2019

# Rigid Equilibriums of a Rotating String

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# RIGID EQUILIBRIUMS OF A ROTATING STRING

BY

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A thesis submitted in partial fulfillment of the requirements

for the

Master of Science

Major in Mathematics

South Dakota State University

2019

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This dissertation is approved as a creditable and independent investigation by a candidate for the Master of Science degree and is acceptable for meeting the dissertation requirements for this degree. Acceptance of this does not imply that the conclusions reached by the candidate are necessarily the conclusions of the major department.

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*This thesis is dedicated to my dog, Mr. Doodle (Dew), who had the biggest heart of any creature I have known…*

*“Last time I asked: ’What does mathematics mean to you?’, and some people answered: ”The manipulation of numbers, the manipulation of structures.’ And if I had asked what music means to you, would you have answered: ’The manipulation of notes?’.”*

Serge Lang

ACKNOWLEDGEMENTS

I would like to thank my friends and family for listening to me talk about strings for years as I evolved my understanding into this thesis. Specifically, my mother Jill Schaffer let me think out loud to her and further my understanding by explaining it to her, and my father Rod Schaffer for taking care of me while I worked day and night. I would also like to thank Dr. Robert Schmidt for his excellent course on differential equations which was the inspiration that started my obsession of describing physics with differential equations (I wonder if he remembers when I first brought him the string equation I derived many years ago and asked him how to solve it). I would also like to thank Dr. Ross Abraham for showing me the supreme precision and elegance of mathematics. Finally, I would like to thank my advisor Dr. Matthew Biesecker for supporting me throughout my project and always taking the time to listen to me and discuss my ideas, all the way back to before he was my advisor and this project was simply a personal interest of mine. . . .

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# ABSTRACT

RIGID EQUILIBRIUMS OF A ROTATING STRING

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This paper describes some possible equilibrium configurations of a string rotating in a certain class of force fields, which have properties motivated by the inverse square gravitational force. Specifically it is shown that for a given number $n$, there exists a sufficient rotational rate such that any equilibrium configuration with no more than $n$ zeros is guaranteed to exist for any force field in the class considered.

# Chapter 1

# INTRODUCTION

## 1.1 A Physical Phenomenon

If a string suspended in Earth's gravitational field by one end is rotated at a constant rate it will appear to settle into an equilibrium in which its shape is no longer changing, but instead it is rotating rigidly. Furthermore, it will appear to the naked eye that this configuration remains in a plane which rotates with the string. Our goal will be to mathematically investigate the truth of these observations using Newton's Laws of physics, however we will not restrict ourself to the gravitational field, but will instead consider arbitrary force fields satisfying some properties motivated by the gravitation example. If one were to experiment more with the gravitational example, it will also become apparent that depending on the rate of rotation different configurations may form with varying numbers of nodes. Later we will make these statements precise in a mathematical theorem, but first we must establish the mathematics to describe our physical situation.

### 1.1.1 Properties of the Force Field

We elaborate on several of the key properties of the gravitational field which makes the rigid rotational configurations possible. First off, at every point in the field, there is a rotational symmetry about the line along which the force acts, i.e. if you move along any direction orthogonal to that line, the resulting change in force depends only on the distance from the line, and not the direction you choose. The string points its axis of rotation along this line (because it is pulled in that direction), and because of the rotational symmetry, if a string was in just the right configuration for the field to provide the rotational acceleration then the string would purely rotate, fixing the rotationaly symmetric field and holding the string in a rigid configuration. We can think of a configuration in which the field is just right as to rigidly rotate the string as a sort of fixed point configuration, one whose motion produces the same configuration indefinetly. Therefore, we begin by assuming our field acts in a single direction on the axis of rotation (which we will think of as the downward direction), and is rotationally symmetric about that axis. Therefore, it either attracts the string towards the axis or it repels it, so it must vanish on the axis of rotation itself, since the force points along the axis for all points on the axis (another way to think of it is that it pulls or pushes evenly from all sides so they all cancel out in the middle).

### 1.1.2 Mathematical Results

We will prove with a few additional requirements, in every field of this type, rigidly rotating planar equilibriums of a string exist for all strings (as we later define them) which satisfy certain requirements. Furthermore, it is also demenstrated that solutions which oscillate more and more times are guaranteed to exist when certain rotational rates are exceeded, and once a given number of oscillations is achieved, any solution with fewer oscillations is also guaranteed. However, it is not established whether the requirements on the rotational rate are neccesary or if they are simply sufficient conditions. In addition, it is not determined whether there is only one unique solution for a given number of oscillations, just that there is at least one. We will first develope our mathematical model of the problem, whcih we will then use to precisely prove these claims, and then we will offer some numerical validation of our results before concluding with what we showed and what is left to be shown. Most of the results of this paper rely on common and famous mathematical results and are therefore stated without reference, however some of the more specific mathematical theorems will be referenced.

## 1.2 Similar Problems

There are many similar problems involving strings in the mathematical literature. The catenary (the shape of a power line hanging between two poles) is a well known and famously studied curve which is one of the simplest string curves to understand since it does not move. Another similar problem which has been studied is the small vibrations of a dangling string, which has identical solutions to the rotating case in the linear approximation, and one such recent example is given by Verbin, 2014. Even more specific considerations have been made of rotating configurations of a string by other authors such as Kolodner, 1955, who considered a constant force field approximation to gravity with a uniformly dense string (a simplified model of the example we used to introduce our problem). Different aspects of this same problem have been studied, such as the stability of the equilibrium configurations, one such example of which was performed by Dmitrochenko, Yoo, and Pogorelov, 2006. In addition, studies with modifications to the problem have been performed as well, such as those which consider nonuniform densities. One such example of which is the study by Gómez et al., 2007, who considered strings with masses concentrated at discrete points. Another similar modification investigated by Noël et al., 2008 assumed a mass attached to the bottom end of an elastic string and also that both endpoints of the string were fixed to the axis of rotation. Many other modifications have been considered in which specific forces were either added or removed from the analysis, most of which are based on the same constant force approximation to gravity, most likely due to the fact that the extra complications of an inverse square force law far outway the negligible differences in solutions obtained in the case of Earth's gravitational field, wich is well approximated as constant near the surface of the Earth. We show more generalized results which consider strings of nonuniform density rotating in a certain class of force fields with properties motivated by the gravitational example, and which include the inverse square force representation of the gravitational field as well as many other fields. The basic idea was to determine what properties a force field needs to have in order for these equilibrium configurations to be possible, and it is a consequence of our results that the configurations are possible in an inverse square force field. We have given reference to these works for the readers own interest, but none of their methods were used and therefore no other reference of them is required throughout this paper. We note also that any similarities in methods used between our work and those of others is merely based on the similarities of the problems studied, and all of the work (aside from the referenced mathematical theorems) is completely original work.

# Chapter 2

# DEVELOPEMENT OF MODEL

## 2.1 Mathematical Definitions

In ordert to develop a mathematical model of our physical system we will need to make some assumptions as well as some definitions. First we assume our string is one-dimensional and that it cannot be stretched or compressed. This suggests we may mathematically define a string as a function and real number pair

$$\rho : [0, L_c] \to \mathbb{R}^+; \quad L_c \in \mathbb{R}^+,$$

where the real number $L_c$ describes the length of the string, and the function $\rho$, which we assume to be differentiable, describes the mass per unit length (i.e. density), of each point along the string and depends only on the point on the string and not time. Now we define a string motion to be a twice differentiable function:

$$\vec{r} : [0, L_c] \times [0, \infty) \to \mathbb{R}^3,$$

where $\vec{r}(s, t)$ is the position vector for the point on the string at $s \in [0, L_c]$ and time $t$. We say a string motion is a rigid rotation if there exists a vector $\vec{\omega} \in \mathbb{R}^3$

such that

$$\vec{r}_t(s,t) = \vec{\omega} \times \vec{r}(s,t) \quad \forall (s,t) \in [0, L_c] \times [0, \infty)$$

where $\vec{\omega}$ points along the axis of rotation and has magnitude equal to the rate of rotation, the $\times$ between vectors refers to the vector cross product in $\mathbb{R}^3$ and we use the subscript notation for partial derivatives. The motion is said to be planar if for each fixed $t \in [0, \infty)$ the curve:

$$\vec{r} : [0, Lc] \to \vec{r}(s,t)$$

lies in a plane.

### 2.1.1 Newton's Law for Strings

Now to determine the relationships between these definitions we must make some assumptions about the physical nature of our system. We assume the string motion obeys Newton's Law of motion, which states that the rate of change of a body's momentum with respect to time is equal to the sum of the forces acting on that object (note that momentum and force are both vector quantities), and can be expressed by the following mathematical equation:

$$\vec{F} = (m\vec{r}_t)_t = m\vec{r}_{tt}$$

where the second equality is the usual form of this law, but only applies if the mass is constant. This equation is oftentimes interpreted as describing the law of motion for a mass that is entirely concentrated to a single point in space $\vec{r}$, however, it is more realistic to interpret it as describing the motion of a point on a massive body that moves in such a way that the velocity is the same for every point on the mass. If the mass is deforming or rotating the assumption about the velocities is no longer valid, in which case we define the momentum of a body to be the integral of the velocity over the mass distribution. Now before we write out the resulting equation for string motions, we must perform an investigation of forces. This equation of motion is actually the definition of force and mass, which seems quite circular since we cannot determine one without the other. However, fortunately enough for us, some special forces exhibit certain properties which allow us to determine their form by other means than this definition. For example the realtionship between gravity and acceleration is known to be independent of the object it is acting on (think falling bowling balls and feathers) and therefore we determine that the force of gravity is equal to the product of this acceleration with whatever mass it acts on i.e. it depends on the mass it acts on. This means describing gravity as a field of force does not capture the intrinsic nature of gravity since the field would be different for every particle of different mass it acted on. Simalarly, there are other forces which have different descriptions, such as the electrical force which depends not on the mass of the object but the electrical charge of the object. In this case the force field would look different relative to every particle of differing electrical charge and so the force field again does not describe the intrinsic nature of this field, but instead the force per unit charge field (the definition of the electric field) gives the description of this force invariant to the particle which it acts on. These facts may be generalized to consider forces which depend on some physical property of an object (such as mass or electrical charge, etc) which we will describe by the force per unit of this quantity and assume it to be a vector field denoted by

