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TOTAL BLOW-UP OF A QUASILINEAR HEAT EQUATION WITH SLOW-DIFFUSION FOR NON-DECAYING INITIAL DATA

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Abstract. We consider solutions of quasilinear equations $u_t = \Delta u^m + u^p$ in $\mathbb{R}^N$ with the initial data u_0 satisfying $0 < u_0 < M$ and $\lim\limits_{|x|\to\infty} u_0(x) = M$ for some constant $M > 0$. It is known that if $0 < m < p$ with $p > 1$, the blow-up set is empty. We find solutions u that blow up throughout $\mathbb{R}^N$ when $m > p > 1$.

Keywords: quasilinear heat equation; total blow-up; blow-up only at space infinity

MSC 2010: 35B44, 35K59

1. Introduction

We consider the nonlinear diffusion equation:

$$\begin{cases} u_t = \Delta u^m + u^p, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x) > 0, & x \in \mathbb{R}^N \end{cases} \tag{1.1}$$

with $m > p > 1$ and $u_0 \in C(\mathbb{R}^N)$ for $N \geqslant 1$. This problem is known to admit a local time solution (see [6], [8]), but it may cease to exist in a finite time. We say that the solution of (1.1) *blows up* in finite time if there is some $T = T(u_0) < \infty$ such that

$$\limsup_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = \infty \tag{1.2}$$

and $T(u_0)$ is called the *blow-up time* of the solution u with the initial value u_0. We define the *blow-up set* by

$$B(u_0) = \Big\{a \in \mathbb{R}^N \colon \limsup_{x\to a,\, t\nearrow T} |u(x,t)| = \infty\Big\}.$$

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Each element of $B(u_0)$ is called a *blow-up point* of u. We say that the solution u of (1.1) blows up *only at space infinity* if, in addition to (1.2), $B(u_0) = \emptyset$. In this case, the *global blow-up profile* $u(x,T) := \lim_{t\to T} u(x,t)$ is defined for every $x \in \mathbb{R}^N$.

Let us recall known results on the blow-up at space infinity. Lacey in [5] considered a one-dimensional problem $u_t = \Delta u + f(u)$ on the half-line and constructed examples of solutions that blow up only at space infinity. He also obtained results of the global blow-up profile. Giga and Umeda in [4] considered the equation $u_t = \Delta u + u^p$ on $\mathbb{R}^N$ and showed that the blow-up at space infinity occurs if the initial data u_0 satisfies

$$0 < u_0 < M \quad \text{and} \quad \lim_{|x|\to\infty} u_0(x) = M$$

for some constant $M > 0$. Shimojō in [12] considered semilinear heat equations on $\mathbb{R}^N$ and calculated the shape of global blow-up profile of solutions at the blow-up time. It is also proved that such blow-up is always complete, that means that the solution cannot extend as a weak solution after blow-up time.

For the case $0 < m < 1$, the heat conductivity mu^{m-1} becomes small as u increases. Hence, we can see that diffusion is very slow when u is large. Thus, the blow-up at space infinity must occur as the result for semilinear heat equation of [3]. This is proved by Seki for $0 < m \leqslant 1 < p$ (see [10]). He also discusses the generalization of the nonlinearity of the form $u_t = \Delta k(u) + f(u)$ including the case $0 < m \leqslant 1 < p$. On the other hand, if $m > 1$, diffusion is very fast when u is just as large. Hence, the speed of heat propagation, from the space infinity to the origin near the blow-up time, becomes much larger compared to the semilinear problem. Thus, a natural question is: "If $m \in (1, \infty)$ is sufficiently large, does the blow-up only at space infinity fail or not?". Partial answer of this problem was obtained by Seki-Suzuki-Umeda (see [11]). Their result implies that if $1 \leqslant m < p$, the blow-up only at space infinity occurs. Motivated by these results, we consider the following problem: Can the blow-up be confined to space infinity even if diffusion is so large that $m > p > 1$?

In this paper, we give a partial answer to this problem and show that the *total blow-up*, which means that $B(u_0) = \mathbb{R}^N$, occurs.

Theorem 1.1. *Let $p > 1$ and $m - p > 2(p-1)/N$. Then problem* (1.1) *has a total blow-up solution with the initial value $u_0 \in C(\mathbb{R}^N)$ satisfying*

$$0 < u_0 < M \quad \textit{and} \quad \lim_{|x|\to\infty} u_0(x) = M \tag{1.3}$$

for a certain positive constant $M \in \mathbb{R}$.

