JP Journal of Algebra, Number Theory and Applications

http://www.pphmj.com
http://dx.doi.org/10.17654/NT039010011
Volume 39, Number 1, 2017, Pages 11-20 **ISSN: 0972-5555**

$*_p$ -MODULES AND A SPECIAL CLASS OF MODULES DETERMINED BY THE ESSENTIAL CLOSURE OF THE CLASS OF ALL $*$-RINGS

Puguh Wahyu Prasetyo[1], **Indah Emilia Wijayanti**[2] **and Halina France-Jackson**[3]

[1,2]Department of Mathematics
Universitas Gadjah Mada
Indonesia
e-mail: ind_wijayanti@ugm.ac.id

[1]Department of Mathematics Education
Ahmad Dahlan University
Indonesia
e-mail: puguhwp@gmail.com

[3]Department of Mathematics
Nelson Mandela Metropolitan University
South Africa
e-mail: cbf@easterncape.co.uk

Abstract

A ring A is called a $*$-ring if A is a prime ring and A has no nonzero proper prime homomorphic image. The $*$-ring was introduced by Korolczuk in 1981. Since $*$-rings have an important role in radical theory of rings, the properties of $*$-ring have been being investigated

Received: August 12, 2016; Accepted: October 19, 2016
2010 Mathematics Subject Classification: 16D60, 16S90, 16N60, 16N80.
Keywords and phrases: $*$-ring, homomorphic image, special class of rings, special class of modules, prime module.

intensively. Since every ring can be viewed as a module over itself, the generalization of $*$-ring into module theory is an interesting investigation. We would like to present the generalization of $*$-rings in module theory named $*_p$-modules. An *A*-module *M* is called a $*_p$-module if *M* is a prime *A*-module and *M* has no nonzero proper prime submodule. According to the result of our investigation, we show that every $*$-ring is a $*_p$-module over itself. Furthermore, let *A* be a ring, let *M* be an *A*-module, and let *I* be an ideal of *A* with $I \subseteq (0 : M)_A$, where $(0 : M)_A = \{a \in A \mid aM = \{0\}\}$. We show that *M* is a $*_p$-module over *A* if and only if *M* is a $*_p$-module over A/I. On the other hand, the essential closure $*_k$ of the class of all $*$-rings is a special class of rings. As the last result of our investigation, we present the special class of modules determined by $*_k$.

1. Introduction

Let *A* be a ring. A ring *A* is called a *prime ring* if $\{0\}$ is a prime ideal of *A* (Gardner and Wiegandt [5]). Any homomorphic image of a ring *A* can be represented as A/I, where *I* is an ideal of *A*. The homomorphic image A/I of *A* is called a *prime homomorphic image* if A/I is a prime ring. The class of rings σ is hereditary if σ contains all ideals of a ring $A \in \sigma$. The class of rings σ is essentially closed if σ is closed under essential extensions. Let π denote the class of all prime rings. A subclass μ of π is called a *special class* if μ is hereditary and μ is essentially closed. For hereditary class of rings ϱ, the upper radical $\mathcal{U}(\varrho)$ is defined as the class of all ring *A* such that *A* has no nonzero homomorphic image in ϱ. The prime radical β is the upper radical determined by the class of all prime rings π.

A prime ring *A* is called a $*$-*ring* if *A* has no nonzero proper ideal *I* of *A* such that A/I is a prime ring (Korolczuk [6]). Some properties of $*$-rings were presented in (France-Jackson [2]). $*$-rings have been being studied intensively in radical theory of rings because of Gardner's question mentioned in (Gardner [4]). Let $*$ denote the class of all $*$-rings and let $*_k$ denote the essential closure of $*$. The essential closure $*_k$ of $*$ is a special class of rings. Gardner asked whether the prime radical β coincide with the upper radical $\mathcal{U}(*_k)$ determined by $*_k$. (France-Jackson et al. [3]) have given an alternative solution of this question to have a positive answer.

On the other hand, let M be an A-module. An A-module M is called a *prime A-module* if $AM \neq \{0\}$ and for $m \in M$ and $J \triangleleft A$ such that $Jm = \{0\}$ implies $m = 0$ or $JM = \{0\}$. The set $(0 : M)_A = \{a \in A | aM = \{0\}\}$ is called an *annihilator* of an A-module M. An A-module is faithful if $(0 : M)_A = \{0\}$ (Gardner and Wiegandt [5]).

