoning as it describes categories of thinking one may see in any elementary or secondary classroom, and for planning future instruction. The van Hiele levels are hierarchical meaning that the thinking at a higher level of the model involves experiences and thinking at the lower levels as well.

***Adapted Math Matters Tile Assignment.*** After the preservice teachers individually worked through the spatial reasoning task, they worked with their practicum partner and created an adapted plan of the original task to use with their elementary student. Through this interaction, preservice teacher teams were better prepared to generate a version of the spatial reasoning task that they would use with an elementary student. Since they both had already solved the task, they would discuss struggles they themselves had and how they could better help their elementary student if the same struggles arose during the enactment of the task. This process resembles Vygotsky's intersubjectivity work and is a key step in constructivist learning situations. Their adaptations included varying the context of the problem, providing triangles for the student to use when solving the problem, using color to help clarify how the triangles fit

together and alter the number of triangles used for the task depending on the age of the elementary student.

Another part of adapting the task required the preservice teachers to create seven or eight questions that would probe their elementary student's thinking while they were engaged in the task (see Appendix A, part 2). These questions helped them in understanding their student's thinking about the spatial reasoning task and not to teach or correct their student's reasoning or answers. These questions were analyzed based on the framework of Boaler & Brodie's nine categories of teacher questions (see Table 3.6) derived from analyzing teaching (2004). "The questions the teacher asks in the classroom, play a critical role in developing mathematical conversations and thinking" (Ulleberg & Solem, 2018). The bulk of the generated questions were placed in three categories: linked and applying, extending thinking, and probing – getting students to explain their thinking since part of the original assignment asked the preservice teachers to write questions that would probe their student's thinking as a way to enhance their spatial reasoning experience.

Table 3.6

*Teacher Question Types with Description and Examples*

<b>Question Type</b>	<b>Description</b>	<b>Examples (Boaler &amp; Brodie, 2004)</b>
Linking and applying	Points to relationships among mathematical ideas and mathematics and other areas of study/life	<ul style="list-style-type: none"> <li>▪ In what other situations could you apply this?</li> <li>▪ Where else have we used this?</li> </ul>
Extending thinking	Extends the situation under discussion to other situations where similar ideas may be used	<ul style="list-style-type: none"> <li>▪ Would this work with other numbers?</li> </ul>
Probing – getting students to explain their thinking	Asks student to articulate, elaborate, or clarify ideas	<ul style="list-style-type: none"> <li>▪ How did you get 10?</li> <li>▪ Can you explain your idea?</li> </ul>

Table 3.6 – Continued

Exploring mathematical meanings and/or relationships	Points to underlying mathematical relationships & meanings	<ul style="list-style-type: none"> <li>▪ Where is this <math>x</math> on the diagram?</li> <li>▪ What does probability mean?</li> </ul>
	Makes links between mathematical ideas and representations	
Gathering information, leading students through a method	Requires immediate answer	<ul style="list-style-type: none"> <li>▪ What is the value of <math>x</math> in this equation?</li> <li>▪ How would you plot that point?</li> </ul>
	Rehearses known facts/procedures	
	Enables students to state facts/procedures	
Generating discussion	Solicits contributions from other members of class	<ul style="list-style-type: none"> <li>▪ Is there another opinion about this?</li> <li>▪ What did you say, Justin?</li> </ul>
Orienting and focusing	Helps to focus on key elements or aspects of the situation in order to enable problem solving	<ul style="list-style-type: none"> <li>▪ What is the problem asking you?</li> <li>▪ What is important about this?</li> </ul>
Inserting terminology	Once ideas are under discussion, enables correct mathematical language to be used to talk about them	<ul style="list-style-type: none"> <li>▪ What is this called?</li> <li>▪ How would we write this correctly?</li> </ul>
Establishing context	Talks about issues outside of math in order to enable links to be made with mathematics	<ul style="list-style-type: none"> <li>▪ What is the lottery?</li> <li>▪ How old do you have to be to play the lottery?</li> </ul>

**Written reflections.** Table 3.7 shows the items included in the written reflections of the preservice teachers and why these are necessary to include. These reflections revealed insights as to how the preservice teachers' spatial reasoning skills relate to their work with their elementary student.

Table 3.7

*Items Included in the Preservice Teachers Written Reflections*

Written Reflections	Why Included
Details of their (preservice teacher) encounter with an elementary student working through the adapted plan	To generate a "play by play" dialogue while the elementary student was engaging in generating solutions to the spatial reasoning task
Responses by the elementary student to the asked predetermined questions	To determine if the predetermined questions helped create a rich learning experience for the elementary student

Table 3.7 – Continued

Preservice teachers in the moment adaptations based on responses, reactions, and expressed thoughts of the elementary student	To determine accommodations and scaffolding that happened while the elementary student engaged in the spatial reasoning task
Prior knowledge by the elementary student concerning the spatial reasoning task	To determine what elementary students know about transformations of shapes (reflections, rotations, and translations)
The mathematical understanding of the preservice teacher	To make a connection between the mathematical knowledge for teaching preservice teachers should possess and how it relates to spatial reasoning
Strategies used by the elementary student when solving the task	To determine if they use guess and check or if they use a base shape (square, parallelogram, or larger triangle)
Preservice teachers analyzing their elementary students work using the van Hiele Levels of Geometric Thought framework	To determine if there is a connection between the van Hiele level of the preservice teacher and the van Hiele level of the elementary student

The characteristics of shapes (mainly a triangle) and how to arrange each of them was a key component of the prior knowledge used by both the preservice teacher and the elementary student. The idea that shapes can be used to create larger, different shapes is also part of the prior knowledge needed to be successful with this spatial reasoning task. The tapped prior knowledge that this spatial reasoning task exposes included Piaget's notion of assimilation and accommodation as well as Vygotsky's beliefs on co-constructed thought for without previous experiences to build upon, limited success would be obtained.

The preservice teachers' mathematical understanding of the spatial reasoning task was significant when it came to working with their elementary student as they were better prepared to help them with the trouble spots since they (the preservice teachers) had already worked the task themselves. Along these same lines, the preservice teacher teams were able to provide a richer learning experience and come up with modifications

that they could use with their student since they had a deeper understanding of the mathematics within the task. This was evident in their individual work and where it was placed on the van Hiele Levels of Geometric Thought since this framework is arranged progressively according to geometrical experiences. Therefore, the more geometric and spatial reasoning experiences one has, the greater his/her level of achievement.

***Phone interviews.*** Seven preservice teachers were selected from the 14 contacted for a phone interview centered around their entire experience with the spatial reasoning task. The selected seven were those that responded to the initial email asking them to participate in an interview. A convenient, scheduled time for both the preservice teacher and me was agreed upon for the interview to take place. The phone interviews were conducted in order to clarify and validate or disprove collected coded data. These interviews were also audio recorded and transcribed for accuracy purposes as well as convenience of use. The preservice teachers were selected based on their individual work with the spatial reasoning task (i.e., did they have the correct number of solutions or were some repeated followed by how they generated their solutions using a base shape, guess and check, or rotating triangles which helped clarify their thinking when attempting to solve the task), the assessment of their work using the van Hiele levels (i.e., Level 1 or Level 2), their adaptations to the original task (i.e., was their practicum partner selected or not as I did not want to interview both members of the same team), the grade level of their elementary student as I wanted various grades represented, and what types of probing questions they generated to ask their elementary student (i.e., those whose questions were mainly in three categories: linking and applying, extending thinking, and probing – getting students to explain their thinking).



Each of the seven preservice teachers were asked the following questions (see Appendix C):

1. Identify a rewarding time when working with your elementary student,
2. Identify a challenging time when working with your elementary student,
3. Explain a time when you had to adapt in the moment,
4. Rate your own comfort level with spatial reasoning, and
5. Is spatial reasoning an important aspect of being a good elementary teacher.

By asking these five questions, collected information would either support or reject the written work each preservice teacher team submitted as data. The first and second questions were asked to help the preservice teacher recall two events which happened during the enactment of their task with their elementary student. Question three was asked to gain insight around Piaget's work in cognitive development through accommodations which the preservice teacher team prepared for within their adapted task. By asking question four, I was determined to find how Vygotsky's work on intersubjectivity as well as the zone of proximal development was evident in each individual preservice teacher as well as their teaching team. I asked question five to define what qualities preservice teachers believe elementary teachers should possess.

**Data Analysis.** In this section I describe how I analyzed the data I collected. When analyzing my collected data, one focus point was on the individual preservice teachers and each of their overall work with the spatial reasoning task. I analyzed this data using the van Hiele Levels of Geometric Thought. I also determined what assumptions were made concerning the type of triangle they think they used to solve the spatial reasoning task based on what written evidence each preservice teacher provided in

their solutions to the task. The ways in which the preservice teacher solved the spatial reasoning task is another key point within the data collected. Specifically, I analyzed the approaches the preservice teachers used when creating their solutions to the task which include: using two triangles to create a base shape in which the other two triangles are manipulated around to find a solution, transformations (rotating, sliding, and reflecting) applied to one or more of the four triangles when creating a solution, and different methods preservice teachers use when creating solutions (e.g., guess and check, pattern making, the use of manipulatives, and comparing created shapes to real world objects).

Another focal point of my data analysis was the adapted plan the preservice teachers created of the spatial reasoning task and used with an elementary student. Precisely, I looked at the ways the preservice teachers adjust the original task (e.g., using less triangles, color coding the sides of the triangle which can be paired together with another triangle, and coloring each complete triangle a different color) in order to help their elementary student be successful with the spatial reasoning task. Besides analyzing the adaptations to the original task made by preservice teachers, I also took a close look at the list of questions generated by the preservice teachers using the framework provided by Boaler and Brodie (2004) as a way to categorize them.

My final point of emphasis within the collected data was the enactment of the spatial reasoning task by an elementary student. I used the van Hiele Levels of Geometric Thought to determine the placement of the work of elementary students as they generated solutions to the spatial reasoning task. Other important points I looked at within the data set included: the use of manipulatives by the elementary student when solving the task, the comparison of real-world objects made by the elementary student

when generating solutions, and the making of “common” shapes (e.g., squares, parallelograms, and larger triangles) by the elementary student with two triangles as they worked on the spatial reasoning task.

*Analysis of the Math Matters Tile Assignment.* Geometric thinking played a significant role in the development of spatial reasoning, problem solving, and critical thinking. The van Hiele Levels of Geometric Thought (see Appendix B) were used as an assessment tool when working with students and preservice teachers (van Hiele, 1959/1985). Once an assessment is made, these levels can also be used as a vehicle to develop geometric thinking and spatial perception. I open coded the work done by the preservice teachers on the Math Matters Tile Assignment. First, I analyzed the number of solutions each preservice teacher found and cross-checked those solutions to see if any of their final solutions were replicas of designs they had already indicated were part of their solution set which signaled a concern with the preservice teachers ability to see transformations and orientations with movement/placement of the four triangles within a created solution.

Next, I considered and open coded any assumptions indicated by the preservice teachers within their written and submitted document concerning the type of triangle used and the different orientations of those triangles. These open codes were compared with the codes from the number of solutions as a way to compare, condense, and begin to determine any patterns/themes concerning the preservice teacher’s work with the spatial reasoning task.

Following this cross-comparison of determined codes, I open coded the approaches the preservice teachers used when solving the task. Approaches ranged from

making a base shape (a square, parallelogram or larger triangle) with two of the triangles and moving the other triangles around it when generating solutions to guess and check (making a shape and then checking to see if it was already created). Some preservice teachers specifically mentioned using manipulatives to generate their solutions where others did not. The open codes created from this part of the data were compared with all the previous codes to help solidify patterns/themes already generated or provided more variation to be considered.

Finally, I open coded the preservice teacher's reasoning, which they explained within their write up of their completion of the task related to their finding of all the solutions to the spatial reasoning task. Preservice teachers were quick to mention they had met the requirements of the spatial reasoning task when stating they had found all the solutions. Others wrote about exhausting all the possible rotations of each triangle and any other design they created would mimic solutions they had already made. These codes were compared to all of the previous codes generated concerning each individual preservice teacher and how he/she completed the spatial reasoning task which determined common patterns/themes among the preservice teachers. It should be noted that a few questions (see Appendix A, part 1 questions 2b and 2c) of the Math Matters Tile Assignment were not included in the data analysis since these questions would not help the researcher gain a better understanding of the preservice teachers spatial reasoning.

The collection of open codes from the individual written work of each preservice teacher as well as the patterns/themes developed from these open codes were used when determining which of the van Hiele levels the work of each preservice teacher was

placed. The van Hiele Levels of Geometric Thought described the ways that preservice teachers' reason about shapes and other geometric ideas.

*Analysis of the adaptations to the written task.* Once the coding and cross-checking of each preservice teacher's individual work on the Math Matters Tile Assignment was complete, I coded the adapted plan that each preservice teacher team created as well as the types of questions they generated to use with an elementary student. Taking into consideration each preservice teacher's van Hiele level, I open coded the adapted task created by the preservice teacher team, specifically analyzing the overall adaptations that were present in the plans the preservice teachers created in order to see the adaptations present as well as if specific grade level modifications were accounted for. Adaptations to the number of triangles used, the setting of the problem and/or its elimination, defining side lengths and their ways of fitting together, using manipulatives, the use of color, and the original question on the Math Matters Tile Assignment were included in the collected data. Throughout each level of analysis, all codes and patterns/themes were compared between preservice teacher teams.

Another part of the adapted plan involved the preservice teacher team creating seven or eight probing questions which exposed their elementary student's thinking while they were engaging in the adapted task. These questions were analyzed using Boaler & Brodie's nine categories of teacher questions (see Appendix D) derived from analyzing teaching (2004). I determined the frequency with which the preservice teachers used the nine different types of questions when preparing questions which probed their elementary student's thinking while they were engaged with the adapted task. The types of questions teachers ask during a class discussion, within group work, and/or while working one-on-

one with a student creates or restricts the opportunities for students to deepen their mathematical understanding. These new codes were kept as a collective unit and used when making inferences about the preservice teachers' van Hiele levels and how this added to or hindered their ability to adapt the spatial reasoning task in order to create a rich mathematical learning experience for their elementary student. By looking across the codes generated by the Math Matters Task and the new codes from the adaptations of the written task, patterns/themes were evident and used to help define current codes or create new points to consider.

*Analysis of written reflections.* On the Math Matters Tile Assignment, preservice teachers were asked to write responses to questions (see Appendix A, part 3) concerning: the prior knowledge of the elementary student, their mathematical understanding, strategies used by the elementary student when solving the task, and questions asked by the preservice teacher during the enactment of the task to gain insight as to how the preservice teachers spatial reasoning skills related to their work with their elementary student. Preservice teacher teams collectively wrote responses to each of the reflection questions. These reflections were analyzed using open coding and cross-checked with each other to determine patterns/themes. The codes that surfaced from the reflection questions about the prior knowledge of the elementary students involved the characteristics of the shapes and the different shapes that can be created when combining shapes were compared with those codes found within the data from the strategies used by the elementary student when solving the task regarding building shapes based on recognized shapes like squares and rectangles, building a shape and then checking to see if it is a new design (guess and check), and rotating triangles when creating a new shape.

These combined codes were cross checked with the preservice teachers' codes concerning strategies used and assumptions when solving the task to see if any patterns/themes emerged.

The codes generated from the reflection questions concerning the mathematical understanding of the preservice teacher about the spatial reasoning task and the probing questions they generated were compared and combined with all of the formed codes created from the preservice teacher and their work with the Math Matters Tile Assignment. This combining of codes helped to solidify the patterns/themes that emerged concerning a rich learning experience for the elementary student based on the depth of the mathematical understanding of the spatial reasoning task by the preservice teacher.

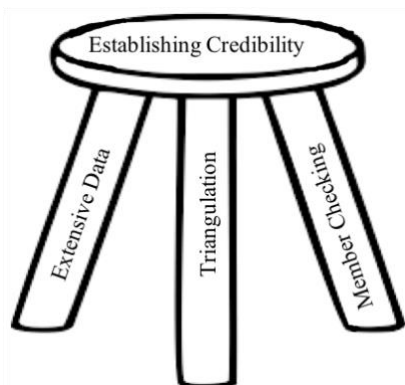
Finally, the data from the preservice teacher team analyzing their elementary student's work based on the van Hiele Levels of Geometric Thought was cross-checked with grade level and assumed experiences due to age. Then the elementary students van Hiele level was analyzed and compared with the van Hiele level of the preservice teachers who were working with that student to see if there were any patterns/themes which emerged.

***Analysis of phone interviews.*** Checking the findings with the case study participants can be a valuable part of the analysis and can enhance validity (Hartley, 2004). Each of the seven selected preservice teachers were asked the following questions (see Appendix C): identify a rewarding time and a challenging time when working with your elementary student, explain a time when you had to adapt in the moment, rate your own comfort level with spatial reasoning, and is spatial reasoning an important aspect of

being a good elementary teacher. Each of their responses was open coded and compared to the codes/categories already established from the collected data to help clarify patterns/themes or expound on unique codes within the data set. In the end, “the ultimate goal of the case study is to uncover patterns, determine meanings, construct conclusions, and build theory” (Patton & Appelbaum, 2003, p. 67).

### **Establishing Credibility**

Shenton (2004) explains that to establish credibility is to “seek to ensure that their study measures or tests what is actually intended” (p. 64). Using the analogy of a three-legged stool, all three legs must be present, the same length, and structurally sound for the stool to work properly and be safe for use. In the same way, I described the ways in which an equal balance of extensive data, triangulation, and member checking supported and established credibility for this study (see Figure 3.1).



*Figure 3.1.* Establishing Credibility with a Three-Legged Stool. (Rygle, n.d.). Retrieved from <https://www.1001freedownloads.com/free-clipart/three-legged-stool-outline>.

The first leg of the stool is extensive data. Extensive data refers to the various comprehensive types of data I collected for this study which included: documents collected from the preservice teachers concerning their individual work on the spatial reasoning task, their adaptations to the original task, their probing questions they planned



to ask their elementary student, their documentation from enacting the adapted task with an elementary student, their written reflections, and their responses to the phone interview questions.

The second leg of the stool is triangulation. Triangulation is a powerful technique that facilitates the validation of data through cross verification from two or more sources (Bogdan & Biklen, 2006). By using multiple methods of data collection, triangulation was employed through the data analysis of the phone interviews, preservice teachers' observations of the elementary student working the adapted task and documents which include both the preservice teachers solutions to the task as well as the adapted task created by the preservice teacher teams used when working with their elementary student (i.e., what was said in the interview can be checked against what was observed during the enactment of the task and/or what was included in the documents from the preservice teachers relevant to solving the spatial reasoning task). Another example of triangulation uses data collected from the reflections of the preservice teachers (research question #3), the data collected from the solutions to the spatial reasoning task generated by the preservice teachers (research question #1), and the data collected from the adapted and enacted task (research question #2) were compared and cross-checked generating patterns/themes which emerged from the data.

