# Hypergraph Ramsey numbers 

David Conlon* ${ }^{*} \quad$ Jacob Fox ${ }^{\dagger} \quad$ Benny Sudakov ${ }^{\ddagger}$


#### Abstract

The Ramsey number $r_{k}(s, n)$ is the minimum $N$ such that every red-blue coloring of the $k$-tuples of an $N$-element set contains either a red set of size $s$ or a blue set of size $n$, where a set is called red (blue) if all $k$-tuples from this set are red (blue). In this paper we obtain new estimates for several basic hypergraph Ramsey problems. We give a new upper bound for $r_{k}(s, n)$ for $k \geq 3$ and $s$ fixed. In particular, we show that


$$
r_{3}(s, n) \leq 2^{2^{s-2} \log n},
$$

which improves by a factor of $n^{s-2} /$ polylog $n$ the exponent of the previous upper bound of Erdős and Rado from 1952. We also obtain a new lower bound for these numbers, showing that there are constants $c_{1}, c_{2}>0$ such that

$$
r_{3}(s, n) \geq 2^{c_{1} s n \log (n / s)}
$$

for all $4 \leq s \leq c_{2} n$. When $s$ is a constant, it gives the first superexponential lower bound for $r_{3}(s, n)$, answering an open question posed by Erdős and Hajnal in 1972. Next, we consider the 3color Ramsey number $r_{3}(n, n, n)$, which is the minimum $N$ such that every 3 -coloring of the triples of an $N$-element set contains a monochromatic set of size $n$. Improving another old result of Erdős and Hajnal, we show that

$$
r_{3}(n, n, n) \geq 2^{n^{\operatorname{cog} n}} .
$$

Finally, we make some progress on related hypergraph Ramsey-type problems.

## 1 Introduction

Ramsey theory refers to a large body of deep results in mathematics whose underlying philosophy is captured succinctly by the statement that "Every large system, contains a large well organized subsystem." This is an area in which a great variety of techniques from many branches of mathematics are used and whose results are important not only to combinatorics but also to logic, analysis, number theory, and geometry. Since the publication of the seminal paper of Ramsey in 1930, this subject experienced tremendous growth, and is currently among the most active areas in combinatorics.

The Ramsey number $r(s, n)$ is the least integer $N$ such that every red-blue coloring of the edges of the complete graph $K_{N}$ on $N$ vertices contains either a red $K_{s}$ (i.e., a complete subgraph all of

[^0]whose edges are colored red) or a blue $K_{n}$. Ramsey's theorem states that $r(s, n)$ exists for all $s$ and $n$. Determining or estimating Ramsey numbers is one of the central problem in combinatorics, see the book Ramsey theory [19] for details. A classical result of Erdős and Szekeres [16], which is a quantitative version of Ramsey's theorem, implies that $r(n, n) \leq 2^{2 n}$ for every positive integer $n$. Erdős [7] showed using probabilistic arguments that $r(n, n)>2^{n / 2}$ for $n>2$. Over the last sixty years, there have been several improvements on these bounds (see, e.g., [5]). However, despite efforts by various researchers, the constant factors in the above exponents remain the same.

Off-diagonal Ramsey numbers, i.e. $r(s, n)$ with $s \neq n$, have also been intensely studied. For example, after several successive improvements, it is known (see [1], [20], [27]) that there are constants $c_{1}, \ldots, c_{4}$ such that

$$
c_{1} \frac{n^{2}}{\log n} \leq r(3, n) \leq c_{2} \frac{n^{2}}{\log n}
$$

and for fixed $s>3$,

$$
\begin{equation*}
c_{3}\left(\frac{n}{\log n}\right)^{(s+1) / 2} \leq r(s, n) \leq c_{4} \frac{n^{s-1}}{\log ^{s-2} n} \tag{1}
\end{equation*}
$$

(For $s=4$, Bohman [3] recently improved the lower bound by a factor of $\log ^{1 / 2} n$.) All logarithms in this paper are base $e$ unless otherwise stated.

Although already for graph Ramsey numbers there are significant gaps between lower and upper bounds, our knowledge of hypergraph Ramsey numbers is even weaker. The Ramsey number $r_{k}(s, n)$ is the minimum $N$ such that every red-blue coloring of the unordered $k$-tuples of an $N$-element set contains either a red set of size $s$ or a blue set of size $n$, where a set is called red (blue) if all $k$-tuples from this set are red (blue). Erdős, Hajnal, and Rado [14] showed that there are positive constants $c$ and $c^{\prime}$ such that

$$
2^{c n^{2}}<r_{3}(n, n)<2^{2^{c^{\prime} n}}
$$

They also conjectured that $r_{3}(n, n)>2^{2^{c n}}$ for some constant $c>0$ and Erdős offered a $\$ 500$ reward for a proof. Similarly, for $k \geq 4$, there is a difference of one exponential between known upper and lower bounds for $r_{k}(n, n)$, i.e.,

$$
t_{k-1}\left(c n^{2}\right) \leq r_{k}(n, n) \leq t_{k}\left(c^{\prime} n\right)
$$

where the tower function $t_{k}(x)$ is defined by $t_{1}(x)=x$ and $t_{i+1}(x)=2^{t_{i}(x)}$.
The study of 3 -uniform hypergraphs is particularly important for our understanding of hypergraph Ramsey numbers. This is because of an ingenious construction called the stepping-up lemma due to Erdős and Hajnal (see, e.g., Chapter 4.7 in [19]). Their method allows one to construct lower bound colorings for uniformity $k+1$ from colorings for uniformity $k$, effectively gaining an extra exponential each time it is applied. Unfortunately, the smallest $k$ for which it works is $k=3$. Therefore, proving that $r_{3}(n, n)$ has doubly exponential growth will allow one to close the gap between the upper and lower bounds for $r_{k}(n, n)$ for all uniformities $k$. There is some evidence that the growth rate of $r_{3}(n, n)$ is closer to the upper bound, namely, that with four colors instead of two this is known to be true. Erdős and Hajnal (see, e.g., [19]) constructed a 4-coloring of the triples of a set of size $2^{2^{c n}}$ which does not contain a monochromatic subset of size $n$. This result shows that the number of colors matters a lot in this problem and leads to the question of what happens in the intermediate case when we use three colors. The 3 -color Ramsey number $r_{3}(n, n, n)$ is the minimum $N$ such that every 3 -coloring of the triples of an $N$-element set contains a monochromatic set of size $n$. Naturally, for $r_{3}(n, n, n)$, one should expect at least some improvement on the $2^{c n^{2}}$ lower bound. Indeed, Erdős and Hajnal provided
such a result (see [13] and [4]), showing that $r_{3}(n, n, n) \geq 2^{c n^{2} \log ^{2} n}$. Here, we substantially improve this bound, extending the above mentioned stepping-up lemma of these two authors to show

Theorem 1.1 There exists a constant c such that

$$
\begin{equation*}
r_{3}(n, n, n) \geq 2^{n^{c \log n}} \tag{2}
\end{equation*}
$$

For off-diagonal Ramsey numbers, a classical argument of Erdős and Rado [15 from 1952 demonstrates that

$$
\begin{equation*}
r_{k}(s, n) \leq 2^{\binom{r_{k-1}(s-1, n-1)}{k-1}} \tag{3}
\end{equation*}
$$

Together with the upper bound in (11) it gives for fixed $s$ that $r_{3}(s, n) \leq 2^{\left(r_{2}(s-1, n-1)\right.}{ }_{2} \leq 2^{c^{\frac{n^{2 s-4}}{\log ^{2 s-6}}}}$. Our next result improves the exponent of this upper bound by a factor of $n^{s-2} / \operatorname{polylog} n$.

Theorem 1.2 For fixed $s \geq 4$ and sufficiently large $n$,

$$
\begin{equation*}
\log r_{3}(s, n) \leq\left(\frac{(s-3)}{(s-2)!}+o(1)\right) n^{s-2} \log n . \tag{4}
\end{equation*}
$$

Clearly, a similar improvement for off-diagonal Ramsey numbers of higher uniformity follows from this result together with (3).

Erdős and Hajnal [12] showed that $\log r_{3}(4, n)>c n$ using the following simple construction. They consider a random tournament on $[N]=\{1, \ldots, N\}$ and color the triples from $[N]$ red if they form a cyclic triangle and blue otherwise. Since it is well known and easy to show that every tournament on four vertices contains at most two cyclic triangles and a random tournament on $N$ vertices with high probability does not contain a transitive subtournament of size $c^{\prime} \log N$, the resulting coloring neither has a red set of size 4 nor a blue set of size $c^{\prime} \log N$. In the same paper from 1972, they suggested that probably $\frac{\log r_{3}(4, n)}{n} \rightarrow \infty$. Here we prove the following new lower bound which implies this conjecture.

Theorem 1.3 There are constants $c_{1}, c_{2}>0$ such that

$$
\log r_{3}(s, n) \geq c_{1} s n \log (n / s)
$$

for all $4 \leq s \leq c_{2} n$.
Combining this result together with the stepping-up lemma of Erdős and Hajnal (see [19]), one can also obtain analogous improvements of lower bounds for off-diagonal Ramsey numbers for complete $k$-uniform hypergraphs with $k \geq 4$.

In view of our unsatisfactory knowledge of the growth rate of hypergraph Ramsey numbers, Erdős and Hajnal [12] started the investigation of the following more general problem. Fix positive integers $k, s$, and $t$. What is the smallest $N$ such that every red-blue coloring of the $k$-tuples of an $N$-element set has either a red set of size $n$ or has a set of size $s$ which contains at least $t$ blue $k$-tuples? Note that when $t=\binom{s}{k}$ the answer to this question is simply $r_{k}(n, s)$.

Let $\binom{X}{k}$ denote the collection of all $k$-element subsets of the set $X$. Define $f_{k}(N, s, t)$ to be the largest $n$ for which every red-blue coloring of $\binom{[N]}{k}$ has a red $n$-element set or a set of size $s$ which contains at least $t$ blue $k$-tuples. Erdős and Hajnal [12] in 1972 conjectured that as $t$ increases from 1 to $\binom{s}{k}, f_{k}(N, s, t)$ grows first like a power of $N$, then at a well-defined value $t=h_{1}^{(k)}(s), f_{k}(N, s, t)$
grows like a power of $\log N$, i.e., $f_{k}\left(N, s, h_{1}^{(k)}(s)-1\right)>N^{c_{1}}$ but $f_{k}\left(N, s, h_{1}^{(k)}(s)\right)<(\log N)^{c_{2}}$. Then, as $t$ increases further, at $h_{2}^{(k)}(s)$ the function $f_{k}(N, s, t)$ grows like a power of $\log \log N$ etc. and finally $f_{k}(N, s, t)$ grows like a power of $\log _{(k-2)} N$ for $h_{k-2}^{(k)}(s) \leq t \leq\binom{ s}{k}$. Here $\log _{(i)} N$ is the $i$-fold iterated logarithm of $N$, which is defined by $\log _{(1)} N=\log N$ and $\log _{(j+1)} N=\log \left(\log _{(j)} N\right)$.

