On Composite Quantum Hypothesis Testing

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We extend quantum Stein's lemma in asymmetric quantum hypothesis testing to composite null and alternative hypotheses. As our main result, we show that the asymptotic error exponent for testing convex combinations of quantum states $\rho^{\otimes n}$ against convex combinations of quantum states $\sigma^{\otimes n}$ is given by a regularized quantum relative entropy distance formula. We prove that in general such a regularization is needed but also discuss various settings where our formula as well as extensions thereof become single-letter. This includes a novel operational interpretation of the relative entropy of coherence in terms of hypothesis testing. For our proof, we start from the composite Stein's lemma for classical probability distributions and lift the result to the non-commutative setting by only using elementary properties of quantum entropy. Finally, our findings also imply an improved Markov type lower bound on the quantum conditional mutual information in terms of the regularized quantum relative entropy – featuring an explicit and universal recovery map.

I. OVERVIEW OF RESULTS

Hypothesis testing is arguably one of the most fundamental primitives in quantum information theory. As such it has found many applications, e.g., in quantum channel coding [23] and quantum illumination [31, 37, 47] or for giving an operational interpretation to abstract quantities [12, 15, 24]. A particular hypothesis testing setting is that of quantum state discrimination where quantum states are assigned to each of the hypotheses and we aim to determine which state is actually given. Several distinct scenarios are of interest which differ in the priority given to different types of error or in how many copies of a system are given to aid the discrimination. Here we investigate the setting of asymmetric hypothesis testing where the goal is to discriminate between two n-party quantum states (strategies or hypotheses) ρ_n and σ_n living on the n-fold tensor product of some finite-dimensional inner product space $\mathcal{H}^{\otimes n}$. That is, we are optimizing over all two-outcome positive operator valued measures (POVMs) with $\{M_n, (1-M_n)\}$ and associate M_n with accepting ρ_n as well as $(1-M_n)$ with accepting σ_n. This naturally gives rise to the two possible errors

$$\alpha_n(M_n) := \mathrm{Tr}\left[\rho_n(1-M_n)\right] \text{ Type 1 error,} \qquad \beta_n(M_n) := \mathrm{Tr}\left[\sigma_n M_n\right] \text{ Type 2 error.} \tag{1}$$

For asymmetric hypothesis testing we minimize the Type 2 error as

$$\beta_n(\varepsilon) := \inf_{0\leq M_n\leq 1}\left\{\beta_n(M_n)\middle|\alpha_n(M_n)\leq\varepsilon\right\} \tag{2}$$

while we require the Type 1 error not to exceed a small constant $\varepsilon \in (0,1)$. We are then interested in finding the optimal asymptotic error exponent (whenever the limit exists)

$$\zeta(\varepsilon) := \lim_{n\to\infty} -\frac{\log\beta_n(\varepsilon)}{n} \text{ and correspondingly } \zeta(0) := \lim_{\varepsilon\to 0}\zeta(\varepsilon). \tag{3}$$

A well studied discrimination setting is that between fixed independent and identical (iid) states $\rho^{\otimes n}$ and $\sigma^{\otimes n}$ where the error exponent $\zeta_{\rho,\sigma}(\varepsilon)$ is determined by quantum Stein's lemma [3, 25, 34] in terms of the quantum relative entropy

$$\forall\varepsilon\in(0,1) \text{ we have } \zeta_{\rho,\sigma}(\varepsilon) = D(\rho\|\sigma) := \begin{cases}\mathrm{Tr}\left[\rho\left(\log\rho-\log\sigma\right)\right] & \mathrm{supp}(\rho)\subseteq\mathrm{supp}(\sigma)\\ \infty & \text{otherwise.}\end{cases} \tag{4}$$

(Here and henceforth the logarithm is defined with respect to the basis 2.) In many applications we aim to solve more general discrimination problems and a prominent example of such is that of composite hypotheses − in which we attempt to discriminate between different sets of states. Previously the case of composite iid null hypotheses $\rho^{\otimes n}$ with $\rho \in \mathcal{S}$ was investigated in [9, 21] leading to the natural error exponent

$$\forall\varepsilon\in(0,1) \text{ we have } \zeta_{\mathcal{S},\sigma}(\varepsilon) = \inf_{\rho\in S} D(\rho\|\sigma)\,. \tag{5}$$

On the other hand the problem of composite alternative hypotheses seems to be more involved in the non-commutative case. In case the set of alternative hypotheses $\sigma_n \in \mathcal{T}_n$ for $n \in \mathbb{N}$ fulfills certain axioms motivated by the framework of resource theories, it was shown in [12] that the error exponent $\zeta_{\rho,\mathcal{T}}(\varepsilon)$ can be written in terms of the regularized relative entropy distance

$$\forall \varepsilon \in (0,1) \text{ we have } \zeta_{\rho,\mathcal{T}}(\varepsilon) = \lim_{n\to\infty} \frac{1}{n} \inf_{\sigma_n \in \mathcal{T}_n} D\left(\rho^{\otimes n} \| \sigma_n\right) . \tag{6}$$

This regularization is in general needed as we know from the case of the relative entropy of entanglement [45] – which might not be too surprising since the set of alternative hypotheses is not required to be iid in general.

For our main result we consider the setting where null and alternative hypotheses are both composite and given by convex combinations of n-fold tensor powers of states from given sets $\rho \in \mathcal{S}$ and $\sigma \in \mathcal{T}$ (see Sect. II for the precise definition). We show that the corresponding asymptotic error exponent $\zeta_{\mathcal{S},\mathcal{T}}(0)$ can be written as

$$\zeta_{\mathcal{S},\mathcal{T}}(0) = \lim_{n\to\infty} \frac{1}{n} \inf_{\substack{\rho\in\mathcal{S} \\ \mu\in\mathcal{T}}} D\left(\rho^{\otimes n} \| \int \sigma^{\otimes n} \, \mathrm{d}\mu(\sigma)\right) \tag{7}$$

where there is a slight abuse of notation with $\mu \in \mathcal{T}$ standing for measures on the set $\mathcal{T}$. We note that even in the case of a fixed null hypothesis $\mathcal{S} = \{\rho\}$ our setting is not a special case of the previous results [12], as our sets of alternative hypotheses are not closed under tensor product – $\sigma_m \in \mathcal{T}_m,\ \sigma_n \in \mathcal{T}_n \not\Rightarrow \sigma_m \otimes \sigma_n \in \mathcal{T}_{mn}$ – which is one of the properties required for the result of [12]. Moreover, we show that the regularization in Eq. (7) is needed, i.e. in contrast to the classical case [10, 28] in general

$$\zeta_{\mathcal{S},\mathcal{T}}(0) \neq \inf_{\substack{\rho\in\mathcal{S} \\ \sigma\in\mathcal{T}}} D(\rho\|\sigma) . \tag{8}$$

Nevertheless, there exist non-commutative cases in which the regularization is not needed and we discuss several such examples. In particular, we give a novel operational interpretation of the relative entropy of coherence in terms of hypothesis testing. The proofs of our results are transparent in the sense that we start from the composite Stein's lemma for classical probability distributions and then lift the result to the non-commutative setting by only using elementary properties of entropic measures.

Finally, we apply the techniques developed in this work to strengthen the previously best known quantum relative entropy Markov type lower bound on the conditional quantum mutual information $I(A:B|C)_\rho := H(AC)_\rho + H(BC)_\rho - H(ABC)_\rho - H(C)_\rho$ [8, 11, 18, 26, 41, 42, 46] with $H(C)_\rho := -\operatorname{Tr}[\rho_C \log \rho_C]$ the von Neumann entropy. We find that

$$I(A:B|C)_\rho \geq \lim_{n\to\infty} \frac{1}{n} D\left(\rho_{ABC}^{\otimes n} \| \int \beta_0(t) \, \mathrm{d}t \left(\mathcal{I}_A \otimes \mathcal{R}_{C\to BC}^{[t]}(\rho_{AC})\right)^{\otimes n}\right) \tag{9}$$

for some universal probability distribution $\beta_0(t)$ and the rotated Petz recovery maps $R_{C\to BC}^{[t]}$ (defined in Sect. IV). In contrast to the previously known bounds in terms of quantum relative entropy distance [11, 42], the recovery map in Eq. (9) takes a specific form only depending on the reduced state on BC. Note that the regularization in Eq. (9) cannot go away in relative entropy distance, as recently shown in [17]. We give a detailed overview how all known Markov type lower bounds on the conditional quantum mutual information compare and argue that Eq. (9) represents the last possible strengthening (see Sect. IV).