The third leg of the stool is member checking. Member checking involves soliciting feedback on one's findings from some of the preservice teachers that were part of the study. "This is the single most important way of ruling out the possibility of misinterpreting the meaning of what participants say and do and the perspective they have on what is going on, as well as being an important way of identifying your own

biases and misunderstandings of what you observed” (Maxwell, 2005, p. 111). By analyzing the written reflections the preservice teachers turned in and then asking questions concerning information/themes that surfaced from analyzing that data, I was able to take these individual findings back to the individual preservice teacher that I interviewed in order to correctly state what was written and spoken.

Therefore within the various data pieces collected, the information/themes that surfaced both strengthened and increased the credibility of my study by complementing one another, shedding light on unexpected findings (Denzin & Lincoln, 1998), or provided disconfirming evidence to better understand and define limitations (Cohen & Crabtree, 2006).

### **Transferability or Naturalistic Generalization**

Lincoln and Guba (1985) point out that it is “not the naturalist’s task to provide an index of transferability; it is his/her responsibility to provide the database that makes transferability judgments possible on the part of potential appliers” (p. 316). Stake (1995) agrees, suggesting that generalization is not the purpose of the case study at all. He prefers the term “particularization.” He favors this term because the purpose of the case study is not to compare multiple cases, but to become intimately aware of the inner workings of a particular case. He suggests that “there is an emphasis on uniqueness, and that implies knowledge of others that the case is different from, but the first emphases is on understanding the case itself” (p. 8). In addition, Stake (2005) proposes that if any generalization is appropriate for qualitative research, it is “naturalistic generalization.” Such generalizations are formed by the readers as the case is unveiled for them. Hence, the purpose of this study is not to define findings that may be transferable to other tasks,

but rather to examine the spatial reasoning skills that preservice elementary teachers demonstrated and how their spatial reasoning skills were used in the enactment of the tasks of teaching.

### **Research Permission and Ethical Considerations**

The University Institutional Review Board (IRB) approval was sought and granted for this study. Since the preservice teachers were over the age of nineteen, they were asked to participate and given a consent form to sign. Their signature signaled their agreement to participate as well as their notification of their rights concerning their participation in this study. Another part of the requirements of IRB was to receive approval from the local school district's IRB committee, which was also sought and granted. Since the adapted spatial reasoning task was given to elementary students, parental informed consent documents, as well as student assent documents, were sent home with students in the participating classrooms. Due to the limited return of signed documents both by parents as well as elementary students, their direct work is not included in this research study.

Individual preservice teachers' names, as well as the names of the elementary students, were not identified during this study. Any names used within this study are pseudonyms which were used in order to conceal the true identities of all participants. All data collected from the preservice teachers, the phone interviews and their transcripts as well as computerized information was kept on an external, password protected device.

## CHAPTER 4: RESULTS OF STUDY

In this chapter, I present the results from my study. First, I describe what the individual preservice teachers did when engaging in a spatial reasoning task. Second, I describe what preservice teachers did when planning for and enacting the spatial reasoning task with elementary students. Finally, I describe the written reflections of the preservice teachers related to their spatial reasoning skills and their working with an elementary student on a spatial reasoning task.

### **Preservice Teachers and the Task**

In this section, I describe the results related to research question one, what do preservice elementary teachers do when engaging in a task that requires spatial reasoning? This involves the overall solution to the task, preservice teachers' solutions and assumptions about the task, the approaches taken by the preservice teachers when solving the task, their reasoning related to finding all the solutions to the task and the van Hiele Levels of Geometric Thoughts concerning the work provided by the preservice teachers. Pseudonyms were used throughout this study as I share work and reflections from the preservice teachers in order to conceal their true identities.

**Solution to the Spatial Reasoning Task.** In the task, I asked preservice teachers to find all the possible different polygonal regions given four unique triangular tile pieces. There were 14 possible solutions to the given problem. Figure 4.1 displays all the possible solutions.

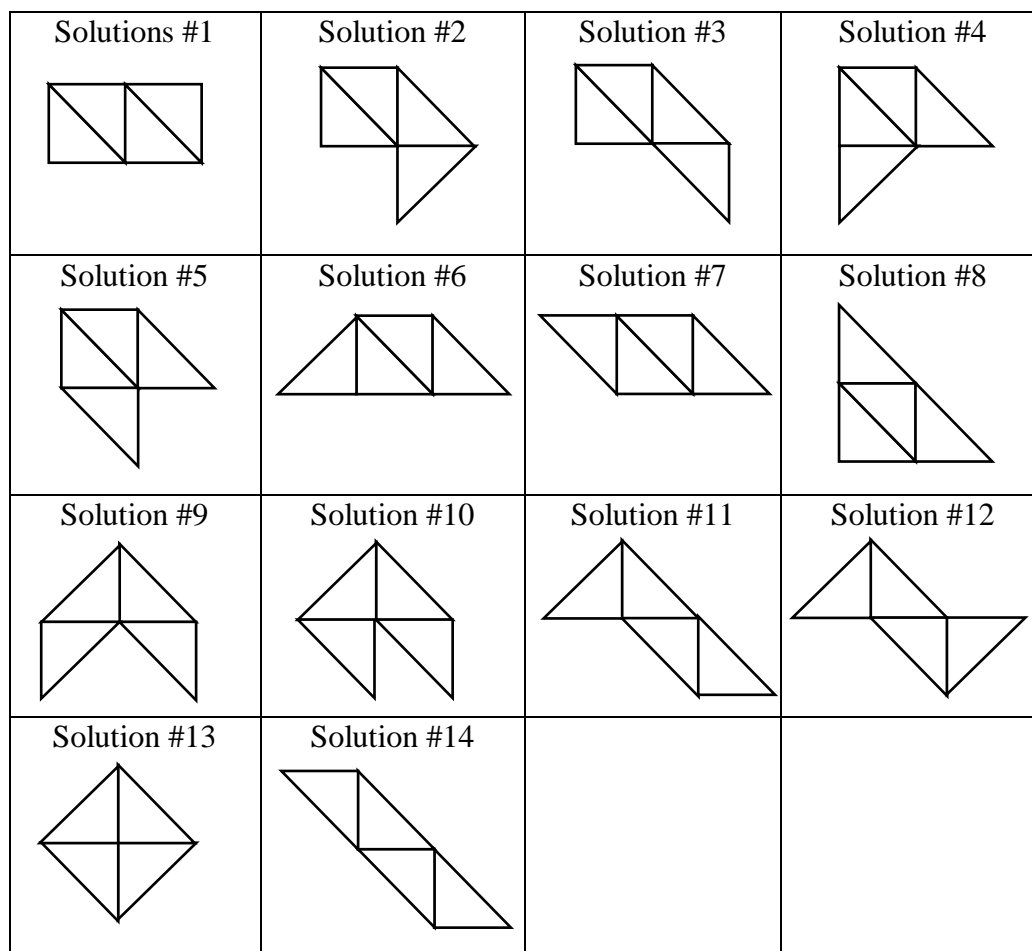


Figure 4.1. Solutions to Math Matters Assessment Task

**Solutions Generated by Preservice Teachers.** The 32 preservice teacher participants each submitted their own work for the spatial reasoning task. The number of solutions found by the preservice teachers falls in a range from five to 36. It should be noted that some preservice teachers were not successful in finding all of the solutions, and others had solutions that were repeated. In the next section, I first describe the assumptions made by preservice teachers about the task. Then, I describe the solution processes.

**Preservice Teachers' Assumptions about the Task.** As described earlier, two preservice teachers found more solutions due to repetition, meaning the preservice teacher assumed that identical polygonal regions which were generated by placing

triangles in different orientations were new solutions (see Figure 4.2). In this instance, the preservice teacher, Kacee, saw two triangles were placed in different orientations within the region and determined that these were two different regions. Consequently, she counted each as a separate solution. She stated,

Although it was not specifically stated, I assumed that the triangle-shaped tiles were isosceles, based on the picture. I also assumed that every shape which contained a different orientation of one or more triangles counted, even if the outline of the shape was the same as another (Kacee, personal communication, November 21, 2017).



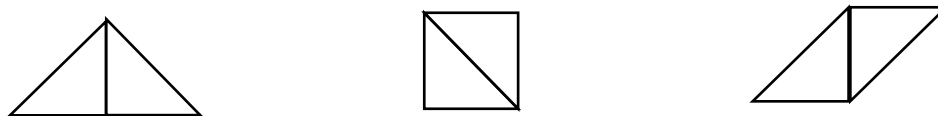
*Figure 4.2. Identical Polygonal Regions that a Preservice Teacher Counted as Two Separate Polygonal Regions*

Another common assumption made by preservice teachers was related to the type of triangle they were given in the original task to use. Seven preservice teachers, including Kacee, assumed that the given triangle was isosceles. This is problematic since many students learn very fast to attend to the visual ‘clues,’ even when these clues are irrelevant or non-reliable; they tend to base their inferences regarding whether two triangles are congruent on how the triangles look, instead of relying on logical inferences (Zodik & Zaslavsky, 2007, p. 269).

In the next section, I describe preservice teachers’ solution approaches.

**Preservice Teachers’ Solution Approaches.** Preservice teachers approached the solving of the task in various ways. Since a triangle by definition is a polygon, preservice teachers needed to find what additional polygons the given triangles could form. The

most commonly used approach by preservice teachers was to create a base shape. Fifteen preservice teachers built their solutions from constructing geometric base shapes (see Figure 4.3).



*Figure 4.3.* Geometric Base Shapes with Two Triangles

For example, Heather describes her approach as follows,

When manipulating the triangles, I was able to see that two triangles combined makes a bigger triangle region. I was also able to see that two triangles form a square with four equal sides. And finally, I was able to see that two triangles form a parallelogram (personal communication, November 21, 2017).

After constructing these base shapes, preservice teachers were able to begin seeing all the options they could generate by moving (e.g., rotating, sliding, and reflecting) the remaining triangles. To better understand these ideas, I will use examples from Mataya's work, which show the movement of the fourth triangle when creating a new polygonal region. Mataya used three triangles to create her base shapes (see Figure 4.4) when generating her 14 solutions to the task. By using Mataya's work, one will have a better idea of how each of the options were generated.



*Figure 4.4.* Mataya's Base Shapes She Used to Create Her Solutions

First, I will show an example from Mataya's work of rotating one triangle to another location on a base shape. In Figure 4.5, Triangle 4 is rotated from one side of Triangle 3 to the other exposed side of Triangle 3, thus creating another polygonal region.

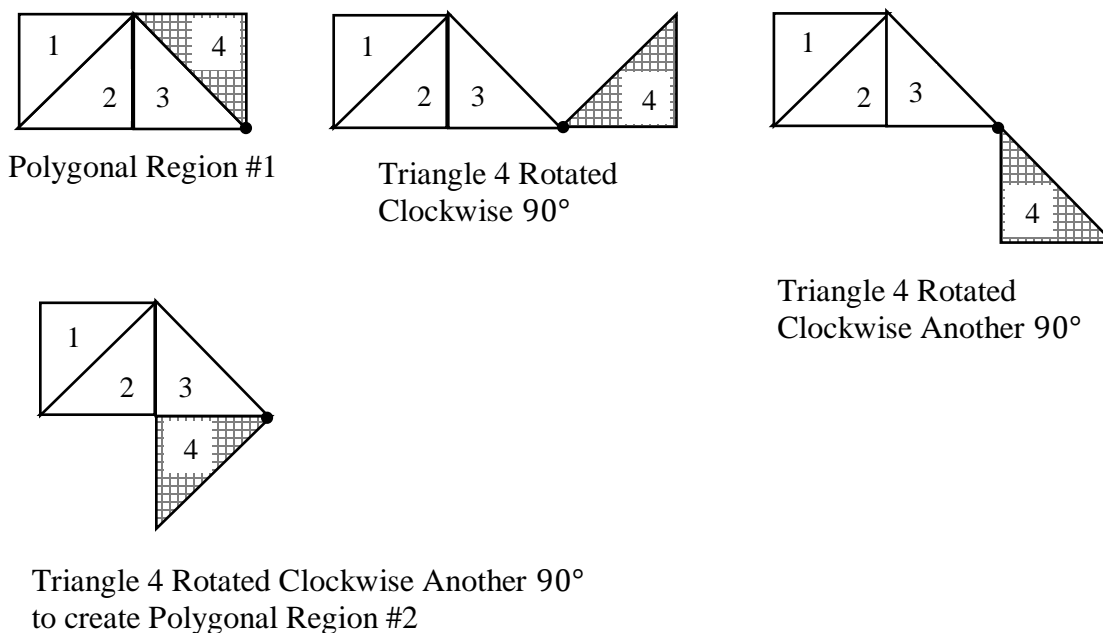


Figure 4.5. An Example of Rotating a Triangle

Second, I will show an example from Mataya's work of sliding one triangle to another location on a base shape. In Figure 4.6, Triangle 4 is slid from beneath Triangle 3 to beneath Triangle 2, thus creating another polygonal region.

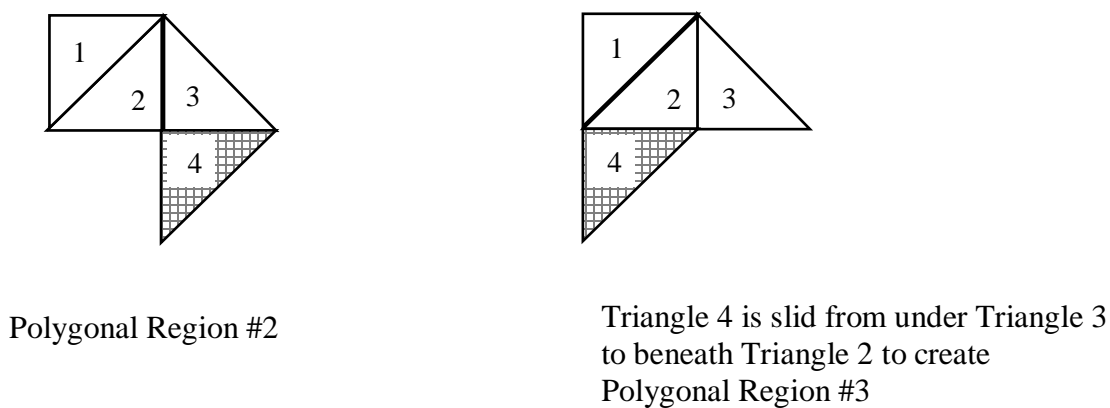
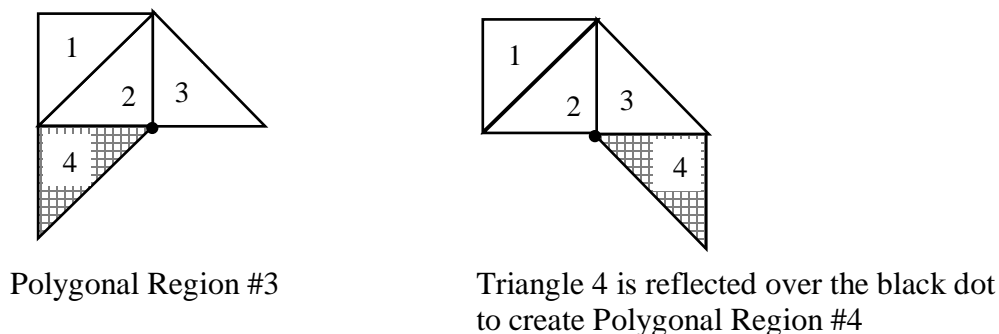


Figure 4.6. An Example of Sliding a Triangle



Third, I will show an example from Mataya's work of reflecting one triangle to another location on a base shape. In Figure 4.7, Triangle 4 is reflected over the point directly under the meeting point of Triangles 2 and 3 (the black dot), thus creating another polygonal region.



*Figure 4.7. An Example of Reflecting a Triangle*

Through Mataya's work, we have a better understanding of the movement made to triangles, which helped create each of the solutions to the task.

Guess and check was another approach identified in their solutions and used by three preservice teachers in order to find all the solutions.

When working with the four triangles, I was mainly using a guess and check method with different shapes, and regions I was making. I started by making simple shapes like a square, rectangle, and a rhombus. After finding those shapes, I changed them by moving different triangles around in a way that still had every piece of tile connected by at least one edge (Adelyn, personal communication, November 21, 2017).

Another preservice teacher made solutions and then compared their created shapes to real-world objects. For example, a preservice teacher made this comment about Figure 4.8 "this is a shape that I look at and think of something in the real world. It

reminds me of a rocket of some sort” (Tina, personal communication, November 21, 2017). If one rotates the same polygonal region counterclockwise 45 degrees, it will resemble the face of a fox (Mike, personal communication, November 21, 2017).



*Figure 4.8.* Polygonal Region that Resembles a Rocket (on the left) or a Fox (on the right)

In order to create these solutions, the original task included four triangles. Preservice teachers had a choice to cut them out (see Appendix A) and use them as manipulatives when creating their solutions but it was not suggested in any way that they had to use manipulatives to solve the spatial reasoning task. Four preservice teachers mentioned using manipulatives in their written solution. It is unclear if only these four preservice teachers used manipulatives or only these four explicitly wrote about it in their written solution. It might be that other preservice teachers used manipulatives but did not describe this when writing up their solution. In the next section, I describe the preservice teachers’ reasons as to why they found all the solutions to the task.

**Preservice Teachers’ Reasoning Related to Finding All Solutions.** When describing how they knew that they found all solutions, preservice teachers’ reasoning was quite similar, mentioning relocating the triangles to connect congruent side lengths and fulfilling the requirements of the task. For example, preservice teachers made comments similar to this one from Sally,

This solution is complete because we have exhausted all possible rotations as well as sides touching to create any sort of polygonal design. Any other designs we could potentially come up with may look different but would essentially be creating a shape we already made. Therefore, these 14 polygonal regions are completely unique and unlike (Sally, personal communication, November 21, 2017).

Erin, another preservice teacher, remarked

I know I found all of the 14 solutions because if I manipulated any of the polygons further, they would either mimic a polygon I had already found or not fit the guidelines of having two sides completely touch each other. I found all of the different ways that all of the different sides of the triangles could touch (personal communication, November 21, 2017).

Jill, a preservice teacher, commented

the solution to this is complete because in order to meet the specific requirements, where all four triangles must be used and at least 2 sides must be touching at all times, if you move any of the 4 triangles to attempt to find another polygonal region, there are no more unique shapes that can be made (personal communication, November 21, 2017).

Another preservice teacher, Sue, stated

I know I have found all of the possible solutions because I started with four different combinations of three triangles and moved the fourth triangle around the starting design. I positioned the fourth triangle on different places on the starting patterns to make a new design. Even though some of them have the same starting

pattern, the fourth triangle is always positioned differently, which makes the polygonal regions all different. I also know they are all different because when they are rotated or reflected, they do not match any of the other designs (personal communication, November 21, 2017).

These four preservice teachers provided similar comments in their statements related to finding all the possible solutions to the task. They mentioned meeting the requirements of the task, which are: similar side lengths of the triangles need to be touching and designs need to be unique (e.g., not matching or mimicking other designs). All four of these preservice teachers stated that they found 14 solutions, but Sally had repeated solutions and really only found 11 solutions.

In the next section, I will discuss the van Hiele Levels of Geometric Thought concerning the work of the preservice teachers on the spatial reasoning task.

### **van Hiele Levels of the Preservice Teachers' Work on the Spatial Reasoning**

**Task.** The van Hiele model is a five-level hierarchy of understanding spatial ideas (van Hiele, 1984a). Table 4.1 lists the five levels of geometric thought.