This problem of Erdős and Hajnal is still widely open. In [12] they started a careful investigation of $h_{1}^{(3)}(s)$ and made several conjectures which would determine this function. We make progress on their conjectures, computing $h_{1}^{(3)}(s)$ for infinitely many values of $s$. We also approximate $h_{1}^{(3)}(s)$ for all $s$.

In the next section, we prove Theorem 1.2 which gives a new upper bound on off-diagonal hypergraph Ramsey numbers. Our lower bound on $r_{3}(s, n)$ appears in Section 3. In Section 4, we study the 3 -color hypergraph Ramsey numbers and prove the lower bound for $r_{3}(n, n, n)$. In Section 5 , confirming a conjecture of Erdős and Hajnal, we determine the function $h_{1}^{(3)}(s)$ for infinitely many values of $s$. Finally, in the last section of the paper, we make several additional remarks on related hypergraph Ramsey problems. Throughout the paper, we systematically omit floor and ceiling signs whenever they are not crucial for the sake of clarity of presentation. We also do not make any serious attempt to optimize absolute constants in our statements and proofs.

## 2 An Upper Bound for $r_{3}(s, n)$

In this section we prove the upper bound (4) on off-diagonal hypergraph Ramsey numbers.
First we briefly discuss a classical approach to this problem by Erdős-Rado and indicate where it can be improved. To prove $\log _{2} r_{3}(s, n) \leq\left({ }_{(s-1, n-1)}^{2}\right)$, given a red-blue coloring $\chi$ of the triples from [ $N$ ], Erdős and Rado greedily construct a set of vertices $\left\{v_{1}, \ldots, v_{r(s-1, n-1)+1}\right\}$ such that for any given pair $1 \leq i<j \leq r(s-1, n-1)$, all triples $\left\{v_{i}, v_{j}, v_{k}\right\}$ with $k>j$ are of the same color, which we denote by $\chi^{\prime}\left(v_{i}, v_{j}\right)$. By definition of the Ramsey number, there is either a red clique of size $s-1$ or a blue clique of size $n-1$ in coloring $\chi^{\prime}$, and this clique together with $v_{r(s-1, n-1)+1}$ forms a red set of size $s$ or a blue set of size $n$ in coloring $\chi$. The greedy construction of the set $\left\{v_{1}, \ldots, v_{r(s-1, n-1)+1}\right\}$ is as follows. First, pick an arbitrary vertex $v_{1}$ and set $S_{1}=S \backslash\left\{v_{1}\right\}$. After having picked $\left\{v_{1}, \ldots, v_{i}\right\}$ we also have a subset $S_{i}$ such that for any pair $a, b$ with $1 \leq a<b \leq i$, all triples $\left\{v_{a}, v_{b}, w\right\}$ with $w \in S_{i}$ are the same color. Let $v_{i+1}$ be an arbitrary vertex in $S_{i}$ and set $S_{i, 0}=S_{i} \backslash\left\{v_{i+1}\right\}$. Suppose we already constructed $S_{i, j} \subset S_{i, 0}$ such that, for every $h \leq j$ and $w \in S_{i, j}$, all triples $\left\{v_{h}, v_{i+1}, w\right\}$ have the same color. If the number of edges $\left(v_{j+1}, v_{i+1}, w\right)$ with $w \in S_{i, j}$ that are red is at least $\left|S_{i, 0}\right| / 2$, then we let

$$
S_{i, j+1}=\left\{w:\left(v_{j+1}, v_{i+1}, w\right) \text { is red and } w \in S_{i, j}\right\}
$$

and set $\chi^{\prime}(i+1, j+1)=$ red, otherwise we let

$$
S_{i, j+1}=\left\{w:\left(v_{j+1}, v_{i+1}, w\right) \text { is blue and } w \in S_{i, j}\right\}
$$

and set $\chi^{\prime}(i+1, j+1)=$ blue. Finally, we let $S_{i+1}=S_{i, i}$. Notice that $\left\{v_{1}, \ldots, v_{i+1}\right\}$ and $S_{i+1}$ have the desired properties to continue the greedy algorithm. Also, for each edge ( $v_{i+1}, v_{j+1}$ ) that we color by $\chi^{\prime}$, the set $S_{i, j}$ is at most halved. So we lose a factor of at most two for each of the $\binom{r(s-1, n-1)}{2}$ edges colored by $\chi^{\prime}{ }^{1}$

[^1]There are two ways we are able to improve on the Erdős-Rado approach. Our first improvement comes from utilizing the fact that we do not need to ensure that for every pair $i<j$, all edges $\left\{v_{i}, v_{j}, v_{k}\right\}$ with $k>j$ are of the same color. That is, the coloring $\chi^{\prime}$ will not necessarily color every pair. Furthermore, the number of edges we color by $\chi^{\prime}$ will be much smaller than the best known estimate for $\binom{r(s-1, n-1)}{2}$, and this is how we will be able to get a smaller upper bound on $r_{3}(s, n)$. This idea is nicely captured using the vertex on-line Ramsey number which we next define. Consider the following game, played by two players, builder and painter: at step $i+1$ a new vertex $v_{i+1}$ is revealed; then, for every existing vertex $v_{j}, j=1, \cdots, i$, builder decides, in order, whether to draw the edge $v_{j} v_{i+1}$; if he does expose such an edge, painter has to color it either red or blue immediately. The vertex on-line Ramsey number $\tilde{r}(k, l)$ is then defined as the minimum number of edges that builder has to draw in order to force painter to create either a red $K_{k}$ or a blue $K_{l}$. In Lemma [2.2, we provide an upper bound on $\tilde{r}(s-1, n-1)$ which is much smaller than the best known estimate on $(\underset{2}{r(s-1, n-1)})$. Since we are losing a factor of at most two for every exposed edge, this immediately improves on the Erdős-Rado bound for $r_{3}(s, n)$.

A further improvement can be made by using the observation that there will not be many pairs $i<j$ for which all triples $\left\{v_{i}, v_{j}, v_{k}\right\}$ with $k>j$ are red. That is, we will be able to show that there are not many red edges in the coloring $\chi^{\prime}$ we construct. Let $0<\alpha \ll 1 / 2$. Suppose we have $\left\{v_{1}, \ldots, v_{i}\right\}$ and a set $S$ and, for a given $j<i$, we want to find a subset $S^{\prime} \subset S$ such that all triples $\left\{v_{j}, v_{i}, w\right\}$ with $w \in S^{\prime}$ are the same color. We pick $S^{\prime}=\left\{w:\left\{v_{j}, v_{i}, w\right\}\right.$ is red and $\left.w \in S\right\}$ if the number of triples $\left\{v_{j}, v_{i}, w\right\}$ with $w \in S$ is at least $\alpha|S|$ and blue otherwise. While the size of $S$ decreases now by a much larger factor for each red edge in $\chi^{\prime}$, there are not many red edges in $\chi^{\prime}$, on the other hand, we lose very little, specifically a factor $(1-\alpha)$, for each blue edge in $\chi^{\prime}$. By picking $\alpha$ appropriately, we gain significantly over taking $\alpha=1 / 2$ for our upper bound on off-diagonal hypergraph Ramsey numbers.

Before we proceed with the proof of our upper bound on $r_{3}(s, n)$, we want to discuss some other Ramsey-type numbers related to our vertex on-line Ramsey game. One variant of Ramsey numbers which was extensively studied in the literature (17) is the size Ramsey number $\hat{r}\left(G_{1}, G_{2}\right)$, which is the minimum number of edges of a graph whose every red-blue edge-coloring contains either a red $G_{1}$ or a blue $G_{2}$. Clearly, $\tilde{r}(k, \ell) \leq \hat{r}\left(K_{k}, K_{l}\right)$ since builder can choose to pick the edges of a graph which gives the size Ramsey number for $\left(K_{k}, K_{l}\right)$. Unfortunately, it is not difficult to show that $\hat{r}\left(K_{k}, K_{l}\right)=\binom{r(k, l)}{2}$ and therefore we cannot obtain any improvement using these numbers. Another on-line Ramsey game which is quite close to ours, was studied in [23]. In this game, there are two players, builder and painter, who move on the originally empty graph with an unbounded number of vertices. At each step, builder draws a new edge and painter has to color it either red or blue immediately. The edge on-line Ramsey number $\bar{r}(k, l)$ is then defined as the minimum number of edges that builder has to draw in order to force painter to create either a red $K_{k}$ or a blue $K_{l}$. A randomized version of the edge on-line Ramsey game was studied in [18]. The authors of [23] proved an upper bound for $\bar{r}(k, l)$ which is similar to our Lemma 2.2, A careful reading of their paper shows that builder first exposes edges from the first vertex to all future vertices, and so on. Thus, this builder strategy cannot be implemented when proving upper bounds for hypergraph Ramsey numbers. Moreover, it is not clear how to use the edge on-line Ramsey game to get an improvement on hypergraph Ramsey numbers. Lemma 2.2 is therefore essential for our proof giving new upper bounds for hypergraph Ramsey numbers.

Using the ideas discussed above, we next prove an upper bound on $r_{3}(s, n)$ which involves some parameters of the vertex on-line Ramsey game.

Theorem 2.1 Suppose in the vertex on-line Ramsey game that builder has a strategy which ensures a red $K_{s-1}$ or a blue $K_{n-1}$ using at most $v$ vertices, $r$ red edges, and in total $m$ edges. Then, for any $0<\alpha \leq 1 / 2$, the Ramsey number $r_{3}(s, n)$ satisfies

$$
\begin{equation*}
r_{3}(s, n) \leq(v+1) \alpha^{-r}(1-\alpha)^{r-m} . \tag{5}
\end{equation*}
$$

Proof. Let $N=(v+1) \alpha^{-r}(1-\alpha)^{r-m}$ and consider a red-blue coloring $\chi$ of the triples of the set $[N]$. We wish to show that the coloring $\chi$ must contain a red set of size $s$ or a blue set of size $n$.

We greedily construct a set of vertices $\left\{v_{1}, \cdots, v_{h}\right\}$ and a graph $\Gamma$ on these vertices with at most $v$ vertices, at most $r$ red edges, and at most $m$ total edges across them such that for any edge $e=v_{i} v_{j}$, $i<j$ in $\Gamma$, the color of any 3-edge $\left\{v_{i}, v_{j}, v_{k}\right\}$ with $k>j$ is the same, say $\chi^{\prime}(e)$. Moreover, this graph will contain either a red $K_{s-1}$ or a blue $K_{n-1}$, which one can easily see will define a red set of size $s$ or a blue set of size $n$..

