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EXISTENCE OF SOLUTIONS FOR SOME QUASILINEAR $\vec{p}(x)$-ELLIPTIC PROBLEM WITH HARDY POTENTIAL

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Abstract. We consider the anisotropic quasilinear elliptic Dirichlet problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + |u|^{s(x)-1}u = f + \lambda \frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ containing the origin. We show the existence of entropy solution for this equation where the data f is assumed to be in $L^1(\Omega)$ and λ is a positive constant.

Keywords: anisotropic variable exponent Sobolev space; quasilinear elliptic equation; Hardy potential; entropy solution; L^1-data

MSC 2010: 35J15, 35J62

1. Introduction

In the recent years, the anisotropic variable exponent Sobolev spaces have taken its place in the mathematical literature. This impulse is essentially due to their applications in nonhomogeneous materials that behave differently in different space directions, we can refer here to the electrorheological and thermoelectric fluids that have multiple applications in brakes shock absorbers, robotics and space technology (see for example [4], [23]).

In [11], Boccardo et al. have studied the nonlinear anisotropic elliptic equation

$$(1.1)\qquad \begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i-2}\frac{\partial u}{\partial x_i}\Big) = f & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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where Ω is an open bounded subset of $\mathbb{R}^N$, $N \geqslant 2$, and the right-hand side f is a bounded Radon measure. They have proved the existence and regularity of solutions in the anisotropic Sobolev spaces $W_0^{1,\vec{p}}(\Omega)$. The critical regularity was obtained under the assumption of $|f|\log(1+|f|) \in L^1(\Omega)$. In [13], Cîrstea and Vétois have proved the existence of weak solutions to the problem (1.1) where the data f is assumed to be a Dirac mass at 0. We refer also to [28] where the author has proved the existence of nonnegative weak solutions in anisotropic Sobolev space for the elliptic and parabolic cases, where the right-hand side is assumed to be a Carathéodory function $f(x,u,\nabla u)$. We refer the reader also to [27].

In [14], Di Nardo and Feo have considered the quasilinear elliptic problem

$$
(1.2) \qquad \begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_i a_i(x,u,\nabla u) + \sum_{i=1}^{N} H_i(x,\nabla u) = f - \sum_{i=1}^{N} \partial_i g_i & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega; \end{cases}
$$

they have proved the existence and uniqueness of weak solutions for this anisotropic elliptic Dirichlet problem, where the data is assumed to be in the dual space.

Di Nardo, Feo and Guibé have studied in [15] the existence of renormalized solutions for some class of nonlinear anisotropic elliptic problems of the type

$$
-\sum_{i=1}^{N} \partial_{x_i}(a_i(x,u)|\partial_{x_i} u|^{p_i-2}\partial_{x_i} u) = f - \operatorname{div} g \quad \text{in } \Omega,
$$

with $f \in L^1(\Omega)$ and $g \in \prod_{i=1}^{N} L^{p_i'}(\Omega)$; the uniqueness of renormalized solution was concluded under some local Lipschitz conditions on the function $a_i(x,s)$ with respect to s (see also [3]).

In the framework of variable exponents Sobolev spaces, Wittbold and Zimmermann have proved in [29] the existence and uniqueness of renormalized solutions for the quasilinear elliptic problem

$$
(1.3) \qquad \begin{cases} \beta(u) - \operatorname{div} a(x,\nabla u) - \displaystyle\sum_{i=1}^{N} \operatorname{div} F(u) \ni f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}
$$

where the data f is assumed to be in $L^1(\Omega)$. In [9], Bendahmane et al. have considered the nonlinear elliptic equation

$$
(1.4) \qquad \begin{cases} -\displaystyle\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\Big(\Big|\frac{\partial u}{\partial x_i}\Big|^{p_i(x)-2}\frac{\partial u}{\partial x_i}\Big) + |u|^{s(x)-1}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}
$$

with $f \in L^1(\Omega)$ and $p_i(\cdot)$ being continuous functions for $i = 1, \ldots, N$; they have shown the existence of solution in the anisotropic variable exponents Sobolev spaces. Also, the authors have proved the corresponding results for the nonlinear anisotropic parabolic case. In [12], Cianchi has considered the quasilinear anisotropic elliptic problem

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where the datum $f(x,s)$ is a Carathéodory function verifying some growth condition. He has proved the existence of weak solutions in the anisotropic Orlicz–Sobolev spaces, which he established via symmetrization. We refer also to [25], where the author has shown the existence and regularity of weak solutions for the data in $L^m(\Omega)$ with $m \geqslant 1$, also to [5] for the existence of weak solutions, and to [2] for the solutions in the sense of distributions, and [21] for the renormalized solutions.

For some elliptic problems with singularity on its right-hand side, we refer the reader to [1] where the authors have studied the nonlinear elliptic problem

$$\begin{cases} -\Delta u \pm |\nabla u|^2 = \lambda \dfrac{u}{|x|^2} + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

with $\lambda > 0$. They have proved the existence of positive solutions for the problem (1.6) in the absorption case $(+|\nabla u|^2)$ with $f \in L^1(\Omega)$. In the reaction case $(-|\nabla u|^2)$, the non-existence of solution is proved even in a very weak sense. Porzio has studied in [26] the existence of weak solutions for the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}(M(x,u)\nabla u) + \nu|u|^{p-1}u = a\dfrac{u}{|x|^2} + f(x) - \operatorname{div}(F) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $p > N/(N-2)$ and a is a positive constant, the Carathéodory function $M(x,s)$ satisfies the growth and coercivity conditions. For the case of nonlinear and non-coercive elliptic problems, we refer the reader to [19], [20].

In this paper, we consider Ω to be an open bounded subset of $\mathbb{R}^N$, $N \geqslant 2$, containing the origin, and let $p_i(\cdot)$ be some measurable functions on Ω for any $i = 0, 1, \ldots, N$ where

$$p_0(x) = \max\{p_i(x),\ i = 1, 2, \ldots, N\} \quad \text{a.e. in } \Omega. \tag{1.8}$$

We will study the existence of entropy solutions for the anisotropic quasilinear elliptic problem

$$\begin{cases} Au + |u|^{s(x)-1}u = f + \lambda \dfrac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

with $\lambda \geqslant 0$, $f \in L^1(\Omega)$ and

$$s(x) > \max\Big(\frac{N(p_0(x)-1)}{N-p_0(x)}, \frac{1}{p_0(x)-1}\Big) \quad \text{a.e. in } \Omega. \tag{1.10}$$

The Leray-Lions operator A acting from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into its dual, is defined by the formula

$$Au = -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) \tag{1.11}$$

where $a_i\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \mapsto \mathbb{R}^N$ are Carathéodory functions for $i=1,\dots,N$ (measurable with respect to x in Ω for every (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$ and continuous with respect to (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$ for almost every x in Ω), which satisfy the following conditions:

$$|a_i(x,s,\xi)| \leqslant \beta\bigg(K_i(x) + |s|^{p_i(x)-1} + \sum_{i=1}^{N} |\xi_i|^{p_i(x)-1}\bigg), \tag{1.12}$$

$$\sum_{i=1}^{N} a_i(x,s,\xi)\xi_i \geqslant \alpha \sum_{i=1}^{N} |\xi_i|^{p_i(x)} \tag{1.13}$$

for any $\xi = (\xi_1,\dots,\xi_N)$ and $\xi' = (\xi'_1,\dots,\xi'_N)$ in $\mathbb{R}^N$, we have

$$(a_i(x,s,\xi) - a_i(x,s,\xi'))(\xi_i - \xi'_i) > 0 \quad \text{for } \xi_i \neq \xi'_i, \tag{1.14}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, where $K_i(x) \in L^{p'_i(\cdot)}(\Omega)$, and α, β are two positive real numbers.

Remark 1.1. The assumption (3.1) is essential to ensure that $|a_i(x,u,\nabla u)|$ belongs to $L^{p'_i(\cdot)}(\Omega)$. In the case of $Au = -\sum\limits_{i=1}^{N} D^i a_i(x,\nabla u)$, the condition (1.12) will be written as

$$|a_i(x,\xi)| \leqslant \beta\bigg(K_i(x) + \sum_{i=1}^{N} |\xi_i|^{p_i(x)-1}\bigg), \tag{1.15}$$

thus the existence of an entropy solution will be guaranteed by following the same way, without using the additional assumption (1.8).