The remainder of the paper is structured as follows. In Section II we prove our main result about composite asymmetric hypothesis testing. This is followed by Section III where we discuss several concrete examples including an operational interpretation of the relative entropy of coherence as well as its Rényi analogues in terms of the Petz divergences [35] and the sandwiched relative entropies [32, 48]. In Section IV we prove the refined lower bound on the conditional mutual information from Eq. (9) and use it to show that the regularization in Eq. (7) is needed in general. Finally, we end in Section V with a discussion of some open questions.

II. PROOF OF MAIN RESULT

In the following all inner product spaces $\mathcal{H}$ are finite-dimensional and $S(\mathcal{H})$ denotes the set of positive semi-definite linear operators on $\mathcal{H}$ of trace one. For $n \in \mathbb{N}$ we attempt the following discrimination problem.

Null hypothesis: the convex sets of iid states $\mathcal{S}_n := \left\{\int \rho^{\otimes n} \, \mathrm{d}\nu(\rho) \middle| \rho \in \mathcal{S}\right\}$ with $\mathcal{S} \subseteq S(\mathcal{H})$

Alternative hypothesis: the convex sets of iid states $\mathcal{T}_n := \left\{ \int \sigma^{\otimes n} \, \mathrm{d}\mu(\sigma) \middle| \sigma \in \mathcal{T} \right\}$ with $\mathcal{T} \subseteq S(\mathcal{H})$

For $\varepsilon \in (0,1)$ the goal is the quantification of the optimal asymptotic error exponent for composite asymmetric hypothesis testing (as we will see the following limit exists)

$$\zeta^n_{\mathcal{S},\mathcal{T}}(\varepsilon) := -\frac{1}{n} \cdot \log \inf_{0 \leq M_n \leq 1} \left\{ \sup_{\mu \in \mathcal{T}} \mathrm{Tr}\left[M_n \sigma_n(\mu)\right] \middle| \sup_{\nu \in \mathcal{S}} \mathrm{Tr}\left[(1-M_n)\rho_n(\nu)\right] \leq \varepsilon \right\} \tag{10}$$

$$\zeta_{\mathcal{S},\mathcal{T}}(\varepsilon) := \lim_{n \to \infty} \zeta^n_{\mathcal{S},\mathcal{T}}(\varepsilon) \quad \text{and} \quad \zeta_{\mathcal{S},\mathcal{T}}(0) := \lim_{\varepsilon \to 0} \zeta_{\mathcal{S},\mathcal{T}}(\varepsilon)\,, \tag{11}$$

where we set

$$\rho_n(\nu) := \int \rho^{\otimes n} \mathrm{d}\nu(\rho) \text{ and } \sigma_n(\mu) := \int \sigma^{\otimes n} \mathrm{d}\mu(\sigma) \tag{12}$$

for the sake of notation, and $\mu \in \mathcal{S}$ and $\nu \in \mathcal{T}$ stand for measures over $\mathcal{S}$ and $\mathcal{T}$, respectively. The following is the main result of this section.

Theorem 1. *For the discrimination problem as above, we have*

$$\zeta_{\mathcal{S},\mathcal{T}}(0) = \lim_{n \to \infty} \frac{1}{n} \inf_{\substack{\rho \in \mathcal{S} \\ \mu \in \mathcal{T}}} D\left(\rho^{\otimes n} \| \sigma_n(\mu)\right)\,. \tag{13}$$

We first prove the $\leq$ bound, i.e. the converse direction, which follows from the following lemma.

Proposition 2. *For $\rho \in \mathcal{S}$, $\mu \in \mathcal{T}$, and $\varepsilon \in (0,1)$ we have*

$$-\frac{1}{n} \log \inf_{0 \leq M_n \leq 1} \left\{ \mathrm{Tr}\left[M_n \sigma_n(\mu)\right] \middle| \mathrm{Tr}\left[(1-M_n)\rho^{\otimes n}\right] \leq \varepsilon \right\} \leq \frac{1}{n} \cdot \frac{D\left(\rho^{\otimes n} \| \sigma_n(\mu)\right) + 1}{1 - \varepsilon}\,. \tag{14}$$

Proof. We follow the original converse proof of quantum Stein's lemma [25] for the states $\rho^{\otimes n}$ and $\sigma_n(\mu)$. By the monotonicity of the quantum relative entropy [30] under POVMs $\{M_n, (1-M_n)\}$ we have

$$D\left(\rho^{\otimes n} \middle\| \sigma^n(\mu)\right) \geq \alpha_n(M_n) \log \frac{\alpha_n(M_n)}{1-\beta_n(M_n)} + (1-\alpha_n(M_n)) \log \frac{1-\alpha_n(M_n)}{\beta_n(M_n)} \geq -\log 2 - (1-\alpha_n(M_n)) \log \beta_n(M_n)\,, \tag{15}$$

where we used the notation from Eq. (1). The claim then follows by a simple rearrangement. □

By taking the appropriate infima as well as the limits $n \to \infty$ and $\varepsilon \to 0$ in Prop. 2 we find

$$\zeta_{\mathcal{S},\mathcal{T}}(0) \leq \liminf_{n \to \infty} \frac{1}{n} \inf_{\substack{\rho \in \mathcal{S} \\ \mu \in \mathcal{T}}} D\left(\rho^{\otimes n} \| \sigma_n(\mu)\right)\,. \tag{16}$$

For the $\geq$ bound, i.e. the achievability direction, we show the following statement.

Proposition 3. *For the discrimination problem as above with $n \in \mathbb{N}$ and $\varepsilon \in (0,1)$, we have*

$$\zeta^n_{\mathcal{S},\mathcal{T}}(\varepsilon) \geq \frac{1}{n} \inf_{\substack{\rho \in \mathcal{S} \\ \mu \in \mathcal{T}}} D\left(\rho^{\otimes n} \| \sigma_n(\mu)\right) - \frac{\log \mathrm{poly}(n)}{n}\,, \tag{17}$$

where $\mathrm{poly}(n)$ *stands for terms of order at most polynomial in* n.

The basic idea for the proof of Prop. 3 is to start from the corresponding composite Stein's lemma for classical probability distributions and lift the result to the non-commutative setting by solely using properties of quantum entropy. For that we need the measured relative entropy defined as [16, 25]

$$D_{\mathcal{M}}(\rho\|\sigma) := \sup_{(\mathcal{X},M)} D\left(\sum_{x \in \mathcal{X}} \mathrm{Tr}\left[M_x \rho\right] |x\rangle\langle x| \middle\| \sum_{x \in \mathcal{X}} \mathrm{Tr}\left[M_x \rho\right] |x\rangle\langle x| \right)\,, \tag{18}$$

where the optimization is over finite sets $\mathcal{X}$ and POVMs M on $\mathcal{X}$ with $\mathrm{Tr}\left[M_x \rho\right]$ a measure on $\mathcal{X}$ for any $x \in \mathcal{X}$. (Henceforth, we write for the classical relative entropy between probability distributions $D(P\|Q)$ – defined via the diagonal embedding of P and Q as on the right-hand side of Eq. (18).) It is known that we can restrict the a priori unbounded supremum to rank-one projective measurements [6, Thm. 2]. We now prove Prop. 3 in several steps and start with an achievability bound in terms of the measured relative entropy.