Theorem 1.1 (Gardner and Wiegandt [5]). *Let A be a ring and let* $I \trianglelefteq A$.

(1) *If* M *is an* A/I*-module, then with scalar multiplication* $am = (a + I)m$, M *forms an A-module with* $I \subseteq (0 : M)_A$.

(2) *If M is an A-module and* $I \subseteq (0 : M)_A$, *then M is an* A/I*-module with the scalar multiplication* $(a + I)m = am$.

(3) *If M is an A-module and* $I \subseteq (0 : M)_A$, *then N is a submodule of the* A/I*-module if and only if N is a submodule of the A-module M.*

(4) $(0 : M)_A/I = (0 : M)_{A/I}$.

(Gardner and Wiegandt [5]) For every ring A, let Σ_A denote the class of all A-modules M with $AM \neq \{0\}$, and $\Sigma = \bigcup \Sigma_A$. Let $\ker(\Sigma_A) = \bigcap ((0 : M)_A | M \in \Sigma_A)$ and we consider the class Σ might satisfy the following conditions:

1. (M1) If $M \in \Sigma_{A/I}$, then $M \in \Sigma_A$.

2. (M2) If $M \in \Sigma_A$ and $I \trianglelefteq A$, $I \subseteq (0 : M)_A$, then $M \in \Sigma_{A/I}$.

3. (M3) If $\ker(\Sigma_A) = \{0\}$, then $\Sigma_B \neq \{\varnothing\}$ for all nonzero ideals B of A.

4. (M4) If $\Sigma_B \neq \{\varnothing\}$ whenever $\{0\} \neq B \trianglelefteq A$, then $\ker(\Sigma_A) = \{0\}$.

Proposition 1.2 (Gardner and Wiegandt [5])**.** *Let A be a ring and let* $I \trianglelefteq A$. *Then there is a prime A-module M such that* $(0 : M)_A = I$ *if and only if I is a prime ideal of A.*

Definition 1.3 (Gardner and Wiegandt [5])**.** For every ring A, let Σ_A be a class of prime A-modules and let $\Sigma = \bigcup \Sigma_A$. The class Σ is called a *special class of modules* if Σ satisfies (M1), (M2), and the following conditions:

1. (SM3) If $M \in \Sigma_A$, $B \trianglelefteq A$ and $BM \neq \{0\}$, then $M \in \Sigma_B$.

2. (SM4) If $B \trianglelefteq A$ and $M \in \Sigma_B$, then $BM \in \Sigma_A$.

If Σ is a special class of modules, then $\mu = \{A \mid A$ has a faithful module in $\Sigma_A\}$ is a special class of rings. Conversely, if μ is a special class of rings and we define $\Sigma_A = \{M \mid M$ is a prime A-module and $A/(0 : M)_A \in \mu\}$, then $\Sigma = \bigcup \Sigma(A)$ is a special class of modules (Nicholson and Watters [7]).

Example 1.4. Let π denote the class of all prime rings and for every ring A let $\Sigma_A = \{M \mid M$ is a prime A-module and $A/(0 : M)_A \in \pi\}$. Since π is a special class of rings, the class $\Sigma = \bigcup \Sigma_A$ is a special class of modules.

These basic theories motivate us to investigate the special class of modules generated by $*_k$.

2. Main Results

Let M be an A-module. A homomorphic image M/N of A-module M is called a *prime homomorphic image* of M if M/N is a prime A-module. Since every ring can be viewed over itself, we will give a new type of module named $*_p$-module. This kind of module is motivated by the existence of $*$-ring.

Definition 2.1. Let M be an A-module. A-module M is called a $*_p$-*module* if M is a prime A-module and M has no nonzero proper prime homomorphic image.

The necessary and sufficient condition for A-module M to be a $*_p$-module is given below.

Lemma 2.2. *Let M be an A-module. The following conditions are equivalent*:

1. *M is a $*_p$-module over A.*

2. *M is a prime A-module and every proper prime submodule N of M implies* $N = \{0\}$.