Note that, in view of the growth condition (1.12), to show that the Carathéodory functions $|a_i(x,u,\nabla u)|$ belong to $L^{p'_i(\cdot)}(\Omega)$ for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, it is necessary to have $u \in L^{p_i(\cdot)}(\Omega)$ for any $i=1,\dots,N$, which is verified by taking the condition (1.8).

For the case where the Carathéodory functions $a_i(x,\xi)$ verify the assumption (1.15), then $|a_i(x,\nabla u)| \in L^{p_i'(\cdot)}(\Omega)$ for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$.

In this paper, we have assumed that the data f belong to $L^1(\Omega)$, then the elliptic equation (1.9) is not in the dual space $W^{-1,\vec{p'}(\cdot)}(\Omega)$, thus the existence of a weak solution have no sense. To overcome this difficulty, some mathematicians have used the notions of entropy and renormalized solutions, which are more adapted for this category of problems. Note that the entropy solutions were introduced by Bénilan et al. in [10], and the notion of renormalized solutions by DiPerna et al. in [17], [18].

The difficulties in proving the existence of entropy solutions stem from the following fact: Since the exponents $p_i(\cdot)$ are assumed to be measurable functions, the Poincaré and Sobolev inequalities are not verified, therefore, the operator Au is not coercive in the anisotropic variable exponent Sobolev space $W_0^{1,\vec{p}(\cdot)}(\Omega)$, defined below. To overcome this difficulty, we use the penalization term $|u|^{p_0(x)-2}u/n$ in approximate problems (3.4). Moreover, the singular term $|u|^{p_0(x)-2}u/|x|^{p_0(x)}$ creates, in general, a hindrance to the existence of solutions. We overpass this difficulty by using the regularizing effect of the term $|u|^{s(x)-1}u$ to remove the non-existence effect produced by the Hardy potential.

This paper is organized as follows. In Section 2 we introduce some preliminary results including a brief discussion on the anisotropic variable exponent Sobolev spaces, and we recall some technical lemmas. Section 3 will be devoted to showing the existence of entropy solutions for our anisotropic $\vec{p}(x)$-quasilinear elliptic equation with Hardy potential (1.9).

2. Preliminary

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geqslant 2$, we denote

$$\mathcal{C}_+(\Omega) = \{\text{measurable function } p(\cdot)\colon\ \Omega \mapsto \mathbb{R} \text{ such that } 1 < p^- \leqslant p^+ < N\},$$

where

$$p^- = \operatorname{ess\,inf}\{p(x)\colon\ x \in \Omega\} \quad \text{and} \quad p^+ = \operatorname{ess\,sup}\{p(x)\colon\ x \in \Omega\}.$$

We define the Lebesgue space with a variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u\colon\ \Omega \mapsto \mathbb{R}$ for which the convex modular

$$\varrho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)}\,\mathrm{d}x$$

is finite. If the exponent is bounded, i.e. if $p^+ < \infty$, then the expression

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0\colon\ \varrho_{p(\cdot)}(u/\lambda) \leqslant 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p^- \leqslant p^+ < \infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. Finally, we have the generalized Hölder type inequality:

$$\Big|\int_\Omega uv\,\mathrm{d}x\Big| \leqslant \Big(\frac{1}{p^-} + \frac{1}{(p')^-}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \tag{2.1}$$

for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

The Sobolev space with a variable exponent $W^{1,p(\cdot)}(\cdot)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \text{ and } |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space equipped with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is a separable and reflexive Banach space. We define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For more details on variable exponent Lebesgue and Sobolev spaces, we refer the reader to [16].

Now, we present the anisotropic variable exponent Sobolev spaces used in the study of our quasilinear anisotropic elliptic problem.

Let $p_0(\cdot), p_1(\cdot), \ldots, p_N(\cdot)$ be $N+1$ variable exponents in $\mathcal{C}_+(\Omega)$. We denote

$$\vec{p}(\cdot) = (p_0(\cdot), \ldots, p_N(\cdot)), \quad D^0u = u \quad \text{and} \quad D^iu = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \ldots, N,$$

and if we define

$$\underline{p} = \min\{p_0^-, p_1^-, \ldots, p_N^-\}, \tag{2.2}$$

then $\underline{p} > 1$. The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as

$$W^{1,\vec{p}(\cdot)}(\Omega) = \{u \in L^{p_0(\cdot)}(\Omega) \text{ and } D^iu \in L^{p_i(\cdot)}(\Omega) \text{ for } i = 1, 2, \ldots, N\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}(\cdot)} = \sum_{i=0}^N \|D^iu\|_{p_i(\cdot)}. \tag{2.3}$$

We define also $W_0^{1,\vec{p}(\cdot)}(\Omega)$ to be the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ with respect to the norm (2.3). The space $(W_0^{1,\vec{p}(\cdot)}(\Omega), \|u\|_{1,\vec{p}(\cdot)})$ is a reflexive Banach space (cf. [24]).

Lemma 2.1. *We have the following continuous and compact embedding:*

- ▷ *if* $\underline{p} < N$ *then* $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *for* $q \in [\underline{p}, \underline{p}^*[$ *where* $\underline{p}^* = N\underline{p}/(N-\underline{p})$,
- ▷ *if* $\underline{p} = N$ *then* $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *for all* $q \in [\underline{p}, \infty[$,
- ▷ *if* $\underline{p} > N$ *then* $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$.

The proof of this lemma follows from the fact that the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W_0^{1,\underline{p}}(\Omega)$ is continuous, and in view of the compact embedding theorem for Sobolev spaces.

Proposition 2.1. *The dual of* $W_0^{1,\vec{p}(\cdot)}(\Omega)$ *is denoted by* $W^{-1,\vec{p}'(\cdot)}(\Omega)$, *where* $\vec{p}'(\cdot) = (p_0'(\cdot), \dots, p_N'(\cdot))$ *and* $1/p_i'(\cdot) + 1/p_i(\cdot) = 1$ *(cf. [8] for the constant exponent case). For each* $F \in W^{-1,\vec{p}'(\cdot)}(\Omega)$ *there exists* $F_i \in L^{p_i'(\cdot)}(\Omega)$ *for* $i = 0, 1, \dots, N$, *such that* $F = F_0 - \sum_{i=1}^{N} D^i F_i$. *Moreover, for any* $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, *we have*

$$\langle F, u\rangle = \sum_{i=0}^{N} \int_\Omega F_i D^i u \,\mathrm{d}x.$$

We define a norm on the dual space by

$$\|F\|_{-1,\vec{p}'(\cdot)} = \inf\left\{\sum_{i=0}^{N} \|F_i\|_{p_i'(\cdot)} \colon\ F = F_0 - \sum_{i=1}^{N} D^i F_i \ \text{with } F_i \in L^{p_i'(\cdot)}(\Omega)\right\}.$$

We set

$$\mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega) := \{u \colon \Omega \mapsto \mathbb{R} \text{ measurable, such that } T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega) \text{ for any } k > 0\}.$$

Note that a measurable function u verifying $T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ for all $k > 0$ does not necessarily belong to $W_0^{1,1}(\Omega)$. However, for any $u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$ it is possible to define the weak gradient of u, still denoted by ∇u.

Proposition 2.2. *Let* $u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$. *For any* $i \in \{1, \dots, N\}$, *there exists a unique measurable function* $v_i \colon \Omega \mapsto \mathbb{R}$ *such that*

$$\forall\, k > 0 \quad D^i T_k(u) = v_i \cdot \chi_{\{|u|<k\}} \quad \text{a.e. } x \in \Omega,$$

where χ_A *denotes the characteristic function of a measurable set* A. *The functions* v_i *are called the weak partial derivatives of* u *and are still denoted by* $D^i u$. *Moreover, if* u *belongs to* $W_0^{1,1}(\Omega)$, *then* v_i *coincides with the standard distributional derivative of* u, *that is,* $v_i = D^i u$.

The proof of Proposition 2.2 follows the usual techniques developed in [10] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [8] and [15].

Lemma 2.2 ([6]). *Let $g \in L^{r(\cdot)}(\Omega)$ and $g_n \in L^{r(\cdot)}(\Omega)$ with $\|g_n\|_{r(\cdot)} \leqslant C$ for $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ in $L^{r(\cdot)}(\Omega)$.*

Lemma 2.3 ([7]). *Assuming that (1.12)–(1.14) hold, and letting $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and*

$$\begin{aligned}\int_\Omega (|u_n|^{p_0(x)-2}u_n - |u|^{p_0(x)-2}u)(u_n-u)\,\mathrm{d}x \\ + \sum_{i=1}^N \int_\Omega (a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u))(D^i u_n - D^i u)\,\mathrm{d}x \to 0,\end{aligned} \tag{2.4}$$

then $u_n \to u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ for a subsequence.