$$\vec{f} : \mathbb{R}^3 \to \mathbb{R}^3.$$

Now if we define

$$\sigma : [0, L_c] \to \mathbb{R}$$

to be the density of this physical property per unit length of our string as a function of the position on the string (we assume this property does not change with respect to time), then we may determine the total force applied to the section of string between $s_1$ and $s_2$ by this field to be the line integral

$$\int_{s_1}^{s_2} \sigma(s)\vec{f}(\vec{r}(s,t))ds,$$

and we oftentimes will abreviate by dropping the functional arguments. However, Newton's Laws of physics also state that when a force is applied there is a reaction force equal in magnitude and opposite in direction (think of a rocket, whose downward thrust is counteracted by an upward lift). In the case of a string the reaction force is the tension which counteracts forces which try to pull the string apart by pulling back (we assume strings can only be pulled along their length and not pushed due to their flexible nature they simply deform under compression). Under these assumptions we find the tension is equal to the force required to hold the string together and it acts at the endpoints of the string pointing outward from the section along the tangent line (since the rest of the string pulls on this section of string). Thus in terms of the string motion, which is parameterized by path length along the string and time, we find that the unit tangent pointing along the string in the direction of the path is given by $\vec{r}_s$ Next, since the tension may vary along the string as well as with respect to time, we define the tension to be a scalar function of path length and time denoted by:

$$T : [0, L_c] \times [0, \infty) \to [0, \infty).$$

These facts imply that the sum of the two tension forces on a section of string is given by:

$$\Big( T(s_2, t)\vec{r}_s(s_2, t) \Big) + \Big( T(s_1, t)(-\vec{r}_s(s_1, t)) \Big),$$

since the tension at the starting point is opposite to the direction along the path. We recognize this as the difference of the product $T\vec{r}_s$ between two points, and therefore assuming the tension is differentiable we may rewrite this expression as

$$\int_{s_1}^{s_2} \frac{\partial}{\partial s} \Big( T(s, t)\vec{r}_s(s, t) \Big) ds$$

Now that we have determined the forces acting on this section of string we will determine its momentum from which we can express Newton's law of motion for strings. Using our definition of momentum we find for the string the momentum is given by:

$$\int_{s_1}^{s_2} \rho(s)\vec{r}_t(s, t) ds$$

which by the Leibniz integral rule has a time derivative equal to

$$\frac{\partial}{\partial t} \Big( \int_{s_1}^{s_2} \rho(s)\vec{r}_t(s, t) ds \Big) = \int_{s_1}^{s_2} \rho(s)\vec{r}_{tt}(s, t) ds$$

since the density is constant in time. Therefore Newton's Law of motion for strings takes the form:

$$\int_{s_1}^{s_2} \rho\vec{r}_{tt} ds = \int_{s_1}^{s_2} \frac{\partial}{\partial s} \Big( T\vec{r}_s \Big) ds + \int_{s_1}^{s_2} \sigma\vec{f} ds$$

and because this equation must hold for every section of string $(s_1, s_2)$ we find it must hold for the integrands and therefore we find:

$$\rho\vec{r}_{tt} = \frac{\partial}{\partial s}(T\vec{r}_s) + \sigma\vec{f} \tag{2.1}$$

### 2.1.2 The Shape of the String in the Rotating Plane

Now under the assumption of a rigidly rotating and planar string motion it is convenient to define a coordinate system as follows. Let $\hat{x}$ be the constant unit vector pointing along the axis of rotation and let $\hat{r}(s,t)$ be the unit vector pointing from the axis of rotation to a point on the string such that it is orthogonal to the axis of rotation. Then we may write:

$$\vec{r}(s,t) = r(s,t)\hat{r}(s,t) + x(s,t)\hat{x}$$

where $r$ and $x$ are the projections of $\vec{r}$ onto their respective basis elements. Using our definition of planar string motion we find that for each fixed $t$ the plane with normal direction equal to:

$$\hat{x} \times \hat{r}(s,t)$$

must be constant with respect to $s$ and therefore we conclude that $\hat{r} = \hat{r}(t)$ is constant with respect to $s$ and we define

$$\hat{\theta}(t) = \hat{x} \times \hat{r}(t),$$

which is also a unit vector, and since it is orthogonal to both $\hat{x}$ and $\hat{r}$ these three vectors form an orthonormal basis for $\mathbb{R}^3$. Now using our definition of rigidly rotating string motion we find there exists a vector $\vec{\omega}$ which can be expressed as

$$\vec{\omega} = \omega \hat{x},$$

where $\omega$ is the scalar magnitude of the rotational rate, such that:

$$\vec{r}_t(s,t) = \vec{\omega} \times \vec{r}(s,t),$$

which we can calculate according to our orthonormal basis to see:

$$\vec{r}_t(s,t) = \omega \hat{x} \times \Big( r(s,t)\hat{r}(t) + x(s,t)\hat{x} \Big) = \omega r(s,t)\hat{x} \times \hat{r}(t) = \omega r(s,t)\hat{\theta}(t),$$

where we used the distributive property of cross products along with the fact that a vector crossed with itself gives the zero vector. However, we also could have simply differentiated our expression for $\vec{r}(s,t)$ to obtain:

$$\vec{r}_t(s,t) = r_t(s,t)\hat{r}(t) + r\hat{r}_t(t) + x_t(s,t)\hat{x},$$

which we can equate to our previous expression to conclude

$$\begin{aligned} r_t(s,t) &= 0 \\ x_t(s,t) &= 0 \\ \hat{r}_t(t) &= \omega \hat{\theta}(t) \end{aligned}$$

These facts make sense for a purely rotational motion, since they imply there is no motion up or down along the axis of rotation as well as inwards or outwards from it. We can now differentiate our velocity expression to derive an equation for the acceleration of the string

$$\begin{aligned}\vec{r}_{tt} &= (\vec{\omega} \times \vec{r})_t \\ &= \vec{\omega} \times \vec{r}_t \\ &= \vec{\omega} \times (\vec{\omega} \times \vec{r}) \\ &= -\omega^2 r \hat{r}\end{aligned}$$

where the last line follows by expanding the vectors in terms of our orthonormal basis and applying the triple vector product identity. Next we compute the spatial derivatives of the string motion:

$$\vec{r}_s = r_s \hat{r} + x_s \hat{x} \qquad \vec{r}_{ss} = r_{ss} \hat{r} + x_{ss} \hat{x}$$

Finally, we substitute these expressions into Newton's Law for string motion to obtain the component equations

$$\begin{aligned}\hat{r}\text{-equation} \quad & -\rho\omega^2 r = T r_{ss} + T_s r_s + \sigma E_r \\ \hat{x}\text{-equation} \quad & 0 = T x_{ss} + T_s x_s + \sigma E_x\end{aligned}$$

Now focusing on the $\hat{x}$ - equation, we integrate along $s$ to yield:

$$0 = T(s,t) x_s(s) - T(0,t) x_s(0) + \int_0^s \sigma E_x$$

which we can differentiate with respect to time to yield:

$$0 = T_t(s,t) x_s(s) - T_t(0,t) x_s(0) + \int_0^s 0$$

and therefore:

$$T_t(s,t) = T_t(0,t)\frac{x_s(0)}{x_s(s)}$$

where $T_t(0,t)$ is the tension at an endpoint of the string, but tension results from the internal forces which hold a string together, and since there is nothing to be held together at a boundary point this tension can only be the result of an external force applied to the end of the string. If we assume $s = 0$ corresponds to the free end of the string then the tension at this point should vanish, however as we will soon see this results in some complications, so for now we assume only that the tension at this point is a constant which we denote as $T_0$. This implies that $T_t(0,t) = 0$ and therefore upon substituting into our previous expression we see $T_t(s,t) = 0$ as well. This fact is also informally verified by our intuition since a rigid rotation of the string is a sort of equilibrium where the tension is just right to provide the rotational acceleration and counteract the external field, so changing the tension at any point would disturb this equilibrium and cause the string to change shapes. This also means that our equations are ODE's (Ordinary Differential Equations) which describe the unchanging shape of the string in the plane of rotation.

We continue our analysis of the $\hat{x}$ equation by noting that the assumption that $E_x < 0 < \sigma$ along with the physical restraint that $T_0 \geq 0$ and the fact that we may assume without loss of generality $x_{s_0} > 0$ (it simply corresponds to reorienting the positive direction for $\hat{x}$) imply that

$$Tx_s = T_0 x_{s_0} - \int_0^s \sigma E_x \geq 0,$$

where equality happens only at $s = 0$. Therefore, neither $T$ nor $x_s$ can vanish for $s > 0$, which immediately implies $x_s > 0$ for all $s \geq 0$ and from this fact and the previous inequality we see that $T \geq 0$ for all $s \geq 0$ as well, with equality only possible at $s = 0$.

This result implies $x$ is strictly monotone in $s$, so we may reparameterize our solutions in terms of $x$ rather than $s$. We therefore solve our equation for $T$ to obtain:

$$T = T_0 \frac{S_x}{S_{x_0}} - S_x \int_{x_0}^{x} \sigma E_x S_x$$

where $x_0$ is the value of $x$ corresponding to $s = 0$. The physical interpretation of this expression is that the tension at each point is such that its component in the $x$-direction is equal to the $x$-component of the applied tension $T_0$ plus the $x$-component of force the field applies to the section of string below that point (which can be thought of as the weight of string below this point). Next we derive an important relation by solving for $T$ in another manner and equating our two expressions. First we take the inner product of (2.1) with the unit tangent vector $\vec{r}_s$

$$< \rho\vec{r}_{tt}, \vec{r}_s > = < T\vec{r}_{ss}, \vec{r}_s > + < T_s\vec{r}_s, \vec{r}_s > + < \sigma\vec{\nabla}E, \vec{r}_s >$$

and apply the fact that it is orthogonal to $\vec{r}_{ss}$ (since it always has unit length it may only change by a rotation so the direction of change is orthogonal to it) to simplify

$$-\rho\omega^2 r r_s = T_s + \sigma E_x x_s + \sigma E_r r_s,$$

and then we integrate along $s$ to obtain:

$$\int_0^s -\rho\omega^2 r r_s = \int_0^s T_s + \int_0^s \sigma E_x x_s + \int_0^s \sigma E_r r_s.$$