Proof. (1) $\Rightarrow$ (2) Let M be a $*_p$-module over A. By the definition, we have M is a prime A-module. Furthermore, M has no nonzero proper prime image. Let N be a proper prime submodule of M. Suppose $N \neq \{0\}$. Then M/N is a nonzero proper prime homomorphic image of M, a contradiction.

(2) $\Rightarrow$ (1) Let M be a prime A-module and every proper prime submodule N of M implies $N = \{0\}$. Suppose M/N is a nonzero prime homomorphic image of M. This gives N is a proper prime submodule of M. This implies that $N = \{0\}$. So, we may conclude that M has no nonzero proper prime homomorphic image. □

Some modules are naturally $*_p$-module. In the next lemma, we show that every simple module M over a ring A is a $*_p$-module.

Lemma 2.3. *Let A be a commutative ring and M be an A-module. If M is a simple A-module, then M is a $*_p$-module over A.*

Proof. Let $a \in A$ and $m \in M$ such that $am = 0$. Suppose $a \neq 0 \Rightarrow a \in (0 : m)$. Thus, $m \in M_r$, where M_r is a torsion submodule of M. Since M is a simple A-module, we have $M_r = \{0\} \Rightarrow m = 0$ or $M_r = M \Rightarrow a \in (0 : M)$. Hence, M is a prime A-module. Since M is a simple A-module, A-module M has no nonzero proper prime homomorphic image. So, M is a $*_p$-module. □

Example 2.4. 1. (Adkins and Weintraub [1]). An abelian group A is a simple $\mathbb{Z}$-module if and only if A is a cyclic group of prime order. Hence, A is a $*_p$-module over the ring $\mathbb{Z}$ of integers if A is a cyclic group of prime order.

2. The integers modulo prime number $\mathbb{Z}_p$ is a simple $\mathbb{Z}$-module. Hence, A is a $*_p$-module over $\mathbb{Z}_p$.

3. (Adkins and Weintraub [1]). Let $V = \mathbb{R}^2 = \{(a, b) | a, b \in \mathbb{R}\}$ and consider the linier transformation $T : V \to V$ defined by $T(u, v) = (-v, u)$. Then the $\mathbb{R}[X]$-module V_T is a simple $\mathbb{R}[X]$-module. So, we may deduce that V_T is $*_p$-module over $\mathbb{R}[X]$–.

The following theorem shows that every $*$-ring is a $*_p$-module.

Theorem 2.5. *Let A be a ring. If A is a $*$-ring, then A is a $*_p$-module over itself.*

Proof. We will show that A is a prime A-module. For this step, we can follow Corollary 3.14.17 in (Gardner and Wiegandt [5]) or we give the other way to proof. Since A is a prime ring, $AA = A^2 \neq \{0\}$. Suppose A is not a prime A-module. Then there exists $J \lhd A$ with $JA \neq \{0\}$ and $0 \neq a \in A$ such that $Ja = \{0\}$. Since $0 \neq a \in A$, we can construct the nonzero ideal $\langle a \rangle$ of A generated by a such that $J\langle a \rangle = \{0\}$, contrary to A is a prime ring.

Suppose *A* is not a $*_p$-module. Then there exists a nonzero proper prime submodule *I* of *A*. In the other words, A/I is a prime *A*-module. Now define $(0 : A/I)_A = \{a \in A | a(A/I) = \{0\}\}$. Clearly $(0 : A/I)_A \neq \{0\}$, because $0 \neq I \subseteq (0 : A/I)_A$. We will show that $(0 : A/I)_A$ is a prime ideal of *A*. Let $J, K \triangleleft A$ such that $JK \subseteq (0 : A/I)_A$. If $K \nsubseteq (0 : A/I)_A$, let $k \in K$, $\bar{a} = a + I \in A/I$ be such that $k\bar{a} = \{\bar{0}\}$. Then $J(k\bar{a}) \subseteq JK\bar{a} \subseteq (0 : A/I)_A \bar{a} = \{\bar{0}\}$. So, $J(A/I) = \{\bar{0}\}$. This gives $J \subseteq (0 : A/I)_A$. Hence, $(0 : A/I)_A$ is a prime ideal of *A*, contrary to *A* is a $*$-ring. □

The converse above is not true in general.