3. Existence of entropy solutions

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geqslant 2$, containing the origin, and let $p_i(\cdot) \in \mathcal{C}_+(\overline{\Omega})$ for $i = 0,1,\ldots,N$, where

$$p_0(x) = \max\{p_i(x),\ i = 1,2,\ldots,N\} \quad \text{a.e. in } \Omega. \tag{3.1}$$

Definition 3.1. A measurable function u is an entropy solution of the Dirichlet problem (1.9) if

$$u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega), \quad |u|^{s(x)-1}u \in L^1(\Omega), \quad \frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} \in L^1(\Omega)$$

and

$$\begin{aligned}\sum_{i=1}^N \int_\Omega a_i(x,u,\nabla u)\cdot D^i T_k(u-\varphi)\,\mathrm{d}x + \int_\Omega |u|^{s(x)-1}uT_k(u-\varphi)\,\mathrm{d}x \\ \leqslant \int_\Omega fT_k(u-\varphi)\,\mathrm{d}x + \lambda\int_\Omega \frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}}T_k(u-\varphi)\,\mathrm{d}x\end{aligned} \tag{3.2}$$

for any $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Our main result is the following:

Theorem 3.1. *Let* $\lambda \geqslant 0$ *and* $f \in L^1(\Omega)$, *assuming that* (1.10) *and* (1.12)–(1.14) *hold true. Then there exists at least one entropy solution* u *for quasilinear elliptic problem* (1.9), *such that* $u \in W_0^{1,\vec{q}(\cdot)}(\Omega)$, *with*

$$\vec{q}(\cdot) = (s(\cdot), q_1(\cdot), \dots, q_N(\cdot)) \quad \text{and} \quad 1 \leqslant q_i(x) < \frac{p_i(x)s(x)}{s(x)+1} \quad \text{for } i = 1, \dots, N. \tag{3.3}$$

P r o o f of Theorem 3.1.

Step 1: Approximate problems. Let $(f_n)_{n\in\mathbb{N}^*}$ be a sequence of smooth functions such that $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \leqslant |f|$ (for example $f_n = T_n(f)$). We consider the approximate problem

$$A_n u_n + |T_n(u_n)|^{s(x)-1} T_n(u_n) = f_n + \lambda \frac{|T_n(u_n)|^{p_0(x)-2} T_n(u_n)}{|x|^{p_0(x)} + 1/n}, \tag{3.4}$$

where $A_n v = -\sum\limits_{i=1}^{N} \partial a_i(x, T_n(v), \nabla v)/\partial x_i + |v|^{p_0(x)-2} v/n$.

We consider the operator $G_n \colon W_0^{1,\vec{p}(\cdot)}(\Omega) \mapsto W^{-1,\vec{p}'(\cdot)}(\Omega)$ given by

$$\langle G_n u, v\rangle = \int_\Omega |T_n(u)|^{s(x)-1} T_n(u) v \,\mathrm{d}x - \lambda \int_\Omega \frac{|T_n(u)|^{p_0(x)-2} T_n(u)}{|x|^{p_0(x)} + 1/n} v \,\mathrm{d}x$$

for any $u, v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$. Thanks to the generalized Hölder's type inequality, we have

$$\begin{aligned} |\langle G_n u, v\rangle| &\leqslant \int_\Omega |T_n(u)|^{s(x)} \, |v| \,\mathrm{d}x + \lambda \int_\Omega \frac{|T_n(u)|^{p_0(x)-1}}{|x|^{p_0(x)} + 1/n} |v| \,\mathrm{d}x \\ &\leqslant n^{s^+} \int_\Omega |v| \,\mathrm{d}x + \lambda n^{p_0^+} \int_\Omega |v| \,\mathrm{d}x \leqslant C_0 \|v\|_{1,\vec{p}(\cdot)}. \end{aligned} \tag{3.5}$$

Lemma 3.1. *The bounded operator* $B_n = A_n + G_n$ *acting from* $W_0^{1,\vec{p}(\cdot)}(\Omega)$ *into* $W^{-1,\vec{p}'(\cdot)}(\Omega)$ *is pseudo-monotone. Moreover,* B_n *is coercive in the following sense:*

$$\frac{\langle B_n v, v\rangle}{\|v\|_{1,\vec{p}(\cdot)}} \to \infty \quad \text{as } \|v\|_{1,\vec{p}(\cdot)} \to \infty \text{ for } v \in W_0^{1,\vec{p}(\cdot)}(\Omega).$$

P r o o f. In view of Hölder's inequality and the growth condition (1.12), it is easy to see that the operator A_n is bounded, and by (3.5) we conclude that B_n is bounded.

For the coercivity, we have for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$,

$$\begin{aligned}
\langle B_n u, u\rangle &= \langle A_n u, u\rangle + \langle G_n u, u\rangle \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) D^i u \, \mathrm{d}x + \int_\Omega |T_n(u)|^{s(x)} |u| \, \mathrm{d}x \\
&\quad + \frac{1}{n} \int_\Omega |u|^{p_0(x)} \, \mathrm{d}x - \lambda \int_\Omega \frac{|T_n(u)|^{p_0(x)-1}}{|x|^{p_0(x)} + 1/n} |u| \, \mathrm{d}x \\
&\geqslant \underline{\alpha} \sum_{i=0}^{N} \int_\Omega |D^i u|^{p_i(x)} \, \mathrm{d}x + \int_\Omega |T_n(u)|^{s(x)+1} \, \mathrm{d}x - 2\lambda n^{p_0^+} \|1\|_{p_0'(\cdot)} \|u\|_{1,\vec{p}(\cdot)} \\
&\geqslant \underline{\alpha} \, \|u\|_{1,\vec{p}(\cdot)}^{\underline{p}} - \underline{\alpha}(N+1) - C_1 \|u\|_{1,\vec{p}(\cdot)}
\end{aligned}$$

with $\underline{\alpha} = \min(\alpha, 1/n)$. It follows that

$$\frac{\langle B_n u, u\rangle}{\|u\|_{1,\vec{p}(\cdot)}} \to \infty \quad \text{as } \|u\|_{1,\vec{p}(\cdot)} \to \infty.$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega), \\ B_n u_k \rightharpoonup \chi_n & \text{in } W^{-1,\vec{p}'(\cdot)}(\Omega), \\ \limsup\limits_{k\to\infty} \langle B_n u_k, u_k\rangle \leqslant \langle \chi_n, u\rangle. \end{cases} \tag{3.6}$$

We will prove that

$$\chi_n = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \to \langle \chi_n, u\rangle \quad \text{as } k \to \infty.$$

In view of the compact embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$, we have $u_k \to u$ in $L^{\underline{p}}(\Omega)$ for a subsequence still denoted as $(u_k)_{k\in\mathbb{N}}$.

As $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, using the growth condition (1.12) it is clear that the sequence $(a_i(x, T_n(u_k), \nabla u_k))_{k\in\mathbb{N}}$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, hence there exists a function $\varphi_i \in L^{p_i'(\cdot)}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i \quad \text{in } L^{p_i'(\cdot)}(\Omega) \quad \text{as } k \to \infty. \tag{3.7}$$

In view of Lebesgue's dominated convergence theorem, we obtain

$$|T_n(u_k)|^{s(x)-1} T_n(u_k) \to |T_n(u)|^{s(x)-1} T_n(u) \quad \text{in } L^{p_0'(\cdot)}(\Omega), \tag{3.8}$$

and

$$\text{(3.9)} \qquad \frac{|T_n(u_k)|^{p_0(x)-2}T_n(u_k)}{|x|^{p_0(x)}+1/n} \to \frac{|T_n(u)|^{p_0(x)-2}T_n(u)}{|x|^{p_0(x)}+1/n} \quad \text{in } L^{p_0'(\cdot)}(\Omega).$$

Also, we have

$$\text{(3.10)} \qquad \frac{1}{n}|u_k|^{p_0(x)-2}u_k \rightharpoonup \frac{1}{n}|u|^{p_0(x)-2}u \quad \text{in } L^{p_0'(\cdot)}(\Omega).$$