This paper is organized as follows. In Section 2, we discuss the condition $m-p > 2(p-1)/N$ of Theorem 1.1 from the point of asymptotic expansion. The rigorous proof of Theorem 1.1 is given in Section 3 by constructing backward self-similar solution.

R e m a r k 1.1. For problem (1.1) with nonnegative initial data satisfying the condition $\lim_{|x|\to\infty} u_0(x) = 0$, it is known that if $p > m > 1$, the blow-up set reduces to finite number of points (see [1], [13]). For $1 < p < m$, total blow-up occurs (see [2]). There is also a third possibility, $B(u_0)$ is a bounded domain for $p = m$. See also Mochizuki and Suzuki [7] for higher dimensional problem. They consider the case when the support of the initial data is compact, and that the support of the solution remains bounded if $p > m$ and it spreads out the whole space if $p < m$ at the blow-up time. The precise behavior of such solutions in one dimensional case is considered in the book [9].

2. Formal asymptotics

We shall explain why the condition $m - p > 2(p-1)/N$ yields total blow-up. We will achieve that by a formal asymptotic calculation. Let $f(u) = u^p$, then the solution of the ODE

$$U' = f(U), \quad U(0) = M, \; M > 0 \tag{2.1}$$

is written as $U(t) = \varphi\big(T(M)-t\big)$, where $\varphi(s) := \kappa s^{-1/(p-1)}$ and $\kappa := (p-1)^{-1/(p-1)}$. Here $T = T(M)$ is the blow-up time for the initial data $U(0) = M$. Substituting $t = 0$ gives $M = \varphi(T(M))$. Furthermore, by a simple calculation, we have

$$\varphi'(s) = -f(\varphi(s)), \qquad \lim_{s\to+0} \varphi(s) = \infty. \tag{2.2}$$

Let us consider (1.1) with initial data $u_0(x) = M - \varepsilon q_0(x)$, where q is a positive function satisfying $\lim_{|x|\to\infty} q_0(x) = 0$ and $\varepsilon > 0$ is a small constant. The first approximation at space infinity must be the flat solution $\varphi(T-t)$. In order to calculate the second term, we shall consider a formal outer expansion

$$u(x,t) = \sum_{i=0}^{\infty} u^{(i)}(x,t)\varepsilon^i$$

and substitute this into $u_t = \Delta k(u) + f(u)$, where $k(u) = u^m$. Then

$$\begin{aligned} u_t^{(0)} &= \Delta k(u^0) + f(u^{(0)}), \\ u_t^{(1)} &= k'(u^{(0)})\Delta u^{(1)} + f'(u^{(0)})u^{(1)}. \end{aligned}$$

Observing the initial condition at space infinity, we assume $u^{(0)}(x,t) = \varphi(T-t)$ as the first approximation of the solution, hence

$$u_t^{(1)} = k'(\varphi(T-t))\Delta u^{(1)} + f'(\varphi(T-t))u^{(1)}. \tag{2.3}$$

Let $q(x,t) = \mathrm{e}^{\Phi(t)\Delta} q_0$ be a solution of $q_t = k'(\varphi(T-t))\Delta q$ with the initial condition $q(x,0) = q_0(x) \in L^1(\mathbb{R}^N)$. In other words,

$$q(x,t) = \mathrm{e}^{\Phi(t)\Delta} q_0, \quad \Phi(t) = \int_0^t k'(\varphi(T-\tau))\,\mathrm{d}\tau.$$

Here we employ the notation

$$(\mathrm{e}^{s\Delta}\, q_0)(x) := \int_{\mathbb{R}^N} G(x-y,s)q_0(y)\,\mathrm{d}y$$

where G is the fundamental solution of the heat equation in $\mathbb{R}^N$:

$$G(x,s) := \frac{1}{(4\pi s)^{N/2}} \exp\Big(-\frac{|x|^2}{4s}\Big).$$

Then the solution of (2.3) is represented as $u^{(1)}(x,t) = -f(\varphi(T-t))q(x,t)$. This can be easily checked from the following calculation.