Example 2.6. The ring $J = \{2x/2y + 1 | \gcd(2x, 2y + 1) = 1, x, y \in \mathbb{Z}\}$ is a $*$-ring. By Theorem 2.5, we have *J* is a $*_p$-module over *J*. However, the module *J* over itself is not a simple module.

Lemma 2.7. *Let A be a ring. If M is a* $*_p$*-module over A, then every nonzero proper homomorphic image of a* $*_p$*-module over A is not a* $*_p$*-module over A.*

Proof. Let *A* be a ring and consider *M* is a $*_p$-module over *A*. Suppose M/N is a nonzero proper homomorphic image of *M*. Clearly, M/N is not a prime *A*-module. Hence, M/N is not a $*_p$-module over *A*. □

In the following theorem, we give a sufficient condition for an *A*-module *M* to be a $*_p$-module over *A*.

Theorem 2.8. *Let I be an ideal of a ring A with* $I \subseteq (0 : M)_A$ *and let M be an A-module such that* $AM \neq \{0\}$. *If M is a* $*_p$*-module over the factor ring* A/I, *then M is a* $*_p$*-module over A.*

Proof. Let M be a $*_p$-module over the factor ring A/I. Then M is a prime A/I-module. By Proposition 3.14.15 in (Gardner and Wiegandt [5]), we have M is a prime A-module. Suppose there exists a nonzero proper prime homomorphic image M/N of M over A. It follows from Proposition 3.14.15 in Gardner and Wiegandt [5], we have M/N is a prime A/I-module. In the other words, M has a nonzero proper prime homomorphic image over A/I, contrary to M is a $*_p$-module. Hence, M has no nonzero proper prime homomorphic image over A. Thus, M is a $*_p$-module over A. □

The following theorem shows the consequence of the existence of a $*_p$-module M over a ring A.

Theorem 2.9. *Let I be an ideal of a ring A such that $I \subseteq (0 : M)_A$ and let M be an A-module such that $AM \neq \{0\}$. If M is a $*_p$-module over the ring A, then M is a $*_p$-module over A/I.*

Proof. Let M be a $*_p$-module over the ring A and let I be an ideal of a ring A such that $I \subseteq (0 : M)_A$. Clearly, M is a prime A/I-module. Suppose there exists a nonzero proper prime homomorphic image M/N of M over A/I. Then M/N is a prime A/I-module. By Proposition 3.14.15 in (Gardner and Wiegandt [5]), we have M/N is a prime A-module, contrary to M is a $*_p$-module. So, we may conclude that M is a $*_p$-module over A/I. □

Theorem 2.10. *Let $*_k$ be the essential closure of the class of all $*$-rings and for every ring A let $\sum_A = \{M \mid M$ is a prime A-module with $A/(0 : M)_A \in *_k\}$. Then the class $\sum = \bigcup \sum_A$ is a special class of modules.*

Proof. We can follow the construction of a special class of modules generated by a special class of rings presented in (Nicholson and Watters [7]) or we will explain the detail of proof by showing that the class $\sum = \bigcup \sum_A$ satisfies (M1), (M2), (SM3), and (SM4). Let M be an A-module such that

$M \in \sum_{A/I}$. Then M is a prime A/I-module with $(A/I)/(0:M)_{A/I} \in *_k$. By Proposition 3.14.15 in (Gardner and Wiegandt [5]), M is a prime A-module. Let $\bar{a} \in (0:M)_{A/I} \Rightarrow \bar{a}M = \{0\}$, where $\bar{a} = a + I$ for some $a \in A$. Since $\{0\} = (a+I)M = aM$, $a \in (0:M)_A$ and by the assumption $I \subseteq (0:M)_A$ implies $\bar{a} = a + I \in (0:M)_A/I$. Hence, $(0:M)_{A/I} \subseteq (0:M)_A/I$. On the other hand, let $a \in (0:M)_A \Rightarrow aM = \{0\}$. Since $\{0\} = aM = (a+I)M \Rightarrow a + I \in (0:M)_{A/I}$. Hence, $(0:M)_A/I \subseteq (0:M)_{A/I}$. So, we may conclude that $(0:M)_A/I = (0:M)_{A/I}$. This gives us the following isomorphism $A/(0:M)_A \cong (A/I)/(0:M)_A/I = (A/I)/(0:M)_{A/I}$ $\in *_k$. We can infer that $M \in \sum_A$.