For any $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we have

$$\begin{aligned}
\text{(3.11)} \qquad \langle \chi_n, v\rangle &= \lim_{k\to\infty}\langle B_n u_k, v\rangle \\
&= \lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i v\,\mathrm{d}x \\
&\quad + \lim_{k\to\infty}\int_\Omega |T_n(u_k)|^{s(x)-1}T_n(u_k)v\,\mathrm{d}x \\
&\quad + \lim_{k\to\infty}\frac{1}{n}\int_\Omega |u_k|^{p_0(x)-2}u_k\; v\,\mathrm{d}x \\
&\quad - \lim_{k\to\infty}\lambda\int_\Omega \frac{|T_n(u_k)|^{p_0(x)-2}T_n(u_k)}{|x|^{p_0(x)}+1/n}v\,\mathrm{d}x \\
&= \sum_{i=1}^{N}\int_\Omega \varphi_i\; D^i v\,\mathrm{d}x + \int_\Omega |T_n(u)|^{s(x)-1}T_n(u)v\,\mathrm{d}x \\
&\quad + \frac{1}{n}\int_\Omega |u|^{p_0(x)-2}u\; v\,\mathrm{d}x - \lambda\int_\Omega \frac{|T_n(u)|^{p_0(x)-2}T_n(u)}{|x|^{p_0(x)}+1/n}v\,\mathrm{d}x.
\end{aligned}$$

Having in mind (3.6) and (3.11), we obtain

$$\begin{aligned}
\text{(3.12)} \quad \limsup_{k\to\infty}\langle B_n(u_k),u_k\rangle &= \limsup_{k\to\infty}\Big\{\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,\mathrm{d}x \\
&\quad + \int_\Omega |T_n(u_k)|^{s(x)-1}T_n(u_k)u_k\,\mathrm{d}x + \frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,\mathrm{d}x \\
&\quad - \lambda\int_\Omega \frac{|T_n(u_k)|^{p_0(x)-2}T_n(u_k)}{|x|^{p_0(x)}+1/n}u_k\,\mathrm{d}x\Big\} \\
&\leqslant \sum_{i=1}^{N}\int_\Omega \varphi_i D^i u\,\mathrm{d}x + \int_\Omega |T_n(u)|^{s(x)-1}T_n(u)u\,\mathrm{d}x \\
&\quad + \frac{1}{n}\int_\Omega |u|^{p_0(x)}\,\mathrm{d}x - \lambda\int_\Omega \frac{|T_n(u)|^{p_0(x)-2}T_n(u)}{|x|^{p_0(x)}+1/n}u\,\mathrm{d}x.
\end{aligned}$$

Thanks to (3.8) and (3.9), we have

$$\int_\Omega |T_n(u_k)|^{s(x)-1}T_n(u_k)u_k\,\mathrm{d}x \to \int_\Omega |T_n(u)|^{s(x)-1}T_n(u)u\,\mathrm{d}x, \tag{3.13}$$

and

$$\int_\Omega \frac{|T_n(u_k)|^{p_0(x)-2}T_n(u_k)}{|x|^{p_0(x)}+1/n}u_k\,\mathrm{d}x \to \int_\Omega \frac{|T_n(u)|^{p_0(x)-2}T_n(u)}{|x|^{p_0(x)}+1/n}u\,\mathrm{d}x. \tag{3.14}$$

Therefore

$$\limsup_{k\to\infty}\Big(\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,\mathrm{d}x + \frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,\mathrm{d}x\Big) \leqslant \sum_{i=1}^N\int_\Omega \varphi_i D^iu\,\mathrm{d}x + \frac{1}{n}\int_\Omega |u|^{p_0(x)}\,\mathrm{d}x. \tag{3.15}$$

On the other hand, in view of (1.14) we have

$$\sum_{i=1}^N\int_\Omega (a_i(x,T_n(u_k),\nabla u_k) - a_i(x,T_n(u_k),\nabla u))(D^iu_k - D^iu)\,\mathrm{d}x + \frac{1}{n}\int_\Omega (|u_k|^{p_0(x)-2}u_k - |u|^{p_0(x)-2}u)(u_k-u)\,\mathrm{d}x \geqslant 0, \tag{3.16}$$

hence

$$\begin{aligned}\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,\mathrm{d}x + \frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,\mathrm{d}x \\ \geqslant \sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu\,\mathrm{d}x + \frac{1}{n}\int_\Omega |u_k|^{p_0(x)-2}u_ku\,\mathrm{d}x \\ + \sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u)(D^iu_k - D^iu)\,\mathrm{d}x + \frac{1}{n}\int_\Omega |u|^{p_0(x)-2}u(u_k-u)\,\mathrm{d}x.\end{aligned}$$

In view of Lebesgue's dominated convergence theorem we have $T_n(u_k)\to T_n(u)$ in $L^{p_i(\cdot)}(\Omega)$, thus $a_i(x,T_n(u_k),\nabla u)\to a_i(x,T_n(u),\nabla u)$ in $L^{p_i'(\cdot)}(\Omega)$, and using (3.7) we get

$$\liminf_{k\to\infty}\Big(\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,\mathrm{d}x + \frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,\mathrm{d}x\Big) \geqslant \sum_{i=1}^N\int_\Omega \varphi_i D^iu\,\mathrm{d}x + \frac{1}{n}\int_\Omega |u|^{p_0(x)}\,\mathrm{d}x.$$

Having in mind (3.15), we conclude that

$$\lim_{k\to\infty}\left(\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,\mathrm{d}x+\frac{1}{n}\int_\Omega |u_k|^{p_0(x)}\,\mathrm{d}x\right) = \sum_{i=1}^{N}\int_\Omega \varphi_i D^iu\,\mathrm{d}x+\frac{1}{n}\int_\Omega |u|^{p_0(x)}\,\mathrm{d}x. \tag{3.17}$$

Therefore, by combining (3.11) and (3.13)–(3.14), we obtain

$$\langle B_nu_k,u_k\rangle \to \langle \chi_n,u\rangle \quad \text{as } k\to\infty.$$

Now, by (3.17) we can prove that

$$\lim_{k\to\infty}\left(\sum_{i=1}^{N}\int_\Omega (a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,\mathrm{d}x + \frac{1}{n}\int_\Omega (|u_k|^{p_0(x)-2}u_k-|u|^{p_0(x)-2}u)(u_k-u)\,\mathrm{d}x\right)=0,$$

and so, by virtue of Lemma 2.3, we get

$$u_k\to u \quad \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega) \quad \text{and} \quad D^iu_k\to D^iu \quad \text{a.e. in } \Omega.$$

Then

$$a_i(x,T_n(u_k),\nabla u_k)\rightharpoonup a_i(x,T_n(u),\nabla u) \quad \text{in } L^{p_i'(\cdot)}(\Omega) \quad \text{for } i=1,\dots,N$$

and thanks to (3.8)–(3.10), we obtain $\chi_n = B_nu$, which concludes the proof of Lemma 3.1. □

In view of Lemma 3.1, there exists at least one weak solution $u_n\in W_0^{1,\vec{p}(\cdot)}(\Omega)$ of the problem (3.4) (cf. [22], Theorem 8.2).

Step 2: A priori estimates.

Lemma 3.2. *Let u_n be a weak solution of the approximate problem (3.4), then the following regularity results hold true:*

$$u\in W_0^{1,\vec{q}(\cdot)}(\Omega) \quad \textit{with } \vec{q}(\cdot)=(s(\cdot),q_1(\cdot),\dots,q_N(\cdot)) \tag{3.18}$$

where the exponent $s(\cdot)$ *verifies the condition* (1.10) *and* $1 \leqslant q_i(x) < p_i(x)s(x)/(s(x)+1)$, *almost everywhere in* Ω. *Then*

$$\sum_{i=1}^{N} \int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^\theta}\,\mathrm{d}x \leqslant C \quad \forall\, 1 < \theta < \frac{s(p_i(x)-q_i(x))}{q_i(x)}, \tag{3.19}$$

$$\sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i(x)}\,\mathrm{d}x \leqslant C(1+k)^\theta \quad \forall\, k > 0, \tag{3.20}$$

with C *a positive constant that does not depend on* k *and* n.