Lemma 4. *For the discrimination problem as above with $n \in \mathbb{N}$ and $\varepsilon \in (0,1)$, we have*

$$\zeta^n_{\mathcal{S},\mathcal{T}}(\varepsilon) \geq \frac{1}{n} \inf_{\substack{\nu \in \mathcal{S} \\ \mu \in \mathcal{T}}} D_{\mathcal{M}}\left(\rho_n(\nu) \| \sigma_n(\mu)\right) . \tag{19}$$

Proof. For sets of classical probability distributions $P \in \mathcal{S}$ and $Q \in \mathcal{T}$ we know from the corresponding commutative result [10, 28] that for $\varepsilon \in (0,1)$

$$\zeta_{\mathcal{S},\mathcal{T}}(\varepsilon) = \inf_{\substack{P \in \mathcal{S} \\ Q \in \mathcal{T}}} D(P\|Q) . \tag{20}$$

Now, the strategy is to first measure the quantum states and then invoke the classical achievability result (20) for the resulting probability distributions. For that fix $n \in \mathbb{N}$ and a POVM $\mathcal{M}_n$ on $\mathcal{H}^{\otimes n}$. For testing the probability distributions $P_n := \mathcal{M}_n\left(\rho^{\otimes n}\right)$ vs. $Q_n := \mathcal{M}_n\left(\sigma^{\otimes n}\right)$ we get an achievability bound

$$\zeta^n_{\mathcal{S},\mathcal{T}}(\varepsilon) \geq \frac{1}{n} \inf_{\substack{\rho \in \mathcal{S} \\ \sigma \in \mathcal{T}}} D\left(\mathcal{M}_n\left(\rho^{\otimes n}\right) \middle\| \mathcal{M}_n\left(\sigma^{\otimes n}\right)\right) \geq \frac{1}{n} \inf_{\substack{\nu \in \mathcal{S} \\ \mu \in \mathcal{T}}} D\left(\mathcal{M}_n\left(\rho_n(\nu)\right) \| \mathcal{M}_n\left(\sigma_n(\mu)\right)\right) , \tag{21}$$

where the second inequality follows since the infimum is taken over a larger set. The claim then follows from applying a minimax theorem for the measured relative entropy (Lem. 15)

$$\sup_{\mathcal{M}_n} \inf_{\substack{\nu \in \mathcal{S} \\ \mu \in \mathcal{T}}} D\left(\mathcal{M}_n\left(\rho_n(\nu)\right) \| \mathcal{M}_n\left(\sigma_n(\mu)\right)\right) = \inf_{\substack{\nu \in \mathcal{S} \\ \mu \in \mathcal{T}}} D_{\mathcal{M}}\left(\rho_n(\nu) \| \sigma_n(\mu)\right) . \tag{22}$$

□

Next, we argue that the measured relative entropy can in fact be replaced by the quantum relative entropy by only paying an asymptotically vanishing penalty term. For this we need the following lemma which can be seen as a generalization of the original technical argument in the proof of quantum Stein's lemma [25].

Lemma 5. *Let $\rho_n, \sigma_n \in S\left(\mathcal{H}^{\otimes n}\right)$ with σ_n permutation invariant. Then, we have*

$$D\left(\rho_n \| \sigma_n\right) - \log \mathrm{poly}(n) \leq D_{\mathcal{M}}\left(\rho_n \| \sigma_n\right) \leq D\left(\rho_n \| \sigma_n\right) . \tag{23}$$

Proof. The second inequality follows directly from the definition of the measured relative entropy in Eq. (18) together with the fact that the quantum relative entropy is monotone under completely positive trace preserving maps [30]. We now prove the first inequality with the help of asymptotic spectral pinching [21]. The pinching map with respect to $\omega \in S(\mathcal{H})$ is defined as

$$\mathcal{P}_\omega(\cdot) := \sum_{\lambda \in \mathrm{spec}(\omega)} P_\lambda(\cdot) P_\lambda \text{ with the spectral decomposition } \omega = \sum_{\lambda \in \mathrm{spec}(\omega)} \lambda P_\lambda . \tag{24}$$

Crucially, we have the pinching operator inequality $\mathcal{P}_\omega[X] \gg \frac{X}{|\mathrm{spec}(\omega)|}$ [21] where $\gg$ denotes the Loewner order. From this we can deduce that (see, e.g., [43, Lem. 4.4])

$$D\left(\rho_n \| \sigma_n\right) - \log\left|\mathrm{spec}\left(\sigma_n\right)\right| \leq D\left(\mathcal{P}_{\sigma_n}\left(\rho_n\right) \| \sigma_n\right) = D_{\mathcal{M}}\left(\rho_n \| \sigma_n\right) , \tag{25}$$

where the equality follows since $\mathcal{P}_{\sigma_n}\left(\rho_n\right)$ and σ_n are diagonal in the same basis. It remains to show that $\left|\mathrm{spec}\left(\sigma_n\right)\right| \leq \mathrm{poly}(n)$. However, since σ_n is permutation invariant we have by Schur-Weyl duality (see, e.g., [20, Sect. 5]) that in the Schur basis

$$\sigma_n = \bigoplus_{\lambda \in \Lambda_n} \sigma_{Q_\lambda} \otimes 1_{P_\lambda} \quad \text{with } |\Lambda_n| \leq \mathrm{poly}(n) \text{ and } \dim\left[\sigma^0_{Q_\lambda}\right] \leq \mathrm{poly}(n). \tag{26}$$

This implies the claim. □

By combining Lem. 4 together with Lem. 5 we immediately find that

$$\zeta^n_{\mathcal{S},\mathcal{T}}(\varepsilon) \geq \frac{1}{n} \inf_{\substack{\rho \in \mathcal{S} \\ \mu \in \mathcal{T}}} D\left(\rho_n(\nu) \| \sigma_n(\mu)\right) - \frac{\log \mathrm{poly}(n)}{n} . \tag{27}$$

Hence, it remains to argue that the infimum over states $\rho_n(\nu)$ can without lost of generality be restricted to iid states $\rho^{\otimes n}$ with $\rho \in \mathcal{S}$.

Lemma 6. *For the same definitions as before and $\omega_n \in S(\mathcal{H}^{\otimes n})$, we have*

$$\frac{1}{n}\inf_{\nu\in\mathcal{S}} D\left(\rho_n(\nu)\|\omega_n\right) \geq \frac{1}{n}\inf_{\rho\in\mathcal{S}} D\left(\rho^{\otimes n}\middle\|\omega_n\right) - \frac{\log\mathrm{poly}(n)}{n}\,. \tag{28}$$

Proof. We observe the following chain of arguments for $\nu \in \mathcal{S}$

$$\begin{aligned}
\frac{1}{n}D\left(\rho_n(\nu)\|\omega_n\right) &= \frac{1}{n}D\left(\sum_{i=1}^{N} p_i\rho_i^{\otimes n}\middle\|\omega_n\right)\\
&= -\frac{1}{n}H\left(\sum_{i=1}^{N} p_i\rho_i^{\otimes n}\right) - \frac{1}{n}\cdot\sum_{i=1}^{N} p_i \operatorname{Tr}\left[\rho_i^{\otimes n}\log\omega_n\right]\\
&\geq -\frac{1}{n}\cdot\sum_{i=1}^{N} p_i H\left(\rho_i^{\otimes n}\right) - \frac{\log\mathrm{poly}\,(n)}{n} - \frac{1}{n}\cdot\sum_{i=1}^{N} p_i \operatorname{Tr}\left[\rho_i^{\otimes n}\log\omega_n\right]\\
&\geq \min_{\rho_i}\frac{1}{n}D\left(\rho_i^{\otimes n}\middle\|\omega_n\right) - \frac{\log\mathrm{poly}\,(n)}{n}\\
&\geq \inf_{\rho\in\mathcal{S}}\frac{1}{n}D\left(\rho^{\otimes n}\middle\|\omega_n\right) - \frac{\log\mathrm{poly}\,(n)}{n}\,,
\end{aligned} \tag{29}$$

where $H(\rho) := -\operatorname{Tr}[\rho\log\rho]$ denotes the von Neumann entropy, the first equality holds by an application of Carathédory's theorem with $N \leq \mathrm{poly}(n)$ (Lem. 16), and the first inequality by a quasi-convexity property of the von Neumann entropy (Lem. 17). (All other steps are elementary.) Since the above argument holds for all $\nu \in \mathcal{S}$ the claim follows. □