Table 4.1

*van Hiele Levels of Geometric Thought (van Hiele, 1984a)*

<b>Level</b>	<b>Name</b>	<b>Descriptor</b>
0	Visualization	Shapes and what they “look like”
1	Analysis	Classes of shapes rather than individual shapes
2	Informal Deduction	Properties of shapes
3	Formal Deduction	Relationships between properties of geometric objects
4	Rigor	Deductive axiomatic systems for geometry

Each level describes the thinking processes used in geometric contexts. Specifically, the levels describe how learners think and what types of geometric ideas they think about as well as what they can do (Van de Walle, Karp, & Bay-Williams, 2016). I coded the

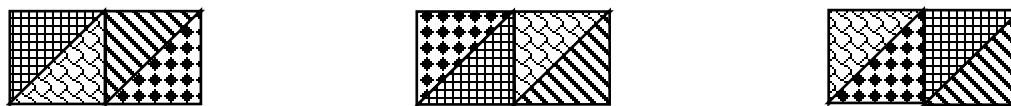
individual work each preservice teacher submitted using the van Hiele Levels of Geometric Thought as a framework and determined that their work was either at Level 0 ( $n = 1$ ), Level 1 ( $n = 16$ ), or Level 2 ( $n = 15$ ). At Level 0, appearance is dominant, which can blur students' thinking about the properties of a shape. They make decisions based on perception, not reasoning. For example, the only preservice teacher at Level 0 used the color (pattern) of the shape of each triangle to determine how many solutions she needed to find. The triangles provided in the original task were white, and Ana decided to use a different color (pattern) to represent each triangle. By doing this, she based her solutions off of the ways in which she could place all the colored (pattern) triangles together to create different polygonal regions.

By using a tree diagram approach, I found there are eight different polygonal shapes I can make with four triangles, along with 192 different colored (pattern filled) triangle combinations (see Figure 4.9). To get the 192 different color (pattern filled) combinations, you pick a start color (pattern) for one shape, and use all the colors (patterns) for the other shapes, exhausting all combinations (Ana, personal communication, November 21, 2017).



*Figure 4.9. Color Choices for the Shapes Represented by Different Patterns*

I determined that Ana used what the shapes looked like as a factor in how many solutions she came up with. By using the visual differences only, created by the arrangement of the colors (patterns) within the polygonal regions, I placed Ana's work at van Hiele Level 0 (see Figure 4.10).



*Figure 4.10.* Ana’s Confusion Related to Colored (pattern filled) Triangles

Based on the individual work provided by 16 preservice teachers, I coded their work on the van Hiele Levels of Geometric Thought at Level 1, which involves seeing figures as collections of properties. By using physical models and drawings of shapes, learners begin to see individual shapes as representative of classes of shapes. One preservice teacher, Katie, describes how she views classes of shapes as “each of the polygonal regions that I have created is different from each other because every ‘side’ connects to a different ‘side’ of a different triangle. No two designs are the same because they are all connected in a different way” (personal communication, November 21, 2017). Another preservice teacher in her explanation of finding all the possible solutions states

I know that each of these are different. In each different region, the triangles have different sides that are touching each other. I also know that I have found all the different possible ways that I can arrange these regions because all of the sides of the triangles have been used in at least one of the designs (Kari, personal communication, November 21, 2017).

This preservice teacher seems to be connecting ideas of triangles used to create regions which leads into designs. In the above quote, Kari is describing why she feels her solutions are all different based on the ways in which the sides of each triangle are rotated and placed next to the sides of another triangle.

Level 2 of the van Hiele levels of Geometric Thought focuses on analyzing the relationships between the properties of shapes. At this level, observations go beyond properties themselves and begin to focus on logical arguments about properties. Fifteen preservice teachers were found to be at this level since they were able to create meaningful definitions and give informal arguments to justify their reasoning. For example, Cole states,

At the start of this process, I created quite a few more shapes than the 14 I ended with. Looking back at it though, for a lot of those figures, I was just manipulating the inside angles and not actually changing the shape as a whole. I had to backtrack my thinking after I realized this because just manipulating the inside angles of say a square doesn't mean it is anything different than the previous square. Knowing this, I know all of my polygons are different because of the fact that their shapes are being manipulated by the outside angles instead of the interior angles (personal communication, November 21, 2017).

In this quote, Cole is describing how he has determined that some of the designs he created were repeats of the other designs he already made. He refers to “manipulating the inside angles of a square” and determining that the shape itself is still a square. Figure 4.11 is an example of “manipulating the inside angles of a square.” In this figure, we see that rotating a square clockwise 90-degrees changes where the 90-degree angle is split into two 45-degree angles but it does not change the fact that the shape is still a square.



*Figure 4.11. A Square Rotated Clockwise 90-degrees Does Not Change the Shape*

Another preservice teacher, Sue, explains her logical thinking process and how she started creating polygonal regions,

I know I have found all of the possible solutions because I started with four different combinations of three triangles and moved the fourth triangle around the starting design. I positioned the fourth triangle on different places on the starting patterns to make a new design (personal communication, November 21, 2017).

A third preservice teacher, Adelyn, mentions transformations she performed on the entire polygonal region she made,

When further looking into the polygonal regions made after my fourteen original solutions I made, I found that even though they may look different in that specific placement, they do make a same polygonal region that has previously been made.

I found this to be true by rotating, flipping, or mirroring the entire polygonal region then comparing it to my original fourteen polygonal regions then found that they did, in fact, match my original fourteen regions (Adelyn, personal communication, November 21, 2017).

The spatial reasoning work submitted by the preservice teachers indicates that their van Hiele Levels of Geometric Thought were either Level 0, Level 1 or Level 2.

The selected examples above demonstrate the differences among levels for the preservice teacher's work revealing, how the product of thought at one level becomes the object of thought at the next level (see Appendix B). The objects (ideas) must be created at one level so that relationships among these objects can become the focus of the next level (Van de Walle, Karp, & Bay-Williams, 2016).



### **Summary of Findings Related to the Preservice Teacher and the Solving of the Task**

In summary, the preservice teachers were able to successfully solve the task even though they generated different total numbers of possible solutions. Most preservice teachers created a base shape and used it as a way to find all the other possible solutions. A few preservice teachers used guess and check as well as comparing their solutions to real-world objects when using four triangles as a way to find all the solutions. There were a few assumptions about the task made by preservice teachers concerning the types of triangles that were part of the task as well as different orientations within the same polygonal region were considered unrelated even though the polygonal region was the same. The preservice teachers exhausted all possibilities as a reason to why they believed they had found all the solutions. Finally, about half of the work completed by the preservice teachers, categorized using the van Hiele Levels of Geometric Thought, was coded at Level 1 since they were seeing figures as a collection of properties. The other half of the work completed by preservice teachers was coded at Level 2 since they were analyzing the relationships between the properties of the shapes. The work completed by one preservice teacher was coded at Level 0 since her work focused on the appearance of the shapes.

### **Preservice Teachers Adapting the Written Task**

In this section, I will describe the results related to research question two about the preservice teachers' planning and adapting the task to use with an elementary student which includes: the task adaptations generated by the preservice teachers and the analysis of the probing questions created by the preservice teachers.

**Preservice Teachers' Written Task Adaptations and Student Enactment.** As part of the assessment assignment, preservice teachers planned how they were going to work with their elementary student. In what follows, I describe the adaptations preservice teachers used with their elementary student as well as I include relevant pieces during the enactment of the adapted task. Adaptions included by preservice teachers encompass using manipulatives (triangles are grouped as pairs or creating a base shape to build from), altering the original task (total number of triangles used or the setting/context of the problem), modifying the rules for generating solutions (color coding sides or coloring each triangle a different color), and difficulty recognizing new and different solutions based on what the elementary student has already created. These adaptations were typical within the data if variations to the original task were used with the elementary student by the preservice teacher teams.

Five preservice teacher teams referenced having manipulatives for their elementary students to use when solving the task as the “shapes will help us make a pattern” (Larry & LaVern, personal communication, November 21, 2017). All of the preservice teachers decided that leaving the original task as is would be overwhelming when trying to have their elementary student do the same task. With their practicum partner, the preservice teachers decided on various adaptations to the original task. The most significant adaptation mentioned by preservice teacher teams was in the total number of triangles the student would be using to solve the task.

Four preservice teacher teams had their elementary students work the task using three triangles instead of four as a way to lower the number of possible solutions the student would be able to find. “We think that using three triangles will be more of a

kindergarten student's understanding of spatial reasoning than using four triangles" (Marcia & Michelle, personal communication, November 21, 2017). One preservice teacher team decided that they would have their elementary student begin the task with four triangles and if their student begins to struggle when figuring things out, they would take one triangle away (Mike & Mataya, personal communication, November 21, 2017). This would alter the total number of polygonal regions a student would be able to create from 14 regions given four triangles down to four regions given three triangles (Marcia & Michelle, personal communication, November 21, 2017).

One kindergarten student was diligently working the problem with three triangles and saw that there was another triangle, so he asked if he could use it. The preservice teacher agrees, and he begins to find more solutions, but after making a few, he discovers that his next solutions are similar to ones that he already created. This illustrates that starting with three triangles and transitioning to four triangles opens up opportunities for students to find more solutions at first, but then they become "stuck" when the most obvious solutions are already generated. Preservice teachers noted that their elementary student rotated and/or flipped the triangles as a way to potentially find more solutions (Marcia & Michelle, personal communication, November 21, 2017). When creating shapes, some elementary students put the triangles together in pairs and then looked to join the two pairs together to make a polygonal region. These pairs made common shapes that the student knew like a square, a triangle, and a parallelogram.

Four preservice teacher teams who were working with students in Grades K-2 made adaptations to the setting. For example, one preservice teacher team adapted the task by giving their elementary student four triangles and asked him to "create different

patterns and shapes using all four triangles” (Kerri & Katie, personal communication, November 21, 2017). Once their student had found as many different shapes as he thought were possible, they had their student pick his favorite design not referencing the bathroom floor plan, which was part of the original task. Two preservice teacher teams decided rephrasing the setting of the problem would make the task more appropriate for their student (e.g., Larry and LaVern adjusted the setting of the problem to be “Mrs. Weber (the classroom teacher) needs your help creating shapes with these four triangles which will help us make a pattern for her bathroom. The only rule we have when making the shapes is the same sides must be touching with no overhang” (personal communication, November 21, 2017) which was more suitable for their first grade student) while one preservice teacher team decided not to include the setting at all (e.g., Sally and Sue decided to ask their student “how he can put the 4 triangles together differently. Leaving out the story about the bathroom will keep him from getting lost in all of the other information and help him to specifically focus on the triangles” (personal communication, November 21, 2017)).

In addition, four preservice teacher teams adapted the rules for generating the solutions. For example, one group who was working with a kindergarten student, slightly adjusted the rule concerning the side lengths of two triangles being the same length when being put together. They decided to “color code the sides, so that on each triangle, the two equal sides will have blue expo marker on them and the one longer side will have red marker on it” (Sally & Sue, personal communication, November 21, 2017). They were certain this color coding would help their student better understand which sides of the triangles can be put together. Another preservice teacher team decided to “tell our student

that when putting the triangles together, he can only put sides together that are the same and completely touch” (Terri & Tina, personal communication, November 21, 2017).

As each elementary student diligently worked to find solutions to the task, some struggled with identifying exactly when they made a new polygonal region or if the polygonal region was the same as one of the solutions they already made. A preservice teacher team adapted in the moment when they were enacting their adapted plan and posed the following question to their student, “Is the shape still the same if we were to turn it upside down?” the student responded, ‘no it would be a different shape.’ The student thought that these two shapes (the ones pointed out) were different shapes, although they were the same” (Marcia & Michelle, personal communication, November 21, 2017). A fourth grade student had trouble understanding that the polygon he just created may be the same polygon just flipped around. A fifth grade student also had a hard time understanding that one shape could be the same as a previous shape “when the triangles are placed a little differently. She made the same shape three different times without realizing it” (Adelyn & Anna, personal communication, November 21, 2017). These examples show how difficult it was for the elementary student to identify if a generated solution was similar to one which was already created.

Another preservice teacher team decided to color each of the four triangles a different color (pattern) as a way to show that each shape is unique even though they are all triangles. When the preservice teacher team was working with their 4<sup>th</sup> grade student, they noticed that their student made a rectangle out of the four triangles, which was a valid start to solving the task. Quickly these colors (patterns) got in the way of the student’s thinking because he made another rectangle and situated the colors (patterns) in

different locations within the rectangle thinking it was a new design he had found (see Figure 4.12). He justified the differences between the two shapes by saying that “the way the colors (patterns) were in a different order means the designs are different” (Terri & Tina, personal communication, November 21, 2017).



*Figure 4.12. The Same Polygonal Region Even Though the Colors (patterns) are not in the Same Location in Each Rectangle*

The problem posed in the original task was altered in a few different ways. One preservice teacher team had their elementary student keep two triangles in a square at all times and then proceeded to ask the student how many polygons they could make. Another preservice teacher team asked their student to create different patterns and shapes using all four of the given triangles. Further data analysis showed that one preservice teacher team included both the number of triangles and the original task limitations in their adaptation of the question, “Using all four triangle pieces, how many unique ways can you arrange the triangle pieces, so each edge of a triangle is touching another edge of a triangle?” (Adelyn & Anna, personal communication, November 21, 2017). These adaptations were representative of all variations to the original task used by preservice teacher teams when working with their elementary students.

In the next section, I will describe the coding of the questions generated by the preservice teachers when they were enacting the task with their elementary student.

### **Questions Used by Preservice Teachers During the Enactment of the Task.**

As part of the assessment assignment, preservice teacher teams prepared seven or eight questions which were meant to make their elementary student’s thinking visible while

they were engaging in the task. In order to understand the types of questions preservice teachers were asking, I coded them using the questioning framework developed by Boaler and Brodie (2004). The types of questions teachers ask during a class discussion, within group work, and/or while working one-on-one with a student creates or restricts the opportunities for students to deepen their mathematical understanding. Table 4.2 provides organization to my research results as well as examples of the questions from my data that I coded for each of the question type categories. By no means are these nine question types exhaustive of all the types of questions teachers might ask during a mathematics lesson, but they provide a significant representation of the questions asked to support the mathematical goals for a given task.

Table 4.2

*Teacher Question Types with Description and Examples from the Data*

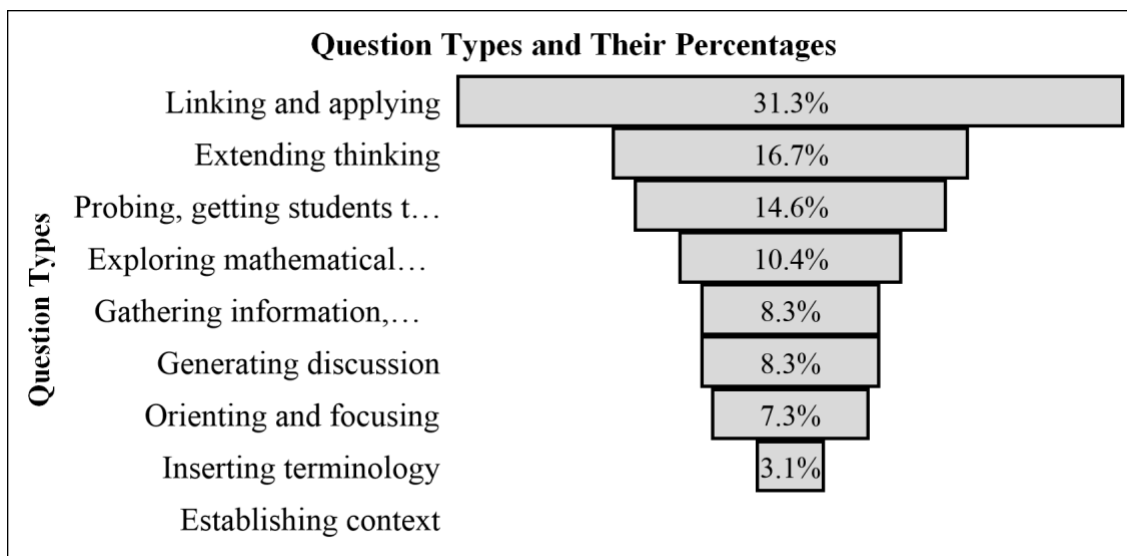
<b>Question Type</b>	<b>Description</b>	<b>Examples from This Study</b>
Linking and applying	Points to relationships among mathematical ideas and mathematics and other areas of study/life	<ul style="list-style-type: none"> <li>▪ How do you know there are not more shapes you can make with the four triangles? (Jill &amp; Jack, personal communication, November 21, 2017)</li> <li>▪ Why do you think that your strategy will allow you to know that you found all the possible shapes? (Mike &amp; Mataya, personal communication, November 21, 2017)</li> </ul>
Extending thinking	Extends the situation under discussion to other situations where similar ideas may be used	<ul style="list-style-type: none"> <li>▪ Does this remind you of anything you have done before? If so, how? (Adelyn &amp; Anna, personal communication, November 21, 2017)</li> <li>▪ Do you think we could do this with a different shape? If so, what shape? (Celia &amp; Connie, personal communication, November 21, 2017)</li> </ul>

Table 4.2 – Continued

Probing – getting students to explain their thinking	Asks student to articulate, elaborate, or clarify ideas	<ul style="list-style-type: none"> <li>▪ Why are the ways you made the only ways to arrange the pieces? (Adelyn &amp; Anna, personal communication, November 21, 2017)</li> <li>▪ What did you do that helped you come up with different ways to arrange the tile pieces? (Adelyn &amp; Anna, personal communication, November 21, 2017)</li> </ul>
Exploring mathematical meanings and/or relationships	Points to underlying mathematical relationships & meanings	<ul style="list-style-type: none"> <li>▪ What happens if you turn that triangle around? (Sally &amp; Sue, personal communication, November 21, 2017)</li> <li>▪ Could you rotate this triangle to make a different looking shape than before? (Sally &amp; Sue, personal communication, November 21, 2017)</li> </ul>
	Makes links between mathematical ideas and representations	
Gathering information, leading students through a method	Requires immediate answer	<ul style="list-style-type: none"> <li>▪ How many ways can you just change one of the triangles to change the shape? (Katie &amp; Kerri, personal communication, November 21, 2017)</li> <li>▪ How many shapes can you make with the triangles if at least two sides are touching? (Jill &amp; Jack, personal communication, November 21, 2017)</li> </ul>
	Rehearses known facts/procedures	
	Enables students to state facts/procedures	
Generating discussion	Solicits contributions from other members of class	<ul style="list-style-type: none"> <li>▪ How would you explain this problem to another student in our class? (Kacee &amp; Kelly, personal communication, November 21, 2017)</li> <li>▪ What was the hardest part for you? Why? (Mickie, personal communication, November 21, 2017)</li> </ul>
Orienting and focusing	Helps to focus on key elements or aspects of the situation in order to enable problem-solving	<ul style="list-style-type: none"> <li>▪ How can you put this problem into your own words to make it easier to understand what it's asking? (Kacee &amp; Kelly, personal communication, November 21, 2017)</li> <li>▪ Is there any unnecessary information in this problem? What is it? (Terri &amp; Tina, personal communication, November 21, 2017)</li> </ul>
Inserting terminology	Once ideas are under discussion, enables correct mathematical language to be used to talk about them	<ul style="list-style-type: none"> <li>▪ What do you think a polygon is? (Erin &amp; Emma, personal communication, November 21, 2017)</li> <li>▪ What can you tell me about triangles and their edges? (Adelyn &amp; Anna, personal communication, November 21, 2017)</li> </ul>
Establishing context	Talks about issues outside of math in order to enable links to be made with mathematics	<ul style="list-style-type: none"> <li>▪ NA</li> </ul>



For my data set, eight question types were asked by preservice teacher teams when working with their elementary students on the adapted task as there were no questions in the category entitled establishing context. About two-thirds of the questions asked fall in one of three categories: linking and applying, extending thinking, and probing – getting students to explain their thinking. The remaining one-third of the question types are represented in the following categories: exploring mathematical meanings and/or relationships, gathering information – leading students through a method, generating discussion, orienting and focusing, and inserting terminology. Figure 4.13 shows the individual percentages for each of the question types used by the preservice teachers involved in my study.