We may then transform these into integrals in terms of $x$ and solve for $T$ to obtain:

$$T = T_0 - \int_{x_0}^{x} \rho\omega^2 r r' - \int_{x_0}^{x} \sigma E_x - \int_{x_0}^{x} \sigma E_r r'$$

which we may equate to our other expression to find:

$$T_0 - \int_{x_0}^{x} \rho\omega^2 r r' - \int_{x_0}^{x} \sigma E_x - \int_{x_0}^{x} \sigma E_r r' = T_0 \frac{S_x}{S_{x_0}} - S_x \int_{x_0}^{x} \sigma E_x S_x$$

and upon rearanging we obtain an important equation:

$$-\int_{x_0}^{x} \left( \rho\omega^2 + \sigma \frac{E_r}{r} \right) r r' = T_0 \left( \frac{S_x}{S_{x_0}} - 1 \right) - S_x \int_{x_0}^{x} \sigma E_x S_x + \int_{x_0}^{x} \sigma E_x$$

Furthermore, from the second expression for tension we find by use of the chain rule and the Fundamental Theorem of Calculus that

$$T_s = (-\rho\omega^2 r r' - \sigma E_x - \sigma E_r r') \frac{1}{S_x},$$

which can be substituted back into (2.1) to yield the equation for the $r$ component

$$\begin{aligned}
-\rho\omega^2 r &= \left( T_0 \frac{S_x}{S_{x_0}} - S_x \int_{x_0}^{x} \sigma E_x S_x \right) \frac{r''}{S_x^4} + \left( -\rho\omega^2 r r' - \sigma E_x - \sigma E_r r' \right) \frac{1}{S_x} \frac{r'}{S_x} + \sigma E_r \\
-\rho\omega^2 r S_x^2 &= \left( T_0 \frac{1}{S_{x_0}} - \int_{x_0}^{x} \sigma E_x S_x \right) \frac{r''}{S_x} + \left( -\rho\omega^2 r r' - \sigma E_x - \sigma E_r r' \right) r' + \sigma E_r S_x^2 \\
0 &= \left( \frac{T_0}{S_{x_0}} - \int_{x_0}^{x} \sigma E_x S_x \right) \frac{r''}{S_x} + \rho\omega^2 r (S_x^2 - r'^2) - \sigma E_x r' + \sigma E_r (S_x^2 - r'^2) \\
0 &= \left( \frac{T_0}{S_{x_0}} - \int_{x_0}^{x} \sigma E_x S_x \right) \frac{r''}{S_x} + \rho\omega^2 r - \sigma E_x r' + \sigma E_r
\end{aligned}$$

$$0 = \left( \frac{T_0}{S_{x_0}} - \int_{x_0}^{x} \sigma E_x S_x \right) r'' - S_x \left( \sigma E_x r' - \left( \rho\omega^2 + \sigma \frac{E_r}{r} \right) r \right). \tag{2.2}$$

We note a useful property of (2.2) is that it may also be written in the form

$$0 = \left[ \left( \frac{T_0}{S_{x_0}} - \int_{x_0}^{x} \sigma E_x S_x dx \right) r'' + \left( -\sigma E_x S_x \right) r' \right] + S_x \left( \rho\omega^2 + \sigma \frac{E_r}{r} \right) r$$

$$0 = \frac{d}{dx} \left[ \left( \frac{T_0}{S_{x_0}} - \int_{x_0}^{x} \sigma E_x S_x \right) r' \right] + S_x \left( \rho\omega^2 + \sigma \frac{E_r}{r} \right) r, \tag{2.3}$$

which can be viewed as a nonlinear analogue of a Sturm-Liouville Equation, a fact we will later take advantage of. Now we note that we may specify any value we wish for $x_0$ since it simply results in shifting the origin of our coordinate system up or down along the axis of rotation such that the bottom of the string ($s = 0$) has an $x$ coordinate equal to $x_0$, and therefore for the sake of simplicity we choose our coordinate system such that $x_0 = 0$. However, because the bottom of the string can move freely in space depending on the string motion, we do not actually where this point is in our rotating plane. It may seem as though we can place it anywhere and just use the resulting coordinate system with no problems, but moving the coordinate system around would change our expression for the external field, and if we do not know where it is then we do not know the new formula for the field. To handle this we use the expression for the field relative to the top end of the string at $s = L_c$, which is fixed in space at a known point and transform to our new coordate system as follows: Let $L$ be the length of the string's projection onto the $x$ axis (a value we will determine later), then in our coordinate system corresponding to $x_0 = 0$ we find the new expression for the field is given by

$$E = E(x - L, r)$$

which we will henceforth write as

$$E = E(x, r, L)$$

Another fact we can extract from this equation is that if $r$ is a solution then so too is $-r$. Physically this makes sense, since our solution has rotational symmetry and this corresponds to flipping the solution which is equivalent to a 180 degree rotation, however to verify this fact mathmatically we note the following properties of the functions in (2.2):

$$\begin{aligned}
E_x(x, -r, L) &= E_x(x, r, L), \\
E_r(x, -r, L) &= -E_r(x, r, L), \\
S_x = \sqrt{1 + (r')^2} &= \sqrt{1 + (-r')^2}.
\end{aligned}$$

The last fact also implies $s$ is unchanged by this substitution, which then implies neither are $\rho$ or $\sigma$, so using these facts and substituting $-r$ into (2.3) we find:

$$\begin{aligned}
&\frac{d}{dx}\left[\left(\frac{T_0}{S_{x_0}} - \int_{x_0}^{x} \sigma E_x S_x dx\right)(-r')\right] + S_x\left(\rho\omega^2 + \sigma\frac{-E_r}{-r}\right) - r \\
&= -\frac{d}{dx}\left[\left[\left(\frac{T_0}{S_{x_0}} - \int_{x_0}^{x} \sigma E_x S_x dx\right)r'\right]' + S_x\left(\rho\omega^2 + \sigma\frac{E_r}{r}\right)r\right],
\end{aligned}$$

so if $r$ satisfies the equation, then so too does $-r$. We therefore can assume without loss of generality that the initial value of $r$ is positive. This fact also raises the question, what should the initial conditions be? It is possible that many different configurations of the string are possible and therefore there would be many different physically meaningful initial conditions. Let us consider what physical constraints a solution must satisfy in order to describe our string motion. First of all our string has a fixed length, so if a solution to (2.2) is to describe our string, then the path length of the solution must equal the length of our string. Mathematically this is expressed by the following equation:

$$\int_0^L \sqrt{1+(r')^2}dx = L_c$$

and furthermore since we assumed the top end of the string is fixed to the axis of rotation our solution must also vanish at this point so therefore

$$r(L) = 0.$$

This is a strange set of conditions, where the latter is some sort of boundary condition only expressed at the unknown point $x = L$, and the former is a somewhat complicated integral relation, so for now we arbitrarily specify initial values

$$r(0) = r_0 \quad \text{and} \quad r'(0) = r_0',$$

which we will attempt to vary such that the physical constraints are satisfied. We previously mentioned if we allow $T_0$ to vainish, as would be the case for a free endpoint of a string, it creates some issues, let us now investigate. The resulting ODE would read

$$\frac{d}{dx}\left[\left(-\int_0^x \sigma E_x S_x dx\right)r'\right] + S_x\left(\rho\omega^2 + \sigma\frac{E_r}{r}\right)r = 0, \tag{2.4}$$

which is singular, since the coefficient of the highest order term vanishes at $x = 0$. This complicates our investigation into the existence of solutions, but another consequence is that if a solution does exist and if it and its derivatives remain finite at $x = 0$ then at this point the differential equation reads

$$-S_{x_0}\left(\sigma_0 E_{x_0} r_0' - \left(\rho_0\omega^2 + \sigma_0 \frac{E_{r_0}}{r_0}\right) r_0\right) = 0,$$

and because $S_x$ cannot vanish it must be true that the other facter does, which implies

$$r_0' = \frac{\rho_0\omega^2 r_0 + \sigma_0 E_{r_0}}{\sigma_0 E_{x_0}}.$$

Therefore, we adopt this along with $r_0$ as our initial conditions for (2.2).

# Chapter 3

# PROOF OF THEOREM

## 3.1 Proof of Main Result

Our main result states that for any functions $Ex$, $E_r$, $\rho$, and $\sigma$ satisfying conditions we soon specify, and for any $L_c$ and natural number $n$, if $\omega$ is chosen such that the conditions

$$\omega^2 > \frac{2\pi^2(n+1)^2}{L_c \inf \rho} \sup |E_x|(1+n\zeta^n)$$
$$\omega^2 > \sup \left| \frac{2E_r}{r \inf \rho} \right|$$

hold, then there exist $n$ solutions to (2.4) for the corresponding pairs of values

$$(r_1, L_1), \ldots, (r_n, L_n),$$

each of which satisfy the conditions that $L_i$ is the $i^{th}$ zero of its corresponding solution $\phi(x, r_i, L_i)$, and the path length of the solution up to that point is equal to the length of the string

$$s(L_i, r_i, L_i) = L_c.$$

Our approach to proving this claim will be to first consider nonzero values of $T_0$, for which case (2.2) is nonsingular and standard ODE theory applies. We will then show solutions exist which are continuous in all parameters, and establish bounds on the solutions which show they can be extended infinitely along $x$. Furthermore, we establish bounds which apply universally to certain ranges of parameters which allows us to trap solution curves in a box, which helps us to show the functions oscillate. Basically, we consider two cases, one in which the length of the curve is bounded for all paramters, in which case we can use a comparison theorem on (2.3) to show it can be made to oscillate any number of times within our box, and if it can be made arbitrarily large, then because the solution is trapped in the box, the only way to make longer and longer paths is to oscillate (and we prove a result which describes how this oscillation must occur with alternating zeros of $r$ and $r'$). Then we use continuity arguments to find paramters which satisfy the physical constraints for each solution in our class of oscillating solutions. And finally we use our universal bounds on these classes of solutions to establish the existence of a uniformly convergent sequence of such functions, whose limit function satisfies (2.4), and satisfies the same constraints as the functions in the sequence.