$$\begin{aligned} u_t^{(1)} &= -f(\varphi(T-t))q_t - \frac{\mathrm{d}f(\varphi(T-t))}{\mathrm{d}t} q \\ &= -f(\varphi(T-t))q_t + f'(\varphi(T-t))\varphi'(T-t)q \\ &= -f(\varphi(T-t))k'(\varphi(T-t))\Delta q - f'(\varphi(T-t))f(\varphi(T-t))q \\ &= k'(\varphi(T-t))\Delta u^{(1)} + f'(\varphi(T-t))u^{(1)}, \end{aligned}$$

where we applied (2.2) and substitute $s = T-t$. By a formal asymptotic expansion, together with $\varphi'(T-t) = -f(\varphi(T-t))$ again, we get

$$u(x,t) = \varphi(T-t) - \varepsilon f(\varphi(T-t))q(x,t) + O(\varepsilon^2) = \varphi(T-t+\varepsilon q(x,t))$$

provided that $|x|$ is sufficiently large so that $T-t \gg q(x,t)$. We shall discuss a sufficient condition for this approach. Note that $\Phi(t)$ is proportional to $(T-t)^{(p-m)/(p-1)} - T^{(p-m)/(p-1)}$, which implies $\Phi(T) = \infty$ if $m > p$. Assume, for simplicity, that the support of q_0 is compact. Then by applying the inequality

$$\sup_{x\in\mathbb{R}^N} |q(x,t)| \leqslant \frac{1}{(4\pi\Phi(t))^{N/2}} \int_{\mathbb{R}^N} q_0(x)\,\mathrm{d}x,$$

we get the following sufficient condition for $T-t \gg q(x,t)$:

$$T-t \gg O((T-t)^{N(m-p)/(2(p-1))}) = O(\Phi(t)^{-N/2}) \geqslant q(x,t).$$

Since we are interested in what happens as $t \to T_-$, we need the restriction below, which appeared in Theorem 1.1.

$$1 < \frac{N(m-p)}{2(p-1)} \ \Leftrightarrow \ m-p > \frac{2}{N}(p-1).$$

Under this condition, we obtain the following approximation:

$$u(x,t) \approx \varphi(T-t+\varepsilon \mathrm{e}^{\Phi(t)\Delta} q_0) \quad \text{if } t \approx T$$

provided that $|x|$ is sufficiently large so that $T-t \gg q(x,t)$. Here $a \approx b$ means that there exist two constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$, where a and b are two positive functions. Taking a limit $t \to T$ and regarding $\mathrm{e}^{\Phi(T)\Delta} q_0 \equiv 0$, we expect that the total blow-up occurs when $m-p > 2(p-1)/N$. On the other hand, the above formal calculation suggests that $m-p < 2(p-1)/N$ yields the blow-up only at space infinity, and the global profile must be

$$u(x,T) \approx \varphi(\varepsilon \mathrm{e}^{\Phi(T)\Delta} q_0) \quad \text{if } t \approx T. \tag{2.4}$$

Note that $\Phi(T) < \infty$ if $m-p < 2(p-1)/N$. This conjecture (2.4) is proved rigorously in [12] for the semi-linear problem ($m=1$), by constructing suitable sub-super solutions.

3. Total blow-up for quasilinear equation

Our aim of this section is to construct a backward self-similar total blow-up solution of problem (1.1) with the initial value $u_0 \in C(\mathbb{R}^N)$ satisfying (1.3).

Assume the solution u of (1.1) blows up in finite time and let $T > 0$ be its blow-up time. We introduce a simple change of variable as described in Section 2:

$$u(x,t) = \varphi(T-t+h(x,t)). \tag{3.1}$$

From this and $\lim_{s\to 0} \varphi(s) = \infty$, we can see that the blow-up of the solution $u(x,t)$ for (1.1) as $t \to T$ corresponds to the extinction of the solution $h(x,t)$ as $t \to T$. By a simple calculation together with (3.1) and (2.2),

$$\partial_t \varphi(T-t+h) = \varphi'(T-t+h)(h_t-1), \quad f(\varphi(T-t+h)) = -\varphi'(T-t+h).$$