*Figure 4.13.* Question Types and Their Percentages

Table 4.3 displays the grade level and frequency where each teacher question type occurred.

Table 4.3

*Grade Level and Frequency Where Each Teacher Question Type Occurred*

<b>Question Type</b>	<b>Grade K</b>	<b>Grade 1</b>	<b>Grade 2</b>	<b>Grade 3</b>	<b>Grade 4</b>	<b>Grade 5</b>	<b>Totals</b>
Linking and applying	6	9	3	3	6	3	30
Extending thinking	6	8	1			1	16
Probing, getting students to explain their thinking	1	3	3	3	1	3	14
Exploring mathematical meanings and/or relationships	7	1			2		10
Gathering information, leading students through a method	3	2			3		8
Generating discussion	2	2	1	2		1	8
Orienting and focusing	1	3		1	1	1	7
Inserting terminology	1				1	1	3
Establishing context							
<b>Totals</b>	27	28	8	9	14	10	96

The question types are spread throughout the data and not specific to one grade level or grade band. In the plans from the preservice teacher teams, there were no establishing context questions. Preservice teachers working with Kindergarten students generated 27 questions spread throughout each of the remaining categories. Preservice teachers working with Grade 1 students wrote 28 questions within seven of the nine categories, excluding the categories inserting terminology and establishing context. Preservice teachers working with Grade 4 students created 14 questions in all categories except extending thinking, generating discussion, and establishing context. Preservice teachers working with Grade 5 students generated ten questions spread throughout all categories

except exploring mathematical meanings and/or relationships, gathering information, leading students through a method, and establishing context. Preservice teachers working with students in Grade 3 wrote nine questions which were placed in four of the question type categories which excluded extending thinking, exploring mathematical meanings and/or relationships, gathering information, leading students through a method, inserting terminology, and establishing context. Preservice teachers working with Grade 2 students created eight questions connected to four categories eliminating the following categories: exploring mathematical meanings and/or relationships, gathering information, leading students through a method, orienting and focusing, inserting terminology, and establishing context.

Since three preservice teacher teams (6 individuals) worked with Kindergarten students and four preservice teacher teams (8 individuals) worked with first grade students, there were more opportunities for these grade levels to include questions in all the teacher question type categories than in the other grade levels. Two preservice teacher teams (4 individuals) worked with students in fourth grade and one preservice teacher team (6 individuals) each worked with students in Grades 2, 3, and 5. Therefore, seven (14 individuals) out of the 12 total preservice teacher teams (24 individuals) worked with students in Kindergarten or first grade which implies that there would potentially be more chances for predetermined question types to cover more categories for these two grade levels than for any of the other grades.

## **Summary of Findings Related to the Preservice Teachers' Written Task**

### **Adaptations and Student Enactment**

In summary, the preservice teacher teams designed a plan for using the original task and enacted this plan with an elementary student. This plan included adapting the task in ways that the preservice teacher teams thought necessary in order to enact the task with elementary students and to adaptations of the plan when enacting the task. These task adaptations included: having manipulatives for the elementary student to use as they engaged with the spatial reasoning task, adjusting the total number of triangles used to solve the task, encouraging the use of connecting two triangles together creating familiar shapes (e.g., square, triangle, and parallelogram) then moving the remaining triangles around to generate a polygonal region, and using different colors to show equivalent side lengths which could be connected to create solutions. Even though these adaptations lead to elementary student success (i.e., success is defined here as an elementary student being able to find at least one solution to the spatial reasoning task), they had a hard time noticing if their new design was actually new or a repeat of one design they already created (Marcia & Michelle, personal communication, November 21, 2017). Finally, each preservice teacher team was asked to generate seven or eight questions which were meant to make their elementary student's thinking visible while they were engaging in the spatial reasoning task. The majority of the questions were in one of three categories: linking and applying, extending thinking, and probing – getting students to explain their thinking.

### **Preservice Teacher Reflections Around Spatial Reasoning Components**

In this section, I will describe the results related to research question three regarding the written reflections of the preservice teachers' concerning spatial reasoning components which includes: their own as well as their elementary students prior knowledge, their mathematical understanding of the task, describing the strategies used by their elementary student when solving the task, the questions they posed to their elementary student while they enacted the task, and their analysis of their elementary student's work with the van Hiele Levels of Geometric Thought.

**Prior Knowledge Needed when Solving a Spatial Reasoning Task.** Since the learning of new information happens within the context of each individual's prior knowledge, understanding how best to build on what learners already know is at the heart of effective instruction (Hattikudur, Sidney, & Alibali, 2016). Both the preservice teacher and the elementary student had to possess knowledge about triangles as shapes and the ways in which they could be arranged in order to begin solving the spatial reasoning task. More specifically, preservice teachers said the elementary students had to know the characteristics of a triangle, namely that it has three sides and three vertices (All Math Words Encyclopedia, 2010). For example, a preservice teacher team was working with a third grade student and they mentioned that when they asked their student about the characteristic of a triangle, the student explained to them that "it had three sides which come in different sizes, some are slanted, and some are straight up and down" (Rashel & Kay, personal communication, November 21, 2017). The characteristics of a shape are one of the main ideas that Kindergarten students learn as identifying features of polygons (Dağlı & Halat, 2016). Similarly, elementary students needed to know about

triangle properties, namely what types of triangles they were and if the edges (sides) were the same length and/or which edges (sides) matched another edge (side) of the triangle (National Governors Association Center for Best Practices, 2010). As students get older, the sides begin to be analyzed as to whether or not they are the same length, or if one side is bigger/longer than the others (NCTM, 1989). Along with the sides, the corners are eventually referred to as angles, and their variations in size start to take meaning (NCTM, 1989).

Another important idea that shapes (in this case, triangles) can be used to create larger, different shapes was prior knowledge that the elementary student had to have experienced at some point in their past learning in order for them to reference and use this insight when solving the spatial reasoning task. For example, a preservice teacher team when working with a second grade student mentioned that their student “needed a basic understanding of how shapes were related and how they can be formed because he had to put shapes (triangles) together to make other polygons and new shapes” (Mike & Mataya, personal communication, November 21, 2017). Ironically, there was a Kindergarten lesson taught the same day before the enactment of the task, which was about shapes and how multiple shapes can come together to make another shape. “Knowing that our Kindergarten students just went over shapes and how they come together, we tried to tie this activity into the same type of material by asking similar questions that were asked in the actual lesson” (Katie & Kerri, personal communication, November 21, 2017). There was also a preservice teacher team working with a 5<sup>th</sup> grade student that referenced

a lot of prior knowledge needed for the lesson is not new material for our student, rather it is material she had already learned, but she may not have put the concepts

together like this before. She noticed that they were all the same triangles which seemed to help her in finding polygonal regions because she started to notice which edges aligned and which didn't (Adelyn & Anna, personal communication, November 21, 2017).

**Preservice Teachers' Mathematical Understanding of the Task.** Within their reflections, preservice teachers described various levels of understanding about the task which may have played a role in how they were able to successfully help their elementary student when completing the adapted task. One preservice teacher team wrote

understanding mathematics is basically the underlying concept within any mathematical question we asked our student. For starters, the more understanding that we had about this task and the mathematics, the more rich of a learning experience we were able to make for our student. Because we had a deeper understanding, we were able to come up with modifications that we could use in order to help our student learn (Mike & Mataya, personal communication, November 21, 2017).

Another preservice teacher team commented, "since we already solved this problem, we knew the struggles that we personally experienced while solving it and we were able to help our student get past them" (Erin & Emma, personal communication, November 21, 2017). "We knew that the triangles could be rotated and flipped, as well as turned to create a multitude of different shape patterns" (Sue & Sally, personal communication, November 21, 2017).

Some preservice teachers acknowledged that their work with the task enriched the questions they created to ask their students, "our questions were summative and asked for

many different formats of reasoning, justification, and problem solving. Throughout our classes this semester, we learned how to make math a learning environment for our students and how to make the content rich” (Celia & Connie, personal communication, November 21, 2017). These questions generated by the preservice teachers provided ways in which they could gain more mathematical insight with a deeper understanding from their elementary student. It also allowed them to empathize with their students who struggled when facing disequilibrium (Jill & Jack, personal communication, November 21, 2017). One preservice teacher admittedly struggled with the task as she wrote, “I did not understand this task when I was asked to complete it. This definitely made an impact on the way I could describe it to my student, as I didn’t fully understand it myself” (Ana & Addie, personal communication, November 21, 2017).

**Preservice Teachers Descriptions of the Strategies Used by Elementary Students When Solving the Task.** All preservice teachers in their reflection commented that their elementary students used guess and check, trial and error, or the process of elimination as methods for solving the task. They would create a design and then check to see if they had created it before or if it was new. Most preservice teachers believed that their students did not know they were using strategies to solve the task, let alone be able to explain them when asked. For example, Mike and Mataya asked their elementary student about the strategies he was using to build the shapes he was creating and he replied, “ I will probably build shapes that I know” (personal communication, November 21, 2017) meaning shapes that he would recognize like a rectangle or a square (Mike & Mataya, personal communication, November 21, 2017). The most commonly used strategy by the elementary students when solving the given task was making a square



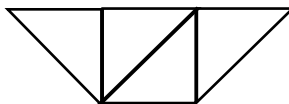
with two of the triangle manipulatives. From there, the two leftover triangles were put around the square figure to create solutions. A fourth grade student “started by moving one triangle at a time until he made all of the polygons that he could before he started moving the other one (triangle) around too” (Erin & Emma, personal communication, November 21, 2017). This square was referred to as a base shape, a diamond, or a rhombus by the preservice teachers. A second grade student “tried to build rectangles, squares, and triangles, which were shapes that he knew rather than random polygonal shapes. After making a couple of different shapes that didn’t look familiar, he was able to expand his thinking and find more shapes” (Mike & Mataya, personal communication, November 21, 2017). One preservice teacher team when working with a Kindergarten student referenced all three strategies,

because we gave the student triangles, he was able to check and see if the sides completely checked, then he used trial and error by putting triangles together and checking to see if the whole side of a triangle was touching another side of another triangle, and he used the process of elimination by looking at the polygonal regions he had already created to decide if he could make more polygonal regions (Terri & Tina, personal communication, November 21, 2017).

Rotating the triangles provided by the preservice teacher was another strategy preservice teachers discussed in their reflections, which demonstrated a more systematic approach to finding solutions to the spatial reasoning task. One preservice teacher when working with a Kindergarten student referenced, “he (the elementary student) knew that in order to make different shapes like we instructed him to, the shapes would need to be turned differently from what he had before which would require a lot of rotation to solve

this task” (Sue & Sally, personal communication, November 21, 2017). A preservice teacher team who worked with a fourth grade elementary student made each of the four given triangles a different color which seemed to add to the complexity of solving the problem. Their student “used a strategy of rotating triangles by the different colors to create a ‘new region’” (Terri & Tina, personal communication, November 21, 2017) which adjusted the order of the triangles by color within a polygonal region but not necessarily created a new region (see Figure 4.12).

A Kindergarten student was struggling on what to do next to get a new polygonal region so the preservice teacher suggested that he leave three triangles as they were and move just one triangle, “Where could we put that one triangle to make the design look different?” (Katie & Kerri, personal communication, November 21, 2017). This student used this strategy multiple times to get a new pattern and created different designs. As their kindergarten student was moving the triangles around, he noticed that the design he created was an image he knew and exclaimed: “See, it’s a boat!” (Katie & Kerri, personal communication, November 21, 2017) (see Figure 4.14). “When he started making the same patterns on accident, he would mix up all the triangles and start fresh” (Katie & Kerri, personal communication, November 21, 2017).



*Figure 4.14.* Kindergarten Student Made this Polygon and Called it a Boat

**Preservice Teachers Question Posing for Elementary Student.** Preservice teachers had generated seven to eight questions to ask their elementary student when they were doing the adapted spatial reasoning task as a way to help enrich the learning of the elementary student. They were to be open-ended questions where the student would be

explaining and expanding on different ideas instead of closed questions which require a single answer without explanation. Preservice teacher teams wrote questions that asked for many different formats of reasoning, justification, and problem solving as a way to make the spatial reasoning content rich. Half of the preservice teachers described the difficulties they experienced in getting their elementary student to fully understand what they were asking him/her to do because the preservice teacher teams were uncertain of the prior knowledge their elementary student possessed concerning a spatial reasoning task. For example, one preservice teacher team stated,

Understanding is a key part to learning anything that you do. You need to build off of prior knowledge and use information that you already know to help you adapt your learning and use that to learn new information, and further your learning as a whole. Our second grade student was determined to build shapes. When we asked him if there was a way he could concretely say and explain whether or not he had made all the polygonal regions, he responded by saying ‘no.’ So our student wasn’t helped at all by our probing questions and we were unsure how to build off of what knowledge he had (Mike & Mataya, personal communication, November 21, 2017).

Another preservice teacher team, Rashel and Kay, were glad that they had completed the task beforehand as it helped them generate questions which would aid their elementary student in drawing her own conclusions and justifications for the solutions she already found (Rashel & Kay, personal communication, November 21, 2017). An important question asked by a preservice teacher team during the enactment of the task was, “Does this problem remind you of any other problems you have done before?”

(Larry & LaVern, personal communication, November 21, 2017) as this was a way to check their student's prior knowledge or previous experiences which they could lean on to solve the task. In this case, one elementary student in first grade referenced using squares to make patterns in kindergarten and that circles and triangles were used in an activity with stacking shapes inside of shapes in preschool. After going through the process of the interview, one preservice teacher team realized some of their questions were not relevant nor were they beneficial to the elementary student in the process of solving this task, so they decided not to ask the questions they had prepared ahead of time and asked other questions instead (Adelyn & Anna, personal communication, November 21, 2017).

A preservice teacher team when working with a first grade student did not want to simply tell their student what to do but rather wanted to promote her thinking and use her own skills to create different designs. In the process of creating these designs, she (the elementary student) believed that she was only going to find four designs. The preservice teachers repeatedly asked her about how many solutions she would be able to find and why she thought she would find that amount. By asking these questions, the preservice teacher team felt like it extended their student's thinking because she had to reason how she was going to accomplish finding more solutions. "These questions also made the student think about the difference between endless possibilities and a limited number of possibilities" (Celia & Connie, personal communication, November 21, 2017). It was helpful for the student to refer back to previous drawings that she had sketched and numbered to see if the design was the same or different. "Towards the end of the interview, the student recognized that there were not endless possibilities, because she

kept recreating designs she had previously made” (Celia & Connie, personal communication, November 21, 2017).

**Preservice Teacher Analysis of Elementary Students Work with the van Hiele Levels of Geometric Thought.** Based on the evidence gathered from the enactment of the adapted task by the elementary student, the preservice teachers used the van Hiele Levels of Geometric Thought to determine which level to place their students’ work and thinking about the spatial reasoning task. Overall, it was determined that each elementary student was at grade level. Grade level here implies that younger students (Grades K-2) are at Level 0 – Visualization or Level 1 – Analysis because of limited geometric experiences and older students (Grades 2-5) are at Level 1 – Analysis or Level 2 – Informal Deduction, because they have had more in-depth geometric experiences as well as mathematical standards, are written to begin the transition from Level 0 to Level 1 to Level 2 as early as fifth grade (Cathcart, Pothier, Vance, & Bezuk, 2011). For example, Jill and Jack, who worked with a first grade student noticed they needed to prompt their student to rotate the triangles to make new, unique shapes since, “those who excel at spatial reasoning often have this ability to create new shapes in their own mind, but a 1<sup>st</sup> grader would most likely not have developed this yet” (Jill & Jack, personal communication, November 21, 2017). Elementary students in grades K-2 were either placed at Level 0 or Level 1. A kindergarten student who was new to the school was chosen to participate in this task. The preservice teacher reported that this student was also an English language learner who was doing well with receptive language processes but struggled with expressive language. When Sue and Sally analyzed their student’s work on the task, they decided he was at Level 0 “since he is working on learning the

right English words for each shape as well as recognizing their defining features such as the number of sides and angles. The appearance of the shape seems dominant because the tilting of a square caused him to see a ‘diamond’ shape” (Sue & Sally, personal communication, November 21, 2017). I do not know if the preservice teacher team was conflating the work done during the enactment of the task because the native language of the student was not known by the preservice teachers.

Larry and LaVern, when working with their first grade student, noticed that he was strictly thinking about the visuals he could find in this task which indicated to them that their elementary student’s work was at Level 0 – Visualization. Level 0 is

thinking what the shapes look like instead of the actual properties. Our student’s spatial reasoning kept coming back to the fact that he wanted to see visuals in the patterns. The student would take the cardboard triangles, and he would put them together and try to make something visual he could recognize. He would make a shape and say, ‘That looks like a dog, a cat, a ramp, a rocket, etc.’ The student’s spatial reasoning had him making shapes or patterns that he recognized while he is having the triangles touching. When students become older spatial reasoning for them is a little different for this problem. Older students mentally flip and turn shapes in their minds and find ways to decide whether they have found all of the solutions (Larry & LaVern, personal communication, November 21, 2017).

Another preservice teacher team, Ana and Addie, placed their first grade student at “Level 0 since he can tell you shapes and what the object ‘looks like’ but his knowledge isn’t quite extensive enough to be at Level 1” (personal communication, November 21, 2017). They commented that their student would need to be able to classify groups of

shapes and not just individual traits if he was going to advance to the next level of thought.

van Hiele describes a student at Level 2 as one that begins to think about geometric objects without focusing on one particular object (shape), and they are able to develop relationships between these properties (van Hiele, 1984a). Elementary students in grades 3-5 were placed at Level 2.

Our third grade student understood the concept of an isosceles triangle and used this knowledge to help her decide which sides would fit together and which sides would not. It was also clear that she understood that the triangles could go together in a variety of ways, but she did not consider flipping the triangles around and trying them in different orientations. Since she did recognize the difference in the sides of the triangles and was able to explain her thought process to us, we considered her to be at an emergent Level 2 (Kacee & Kelly, personal communication, November 21, 2017).