Let $M \in \sum_A$. Then M is a prime A-module with $A/(0:M)_A \in *_k$. By following Proposition 3.14.15 in (Gardner and Wiegandt [5]), we have M is a prime A/I-module, where $I \subseteq (0:M)_A$. Since $A/(0:M)_A \in *_k$ and $(A/I)/(0:M)_{A/I} \cong A/(0:M)_A$, we have $M \in \sum_{A/I}$.

Let $M \in \sum_A$ and let $B \triangleleft A$ such that $BM \neq \{0\}$.

By Proposition 3.14.13 in (Gardner and Wiegandt [5]), we have M is a prime B-module. Since $B/(0:M)_B = B/(B \cap (0:M)_A) \cong (B + (0:M)_A)$ $/(0:M)_A \triangleleft A/(0:M)_A \in *_k$ and $*_k$ is a special class of rings, we have $B/(0:M)_B \in *_k$.

Let $B \triangleleft A$ and let $M \in \sum_M$. Then M is a prime B-module with $B/(0:M)_B \in *_k$. By Proposition 3.14.14 in (Gardner and Wiegandt [5]), we have BM is a prime A-module with respect to a $\sum b_i m_i = \sum (ab_i) m_i$, $a \in A, b_i \in B, m_i \in M$. We will show that $A/(0:BM)_A \in *_k$. Furthermore, $B/(0:M)_B = B/(B \cap (0:BM)_A) \cong (B + (0:BM)_A)/(0:BM)_A \in *_k$. On the other hand, $(B + (0:BM)_A)/(0:BM)_A \triangleleft A/(0:BM)_A$. Since $*_k$ is a special class of rings, $*_k$ satisfies the following condition:

If $\{0\} \neq I \lhd A$, $I \in *_k$ and A is a prime ring, then $A \in *_k$.

We have the following facts:

$$(B + (0 : BM)_A)/(0 : BM)_A \lhd A/(0 : BM)_A \text{ and}$$

$$(B + (0 : BM)_A)/(0 : BM)_A \text{ is a prime ring.}$$

So, we may conclude that $A/(0 : BM)_A \in *_k$. This implies $BM \in \sum_A$. Hence, the class $\sum = \bigcup \sum_A$, where $\sum_A = \{M \mid M$ is a prime A-module such that $A/(0 : M)_A \in *_k$, is the special class of modules determined by $*_k$. □

Acknowledgement

The research of the first author was supported by Lembaga Pengelola Dana Pendidikan, Ministry of Finance of The Republic Indonesia. The first author wishes to express gratitude and thank to Professor Joe Repka and all members and students of the Department of Mathematics of University of Toronto for their supports during the visiting program of the first author in University of Toronto which is supported by Directorate General of Resources for Science, Technology, and Higher Education.

References

[1] W. A. Adkins and S. H. Weintraub, Algebra: An Approach via Module Theory, Springer-Verlag, New York, 1992.

[2] H. France-Jackson, ∗-rings and their radicals, Quaestiones Mathematicae 8 (1985), 231-239.

[3] H. France-Jackson, S. Wahyuni and I. E. Wijayanti, Radical related to special atoms revisited, Bull. Austral. Math. Soc. 91 (2015), 202-210.

[4] B. J. Gardner, Some recent results and open problems concerning special radicals, Radical Theory, Proceedings of the 1988 Sendai Conference, 1988, pp. 25-56.

[5] B. J. Gardner and R. Wiegandt, Radical Theory of Rings, Marcel Dekker, New York, 2004.

[6] H. Korolczuk, A note on the lattice of special radicals, Bulletin De L'Academie Polonaise Des Sciences XXIX (1981), 3-4.

[7] W. K. Nicholson and J. F. Watters, Normal radicals and normal classes of modules, Glasgow Math. J. 30 (1988), 97-100.