We begin the construction of this set of vertices by arbitrarily first choosing vertices $v_{1} \in[N]$ and setting $S_{1}=[N] \backslash\left\{v_{1}\right\}$. Given a set of vertices $\left\{v_{1}, \cdots, v_{a}\right\}$, we have a set $S_{a}$ such that for each edge $e=v_{i} v_{j}$ of $\Gamma$ with $i, j \leq a$, the color of the 3 -edge $\left\{v_{i}, v_{j}, w\right\}$ is the same for every $w$ in $S_{a}$.

Now let $v_{a+1}$ be a vertex in $S_{a}$. We play the vertex on-line Ramsey game, so that builder chooses the edges to be drawn according to his strategy. Painter then colors these edges. For the first edge $e_{1}$ chosen, painter looks at all triples containing this edge and a vertex from $S_{a} \backslash\left\{v_{a+1}\right\}$. The 2-edge is colored red in $\chi^{\prime}$ if it there are more than $\alpha\left(\left|S_{a}\right|-1\right)$ such triples that are red and blue otherwise. This defines a new subset $S_{a, 1}$, which are all vertices in $S_{a} \backslash\left\{v_{a+1}\right\}$ such that together with edge $e_{1}$ form a triple of color $\chi^{\prime}\left(e_{1}\right)$. For the next drawn edge $e_{2}$ we color it red if there more than $\alpha\left|S_{a, 1}\right|$ red triples containing it and a vertex from $S_{a, 1}$ and blue otherwise. This will define an $S_{a, 2}$ and so forth. After we have added all edges from $v_{a+1}$, the remaining set will be $S_{a+1}$. Let $m_{a}$ be the number of edges $e=v_{i} v_{a}$ with $i<a$ in $\Gamma$ and $r_{a}$ be the number of such edges that are red.

We now show by induction that

$$
\left|S_{a}\right| \geq(v+1-a) \alpha^{-r+\sum_{i=1}^{a} r_{i}}(1-\alpha)^{r-m+\sum_{i=1}^{a} m_{i}-r_{i}} .
$$

For the base case $a=1$, we have

$$
\left|S_{1}\right|=N-1=(v+1) \alpha^{-r}(1-\alpha)^{r-m}-1 \geq v \alpha^{-r}(1-\alpha)^{r-m} .
$$

Suppose we have proved the desired inequality for $a$. When we draw a vertex $v_{a+1}$, the size of our set $S_{a}$ decreases by 1. Each time we draw an edge from $v_{a+1}$ we have the size of our set $S$ goes down by a factor $\alpha$ or $1-\alpha$. Therefore,

$$
\begin{aligned}
\left|S_{a+1}\right| & \geq \alpha^{r_{i}}(1-\alpha)^{m_{i}-r_{i}}\left(\left|S_{a}\right|-1\right) \geq \alpha^{r_{i}}(1-\alpha)^{m_{i}-r_{i}}\left|S_{a}\right|-1 \\
& \geq(v+1-a) \alpha^{-r+\sum_{i=1}^{a+1} r_{i}}(1-\alpha)^{r-m+\sum_{i=1}^{a+1} m_{i}-r_{i}}-1 \\
& \geq((v+1)-(a+1)) \alpha^{-r+\sum_{i=1}^{a+1} r_{i}}(1-\alpha)^{r-m+\sum_{i=1}^{a+1} m_{i}-r_{i}} .
\end{aligned}
$$

By our assumption on the vertex on-line Ramsey game, by the time we will construct graph $\Gamma$ containing either a read $K_{s-1}$ or a blue $K_{n-1}$, this graph will have $a \leq v$ vertices, at most $r$ red edges, and at most $m$ total edges. Therefore at this time we have

$$
\left|S_{a}\right| \geq(v+1-a) \alpha^{-r+\sum_{i=1}^{a} r_{i}}(1-\alpha)^{r-m+\sum_{i=1}^{a} m_{i}-r_{i}} \geq 1,
$$

i.e., $S_{a}$ is not empty. Thus a vertex from $S_{a}$ together with the red $K_{s-1}$ or blue $K_{n-1}$ in edge-coloring $\chi^{\prime}$ of $\Gamma$ make either a red set of size $s$ or a blue set of size $n$ in coloring $\chi$, completing the proof.

Lemma 2.2 In the vertex on-line Ramsey game builder has a strategy which ensures a red $K_{s}$ or a blue $K_{n}$ using at most $\binom{s+n-2}{s-1}$ vertices, $(s-2)\binom{s+n-2}{s-1}+1$ red edges, and $(s+n-4)\binom{s+n-2}{s-1}+1$ total edges. In particular,

$$
\tilde{r}(s, n) \leq(s+n-4)\binom{s+n-2}{s-1}+1 .
$$

Proof. We are going to define a set of vertices labeled by strings and the associated set of edges to be drawn during the game as follows. The first vertex exposed, will be labeled as $w_{\emptyset}$. Every other vertex which we expose during the game will be connected by edge to $w_{\emptyset}$. Recall that immediately after the edge is exposed it is colored by painter. The first vertex which is connected to $w_{\emptyset}$ by red (blue) edge is labeled $w_{R}\left(w_{B}\right)$. Successively, we connect vertex $v$ to $w_{R}$ or $w_{B}$ if and only if this vertex is already connected to $w_{\emptyset}$ by a red or respectively blue edge.

More generally, if we have defined $w_{a_{1} a_{2} \cdots a_{p}}$ with each $a_{i}=R$ or $B$ and $v$ is the first exposed vertex which is connected to $w_{a_{1} \cdots a_{j}}$ in color $a_{j+1}$ for each $j=0, \cdots, p$, we label $v$ as $w_{a_{1} \cdots a_{p+1}}$. (When $j=0$, $w_{a_{1} \cdots a_{j}}=w_{\emptyset}$.) The only successively chosen vertices which we join to $w_{a_{1} \cdots a_{p+1}}$ by an edge will be those edges $v$ which are also joined to $w_{a_{1} \cdots a_{j}}$ in color $a_{j+1}$ for each $j=0, \cdots, p$.

Suppose now that we have exposed $\binom{s+n-2}{s-1}=\binom{s+n-3}{s-2}+\binom{s+n-3}{s-1}$ vertices in total. Since $w_{\emptyset}$ is connected to all vertices, its degree is $\binom{s+n-2}{s-1}-1$. Thus $w_{\emptyset}$ is connected either to $\binom{s+n-3}{s-2}$ vertices in red or $\binom{s+n-3}{s-1}$ vertices in blue. If the former holds we look at the neighbors of $w_{R}$, which is all vertices which labeled by string with first letter $R$. Otherwise we look at neighbors of $w_{B}$. Suppose now that we are looking at the neighbors of $w_{a_{1} \cdots a_{p}}$, where $r$ of the $a_{i}$ are red and $b$ of them are blue. Then, by our construction, $w_{a_{1} \cdots a_{p}}$ will have been joined to $\binom{s+n-r-b-2}{s-r-1}-1$ vertices. Now, either $w_{a_{1} \cdots a_{p}}$ is joined to $\binom{s+n-r-b-3}{s-r-2}$ vertices in red or $\binom{s+n-r-b-3}{s-r-1}$ vertices in blue. In the first case we look at $w_{a_{1} \cdots a_{p} R}$ and its neighbors and in the second case at $w_{a_{1} \cdots a_{p} B}$ and its neighbors. By the time we reach a string of length $s+n-3$, we will have either $s-1$ reds or $n-1$ blues. If $s-1$ of the $a_{i}$, say $a_{j_{1}+1}, \cdots, a_{j_{s-1}+1}$, are $R$, then we know that the collection of vertices $w_{a_{1} \cdots a_{j_{1}}}, \cdots, w_{a_{1} \cdots a_{j_{s-1}}}, w_{a_{1} \cdots a_{j_{s-1}+1}}$ forms a red clique of size $s$. Similarly, were $n-1$ of the $a_{i}$ blue, we would have a blue clique of size $n$.

All that remains to do is to estimate how many edges builder draws. Look on the vertices in the order they were exposed. Clearly, for every vertex we can only look on the edges connecting it to preceeding vertices. Notice that a vertex $w_{a_{1} \cdots a_{p}}$ is adjacent to precisely $p$ vertices which were exposed before it. Moreover the number of red edges connecting $w_{a_{1} \cdots a_{p}}$ to vertices before it is precisely the number of $a_{i}$ which are $R$. Since all but the last vertex are labeled by strings of length at most $s+n-4$, we have at most $(s+n-4)\binom{s+n-2}{s-1}+1$ total edges. Similarly, all but the last vertex have at most $s-2$ symbols $R$ in their string, which shows that the number of edges colored red during the game is at most $(s-2)\binom{s+n-2}{s-1}+1$.

The following result implies (4).
Corollary 2.3 The Ramsey number $r_{3}(s, n)$ with $4 \leq s \leq n$ satisfies

$$
\begin{equation*}
r_{3}(s, n) \leq 2^{\frac{(s-3)}{(s-2)!}(s+n)^{s-2} \log _{2}(64 n / s)} \tag{6}
\end{equation*}
$$

Proof. By Lemma 2.2, in the vertex on-line Ramsey game builder has a strategy which ensures a red $K_{s-1}$ or a blue $K_{n-1}$ using at most $v=\binom{s+n-4}{s-2}$ vertices, $r=(s-3)\binom{s+n-4}{s-2}+1$ red edges, and $m=(s+n-6)\binom{s+n-4}{s-2}+1$ total edges. To minimize the function $\alpha^{-r}(1-\alpha)^{r-m}$, one should take $\alpha=\frac{r}{m}$. Note that $m / r \leq(s+n-6) /(s-3) \leq 4 n / s, v<r \leq m / 2$ and $r \leq \frac{(s-3)}{(s-2)!}(s+n)^{s-2}$. Hence, the Ramsey number $r_{3}(s, n)$ satisfies

$$
\begin{aligned}
r_{3}(s, n) & \leq(v+1)(m / r)^{r}(1-r / m)^{r-m} \leq(v+1)(4 n / s)^{r}(1-r / m)^{-m} \\
& \leq r(4 n / s)^{r}(1+2 r / m)^{m} \leq r\left(4 e^{2} n / s\right)^{r}<(64 n / s)^{r} \\
& \leq 2^{\frac{(s-3)}{(s-2)!}(s+n)^{s-2} \log _{2}(64 n / s)}
\end{aligned}
$$

Also, taking $\alpha=1 / 2$ in Theorem 2.1, it is worth noting that in the diagonal case our results easily imply the following theorem, which improves upon the bound $r_{3}(k, k) \leq 2^{2^{4 k}}$ due to Erdős and Rado.