Combining Lem. 6 with Eq. (27) leads to Prop. 3 and then taking the limits $n \to \infty$ and $\varepsilon \to 0$ we find

$$\zeta_{\mathcal{S},\mathcal{T}}(0) \geq \limsup_{n\to\infty}\frac{1}{n}\inf_{\substack{\rho\in\mathcal{S}\\ \mu\in\mathcal{T}}} D\left(\rho^{\otimes n}\|\sigma_n(\mu)\right)\,. \tag{30}$$

Together with the converse from Eq. (16) we get

$$\liminf_{n\to\infty}\frac{1}{n}\inf_{\substack{\rho\in\mathcal{S}\\ \mu\in\mathcal{T}}} D\left(\rho^{\otimes n}\|\sigma_n(\mu)\right) \geq \zeta_{\mathcal{S},\mathcal{T}}(0) \geq \limsup_{n\to\infty}\frac{1}{n}\inf_{\substack{\rho\in\mathcal{S}\\ \mu\in\mathcal{T}}} D\left(\rho^{\otimes n}\|\sigma_n(\mu)\right) \Rightarrow \zeta_{\mathcal{S},\mathcal{T}}(0) = \lim_{n\to\infty}\frac{1}{n}\inf_{\substack{\rho\in\mathcal{S}\\ \mu\in\mathcal{T}}} D\left(\rho^{\otimes n}\|\sigma_n(\mu)\right)\,, \tag{31}$$

which finishes the proof of Thm. 1. □

III. EXAMPLES AND EXTENSIONS

Here we discuss several concrete examples of composite discrimination problems – some of which have a single-letter solution.

A. Relative entropy of coherence

Following the literature around [4] the set of states diagonal in a fixed basis $\{|c\rangle\}$ is called incoherent and denoted by $\mathcal{C} \subseteq S(\mathcal{H})$. The relative entropy of coherence of $\rho \in S(\mathcal{H})$ is defined as

$$D_{\mathcal{C}}(\rho) := \inf_{\sigma\in\mathcal{C}} D(\rho\|\sigma)\,. \tag{32}$$

Using the result from Sect. II we can characterize the following discrimination problem.

Null hypothesis: the fixed state $\rho^{\otimes n}$

Alternative hypothesis: the convex sets of iid coherent states $\bar{\mathcal{C}}_n := \left\{\int \sigma^{\otimes n}\,\mathrm{d}\mu(\sigma)\middle|\sigma\in\mathcal{C}\right\}$

Namely, as a special case of Thm. 1 we immediately find

$$\bar{\zeta}_{\bar{\mathcal{C}}}(0) = \lim_{n\to\infty} \frac{1}{n} \inf_{\mu\in\mathcal{C}} D\left(\rho^{\otimes n} \| \int \sigma^{\otimes n}\, \mathrm{d}\mu(\sigma)\right) = D_{\mathcal{C}}(\rho)\,, \tag{33}$$

where the last equality follows from (Lem. 18). In fact there is even a single-letter solution for the following less restricted discrimination problem.

Null hypothesis: the fixed state $\rho^{\otimes n}$

Alternative hypothesis: the convex set of coherent states $\sigma_n \in \mathcal{C}_n$

It is straightforward to check that this hypothesis testing problem fits the general framework of [12] leading to

$$\zeta_{\mathcal{C}}(0) = \lim_{n\to\infty} \frac{1}{n} \inf_{\sigma_n\in\mathcal{C}_n} D\left(\rho^{\otimes n} \| \sigma_n\right) = D_{\mathcal{C}}(\rho)\,, \tag{34}$$

where the last step again follows from Lem. 18. We have therefore two a priori different hypothesis testing scenarios that both give an operational interpretation to the relative entropy of coherence. We remark that our results also easily extend to the relative entropy of frameness [19].

In the following we give a simple self-contained proof of Eq. (34) that is different from the proof in [12] and follows ideas from [3, 24, 44]. The goal is the quantification of the optimal asymptotic error exponent (as we will see the following limit exists)

$$\zeta^n_{\mathcal{C}}(\varepsilon) := -\frac{1}{n} \cdot \log \inf_{\substack{0\leq M_n\leq 1\\ \mathrm{Tr}\left[M_n\rho^{\otimes n}\right]\geq 1-\varepsilon}} \sup_{\sigma_n\in\mathcal{C}_n} \mathrm{Tr}\left[M_n\sigma_n\right] \quad \text{with} \quad \zeta_{\mathcal{C}}(\varepsilon) := \lim_{n\to\infty} \zeta^n_{\rho,\mathcal{C}}(\varepsilon) \quad \text{and} \quad \zeta_{\mathcal{C}}(0) := \lim_{\varepsilon\to 0} \zeta_{\rho,\mathcal{C}}(\varepsilon)\,. \tag{35}$$

Proposition 7. *For the discrimination problem as above we have $\zeta_{\mathcal{C}}(0) = D_{\mathcal{C}}(\rho)$.*

The converse direction $\leq$ follows exactly as in Lem. 2, together with Lem. 18 to make the expression single-letter. For the achievability direction $\geq$ we make use of a general family of quantum Rényi entropies: the Petz divergences [35]. For $\rho, \sigma \in S(\mathcal{H})$ and $s \in (0,1)\cup(1,\infty)$ they are defined as

$$D_s\left(\rho\|\sigma\right) := \frac{1}{s-1} \log \mathrm{Tr}\left[\rho^s \sigma^{1-s}\right]\,, \tag{36}$$

whenever either $s < 1$ and ρ is not orthogonal to σ in Hilbert-Schmidt inner product or $s > 1$ and the support of ρ is contained in the support of σ. (Otherwise we set $D_s(\rho\|\sigma) := \infty$.) The corresponding Rényi relative entropies of coherence are given by [13]

$$D_{s,\mathcal{C}}(\rho) := \inf_{\sigma\in\mathcal{C}} D_s(\rho\|\sigma) \text{ with the additivity property } D_{s,\mathcal{C}}\left(\rho^{\otimes n}\right) = n\cdot D_{s,\mathcal{C}}(\rho). \tag{37}$$

Prop. 7 follows by taking the limits $n \to \infty$, $s \to 1$, and $\varepsilon \to 0$ in the following lemma.

Lemma 8. *For the discrimination problems as above with $n \in \mathbb{N}$ and $\varepsilon \in (0,1)$ we have for $s \in (0,1)$ that*

$$\zeta^n_{\mathcal{C}}(\varepsilon) \geq D_{s,\mathcal{C}}(\rho) - \frac{1}{n}\cdot\frac{s}{1-s}\log\frac{1}{\varepsilon}\,. \tag{38}$$

Proof. It is straightforward to check with Sion's minimax theorem (Lem. 14) that

$$\inf_{\substack{0\leq M_n\leq 1\\ \mathrm{Tr}\left[M_n\rho^{\otimes n}\right]\geq 1-\varepsilon}} \sup_{\sigma_n\in\mathcal{C}_n} \mathrm{Tr}\left[M_n\sigma_n\right] = \sup_{\sigma_n\in\mathcal{C}_n} \inf_{\substack{0\leq M_n\leq 1\\ \mathrm{Tr}\left[M_n\rho^{\otimes n}\right]\geq 1-\varepsilon}} \mathrm{Tr}\left[M_n\sigma_n\right]\,. \tag{39}$$

Now, for $\lambda_n \in \mathbb{R}$ with $n \in \mathbb{N}$ we choose $M_n(\lambda_n) := \left\{\rho^{\otimes n} - 2^{\lambda_n}\sigma_n\right\}_+$ where $\{\cdot\}_+$ denotes the projector on the eigenspace of the positive spectrum. We have $0 \ll M_n(\lambda_n) \ll 1$ and by Audenaert's inequality (Lem. 19) with $s \in (0,1)$ we get

$$\mathrm{Tr}\left[(1-M_n(\lambda_n))\rho^{\otimes n}\right] \leq 2^{(1-s)\lambda_n}\, \mathrm{Tr}\left[\left(\rho^{\otimes n}\right)^s \sigma_n^{1-s}\right] = 2^{(1-s)\left(\lambda_n - D_s\left(\rho^{\otimes n}\|\sigma_n\right)\right)}\,. \tag{40}$$