A fourth grade student

wasn't quite at a Level 3 yet, because he struggles when trying to explain how he was manipulating the different shapes. He said things like, 'Yeah um you just switch the colors and don't do the same colors that you did last time,' when he was asked what strategies he used to come up with the different designs. He placed most of his focus on the location and transformation of the triangles. Most of the time he worked, he created two separate designs using two triangles for each, rather than one design that used all four triangles together (Terri & Tina, personal communication, November 21, 2017).

### **Summary of Findings Related to Preservice Teacher Reflections Around Spatial Reasoning Components**

In summary, preservice teacher reflections around the spatial reasoning components of the task involved theirs as well as the elementary student's prior knowledge of a triangle as a shape, and the properties and/or characteristics of a triangle. By solving the spatial reasoning task first, the preservice teacher's mathematical understanding of the task allowed them to know where the elementary student might struggle and how to best help them. Elementary students used a base shape (e.g., square, rectangle, and triangle) as well as shapes which reminded them of real-world objects as strategies for solving the spatial reasoning task. Preservice teachers generated probing questions to ask their elementary student as a way to examine and extend their student's thinking about the spatial reasoning task. Finally, the preservice teachers used the van Hiele Levels of Geometric Thought as a way to categorize the work of their elementary student, which was at Level 0, Level 1, or Level 2.



## CHAPTER 5: DISCUSSION OF STUDY FINDINGS AND IMPLICATIONS

In this study, I examined the spatial reasoning skills that preservice elementary teachers demonstrated and how their spatial reasoning skills may influence how they planned for and enacted a spatial reasoning task with elementary students. Specifically, I focused on how they adapted and used the same task with an elementary student. In this chapter, I discuss the results presented in chapter four and provide answers to the research questions. I conclude with a description of the implications of this study, provide recommendations for future research and teacher education programs, and offer a conclusion.

Table 5.1 summarizes the central claims of this study. This chapter presents evidence in support of the five claims that align with the research questions of this study. The research questions provide a frame for the discussion of my results and the associated claims.

Table 5.1

*Alignment of Research Steps, Research Purpose, Connection to Research Question, and Claims*

<b>Research Steps</b>	<b>Research Purpose</b>	<b>Connection to Research Question</b>	<b>Claims</b>
1. Preservice teachers complete the Math Matters Tile Assignment	To determine individual spatial reasoning level using van Hiele Levels of Geometric Thought	The spatial reasoning level of a preservice teacher will influence what they do when engaging in a task that requires spatial reasoning.	1. van Hiele Levels vary among preservice teachers

Table 5.1 – Continued

2. Preservice teacher teams adapted the Math Matters Tile Assignment for their teaching experience	To observe the variety of adaptations generated and the interaction of elementary students with the task as recorded in their journal and shared in interviews	This activity will provide insight into what preservice teachers do when adapting and enacting the spatial reasoning task with an elementary student.	2. Task adaptations minimize the deep mathematical understandings
3. Preservice teacher teams enact plan with an elementary student			3. Preservice teacher questions appear to get at probing student thinking
4. Preservice teachers record written reflections of teaching experience	To observe examples as recorded in their journal of the thought processes of the preservice teacher teams interactions with an elementary student as well as provide an opportunity for the researcher to clarify and validate data	This activity will provide insight into what preservice teachers write about in written reflections related to their spatial reasoning skills and their working with an elementary student on a spatial reasoning task.	4. One task is not a true indicator of the overall spatial reasoning a preservice teacher possesses
5. Phone Interview			5. The van Hiele Levels of elementary students will vary greatly compared to the preservice teachers' level

### **What did preservice teachers do when engaging in tasks that require spatial reasoning?**

In this section, I summarize the findings related to what preservice teachers did when engaging in a task that required spatial reasoning. I then discuss my findings in relation to the findings of other researchers. The data I analyzed from the written work submitted by each preservice teacher on the Math Matters Tile Assignment provided insight into what preservice teachers do when engaging with a spatial reasoning task. Preservice teachers had to generate all the solutions possible when using four triangles.

Seventeen out of 32 preservice teachers successfully found all fourteen solutions to the spatial reasoning task. The triangles all had the same height and base length. I intentionally provided only these two measurements which made solving for the other two side lengths mathematically impossible as I wanted to determine if any preservice teachers would assume the given triangles were isosceles triangles, which seven did.

Preservice teachers either made their own triangles or used the given ones from the assignment as manipulatives to move around, which helped the four who included this in their written explanation find their solutions to the task. Other preservice teachers may have used manipulatives to help them generate all the solutions to the spatial reasoning task but did not include this in their written explanation. Fifteen preservice teachers used a base shape as a starting point for each solution while three others exhausted all possibilities using a guess and check method. Fourteen preservice teachers solved the task by either rotating a triangle, moving triangles around, connecting the triangles in every possible way, or comparing the shapes they created to real-world objects.

Preservice teachers indicated that they knew they had found all the possible solutions because they had unique non-repeated designs, and the side lengths, which were comparable on the triangles were mapped to each other. I categorized the written work submitted by the preservice teachers in this study concerning the spatial reasoning task to be at Level 1 or Level 2 on the van Hiele Levels of Geometric Thought. The work of 15 preservice teachers was placed at Level 2 since they created meaningful definitions and provided informal arguments to justify their reasoning. The work of 16 preservice teachers was placed at Level 1 because they implied seeing figures as collections of

shapes indicating that the side lengths of the triangles which were the same were matched up when generating a new polygonal design. The work by one preservice teacher was placed at Level 0 since her work focused solely on the appearance of the shapes.

**Relationship to Other Researchers' Findings.** In this section, I describe my results with respect to findings from other researchers.

***Shape assumptions.*** Preservice teachers assumed that the given triangles were isosceles, since just looking at an object or a set of objects is not proof enough to say they are or are not the same (Vinner & Dreyfus, 1989; Thomas & Holton, 2003). It is not safe to claim that something is true about a figure merely because it looks true; instead, one must prove it is true by airtight mathematical logic (Ryan, 2016). Vinner and Hershkowitz (1980) make a connection between a student's concept image and their understanding of the formal definitions of shapes. They explained that students would remember prior experiences with diagrams, attributes, and examples associated with the shape, instead of the formal definition. Mathematics is not about beliefs, but about reasoning, creativity, and inquiry. Hence the need for using manipulatives to solve this spatial reasoning task is discussed below since the movement of these triangles would help preservice teachers build concrete solving methods as they are solving an abstract task.

***Manipulatives.*** The Math Matters Tile Assignment included four triangles which preservice teachers could cut out and use as manipulatives, but they were not required to use them to solve this task. The use of manipulatives is extremely important when attempting to take one's thinking from concrete to abstract. Kaplan (2000) found that children between the ages of five and eight rely on active manipulation of real materials

to connect abstract materials, such as ideas and statements, to something observable and imaginable. The use of Cuisenaire Rods and Pattern Blocks in early elementary classrooms is a testament to the importance of concrete manipulative materials to support students as they develop abstract concepts (Resnick, 1998). The use of manipulatives as stated in the data by four preservice teachers when solving the task seemed to not be helpful since the idea of rotations and transformations was a struggle for some making solving the task difficult. The idea of seeing the movement of the triangles in one's mind before manipulating the individual pieces is more of an advanced skill which is developed with practice over time. Preservice teachers should be provided with hands-on activities using manipulative concrete materials for discovering the properties of simple geometric shapes in different orientations (Armah, Cofie, & Okpoti, 2018). Sekiyama, Kinoshita, and Soshi (2014) support this idea as they suggested that children aged seven to eight years fall into a transition period for spatial thinking, wherein more mental processing emerges from the earlier physical and illustrated approaches.

*van Hiele Levels of Geometric Thought.* Since geometric thinking plays a significant role in the development of spatial reasoning and visualization, it is important to try and understand how preservice teachers reason about shapes and other geometric ideas. The van Hiele Levels of Geometric Thought are a product of experiences and instruction rather than age which may be a place to begin when explaining why the work of preservice teachers in this study was placed at Level 1 or Level 2 as an individual must have enough experiences (classroom or otherwise) with these geometric ideas in order to move to a higher level of sophistication. More questions about the mathematical background of each preservice teacher would need to be asked in order to determine more

concrete connections between their experiences and their level on van Hiele Levels of Geometric Thought. “All learners are capable of growing and developing the ability to think and reason in geometric contexts, but this ability requires ongoing and significant experiences across a developmental progression” (Van de Walle, Karp, & Bay-Williams, 2016, p. 489). Through rich experiences, students can reach Level 2 in elementary school. This is where most preservice teachers placed the work of their elementary students who completed the task. Without these experiences, many adults (including teachers) remain in Level 1 all their lives, even if they take a formal geometry course in high school (Mayberry, 1983).

It is common in many US schools for students to take geometry in high school. High school geometry builds on geometry instruction that has occurred throughout elementary and middle school, but with the key difference being that students must prove and explain concepts they learned in prior years (National Governors Association Center for Best Practices, 2010). In elementary school, students learned about the attributes of shapes, compared and categorized these attributes, and learned to compose and decompose shapes. In middle school, students developed conceptual understanding of angle relationships in parallel line diagrams and angle relationships within and outside of triangles. They have also learned to describe geometric features, measure circumference and area of circles, and make observations and conjectures about geometric shapes using sound reasoning and evidence. Students have learned to construct a triangle using different side lengths and that the properties of a triangle are based on the relationship between the side lengths and the interior angle measures. These foundational understandings are essential to student success as they build chains of reasoning to

explain, model, and prove geometric relationships and situations (National Governors Association Center for Best Practices, 2010).

Typically, all high school students take four years of mathematics, including algebra, geometry, and other advanced math classes such as precalculus, calculus, or practical mathematics such as statistics, financial literacy or data science (NCTM, 2018). As high school students graduate and head to college, they begin to build the knowledge they need to be successful in their future career of choice. Within this preparation to be an elementary school teacher, preservice teachers find themselves taking math methods classes which build their content knowledge as well as their pedagogical knowledge. In their preparation to teach geometry, preservice teachers are exposed to the van Hiele Levels of Geometric Thought which describe how we think, what types of geometric ideas we think about (called objects of thought), and what students can do (products of thought) as they are now faced with teaching geometric concepts to their potential students.

Looking more closely at the van Hiele Levels of Geometric Thought, Levels 3 and 4 would be considered advanced levels and preservice teachers did not demonstrate work at either of these levels. At Level 3, students can create deductive geometric proofs, which is part of the work completed in high school geometry classes. They understand the role of definitions, theorems, axioms, corollaries, postulates, and proofs as a means to establish geometric truth (van Hiele, 1986). At Level 4, students understand how mathematical systems are established as they are able to use all types of proofs. They comprehend Euclidean and non-Euclidean geometry and are capable of describing the effect of adding or removing an axiom on a given geometric system which is

generally at the level of a college mathematics major (van Hiele, 1986). van Hiele wrote (1999), “My experience as a teacher of geometry convinces me that all too often, students have not yet achieved this level (Level 2) of informal deduction. Consequently, they are not successful in their study of the kind of geometry that Euclid created, which involves formal deduction (Level 3)” (p. 311). This offers a potential beginning place for why the work of the preservice teachers was placed at Level 1 or at most Level 2 when they are preparing to be an elementary teacher as an individual must have enough experiences (classroom or otherwise) with these geometric ideas in order to move to a higher level of sophistication. It is also important to point out here that when materials and instruction are operating at a higher level than a student’s level of understanding the lack of alignment between the materials and instruction often prevents student growth in understanding as measured by the van Hiele levels (van Hiele, 1999).

### **What did preservice teachers do when planning for and enacting the spatial reasoning task with elementary students?**

In this section, I summarize the findings related to the adaptations preservice teachers made and the ways in which they enacted the task with an elementary student. I then discuss my findings in relation to the findings of other researchers.

I used the written work submitted by each preservice team concerning the second part of the Math Matters Tile Assignment to analyze what preservice teachers do when planning for and enacting the spatial reasoning task with elementary students. Preservice teachers worked with their practicum partner to adapt the task in order to make it more suitable for their elementary student to find solutions. These adaptations, made during the planning and the enactment of the task, involved: eliminating the setting of the original



problem as a way to lessen confusion (Sally & Sue, personal communication, November 21, 2018), reducing the number of triangles used from four to three, color coding sides of the triangle which are equal in length, forgoing the bathroom floor template, providing manipulatives for the elementary student to use when solving the task, and helping the student keep track of the solutions they created.

Preservice teachers were also asked to generate seven to eight questions to probe their elementary student's thinking while they were doing the task. Most of the questions were in one of three categories: linking and applying (31.3%), extending thinking (16.7%), and probing – getting students to explain their thinking (14.6%). The rest of the questions created by the preservice teachers fit into one of five categories: exploring mathematical meanings and/or relationships (10.4%), gathering information, leading students through a method (8.3%), generating discussion (8.3%), orienting and focusing (7.3%), and inserting terminology (3.1%). I did not code any questions as establishing context.

**Relationship to Other Researchers' Findings.** In the section I describe my results with respect to findings from other researchers.

***Manipulatives.*** Children construct much of their knowledge through active manipulation of their environment (Beatty, 1984; Bredekamp & Copple, 1997; Lee, 1992; McInerney & McInerney, 2002). Piaget demonstrated that young children learn about geometric shapes, not from taking mental pictures of objects, but from actions they perform on objects. Therefore, it was important that one adaptation preservice teachers had for elementary students was to provide manipulatives for their use during the task as learning occurs when constructive play enables children to combine their repetitive

sensorimotor ideas with the symbolic representation of ideas (Fenson & Schell, 1985; Santrock, 1998). Research has shown, however, that “manipulatives themselves do not magically carry mathematical understanding. Rather, they provide concrete ways for students to give meaning to new knowledge” (Ontario Ministry of Education, 2003, p. 19).

The data showed that manipulatives were only being used by four preservice teachers when they individually solved the spatial reasoning task and all preservice teacher teams made sure their elementary students had manipulatives to use as they also engaged in solving the task.

It was very helpful in her (the elementary student) spatial awareness skills to have the pieces sitting out in front of her so that she could work through them with her hands. This allowed for her to be able to try more possibilities than if she had to draw them out. Having physical manipulatives for students is something that I believe is often undervalued. Being able to work through the problem using shapes really allowed her to be able to understand why some of the shapes would make sense, but also why some of the shape combinations did not make too much sense (Rashel & Kay, personal communication, November 21, 2018).

It is evident from the data that the preservice teacher teams provided only one set of four triangles for the students to use when solving the task instead of providing enough triangle sets that each student's solution could be left alone while they used another set of four triangles to generate the next solution. However, when the task was implemented by the preservice teacher teams, only one preservice teacher team chose to have enough sets

of triangles so that their student could work with a different set of triangles each time they found a solution.

***Adjustments to the problem.*** All preservice teachers who participated in this study decided that they needed to adjust the given task. When looking at the potential task, I intentionally had the preservice teachers work the task first. This provided them with insights as to potential problems that their elementary student might have when solving the task and how they could best support their student through the difficult spots in the task but not eliminate these obstacles altogether. There are various reasons why these adaptations were made none more important than improving the opportunities students' have for learning mathematics. With only 10-15 minutes allotted for the student to work on the task, preservice teachers decided to eliminate the floor plan part of the task and focus on students finding a variety of solutions to the task. By stripping the task back to the bare essentials (no context and only asking the elementary student to find solutions), preservice teachers attempted to avoid the cognitive overload their elementary student might encounter while solving this task. One must be careful with getting rid of the context clues of the problem, as noted in the sentence above, since "contexts may help students to make sense of problems and they may motivate them by helping them to see its application. This application led us to question the relationship between mathematics and real-life" (Back, Foster, Tomalin, Mason, Swan, & Watson, 2013). Working on solutions to real-world problems is at the heart of any STEM investigation (Nadelson & Siefert, 2017). These solutions may include devices and designs that improve our lives, fulfill our needs or wants, and make our world better. From designing a better pencil to figuring out how to assist areas lacking clean drinking water, the

opportunity to search for solutions to real-world problems fuels students' critical thinking, curiosity, and sparks their investigative interests in solving problems (Jolly, 2016).

Another adjustment by the preservice teachers included how they explained to their student the ways in which the triangle pieces could fit together. Four preservice teachers decided to only use three triangles instead of four as a way to reduce the cognitive load of their younger elementary student. Another preservice teacher began by verbally explaining to their student what they were to do to solve the task (without modeling). Their student was very confused in the beginning by what she was to do to solve the task. After they modeled one solution for their student, she was able to grasp what was being asked of her. By giving the information in a way that their student could relate to (through modeling), they were able to show their student what it meant to use all the triangles and have the same side lengths completely touching.

Another adaptation that preservice teachers used when adapting the task was color. They decided to "color code the sides so that on each triangle, the two equal sides will have blue expo marker on them, and the one longer side will have red marker on it. This will help our student understand which sides can be put together" (Sally & Sue, personal communication, November 21, 2018). While choosing to color code equal side lengths might be helpful, caution should be taken since it is possible some people have a color vision deficiency (with versions of the color red, blue, and/or green), which means their perception of colors is different from what most of us see. "The most severe forms of these deficiencies are referred to as color blindness. People with color blindness aren't aware of differences among colors that are obvious to the rest of us" (NEI, 2015). By

using blue and red to color code the side lengths, it might have disadvantaged some students who struggle with seeing color.

Along with the choice of color used, coloring each triangle a different color also caused some confusion for students and seemed to interrupt their ability to successfully solve the spatial reasoning task as their eyes were drawn to changing their design based on color instead of the locations of each triangle. An elementary student in this study was drawn to rotating triangles by different colors to create a “new” region but in reality, adjusting the order of the triangles by color within the same polygonal region did not make a new region (Terri & Tina, personal communication, November 21, 2017). Here we see that the color of the triangles and the order they were placed was overpowering the idea of finding new and different polygonal regions in comparison to the locations of the triangle shapes themselves.

This is similar to the study findings by Pan and Soto (2010), who asked participants to identify if the color or the shape of the two objects presented were the same. In the first experiment, the colors of the two objects were the same, but the shapes were different, while in the second experiment, the conditions were reversed. The result showed that the participants’ response times were faster in identifying the differences in colors compared to differences in the shapes of the objects in both experimental conditions (Dutta & Baruah, 2018). This finding can be interpreted to show that colors have a greater ability to capture attention than other variables. The collected data supports the idea that students will be drawn to color first over the shape of the object which may interfere with their ability to solve a spatial reasoning task due to the distraction caused by coloring the shapes.

*Types of teacher questions.* I coded 60 of the 96 questions generated by preservice teachers in three categories namely, linking and applying (31.3% or 30 questions), extending thinking (16.7% or 16 questions), and probing – getting students to explain their thinking (14.6% or 14 questions). Thirty-six of the questions the preservice teachers wrote were encased within six of the total categories. There were no questions placed in the establishing context (0%) category which “talks about issues outside of math in order to enable links to be made with mathematics” (Boaler & Brodie, 2004, p. 777) since all of the preservice teachers chose not to include the floor plan application part of the assignment for their student to wrestle with and they did not use a different context. Only three questions dealt with inserting terminology (3.1%) since most students would have a good idea of what a triangle was and how equal sides can be matched together, so this was also not a category that was significantly targeted (Kacee & Kelly, personal communication, November 21, 2017). Because the original task included four identical shapes thus limiting the key aspects of the task, it is also not alarming that seven questions were categorized in the orienting and focusing category (7.3%) as the intent of this category is to “focus on key elements or aspects of the situation in order to enable problem-solving” (Boaler & Brodie, 2004).