Proof. Let $\theta > 1$ which will be chosen later. We consider the function $\varphi(t)\colon \mathbb{R} \mapsto \mathbb{R}$ defined by

$$\varphi(t) = \Big(1 - \frac{1}{(1+|t|)^{\theta-1}}\Big)\operatorname{sign}(t).$$

It is clear that $\varphi(u_n) \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$. By taking $\varphi(u_n)$ as a test function in (3.4) we get

$$\begin{aligned}(\theta-1)\sum_{i=1}^{N} \int_\Omega \frac{a_i(x,T_n(u_n),\nabla u_n)\cdot D^i u_n}{(1+|u_n|)^\theta}\,\mathrm{d}x + \int_\Omega |T_n(u_n)|^{s(x)-1}T_n(u_n)\varphi(u_n)\,\mathrm{d}x \\ + \frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_n\varphi(u_n)\,\mathrm{d}x \\ = \int_\Omega f_n\varphi(u_n)\,\mathrm{d}x + \lambda\int_\Omega \frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+1/n}\varphi(u_n)\,\mathrm{d}x.\end{aligned}$$

Since $\varphi(u_n)$ have the same sign as u_n, the third term on the left-hand side of the previous inequality is positive. Also, we have $|\varphi(\cdot)| \leqslant 1$ and in view of (1.13), we obtain

$$\begin{aligned}\alpha(\theta-1)\sum_{i=1}^{N} \int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^\theta}\,\mathrm{d}x + \int_\Omega |T_n(u_n)|^{s(x)}|\varphi(u_n)|\,\mathrm{d}x \\ \leqslant \lambda\int_\Omega \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+1/n}\,\mathrm{d}x + \int_\Omega |f|\,\mathrm{d}x.\end{aligned} \tag{3.21}$$

It is clear that

$$\frac{1}{2} \leqslant 1 - \frac{1}{(1+|u_n|)^{\theta-1}} \quad \text{for } |u_n| \geqslant R = \max(2^{1/(\theta-1)}-1, 1).$$

Thus, we have

$$\begin{aligned}\frac{1}{2}\int_{\{|u_n|\geqslant R\}} |T_n(u_n)|^{s(x)}\,\mathrm{d}x &\leqslant \int_{\{|u_n|\geqslant R\}} |T_n(u_n)|^{s(x)}\Big(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\Big)\,\mathrm{d}x \\ &\leqslant \int_\Omega |T_n(u_n)|^{s(x)}\Big(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\Big)\,\mathrm{d}x,\end{aligned}$$

which implies

$$\frac{1}{2}\int_\Omega |T_n(u_n)|^{s(x)}\,\mathrm{d}x = \frac{1}{2}\int_{\{|u_n|<R\}} |T_n(u_n)|^{s(x)}\,\mathrm{d}x + \frac{1}{2}\int_{\{|u_n|\geqslant R\}} |T_n(u_n)|^{s(x)}\,\mathrm{d}x$$
$$\leqslant \frac{1}{2}R^{s^+}|\Omega| + \int_\Omega |T_n(u_n)|^{s(x)}\Big(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\Big)\,\mathrm{d}x.$$

Using (3.21), we deduce that

$$\alpha(\theta-1)\sum_{i=1}^N \int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^\theta}\,\mathrm{d}x + \frac{1}{2}\int_\Omega |T_n(u_n)|^{s(x)}\,\mathrm{d}x \leqslant \frac{1}{2}R^{s^+}|\Omega| + \lambda\int_\Omega \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}}\,\mathrm{d}x + \int_\Omega |f|\,\mathrm{d}x. \tag{3.22}$$

We have $s(x) > p_0(x) - 1$, in view of Young's inequality we obtain

$$\lambda\int_\Omega \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}}\,\mathrm{d}x \leqslant \frac{1}{4}\int_\Omega |T_n(u_n)|^{s(x)}\,\mathrm{d}x + C_2\int_\Omega \frac{\mathrm{d}x}{|x|^{s(x)p_0(x)/(s(x)-p_0(x)+1)}}$$

with C_2 a positive constant depending only on $s(\cdot)$, $p_0(\cdot)$ and λ. Thus, we obtain

$$\alpha(\theta-1)\sum_{i=1}^N \int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^\theta}\,\mathrm{d}x + \frac{1}{4}\int_\Omega |T_n(u_n)|^{s(x)}\,\mathrm{d}x \leqslant \frac{1}{2}R^{s^+}|\Omega| + C_2\int_\Omega \frac{\mathrm{d}x}{|x|^{s(x)p_0(x)/(s(x)-p_0(x)+1)}} + \int_\Omega |f|\,\mathrm{d}x. \tag{3.23}$$

Under the assumption $s(x) > N(p_0(x)-1)/(N-p_0(x))$, the integral

$$\int_\Omega \frac{\mathrm{d}x}{|x|^{s(x)p_0(x)/(s(x)-p_0(x)+1)}}$$

is finite. Therefore (3.19) is deduced. Moreover, we have

$$\int_\Omega |T_n(u_n)|^{s(x)}\,\mathrm{d}x \leqslant C. \tag{3.24}$$

Taking $q_i(\cdot) \in C_+(\Omega)$ such that $1 \leqslant q_i(x) < p_i(x)$ for $i = 1,\dots,N$, by virtue of the generalized Hölder's inequality we get

$$\begin{aligned}\sum_{i=1}^N \int_\Omega |D^i u_n|^{q_i(x)}\,\mathrm{d}x &\leqslant 2\sum_{i=1}^N \Big\|\frac{|D^i u_n|^{q_i(x)}}{(1+|u_n|)^{\theta q_i(x)/p_i(x)}}\Big\|_{p_i(\cdot)/q_i(\cdot)} \\ &\quad\times \|(1+|u_n|)^{\theta q_i(x)/p_i(x)}\|_{p_i(\cdot)/(p_i(\cdot)-q_i(\cdot))} \\ &\leqslant 2\sum_{i=1}^N \Big(\int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^\theta}\,\mathrm{d}x + 1\Big)^{q_i^+/p_i^-} \\ &\quad\times \Big(\int_\Omega (1+|u_n|)^{q_i(x)\theta/(p_i(x)-q_i(x))}\,\mathrm{d}x + 1\Big)^{1-q_i^-/p_i^+}.\end{aligned} \tag{3.25}$$

We now choose $\theta > 1$ such that $q_i(x)\theta/(p_i(x) - q_i(x)) < s(x)$ a.e. in Ω, such a real number θ exists if

$$1 < \frac{s(x)(p_i(x) - q_i(x))}{q_i(x)} \quad \text{that is} \quad q_i(x) < \frac{p_i(x)s(x)}{s(x)+1}.$$

Combining (3.23)–(3.25), we obtain the desired estimates (3.18).

To get (3.20), we have thanks to (3.19) that

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega} |D^i T_k(u_n)|^{p_i(x)}\,\mathrm{d}x &= \sum_{i=1}^{N}\int_{\{|u_n|<k\}} |D^i u_n|^{p_i(x)}\,\mathrm{d}x\\ &\leqslant (1+k)^{\theta}\sum_{i=1}^{N}\int_{\Omega}\frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}\,\mathrm{d}x.\end{aligned}$$

Step 3: The weak convergence of $(T_k(u_n))_n$ *in* $W_0^{1,\vec{p}(\cdot)}(\Omega)$. To show the weak convergence of $(T_k(u_n))_n$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, we begin by proving that $(u_n)_n$ is a Cauchy sequence. Indeed, thanks to (3.20), we can obtain

$$\sum_{i=0}^{N}\int_{\Omega} |D^i T_k(u_n)|^{p_i(x)}\,\mathrm{d}x \leqslant C(1+k)^{\theta} + k^{p_0^+}|\Omega| \quad \text{for } k \geqslant 1.$$

Therefore, the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, and there exists a subsequence still denoted by $(T_k(u_n))_n$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup \eta_k & \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega),\\ T_k(u_n) \to \eta_k & \text{in } L^{\underline{p}}(\Omega) \text{ and a.e. in } \Omega.\end{cases} \tag{3.26}$$

On the other hand, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega} |D^i T_k(u_n)|^{p_i(x)}\,\mathrm{d}x &\geqslant \sum_{i=1}^{N}\int_{\Omega} (|D^i T_k(u_n)|^{\underline{p}} - 1)\,\mathrm{d}x\\ &= \|\nabla T_k(u_n)\|_{\underline{p}}^{\underline{p}} - N|\Omega|.\end{aligned}$$

Thanks to (3.20), we deduce that there exists a constant C_3 that does not depend on k and n, such that

$$\|\nabla T_k(u_n)\|_{\underline{p}} \leqslant C_3 k^{\theta/\underline{p}} \quad \text{for } k \geqslant 1. \tag{3.27}$$

Thanks to the Poincaré type inequality, we obtain

$$k\ \text{meas}\{|u_n|>k\} = \int_{\{|u_n|>k\}} |T_k(u_n)|\,\mathrm{d}x \leqslant \int_\Omega |T_k(u_n)|\,\mathrm{d}x \leqslant C_4\|T_k(u_n)\|_{\underline{p}} \leqslant C_5\|\nabla T_k(u_n)\|_{\underline{p}} \leqslant C_6 k^{\theta/\underline{p}}. \tag{3.28}$$