By substituting (3.1) into $\Delta u^m = m(m-1)u^{m-2}|\nabla u|^2 + mu^{m-1}\Delta u$, we have

$$\begin{aligned}
&\Delta\varphi^m(T-t+h)\\
&\quad = m(m-1)\varphi^{m-2}(T-t+h)|\varphi'(T-t+h)\nabla h|^2\\
&\qquad + m\varphi^{m-1}(T-t+h)(\varphi'(T-t+h)\Delta h + \varphi''(T-t+h)|\nabla h|^2)\\
&\quad = m(m-1)\varphi^{m-2}(T-t+h)|\varphi'(T-t+h)\nabla h|^2\\
&\qquad + m\varphi^{m-1}(T-t+h)(\Delta h - f'(\varphi(T-t+h))|\nabla h|^2)\varphi'(T-t+h).
\end{aligned}$$

Here we apply the relation $\varphi''(s) = -f'(\varphi(s))\varphi'(s)$, which can be shown by differentiating (2.2). Substituting (3.1) into (1.1) and dividing it by $\varphi'(T-t+h)$, we obtain

$$h_t = m\varphi^{m-1}(T-t+h)\Big(\Delta h + \Big((m-1)\frac{\varphi'(T-t+h)}{\varphi(T-t+h)} - f'(\varphi(T-t+h))\Big)|\nabla h|^2\Big).$$

Applying $\varphi'(s)/\varphi(s) = -s^{-1}/(p-1)$ and $f'(\varphi(s)) = ps^{-1}/(p-1)$, we get the equation

$$h_t = \frac{m\kappa^{m-1}}{(T-t+h)^{(m-1)/(p-1)}}\Big(\Delta h - \frac{(m+p-1)|\nabla h|^2}{(p-1)(T-t+h)}\Big) \tag{3.2}$$

with the initial data $h(\cdot,0) = \varphi^{-1}(u_0) - T$.

Next we introduce new space and time variables and a function

$$w(y,\sigma) := \frac{h(x,t)}{T-t}, \quad y := (T-t)^{\beta}x, \quad \sigma = \log\frac{1}{T-t},$$

where $\beta := (m-p)/(2(p-1))$ and h is the solution of (3.2). By the chain rule, together with

$$y_t(x,t) = -\mathrm{e}^{\sigma}\beta y(x,t), \quad y_x(x,t) = \mathrm{e}^{-\beta\sigma}, \quad \sigma_t(t) = \mathrm{e}^{\sigma},$$

we obtain

$$h_t(x,t) = \partial_t((T-t)w(y,\sigma)) = -\beta y\cdot\nabla w(y,\sigma) + w_\sigma(y,\sigma) - w(y,\sigma)$$

and

$$\nabla h(x,t) = \mathrm{e}^{-(\beta+1)\sigma}\nabla w(y,\sigma), \quad \Delta h(x,t) = \mathrm{e}^{-(2\beta+1)\sigma}\Delta w(y,\sigma).$$

Substituting these into (3.2), we have

$$\begin{aligned}
&-\beta y\cdot\nabla w(y,\sigma) + w_\sigma(y,\sigma) - w(y,\sigma)\\
&\qquad = \frac{m\kappa^{m-1}}{(1+w(y,\sigma))^{(m-1)/(p-1)}}\mathrm{e}^{((m-1)/(p-1)-(2\beta+1))\sigma}\\
&\qquad\quad \times\Big(\Delta w(y,\sigma) - \frac{m+p-1}{p-1}\frac{|\nabla w(y,\sigma)|^2}{1+w(y,\sigma)}\Big).
\end{aligned}$$

Therefore, the function w satisfies the rescaled equation

$$w_\sigma = \frac{m\kappa^{m-1}}{(1+w)^{2\beta+1}}\Big(\Delta w - \frac{m+p-1}{p-1}\frac{|\nabla w|^2}{1+w}\Big) + (\beta y\cdot\nabla w + w) \tag{3.3}$$

for $y \in \mathbb{R}^N$ and $s > 0$. We can easily see that

$$\lim_{\sigma\to\infty} \|\mathrm{e}^{-\sigma} w(\cdot,\sigma)\|_{L^\infty(\mathbb{R}^N)} = 0 \quad \text{if and only if} \quad B(u_0) = \mathbb{R}^N. \tag{3.4}$$