Generating discussion (8.3%) and gathering information, leading students through a method (8.3%) each had eight questions that fell within these two categories. Because this was an individual project, there was little to no chance that contributions to the interview would be added from members of the elementary student’s class even though a few questions asked how they might describe the task to another student in their class. Preservice teachers wrote questions that did not ask for basic facts or procedures dealing

with the task; in other words, they tried to generate questions that involved higher-order thinking skills.

Ten of the questions preservice teachers wrote were categorized into the exploring mathematical meanings and/or relationships (10.4%) category which dealt with links between mathematical ideas and representations as well as underlying mathematical relationships and meanings. A lot of the questions placed in this category used triangle rotations as well as reflections. All in all, student learning happens when close attention to what students say and do is in relation to what the teacher does and says (Darling-Hammond, 2000). Preservice teachers need to develop a questioning style that guides, supports, and stimulates the thinking of the student and they need to allow students to struggle through disequilibrium in order for real learning and deeper understanding to happen. Many researchers claimed that effective employment of questioning strategies can be very challenging because asking productive questions is such a highly sophisticated art that requires considerable teaching experience and pedagogical content knowledge (Boaler & Brodie, 2004; Franke, Webb, Chan, Ing, Freund, & Battey, 2009), two areas that preservice teachers will at first struggle with.

***School year timing.*** The timing of the enactment of the spatial reasoning task with elementary students is a potential contributing factor to this research question. Since my data collection was during second quarter and based on the mathematics pacing guides for each grade K-5 at the local school district, the elementary students were working this task from their prior knowledge as the geometry units for these schools happens at the end of the school year during quarter four. Kindergarten is the exception as geometry topics, as seen within the mathematics pacing guide, were sprinkled in

throughout the entire school year. The day in which one Kindergarten student completed the task, their topic for the math lesson was on combining shapes to make new shapes. This student recognized the similarities between what he had just learned in class and what the preservice teachers were asking him to do with the triangles which gave him a slight advantage as well as a boost of confidence.

Other students in grades three and four, had trouble recognizing that not all of the sides of the triangles used were the same length, so they had to be very careful as to the ways in which they tried to put them together in order to stay within the given guidelines of the task. In one case, the preservice teachers had to put the triangles next to each other for the student to realize that they were using triangles that were all the same. “I think if 3<sup>rd</sup> grade had worked at all with shapes and geometry at this point in the year, she would have been able to excel more. It took her awhile to understand that not all the sides of the triangle were the same length, so that meant that she had to be very careful in the ways in which she tried to put the sides together” (Rashel & Kay, personal communication, November 21, 2018).

Geometry is one of the focus areas for the NCTM (2000) Content Standards, NCTM (2006) Curriculum Focal Points, and the Common Core State Standards for Mathematics (2010) but yet when we see it being taught in schools, it takes on a more subsidiary role compared to numeric relationships and operations, algebraic relationships and processes, and algebraic applications. Being taught at the end of the school year, also makes geometry a topic that may be less emphasized because of the crowded schedule during the last months of the school year as well as the standardized testing which must also happen during this same time frame. In some cases, topics and the percentages of



those topics which are included on the mathematics standardized tests drives what gets emphasized in classrooms at the end of the school year. For example, if the minority of the entire standardized test for mathematics includes geometry topics and the majority of the test deals with numbers and operation as well as algebra concepts and applications, chances are teachers will be more focused on reviewing topics that cover the majority of the test and leave off the rest. And if the scores that students received on these high stakes tests are associated with teacher worth, values, promotions, and effectiveness, teachers are going to make sure their students are well prepared to do their best work on the bulk of what the test will cover and not worry about the geometry part at all.

**What did preservice teachers write about in written reflections related to their spatial reasoning skills and their working with an elementary student on a spatial reasoning task?**

In this section, I summarize the findings related to the adaptations preservice teachers made and the ways in which they enacted the task with an elementary student. I then discuss my findings in relation to the findings of other researchers.

Preservice teachers had to write responses to predetermined questions (see Appendix A, part 3) based on these specific topics: prior knowledge, their mathematical understanding of the task, the strategies they watched their elementary student use when solving the task, questions written by the preservice teachers to help extend the elementary students' thinking, and their placement of the work of their elementary student with the van Hiele Levels of Geometric Thought. The prior knowledge that preservice teachers included in their written response included the characteristics of a triangle, namely a shape with three sides and three vertices. Going a bit farther, they

needed to see which sides of the triangle were the same length and how that corresponded to the size of the angles opposite those sides. Preservice teachers also had to be familiar with the idea that shapes (in this case, four triangles) can be used to create larger, different shapes.

Next, preservice teachers had various levels of mathematical understanding from working the task individually, which both helped and hindered their ability to support their elementary student as they engaged in the task. The planning process of the preservice teachers created a rich learning experience for their elementary student as they had already contemplated what modifications would be beneficial (Mike & Mataya, personal communication, November 21, 2017). Their mathematical understanding also helped the preservice teachers generate higher-level questions which enriched the experience the elementary student encountered (Jill & Jack, personal communication, November 21, 2018).

When solving the task, the preservice teachers observed their elementary students using trial and error as a way in which they attempted to find all the solutions to the task. Within this trial and error process, the elementary student rotated the triangles in an effort to observe the triangles facing a different direction and possibly creating a solution. Using two triangles to make a square was another strategy that preservice teachers observed elementary students gravitating towards when finding solutions to the task. While the elementary student was interacting with the task, preservice teachers strategically asked seven or eight probing questions they had generated as a way to extend the thinking of their student about the spatial reasoning task. Some of the questions the preservice teachers created generated accurate responses, but other

questions had to be altered (adapted in the moment) while the elementary student was working the task since the questions were not relevant for the work the student was producing.

Preservice teachers within their reflections included their assessment of their student's spatial reasoning understanding, which involved placing the work of their elementary student on one of the van Hiele Levels of Geometric Thought. I did not collect data concerning the instruction the elementary students had received in prior years or the current school year. Preservice teachers placed the work of their students at Level 0, Level 1, or Level 2 as these levels described ways that their elementary students reasoned about shape and other geometric ideas. Since these levels are a product of experiences and instruction rather than age, the younger students who have had less exposure to geometric ideas would be at a lower level on the van Hiele Levels of Geometric Thought and those students, who are usually older and have had more instruction and exposure to geometric ideas, would be at a higher level.

**Relationship to Other Researchers' Findings.** In this section I describe my results with respect to findings from other researchers.

***Composing and decomposing shapes.*** Composing and decomposing shapes is an important part of early geometry (Clements & Sarama, 2014). By definition, composition involves arranging shapes together to form a whole. For example, taking six equilateral triangles and creating a hexagon or using two trapezoids to make a hexagon. On the other hand, decomposition is taking shapes apart and separating them into smaller pieces. For example, decomposing a rectangle into two right triangles or one trapezoid can be decomposed into three equilateral triangles. The mathematical power associated with

composing and decomposing shapes is foundational to the understanding of many other areas of math, especially number and arithmetic, such as part-whole relationships and fractions (Clements & Sarama, 2014).

When working with a fourth grade student, Erin and Emma reflected on the spectrum of geometric knowledge they watched their student complete. First, their elementary student took the four triangular shapes and laid them out on the table followed by moving them around, sorting through them, and classifying the triangles as all being the same since they had all the same side lengths and angles. Next, Erin and Emma watched as their student took two triangles and pushed them together to make a square which triggered in his mind that the other two triangles could also make a square, so he pushed those triangles together as well. Erin and Emma were not surprised with their student's next move of pushing both squares together to make a rectangle. When asked by the preservice teacher what he knew about what he just did, the student responded, "I have one big rectangle which is made up of two squares and to make those squares, I used four triangles" (Erin & Emma, personal communication, November 21, 2018). As the student took apart the rectangle, Erin and Emma quickly saw how their student decomposed the rectangle into two squares and then breaking apart the two squares into the four original triangles.

This is an example of providing a student with the opportunity to play, explore, and encounter mathematical patterns and structures as a way to help them develop mathematical understandings (Ginsburg, Inoue, & Seo, 1999; Sarama & Clements, 2009). Considering how children's play naturally employs skills of observation and experimentation, it can also lead to the development of specific process models for how

things should be constructed and how things work, thus signaling important elements of engineering thinking. Children's ability to think, reason, and use information allows them to acquire knowledge, understand the world around them, and make appropriate decisions. It is this open-minded curiosity that teachers need to harness and use when teaching mathematics and funnel it in such a way that visualizing, mentally transforming, and composing and decomposing shapes are skills that students develop over time.

*van Hiele Levels of Geometric Thought and elementary students.* In their assessment of the elementary students' spatial reasoning knowledge, preservice teachers placed the work of their student on one of the van Hiele Levels of Geometric Thought. Adelyn and Anna, when working with their elementary student, placed her work at Level 2 "since this is where she needs to be based on grade-level" (personal communication, November 21, 2018). This comment tends to go against the structure of the van Hiele Levels of Geometric Thought. First of all, the progression of individuals through these levels is based on the experiences and instruction the individual encounters rather than age. In order to move to a higher level of sophistication, one must have enough experiences (classroom or otherwise) with these geometric ideas to indicate knowledge gained (Mayberry, 1983). It would reason that preservice teachers would mistakenly confuse the van Hiele Levels with the age of their students because they are being taught, within their university classes, about age-appropriate problems and understandings about the specific grade level they are working with. For example, it is age-appropriate for fourth grade students to be working with operations involving fractions, but this would not be appropriate for Kindergarten students since they are not exposed to fractions or fraction language like halves, fourths or quarters and thirds.

Second, preservice teachers could also draw on the number of geometric experiences a fifth grade student has had in comparison to a first grade student.

Preservice teachers would assume that the older a student is, the more experiences they have encountered both in school and in real life which would give them an advantage and would indirectly place the older student at a higher van Hiele Level. Preservice teachers must be careful about this assumption since children can reach Level 2 in elementary school through rich experiences, but without these experiences, many adults (including teachers) remain at Level 1 all their lives, even if they take a formal geometry course in high school (Mayberry, 1983).

Of the preservice teachers who directly mentioned the van Hiele Level of their student, students in Grades K-1 were at Level 0, students in Grades 1-2 were at Level 1, and students in Grades 3-5 were at Level 2. I envision these levels as an all-encompassing layout where each level is both part of the previous level and a springboard into the next level (see Figure 5.1).



*Figure 5.1.* Elementary Students Placed within the van Hiele Levels of Geometric Thought

From the diagram, we see that Level 0 is the innermost point of the diagram. As you work from the inside of the diagram to the outer most edge, Level 1 is in the middle of the diagram, and it encases Level 0. Level 2 is the outermost level, and it encloses both

Level 0 and Level 1. Elementary students begin at Level 0 and then progress their way up through each level but only if the instruction provided by the teacher is intentional, appropriate, sequenced, and mathematical (Brown, 2009; Clements, Wilson, & Sarama, 2004).

***The classroom as a mathematical learning environment.*** In order for students to maximize their learning, they must feel comfortable within their classroom to: engage with the task, collaborate with their classmates, share their mathematical ideas, listen carefully to what is being said and shared, as well as persevere through struggles and disequilibrium moments (Dance & Kaplan, 2018). They need to be prepared to pose strategies for solving problems, to provide explanations for why things work as they do, and to make conjectures for the consideration of their classmates. Students should be engaging in well-chosen, purposeful, problem-based tasks. A good mathematics problem can be defined as any task or activity for which the students have no prescribed or memorized rules or methods, nor is there a perception by students that there is a specific correct solution method (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murry, Olivier, & Piet, 1997). A good mathematics problem will have multiple entry points and require students to make sense of the mathematics. It should also foster the development of efficient computations strategies as well as require justifications or explanations for answers and methods (Ward, Schoenbrodt, & Riggs, 2010) .

The preservice teacher asked elementary students who participated in this study various questions in order to gain insight into what they knew about the geometric task they were solving. Some elementary students referenced real-life objects they could “see” in the shape they built. Preservice teachers in following the thinking of their student

“must be particularly careful not to assume that children see situations, problems, or solutions as adults do. Instead, good teachers interpret what the child is doing and thinking and attempts to see the situation from the child’s point of view” (Clements & Sarama, 2009, p. 4) and not pushing their own solutions or solving methods on the student.

Ultimately, the quality of mathematics teaching depends on the teachers’ mathematical knowledge (Ball, Hill, & Bass, 2005), and it has a direct effect on student achievement. This knowledge is necessary not only for understanding mathematics but also for being able to impart that understanding to others; it requires the ability to unpack mathematical concepts, “making features of particular content visible to and learnable to students” (Ball, Thames, & Phelps, 2008, p. 400). The teacher plays an integral role in making meaningful connections between the mathematical strands, the real work and other disciplines, and most importantly, “between the intuitive informal mathematics that students have learned through their own experiences and the mathematics they are learning in school” (Ontario Ministry of Education, 2003, p. 14).

When looking at the van Hiele Levels of Geometric Thought of both the preservice teachers as well as the elementary students, it is worth noting that I coded the preservice teachers’ work and placed them at Level 0, Level 1, or Level 2. The random pairing of preservice teachers as practicum partners showed that each practicum team had work from two preservice teachers, which was categorized at two different van Hiele Levels. The work of eight preservice teacher teams had one person at Level 1 and the other person at Level 2. The work of three preservice teacher teams had both individuals at Level 1, and the work of one preservice teacher team had one person at Level 0 and the

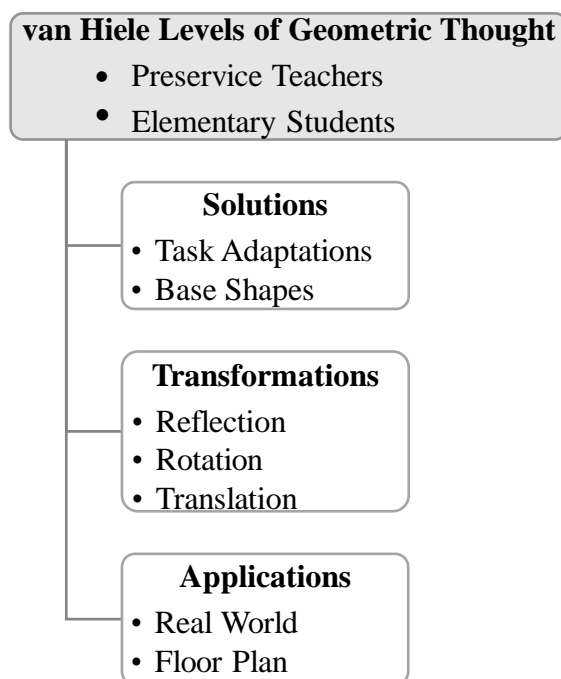


other person at Level 1. The preservice teachers assessed their elementary students work on the task and placed them at Level 0, Level 1, or Level 2. There seems to be a similarity between the van Hiele Levels of the preservice teachers and the elementary students. It would be interesting to determine if the differences in the levels of the preservice teacher teams had any impact on the mathematical experience their elementary student had when solving the task or in how they categorized their elementary students' work. Gaining knowledge for teaching mathematics is by no means an easy task, and it requires practice in applying it. As preservice teachers gain the mathematical knowledge for teaching, they become more capable and confident in helping students extend and formalize their understanding of mathematical concepts ("Maximizing Student," 2011).

### **Cross-Sectional Analysis of My Data**

Spatial reasoning plays a unique role in learning and developing expertise in STEM disciplines (Wai, et al., 2009). Developing spatial skills well before high school may have a more pronounced impact on STEM outcomes. Despite the urgent need for strong spatial reasoning skills in our technology-driven world, our current education system spends little time fostering students' innate visual and spatial reasoning skills as more time is spent on things such as rote memorization of facts and learning how to follow routine procedures (Entwistle & Entwistle, 2003). This is becoming increasingly problematic as the need to become literate in the STEM fields has never been greater. As educators, our goal is to "help students refine the way they think and reason about direction, distance, and location which enhances spatial understanding" (Van de Walle, Karp, & Bay-Williams, 2016, p. 514). Hence, the urgency of change within our classrooms and our preparation of preservice teachers.

Figure 5.2 is a visual representation of the aspects of my data that allowed me to gain insight into the van Hiele levels for both preservice teachers and elementary students. Each aspect has two or more points which were revealed during my analysis that supported my categorization of preservice teachers and elementary students within the van Hiele Level of Geometric Thought Framework. In what follows, I discuss each aspect and make connections to spatial reasoning.



*Figure 5.2.* van Hiele Levels of Geometric Thought with Relevant Aspects from my Data

**van Hiele Levels of Geometric Thought.** The van Hiele Levels of Geometric Thought are critical levels in which geometric understanding is categorized. Students reasoning about geometry develops through five sequential levels in relation to understanding spatial ideas. In order for students to progress through the levels, preservice teachers must provide instruction that is sequential and intentional. Not only are these levels a starting point, they also provide idea locators for activities that will help move the thinking of preservice teachers and elementary students forward. For example,

“children need to go beyond the use of superficial shape labels to recognizing and specifying the defining attributes of shapes. Teachers need to design activities that demonstrate shape distinctions, since these are not immediately obvious to younger children” (Erikson Institute Early Math Collaborative, 2014). As students sort and classify shapes with knowledgeable others, which is connected to the Vygotsky’s work on intersubjectivity, they become aware of rules about shapes, such as a triangle has three sides and three angles (corners) or a cylinder is a rounded form with two flat ends that are in the shape of a circle.

Both the preservice teachers and the elementary students’ work was placed at one of the levels between zero and two using the van Hiele levels of Geometric Thought. The placement of the work of elementary students between Levels 0-2 was expected since they are young learners and have not had as many experiences (classroom or otherwise) with these geometric ideas to move to a higher level of sophistication. In order for students to advance levels, the collection of geometric experiences provided by the classroom teacher is vitally important. To move from Level 0 to Level 1, instructional considerations should include: moving from simple shape identification to identifying properties of figures, shifting from individual models to an entire class of figures (such as all rectangles), providing frequent interactions to draw, build, make, put together (compose) shapes, and take apart (decompose) shapes in both two and three dimensions, and challenge students to test their ideas about observations of a particular shape and if they apply to other shapes of the same kind (van Hiele, 1999; Van de Walle, Karp, & Bay-Williams, 2016).

For the preservice teacher's work, I would have assumed that their van Hiele Levels would have been higher because of the large number of general experiences they have had, but potentially only a few of these experiences may have focused explicitly on geometric thought and ideas which is critical. For example, people might be looking at pictures of the family at a recent wedding. These photos are sized 4 inches by 6 inches, which is definitely a rectangle, but they are not doing anything involving an in-depth mathematical understanding of this geometric shape. Instead, they are focusing on who is in the photo. I believe it is safe to assume that people have geometric shape experiences every day, but it is the depth of these geometric experiences that will help develop mathematical understanding. To move from Level 1 to Level 2, instructional considerations should include: encouraging the making and testing of hypotheses or conjectures as well as explore and test examples, examining properties of shapes to determine necessary and sufficient conditions for a shape to be a particular category of shape, and encouraging students to attempt informal proofs as well as focus on the language of informal deduction (van Hiele, 1999; Van de Walle, Karp, & Bay-Williams, 2016). It is worth mentioning that according to the Common Core State Standards of Mathematics (2010), Level 2 thinking should begin in Grade 5 when students are to classify two-dimensional figures into categories based on their properties.