Choosing θ small enough $(1<\theta<\underline{p})$, we conclude that

$$\text{meas}\{|u_n|>k\} \leqslant C_6\frac{1}{k^{1-\theta/\underline{p}}} \to 0 \quad \text{as } k\to\infty. \tag{3.29}$$

For all $\delta>0$, we have

$$\text{meas}\{|u_n-u_m|>\delta\} \leqslant \text{meas}\{|u_n|>k\} + \text{meas}\{|u_m|>k\} + \text{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}.$$

Let $\varepsilon>0$, using (3.29) we can choose $k=k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n|>k\} \leqslant \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\{|u_m|>k\} \leqslant \frac{\varepsilon}{3}. \tag{3.30}$$

On the other hand, thanks to (3.26) we can assume that $(T_k(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Thus, for any $k>0$ and $\delta,\varepsilon>0$, there exists $n_0=n_0(k,\delta,\varepsilon)$ such that

$$\text{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\} \leqslant \frac{\varepsilon}{3} \quad \forall\, m,n \geqslant n_0(k,\delta,\varepsilon). \tag{3.31}$$

In view of (3.30) and (3.31), we deduce that for any $\delta,\varepsilon>0$, there exists $n_0=n_0(\delta,\varepsilon)$ such that

$$\text{meas}\{|u_n-u_m|>\delta\} \leqslant \varepsilon \quad \forall\, n,m \geqslant n_0(\delta,\varepsilon),$$

which proves that the sequence $(u_n)_n$ is a Cauchy sequence in measure and then converges almost everywhere to some measurable function u. Consequently, we have

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega), \tag{3.32}$$

and in view of Lebesgue's dominated convergence theorem, we obtain

$$T_k(u_n) \to T_k(u) \quad \text{in } L^{p_0(\cdot)}(\Omega) \text{ and a.e in } \Omega. \tag{3.33}$$

Step 4: Strong convergence of truncations. In the sequel, we denote by $\varepsilon_i(n)$, $i=1,2,\dots$, various real-valued functions of real variables that converge to 0 as n tends to infinity.

Let $h > k > 0$, take $z_n := u_n - T_h(u_n) + T_k(u_n) - T_k(u)$ and $\omega_n := T_{2k}(z_n)$. By using ω_n as a test function in the approximate problem (3.4) we obtain

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i\omega_n \,\mathrm{d}x + \int_{\Omega} |T_n(u_n)|^{s(x)-1} T_n(u_n)\omega_n \,\mathrm{d}x$$
$$+ \frac{1}{n}\int_{\Omega} |u_n|^{p_0(x)-2} u_n \omega_n \,\mathrm{d}x = \lambda \int_{\Omega} \frac{|T_n(u_n)|^{p_0(x)-2} T_n(u_n)}{|x|^{p_0(x)} + 1/n}\omega_n \,\mathrm{d}x + \int_{\Omega} f_n \omega_n \,\mathrm{d}x.$$

For $M = 4k + h$, it is clear that $D^i\omega_n = 0$ on the set $\{|u_n| \geqslant M\}$, and ω_n have the same sign as u_n on the set $\{|u_n| > k\}$, therefore

$$\sum_{i=1}^{N}\int_{\{|u_n|\leqslant M\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) D^i\omega_n \,\mathrm{d}x + \int_{\Omega} |T_n(u_n)|^{s(x)-1} T_n(u_n)\omega_n \,\mathrm{d}x$$
$$+ \frac{1}{n}\int_{\{|u_n|\leqslant k\}} |u_n|^{p_0(x)-2} u_n \omega_n \,\mathrm{d}x$$
$$\leqslant \lambda \int_{\Omega} \frac{|T_n(u_n)|^{p_0(x)-2} T_n(u_n)}{|x|^{p_0(x)} + 1/n}\omega_n \,\mathrm{d}x + \int_{\Omega} f_n \omega_n \,\mathrm{d}x.$$

In view of Young's inequality, we have

$$\lambda \int_{\{|u_n|>k\}} \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)} + 1/n}|\omega_n| \,\mathrm{d}x$$
$$\leqslant \int_{\{|u_n|>k\}} |T_n(u_n)|^{s(x)}|\omega_n| \,\mathrm{d}x + C_7 \int_{\{|u_n|>k\}} \frac{|\omega_n|}{|x|^{p_0(x)s(x)/(s(x)-p_0(x)+1)}} \,\mathrm{d}x,$$

and since $\omega_n = T_k(u_n) - T_k(u)$ on the set $\{|u_n| \leqslant k\}$, we have

$$\sum_{i=1}^{N}\int_{\{|u_n|\leqslant M\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) D^i\omega_n \,\mathrm{d}x \tag{3.34}$$
$$+ \int_{\{|u_n|\leqslant k\}} |T_k(u_n)|^{s(x)-1} T_k(u_n)(T_k(u_n) - T_k(u)) \,\mathrm{d}x$$
$$+ \frac{1}{n}\int_{\{|u_n|\leqslant k\}} |T_k(u_n)|^{p_0(x)-2} T_k(u_n)(T_k(u_n) - T_k(u)) \,\mathrm{d}x$$
$$\leqslant \lambda \int_{\{|u_n|\leqslant k\}} \frac{|T_k(u_n)|^{p_0(x)-2} T_k(u_n)}{|x|^{p_0(x)} + 1/n}(T_k(u_n) - T_k(u)) \,\mathrm{d}x$$
$$+ \int_{\Omega} f_n \omega_n \,\mathrm{d}x + C_7 \int_{\{|u_n|>k\}} \frac{|\omega_n|}{|x|^{p_0(x)s(x)/(s(x)-p_0(x)+1)}} \,\mathrm{d}x.$$

Now, we will study each terms in the previous inequality.

For the second and third terms on the left-hand side of (3.34), in view of Lebesgue's dominated convergence theorem, we have

$$|T_k(u_n)|^{s(x)-1}T_k(u_n) \to |T_k(u)|^{s(x)-1}T_k(u) \quad \text{in } L^1(\Omega),$$

and

$$|T_k(u_n)|^{p_0(x)-2}T_k(u_n) \to |T_k(u)|^{p_0(x)-2}T_k(u) \quad \text{in } L^1(\Omega),$$

and since $T_k(u_n) \rightharpoonup T_k(u)$ weak-$*$ in $L^\infty(\Omega)$, we have

$$\varepsilon_1(n) = \int_{\{|u_n|\leqslant k\}} |T_k(u_n)|^{s(x)-1}T_k(u_n)(T_k(u_n) - T_k(u))\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty, \tag{3.35}$$

and

$$\varepsilon_2(n) = \frac{1}{n}\int_{\{|u_n|\leqslant k\}} |T_k(u_n)|^{p_0(x)-2}T_k(u_n)(T_k(u_n) - T_k(u))\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{3.36}$$

Concerning the terms on the right-hand side of (3.34), we have

$$\begin{aligned}\varepsilon_3(n) &= \left|\int_{\{|u_n|\leqslant k\}} \frac{|T_k(u_n)|^{p_0(x)-2}T_k(u_n)}{|x|^{p_0(x)} + 1/n}(T_k(u_n) - T_k(u))\,\mathrm{d}x\right| \\ &\leqslant k^{p_0^+-1}\int_{\{|u_n|\leqslant k\}} \frac{|T_k(u_n) - T_k(u)|}{|x|^{p_0(x)}}\,\mathrm{d}x \to 0 \quad \text{as } n \to \infty;\end{aligned} \tag{3.37}$$

also, we have

$$\int_\Omega f_n\omega_n\,\mathrm{d}x = \int_\Omega f\ T_{2k}(u - T_h(u))\,\mathrm{d}x + \varepsilon_4(n), \tag{3.38}$$

and

$$\begin{aligned}&\int_{\{|u_n|>k\}} \frac{|\omega_n|}{|x|^{p_0(x)s(x)/(s(x)-p_0(x)+1)}}\,\mathrm{d}x \\ &\qquad = \int_{\{|u|>h\}} \frac{|T_{2k}(u - T_h(u))|}{|x|^{p_0(x)s(x)/(s(x)-p_0(x)+1)}}\,\mathrm{d}x + \varepsilon_5(n).\end{aligned} \tag{3.39}$$