The simplest example of a solution of (3.3) is a constant $w \equiv 0$, which corresponds to a flat solution $u(x,t) = U(t)$ of the original problem (1.1). Here $U(t)$ is the solution of (2.1). Another typical example is the self-similar solution. In our case, it has the form $h(x,t) = (T-t)g((T-t)^\beta x)$, where $g = g(y)$ satisfies

$$\Delta g - \frac{m+p-1}{p-1}\frac{|\nabla g|^2}{1+g} + \frac{(1+g)^{2\beta+1}}{m\kappa^{m-1}}(\beta y\cdot\nabla g + g) = 0 \tag{3.5}$$

with $y = (T-t)^\beta x$. In other words, a solution h is self-similar if its rescaled function $w(y,\sigma)$ is independent of σ. If we assume that $g(y)$ is a radial function, $g = g(r)$ is the solution of the following ordinary differential equation:

$$g_{rr} + \frac{N-1}{r}g_r - \frac{m+p-1}{p-1}\frac{g_r^2}{1+g} + \frac{(1+g)^{2\beta+1}}{m\kappa^{m-1}}(\beta r g_r + g) = 0, \tag{3.6}$$

$$g(0) = \mu, \quad g_r(0) = 0, \tag{3.7}$$

where $r = |y|$ and $\mu > 0$ is a constant.

Let us note that equation (3.6) has a trivial solution $g \equiv 0$, as well as the spatially homogeneous solution $g \equiv -1$. Let us also note that problem (3.6)–(3.7) admits a solution $g(r)$ with asymptotic behavior:

$$g(r) = \mu - \frac{\mu(1+\mu)^{2\beta+1}}{2m\kappa^{m-1}N}r^2 + o(r^2) \quad \text{as } r \to 0. \tag{3.8}$$

This asymptotics is obtained by solving an approximated ordinary differential equation:

$$g_{rr} + \frac{(1+\mu)^{2\beta+1}}{m\kappa^{m-1}}g \approx 0 \quad \text{for } r \approx 0,$$

which comes from the even symmetric assumption $g_r(0) = 0$ and $g(0) = \mu$.

We must find a value μ with the corresponding solution of the above problem (3.6)–(3.7) that is nonnegative and decreasing at space infinity.

Proposition 3.1. *Let $p > 1$ and $m - p > 2(p-1)/N$. Then problem* (3.6)–(3.7) *has a strictly positive monotone solution satisfying $g(\infty) = 0$ if $\mu > 0$ is sufficiently small.*

If we assume this Proposition, by (3.1), the corresponding solution u of problem (1.1) is written in the form:

$$u_s(x,t) = \varphi((T-t)(1+g((T-t)^\beta x))), \quad \beta > 0.$$

Combining this with $\varphi(0) = \infty$, we obtain $u_s(x,T) = \infty$ for any $x \in \mathbb{R}^N$. Thus $B(u_s(\cdot,0)) = \mathbb{R}^N$. Furthermore, condition (1.3) of the initial value can be easily checked and our result is obtained. Now we shall prove the existence of strictly positive solution $g = g(r)$ for problem (3.6)–(3.7).

Lemma 3.1. *Let $g = g(r)$ be the solution of problem* (3.6)–(3.7). *If $g > 0$ on an interval* $[0, R_0)$, *then g is strictly decreasing on* $[0, R_0)$.

Proof. Define

$$r_0 = \sup\{r > 0\colon \ g \text{ is strictly decreasing on } [0,r]\}$$

and assume $r_0 < R_0$. Then the definition of r_0 implies $g_r(r_0) = 0$ (both $g_r(r_0) > 0$ and $g_r(r_0) < 0$ easily lead to a contradiction) and (3.6) implies $g_{rr}(r_0) < 0$. This in turn means that g is strictly decreasing on a right neighborhood of r_0, a contradiction with the definition of r_0. Hence $r_0 \geqslant R_0$. □

By Lemma 3.1, one can distinguish the following two cases:

(a) $g > 0$ on $[0,\infty)$ and g is strictly decreasing on $[0,\infty)$.

(b) There exists $R \in (0,\infty)$ such that $g > 0$ on $[0,R)$ and $g(R) = 0$. This implies that g is strictly decreasing on $[0,R)$; thus, by continuity, it is strictly decreasing on $[0,R]$. In particular, $g_r(R) < 0$.

Now we exclude the second case (b) using the following lemma.

Lemma 3.2. *Assume that $\beta N > (1+\mu)^{2\beta+1}$. Let $g = g(r)$ be the solution of problem* (3.6)–(3.7). *Then $g > 0$ on* $[0,\infty)$.