## Theorem 2.4

$$
\log _{2} \log _{2} r_{3}(k, k) \leq(2+o(1)) k .
$$

Our methods can also be used to study Ramsey numbers of non-complete hypergraphs. To illustrate this, we will obtain a lower bound on $f_{3}(N, 4,3)$, slightly improving a result of Erdős and Hajnal. Let $K_{t}^{(3)}$ denote the complete 3-uniform hypergraph with $t$ vertices, and $K_{t}^{(3)} \backslash e$ denote the 3-uniform hypergraph with $t$ vertices formed by removing one triple. For $k$-uniform hypergraphs $H$ and $G$, the Ramsey number $r(H, G)$ is the minimum $N$ such that every red-blue coloring of $\binom{[N]}{k}$ contains either a red copy of $H$ or a blue copy of $G$. Note that an upper bound on $f_{3}(N, 4,3)$ is equivalent to a lower bound on the Ramsey number $r\left(K_{4}^{(3)} \backslash e, K_{n}^{(3)}\right)$ because they are inverse functions of each other. Erdős and Hajnal [12] proved the following bounds:

$$
\frac{1}{2} \frac{\log _{2} N}{\log _{2} \log _{2} N} \leq f_{3}(N, 4,3) \leq\left(2 \log _{2} N\right)+1
$$

The upper bound follows from the same coloring (discussed in the introduction) based on tournaments which gives a lower bound on $r_{3}(4, n)$. We will use our approach to improve the lower bound by an asymptotic factor of 2 .
Proposition 2.5 We have $r\left(K_{4}^{(3)} \backslash e, K_{n}^{(3)}\right) \leq(2 e n)^{n}$.
Sketch of proof. We apply the exact same proof technique as we did for Theorem 2.1 except that we will expose all edges. We have a coloring of the complete 3 -uniform hypergraph with $N$ vertices which neither contains a red $K_{4}^{(3)} \backslash e$ nor a blue set of size $n$. Note that the coloring $\chi^{\prime}$ of the edges of the complete graph with vertex set $V=\left\{v_{1}, \ldots, v_{h-1}\right\}$ we get in the proof does not contain a pair of monochromatic red edges $\left(v_{j}, v_{i}\right)$ and $\left(v_{j}, v_{k}\right)$ with $1 \leq j<i<k<h$ or $1 \leq i<j<k<h$, otherwise $v_{i}, v_{j}, v_{k}, v_{h}$ are the vertices of a red $K_{4}^{(3)} \backslash e$. Therefore, the red graph in the coloring $\chi^{\prime}$ is just a disjoint union of stars. Let $m$ be the number of edges in the red graph. Note that disjoint union of stars with $m$ edges has an independent set of size $m$ and forms a bipartite graph. Therefore the red graph has an independent set of size at least $\max \{m,(h-1) / 2\}$. Such an independent set in the red graph is a clique in the blue graph in the coloring $\chi^{\prime}$, and together with $v_{h}$ make a blue complete

3-uniform hypergraph in the coloring $\chi$. This, gives us inequalities $m+1<n$ and $(h-1) / 2+1<n$. With hindsight, we pick $\alpha=1 /(2 n)$. By Theorem 2.1, this implies that

$$
r\left(K_{4}^{(3)} \backslash e, n\right) \leq(1+h) \alpha^{-m}(1-\alpha)^{m-\binom{h-1}{2}} \leq(2 n-1)(2 n)^{n-2}\left(1-\frac{1}{2 n}\right)^{-h^{2} / 2} \leq(2 e n)^{n}
$$

where we use that $3 \leq h \leq 2 n-2, m \leq n-2$ and that $(1-1 / x)^{1-x} \leq e$ for $x>1$.
Theorem 1.3 shows that $\log r_{3}(4, n)>c n \log n$ for an absolute constant $c$. It would be also nice to give a similar lower bound (if it is true) for $r\left(K_{4}^{(3)} \backslash e, K_{n}^{(3)}\right)$ since then we would know that $\log r\left(K_{4}^{(3)} \backslash e, K_{n}^{(3)}\right)$ has order of $n \log n$.

## 3 A lower bound construction

The purpose of this section is to prove Theorem 1.3 which gives a new lower bound on $r_{3}(s, n)$. To do this, we need to recall an estimate for graph Ramsey numbers. As we already mentioned in (1), for sufficiently large $n$ and fixed $s, r(s, n)>c(n / \log n)^{(s+1) / 2}>n^{3 / 2}$. Also, for all $4 \leq s \leq n$ and $n$ sufficiently large, one can easily show that $r(s, n)>\left(\frac{n+s}{s}\right)^{s / 3}$. (This is actually not the best lower bound for $r(s, n)$ but it is enough for our purposes.) Indeed, if $s=3$, this bound is trivial. For $s \geq 4$, consider a random red-blue edge-coloring of the complete graph on $N=\left(\frac{n+s}{s}\right)^{s / 3}$ vertices in which each edge is red with probability $p=\left(\frac{s}{n+s}\right)^{0.9}$. It is easy to check that the expected number of monochromatic red $s$-cliques and blue $n$-cliques in this coloring is $\binom{N}{s} p\binom{s}{2}+\binom{N}{n}(1-p)^{\binom{n}{2}}<1$. These estimates together with the next theorem clearly imply Theorem 1.3.

Theorem 3.1 For all sufficiently large $n$ and $4 \leq s \leq n$,

$$
r_{3}(s, n)>(r(s-1, n / 4)-1)^{n / 24}
$$

Proof. Let $\ell=n / 4, r=r(s-1, \ell)-1, N=r^{n / 24}$, and $c_{1}:\binom{[r]}{2} \rightarrow\{$ red, blue\} be a red-blue edge-coloring of the complete graph on $[r]$ with no red clique of size $s-1$ and no blue clique of size $\ell$. Consider a coloring $c_{2}:\binom{[N]}{2} \rightarrow[r]$ picked uniformly at random from all $r$-colorings of $\binom{[N]}{2}$, i.e., each edge has probability $\frac{1}{r}$ of being a particular color independent of all other edges. Using the auxiliary colorings $c_{1}$ and $c_{2}$, we define the red-blue coloring $c:\binom{[N]}{3} \rightarrow$ \{red, blue \} where the color of a triple $\{a, b, c\}$ with $a<b<c$ is $c_{1}\left(c_{2}(a, b), c_{2}(a, c)\right)$ if $c_{2}(a, b) \neq c_{2}(a, c)$ and is blue if $c_{2}(a, b)=c_{2}(a, c)$. We next show that in coloring $c$ there is no red set of size $s$ and with positive probability no blue set of size $n$, which implies the theorem.

First, suppose that the coloring $c$ contains a red set $\left\{u_{1}, \ldots, u_{s}\right\}$ of size $s$ with $u_{1}<\ldots<u_{s}$. Then all the colors $c_{2}\left(u_{1}, u_{j}\right)$ with $2 \leq j \leq s$ are distinct and form a red clique of size $s-1$ in $c_{1}$, a contradiction.

Next, we estimate the expected number of blue cliques of size $n$ in coloring $c$. Let $\left\{v_{1}, \ldots, v_{n}\right\}$ with $v_{1}<\ldots<v_{n}$ be a set of $n$ vertices. Fix for now $1 \leq i \leq n$. If all triples $\left\{v_{i}, v_{j}, v_{k}\right\}$ with $i<j<k$ are blue, then the distinct colors among the colors $c_{2}\left(v_{i}, v_{j}\right)$ for $i<j \leq n$ must form a blue clique in coloring $c_{1}$. Therefore the number of distinct colors $c_{2}\left(v_{i}, v_{j}\right)$ with $i<j \leq n$ is less than $\ell$. Every such subset of distinct colors is contained in at least one of the $\binom{r}{\ell-1}$ subsets of $[r]$ of size $\ell-1$. If we fix a
set of $\ell-1$ colors, the probability that each of the colors $c_{2}\left(v_{i}, v_{j}\right)$ with $i<j \leq n$ is one of these $\ell-1$ colors is $\left(\frac{\ell-1}{r}\right)^{n-i}$. Therefore the expected number of blue cliques of size $n$ in coloring $c$ is at most

$$
\begin{aligned}
\binom{N}{n} \prod_{i=1}^{n}\binom{r}{\ell-1}\left(\frac{\ell-1}{r}\right)^{n-i} & \leq N^{n}\binom{r}{\ell-1}^{n}\left(\frac{\ell-1}{r}\right)^{\binom{n}{2}} \leq N^{n}\left(\frac{e r}{\ell-1}\right)^{(\ell-1) n}\left(\frac{\ell-1}{r}\right)^{\binom{n}{2}} \\
& =\left(N e^{\ell-1}\left(\frac{\ell-1}{r}\right)^{\frac{n-1}{2}-(\ell-1)}\right)^{n}<\left(N e^{\ell-1}\left(r^{-1 / 3}\right)^{\frac{n-1}{2}-(\ell-1)}\right)^{n} \\
& <\left(N^{2}\left(r^{-1 / 3}\right)^{n / 4}\right)^{n}=1
\end{aligned}
$$

where we use that $\ell-1<r^{2 / 3}, \ell=n / 4$, and $N=r^{n / 24}=r^{\ell / 6}>e^{\ell}$. Hence, there is a coloring $c$ with no red set of size $s$ and no blue set of size $n$.

An additional feature of our new lower bound on $r_{3}(s, n)$ is that it increases continuously with growth of $s$ and for $s=n$ coincides with the bound $r_{3}(n, n) \geq 2^{c n^{2}}$, which was given by Erdős, Hajnal, and Rado [14]. For example, for $n^{1 / 2} \ll s \ll n$, the previously best known bound for $r_{3}(s, n)$ was essentially $r_{3}(s, n) \geq r_{3}(s, s) \geq 2^{c s^{2}}$.

## 4 Bounding $r_{3}(n, n, n)$

We now prove the lower bound, $r_{3}(n, n, n) \geq 2^{n^{c \log n}}$, mentioned in the introduction. Though our method follows the stepping-up tradition of Erdős and Hajnal, it is curious to note that their own best lower bound on the problem, $r_{3}(n, n, n) \geq 2^{c n^{2} \log ^{2} n}$, is not proven in this manner. In Erdős and Hajnal's proof that the $r_{3}(n, n, n, n)>2^{2^{c n}}$, they use the stepping up lemma starting from a 2-coloring of a complete graph with $r(n-1, n-1)-1$ vertices not containing a monochromatic clique of size $n-1$ to obtain a 4 -coloring of the triples of a set of size $2^{r(n-1, n-1)-1}$ without a monochromatic set of size $n$. Our proof that $r_{3}(n, n, n)>2^{n^{c \log n}}$ is also based on the stepping-up lemma, using essentially the following idea. We start with a 2 -coloring of the complete graph on $r\left(\log _{2} n, n-1\right)-1$ vertices which contains neither a monochromatic red clique of $\operatorname{size} \log _{2} n$ nor a monochromatic blue clique of size $n-1$. Then we obtain a 4 -coloring of the triples of a set of size $2^{r\left(\log _{2} n, n-1\right)-1} \geq 2^{n^{c \log n}}$ as in the Erdős-Hajnal proof. Next we combine two of the four color classes to obtain a 3-coloring of the triples. Finally, we carefully analyze this 3-coloring to show that it does not contain a monochromatic set of size $n$.