Moreover, again Audenaert's inequality (Lem. 19) for $s \in (0,1)$ implies

$$\mathrm{Tr}\left[M_n(\lambda_n)\sigma_n\right] \leq 2^{-s\lambda_n}\,\mathrm{Tr}\left[\left(\rho^{\otimes n}\right)^s \sigma_n^{1-s}\right] = 2^{-s\lambda_n-(1-s)D_s\left(\rho^{\otimes n}\|\sigma_n\right)}\,. \tag{41}$$

Hence, choosing

$$\lambda_n := D_s\left(\rho^{\otimes n}\middle\|\sigma_n\right) + \log \varepsilon^{\frac{1}{1-s}} \text{ with } M_n := M_n(\lambda_n) \tag{42}$$

leads with Eq. (40) to $\mathrm{Tr}\left[M_n\rho^{\otimes n}\right] \geq 1-\varepsilon$. Finally, Eq. (39) together with Eq. (41) and the additivity property from Eq. (37) leads to the claim. ☐

A more refined analysis of the above calculation also allows to determine the Hoeffding bound as well as the strong converse exponent (cf. [3, 24]). The former gives an operational interpretation to the Rényi relative entropy of coherence $D_{s,\mathcal{C}}(\rho)$, whereas the latter gives an operational interpretation to the sandwiched Rényi relative entropies of coherence [13]

$$\tilde{D}_{s,\mathcal{C}}(\rho) := \inf_{\sigma\in\mathcal{C}} \tilde{D}_s(\rho\|\sigma) \text{ with the sandwiched Rényi relative entropies } \tilde{D}_s(\rho\|\sigma) := \frac{1}{s-1}\log \mathrm{Tr}\left[\left(\sigma^{\frac{1-s}{2s}}\rho\sigma^{\frac{1-s}{2s}}\right)^s\right] \tag{43}$$

whenever either $s<1$ and ρ is not orthogonal to σ in Hilbert-Schmidt inner product or $s>1$ and the support of ρ is contained in the support of σ [32, 48]. (Otherwise we set $D_s(\rho\|\sigma) := \infty$.) The crucial insight for the proof is again the additivity property $\tilde{D}_{s,\mathcal{C}}\left(\rho^{\otimes n}\right) = n\cdot\tilde{D}_{s,\mathcal{C}}(\rho)$ that was already shown in [13].

B. Relative entropy of recovery

The relative entropy of recovery of $\rho_{ABC} \in S(\mathcal{H}_{ABC})$ and its regularized version are defined as [6, 11, 38]

$$D(A;B|C)_\rho := \inf_{\mathcal{R}} D\left(\rho_{ABC}\|\mathcal{R}_{C\to BC}\left(\rho_{AC}\right)\right) \quad \text{and} \quad D^\infty(A;B|C)_\rho := \lim_{n\to\infty}\frac{1}{n}D(A;B|C)_{\rho^{\otimes n}}\,, \tag{44}$$

where the infimum goes over all completely positive and trace preserving maps $\mathcal{R}_{C\to BC}$. It was recently shown that in general $D^\infty(A;B|C)_\rho \neq D(A;B|C)_\rho$ [17]. Using the framework from [12] the following discrimination problem was linked to the regularized relative entropy of recovery [15].

Null hypothesis: the fixed state $\rho_{ABC}^{\otimes n}$

Alternative hypothesis: the convex sets of states $\mathcal{R}^n := \left\{\left(\mathcal{I}_{A^n}\otimes\mathcal{R}_{C^n\to B^nC^n}\right)\left(\rho_{AC}^{\otimes n}\right)\right\}$ with $\mathcal{R}_{C^n\to B^nC^n}$ completely positive and trace preserving

Namely, we have

$$\zeta_{A:B|C}(0) = \lim_{n\to\infty}\frac{1}{n}\inf_{\mathcal{R}} D\left(\rho_{ABC}^{\otimes n}\middle\|\mathcal{R}_{C^n\to B^nC^n}\left(\rho_{AC}^{\otimes n}\right)\right)\,. \tag{45}$$

In contrast, our result from Sect. II covers the following discrimination problem.

Null hypothesis: the fixed state $\rho_{ABC}^{\otimes n}$

Alternative hypothesis: the convex sets of iid states $\bar{\mathcal{R}}^n := \left\{\int\left((\mathcal{I}_A\otimes\mathcal{R}_{C\to BC})(\rho_{AC})\right)^{\otimes n}\,\mathrm{d}\mu(\mathcal{R})\right\}$ with $\mathcal{R}_{C\to BC}$ completely positive and trace preserving

Namely, we have

$$\bar{\zeta}_{A:B|C}(0) = \lim_{n\to\infty}\frac{1}{n}\inf_{\mu\in\mathcal{R}} D\left(\rho_{ABC}^{\otimes n}\middle\|\int\left((\mathcal{I}_A\otimes\mathcal{R}_{C\to BC})(\rho_{AC})\right)^{\otimes n}\mathrm{d}\mu(\mathcal{R})\right)\,. \tag{46}$$

Interestingly, we can again show that both rates are actually identical.

Proposition 9. *For the discrimination problems as above we have* $\zeta_{A:B|C}(0) = \bar{\zeta}_{A:B|C}(0)$.

Proof. We have by definition that $\zeta_{A:B|C}(0) \leq \bar{\zeta}_{A:B|C}(0)$ and for the other direction we use a de Finetti reduction for quantum channels from [11, Lem. 8] (first derived in [18]). Namely, we have for $\omega_{C^n} \in S\left(\mathcal{H}_C^{\otimes n}\right)$ and permutation invariant $\mathcal{R}_{C^n \to B^n C^n}$ that

$$\mathcal{R}_{C^n \to B^n C^n}\left(\omega_{C^n}\right) \ll \operatorname{poly}(n) \cdot \int \left(\mathcal{R}_{C \to BC}\right)^{\otimes n}\left(\omega_{C^n}\right) \mathrm{d}\nu(\mathcal{R}) \tag{47}$$

for some measure $\nu(\mathcal{R})$ over the completely positive and trace preserving maps on $C \to BC$. As explained in the proof of [11, Prop. 9], the joint convexity of the quantum relative entropy together with the operator monotonicity of the logarithm then imply that

$$D\left(\rho_{ABC}^{\otimes n} \middle\| \mathcal{R}_{C^n \to B^n C^n}\left(\rho_{AC}^{\otimes n}\right)\right) \geq D\left(\rho_{ABC}^{\otimes n} \middle\| \int \left((\mathcal{I}_A \otimes \mathcal{R}_{C \to BC})(\rho_{AC})\right)^{\otimes n} \mathrm{d}\nu(\mathcal{R})\right) - \log \operatorname{poly}(n)\,. \tag{48}$$

By inspection this leads to $\zeta_{A:B|C}(0) \geq \bar{\zeta}_{A:B|C}(0)$ and hence implies the claim. □

C. Quantum mutual information

The quantum mutual information of $\rho_{AB} \in S(\mathcal{H}_{AB})$ is defined as

$$I(A:B)_\rho := H(A)_\rho + H(B)_\rho - H(AB)_\rho\,. \tag{49}$$

Our main result from Sect. II provides a solution to the following discrimination problem.

Null hypothesis: the state $\rho_{AB}^{\otimes n}$

Alternative hypothesis: the convex set of iid states $\bar{\mathcal{T}}_{A^n:B^n} := \left\{\rho_A^{\otimes n} \otimes \int \sigma_B^{\otimes n}\,\mathrm{d}\mu(\sigma) \middle| \sigma_B \in S(\mathcal{H}_B)\right\}$

Namely, we have

$$\bar{\zeta}_{A:B}(0) = \lim_{n\to\infty} \frac{1}{n} \inf_{\mu \in \bar{\mathcal{T}}} D\left(\rho_{AB}^{\otimes n} \middle\| \rho_A^{\otimes n} \otimes \int \sigma_B^{\otimes n}\,\mathrm{d}\mu(\sigma)\right) = I(A:B)_\rho\,. \tag{50}$$

Here the last equality follows from the easily checked identity

$$I(A:B)_\rho = \inf_{\sigma \in S(\mathcal{H})} D(\rho_{AB} \| \sigma_A \otimes \sigma_B)\,. \tag{51}$$

More general composite discrimination problems leading to the quantum mutual information were solved in [24] and here we further extend these results to the following (cf. the classical work [44]).