***Solutions.*** Part two of the Math Matters Tile Assignment required preservice teachers to adapt the original task so that an elementary student in their practicum class (Grades K-5) would be able to solve the task. Most preservice teachers removed things they felt were too difficult for their student to do. For example, most preservice teachers removed the bathroom floor plan, which meant that the elementary students' solutions

were not analyzed more deeply after they were created. Elementary students checked to see if their new solution resembled any of the other solutions they had already made but nothing more. It is unclear if the preservice teachers removed the bathroom floor plan extension because of time constraints when interviewing their elementary student or if this was what they found to be the most difficult and they would have been challenged answering questions that the elementary student might ask when seeing if their newly created polygonal region would fit within the bathroom space.

Preservice teachers have limited experience selecting and constructing worthwhile mathematical tasks, yet this is one of the most important pedagogical decisions they need to make. The tasks teachers pose in their classrooms deserve significant attention because they open or close the students' opportunities for meaningful mathematics learning (Crespo, 2003). Worthwhile tasks, however, are not necessarily one-of-a-kind, innovative, colorful, and complexly designed tasks. Even the most routine of mathematical activities can be constructed into a worthwhile mathematical experience when posed in such a way as to engage students in mathematical inquiry (Schoenfeld, 1989).

Part one from the Math Matters Tile Assignment had the preservice teachers individually work through the task and find the total number of solutions. By having the preservice teachers complete the task prior to adapting the task for an elementary student, this prevented the preservice teachers from posing problems blindly, in other words, ones that they had not themselves solved. This is common for preservice teachers early in their practice, due to lack of experiences and overall understanding, to choose problems without fully exploring or understanding their mathematical and pedagogical potential

(Crespo, 2003). By not working the problem ahead of time, it limits the ways in which the preservice teachers are prepared to anticipate student thinking and the various strategies their students might use to solve the problem. It should also be noted that being a good problem solver does not directly translate into the preservice teacher posing better problems than a preservice teacher who is a poor problem solver (Gonzales, 1996).

When solving the task, most preservice teachers used a “base shape” approach. This base shape consisted of two triangles aligned in such a way that they create another recognizable shape. These recognizable shapes were a square, a triangle, and a parallelogram. Preservice teachers used these base shapes as starting points for creating their solutions to the task. The data is inconclusive as to whether the preservice teachers let their elementary student solve the task using their own method, or if the preservice teacher influenced the ways in which their elementary student solved the task by what they said or by emphasizing keeping a “shape” and building from there. Four of the preservice teachers adapted the task by giving their elementary student three triangles to work with instead of four. This adaptation drastically changed the outcome of the task because it significantly reduced the total number of solutions one could find.

***Transformations of the shapes.*** The idea of shape transformations when creating solutions were made both by preservice teachers and elementary students and reflected on by the preservice teachers. Transforming shapes implies some action being applied to the shape which moves it to a different location or changes its size. Preservice teachers who participated in this study used general terms like flipped, moved, rotated, manipulated and shifted, to describe the transformations of the triangles as they found solutions to the spatial reasoning task. The data is inconclusive concerning the lack of precision with

geometric vocabulary by the preservice teachers as to whether they knew the appropriate geometric vocabulary concerning transformations and chose not to use them or if they did not know there was geometric vocabulary associated with transformations to use. The spatial reasoning task only worked with movements of the shapes (rigid motions) and did not change its size (dilations).

On the other hand, elementary students are not formally exposed to transformations until eighth grade. One of the nine geometry standards in eighth grade from the Common Core State Standards of Mathematics (2010) states “Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates” which deals with transformations of shapes. The Nebraska State Math Standards (2015) agree with the Common Core Standards of Mathematics and have rotations, translations, reflections, and dilations under single transformations as an eighth grade standard. Within the school setting, elementary students in Grades K-5 are not formally exposed to these ideas, but some might have had various experiences elsewhere which revealed transformations about shapes. For example, real-life transformations are happening when you see the reflection of the mountains on top of the water, or the passenger car of a Ferris wheel changing its position as the Ferris wheel rotates, or the translation of an airplane as it moves across the sky.

Indirectly, students in grade four are exposed to a line of symmetry. One of the three geometry standards in fourth grade from the Common Core State Standards for Mathematics (2010) states, “Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry,” which implies a

reflection of the object. The Nebraska State Math Standards (2015) agree with the Common Core State Standards for Mathematics and have lines of symmetry as a fourth grade standard. Erin and Emma (personal communication, November 21, 2018) when working with their fourth grade student commented that they often had to help guide their elementary student through the use of rigid motions (translations, reflections, and rotations) when he was solving the task as he had not been exposed to those concepts yet (as the local school district's mathematics pacing guides have the learning of geometry at the end of the school year). Since the preservice teachers were the ones conducting the interview with the elementary student and reflecting on what happened during it, the data is inconclusive concerning reflections that the elementary student might have done while solving the task.

***Applications.*** Part one of the Math Matters Tile Assignment required the preservice teachers to find the number of solutions they thought were possible with the four given triangles. Next, they had to trim their solutions down to three creative ones that they would present to Mrs. Solid concerning tiling her bathroom and then of those three, pick one that would be their top choice. The bathroom had normal things that one has to work around like a tub/shower, a toilet, a vanity, and a doorway. Measurements accompanied the bathroom floor plan so that the preservice teachers would have a better idea about the size of the open floor space they had for laying these unique tile arrangements and potentially what might fit and what might not. The original problem was attached to a real-life situation that involves deeper mathematical thinking and reasoning than just solving the problem.



The adaptations to the original task by most preservice teachers included removing the application part of the task involving the bathroom floor plan. This changed the mathematical depth of the problem since the application part of the problem was removed. One must be careful with getting rid of the context of the problem since “contexts may help students to make sense of problems, and they may motivate them by helping them to see its application. This application led us to question the relationship between mathematics and real-life” (Back, Foster, Tomalin, Mason, Swan, & Watson, 2013). The other main adaption to the original task was reducing the number of triangles used to find the polygonal regions from four down to three. The data showed that preservice teachers working with younger students in Grades K-1 reduced the number of triangles to three. Reducing the number of triangles does not interfere with the real-world application of the problem. It just significantly reduces the total number of solutions that can be found.

Preservice teachers, Celia and Connie, when working with their first grade student, decided to include the application part of the task. Their student found a solution, and then they would ask their student the following question, “How do you know it will fit in the bathroom?” (personal communication, November 21, 2018). Their student usually responded with describing the middle space on the floor, which was the biggest open space he could see. Celia and Connie were impressed with their student’s ability to look at a given space and decipher if an object/design would fit in that area. Their first grade student had a difficult time understanding that the line used at the top of the floor plan was occupied by the tub/shower since it “looked open” (personal communication, November 21, 2018). Not having the tub/shower colored in indicating that the space was

full and not available confused the first grade student. It would have given a clearer visual of the dimensions of the open spaces on the bathroom floor as well as reduced some of the struggles the first grade student encountered trying to fit his solutions on the floor plan had the items taking up space in the bathroom been colored in.

Another set of preservice teachers, Rashel & Kay, used the bathroom floor plan when working with their third grade student. Their student could visualize how her solution would fit within the open space of the floor plan, and she only tried those shapes that she “saw” fit. Very few of the shapes she built and tried did not fit in the open space on the bathroom floor. When asked by one of the preservice teachers why the shape she made wouldn’t fit a certain way in the floor plan, she responded, “well because if it did the shape would be too big to fit in the area” (personal communication, November 21, 2018). This student realized that the shapes she was making were suppose to be placed into a set area, which was a big indicator that this student understood the spatial restraints that were in place given the floor plan.

### **Implications of the Study**

The purpose of this qualitative descriptive case study was to examine the spatial reasoning skills that preservice teachers possess and how their spatial reasoning skills get used in the enactment of the tasks of teaching; specifically, how they plan for and enact a spatial reasoning task with elementary students. Spatial reasoning plays a unique role in learning and developing expertise in STEM disciplines (Wai, et al., 2009). Developing spatial skills well before high school may have a more pronounced impact on STEM outcomes. Despite the urgent need for strong spatial reasoning skills in our technology driven world, our current education system spends little time fostering students’ innate

visual and spatial reasoning skills as more time is spent on things such as rote memorization of facts and learning how to follow routine procedures (Entwistle & Entwistle, 2003). This is becoming increasingly problematic as the need to become literate in the STEM fields has never been greater.

The following three implications from my research are critical in how educators move forward when preparing elementary preservice teachers to teach mathematics. First, I will discuss why preservice teachers need to develop their geometric thought. Next, I will explain the importance of opportunities for preservice teachers to work on and adapt spatial reasoning tasks. Finally, I will describe why preservice teachers should continually be engaged in spatial reasoning tasks since the learning of such mathematical knowledge is malleable.

**Implication #1 – Preservice teachers need to develop their geometric thought.**

Spatial sense is, “an intuitive feel for one’s surroundings and the objects in them” (NCTM, 1989, p. 49). It is necessary for understanding and appreciating the many geometric aspects of our world. Insights and intuitions about the characteristics of two-dimensional shapes and three-dimensional figures, the interrelationships of shapes, and the effects of changes to shapes are important aspects of spatial reasoning (NCTM, 1989). Students develop their spatial sense by visualizing, drawing, and comparing shapes and figures in various positions (Bennie & Smit, 1999). For a student to possess spatial reasoning and to use it effectively to move to understanding, they would need to be able to visualize how objects can be turned in one’s mind. They need to freely explore how shapes fit together to form larger shapes. These ideas around shapes are critical to the development of one’s geometric thought. By developing geometric thought,

preservice teachers and students make progress on the van Hiele Levels of Geometric Thought. In the enactment of the task, Adelyn and Anna asked their student,

Why did you arrange these tile pieces like this (referring to a square made by two triangles)? She responded with, ‘I made the box and then I kind of put the other two shapes around it.’ Watching her explain this visually, she showed us that she made a square with two of the triangle manipulatives, and then added triangles around it. ... We thought this was an effective approach because this is similar to how we came up with our polygonal regions” (personal communication, November 21, 2017).

This quote makes the connection that the similar approaches both by the preservice teachers and the elementary student are at similar van Hiele Levels of Geometric Thought.

Zavlavsky (1994), in her study, observed that the students’ difficulties with symmetry are often related to the teachers’ misunderstandings of the concept. She goes on to say that the spatial reasoning abilities of students should be a focus as some student participants could not draw the symmetry of the figure according to the symmetry line. Turgut, Yenilmez, and Anapa (2014) believe that deficiencies in these concepts, which are taught in primary education, stem from poor spatial visualization and mental rotation. Studies showed that the spatial abilities of the elementary mathematics teacher and preservice teachers were low (Turgut, 2007). Similar findings in the literature reported that preservice elementary teachers experienced problems related to translation, reflection, rotation, and forming as well as recognizing transformations (Edwards &

Zazkis, 1993; Leikin, Berman, & Zaslavsky, 2000a; Leikin, Berman, & Zaslavsky, 2000b; Yanik & Flores, 2009).

According to Desmond (1997), they also had difficulties determining the correct transformation and motion attributes required to move an object from one location to another, and the results of transformations involving multiple combinations of figures. In analyzing the work of their elementary student, Ana and Addie, reference the difficulty their student had, “with regards to transformation, our student was unfamiliar with translations, reflections, rotations, dilations, symmetry or similarity. This was a topic that was completely foreign to him” (personal communication, November 21, 2017). The conceptualization of these skills can improve the student’s ability to handle an object as a whole, to manipulate that object, and improve their spatial reasoning skills. Skills like symmetry, rotation, and spatial reasoning, which require visualization, integration and rotation skills can be improved using the appropriate media, materials, and technologies (Kurtulus, 2011; Kurtulus & Uygan, 2010; Yolcu & Kurtukus, 2010).

**Implication #2 – Preservice teachers need the opportunity to work on and adapt spatial reasoning tasks.** Preservice teachers should always work each problem they plan on assigning before giving it to students to wrestle with. Jill and Jack, when working with their first grade student, remarked, “after doing this task ourselves, we were able to empathize with our student when she faced disequilibrium. This also enabled us to offer her techniques that we used when we were confused to help get through the problem” (personal communication, November 21, 2017). By working the problem first, the preservice teacher begins to learn what the focus of the problem is and how this might lead them when making adaptations to the problem (Ginsburg, 2016). These adaptations

usually involve narrowing the problem to better get at the heart of what the student knows or needs to know about the problem. According to Crespo, what the preservice teacher focuses on in their lesson will drive what they ask their students (1999). If the preservice teacher has limited knowledge about the topic, this significantly limits the depth of the questions they are prepared to ask during their lesson (Crespo, 1999). If the preservice teacher selects and adapts problems which are unproblematic (e.g., those which could be solved easily and quickly), they eliminate the exposure of their students to deep mathematical thinking (Crespo, 1999).

It is also worth noting that teacher math anxiety is a possible reason associated with a general avoidance of math thus limiting the preparedness of preservice teachers when adapting the task (Crespo, 1999; Ball, 1990; Morris, 1981). Teachers who have high math anxiety, for example, spend less time preparing for math lessons and even use math instruction time for other subjects (Swetman, Munday, & Windham, 1993). Bursal & Paznokas (2006) showed that mathematics anxiety often manifests itself as a lack of understanding – often leading to avoidance of the subject – thus creating a negative attitude toward the subject (Zettle & Raines, 2002). Sells (1973) provides evidence that students who avoid mathematics courses limit their career opportunities.

Since spatial reasoning is not a part of the curriculum, teacher avoidance of spatial reasoning based on anxiety might be even more extreme. Anxiety and attitude may have a direct impact on the teaching methods teachers use, much of which is decontextualized, and goes against the recommendations made by NCTM and others (Bryant, 2009). By increasing the comfort level of teachers as well as preservice teachers with spatial

activities and specifically designed interventions, student achievement in spatial learning could benefit.

**Implication #3 – Preservice teachers should continually be engaged in spatial reasoning tasks.** Visual and spatial reasoning skills are highly malleable (Uttal, et al., 2013). To develop these critical skills, we need to provide preservice teachers with carefully designed, meaningful learning experiences where they can explore, play, and interact with the world around them as they learn (Cherkowski, 2015). This constant engagement helps build the skills necessary for preservice teachers to be confident when assigning a similar type of problem to their students. The more one is intentional and deliberate about practicing a skill, the better you become at accomplishing it (Barr, 2012). For example, one who plays the video game Tetris 30 minutes a day will drastically become better at it compared to someone who plays Tetris 30 minutes once a month (Okagaki & Frensch, 1994).

Spatial reasoning skills are linked to STEM problem solving which relies primarily on spatial thinking; therefore, success in STEM relies primarily on a student's spatial reasoning skills and critical thinking (Stieff & Uttal, 2015). In order for preservice teachers to enhance their own spatial reasoning skills, they need to engage in tasks that involve spatial thinking to solve them (Erkek, Isiksal, & Çakiroglu, 2017). As the spatial reasoning skills of preservice teachers are increased, so also are the spatial reasoning skills of their students both within the problems the preservice teacher chooses to have their student engage with as well as the in-depth questioning that students will be answering while working on the spatial reasoning task (Otumfuor & Carr, 2017). A student's achievement in STEM rests partially on how capable they are at solving

problems that involve reasoning about spatial information. It stands to reason that interventions that improve an individual student's spatial reasoning ability should translate to increased STEM achievement.

### **Limitations**

As with any research project, limitations exist and need to be disclosed (Price & Murnan, 2004). There are four limitations to the current study. First, preservice teachers are working with elementary students in one school district, which may limit the variability within the data collected. Since the scope and sequence of the mathematical units throughout each grade level and the school district are intentionally linked and planned, little variation occurs within the classroom setting. It is possible that elementary students in another school district might have different geometrical experiences, thus yielding different spatial reasoning results.

Second, preservice teachers are just beginning their professional semesters, which might cause discomfort interviewing elementary students at some grade levels due to their own math anxiety. Preservice teachers are new to the demands of the profession and are daily challenged by the mathematical thinking of students who are wrestling with learning mathematical concepts. If preservice teachers are not solid in their own mathematical understanding, they are not able to be flexible in their own thinking when it comes to helping students with their mathematical understanding thus guiding them to connected, efficient strategies when solving math problems.

Third, the timing of the school year when the data was collected is also a limitation. Since my data was collected at the end of second quarter and few geometric concepts had been covered in the classrooms of the participating school district up to this



point, this may have hindered the depth of the spatial reasoning ideas that both elementary students and preservice teachers recalled and used in solving the task. If this assessment project was given during the fourth quarter when the majority of the geometry standards would have been covered in the participating school district, I believe the results from both the elementary students and the preservice teachers would be different. Also, my data being collected within a short period of time at the end of the semester which may result in limited time to work with individual elementary students based on the demands of the school curriculum needing to be met.

Finally, the data collected and the phone interviews were conducted and analyzed by one researcher, which may limit the understanding of the data to the researcher's own biases. By having more than one person analyzing the data, the interrater reliability would be another way in which the coded data would be better defined and explained, thus neutralizing the researcher's biases.

### **Recommendations for Future Research**

Because this is a limited look at the spatial reasoning skills of one group of preservice teachers, the findings will require more corroborative studies to be generalizable. Therefore, studying preservice teachers as they are engaging in multiple spatial reasoning tasks could strengthen the findings and help confirm study claims. A larger sample size could also potentially make it easier to define where preservice teachers lie on the continuum of the van Hiele levels. Since it takes rich and calculated geometric experiences in order to move to higher levels of sophistication, analyzing more samples of completed work helps one become more skilled knowing what work belongs to which van Hiele level. Additional insights could be gained from examining preservice

teachers who are enrolled in a mathematics course focused on geometry instead of preservice teachers enrolled in a mathematics course whose focal point is number and operation as was the case in this study.

Finally, it would be worth investigating a school which incorporates the components of mathematical play and how these concepts enhance the flexible thinking needed to boost one's spatial reasoning. Mathematical play involves tasks that stretch the minds of the students who are engaging with them. Components of mathematical play were used when solving the Math Matters Tile Assignment and promoted spatial reasoning through: classifying (sorting) objects, exploring magnitude (describing and comparing the size of objects), enumerating (counting or subitizing), investigating dynamics (putting things together, taking them apart, or exploring motions such as flipping an object), studying pattern and shape (exploring geometry properties or identifying or creating patterns/shapes), and exploring spatial relations (describing or drawing a location or direction) (Ginsburg, Lee, & Boyd, 2008). These tasks provide an opportunity for creative solutions, in-depth mathematical application to real-life work, and plenty of time for trial and error. As students wrestle with the task, they gain deeper mathematical intelligence and build upon their spatial reasoning skills.