We begin our analysis of (2.2) with some results from standard ODE theory. Specifically we display existence, uniqueness, and continuity properties of solutions. The theory on ODE's is usually specified in terms of first order systems of ODE's, so we begin by transforming our equation into a first order system as follows. First we define:

$$u_1 = r$$
$$u_2 = r'$$
$$u_3 = s$$
$$u_4 = -\int_0^x E_x S_x dx$$

which converts (2.2) to the following system of equations

$$\begin{aligned}
u_1' &= u_2\\
u_2' &= \frac{\sqrt{1+u_2^2}}{\frac{T_0}{\sqrt{1+r_0^2}} + u_4}\Big[E_x(x,u_1,L)u_2 - \rho(u_3)\omega^2 u_1 - \sigma(u_3)E_r(x,u_1,L)\Big]\\
u_3' &= \sqrt{1+u_2^2}\\
u_4' &= -E_x(x,u_1,L)\sqrt{1+u_2^2}
\end{aligned}$$

and our first goal will be to show that this system has the existence, uniqueness, and continuity properties on a set of paramters and initial conditions. Before we begin we must handle the fact that $\rho(s)$ and $\sigma(s)$ are both only defined on $[0, L_c]$, but it will be convenient in our theory to allow $s$ to become arbitrarily large. Therefore, since they are both Lipschitz continuous on $[0, L_c]$ we may extend them to Lipschitz continuous functions on $[0, \infty)$ such that they still satisfy the requirement

$$\inf \rho, \inf \sigma > 0.$$

Next we define the following sets

$$J = \mathbb{R} \times \mathbb{R} \times [0,\infty) \times [0,\infty)$$
$$D = [0,\infty) \times J \times [0,\infty) \times [0,\infty) \times (0,\infty) \times (0,\infty)$$

and rewrite our system of equations in vector form

$$\vec{u}' = \vec{F}(x, \vec{u}, L, r_0, T_0, \omega),$$

defining $F_1, F_2, F_3, F_4$ to be the component equations of $\vec{F}$ which define our system of equations. We see immediately that $F_1$ and $F_3$ are continuously differentiable on $D$ and therefore they are both locally Lipschitz in $\vec{U}$ on D. Furthermore, by assumption we have that the functions $E_x$ , $E_r$ , $\rho$ , and $\sigma$ are all locally Lipschitz in $\vec{u}$ on $D$ and therefore $F_4$ is a product of locally Lipschitz functions on $D$ so it follows that so too is $F_4$. Similarly for $F_3$, the factor

$$\Big[E_x(x, u_1, L)u_2 - \rho(u_3)\omega^2 u_1 - \sigma(u_3)E_r(x, u_1, L)\Big]$$

is sums and products of locally Lipschitz functions on $D$, so it is locally Lipschitz on $D$ as well, and the other factor,

$$\frac{\sqrt{1 + u_2^2}}{\frac{T_0}{\sqrt{1+r_0^2}} + u_4}$$

is easily verified to be continuously differentiable on $D$ so it too is locally Lipschitz, and therefore $F_3$ is a product of locally Lipschitz functions on $D$, so it too is locally Lipschitz on $D$. It follows then that $\vec{F}$ is locally Lipschitz on $D$. We also have that $F_3$ and $F_4$ are both positive on $D$, so it is true that $\vec{F}$ maps $D$ into $J$ and therefore for any point $(x_0, \vec{u}_0) \in [0, \infty) \times J$ the famous Picard iteration has a convergent sequence of functions whose limit is the unique solution to the IVP

$$\vec{u}' = \vec{F}(x, \vec{u}, L, r_0, T_0, \omega) \quad , \quad \vec{u}(x_0) = \vec{u}_0$$

defined on some interval

$$\vec{u} : [0, d] \to J.$$

Next we prove an oscillating nature of the solutions which helps us to extend our solutions to all of $D$ as well as to show later on that the solutions can be made to oscillate with arbitrarily many zeros in a given interval. We must first impose the condition that

$$\omega > \sqrt{\sup \frac{2\sigma E_r}{r \inf \rho}}, \tag{3.1}$$

in which case we can see that

$$\left(\rho\omega^2 + \sigma\frac{E_r}{r}\right) > \left(\inf\rho - \left|\frac{\sigma E_r}{\sup|\frac{2\sigma E_r}{r\inf\rho}|r}\right|\right)\omega^2 \geq \left(\inf\rho - \frac{\inf\rho}{2}\right)\omega^2 = \frac{\inf\rho}{2}\omega^2 > 0,$$

and by a similar argument we can bound it from above as well, so that

$$\frac{\inf\rho}{2}\omega^2 < \left(\rho\omega^2 + \sigma\frac{E_r}{r}\right) < \frac{2\sup\rho + \inf\rho}{2}\omega^2. \tag{3.2}$$

In addition, the following properties that we assume will come up quite often as well:

$$\sup E_x < 0; \quad |\frac{\sigma E_r}{r}| \leq M; \quad E_x, E_r \quad \text{are locally Lipschitz with respect to r,} \tag{3.3}$$

for some bound $M$. We assume all of the conditions (3.1) , (3.2) , and (3.3) hold hereafter.

**Proposition 1.** *For any nontrivial solution to* (2.2)*, with $\omega$ chosen to satisfy* (3.1)*, the zeros of $r(x)$ and $r'(x)$ all have multiplicity one and they alternate, where a zero of $r(x)$ must come first.*

*Proof.* If either $r(x)$ or $r'(x)$ had a zero of greater multiplicity, then $r(x)$ and

$r'(x)$ would vanish at a point, or else $r'(x)$ and $r''(x)$ would, and in either case we can see from (2.2) we would have only the trivial solution. To show the zeros alternate we consider two cases.

Case 1: $r(x)r'(x) < 0$

In this case we claim the only way for $r(x)r'(x)$ to change signs is if $r(x)$ changes signs. Assume instead that $r'(x)$ changes signs, then there is a point $a$ at which $r'(x)$ must vanish. Also the fact that $E_x < 0$ and (3.2) holds, we see from the (2.2) that

$$\operatorname{sgn}(r'') = -\operatorname{sgn}(|E_x|r' + |\rho\omega^2 + \frac{E_r}{r}|r)$$

and thus at $a$ we find

$$\begin{aligned}\operatorname{sgn}(r'') &= -\operatorname{sgn}(|\rho\omega^2 + \frac{E_r}{r}|r)\\ &= -\operatorname{sgn}(r)\end{aligned}$$

This implies in some left neighborhood of the point a, $\operatorname{sgn}(r'') = -\operatorname{sgn}(r) = \operatorname{sgn}(r')$ so that $r'(x)r''(x) > 0$ in this neighborhood. However this is a contradiction since it implies $|r'(x)|$ is increasing as $x$ approaches $a$, the point at which it must vanish. Therefore $r'(x)$ cannot change signs in case 1, so $r(x)r'(x)$ can only change from negative to positive if $r(x)$ changes signs, which leads to the next case.

Case 2: $r(x)r'(x) > 0$

This would imply that $|r(x)|$ is increasing, so it could not vanish, and thus the only way to go from Case 2 back to Case 1 is if $r'(x)$ were to change signs.

Finally, the initial conditions show we begin in case 1 and any zeros of $r(x)$

or $r'(x)$ must therefore alternate through the cases beginning with a zero of $r(x)$. □

Next we show we can continuously extend our solutions on all of $D$ through the following series of propositions, which are also useful for later results.

**Proposition 2.** *The solution $\vec{u}$ can be extended to all $x > 0$.*

*Proof.* We show it is true for $u_1$ and $u_2$, and this fact along with our conditions (3.3) would imply it holds for $u_3$ and $u_4$ as well. Standard ODE theory tells us we can extend in $x$ from $[0, d]$ to the right to an open interval, so either we can extend to $[0, \infty)$ or else there is some finite value $a$ such that the function cannot be extended to this point. The latter condition can only occur if either function becomes unbounded as $x$ approaches $a$. We first show that $u_2$ cannot become unbounded unless $u_1$ does simultaneously. Recall our sign relation we used in Proposition 1, which states in terms of our transformed system of equations that

$$\mathrm{sgn}(u_2 u_2') = -\mathrm{sgn}(|\sigma E_x| u_2^2 + |\rho\omega^2 + \sigma \frac{E_r}{u_1}| u_1 u_2).$$

From which we can see

$$|E_x| u_2^2 + |\rho\omega^2 + \sigma \frac{E_r}{u_1}| u_1 u_2| \geq |\sigma E_x| u_2^2 - |\rho\omega^2 + \sigma \frac{E_r}{u_1}||u_1||u_2|,$$

which is positive whenever

$$|u_2| > \frac{|\rho\omega^2 + \sigma \frac{E_r}{u_1}||u_1|}{|\sigma E_x|},$$

and therefore in this case $u_2 u_2'$ would be negative, i.e. $|u_2|$ would be decreasing. This implies that

$$\begin{aligned} |u_2| &< \frac{\sup|\rho\omega^2 + \sigma\frac{E_r}{u_1}||u_1|}{\inf|\sigma E_x|} \\ &< \frac{\sup|\rho\omega^2 + \sigma\frac{E_r}{u_1}|}{\inf|\sigma E_x|}\sup|u_1| \\ &< C\sup|u_1|, \end{aligned}$$

$$|u_2| < C\sup|u_1|, \tag{3.4}$$

since $|u_2|$ would be decreasing if not, so it would never have been able to increase in magnitude beyound $C\sup|u_1|$. This implies if $|u_1|$ is bounded then so too is $|u_2|$. Therefore, we only need to show $u_1$ remains bounded. Consider the case where $u_1$ has a vertical asymptote at $x = a$. In this case it is clear by the mean value theorem that there is an increasing sequence of values converging to $a$ where $u_1' = u_2$ must diverge in the same direction simultaneously with $u_1$, so without loss of generality assume they diverge in the positive direction. Then $u_1$ has no zeros in a left neighborhood of $a$ so by Proposition 1 neither does $u_2$ on the same interval, which implies they are both positive on this interval and we may assume our sequence lies in an interval where both $u_1$ and $u_2$ are positive. Now note that using the sign relation in Proposition 1 along with (3.1) we have that

$$\operatorname{sgn}(u_2') = -\operatorname{sgn}(|E_x|u_2 + |\rho\omega^2 + \sigma\frac{E_r}{u_1}|u_1)$$

must be negative whenever both $u_1$ and $u_2$ are positive. However, this is a contradiction since $u_2$ cannot diverge on an increasing sequence of points in an interval where it is decreasing. This fact then implies that if $u_1$ diverges at $a$, then it must vanish infinitely many times as $x$ approaches $a$ to avoid the asymptotic impossibility. However, we show that the function can only vanish finitely many times in any finite interval. Our assumptions imply that initially $u_1 u_2 < 0$, so by Proposition 1 we find every zero of $u_1$ is preceeded by an interval where $u_1 u_2 < 0$, and $u_1$ decreases from a local max (including $r_0$) to zero. Furthermore, from (3.4) it is clear that on any of these intervals, if $r_k$ is the associated critical value with corresponding interval $[a_k, b_k]$, then

$$|u_2| < C|r_k|.$$

Thus by the fundamental theorem of calculus we see

$$\begin{aligned} |r_k| &= \int_{a_k}^{b_k} |u_2| dx \\ &\leq (b_k - a_k) C \sup |u_1| \\ &= (b_k - a_k) C |r_k|, \end{aligned}$$

which we may rearrange to yield the inequality

$$(b_k - a_k) > \frac{1}{C},$$

so that $x$ increases by an amount greater than $\frac{1}{C}$ everytime $|u_1|$ vanishes, which is a contradiction since $x$ would surpass $a$ after the max was increased more than

$$\frac{a - 0}{\frac{1}{C}} = aC$$

times. Therefore, no such value $a$ exists and the funciton can be extended in $x$ to $[0, \infty)$. □

### 3.1.1 Bounding Solutions

The next two propositions serve to establish a useful bound on the solutions to (2.2). The first proposition shows that a weighted average of $rr'$, over any bounded interval $[0, L_c]$ for which the functions are defined, is negative, where

$$rr' = \frac{(r^2)'}{2}$$

describes the rate of change of the magnitude of $r$. Depending on the weighting function it may turn out that $r$ is increasing even though the corresponding weighted average of $rr'$ is negative, however, as we will see this fact will limit the extent to which $r$ can grow. First we prove the proposition regarding this weighted average.