Null hypothesis: the state $\rho_{AB}^{\otimes n}$

Alternative hypothesis: the set of states $\mathcal{T}_{A^n:B^n} := \left\{\sigma_{A^n} \otimes \sigma_{B^n} \in S\left(\mathcal{H}_{AB}^{\otimes n}\right) \middle| \sigma_{A^n} \vee \sigma_{B^n} \text{ permutation invariant}\right\}$.

The goal is again the quantification of the optimal asymptotic error exponent (as we will see the following limit exists)

$$\zeta_{A:B}^n(\varepsilon) := -\frac{1}{n} \cdot \log \inf_{\substack{0 \leq M_n \leq 1 \\ \operatorname{Tr}\left[M_n \rho^{\otimes n}\right] \geq 1-\varepsilon}} \sup_{\sigma \otimes \sigma \in \mathcal{T}_n} \operatorname{Tr}\left[M_{A^n B^n} \sigma_{A^n} \otimes \sigma_{B^n}\right] \tag{52}$$

$$\text{with} \quad \zeta_{A:B}(\varepsilon) := \lim_{n \to \infty} \zeta_{A:B}^n(\varepsilon) \quad \text{and} \quad \zeta_{A:B}(0) := \lim_{\varepsilon \to 0} \zeta_{A:B}(\varepsilon)\,. \tag{53}$$

Note that the sets $\mathcal{T}_{A^n B^B}$ are not convex and hence the minimax technique used in Sect. III A does not work here. However, following the ideas in [24, 44] we can exploit the permutation invariance and use de Finetti reductions of the form [14, 22] to find the following.

Proposition 10. *For the discrimination problem as above we have $\zeta_{A:B}(0) = I(A:B)_\rho$.*

The converse direction $\leq$ follows exactly as in Lem. 2, together with Eq. (51) to make the expression single-letter. The achievability direction $\geq$ follows from the following lemma by taking the limits $n \to \infty$, $s \to 1$, $\varepsilon \to 0$ and then applying Eq. (51).

Lemma 11. *For the discrimination problem as above with $n \in \mathbb{N}$ and $\varepsilon \in (0,1)$ we have for $s \in (0,1)$ that*

$$\zeta^n_{A:B}(\varepsilon) \geq \inf_{\sigma \in S(\mathcal{H})} D_s\left(\rho_{AB} \| \sigma_A \otimes \sigma_B\right) - \frac{1}{n} \cdot \frac{s}{1-s} \log\frac{1}{\varepsilon} - \frac{\log \mathrm{poly}(n)}{n}. \tag{54}$$

Proof. We choose

$$M_{A^nB^n}(\lambda_n) := \left\{\rho_{AB}^{\otimes n} - 2^{\lambda_n}\omega_{A^n} \otimes \omega_{B^n}\right\}_+ \quad \text{with} \quad \omega_{A^n} := \binom{n + |A|^2 - 1}{n}^{-1} \cdot \mathrm{Tr}_{\tilde{A}^n}\left[P^{\mathrm{Sym}}_{A^n\tilde{A}^n}\right], \tag{55}$$

where $P^{\mathrm{Sym}}_{A^n\tilde{A}^n}$ denotes the projector onto the symmetric subspace of $\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_{\tilde{A}}^{\otimes n}$ with $|A| = |\tilde{A}|$ (denoting the dimension of $\mathcal{H}_A$ by $|A|$), and similarly for B^n. Since $\omega_{A^n} \otimes \sigma_{B^n}$ is permutation invariant we get together with Audenaert's inequality (Lem. 19) that

$$\mathrm{Tr}\left[(1 - M_{A^nB^n}(\lambda_n))\rho_{AB}^{\otimes n}\right] \leq 2^{(1-s)\lambda_n}\,\mathrm{Tr}\left[\left(\rho_{AB}^{\otimes n}\right)^s \left(\omega_{A^n} \otimes \omega_{B^n}\right)^{1-s}\right] \leq 2^{(1-s)\left(\lambda_n - \inf_{\sigma\otimes\sigma \in \mathcal{T}_n} D_s\left(\rho_{AB}^{\otimes n} \| \sigma_{A^n} \otimes \sigma_{B^n}\right)\right)}. \tag{56}$$

Furthermore we have by Schur-Weyl duality that $\sigma_{A^n} \leq \binom{n+|A|^2-1}{n} \cdot \omega_{A^n}$ for all permutation invariant σ_{A^n} (see, e.g., [24, Lem. 1]) and thus by again using Audenaert's inequality (Lem. 19) we find

$$\begin{aligned}
\mathrm{Tr}\left[M_{A^nB^n}(\lambda_n)\left(\sigma_{A^n} \otimes \sigma_{B^n}\right)\right] &= \mathrm{Tr}\left[M_{A^nB^n}(\lambda_n)\left(\sigma_{A^n} \otimes \left(\sum_{\pi \in S_n} U_{B^n}(\pi)\sigma_{B^n}U^\dagger_{B^n}(\pi)\right)\right)\right] \quad (S_n\text{: symmetric group})\\
&\leq \underbrace{\binom{n+|A|^2-1}{n}\binom{n+|B|^2-1}{n}}_{=:\,p(n)\,\leq\,\mathrm{poly}(n)} \cdot \mathrm{Tr}\left[M_{A^nB^n}(\lambda_n)\left(\omega_{A^n} \otimes \omega_{B^n}\right)\right]\\
&\leq p(n) \cdot 2^{-s\lambda_n}\,\mathrm{Tr}\left[\left(\rho_{AB}^{\otimes n}\right)^s \left(\omega_{A^n} \otimes \omega_{B^n}\right)^{1-s}\right]\\
&\leq p(n) \cdot 2^{-s\lambda_n - (1-s)\inf_{\sigma\otimes\sigma \in \mathcal{T}_n} D_s\left(\rho_{AB}^{\otimes n} \| \sigma_{A^n} \otimes \sigma_{B^n}\right)}.
\end{aligned} \tag{57}$$

We now choose

$$\lambda_n := \inf_{\sigma\otimes\sigma \in \mathcal{T}_n} D_s\left(\rho_{AB}^{\otimes n} \| \sigma_{A^n} \otimes \sigma_{B^n}\right) + \log \varepsilon^{\frac{1}{1-s}} \text{ with } M_{A^nB^n} := M_{A^nB^n}(\lambda_n), \tag{58}$$

from which we get $\mathrm{Tr}\left[M_{A^nB^n}\rho_{AB}^{\otimes n}\right] \geq 1 - \varepsilon$ and together with Eq. (52) and Eq. (57) that

$$\zeta^n_{A:B}(\varepsilon) \geq \inf_{\sigma\otimes\sigma \in \mathcal{T}_n} D_s\left(\rho_{AB}^{\otimes n} \| \sigma_{A^n} \otimes \sigma_{B^n}\right) - \frac{1}{n} \cdot \frac{s}{1-s} \log\frac{1}{\varepsilon} - \frac{\log p(n)}{n}. \tag{59}$$

To deduce the claim it is now sufficient to argue that the Rényi quantum mutual information[1]

$$I_s(A:B)_\rho := \inf_{\sigma\otimes\sigma \in S(\mathcal{H})} D_s\left(\rho_{AB} \| \sigma_A \otimes \sigma_B\right) \tag{60}$$

is additive on tensor product states. This, however, follows exactly as in the classical case [44, App. A-C] from the (quantum) Sibson identity [39, Lem. 3]

$$D_s\left(\rho_{AB} \| \sigma_A \otimes \sigma_B\right) = D_s\left(\rho_{AB} \| \sigma_A \otimes \bar{\sigma}_B\right) + D_s\left(\bar{\sigma}_B \| \sigma_B\right) \quad \text{with} \quad \bar{\sigma}_B := \frac{\left(\mathrm{Tr}_A\left[\rho_{AB}^s \sigma_A^{1-s}\right]\right)^{\frac{1}{s}}}{\mathrm{Tr}\left[\left(\mathrm{Tr}_A\left[\rho_{AB}^s \sigma_A^{1-s}\right]\right)^{\frac{1}{s}}\right]}. \tag{61}$$

□

A more refined analysis of the above calculation along the work [24] also allows to determine the Hoeffding bound for the product testing discrimination problem as above. However, for the strong converse exponent we are missing the additivity of the sandwiched Rényi quantum mutual information

$$\tilde{I}_s(A:B)_\rho := \inf_{\sigma\otimes\sigma \in S(\mathcal{H})} \tilde{D}_s\left(\rho_{AB} \| \sigma_A \otimes \sigma_B\right) \tag{62}$$

on tensor product states.