## Theorem 4.1

$$
r_{3}(n, n, n)>2^{r\left(\log _{2} n, n-1\right)-1} .
$$

Proof. Let $G$ be a graph on $m=r\left(\log _{2} n, n-1\right)-1$ vertices which contains neither a clique of size $n-1$ nor an independent set of $\operatorname{size} \log _{2} n$ and let $\bar{G}$ be the complement of $G$. We are going to consider the complete 3 -uniform hypergraph $H$ on the set

$$
T=\left\{\left(\gamma_{1}, \cdots, \gamma_{m}\right): \gamma_{i}=0 \text { or } 1\right\} .
$$

$$
\begin{aligned}
& \text { If } \epsilon=\left(\gamma_{1}, \cdots, \gamma_{m}\right), \epsilon^{\prime}=\left(\gamma_{1}^{\prime}, \cdots, \gamma_{m}^{\prime}\right) \text { and } \epsilon \neq \epsilon^{\prime} \text {, define } \\
& \qquad \delta\left(\epsilon, \epsilon^{\prime}\right)=\max \left\{i: \gamma_{i} \neq \gamma_{i}^{\prime}\right\},
\end{aligned}
$$

that is, $\delta\left(\epsilon, \epsilon^{\prime}\right)$ is the largest coordinate at which they differ. Given this, we can define an ordering on $T$, saying that

$$
\begin{aligned}
& \epsilon<\epsilon^{\prime} \text { if } \gamma_{i}=0, \gamma_{i}^{\prime}=1, \\
& \epsilon^{\prime}<\epsilon \text { if } \gamma_{i}=1, \gamma_{i}^{\prime}=0
\end{aligned}
$$

Equivalently, associate to any $\epsilon$ the number $b(\epsilon)=\sum_{i=1}^{m} \gamma_{i} 2^{i-1}$. The ordering then says simply that $\epsilon<\epsilon^{\prime}$ iff $b(\epsilon)<b\left(\epsilon^{\prime}\right)$.

We will further need the following two properties of the function $\delta$ which one can easily prove.
(a) If $\epsilon_{1}<\epsilon_{2}<\epsilon_{3}$, then $\delta\left(\epsilon_{1}, \epsilon_{2}\right) \neq \delta\left(\epsilon_{2}, \epsilon_{3}\right)$ and
(b) if $\epsilon_{1}<\epsilon_{2}<\cdots<\epsilon_{p}$, then $\delta\left(\epsilon_{1}, \epsilon_{p}\right)=\max _{1 \leq i \leq p-1} \delta\left(\epsilon_{i}, \epsilon_{i+1}\right)$.

In particular, these properties imply that there is a unique index $i$ which achieves maximum of $\delta\left(\epsilon_{i}, \epsilon_{i+1}\right)$. Indeed suppose that there are indices $i<i^{\prime}$ such that

$$
\ell=\delta\left(\epsilon_{i}, \epsilon_{i+1}\right)=\delta\left(\epsilon_{i^{\prime}}, \epsilon_{i^{\prime}+1}\right)=\max _{1 \leq j \leq p-1} \delta\left(\epsilon_{j}, \epsilon_{j+1}\right) .
$$

Then, by property (b) we also have that $\ell=\delta\left(\epsilon_{i}, \epsilon_{i^{\prime}}\right)=\delta\left(\epsilon_{i^{\prime}}, \epsilon_{i^{\prime}+1}\right)$. This contradicts property (a) since $\epsilon_{i}<\epsilon_{i^{\prime}}<\epsilon_{i^{\prime}+1}$.

We are now ready to color the complete 3 -uniform hypergraph $H$ on the set $T$. If $\epsilon_{1}<\epsilon_{2}<\epsilon_{3}$, let $\delta_{1}=\delta\left(\epsilon_{1}, \epsilon_{2}\right)$ and $\delta_{2}=\delta\left(\epsilon_{2}, \epsilon_{3}\right)$. Note that, by property (a) above, $\delta_{1}$ and $\delta_{2}$ are not equal. Color the edge $\left\{\epsilon_{1}, \epsilon_{2}, \epsilon_{3}\right\}$ as follows:
$C_{1}$, if $\left(\delta_{1}, \delta_{2}\right) \in e(G)$ and $\delta_{1}<\delta_{2}$;
$C_{2}$, if $\left(\delta_{1}, \delta_{2}\right) \in e(G)$ and $\delta_{1}>\delta_{2}$;
$C_{3}$, if $\left(\delta_{1}, \delta_{2}\right) \notin e(G)$, i.e., it is an edge in $\bar{G}$.
Suppose that $C_{1}$ contains a clique $\left\{\epsilon_{1}, \cdots, \epsilon_{n}\right\}_{<}$of size $n$. For $1 \leq i \leq n-1$, let $\delta_{i}=\delta\left(\epsilon_{i}, \epsilon_{i+1}\right)$. Note that the $\delta_{i}$ form a monotonically increasing sequence, that is $\delta_{1}<\delta_{2}<\cdots<\delta_{n-1}$. Also, note that since, for any $1 \leq i<j \leq n-1,\left\{\epsilon_{i}, \epsilon_{i+1}, \epsilon_{j+1}\right\} \in C_{1}$, we have, by property (b) above, that $\delta\left(\epsilon_{i+1}, \epsilon_{j+1}\right)=\delta_{j}$, and thus $\left\{\delta_{i}, \delta_{j}\right\} \in e(G)$. Therefore, the set $\left\{\delta_{1}, \cdots, \delta_{n-1}\right\}$ must form a clique of size $n-1$ in $G$. But we have chosen $G$ so as not to contain such a clique, so we have a contradiction. A similar argument shows that $C_{2}$ also cannot contain a clique of size $n$.

For $C_{3}$, assume again that we have a monochromatic clique $\left\{\epsilon_{1}, \cdots, \epsilon_{n}\right\}_{<}$of size $n$, and, for $1 \leq i \leq n-1$, let $\delta_{i}=\delta\left(\epsilon_{i}, \epsilon_{i+1}\right)$. Not only can we no longer guarantee that these $\delta_{i}$ form a monotonic sequence, but we can no longer guarantee that they are distinct. Suppose that there are $d$ distinct values of $\delta$, given by $\left\{\Delta_{1}, \cdots, \Delta_{d}\right\}$, where $\Delta_{1}>\cdots>\Delta_{d}$. We will consider the subgraph of $\bar{G}$ induced by this vertices. Note that, by definition of the coloring $C_{3}$, the vertices $\Delta_{i}$ and $\Delta_{j}$ are adjacent in $\bar{G}$ if there exists $\epsilon_{r}<\epsilon_{s}<\epsilon_{t}$ with $\Delta_{i}=\delta\left(\epsilon_{r}, \epsilon_{s}\right)$ and $\Delta_{j}=\delta\left(\epsilon_{s}, \epsilon_{t}\right)$. We show that this set necessarily has a complete subgraph on $\log _{2} n$ vertices, contradicting our assumptions on $\bar{G}$.

Since $\Delta_{1}$ is the largest of the $\Delta_{j}$, there is a unique index $i_{1}$ such that $\Delta_{1}=\delta_{i_{1}}$. Note that $\Delta_{1}$ is adjacent in $\bar{G}$ to all $\Delta_{j}, j>1$. Indeed, every such $\Delta_{j}=\delta\left(\epsilon_{i^{\prime}}, \epsilon_{i^{\prime}+1}\right)$ for some index $i^{\prime} \neq i_{1}$ and suppose
$i_{1}<i^{\prime}$ (the other case is similar). Then $\epsilon_{i_{1}}<\epsilon_{i^{\prime}}<\epsilon_{i^{\prime}+1}$. Also, by property (b) and maximality of $\Delta_{1}$, we have that $\Delta_{1}=\delta\left(\epsilon_{i_{1}}, \epsilon_{i_{1}+1}\right)=\delta\left(\epsilon_{i_{1}}, \epsilon_{i^{\prime}}\right)$, and therefore it is connected to $\Delta_{j}$ in $\bar{G}$. Now either there are $(n-2) / 2=n / 2-1$ values of $j$ greater than $i_{1}$ or less than $i_{1}$. Let $V_{1}$ be the larger of these two intervals.

Suppose, inductively, that one has been given an interval $V_{j-1}$ in $[n-1]$. Look at the set $\left\{\delta_{a} \mid a \in\right.$ $\left.V_{j-1}\right\}$. One of these $\delta$, say $\delta_{i_{j}}$, will be the largest and as we explain above, will be connected to every other $\delta_{a}$ with $a \in V_{j-1}$. There are at least $\left(\left|V_{j-1}\right|-1\right) / 2$ indices in either $\left\{a \in V_{j-1} \mid a<i_{j}\right\}$ or $\left\{a \in V_{j-1} \mid a>i_{j}\right\}$. Let $V_{j}$ be the larger of these two intervals, so in particular $\left|V_{j}\right| \geq\left(\left|V_{j-1}\right|-1\right) / 2$. By induction, it is easy to show that $\left|V_{j}\right| \geq \frac{n}{2^{j}}-1$. Therefore, for $j \leq \log _{2} n-1,\left|V_{j}\right| \geq 1$ and, hence, the set $\delta_{i_{1}}, \cdots, \delta_{i_{\log _{2} n}}$ forms a clique in $\bar{G}$, as required. This contradicts the fact that $G$ has no independent set of this size and completes the proof.

As discussed in the beginning of Section 3, the probabilistic method demonstrates that, for $s \leq n$, $r(s, n) \geq\left(\frac{n+s}{s}\right)^{c^{\prime} s}$. Substituting this bound with $s=\log _{2} n$ into Theorem4.1]implies the desired result $r_{3}(n, n, n) \geq 2^{n^{c \log n}}$.