By combining (3.34)–(3.39), we deduce that

$$\begin{aligned}\sum_{i=1}^N \int_{\{|u_n|\leqslant M\}} &a_i(x, T_M(u_n), \nabla T_M(u_n))D^i\omega_n\,\mathrm{d}x \\ &\leqslant \int_\Omega fT_{2k}(u - T_h(u))\,\mathrm{d}x \\ &\quad + C_7\int_{\{|u|>h\}} \frac{|T_{2k}(u - T_h(u))|}{|x|^{p_0(x)s(x)/(s(x)-p_0(x)+1)}}\,\mathrm{d}x + \varepsilon_6(n).\end{aligned} \tag{3.40}$$

On the other hand, we have $\omega_n = T_k(u_n) - T_k(u)$ on $\{|u_n| \leqslant M\}$, then

$$(3.41)\qquad \sum_{i=1}^{N} \int_{\{|u_n|\leqslant M\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) D^i \omega_n \,\mathrm{d}x$$
$$= \sum_{i=1}^{N} \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)))$$
$$\times (D^i T_k(u_n) - D^i T_k(u)) \,\mathrm{d}x$$
$$+ \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u))(D^i T_k(u_n) - D^i T_k(u)) \,\mathrm{d}x$$
$$+ \sum_{i=1}^{N} \int_{\{|u_n|>k\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u) \,\mathrm{d}x$$
$$+ \sum_{i=1}^{N} \int_{\{k<|u_n|\leqslant M\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) D^i \omega_n \,\mathrm{d}x.$$

For the second and third terms on the right-hand side of (3.41), thanks to Lebesgue's dominated convergence theorem, we have $T_k(u_n) \to T_k(u)$ in $L^{p_i(\cdot)}(\Omega)$, then $a_i(x, T_k(u_n), \nabla T_k(u)) \to a_i(x, T_k(u), \nabla T_k(u))$ in $L^{p'_i(\cdot)}(\Omega)$, and since $D^i T_k(u_n) \rightharpoonup D^i T_k(u)$ in $L^{p_i(\cdot)}(\Omega)$ it follows that

$$(3.42)\quad \varepsilon_7(n) = \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u))(D^i T_k(u_n) - D^i T_k(u)) \,\mathrm{d}x \to 0 \quad \text{as } n \to \infty,$$

and since $a(x, s, 0) = 0$, we get

$$(3.43)\qquad \int_{\{|u_n|>k\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u) \,\mathrm{d}x$$
$$= \int_{\{|u_n|>k\}} a_i(x, T_k(u_n), 0) D^i T_k(u) \,\mathrm{d}x = 0.$$

Concerning the last term on the right-hand side of (3.41), thanks to (1.12) we have that $(a_i(x, T_M(u_n), \nabla T_M(u_n)))_n$ is bounded in $L^{p'_i(\cdot)}(\Omega)$, then there exists a function $\varphi_i \in L^{p'_i(\cdot)}(\Omega)$ such that $|a_i(x, T_M(u_n), \nabla T_M(u_n))| \rightharpoonup \varphi_i$ in $L^{p'_i(\cdot)}(\Omega)$. It follows that

$$(3.44)\qquad \lim_{n\to\infty} \int_{\{k<|u_n|\leqslant M\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) D^i \omega_n \,\mathrm{d}x$$
$$= \lim_{n\to\infty} \int_{\{k<|u_n|\leqslant M\}\cap\{|z_n|\leqslant 2k\}} a_i(x, T_M(u_n), \nabla T_M(u_n))$$
$$\times (D^i u_n - D^i T_h(u_n) - D^i T_k(u)) \,\mathrm{d}x$$

$$\geqslant -\lim_{n\to\infty}\int_{\{k<|u_n|\leqslant M\}} |a_i(x,T_M(u_n),\nabla T_M(u_n))|\,|D^iT_k(u)|\,\mathrm{d}x$$
$$\geqslant -\int_{\{k<|u|\leqslant M\}} \varphi_i\,|D^iT_k(u)|\,\mathrm{d}x = 0.$$

By combining (3.40) and (3.41)–(3.44), we get

$$\sum_{i=1}^N\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))) \times (D^iT_k(u_n)-D^iT_k(u))\,\mathrm{d}x \leqslant 2k\int_{\{|u|>h\}}|f|\,\mathrm{d}x + 2kC_7\int_{\{|u|>h\}}\frac{\mathrm{d}x}{|x|^{p_0(x)s(x)/(s(x)-p_0(x)+1)}} + \varepsilon_8(n). \tag{3.45}$$

Since $N(p_0(x)-1)/(N-p_0(x)) < s(x)$, we have $p_0(x)s(x)/(s(x)-p_0(x)+1) < N$, then $1/|x|^{p_0(x)s(x)/(s(x)-p_0(x)+1)} \in L^1(\Omega)$.

By letting n and then h tend to infinity in the inequality above, thanks to (3.33) we can obtain

$$\lim_{n\to\infty}\biggl(\sum_{i=1}^N\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))) \times (D^iT_k(u_n)-D^iT_k(u))\,\mathrm{d}x + \int_\Omega (|T_k(u_n)|^{p_0(x)-2}T_k(u_n) - |T_k(u)|^{p_0(x)-2}T_k(u)) \times (T_k(u_n)-T_k(u))\,\mathrm{d}x\biggr) = 0. \tag{3.46}$$

In view of Lemma 2.3, we conclude that

$$\begin{cases} T_k(u_n)\to T_k(u) & \text{strongly in } W_0^{1,\vec{p}(\cdot)}(\Omega),\\ D^iu_n\to D^iu & \text{a.e. in } \Omega \text{ for } i=1,\dots,N. \end{cases} \tag{3.47}$$

Step 5: The equi-integrability of the nonlinear functions. Now, we shall show that

$$|T_n(u_n)|^{s(x)-1}T_n(u_n)\to |u|^{s(x)-1}u \quad \text{in } L^1(\Omega), \tag{3.48}$$

$$\frac{1}{n}|u_n|^{p_0(x)-2}u_n\to 0 \quad \text{in } L^1(\Omega), \tag{3.49}$$

and

$$\frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+1/n}\to\frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} \quad \text{in } L^1(\Omega). \tag{3.50}$$

In view of Vitali's theorem, it suffices to prove the uniform equi-integrability of these functions. By taking $T_1(u_n - T_h(u_n))$ as a test function in (3.4), we can obtain

$$
\begin{aligned}
&\alpha \sum_{i=1}^{N} \int_{\{h<|u_n|\leqslant h+1\}} |D^i u_n|^{p_i} \,\mathrm{d}x + \int_{\{|u_n|\geqslant h\}} |T_n(u_n)|^{s(x)} |T_1(u_n - T_h(u_n))| \,\mathrm{d}x \\
&\qquad + \frac{1}{n} \int_{\{|u_n|\geqslant h+1\}} |u_n|^{p_0(x)-1} \,\mathrm{d}x \\
&\qquad \leqslant \lambda \int_{\{|u_n|\geqslant h\}} \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)} + 1/n} |T_1(u_n - T_h(u_n))| \,\mathrm{d}x + \int_{\{|u_n|\geqslant h\}} |f_n| \,\mathrm{d}x.
\end{aligned}
$$

Thanks to Young's inequality, we have

$$
\begin{aligned}
&\lambda \int_{\{|u_n|\geqslant h\}} \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)} + 1/n} |T_1(u_n - T_h(u_n))| \,\mathrm{d}x \\
&\qquad\qquad \leqslant \frac{1}{3} \int_{\{|u_n|\geqslant h\}} |T_n(u_n)|^{s(x)} |T_1(u_n - T_h(u_n))| \,\mathrm{d}x \\
&\qquad\qquad\quad + C_8 \int_{\{|u_n|\geqslant h\}} \frac{|T_1(u_n - T_h(u_n))|}{|x|^{s(x)p_0(x)/(s(x)-p_0(x)+1)}} \,\mathrm{d}x,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\frac{1}{3} \int_{\{|u_n|\geqslant h+1\}} |T_n(u_n)|^{s(x)} \,\mathrm{d}x + \lambda \int_{\{|u_n|\geqslant h+1\}} \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)} + 1/n} \,\mathrm{d}x \\
&\qquad + \frac{1}{n} \int_{\{|u_n|\geqslant h+1\}} |u_n|^{p_0(x)-1} \,\mathrm{d}x \\
&\qquad \leqslant 2C_8 \int_{\{|u_n|\geqslant h\}} \frac{|T_1(u_n - T_h(u_n))|}{|x|^{s(x)p_0(x)/(s(x)-p_0(x)+1)}} \,\mathrm{d}x + \int_{\{|u_n|\geqslant h\}} |f_n| \,\mathrm{d}x.
\end{aligned}
$$