**Proposition 3.** *For every $r_0, L_c > 0$ there exist a value $\delta$ such that for $0 < T_0 < \delta$ the corresponding solution to* (2.2) *is such that the equation*

$$-\int_0^x \left(\rho\omega^2 + \sigma\frac{E_r}{r}\right) rr' = T_0\left(\frac{S_x}{S_{x_0}} - 1\right) - S_x\int_0^x \sigma E_x S_x + \int_0^x \sigma E_x \tag{3.5}$$

*is greater than zero for all $x$ with $0 < x < L_c$.*

*Proof.* From the condition (3.2), we see that

$$\rho\omega^2 + \sigma\frac{E_r}{r} > 0$$

and since $r_0 > 0 > r_0'$, Proposition 1 tells us that $rr' < 0$ up to the first zero of $r$.

These two facts show that (3.5) is positive at least until $r$ vanishes. Assume there is a point

$$0 < a < L_c$$

where $r$ vanishes (This point may depend on $r_0$ and/or any other parameters in the ODE). Then for $a < x < L_c$ we write

$$-\int_0^x \left(\rho\omega^2 + \sigma\frac{E_r}{r}\right) rr' = T_0\left(\frac{S_x}{S_{x_0}} - 1\right) - S_x\int_0^x \sigma E_x S_x + \int_0^x \sigma E_x$$

$$= T_0\left(\frac{S_x}{S_{x_0}} - 1\right) - S_x\int_0^a \sigma E_x S_x + \int_0^a \sigma E_x - S_x\int_a^x \sigma E_x S_x + \int_a^x \sigma E_x$$

$$> -T_0 - \int_0^a \sigma E_x S_x + \int_0^a \sigma E_x - S_x\int_a^x \sigma E_x S_x + \int_a^x \sigma E_x$$

where the inequality follows from the fact that $S_x, S_{x_0} \geq 1$. This fact along with the assumption that $\sup E_x < 0$ implies we may continue as follows

$$= -T_0 - \int_0^a \sigma E_x(S_x - 1) - S_x\int_a^x \sigma E_x S_x + \int_a^x \sigma E_x$$

$$= -T_0 - \int_0^a \sigma E_x(S_x - 1) + \gamma^2$$

$$\geq -T_0 + \inf|E_x|\int_0^a (S_x - 1) + \gamma^2$$

$$= -T_0 + \inf|\sigma E_x|(s(a) - a) + \gamma^2$$

for some $\gamma(x) \geq 0$. Furthermore, because a straight line is the shortest path between two points we find the path length $s(a)$ between the initial value $(0, r_0)$ and the first zero $(a, 0)$ is such that $s(a) \geq \sqrt{r_0^2 + a^2}$ and therefore we continue again to show

$$\geq -T_0 + \inf|\sigma E_x|(\sqrt{r_0^2 + a^2} - a) + \gamma^2$$

$$\geq -T_0 + \inf|\sigma E_x|(\sqrt{r_0^2 + L_c^2} - L_c) + \gamma^2$$

where the second inequality follows from the fact that $\sqrt{r_0^2 + x^2} - x$ is strictly decreasing (the derivative $\frac{x}{\sqrt{r_0^2+x^2}} - 1 < 0$ for all $x$), and $a < L < L_c$ where $L_c$ is an upper bound on $L$ for all solutions considered. Finally it is clear that $\delta = \inf |\sigma E_x| (\sqrt{r_0^2 + L_c^2} - L_c)$ will work since if

$$T_0 < \delta = \inf |\sigma E_x| (\sqrt{r_0^2 + L_c^2} - L_c)$$

then the last inequality above will be greater than zero and the result follows.

□

With this result along with Proposition 1 we are now ready to prove a certain bound on certain solutions $r$ to (2.2).

**Proposition 4.** *Let $r_n$ be the $n^{th}$ local extreme value of $r(x)$ or $r_n = r(L_c)$ if no such value exists, then if* (3.1) *holds then there exists a value $\alpha > 0$ such that $|r_n| \leq \zeta^n r_0$ for all $n$.*

*Proof.* We assume (3.2) holds, and therefore

$$\frac{\inf \rho}{2} \omega^2 < \rho \omega^2 + \sigma \frac{E_r}{r} < \frac{2 \sup \rho + \inf \rho}{2} \omega^2,$$

and we claim that $\alpha = \sqrt{\frac{2 \sup \rho + \inf \rho}{\inf \rho}}$ will work. We begin with the result of Proposition 2 which showed

$$\int_0^x (\rho \omega^2 + \sigma \frac{E_r}{r}) r r' < 0$$

and note that with the following definitions

$$(rr')^{+} = \begin{cases} rr', & rr' > 0 \\ \\ 0, & rr' \leq 0 \end{cases} \qquad (rr')^{-} = \begin{cases} 0, & rr' > 0 \\ \\ rr', & rr' \leq 0 \end{cases}$$

it is true that $rr' = (rr')^{+} + (rr')^{-}$ and therefore we may write

$$\begin{aligned} \int_0^x \left(\rho\omega^2 + \sigma\frac{E_r}{r}\right) rr' &= \int_0^x \left(\rho\omega^2 + \sigma\frac{E_r}{r}\right)(rr')^{+} + \int_0^x \left(\rho\omega^2 + \sigma\frac{E_r}{r}\right)(rr')^{-} \\ &\geq \frac{\inf\rho}{2}\omega^2 \int_0^x (rr')^{+} + \frac{2\sup\rho + \inf\rho}{2}\omega^2 \int_0^x (rr')^{-} \end{aligned}$$

So we find the equation in the last line must also be negative which implies that

$$\int_0^x (rr')^{+} < -\frac{2\sup\rho + \inf\rho}{\inf\rho}\int_0^x (rr')^{-} = \alpha^2 \left| \int_0^x (rr')^{-} \right| \tag{3.6}$$

Now we proceed by induction: for $n = 0$ it is clear that $|r_0| \leq \alpha^0 r_0 = r_0 = |r_0|$. Assume the result holds for $n = k$ and consider $r_{k+1}$. There are two cases:

$$\operatorname{sgn}(r_{k+1}) = \operatorname{sgn}(r_k) \quad \text{or} \quad sgn(r_{k+1}) = -\operatorname{sgn}(r_k)$$

In the former case the definition of $r_{k+1}$ implies that $|r_{k+1}| \leq |r_k|$. Therefore our induction hypothesis along with the fact that

$$\alpha = \sqrt{\frac{2\sup\rho + \inf\rho}{\inf\rho}} \geq \sqrt{\frac{2\inf\rho + \inf\rho}{\inf\rho}} = \sqrt{3} > 1$$

implies that

$$|r_{k+1}| \leq |r_k| \leq \alpha^k r_0 < \alpha^{k+1} r_0$$

In the case where the signs are opposite, Proposition 1 implies that there is exactly one zero of $r$ between any two $r_i$ and $r_{i+1}$ for $i = 0, \ldots, k$ and furthermore, that between $r_i$ and each respective zero the quantity $rr' \leq 0$ and also $rr' \geq 0$ from that zero to $r_{i+1}$. Therefore a direct computation of the inequality (3.6) along with our induction hypothesis shows that

$$\frac{r^2}{2}\bigg|_{r=0}^{r=r_1} + \cdots + \frac{r^2}{2}\bigg|_{r=0}^{r=r_{k+1}} < \alpha^2 \left| \frac{r^2}{2}\bigg|_{r=r_0}^{r=0} + \cdots + \frac{r^2}{2}\bigg|_{r=r_k}^{r=0} \right|$$

$$r_1^2 + \cdots + r_{k+1}^2 < \alpha^2(r_0^2 + \cdots + r_k^2)$$

$$\begin{aligned}
r_{k+1}^2 &< \alpha^2 r_0^2 + (\alpha^2 - 1)(r_1^2 + \cdots + r_k^2) \\
&< \alpha^2 r_0^2 + (\alpha^2 - 1)(\alpha^2 r_0^2 + \cdots + \alpha^{2k} r_0^2) \\
&= \alpha^2 r_0^2 + (\alpha^2 - 1)(1 + \cdots + \alpha^{2(k-1)})\alpha^2 r_0^2 \\
&= \alpha^2 r_0^2 + (\alpha^2 - 1)\frac{\alpha^{2k} - 1}{\alpha^2 - 1}\alpha^2 r_0^2 \\
&= \alpha^{2(k+1)} r_0^2
\end{aligned}$$

where the second-to-last step follows from the partial sum formula for a geometric series. Finally, upon taking the square root of the total inequality we find the result is true.

□

The last three propositions imply that for a compact set of initial data (conditions and paramters) since $C$ and $r_0$ will both be bounded, it will be true on any given finite interval that the number of zeros of $u_1$ is uniformly bounded, which uniformly bounds $u_1$ in terms of $r_0$ by Proposition 4, which then uniformly bounds $u_2$ by (3.4), and therefore $\vec{u}$ is uniformly bounded on this set, so it follows (using the Gronwall indequality method) that it can be extended to a continuous function on all of $D$.

### 3.1.2 Oscillation

Now as we described in the beginning of this chapter, we will show that we can squeeze arbitrarily many zeros into a given interval simply by choosing large enough values of $\omega$. This is consistent with our physical intuition that twirling the string faster should cause it to become more wavy, and is made precise with the following proposition.