## 5 A hypergraph problem of Erdős and Hajnal

In this section we determine the function $h_{1}^{(3)}(s)$ for infinitely many values of $s$ and find a small interval containing $h_{1}^{(3)}(s)$ for all values of $s$. Recall that $f_{3}(N, s, t)$ is the largest integer $n$ for which every red-blue coloring of $\binom{[N]}{3}$ has a red $n$-element set or a set of size $s$ with at least $t$ blue triples. Also recall that $h_{1}^{(3)}(s)$ is the least $t$ for which $f_{3}(N, s, t)$ stops growing like a power of $N$ and starts growing like a power of $\log N$, i.e., $f_{3}\left(N, s, h_{1}^{(3)}(s)-1\right)>N^{c_{1}}$ but $f_{3}\left(N, s, h_{1}^{(3)}(s)\right)<(\log N)^{c_{2}}$.

Consider the minimal family $\mathcal{F}$ of 3 -uniform hypergraphs defined as follows. The empty hypergraphs on 1 and 2 vertices and an edge are elements of $\mathcal{F}$. If $H, G \in \mathcal{F}$ and $v$ is a vertex of $H$, then the following 3-uniform hypergraph $H(G, v)$ is in $\mathcal{F}$ as well. The vertex set of $H(G, v)$ is $(V(H) \backslash\{v\}) \cup V(G)$ and its edges consist of the edges of $G$, the edges of $H$ not containing $v$, and all triples $\{a, b, c\}$ with $a, b \in H$ and $c \in G$ for which $\{a, b, v\}$ is an edge of $H$. Erdős and Hajnal showed that for every hypergraph $H \in \mathcal{F}$ on $s$ vertices, every red-blue coloring of the triples of a set of size $N$ has either a red copy of $H$ or a blue set of size $N^{\epsilon_{s}}$. This can be shown by induction on $s$ using the following claim which can be proved using a simple counting argument. If $H, G \in \mathcal{F}$, then any red-blue coloring of the triples of a set of size $N$ without a blue set of size $N^{\epsilon_{1}}$ has $N^{\delta}$ copies of $H$ all sharing the same copy of $H \backslash v$. In these $N^{\delta}$ vertices, we either get a red copy of $G$ which together with the copy of $H \backslash v$ make a copy of $H(G, v)$ or a blue set of size $\left(N^{\delta}\right)^{\epsilon_{2}}$. By choosing $\epsilon=\min \left(\epsilon_{1}, \delta \epsilon_{2}\right)$, we get the desired result.

Let $g_{1}^{(3)}(s)$ be the maximum number of edges in a hypergraph in $\mathcal{F}$ with $s$ vertices. One can check that every hypergraph $H \in \mathcal{F}$ on $s$ vertices has the following structure. Its vertex set can be partitioned into three parts $A, B, C$ (one of which might be empty) such that all triples intersecting $A, B$, and $C$ are edges of $H$ and subhypergraphs induced by sets $A, B$, and $C$ are also members of $\mathcal{F}$. This implies that the function $g_{1}^{(3)}(s)$ can also be defined recursively. Put $g_{1}^{(3)}(1)=g_{1}^{(3)}(2)=0$. Assume that $g_{1}^{(3)}(m)$ has already been defined for all $m<s$. Then

$$
g_{1}^{(3)}(s)=\max _{a+b+c=s} g_{1}^{(3)}(a)+g_{1}^{(3)}(b)+g_{1}^{(3)}(c)+a b c
$$

It is not difficult to see that the maximum is obtained when $a, b$, and $c$ are as nearly equal as possible. It follows from the definition of $h_{1}^{(3)}$ and the result in the previous paragraph that $h_{1}^{(3)}(s)>g_{1}^{(3)}(s)$. Erdős and Hajnal further conjectured that this bound is tight.

Conjecture 5.1 For all positive integers $s, h_{1}^{(3)}(s)=g_{1}^{(3)}(s)+1$.
Consider an edge-coloring $c$ of the complete graph on $[N]$ with colors $I, I I, I I I$ picked uniformly at random. From this coloring, we get a red-blue coloring $C$ of the triples from [ $N$ ] as follows: if $a<b<c$ has $(a, b)$ color $I,(b, c)$ color $I I$, and $(a, c)$ color $I I I$, then color $\{a, b, c\}$ red, otherwise color the triple blue. Since every complete graph of order $q$ contains $\Theta\left(q^{2}\right)$ edge-disjoint triangles, the probability that a given set of size $q$ contains only blue triples is at most $2^{-\Theta\left(q^{2}\right)}$. Thus, it is straightforward to check that with high probability, in the coloring $C$ the largest blue set has size $O(\log N)$. Over all edge-colorings of the complete graph on $[s]$ with colors $I, I I, I I I$, let $F_{1}(s)$ denote the maximum number of triples $(a, b, c)$ with $1 \leq a<b<c \leq s$ such that $(a, b)$ is color $I,(b, c)$ is color $I I$, and $(a, c)$ is color $I I I$. Note that in the coloring $C$ we constructed above whose largest blue set has size $O(\log N)$, every set of size $s$ has at most $F_{1}(s)$ red triples. Therefore, by definition, $h_{1}^{(3)} \leq F_{1}(s)+1$. Since also $h_{1}^{(3)}(s)>g_{1}^{(3)}(s)$, it implies that $F_{1}(s) \geq g_{1}^{(3)}(s)$. Erdós and Hajnal conjectured that these two functions are actually equal, which would imply $h_{1}^{(3)}(s)=F_{1}(s)+1=g_{1}^{(3)}(s)+1$ and hence Conjecture 5.1.

Conjecture 5.2 For all positive integers $s, F_{1}(s)=g_{1}^{(3)}(s)$.
Erdős and Hajnal verified Conjectures 5.1 and 5.2 for $s \leq 9$.
To attack these conjectures, we use a new function which was not considered in [12]. Let $T(s)$ be the maximum number of directed triangles in all tournaments on $s$ vertices. It is an exercise (see, e.g., [24]) to check that every tournament with $n$ vertices of outdegrees $d_{1}, \ldots, d_{n}$ has exactly $\binom{n}{3}-\sum_{i=1}^{n}\binom{d_{i}}{2}$ cyclic triangles. This number is maximized when all the $d_{i}$ are as equal as possible, that is, if $n$ is odd, $d_{i}=\frac{n-1}{2}$ for all $i$ and, if $n$ is even, half of the $d_{i}$ are $\frac{n-2}{2}$ and the other half are $\frac{n}{2}$. It is easy to see that there is a tournament with such outdegrees, and therefore we have the following formula for $T(s)$ :

$$
T(s)= \begin{cases}\frac{(s+1) s(s-1)}{24} & \text { if } s \text { is odd }  \tag{7}\\ \frac{(s+2) s(s-2)}{24} & \text { if } s \text { is even }\end{cases}
$$

It appears that $T(s)$ and $F_{1}(s)$ are closely related. Indeed, given an edge-coloring of the complete graph on $[s]$ with colors $I, I I, I I I$, construct the following tournament on [s]. If $(a, b)$ with $a<b$ is color $I$ or $I I$, then direct the edge from $a$ to $b$ and otherwise direct the edge from $b$ to $a$. Note that any triple $(a, b, c)$ with $a<b<c$ and $(a, b)$ color $I,(b, c)$ color II, and $(a, c)$ color III makes a cyclic triangle in our tournament. We therefore have $F_{1}(s) \leq T(s)$. Let us summarize the inequalities we have seen so far:

$$
\begin{equation*}
g_{1}^{(3)}(s) \leq h_{1}^{(3)}(s)-1 \leq F_{1}(s) \leq T(s) \tag{8}
\end{equation*}
$$

Let $d(s)=g_{1}^{(3)}(s)-T(s)$. We have $d(s)=0$ if and only if all the inequalities in (8) are equalities. We call such a number $s$ nice. Note that Conjectures 5.1 and 5.2 necessarily hold in the case $s$ is nice. Using this fact, we next find infinitely many values of $s$ for which Conjectures 5.1 and 5.2 hold.

Proposition 5.3 If $s$ is a power of 3, then

$$
g_{1}^{(3)}(s)=h_{1}^{(3)}(s)-1=F_{1}(s)=T(s)=\frac{1}{4}\binom{s+1}{3} .
$$

Proof. We easily see that $s=1$ is nice. By induction, the proposition follows from checking that if $s$ is odd and nice, then so is $3 s$. Since, by definition, $g_{1}^{(3)}(3 s)=s^{3}+3 g_{1}^{(3)}(s)$, we indeed have

$$
d(3 s)=T(3 s)-g_{1}^{(3)}(3 s)=\frac{1}{4}\binom{3 s+1}{3}-3 g_{1}^{(3)}(s)-s^{3}=\frac{3}{4}\binom{s+1}{3}-3 g_{1}^{(3)}(s)=3 d(s) .
$$

The computation in the proof of the proposition above shows that if $s=6 x+3$ with $x$ a nonnegative integer, then $d(s)=3 d(2 x+1)$. One can check the other cases of $s(\bmod 6)$ rather easily.

Lemma 5.4 If $x$ is a positive integer, then

$$
\begin{aligned}
d(6 x-2) & =2 d(2 x-1)+d(2 x), \\
d(6 x-1) & =d(2 x-1)+2 d(2 x)+x, \\
d(6 x) & =3 d(2 x), \\
d(6 x+1) & =2 d(2 x)+d(2 x+1)+x, \\
d(6 x+2) & =d(2 x)+2 d(2 x+1), \\
d(6 x+3) & =3 d(2 x+1) .
\end{aligned}
$$

Note that from this lemma, we can easily determine which values of $s$ are nice. In particular, the nice positive integers up to 100 are

$$
1,2,3,4,6,8,9,10,12,18,24,26,27,28,30,36,54,72,78,80,81,82,84,90 .
$$

Also, from Lemma [5.4, we can easily prove an upper bound on $d(s)$.
Proposition 5.5 For all positive integers $s, d(s)=O(s \log s)$.
Proof. Let $D(s)=d(s)-c s \log s$ with $c$ a sufficiently large constant. Using induction on $s$ and the recursive formula for $d(s)$ depending on $s(\bmod 6)$ in Lemma 5.4, we get that $D(s)$ is negative for $s>1$. Indeed, assuming $s=6 x+1$ with $x$ a positive integer (the other five cases are handled similarly), we get

$$
\begin{aligned}
D(s) & =d(6 x+1)-c(6 x+1) \log (6 x+1)=2 d(2 x)+d(2 x+1)+x-c(6 x+1) \log (6 x+1) \\
& <2 d(2 x)+d(2 x+1)-c(6 x+1) \log (2 x+1)<2 D(2 x)+D(2 x+1)<0 .
\end{aligned}
$$

The above proposition demonstrates that $T(s)$ and $g_{1}^{(3)}(s)$, which are cubic in $s$, are always fairly close together. Therefore, using (7), we have that $h_{1}^{(3)}(s)$ always lies in an interval of length $O(s \log s)$ around $s^{3} / 24$.