Thus, for any $\eta > 0$, there exists $h(\eta) > 0$ such that

$$
\begin{aligned}
\int_{\{|u_n|\geqslant h(\eta)\}} |T_n(u_n)|^{s(x)} \,\mathrm{d}x &+ \int_{\{|u_n|\geqslant h(\eta)\}} \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)} + 1/n} \,\mathrm{d}x \\
&+ \frac{1}{n} \int_{\{|u_n|\geqslant h(\eta)\}} |u_n|^{p_0(x)-1} \,\mathrm{d}x \leqslant \frac{\eta}{2}.
\end{aligned} \tag{3.51}
$$

On the other hand, for any measurable subset $E \subseteq \Omega$, we have

$$
\int_E |T_n(u_n)|^{s(x)} \,\mathrm{d}x + \int_E \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)} + 1/n} \,\mathrm{d}x + \frac{1}{n} \int_E |u_n|^{p_0(x)-1} \,\mathrm{d}x \tag{3.52}
$$

$$
\begin{aligned}
&\leqslant \int_E |T_{h(\eta)}(u_n)|^{s(x)}\,\mathrm{d}x + \int_E \frac{|T_{h(\eta)}(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+1/n}\,\mathrm{d}x \\
&\quad + \frac{1}{n}\int_E |T_{h(\eta)}(u_n)|^{p_0(x)-1}\,\mathrm{d}x + \int_{\{|u_n|\geqslant h(\eta)\}} |T_n(u_n)|^{s(x)}\,\mathrm{d}x \\
&\quad + \int_{\{|u_n|\geqslant h(\eta)\}} \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+1/n}\,\mathrm{d}x + \frac{1}{n}\int_{\{|u_n|\geqslant h(\eta)\}} |u_n|^{p_0(x)-1}\,\mathrm{d}x.
\end{aligned}
$$

Due to (3.47), there exists $\beta(\eta) > 0$ such that: for any $E \subseteq \Omega$ with $\mathrm{meas}(E) \leqslant \beta(\eta)$

$$
\begin{aligned}
\text{(3.53)} \qquad &\int_E |T_{h(\eta)}(u_n)|^{s(x)}\,\mathrm{d}x + \int_E \frac{|T_{h(\eta)}(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+1/n}\,\mathrm{d}x \\
&\qquad + \frac{1}{n}\int_E |T_{h(\eta)}(u_n)|^{p_0(x)-1}\,\mathrm{d}x \leqslant \frac{\eta}{2}.
\end{aligned}
$$

Finally, by combining (3.51), (3.52) and (3.53), one easily has

$$
\text{(3.54)} \qquad \int_E |T_n(u_n)|^{s(x)}\,\mathrm{d}x + \int_E \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+1/n}\,\mathrm{d}x + \frac{1}{n}\int_E |u_n|^{p_0(x)-1}\,\mathrm{d}x \leqslant \eta,
$$

with $\mathrm{meas}(E) \leqslant \beta(\eta)$. We deduce that $(|T_n(u_n)|^{s(x)-1}T_n(u_n))_n$, $(|u_n|^{p_0(x)-2}u_n)_n$ and $(|T_n(u_n)|^{p_0(x)-2}T_n(u_n)/(|x|^{p_0(x)}+1/n))_n$ are equi-integrable, and

$$
\begin{aligned}
|T_n(u_n)|^{s(x)-1}T_n(u_n) &\to |u|^{s(x)-1}u \quad \text{a.e. in } \Omega, \\
\frac{1}{n}|u_n|^{p_0(x)-2}u_n &\to 0 \quad \text{a.e. in } \Omega
\end{aligned}
$$

and

$$
\frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+1/n} \to \frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} \quad \text{a.e. in } \Omega.
$$

In view of Vitali's theorem, the convergences (3.48)–(3.50) are concluded.

Step 6: Passage to the limit. Let $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $M = k + \|\varphi\|_\infty$. By taking $T_k(u_n - \varphi)$ as a test function in (3.4), we get

$$
\begin{aligned}
\text{(3.55)} \qquad &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi)\,\mathrm{d}x \\
&\qquad + \int_\Omega |T_n(u_n)|^{s(x)-1}T_n(u_n)T_k(u_n-\varphi)\,\mathrm{d}x \\
&\qquad + \frac{1}{n}\int_\Omega |u_n|^{p_0(x)-2}u_n T_k(u_n-\varphi)\,\mathrm{d}x \\
&= \lambda \int_\Omega \frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+1/n} T_k(u_n-\varphi)\,\mathrm{d}x + \int_\Omega f_n T_k(u_n-\varphi)\,\mathrm{d}x.
\end{aligned}
$$

On the one hand, we have $\{|u_n - \varphi| \leqslant k\} \subseteq \{|u_n| \leqslant M\}$, hence

$$\begin{aligned}
&\int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi)\,\mathrm{d}x \\
&\qquad = \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n))(D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leqslant k\}}\,\mathrm{d}x \\
&\qquad = \int_\Omega (a_i(x, T_M(u_n), \nabla T_M(u_n)) - a_i(x, T_M(u_n), \nabla\varphi)) \\
&\qquad\qquad\qquad \times (D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leqslant k\}}\,\mathrm{d}x \\
&\qquad\quad + \int_\Omega a_i(x, T_M(u_n), \nabla\varphi)(D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leqslant k\}}\,\mathrm{d}x.
\end{aligned}$$

It is clear that

$$\begin{aligned}
&\lim_{n\to\infty} \int_\Omega a_i(x, T_M(u_n), \nabla\varphi)(D^i T_M(u_n) - D^i\varphi)\chi_{\{|u_n-\varphi|\leqslant k\}}\,\mathrm{d}x \\
&\qquad\qquad = \int_\Omega a_i(x, T_M(u), \nabla\varphi)(D^i T_M(u) - D^i\varphi)\chi_{\{|u-\varphi|\leqslant k\}}\,\mathrm{d}x.
\end{aligned}$$

According to Fatou's lemma, we obtain

$$\begin{aligned}
(3.56)\qquad &\liminf_{n\to\infty} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n - \varphi)\,\mathrm{d}x \\
&\geqslant \sum_{i=1}^N \int_\Omega (a_i(x, T_M(u), \nabla T_M(u)) - a_i(x, T_M(u), \nabla\varphi)) \\
&\qquad\qquad \times (D^i T_M(u) - D^i\varphi)\chi_{\{|u-\varphi|\leqslant k\}}\,\mathrm{d}x \\
&\quad + \sum_{i=1}^N \int_\Omega a_i(x, T_M(u), \nabla\varphi)(D^i T_M(u) - D^i\varphi)\chi_{\{|u-\varphi|\leqslant k\}}\,\mathrm{d}x \\
&= \sum_{i=1}^N \int_\Omega a_i(x, T_M(u), \nabla T_M(u))(D^i T_M(u) - D^i\varphi)\chi_{\{|u-\varphi|\leqslant k\}}\,\mathrm{d}x \\
&= \sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u) D^i T_k(u - \varphi)\,\mathrm{d}x.
\end{aligned}$$

On the other hand, we have $T_k(u_n - \varphi) \rightharpoonup T_k(u - \varphi)$ weak-$*$ in $L^\infty(\Omega)$ and thanks to (3.48)–(3.50), we deduce that

$$(3.57)\quad \int_\Omega |T_n(u_n)|^{s(x)-1} T_n(u_n) T_k(u_n - \varphi)\,\mathrm{d}x \to \int_\Omega |u|^{s(x)-1} u T_k(u - \varphi)\,\mathrm{d}x,$$

$$(3.58)\qquad \frac{1}{n}\int_\Omega |u_n|^{p_0(x)-1} u_n T_k(u_n - \varphi)\,\mathrm{d}x \to 0,$$

$$(3.59)\quad \int_\Omega \frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+1/n} T_k(u_n-\varphi)\,\mathrm{d}x \to \int_\Omega \frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} T_k(u-\varphi)\,\mathrm{d}x,$$

and

$$(3.60)\qquad \int_\Omega f_n T_k(u_n-\varphi)\,\mathrm{d}x \to \int_\Omega f T_k(u-\varphi)\,\mathrm{d}x.$$

Hence, putting all the terms together, we conclude the proof of Theorem 3.1. □

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