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[1] This definition is slightly different from the Rényi quantum mutual information discussed in [24].

IV. CONDITIONAL QUANTUM MUTUAL INFORMATION

Here we discuss how our results are related to the conditional quantum mutual information. This allows us to show that the regularization in our formula for composite convex iid testing from Sect. II is needed in general.

A. Markov type lower bounds

The following is a proof of the lower bound on the conditional quantum mutual information from Eq. (9).

Theorem 12. *For $\rho_{ABC} \in S(\mathcal{H}_{ABC})$ we have*[2]

$$I(A:B|C)_\rho \geq \limsup_{n\to\infty} \frac{1}{n} D\left(\rho_{ABC}^{\otimes n} \middle\| \int \beta_0(t) \left(\mathcal{I}_A \otimes \mathcal{R}_{C\to BC}^{[t]}(\rho_{AC})\right)^{\otimes n} \mathrm{d}t\right), \tag{63}$$

where $\beta_0(t) := \frac{\pi}{2}\left(\cosh(\pi t)+1\right)^{-1}$ *and* $R_{C\to BC}^{[t]}(\cdot) := \rho_{BC}^{\frac{1+it}{2}}\left(\rho_C^{\frac{-1-it}{2}}(\cdot)\rho_C^{\frac{-1+it}{2}}\right)\rho_{BC}^{\frac{1-it}{2}}$.

Proof. We start from the lower bound [41, Thm. 4.1] applied to $\rho_{ABC}^{\otimes n}$

$$I(A:B|C)_\rho \geq \frac{1}{n} D_M\left(\rho_{ABC}^{\otimes n}\|\sigma_{A^nB^nC^n}\right) \text{ with } \sigma_{A^nB^nC^n} := \int \beta_0(t)\left(\sigma_{ABC}^{[t]}\right)^{\otimes n} \mathrm{d}t \text{ and } \sigma_{ABC}^{[t]} := \left(\mathcal{I}_A \otimes R_{C\to BC}^{[t]}\right)(\rho_{AC}), \tag{64}$$

where we have used that the conditional quantum mutual information is additive on tensor product states. Now, we simply observe that $\sigma_{A^nB^nC^n}$ is permutation invariant and hence the claim can be deduced from Lem. 5 together with taking the limit $n \to \infty$. □

Together with previous work we find the following corollary that encompasses all known Markov type lower bounds on the conditional quantum mutual information.

Corollary 13. *For $\rho_{ABC} \in S(\mathcal{H}_{ABC})$ the conditional quantum mutual information $I(A:B|C)_\rho$ is lower bounded by the three incomparable bounds*

$$-\int \beta_0(t) \log\left\|\sqrt{\rho_{ABC}}\sqrt{\sigma_{ABC}^{[t]}}\right\|_1^2 \mathrm{d}t,\ D_{\mathcal{M}}\left(\rho_{ABC}\middle\|\int \beta_0(t)\sigma_{ABC}^{[t]}\,\mathrm{d}t\right),\ \limsup_{n\to\infty}\frac{1}{n}D\left(\rho_{ABC}^{\otimes n}\middle\|\int\beta_0(t)\left(\sigma_{ABC}^{[t]}\right)^{\otimes n}\mathrm{d}t\right). \tag{65}$$

In contrast to the second and third bound, the first lower bound is not tight in the commutative case but has the advantage that the average over $\beta_0(t)$ stands outside of the distance measure used. All the lower bounds are typically strict – whereas in the commutative case the second and third bound both become equalities.

Proof. The first bound was shown in [26, Sect. 3], the second one in [41, Thm. 4.1], and the third one is Thm. 12. To see that the bounds are incomparable notice that the distribution $\beta_0(t)$ cannot be taken outside the relative entropy measure in the second and the third bound since quantum Stein's lemma would then lead to a contradiction to a recent counterexample from [17, Sect. 5]: there exists $\theta \in [0, \pi/2]$ such that

$$I(A:B|C)_\rho \not\geq \inf_{\mathcal{R}} D\left(\rho_{ABC}\|(\mathcal{I}_A\otimes\mathcal{R}_{B\to BC})(\rho_{AC})\right) \text{ for the pure state } \rho_{ABC} = |\rho\rangle\langle\rho|_{ABC} \tag{66}$$

$$\text{with } |\rho\rangle_{ABC} = \frac{1}{\sqrt{2}}|0\rangle_A\otimes|0\rangle_B\otimes|0\rangle_C + \frac{1}{\sqrt{2}}\left(\cos(\theta)|0\rangle_A\otimes|1\rangle_C + \sin(\theta)|1\rangle_A\otimes|0\rangle_C\right)\otimes|1\rangle_B\,. \tag{67}$$

That the lower bounds are typically strict can be seen from numerical work (see, e.g., [11]). □

It seems that the only remaining conjectured strengthening that is not known to be wrong is the lower bound in terms of the non-rotated Petz map [7, Sect. 8]

$$I(A:B|C)_\rho \geq -\log\left\|\sqrt{\rho_{ABC}}\sqrt{\sigma_{ABC}^{[0]}}\right\|_1^2. \tag{68}$$

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[2] The inverses are understood as generalized inverses.

We refer to [27] for the latest progress in that direction. All the same arguments can also be applied to lift the strengthened monotonicity from [41, Cor. 4.2]. For $\rho \in S(\mathcal{H})$, σ a positive semi-definite operator on $\mathcal{H}$, and $\mathcal{N}$ a completely positive trace preserving map on the same space this leads to

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geq \limsup_{n\to\infty} \frac{1}{n} D\left(\rho^{\otimes n}\middle\| \int \beta_0(t) \left(\mathcal{R}^{[t]}_{\sigma,\mathcal{N}}(\rho)\right)^{\otimes n} \mathrm{d}t\right), \tag{69}$$

where $\mathcal{R}^{[t]}_{\sigma,\mathcal{N}}(\cdot) := \sigma^{\frac{1+it}{2}} \mathcal{N}^\dagger \left(\mathcal{N}(\sigma)^{\frac{-1-it}{2}} (\cdot) \mathcal{N}(\sigma)^{\frac{-1+it}{2}}\right) \sigma^{\frac{1-it}{2}}$. Together with [26, Sect. 3] and [41, Cor. 4.2] we then again have three incomparable lower bounds as in Cor. 13.