### **Recommendations for Teacher Education Programs**

Findings from this study reveal that teacher education programs have a significant role in developing knowledge and providing preservice teachers with experiences to develop spatial reasoning skills. Support for preservice teacher learning in this area is needed, not only to bring awareness of what spatial reasoning is and its importance, but to provide guidance on how to support student spatial reasoning to foster their

mathematical development. As an area of research and learning, more needs to be understood about how spatial reasoning benefits students' mathematics learning, and how to implement this skill into classrooms and lesson design. With the absence of spatial reasoning in textbooks and classroom instruction, Wheatley (2002) suggests the need to develop lessons and activities to supplement instruction.

Another important consideration for the development of preservice teachers is being intentional about what mathematics preservice teachers are learning and the manner in which they are learning it because these strategies carry over into their ability to teach students who have real ideas (both correct and incorrect) about solving mathematical problems which might be contrary to the strategy that the preservice teacher feels comfortable teaching. For example, preservice teachers can often comfortably carry out algorithms well (i.e., have sufficient procedural knowledge), yet struggle or are unable to explain why algorithms work (i.e., have little underlying conceptual knowledge) making it difficult to attend to students' misconceptions. All in all, preservice teachers must possess excellent problem solving and mathematical reasoning skills, a deep understanding of the mathematics they will teach including basic math ideas, and the habits of mind of a mathematical thinker (National Research Council, 2010).

Finally, teacher education programs need to regularly incorporate problems within all elementary preservice teacher mathematical preparatory classes which provide deep mathematical understanding wrapped with spatial reasoning ideas and concepts. By including these types of problems within their preparatory classes, preservice teachers would have a resource to draw from when they are responsible for the learning within

their own classroom. Through regular fusion during the school year, these spatial reasoning problems would help bridge mathematical connections of the many standards and objectives their students need to master. “Improving the mathematics learning of every child depends on making central the learning opportunities of our teachers” (Ball, 2003, p. 9).

### **Conclusion**

Based on this qualitative descriptive case study, I conclude that spatial reasoning is often difficult for preservice teachers to grasp. The flexible thinking that is required when working with a spatial reasoning task may be difficult for preservice teachers due to their limited experiences working with these types of problems. It might be possible this lack of exposure by the preservice teachers creates a lack of confidence in their own mathematical ability. Strong mathematics learners are those who think deeply, make connections, and visualize (Boaler, Chen, Williams, & Cordero, 2016). A focus on spatial reasoning allows mathematics to become a more visual endeavor. How learners represent and connect pieces of knowledge is a key factor in whether they will understand it deeply and can use it in problem solving (NRC, 2001). Thus, learning with understanding makes one mathematically powerful since they have developed the skills necessary to begin solving an unfamiliar problem.

Geometry and measurement are often difficult topics for young math students. Clements argues that part of this problem stems from classroom materials that are not exact and misleading (Erikson Institute Early Math Collaborative, 2013). For example, in the math class, students are taught that a triangle has three sides and three angles (corners) known as vertices. When these same students go to music class and the teacher

talks about a musical instrument called a “triangle,” this instrument does not fit the definition they know because this “triangle” has three sides, but they do not connect and it has three rounded corners, but they are not vertices. Notice the confusion we create for students by not being precise and using language for images that resemble something they are not. It is this blurry vision we construct when picking words to describe what we see that confuse students since mathematics is about the precisions of thinking and reasoning.

Ultimately, it is important to develop the spatial reasoning skills of preservice teachers using various methods since these skills are malleable. Over time and through intentional practice, one should see an increase in the following actions which involve spatial reasoning: perspective taking, visualizing, locating, orienting, dimension shifting, pathfinding, sliding, rotating, reflecting, diagramming, modeling, symmetrizing, composing, decomposing, scaling, map-making, and designing (Davis, Okamoto, & Whiteley, 2015). Complex mathematical problem solving rests on spatial reasoning skills and links between spatial and mathematical skills being established (Gunderson, Ramirez, Beilock, & Levine, 2012). Spatial thinking, or mentally manipulating information about the structure of the shapes and spaces in one’s environment, is critical for developing skills that support STEM learning (Newcombe, 2010; Wolfgang, Stannard, & Jones, 2001). By developing the spatial reasoning skills of preservice teachers, these skills should carry over into their work in the classroom with their elementary students.

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## Appendix A

### Assessment Project

Completed by Mon, Dec 4

#### **Purpose:**

The purpose of this assignment is to analyze a child's understanding of a mathematics topic through a diagnostic task-based interview. You will solve a mathematics task, adapt the task for suitability with a child in your practicum class, work with your practicum partner to interview a child, and analyze what the interview reveals about the child's understanding of mathematics.

#### **The Setting for the Task:**

You have recently been hired by the company Math Matters. This company is one that specializes in the designing and installing of unique tile floors. You have been assigned to tile the main bathroom floor of a new home your client, Mrs. Solid, is building. See Figure 1 for the floor plan of the bathroom you have been assigned to tile. Mrs. Solid has a special request. She has four uniquely designed (different colors/pattern) tile pieces she wants to incorporate (see Figure 2). You are not concerned about the tile that will surround these four tile pieces as you would be allowed to cut those to fit. The four unique tile pieces cannot be cut and must be used as they are. Mrs. Solid also specified when putting two triangles together, two sides must completely touch each other (not just corners or partial sides).

#### **Part 1: Completing the Task (Individually)**

1. Using all four unique tiles each time, figure out how many different polygonal regions you can make.
  - a. Write a convincing statement which proves that you have found all the possibilities using these four unique tiles. Your statement should include:
    - i. How each of your polygonal regions is different from the others
    - ii. Show that your solution is complete, i.e., there are no more polygonal regions to be found.
2. Show Mrs. Solid the designs you created.
  - a. Provide a diagram for each polygonal region you have created.
  - b. Narrow the number of creative designs down to three. Explain to Mrs. Solid why your top three choices are the most creative.
  - c. Finally, pick your top design. Write an argument convincing Mrs. Solid that it is the best option for her bathroom floor.

**Upload all of Part 1 to Canvas on or before Fri, Nov. 10**

#### **Part 2: Child Interview (With Practicum Partner)**

3. Before the Interview
  - a. Read p. 84-95 in Van de Walle textbook
  - b. Work with your Cooperating Teacher to identify a student to interview
  - c. Work with your practicum partner to adapt the Bathroom Tiling Task so that you will be able to use the task as a way to assess your student's current understanding of spatial reasoning.
  - d. Prepare 7-8 specific questions you will ask to probe the student's thinking while they are doing the task

**Upload Part 2 to Canvas on or before Fri, Nov. 17**

### Part 3: During/After the Child Interview

4. During the Interview
  - a. Work with your CT to schedule a time to administer the diagnostic interview (Nov 21, Nov 28, or Nov 30). The interview should take about 10-15 minutes and should be completed in an environment with minimal distractions (hallway, study space, library, etc.). This is a diagnostic interview and it is important that you NOT teach or correct the student's reasoning or answers. Your goal is to understand the student's reasoning. You may wish to begin the interview by explaining your goal and reassuring the student that you will ask many questions regardless of whether the answer is correct or not.
  - b. Provide the student with paper, pencil, and/or manipulatives/technology appropriate for the task. If they want to change one of their solutions, ask them NOT to erase their work, but instead draw an X through it. Keep their paper as documentation of their work and/or take photographs of their work (be careful to avoid photographs of the student's face or student's name).
  - c. Audio record the interview with your student. You will use direct quotes from the transcript as evidence to support your thoughts about what your student understands about spatial reasoning.
  - d. During the interview, one of you should interview/interact with the student and the other should take detailed notes about their strategies, solutions, and responses to the questions being asked. Be sure to include the order in which the student creates their polygonal regions.
5. After the Interview – Your report should include:
  - a. Type up your notes as soon as possible after the interview (while the interview is still fresh in your mind)
  - b. Pictures – Be sure to include a brief explanation of what the picture(s) is showing and what the student was saying or doing during this moment.
  - c. A link to the audio recording from your interview
  - d. Transcribe your audio recording of the interview so you can refer to it and use direct quotes as evidence when answering the prompts below.
  - e. Respond to the following prompts:
    - i. What prior knowledge did the student need in order to solve the problem?
    - ii. Did your understanding of mathematics have any impact on your ability to use this task to provide your student with a rich learning experience? Explain.
    - iii. What strategies did the student use to solve the task?
      1. Did the student try different approaches? If so, describe them using direct quotes from the transcript of the interview.
    - iv. In what ways did the questions you asked extend the student's thinking? Use direct quotes from the transcript to support your reasoning.
  - f. Summarize the student's spatial reasoning understanding demonstrated in the interview. Your assessment should reference the student's understanding (use direct quotes from the transcript) of specific mathematics ideas, models, and representations (reference Van de Walle Chap. 20). Accuracy counts when assessing what a student understands about a particular topic. Your analysis of the student's spatial reasoning understanding should be approximately 1 page in length.

Upload Part 3 to Canvas on or before Mon, Dec 4

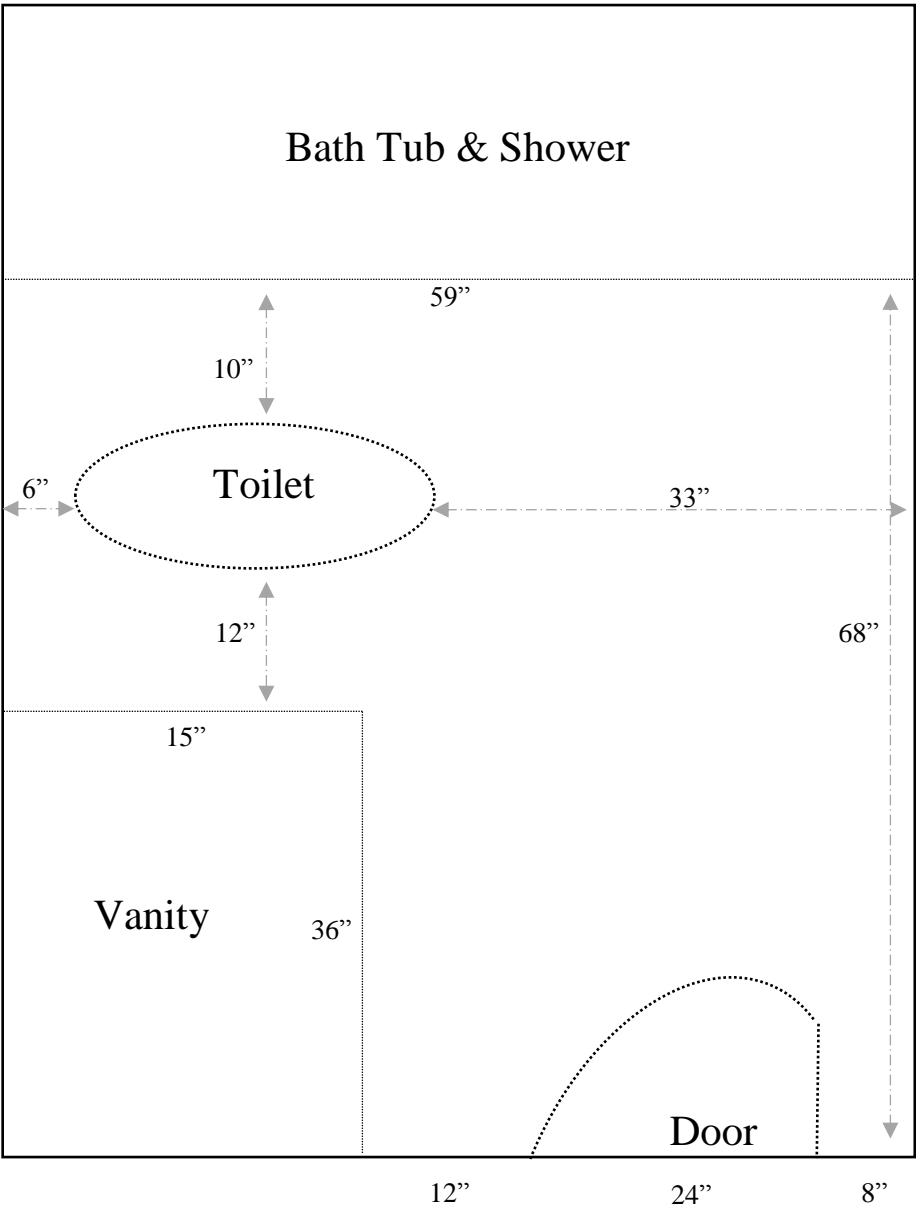


Figure 1 – Bathroom Floor Plan

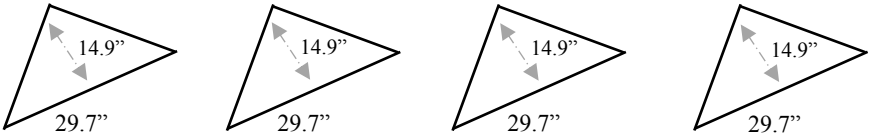
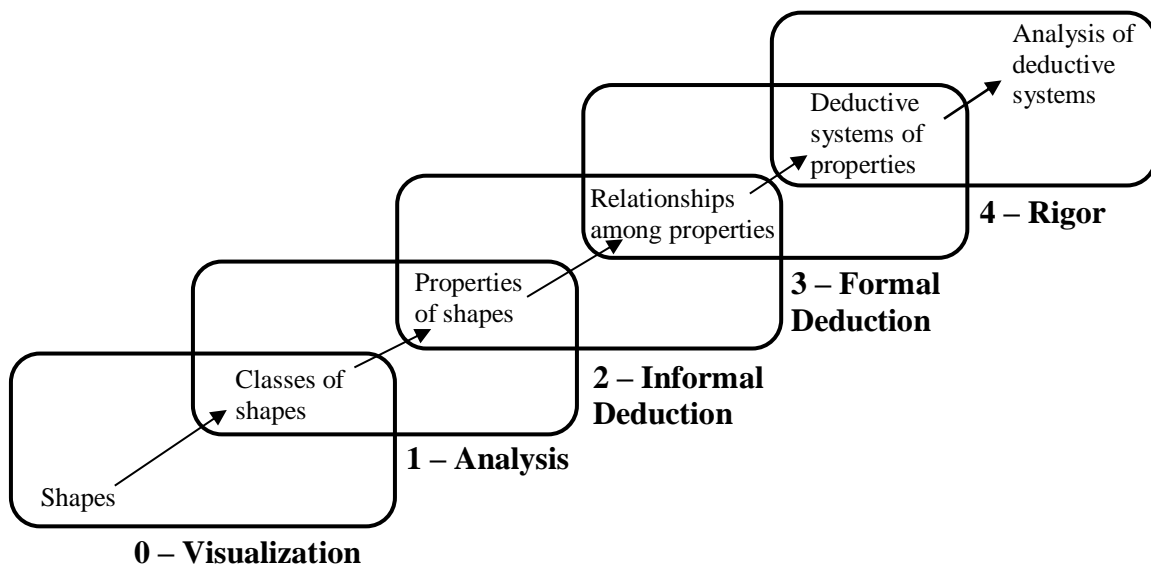


Figure 2 – The Four Distinct (different colors/patterns) Tile Pieces

## Appendix B

### The van Hiele Theory of Geometric Thought



Level	Name	Description
0	Visualization	Students recognize figure by appearance alone, often by comparing them to a known prototype. At this level, students make decisions based on perception, not reasoning.
1	Analysis	Students see figures as collections of properties. They can recognize and name properties of geometric figures, but don't see the relationships between these properties. When describing an object, a student might list all the properties the student knows, but not discern which properties are necessary and which are sufficient to describe the object.
2	Informal deduction	Students perceive relationships between properties and between figures. At this level, students can create meaningful definitions and give informal arguments to justify their reasoning. Logical implications and class inclusions, such as squares being a type of rectangle, are understood.
3	Formal deduction	Students can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions. At this level, students should be able to construct proofs such as those typically found in a high school geometry class.
4	Rigor	Students at this level understand the formal aspects of deduction, such as establishing and comparing mathematical systems. Students at this level can understand the use of indirect proof, proof by contrapositive, and non-Euclidean systems.

Mason, M. (2002). The van Hiele Levels of Geometric Understanding. *Professional Handbook for Teachers*. Geometry: Explorations and Applications. MacDougal Littell Inc.

## **Appendix C**

### **Semi-Structured Phone Interview Questions**

Start by saying – Thank you for returning my call. Today’s interview is not evaluative in any way, I would just like to gain some insight into your thinking about the Assessment Project and your work with your elementary student.

1. Identify an instance when working with your elementary student that was rewarding? Why?
2. Identify an instance when working with your elementary student that was challenging? Why?
3. When working with your elementary student, can you explain a time where you had to adapt in the moment while they were engaged in doing the task?
4. On a scale from 1 to 10, where 1 is not comfortable at all to 10 which is extremely comfortable, what is your comfort level with spatial reasoning? Why?
5. Do you think spatial reasoning is an important aspect of being a good elementary teacher? Why or why not?
6. Is there anything else you would like to add?

## Appendix D

### Teacher Question Types

Question Type	Description	Examples
1. Linking and applying	Points to relationships among mathematical ideas and mathematics and other areas of study/life	<ul style="list-style-type: none"> <li>• In what other situations could you apply this?</li> <li>• Where else have we used this?</li> </ul>
2. Extending thinking	Extends the situation under discussion to other situations where similar ideas may be used	<ul style="list-style-type: none"> <li>• Would this work with other numbers?</li> <li>•</li> </ul>
3. Probing, getting students to explain their thinking	Asks student to articulate, elaborate or clarify ideas	<ul style="list-style-type: none"> <li>• How did you get 10?</li> <li>• Can you explain your idea?</li> </ul>
4. Exploring mathematical meanings and/or relationships	Points to underlying mathematical relationships and meanings	<ul style="list-style-type: none"> <li>• Where is this x on the diagram?</li> <li>• What does probability mean?</li> </ul>
	Makes links between mathematical ideas and representations	
5. Gathering information, leading students through a method	Requires immediate answer	<ul style="list-style-type: none"> <li>• What is the value of x in this equation?</li> <li>• How would you plot that point?</li> </ul>
	Rehearses known facts/procedures	
	Enables students to state facts/procedures	
6. Generating discussion	Solicits contributions from other members of class	<ul style="list-style-type: none"> <li>• Is there another opinion about this?</li> <li>• What did you say, Justin?</li> </ul>
7. Orienting and focusing	Helps to focus on key elements or aspects of the situation in order to enable problem-solving	<ul style="list-style-type: none"> <li>• What is the problem asking you?</li> <li>• What is important about this?</li> </ul>
8. Inserting terminology	Once ideas are under discussion, enables correct mathematical language to be used to talk about them	<ul style="list-style-type: none"> <li>• What is this called?</li> <li>• How would we write this correctly?</li> </ul>
9. Establishing context	Talks about issues outside of math in order to enable links to be made with mathematics	<ul style="list-style-type: none"> <li>• What is the lottery?</li> <li>• How old do you have to be to play the lottery?</li> </ul>

Boaler, J., & Brodie, K. (2004). The importance, nature and impact of teacher questions. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the 26<sup>th</sup> annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* – Vol. 2 (pp. 773-782). Toronto, Canada.