Proof. The decay rate of the solution is given by the solution of $\beta r\overline{g}_r + \overline{g} = 0$, which is the dominant term of the ODE (3.6). Thus, we introduce a function

$$v := -\frac{\beta r g_r}{g}\colon \ [0,R) \to [0,\infty). \tag{3.9}$$

By the definition of R, the function v is a nonnegative function and is well-defined. Assume that $R < \infty$. Then case (b) of Lemma 3.1 implies that $\lim\limits_{r\to R} v(r) = \infty$.

Differentiating (3.9) and using (3.6), we get

$$
\begin{aligned}
v_r &= -\frac{\beta r}{g}\Big(g_{rr} + \frac{1}{r}g_r\Big) + \beta r\Big(\frac{g_r}{g}\Big)^2 \\
&= \beta(N-2)\frac{g_r}{g} + \beta r\Big(\frac{g_r}{g}\Big)^2 - \frac{m+p-1}{p-1}\frac{\beta r g_r^2}{g(1+g)} + \frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}}(1-v) \\
&= -(N-2)\frac{v}{r} + \frac{v^2}{\beta r} - \frac{m+p-1}{p-1}\frac{g}{1+g}\frac{v^2}{\beta r} + \frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}}(1-v) \\
&= -(N-2)\frac{v}{r} + \Big(1 - \frac{m+p-1}{p-1}\frac{g}{1+g}\Big)\frac{v^2}{\beta r} + \frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}}(1-v).
\end{aligned}
$$

From (3.8) and (3.9), we see that

$$
v(r) = \frac{\beta(1+\mu)^{2\beta+1}}{m\kappa^{m-1}N}r^2 + o(r^2) \quad \text{as } r \to 0.
$$

We will use this asymptotics in order to estimate the function v from above. Next we shall check that the function $\overline{v}(r) := \beta(1+\mu)^{2\beta+1}/m\kappa^{m-1}Nr^2$ is a super-solution of the above ODE provided that

$$
1 \leqslant \beta N\frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}} + \frac{m+p-1}{p-1}\frac{g}{1+g} \tag{3.10}
$$

for all $r \in [0, R)$. In fact, under condition (3.10), we get

$$
\begin{aligned}
&\overline{v}_r + (N-2)\frac{\overline{v}}{r} - \Big(1 - \frac{m+p-1}{p-1}\frac{g}{1+g}\Big)\frac{\overline{v}^2}{\beta r} - \frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}}(1-\overline{v}) \\
&\quad = \frac{N\overline{v}}{r}\Big(1 - \frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}}\Big) - \Big(1 - \frac{m+p-1}{p-1}\frac{g}{1+g} - \beta N\frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}}\Big)\frac{\overline{v}^2}{\beta r} \\
&\quad \geqslant -\Big(1 - \frac{m+p-1}{p-1}\frac{g}{1+g} - \beta N\frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}}\Big)\frac{\overline{v}^2}{\beta r} \geqslant 0.
\end{aligned}
$$

Here we used the relations $\overline{v}_r = 2\overline{v}/r$ together with

$$
\frac{\beta r(1+g)^{2\beta+1}}{m\kappa^{m-1}} = \frac{N\overline{v}}{r}\frac{(1+g)^{2\beta+1}}{(1+\mu)^{2\beta+1}}
$$

and the inequality $g(r) \leqslant \mu$ for $r \in [0, R]$. Condition (3.10) is satisfied because the function g is nonnegative on $[0, R)$ and $\beta N > (1+\mu)^{2\beta+1}$. Therefore, by the comparison argument, $v \leqslant \overline{v}$ for all $r \in [0, R)$ and $\lim_{r\to r_1} v(r) \leqslant \overline{v}(R) < \infty$. This yields a contradiction. □

P r o o f of Proposition 3.1. Let $p > 1$ and $m - p > 2(p-1)/N$, then $\beta N > 1$. By Lemma 3.2, problem (3.6)–(3.7) has a positive solution if we choose $\mu > 0$ sufficiently small such that $\beta N > (1+\mu)^{2\beta+1}$. Lemma 3.1 implies that this solution is strictly decreasing. Furthermore, since there exists no positive spatially homogeneous solution of equation (3.6), we obtain $g(\infty) = 0$. Hence we obtain the result. □

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