**Proposition 5.** *For any $0 < L < L_c$ and for any Natural number $n$ and assuming* (3.1) *,* (3.2) *, and* (3.3) *all hold, there exists a value $\omega_n$ such that for $\omega > \omega_n$ the corresponding solution to* (2.2) *will have at lease $n$ zeros in $[0, L]$.*

*Proof.* We begin with the solution $r(x)$ of (2.2) and using the fact that this solution and its derivative are both continuous functions as well as $\rho$, $E_r$, and $E_x$ we construct the following continuous functions:

$$P(x) = \frac{T_0}{S_{x_0}} - \int_0^x \sigma E_x S_x \quad , \quad Q(x) = S_x(\rho\omega^2 + \sigma\frac{E_r}{r})$$

Now we note with these functions the solution $r(x)$ to (2.2) is also a solution to the linear Sturm-Liouville equation:

$$(P(x)r'(x))' + Q(x)r(x) = 0$$

$$r(0) = r_0 \quad ; \quad r'(0) = \frac{\rho_0\omega^2 r_0 + \sigma_0 E_{r_0}}{\sigma_0 E_{x_0}}$$

Next we define

$$B = \frac{T_0}{S_{x_0}} + \sup|\sigma E_x|[L + n\zeta^n r_0] \quad , \quad R = \frac{\pi^2(n+1)^2}{L^2}B,$$

and note by assumption

$$Q > \omega^2 \frac{\inf \rho}{2}$$

which means we may choose $\omega$ such that $Q > R$. Now we proceed by cases:

Case 1: $0 < P(x) \leq B$

Consider the Sturm-Liouville equation

$$(By'(x))' + Ry(x) = 0$$

with the same initial conditions as the former Sturm equation. Using the inequalities on the coefficients of the equations we may apply the Sturm Comparison Theorem (We offer reference to the textbook by Teschl, 2012 which outlines the theorem) which specifies that between any two zeros of $y(x)$ there must be a zero of $r(x)$. Where the general solution for $y(x)$ is

$$\begin{aligned} y(x) &= c_1 \sin\sqrt{\frac{R}{B}}x + c_2 \cos\sqrt{\frac{R}{B}}x \\ &= c_1 \sin\frac{\pi(n+1)}{L}x + c_2 \cos\frac{\pi(n+1)}{L}x \end{aligned}$$

which is easily verified to have $n+1$ zeros in $[0, L]$, implying $r(x)$ must have at least $n$ zeros in $[0, L]$.

Case 2: $P(x) > B$

We begin by noting that

$$P(x) \leq \frac{T_0}{S_{x_0}} + \sup |\sigma E_x| \int_0^L S_x$$

$$= \frac{T_0}{S_{x_0}} + \sup |\sigma E_x| s(L)$$

implying the later expression is also greater than $B$ which yields the result

$$s(L) > L + n\zeta^n r_0 \tag{3.7}$$

Now let $k$ be the number of local extreme values of $r(x)$, including $r_0$. Proposition 1 implies $r(x)$ is monotone between any consecutive extreme values and therefore we find

$$s(L) = \int_0^L \sqrt{1 + (r'(x))^2}$$

$$\leq \int_0^L 1 + |r'(x)|$$

$$\leq L + r_0 + r_1 + \cdots + r_{k-1}$$

where $r_0, r_1, \ldots, r_{k-1}$ are the extreme values, and upon applying proposition 3 and the fact that $\alpha > 1$ we find

$$s(L) \leq L + r_0 + \alpha r_0 + \cdots + \alpha^{k-1} r_0$$

$$< L + k\zeta^k r_0$$

which when combined with (3) yields the result that $k > n$. Finally we apply proposition 1 to show that $r(x)$ must have more than $k$ (and therefore $n$) zeros in the interval $[0, L]$ which concludes the proof. □

### 3.1.3 Satisfying the Contraints

The next series of propositions serves to show the existence of paramaters and initial conditions which produce solutions to (2.2) that satisfy the physical constraints. The basic idea is to first show that the zeros of a given solution have continuous dependence on the parameters and initial conditions, which implies the length of the curve up to any zero also has continuous dependence. Then we show that there exist parameters which produce curve lengths on either side of $L_c$ and apply the continuity to find a point in between, such that the length is exactly equal to $L_c$. We begin with the continuity of the zeros.

**Proposition 6.** *Let $A$ be a set, $L_c > 0$ and let $\phi(x, \vec{\lambda})$ be a continuous function such that $\forall \vec{\lambda} \in A$ there are at least $n$ values of $x$, $\alpha_1(\vec{\lambda}), \ldots, \alpha_n(\vec{\lambda}) \in [0, L_c]$ where the function vanishes with multiplicity one, then they are all continuous functions of $\vec{\lambda}$ on $A$.*

*Proof.* Let $r, L \in (0, L_c]$ and $\vec{\lambda}_k$ be any sequence which converge to $\vec{\lambda}$, and let

$$\alpha_1^{(k)}, \ldots, \alpha_n^{(k)}$$

be the first $n$ zeros of each corresponding $\phi(x, \vec{\lambda}_k)$. By assumption we know each of these sequences is contained in $[0, L_c]$ and therefore they all have convergent subsequences. Our strategy will be to show that every convergent subsequence of $\alpha_i^{(k)}$ converges to $\alpha_1$, which would imply the sequence itself converges to $\alpha_i$. To accomplish this we begin by reindexing $k$ such that we have a convergent subsequence of $\alpha_i^{(k)}$, next we reindex $k$, $n-1$ more times such that we have convergent subsequences of all other $\alpha_j^{(k)}$'s while preserving the limit of our original subsequence of $\alpha_i^{(k)}$. Define these respective limits to be

$$\beta_1, \ldots, \beta_n$$

then by the continuity of $\phi(x, \vec{\lambda})$ and the fact that

$$\phi(\alpha_j^{(k)}, \vec{\lambda}_k) = 0$$

we find that,

$$\phi(\beta_j, \vec{\lambda}) = 0$$

i.e. each $\beta_j$ is a zero of $\phi(x, \vec{\lambda})$. Next we show that

$$0 < \beta_1 < \cdots < \beta_n$$

and there are no other zeros of $\phi(x, \vec{\lambda})$ between any two of these values. By definition of our sequences we know that

$$0 \leq \beta_1 \leq \cdots \leq \beta_n$$

and since $\phi(0, \vec{\lambda}) = r \neq 0$ we find $0 < \beta_1$. Now assume $\beta_j = \beta_{j+1}$, then by Rolle's Theorem there exists a sequence of values $\gamma_k$ such that

$$\phi_x(\gamma_k, \vec{\lambda}_k) = 0 \qquad , \qquad \alpha_j^{(k)} < \gamma_k < \alpha_{j+1}^{(k)}$$

but then by continuity and the squeeze theorem we find

$$\phi_x(\beta_j, \vec{\lambda}) = 0$$

which is a contradiction, since all zeros of $\phi(x, \vec{\lambda})$ have multiplicity one. Therefore we find

$$0 < \beta_1 < \cdots < \beta_n$$

Now to show there are no other zeros in this chain, define $\beta_0 = 0$ and assume there is another zero, $\alpha$, between any two consecutive values of $\beta_j$. Then since each $r_k$ is positive and there are the same number of sign changes of $\phi(x, \vec{\lambda}_k)$ from 0 to any point in the interval

$$\left(\alpha_j^{(k)}, \alpha_{j+1}^{(k)}\right)$$

we may assume without loss of generality that

$$\phi(x, \vec{\lambda}_k) > 0 \qquad \forall x \in \left(\alpha_j^{(k)}, \alpha_{j+1}^{(k)}\right)$$

as well as that we have indexed $k$ such that

$$\alpha_j^{(k)} < \beta_j + \epsilon < \alpha < \beta_{j+1} - \epsilon < \alpha_{j+1}^{(k)}$$

for some $\epsilon > 0$. Therefore

$$\phi(x, \vec{\lambda}_k) > 0 \qquad \forall x \in \left(\beta_j + \epsilon, \beta_{j+1} - \epsilon\right)$$

and in the limit we find

$$\phi(x, \vec{\lambda}) \geq 0 \qquad \forall x \in \left(\beta_j + \epsilon, \beta_{j+1} - \epsilon\right)$$

but this is a contradiction since $\alpha$ is in this interval it would imply it was a zero of higher multiplicity. This implies

$$\beta_1 = \alpha_1 \quad , \quad \dots \quad , \quad \beta_n = \alpha_n$$

and therefore we can conclude the limit of our original sequence $\alpha_i^{(k)}$ is $\alpha_i$, where $i$ is arbitrary, so we find each other original sequence converges to its corresponding zero as well so the continuity of the zeros follows. □

Next we show that for any natural number $n$ we may find parameters such that $L$ is the $n^{th}$ zero of $r$ and the path length of the solution $s(L) = L_c$. Let $\phi(x, r, L)$ be the solution to (2.2) with its dependence on the initial value $r(0) = r$ and the paramater $L$, let $s(x, r, L)$ be the path length function for the same corresponding solution, and let $\alpha_i(r, L)$ be the $x$ value of the $i^{\text{th}}$ zero of $\phi(x, r, L)$.

**Proposition 7.** *For any family of solutions to* (2.2) *such that* $\forall r, L \in (0, L_c]$, $\phi(x, r, L)$ *has* $n$ *zeros* $\alpha_1(r, L), \dots, \alpha_n(r, L) \in [0, L_M] \subset [0, L_c]$ *there exists values* $r_i, L_i \in (0, L_c), i = 1, \dots, n$ *such that* $L_i = \alpha_i(r_i, L_i)$ *and* $s(L_i, r_i, L_i) = L_c$.