B. Regularization needed for composite convex iid testing

Here we use our bound on the conditional quantum mutual information (Thm. 12) to show that the regularization in Thm. 1 is in general needed (see also [9]). That is, we give a proof for Eq. (8). Namely, by Thm. 12 we have[3]

$$\begin{aligned} I(A:B|C)_\rho &\geq \limsup_{n\to\infty} \frac{1}{n} D\left(\rho_{ABC}^{\otimes n}\middle\| \int \beta_0(t) \left(\mathcal{I}_A \otimes \mathcal{R}^{[t]}_{C\to BC}(\rho_{AC})\right)^{\otimes n} \mathrm{d}t\right) \quad (70)\\ &\geq \limsup_{n\to\infty} \frac{1}{n} \inf_{\mu\in\mathcal{R}} D\left(\rho_{ABC}^{\otimes n}\middle\| \int \left(\mathcal{R}_{C\to BC}\left(\rho_{AC}\right)\right)^{\otimes n} \mathrm{d}\mu(\mathcal{R})\right). \quad (71)\end{aligned}$$

From the first composite discrimination problem described in Sect. III B we see that the latter quantity is equal to the optimal asymptotic error exponent $\bar{\zeta}_{A:B|C}(0)$ for testing $\rho_{ABC}^{\otimes n}$ against $\int ((\mathcal{I}_A \otimes \mathcal{R}_{B\to BC})(\rho_{AC}))^{\otimes n} \,\mathrm{d}\mu(\mathcal{R})$. Now, if the regularization in the formula for $\bar{\zeta}_{A:B|C}(0)$ would actually not be needed this would imply that

$$I(A:B|C)_\rho \geq \inf_{\mathcal{R}} D\left(\rho_{ABC}\|(\mathcal{I}_A \otimes \mathcal{R}_{B\to BC})(\rho_{AC})\right), \tag{72}$$

However, this is in contradiction with the counterexample from [17, Sect. 5] as discussed in Eq. (66). Hence, we conclude that the regularization for composite convex iid testing is needed in general. □

V. CONCLUSION

We extended quantum Stein's lemma in asymmetric quantum hypothesis testing by showing that the asymptotic error exponent for testing convex combinations of quantum states $\rho^{\otimes n}$ against convex combinations of quantum states $\sigma^{\otimes}$ is given by a regularized quantum relative entropy distance formula (which does not become single-letter in general). Moreover, we also gave various examples where our formula as well as extensions thereof become single-letter. It remains interesting to find more non-commutative settings that allow for a single-letter solution.

Another closely related problem is that of composite symmetric hypothesis testing where it is well known that in the case of fixed iid states $\rho^{\otimes n}$ vs. $\sigma^{\otimes n}$ the optimal error exponent is given by the quantum Chernoff bound [1, 33]

$$C(\rho,\sigma) = \sup_{0\leq s\leq 1} -\log \mathrm{Tr}\left[\rho^s \sigma^{1-s}\right]. \tag{73}$$

However, the discrimination problem of testing convex combinations of iid states $\rho^{\otimes}$ with $\rho \in \mathcal{S}$ against convex combinations of iid states $\sigma^{\otimes}$ with $\sigma \in \mathcal{T}$ is still unsolved and it was conjectured [2] that as in the commutative case we have

$$C_{\mathcal{S},\mathcal{T}} = \inf_{\substack{\rho\in\mathcal{S}\\ \sigma\in\mathcal{T}}} C(\rho,\sigma). \tag{74}$$

The most recent progress [2] states that in the case of a fixed null hypothesis $\mathcal{S} = \{\rho\}$ the rate in Eq. (74) is achievable up to a factor of two (see also [29] for a very related problem that allows for an exact single-letter solution). We

[3] Alternatively we could use the implicitly stated bound from [11, Eq. 38].

note that extending the proof of the fixed state iid setting one can show that the following rate is achievable in the composite setting (assuming that the limit exists)

$$C_{\mathcal{S},\mathcal{T}} = \sup_{0\leq s\leq 1} \lim_{n\to\infty} \frac{1}{n} \inf_{\substack{\nu\in\mathcal{S}\\ \mu\in\mathcal{T}}} -\log \mathrm{Tr}\left[\left(\int \rho^{\otimes n}\,\mathrm{d}\nu(\rho)\right)^{s}\left(\int \sigma^{\otimes n}\,\mathrm{d}\mu(\sigma)\right)^{1-s}\right]. \tag{75}$$

However, our results about composite asymmetric hypothesis testing raise the question whether it is indeed possible to simplify Eq. (75) to the conjecture in Eq. (74).

Finally, we note that finding single-letter achievability results for composite hypothesis testing problems would allow to make progress on some long-standing open problems in network quantum Shannon theory [36, Sect. 5.2].

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Appendix A: Some Lemmas

Here we present several lemmas that are used in the main part. We start with Sion's minimax theorem.

Lemma 14. *[40] Let X be a compact convex subset of a linear topological space and Y a convex subset of a linear topological space. If a real-valued function on $X \times Y$ is such that*

- *$f(x,\cdot)$ is upper semi-continuous and quasi-concave on Y for every $x \in X$*
- *$f(\cdot,y)$ is lower semi-continuous and quasi-convex on X for every $y \in Y$,*

then we have

$$\min_{x\in X}\sup_{y\in Y} f(x,y) = \sup_{y\in Y}\min_{x\in X} f(x,y)\,. \tag{A1}$$

If the measured relative entropy is optimized over closed, convex sets then Sion's minimax theorem can be applied.

Lemma 15. *[10, Lem. 20] Let $\mathcal{S},\mathcal{T} \subseteq S(\mathcal{H})$ be closed, convex sets. Then, we have*

$$\min_{\substack{\rho\in\mathcal{S}\\ \sigma\in\mathcal{T}}} D_{\mathcal{M}}(\rho\|\sigma) = \sup_{(\mathcal{X},M)} \min_{\substack{\rho\in\mathcal{S}\\ \sigma\in\mathcal{T}}} D\left(\sum_{x\in\mathcal{X}} \mathrm{Tr}\left[M_x\rho\right]|x\rangle\langle x| \middle\| \sum_{x\in\mathcal{X}} \mathrm{Tr}\left[M_x\rho\right]|x\rangle\langle x|\right). \tag{A2}$$

We have the following discretization result.

Lemma 16. *For every measure μ over a subset $\mathcal{S} \subseteq S(\mathcal{H})$ with the dimension of $\mathcal{H}$ given by d, there exists a probability distribution $\{p_i\}_i^N$ with $N \leq (n+1)^{2d^2}$ and $\rho_i \in \mathcal{S}$ such that*

$$\int \rho^{\otimes n}\,\mathrm{d}\mu(\rho) = \sum_{i=1}^{N} p_i\rho_i^{\otimes n}\,. \tag{A3}$$

Proof. The idea is to use Carathéodory theorem together with the smallness of the symmetric subspace. For pure states the proof from [5, Cor. D.6] applies and the general case follows immediately by considering purifications and taking the partial trace over the purifying system. □

The von Neumann entropy has the following quasi-convexity property (besides its well-known concavity).

Lemma 17. *Let $\rho_i \in S(\mathcal{H})$ for $i = 1,\ldots,N$ and $\{p_i\}$ be a probability distribution. Then, we have*

$$H\left(\sum_{i=1}^{N} p_i\rho_i\right) \leq \sum_{i=1}^{N} p_i H(\rho_i) + \log N\,. \tag{A4}$$

Proof. This follows from elementary entropy inequalities:

$$H\left(\sum_{i=1}^{N} p_i\rho_i\right) \leq \sum_{i=1}^{N} p_i H\left(\rho_i\right) + H(p_i) \leq \sum_{i=1}^{N} p_i H(\rho_i) + \log N\,. \tag{A5}$$

□

The following is a property of the quantum relative entropy.

Lemma 18. *[19, Thm. 3] Let $\mathcal{N}$ be a trace-preserving, completely positive map with $\mathcal{N}(1) = 1$ (unital) and $\mathcal{N}^2 = \mathcal{N}$ (idempotent). Then, the minimum relative entropy distance between $\rho \in S(\mathcal{H})$ and $\sigma \in S(\mathcal{H})$ in the image of $\mathcal{N}$ satisfies*

$$\inf_{\sigma\in\mathrm{Im}(\mathcal{N})} D(\rho\|\sigma) = H(\mathcal{N}(\rho)) - H(\rho) = D(\rho\|\mathcal{N}(\rho))\,. \tag{A6}$$

In particular, we have for the relative entropy of coherence that $D_{\mathcal{C}}(\rho) = D(\rho\|\rho_{\mathrm{diag}})$, where ρ_{diag} denotes the state obtained from ρ by deleting all off-diagonal elements.

Audenaert's matrix inequality originally used to derive the quantum Chernoff bound can be stated as follows.

Lemma 19. *[1, Thm. 1] Let $X, Y \gg 0$ and $s \in (0,1)$. Then, we have*

$$\mathrm{Tr}\left[X^sY^{1-s}\right] \geq \mathrm{Tr}\left[X\left(1 - \{X-Y\}_+\right)\right] + \mathrm{Tr}\left[Y\{X-Y\}_+\right]\,. \tag{A7}$$

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