In their attempt to determine $h_{1}^{(3)}(s)$, Erdős and Hajnal consider yet another function. Consider a coloring of the edges of the complete graph on $s$ vertices labeled $1, \ldots, s$ by two colors I and II which
maximizes the number of triangles $(a, b, c)$ with $1 \leq a<b<c \leq s$ such that $(a, b)$ and $(b, c)$ has color I, and $(a, c)$ has color II. Denote this maximum by $F_{2}(s)$. Trivially, $F_{2}(s) \geq F_{1}(s)$. Erdős and Hajnal thought that "perhaps $F_{2}(s)=F_{1}(s)$ ". As we will show, this is indeed the case for some values of $s$, but is not true in general. For example, it is false already for $s=5$ and $s=7$. Moreover, we precisely determine the $F_{2}(s)$ for all values of $s$.

Lemma 5.6 For all positive integers $s, F_{2}(s)=T(s)$.
Proof. We first show that $T(s) \geq F_{2}(s)$. Indeed, from a two coloring with colors I and II of the edges of the complete graph with vertices $1, \ldots, s$ we get a tournament on $s$ vertices as follows: if $(a, b)$ with $a<b$ is color I, then orient the edge from $a$ to $b$, otherwise $(a, b)$ is color II and orient the edge from $b$ to $a$. Any triangle ( $a, b, c$ ) with $a<b<c$ with $(a, b)$ and $(b, c)$ color I and ( $a, c$ ) color II is a cyclic triangle in the tournament, and the inequality $T(s) \geq F_{2}(s)$ follows.

We next show that actually $T(s)=F_{2}(s)$. Consider the two coloring of the edges of the complete graph on $s$ vertices where $(a, b)$ is color II if and only if $b-a$ is even. A simple calculation shows that the number of triangles $(a, b, c)$ with $a<b<c$ with $(a, b)$ and $(b, c)$ color I and $(a, c)$ color II in this coloring is precisely the formula (77) for $T(s)$. Assume $s$ is even (the case $s$ is odd can be treated similarly). For fixed $a$ and $c$ with $c-a$ even, the number of such triangles containing edge ( $a, c$ ) is $\left\lfloor\frac{c-a}{2}\right\rfloor$. Letting $c=a+2 i$, we thus have

$$
F_{2}(s) \geq \sum_{a=1}^{s} \sum_{1 \leq i \leq\left\lfloor\frac{s-a}{2}\right\rfloor} i=\sum_{a=1}^{s}\binom{\left\lfloor\frac{s-a}{2}\right\rfloor+1}{2}=\sum_{j=1}^{s / 2} 2\binom{j}{2}=2\binom{\frac{s}{2}+1}{3}=T(s),
$$

and hence $F_{2}(s)=T(s)$.

## 6 Odds and ends

### 6.1 Polynomial versus Exponential Ramsey numbers

As we discussed in Section 2, the Ramsey number of $K_{4}^{(3)} \backslash e$ versus $K_{n}^{(3)}$ is at least exponential in $n$. The hypergraph $K_{4}^{(3)} \backslash e$ is a special case of the following construction. Given an arbitrary graph $G$, let $H_{G}$ be the 3-uniform hypergraph whose vertices are the vertices of $G$ plus an auxiliary vertex $v$. The edges of $H_{G}$ are all triples obtained by taking the union of an edge of $G$ with vertex $v$. For example, by taking $G$ to be the triangle, we obtain $K_{4}^{(3)} \backslash e$. It appears that the Ramsey numbers $r\left(H_{G}, K_{n}^{(3)}\right)$ have a very different behavior depending on the bipartiteness of $G$.

Proposition 6.1 If $G$ is a bipartite graph, then there is a constant $c=c(G)$ such that $r\left(H_{G}, K_{n}^{(3)}\right) \leq$ $n^{c}$. On the other hand, for non-bipartite $G, r\left(H_{G}, K_{n}^{(3)}\right) \geq 2^{c^{\prime} n}$ for an absolute constant $c^{\prime}>0$.

Proof. Let $G$ be a bipartite graph with $t$ vertices. The classical result of Kövari, Sós, and Turán [22] states that a graph with $N$ vertices and at least $N^{2-1 / t}$ edges contains the complete bipartite graph $K_{t, t}$ with two parts of size $t$. Therefore, any 3 -uniform hypergraph of order $N$ which contains a vertex of degree at least $N^{2-1 / t}$ contains also a copy of $H_{K_{t, t}}$ and hence also $H_{G}$. Consider a red-blue edgecoloring $C$ of the complete 3 -uniform hypergraph on $N=(3 n)^{2 t}$ vertices, and let $m$ denote the number
of red edges in $C$. If $m \geq N^{3-1 / t}$, then there is a vertex whose red degree is at least $3 m / N \geq N^{2-1 / t}$, which by the above remark gives a red copy of $H_{G}$. Otherwise, $m<N^{3-1 / t}$ and we can use a well known Turán-type bound to find a large blue set in coloring $C$. Indeed, it is well known (see, e.g., Chapter 3, Exercise 3 in [2]) that a 3-uniform hypergraph with $N$ vertices and $m \geq N$ edges has an independent set (i.e., set with no edges) of size at least $\frac{N^{3 / 2}}{3 m^{1 / 2}}$. Thus, the hypergraph of red edges has an independent set of size at least

$$
\frac{N^{3 / 2}}{3 m^{1 / 2}}>\frac{N^{3 / 2}}{3\left(N^{3-1 / t}\right)^{1 / 2}}=\frac{1}{3} N^{1 /(2 t)}=n,
$$

which clearly is a blue set.
To prove the second part of this proposition, we use a construction of Erdős and Hajnal mentioned in the introduction. Suppose that $G$ is not bipartite, so it contains an odd cycle with vertices $\left\{v_{1}, \ldots, v_{2 i+1}\right\}$ and edges $\left\{v_{j}, v_{j+1}\right\}$ for $1 \leq j \leq 2 i+1$, where $v_{2 i+2}:=v_{1}$. We start with a tournament $T$ on $[N]$ with $N=2^{c^{\prime} n}$ which contains no transitive tournament of order $n$. As we already mentioned, for sufficiently small $c^{\prime}$, a random tournament has this property with high probability. Color the triples from $[N]$ red if they form a cyclic triangle in $T$ and blue otherwise. Clearly, this coloring does not contain a blue set of size $n$. Suppose it contains a red copy of $H_{G}$. This implies that $T$ contains $2 i+2$ vertices $v, u_{1}, \ldots, u_{2 i+1}$ such that all the triples $\left(v, u_{j}, u_{j+1}\right)$ form a cyclic triangle. Then, the edges $\left(v, u_{j}\right)$ and $\left(v, u_{j+1}\right)$ have opposite orientation (one edge oriented towards $v$ and the other oriented from $v$ ). Coloring the vertices $u_{j}$ by 0 or 1 depending on the direction of edge $\left(v, u_{j}\right)$ gives a proper 2 -coloring of an odd cycle, contradiction.

### 6.2 Discrepancy in hypergraphs

Despite the fact that Erdős [11] (see also the book [4]) believed $r_{3}(n, n)$ is closer to $2^{2^{c n}}$, together with Hajnal [13] they discovered the following interesting fact about hypergraphs which maybe indicates the opposite. They proved that there are $c, \epsilon>0$ such that every 2 -coloring of the triples of an $N$-set contains a set of size $s>c(\log N)^{1 / 2}$ which contains at least $(1 / 2+\epsilon)\binom{s}{3} 3$-sets in one color. That is, the set of size $s$ deviates from having density $1 / 2$ in each color by at least some fixed positive constant. Erdős further remarks that he would begin to doubt that $r_{3}(n, n)$ is double-exponential in $n$ if one can prove that in any 2-coloring of the triples of the $N$-set, contains some set of size $s=c(\eta)(\log N)^{\epsilon}$ for which at least $(1-\eta)\binom{s}{3}$ triples have the same color. We prove the following result, which demonstrates this if we allow $\epsilon$ to decrease with $\eta$.

Theorem 6.2 For $\eta>0$ and all positive integers $r$ and $k$, there is a constant $\beta=\beta(r, k, \eta)>0$ such that every $r$-coloring of the $k$-tuples of an $N$-element set has a subset of size $s>(\log N)^{\beta}$ which contains more than $(1-\eta)\binom{s}{k} k$-sets in one color.

These results can be conveniently restated in terms of another function introduced by Erdős in [11. Denote by $F^{(k)}(N, \alpha)$ the largest integer for which it is possible to split the $k$-tuples of a $N$-element set $S$ into two classes so that for every $X \subset S$ with $|X| \geq F^{(k)}(N, \alpha)$, each class contains more than $\alpha\binom{|X|}{k} k$-tuples of $X$. Note that $F^{(k)}(N, 0)$ is essentially the inverse function of the usual Ramsey function $r_{k}(n, n)$. It is easy to show that for $0 \leq \alpha<1 / 2$,

$$
c(\alpha) \log N<F^{(2)}(N, \alpha)<c^{\prime}(\alpha) \log N .
$$

As Erdős points out, for $k \geq 3$ the function $F^{(k)}(N, \alpha)$ is not well understood. If $\alpha=1 / 2-\epsilon$ for sufficiently small $\epsilon>0$, then the result of Erdős and Hajnal from the previous paragraph (for general $k)$ demonstrates

$$
c_{k}(\epsilon)(\log N)^{1 /(k-1)}<F^{(k)}(N, \alpha)<c_{k}^{\prime}(\epsilon)(\log N)^{1 /(k-1) .}
$$

On the other hand, since $F^{(k)}(N, 0)$ is the inverse function of $r_{k}(n, n)$, then the old conjecture of Erdős, Hajnal, and Rado would imply that

$$
c_{1} \log _{(k-1)} N<F^{(k)}(N, 0)<c_{2} \log _{(k-1)} N,
$$

where we recall that $\log _{(t)} N$ denotes the $t$ times iterated logarithm function. Assuming the conjecture, as $\alpha$ increases from 0 to $1 / 2, F^{(k)}(N, \alpha)$ increases from $\log _{(k-1)} n$ to $(\log N)^{(1 /(k-1)}$. Erdős [4] asked (and offered a $\$ 500$ cash reward) if the change in $F^{(k)}(N, \alpha)$ occurs continuously, or there are jumps? He suspected the only jump occurs at $\alpha=0$. If $\alpha$ is bounded away from 0 , Theorem 6.2 demonstrates that $F^{(k)}(N, \alpha)$ already grows as some power of $\log N$. That is, for each $\alpha>0$ and $k$ there are $c, \epsilon>0$ such that $F^{(k)}(N, \alpha)>c(\log N)^{\epsilon}$.