*Proof.* We find these values as fixed points of an iterative process. Begin with any $L_i^{(1)} \in (0, L_c)$ and choose $r_i^{(1)}$ such that $s\left(\alpha_i\left(r_i^{(1)}, L_i^{(1)}\right), r_i^{(1)}, L_i^{(1)}\right) = L_c$ which is possible since

$$\sqrt{r^2 + (\alpha_i\,(r, L))^2} \leq s(\alpha_i(r, L), r, L) \leq \alpha_i(r, L) + i\zeta^i r$$

from the fact that the straight line distance between two points is a minimal path length and also from (3.7). Therefore if $r = L_c$ then

$$L_c \leq \sqrt{r^2 + (\alpha_i\,(r, L))^2} \leq s(\alpha_i(r, L), r, L)$$

and if

$$r = \frac{L_c - L_M}{2i\zeta^i}$$

then

$$s(\alpha_i(r, L), r, L) \leq \alpha_i(r, L) + i\zeta^i r < L_M + 2i\zeta^i r = L_c$$

and because $\alpha_i$ and $s$ are both continuous functions, their composition is continuous in $r$ and therefore by the Intermediate Value Theorem there exists an $r_1$ such that

$$s(\alpha_i(r_1, L_1), r_1, L_1) = L_c.$$

Then define $L_2 = \alpha_i(r_1, L_1)$ and then repeat the previous process with $L_2$, and continuing in this manner, we generate the bounded sequences of values

$$r_k, L_k \in (0, L_c)$$

which we assume to converge without loss of generality. Define $r_i$ and $L_i$ to be the respective limits and note by definition the sequences satisfy the following recursive relations:

$$L_{k+1} = \alpha_i(r_k, L_k) \quad \text{and} \quad s(L_{k+1}, r_k, L_k) = L_c.$$

Therefore by the continuity of $s$ and $\alpha_i$, we find in the limit:

$$L_i = \alpha_i(r_i, L_i) \quad \text{and} \quad s(L_i, r_i, L_i) = L_c.$$

□

### 3.1.4 Solution to the Singular IVP

The next set of propositions we prove shows that given a collection of solutions which satisfy the criteria of the previous propositions, we may construct a convergent sequence of solutions such that the limit is a solution to (2.4). Furthermore, this limit will also satisfy the physical constraints we imposed on each solution in the sequence. We begin with

**Proposition 8.** *There exists a convergent sequence of solutions whose limit function is a solution to* (2.4)*.*

*Proof.* Let $T_k$ be a sequence of positive numbers which converges to 0 and let

$$\phi(x, 0, r_k, L_k, T_k)$$

be a corresponding sequence of solutions from the family described in proposition 6 with initial data specified at $x = 0$, and $r_k, L_k$ chosen as in proposition 6 so that its conclusion holds for each of these pairs. Now since these sequences of parameters are bounded we assume without loss of generality that the sequences converge. Then we first show that there is a uniformly convergent subsequence of solutions by showing the functions and derivatives are universally bounded which together with the continuity implies equicontinuity and therefore the existence is guaranteed by the Arzelá–Ascoli theorem, which is outlined in an ODE textbook by Teschl, 2012. Proposition 3 tells us that

$$|\phi(x, 0, r_k, L_k, T_k)| \leq \zeta^n r_k < \zeta^n L_c$$

so we only need to show the derivatives are universally bounded. To this end we note that each derivative is continuous in their respective domains $[0, L_k]$, so they must take on a max value. At the endpoint where the function vanishes, we find from (2.2) that

$$\operatorname{sgn}(r'') = -\operatorname{sgn}(|E_x|r') = -\operatorname{sgn}(r')$$

which implies $r'r'' < 0$ in a neighborhood of this point, i.e. $|r'|$ is decreasing in this neighborhood, and therefore the max value does not occur here. The other endpoint as well as any critical points in the interior will all occur when

$$r' = \frac{\rho\omega^2 r + \sigma E_r}{E_x}$$

so

$$r' \leq \frac{\sup|\rho\omega^2 r + \sigma E_r|}{\inf|E_x|} = M$$

which shows the derivatives are universally bounded. Now assume without loss of generality that the sequence of functions converges. This fact also implies

$$r = |\int_0^{\alpha_i} \phi_x(x, 0, r, L, T)dx| \leq M\alpha_i$$

so for any zero of any solution to (2.2) we find

$$\alpha_i \geq \frac{r}{M} = \delta$$

This implies that $\delta$ is in the nontrivial domain of these functions. We therefore consider the sequences of values

$$r_\delta^{(k)} = \phi(\delta, 0, r_k, r_k', L_k, T_k) \quad \text{and} \quad r_{x_\delta}^{(k)} = \phi_x(\delta, 0, r_k, r_k', T_k)$$

where the former sequence converges since the function converges uniformly, and the second sequence is bounded so we assume without loss of generality that it converges. Now we note that by the uniqueness theorem of differential equations we have that

$$\phi(x, 0, r_k, r_k', L_k, T_k) = \phi(x, \delta, r_\delta^{(k)}, r_{x_\delta}^{(k)}, T_k)$$

everywhere that their domains overlap. Furthermore, because $\phi$ is a continuous function and because $F$ is continuous for all $x$ with the parameters $\delta, r_\delta, r_{x_\delta}, 0$ we find from ODE theory that the solution is continuous in a neighborhood of this point and therefore

$$\phi(x, \delta, r_\delta^{(k)}, r_{x_\delta}^{(k)}, T_k) \quad \text{converges uniformly to} \quad \phi(x, \delta, r_\delta, r_{x_\delta}, 0)$$

but then so does $\phi(x, 0, r_k, r_k', L_k, T_k)$ which shows this equation satisfies (2.4) everywhere on the interval of overlap and since it also satisfies the initial conditions at $x = 0$ it is a solution to the singular I.V.P. □

**Proposition 9.** *The limit solution to* (2.4) *described in proposition 7 inherits the properties described in proposition 6.*

*Proof.* Define the set $A$ to be the closure of

$$\{(L_k, r_k, r_k', T_k) | k = 1, 2, \dots\},$$

and note that for every isolated point $\vec{\lambda}$ in $A$

$$\phi(x, \vec{\lambda})$$

is a solution of (2.2) such that $L_k$ is equal to the $n^{th}$ zero, and we claim the limit point of $A$ also has $n$ zeros. To see this we note that for each $k$ the corresponding $n$ zeros are all bounded, so we assume we have chosen convergent sequences. Then the fact that the images of these sequences are all sequences of zeros implies each of these sequences converges to a zero of the singular solution corresponding to the limit point of $A$. Furthermore by the same argument used in Proposition 6 these must all converge to unique zeros, so there are at least $n$ of them. Thus we may apply Proposition 5 to see that the first $n$ zeros are all continuous functions of $\vec{\lambda}$ on $A$. So if $\vec{\lambda}_k$ is a sequence in $A$ converging to the limit point of $A$ then the corresponding sequence of values for the $n^{th}$ zero $\alpha_n(\vec{\lambda}_k)$, must converge to the $n^{th}$ zero of the singular solution, but $\alpha_n(\vec{\lambda}_k) = L_k$, so $L_k$ converges to the $n^{th}$ zero of the singular solution.

Next we show that the path length of the function from $x = 0$ to $x = L$ is equal to $L_c$. To see this note that the path length function $S$ is continuous by Proposition 1, and since

$$s(L_k, \vec{\lambda}_k) = L_c,$$

and the function $s$ itself converges, in the limit we see that

$$s(L, \vec{\lambda}) = L_c$$

which concludes the proof. □

We are now ready to prove our main result.

**Main Theorem.** *For any functions $E_x$, $E_r$, $\rho$, and $\sigma$ satisfying their respectively defined conditions, and for any $L_c$ and natural number $n$, if $\omega$ is chosen such that the* *conditions*

$$\omega^2 > \frac{2\pi^2(n+1)^2}{L_c \inf \rho} \sup |E_x|(1+n\zeta^n)$$
$$\omega^2 > \sup \left| \frac{2E_r}{r \inf \rho} \right|$$

*hold, then there exist n solutions to* (2.4) *for the corresponding pairs of values*

$$(r_1, L_1), \ldots, (r_n, L_n),$$

*each of which satisfy the conditions that* $L_i$ *is the* $i^{th}$ *zero of its corresponding solution* $\phi(x, r_i, L_i)$ *and the length of the curve up to* $L_i$ *is equal to* $L_c$.

$$s(L_i, r_i, L_i) = L_c.$$

*Proof.* Let

$$E_x \quad E_r \quad \rho \quad \sigma$$

all sitisfy their associated constraints, and let $L_c$ be any positive number and $n$ any natural number. Next define the value

$$\delta = \omega^2 - \frac{2\pi^2(n+1)^2}{L_c \inf \rho} \sup |E_x|(1+n\zeta^n),$$

which is positive for any $\omega$ satisfying the conditions of the theorem, and let $T_k$ be any sequence of positive numbers converging to zero, with the additional property that

$$T_k < \max \left\{ \frac{L_c^2 \delta^2 \inf \rho S_{x_0}}{2\pi^2(n+1)^2}, \inf |E_x| \left( \sqrt{\left(\frac{\delta}{2n\zeta^n}\right)^2 + L_c^2} - L_c \right) \right\} = T.$$

Define the function

$$f(x) = \frac{2\pi^2(n+1)^2}{x^2 \inf \rho} \left( \frac{T_k}{S_{x_0}} + \sup |E_x| \left[ x + n\zeta^n L_c \right] \right)$$

which is clearly continuous for all $x \in (0, \infty)$, and note that for every value $T_k$ it is true that

$$\begin{aligned} f(L_c) &= \frac{2\pi^2(n+1)^2}{L_c^2 \inf \rho} \sup |E_x| \left[ L_c + n\zeta^n L_c \right] + \frac{2\pi^2(n+1)^2}{L_c^2 \inf \rho} \frac{T_k}{S_{x_0}} \\ &< \frac{2\pi^2(n+1)^2}{L_c^2 \inf \rho} \sup |E_x| \left[ L_c + n\zeta^n L_c \right] + \delta \\ &= \omega^2. \end{aligned}$$