We will deduce Theorem 6.2 from a result about the $r$-color Ramsey number of a certain $k$-uniform hypergraph with $n$ vertices and edge density almost one. The Ramsey number $r(H ; r)$ of a $k$-uniform hypergraph $H$ is the minimum $N$ such that every $r$-edge-coloring of the $k$-tuples of a $N$-element set contains a monochromatic copy of $H$. The blow-up $K_{\ell}^{(k)}(n)$ is the $k$-uniform hypergraph whose vertex set consists of $\ell$ parts of size $n$ and whose edges are all $k$-tuples that have their vertices in some $k$ different parts. Note that $K_{\ell}^{(k)}(n)$ has $\ell n$ vertices and $\binom{\ell}{k} n^{k} \geq\left(1-\binom{k}{2} / \ell\right)\binom{l n}{k}$ edges. In particular, as $\ell$ grows with $k$ fixed, the edge density of $K_{\ell}^{(k)}(n)$ goes to 1 . Therefore, Theorem 6.2 is a corollary of the following result.

Theorem 6.3 For all positive integers $r, k, \ell$, there is a constant $c=c(r, k, \ell)$ such that

$$
r\left(K_{\ell}^{(k)}(n) ; r\right) \leq e^{c n^{\ell}}
$$

 color Ramsey number $r\left(K_{\ell}^{(k)} ; r\right)$. The proof uses a simple trick which appears in [10] and (see also [21]). By definition, every vertex subset of size $t$ contains a monochromatic set of size $\ell$. Since each monochromatic set of size $\ell$ is contained in $\binom{N-\ell}{t-\ell}$ subsets of size $t$, the number of monochromatic sets of size $\ell$ is at least

$$
\binom{N}{t} /\binom{N-\ell}{t-\ell}=\binom{t}{\ell}^{-1}\binom{N}{\ell}
$$

By the pigeonhole principle, there is a color $1 \leq i \leq r$ for which there are at least $\frac{1}{r}\binom{t}{\ell}^{-1}\binom{N}{\ell}$ monochromatic sets of size $\ell$ in color $i$. Define the $\ell$-uniform hypergraph $G$ with vertex set [ $N$ ] whose edges consist of the monochromatic sets of size $\ell$ in color $i$ in our $r$-coloring. We have just shown that hypergraph $G$ with $N$ vertices has at least $\frac{1}{r}\binom{t}{\ell}^{-1}\binom{N}{\ell} \geq \epsilon \frac{N^{\ell}}{\ell!}$ edges with $\epsilon=\frac{1}{2 r}\binom{t}{\ell}^{-1}$. A standard extremal lemma for hypergraphs (see, e.g., [8], [25]) demonstrates that any $\ell$-uniform hypergraph with $N$ vertices and at least $\epsilon \frac{N^{\ell}}{\ell!}$ edges with $(\ln N)^{-1 /(\ell-1)} \leq \epsilon \leq \ell^{-3}$ contains a complete $\ell$-uniform $\ell$-partite hypergraph with parts of size $\left\lfloor\epsilon(\ln N)^{1 /(\ell-1)}\right\rfloor$. (An $l$-uniform hypergraph is $l$-partite if there is a partition of the vertex set into $l$ parts such that each edge has exactly one vertex in each part.) In particular, $G$ contains a complete $\ell$-uniform $\ell$-partite hypergraph with parts of size $\left\lfloor\epsilon(\ln N)^{1 /(\ell-1)}\right\rfloor=n$, where we
use that $\epsilon=c^{-1 /(\ell-1)}$. The vertices of this complete $\ell$-uniform $\ell$-partite hypergraph with $n$ vertices in each part in $G$ are the vertices of a monochromatic $K_{\ell}^{(k)}(n)$ in color $i$, completing the proof.

Finally we want to mention another problem of Erdős related to the growth of Ramsey numbers of complete 3 -uniform hypergraphs. Erdős [9] (see also [11] and [4) asked the following problem.

Question 6.4 Suppose $|S|=N$ and the triples from $S$ are split into two classes. Does there exist a pair of subsets $A, B \subset S$ with $|A|=|B| \geq c(\log N)^{1 / 2}$ such that all triples from $A \cup B$ that hit both $A$ and $B$ are in the same class?

Erdős showed that the answer is yes under the weaker assumption that only the triples with two vertices in $A$ and one vertex in $B$ must be monochromatic. Although this question is still open we would like to mention that the answer to it is no if the triples of $S$ are split into four classes instead of two. Indeed, in [6], we found a 3-uniform hypergraph $C_{n}$ on $n$ vertices which is much sparser than the complete hypergraph $K_{n}^{(3)}$ and whose four-color Ramsey number satisfies $r\left(C_{n} ; 4\right)>2^{2^{c 1^{n}}}$. Let $V=\left\{v_{1}, \cdots, v_{n}\right\}$ be a set of vertices and let $C_{n}$ be the 3 -uniform hypergraph on $V$ whose edge set is given by $\left\{v_{i}, v_{i+1}, v_{j}\right\}$ for all $1 \leq i, j \leq n$. (Note that when $i=n$, we consider $i+1$ to be equal to 1.) When $n$ is even, the vertices of $C_{n}$ can be partitioned into two subsets $A$ and $B$ (with $v_{i} \in A$ if and only if $i$ is even) of size $n / 2$ such that all edges of $C_{n}$ hit both $A$ and $B$. Thus, a four-coloring of the triples of $[N]$ with $N=2^{2^{c_{1} n}}$ and with no monochromatic copy of $C_{n}$ also does not contain a pair $A, B \subset[N]$ with $|A|=|B|=\frac{1}{2 c_{1}} \log \log N$ such that all triples that hit both $A$ and $B$ are in the same class.
Acknowledgments. The results in Section 6.1 were obtained in collaboration with Noga Alon, and we thank him for allowing us to include them here. We also thank N. Alon and D. Mubayi for interesting discussions.

## References

[1] M. Ajtai, J. Komlós, and E. Szemerédi, A note on Ramsey numbers, J. Combinatorial Theory, Ser. A 29 (1980), 354-360.
[2] N. Alon and J. H. Spencer, The probabilistic method, 2nd ed., Wiley, 2000.
[3] T. Bohman, The Triangle-Free Process, preprint.
[4] F. Chung and R. Graham, Erdős on Graphs. His Legacy of Unsolved Problems, A K Peters, Ltd., Wellesley, MA, 1998.
[5] D. Conlon, A new upper bound for diagonal Ramsey numbers, Annals of Mathematics, to appear.
[6] D. Conlon, J. Fox, and B. Sudakov, Ramsey numbers of sparse hypergraphs, Random Structures and Algorithms, to appear.
[7] P. Erdős, Some remarks on the theory of graphs, Bull. Amer. Math. Soc. 53 (1947), 292-294.
[8] P. Erdős, On extremal problems of graphs and generalized graphs, Israel J. Math 2 (1964), 183-190.
[9] P. Erdős, Topics in combinatorial analysis, Proceedings of the Second Louisiana Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, 1971), pp. 2-20, Louisiana State University, Baton Rouge, LA, 1971.
[10] P. Erdős, On some extremal problems on $r$-graphs, Discrete Math 1 (1971/72), 1-6.
[11] P. Erdős, Problems and results on graphs and hypergraphs: similarities and differences, in Mathematics of Ramsey theory, Algorithms Combin., Vol. 5 (J. Nešetřil and V. Rödl, eds.) 12-28. Berlin: Springer-Verlag, 1990.
[12] P. Erdős, A. Hajnal, On Ramsey like theorems, Problems and results, Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972) , pp. 123-140, Inst. Math. Appl., Southend-on-Sea, 1972.
[13] P. Erdős, A. Hajnal, Ramsey-type theorems, Discrete Appl. Math. 25 (1989), 37-52.
[14] P. Erdős, A. Hajnal, R. Rado, Partition relations for cardinal numbers, Acta Math. Acad. Sci. Hungar. 16 (1965), 93-196.
[15] P. Erdős, R. Rado, Combinatorial theorems on classifications of subsets of a given set, Proc. London Math. Soc. 3 (1952), 417-439.
[16] P. Erdős and G. Szekeres, A combinatorial problem in geometry, Compositio Math. 2 (1935), 463-470.
[17] R. J. Faudree and R. H. Schelp, A survey of results on size Ramsey numbers, Paul Erdős and his mathematics, II (Budapest, 1999), Bolyai Soc. Math. Stud., Vol. 11, János Bolyai Math. Soc., Budapest, 2002, pp. 291-309.
[18] E. Friedgut, Y. Kohayakawa, V. Rödl, A. Ruciński, and P. Tetali, Ramsey games against a onearmed bandit, Combin. Prob. Comp. 12 (2003), 515-545.
[19] R.L. Graham, B.L. Rothschild, J.L. Spencer, Ramsey theory, 2nd edition, John Wiley \& Sons (1980).
[20] J. H. Kim, The Ramsey number $R(3, t)$ has order of magnitude $t^{2} / \log t$, Random Structures and Algorithms 7 (1995), 173-207.
[21] A. V. Kostochka and V. Rödl, On Ramsey numbers of uniform hypergraphs with given maximum degree, J. Combin. Theory Ser. A 113 (2006), 1555-1564.
[22] T. Kövari, V. Sós, and P. Turán, On a problem of K. Zarankiewicz, Colloq Math. 3 (1954), 50-57.
[23] A. Kurek and A. Ruciński, Two variants of the size Ramsey number, Discuss. Math. Graph Theory 25 (2005), 141-149.
[24] L. Lovász, Combinatorial problems and exercises, Corrected reprint of the 1993 second edition. AMS Chelsea Publishing, Providence, RI, 2007.
[25] V. Nikiforov, Complete $r$-partite subgraphs of dense $r$-graphs, preprint.
[26] F.P. Ramsey, On a problem of formal logic, Proc. London Math. Soc. Ser. 230 (1930), 264-286.
[27] J. Spencer, Asymptotic lower bounds for Ramsey functions, Discrete Math. 20 (1977/78), 69-76.


[^0]:    *St John's College, Cambridge, United Kingdom. E-mail: D.Conlon@dpmms.cam.ac.uk. Research supported by a Junior Research Fellowship at St John's College, Cambridge.
    ${ }^{\dagger}$ Department of Mathematics, Princeton, Princeton, NJ. Email: jacobfox@math.princeton.edu. Research supported by an NSF Graduate Research Fellowship and a Princeton Centennial Fellowship.
    ${ }^{\ddagger}$ Department of Mathematics, UCLA, Los Angeles, CA 90095. Email: bsudakov@math.ucla.edu. Research supported in part by NSF CAREER award DMS-0546523 and by USA-Israeli BSF grant.

[^1]:    ${ }^{1}$ We also lose one element from $S_{i}$ when we pick $v_{i+1}$, but this loss is rather insubstantial.