Furthermore, because this function diverges to positive infinity as $x$ converges to 0 there are values of $x$ for which $f(x) > \omega^2$, so by the continuity of $f$, we find a value

$$L_M \in (0, L_c)$$

such that

$$f(L_M) = \omega^2.$$

Now assume we only consider intial values $r_0$ such that

$$r_0 \in [\frac{L_c - L_M}{2n\zeta^n}, L_c),$$

then if we substitute the lower bound for $r_0$ into the following expression

$$\inf |E_x| \left( \sqrt{r_0^2 + L_c^2} - L_c \right)$$

we see that it equals $T$, so therefore

$$T_k < \inf|E_x| \left(\sqrt{r_0^2 + L_c^2} - L_c\right)$$

for all $k$ and all such values of $r_0$. Finally note that

$$\begin{aligned} f(L_M) &= \frac{2\pi^2(n+1)^2}{L_M^2 \inf\rho} \sup|E_x| \left[L_M + n\zeta^n L_c\right] + \frac{2\pi^2(n+1)^2}{L_M^2 \inf\rho} \frac{T_k}{S_{x_0}} \\ &> \frac{2\pi^2(n+1)^2}{L_M^2 \inf\rho} \sup|E_x| \left[L_M + n\zeta^n r0\right] + \frac{2\pi^2(n+1)^2}{L_M^2 \inf\rho} \frac{T_k}{S_{x_0}} \end{aligned}$$

but $f(L_M) = \omega^2$, so it follows that

$$\omega^2 > \frac{2\pi^2(n+1)^2}{L_M^2 \inf\rho} \sup|E_x| \left[L_M + n\zeta^n r0\right] + \frac{2\pi^2(n+1)^2}{L_M^2 \inf\rho} \frac{T_k}{S_{x_0}}$$

and therefore we may apply Propositions 5 and 6 to show that for any values of $L$, any $T_k$ in our sequence and all

$$r_0 \in [\frac{L_c - L_M}{2n\zeta^n}, L_c),$$

the corresponding solutions to (2.2) will have at least $n$ zeros in $[0, L_M]$ and they are all continuous functions of these values. Now consider a solution for fixed $T_k$, and for every value $i = 1, \ldots, n$ apply Proposition 7 to find the associated values

$$r_i^{(k)} \quad \text{and} \quad L_i^{(k)}$$

such that the corresponding solutions all satisfy the conditions

$$\alpha_i\left(r_i^{(k)}, L_i^{(k)}, T_k\right) = L_i^{(k)} \quad \text{and} \quad s\left(\alpha_i^{(k)}, r_i^{(k)}, L_i^{(k)}\right) = L_c.$$

Then we may apply Propositions 8 and 9 to show that there are convergent subsequence of each of these solutions, which converge uniformly to solutions of the (2.4), and furthermore the limit solutions will inherit the properties

$$\alpha_i\left(r_i, L_i, 0\right) = L_i \quad \text{and} \quad s\left(\alpha_i, r_i, L_i, 0\right) = L_c.$$

$\square$

# Chapter 4

# NUMERICAL VALIDATION

## 4.1 Gravitaional Example

We now wish to see what these solutions look like for a particular force field. We will show the numerical solution curves for constant force field (which approximates gravity on the surface of the Earth). In this case the force depends on mass so that

$$\sigma = \rho.$$

For a constant field pointing downwards with acceleration $g = 9.81$ (units are not important for mathematical verification, so assume any units you want) we find the components of the field are simply

$$E_x = -9.81 \quad \text{and} \quad E_r = 0,$$

which can easily be verified to satisfy the requirements from the theory. Finally we define our string to have length $L_c = 1$ and density function

$$\rho = 1.$$

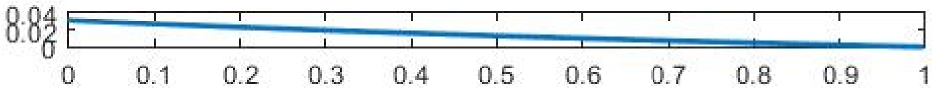


FIGURE 4.1: n = 1

These functions as well can be verified to satisfy the requirements of the theorem and for a fixed value of $T_0 = 0.000001$ and $\omega = 20.944$ they produce the following curves for all $n$ up to 4. The resulting values of $r$ and $L$ which make these solutions satisfy the physical constraints were $(0.9974, 0.0311)$ for $n = 1$, $(0.3211, 0.3326)$ for $n = 2$, $(0.1610, 0.6663)$ for $n = 3$, and $(0.0631, 0.9256)$ for $n = 4$. These solutions were generated using a fourth order Runge-Kutta method with a step size of 0.0001. The following figures display these curves, which are the rigid shapes a string with the given density function and length would form in this given field.

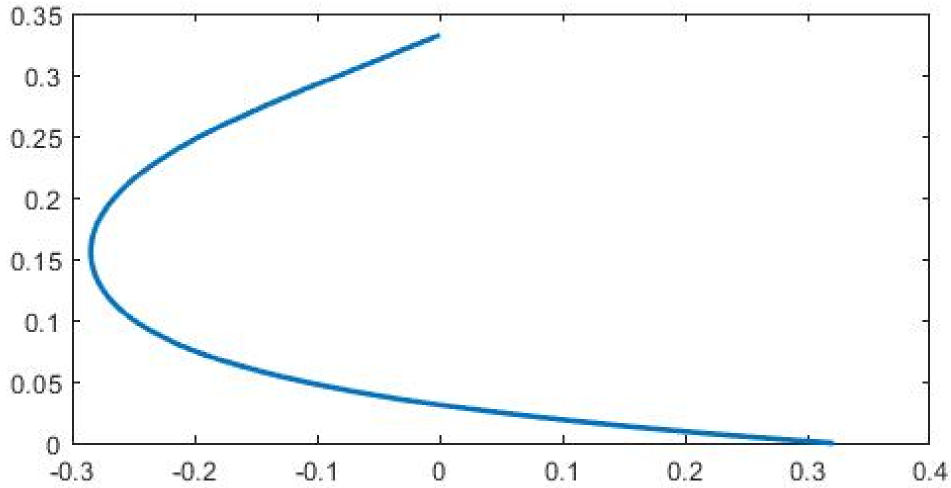


FIGURE 4.2: n = 2

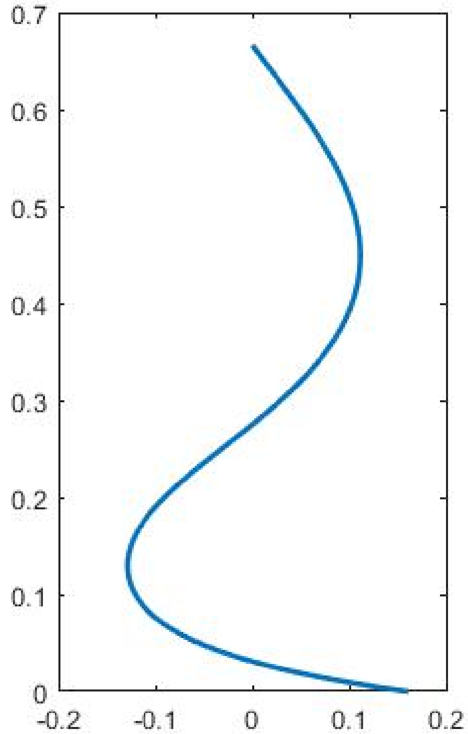


FIGURE 4.3: n = 3

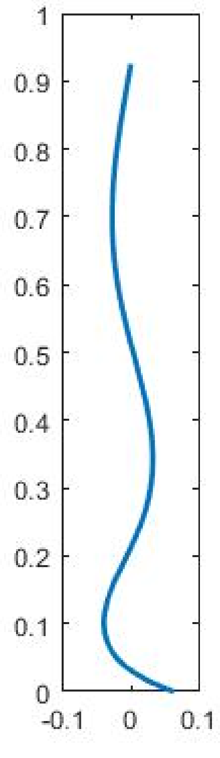


FIGURE 4.4: n = 4

# Chapter 5

# CONCLUSION

## 5.1 Overview of Results

We have demonstrated that for any force field satisfying certain conditions there are rigid equilibriums of rotating strings. Furthermore, we demonstrated that there exist sufficient conditions on the rotational rate of the string which will guaruntee solutions of any number of oscillations along the length of the string. Next we elaborate on the results which we did not cover in this paper.

### 5.1.1 The Closing Picture

The results of this paper display sufficient conditions for the rigid equilibriums, and therefore there is the question of whether or not these conditions are necessary. It turns out that the results of Proposition 2, which showed a given solution can only vanish finitely many times in any finite interval, shows that depending on the radial component of the force field, there exist certain necessary conditions on the rotational rate required for configurations with a given number of nodes. We did not have time to cover these results as they depend on the radial component of the field for this thesis, so we do not offer a proof of this claim. Another question is whether or not the solutions are unique for a given number of oscillations of a particular string in a particular field. One idea for how this may be added to our argument, would be to show that the path length to any one of the zeros of a solution is a monotone function of the initial value $r_0$. This would imply for any fixed parameters there would be only one value at which $r_0$ would be the right size to yield the correct path length $L_c$ of the solution, but since we are also varying $T_0$, and $L$ in the process of creating solutions, it is not clear whether we could find multiple values of $L$ and/or $T_0$ such that there were seperate (and unique to their corresponding parameters) values of $r_0$ which each produced different curves with the same number of oscillations. Other areas this could be extended would be to investigate fields where the force is allowed to vanish or even change directions. It would also be interesting to consider the case in which the bottom end of the string was not free (i.e. $T_0 \neq 0$) which could describe a string with a weight attached to the end or some kind of force being applied directly. In this case we would gain an extra degree of freedom in our initial conditions, since the singular condition specified $r_0'$, and therefore it seems more likely that there could be more than one configuration with a given number of oscillations (although physical intuition also seems to suggest they should behave similarly to the $T_0$ case just with the endpoint pulled down farther). Also the stability of these sort of fixed points of the PDE could be investigated in order to determine which configuration it was most likely to settle into under given conditions. There are many directions which this paper could be extended, indeed strings have served as a fruitful source of mathematics for eager minds of past and present.

# Bibliography

Dmitrochenko, Oleg, WanSuk Yoo, and Dmitry Pogorelov (2006). “Helicoseir as Shape of a Rotating String (I): 2D Theory and Simulation Using ANCF”. In: *Multibody System Dynamics* 15.2, pp. 135–158.

Gómez, B J et al. (2007). “Oscillations of a string with concentrated masses”. In: *European Journal of Physics* 28.5, pp. 961–975. DOI: 10.1088/0143-0807/28/5/019. URL: https://doi.org/10.1088%2F0143-0807%2F28%2F5%2F019.

Kolodner, Ignace I. (1955). “Heavy rotating string—a nonlinear eigenvalue problem”. In: *Communications on Pure and Applied Mathematics* 8.3, pp. 395–408.

Noël, Jean-Marc et al. (2008). “Natural configurations and normal frequencies of a vertically suspended, spinning, loaded cable with both extremities pinned”. In: *European Journal of Physics* 29.5, N47–N53. DOI: 10.1088/0143-0807/29/5/n02. URL: https://doi.org/10.1088%2F0143-0807%2F29%2F5%2Fn02.

Teschl, Gerald (2012). *Ordinary differential equations and dynamical systems*. American Mathematical Society.

Verbin, Y (2014). “Boundary conditions and modes of the vertically hanging chain”. In: *European Journal of Physics* 36.1, p. 015005.