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# Effect of cross-diffusion on the stability of a triple-diffusive Oldroyd-B fluid layer

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**Abstract.** The onset and stability of a triple cross-diffusive viscoelastic fluid layer is investigated. The rheology of viscoelastic fluid is approximated by the nonlinear Oldroyd-B constitutive equation which encompasses Maxwell and Newtonian fluid models as special cases. By performing the linear instability analysis, analytical expression for the occurrence of stationary and oscillatory convection is obtained. The numerical results show that the elasticity and cross-diffusion effects reinforce together in displaying complex dynamical behavior on the system. The presence of cross-diffusion is found to either stabilize or destabilize the system depending on the strength of species concentration as well as elasticity of the fluid and also alters the nature of convective instability. The disconnected closed oscillatory neutral curve lying well below the stationary neutral curve is observed to be convex in its shape in contrast to quasiperiodic bifurcation from the quiescent basic state noted in the case of Newtonian fluids. This striking feature is attributed to the viscoelasticity of the fluid. By performing a weakly nonlinear stability analysis, the stability of bifurcating solution is discussed. It is worth reporting that the viscoelastic parameters significantly influence the stability of stationary bifurcation though the stationary onset is unaffected by viscoelasticity. Besides, subcritical instability is occurs and the critical Rayleigh number at which such an instability is possible decreases in the presence of cross-diffusion terms. The results of Maxwell and Newtonian fluids are delineated as particular cases from the present study.

**Mathematics Subject Classification.** 70K20, 70K50, 34A34, 76E06, 76R50, 76A10.

**Keywords.** Instability, Cross-diffusion terms, Triple-diffusive convection, Nonlinear stability, Bifurcation, Perturbation method.

## 1. Introduction

In liquid mixtures, the diffusion of any species depends merely on its concentration gradient rather than on the spatial distribution of other species. In single component systems, the diffusion seems to be a simple process while in multicomponent systems it is not so. A range of possibilities becomes obvious when the diffusion process involves the contribution of two or more diffusive agents. In such cases, variety of phenomena like self-diffusion, intra-diffusion, inter-diffusion, tracer diffusion, uphill diffusion, mutual diffusion and cross-diffusion are possible to occur. The cross-diffusion is a phenomenon in which the concentration gradient of one species induces a flux of the other species. In the absence of chemical reactions, the cross-diffusion induces convective motions around liquid interfaces. On the other hand, cross-diffusion is also responsible for the processes like Chemotaxis, weakly non-bonding solute–solute interactions, electrostatic, etc. The possibility of cross-diffusion terms in multicomponent systems was suggested by Onsager and Fuoss [1], while Baldwin et al. [2] undertook the experimental verification of the existence of cross-diffusion and also observed that the cross-diffusion coefficients can be quite significant. The motion of particles in the solution mainly depends on the magnitude of diffusion coefficients, which in turn depends on the composition of solution and solute concentration gradients. For determining the diffusion coefficients, a suitable frame of reference is essential. The solvent-fixed reference frame, the mass-fixed reference frame and the volume-fixed reference frame are three main reference frames, out of which the last one is most suitable for laboratory experiments. When the concentrations are low the interactions

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between particles become negligible; then, the self-diffusion occurs and in the diffusion coefficients matrix the diagonal elements tend to self-diffusion coefficients. These coefficients do not depend on the choice of reference frame. The relationship between self and cross-diffusion coefficients was first determined by Mimura and Kawasaki [3].

The study of convective instability in a system having two diffusing components with dissimilar molecular diffusivities has been a topic of great theoretical and experimental interest. Excellent documentation of the studies pertaining to convection in two component fluid systems was done by Turner [4], Huppert and Turner [5], Platten and Legros [6] and Garaud [7]. The presence of more than two diffusing components with different molecular diffusivities is witnessed in several natural and industrial fluid systems which give rise to convective instabilities called multicomponent convection. The study of onset of convection in the multicomponent solutions (solvent with multiple solutes) finds comprehensive applications in numerous fields like geophysics, soil sciences, oceanography, limnology, geothermally heated lakes, magmas, sea water, food processing, high-quality material production, solidification of molten alloys, chemical engineering, oil reservoir engineering and so on [8–11].

In any fluid system with $n$-species, the diffusion processes can be described by generalized Fick's law

$$F_i = -\sum_{j=1}^{n} D_{ij} \Delta S_j,$$

which indicates that the flux $F_i$ of $i$th species depends on the concentration gradients ($\Delta S_j$ ) of all the species. For $j = i$, $D_{ii}$ the diagonal elements of the diffusion matrix, represents the self-diffusion coefficients, while the off-diagonal elements $D_{ij}$ corresponding to $j \neq i$ specify the cross-diffusion coefficients. The motion of a species due to its concentration gradient leads to the flux of the other species either along or against its direction of motion. Based on this, the cross-diffusion coefficients can either be positive (co-flux) or negative (counter-flux). The cross-diffusion terms in the diffusivity matrix control the instability of the system considerably, they are commonly unnoticed on the basis of the often repeated slogan that these are of smaller magnitude compared with the main diagonal elements. But in some liquid mixtures, cross-diffusion (off-diagonal elements) terms are found to be much larger than self-diffusion (main diagonal elements) terms [12,13].

There is an abundant of literature present on triple-diffusive convection in a horizontal layer of Newtonian fluid [14–18]. The multicomponent convection in a Newtonian fluid layer with the Soret effect was studied by Ryzhkov and Shevtsova [19]. Shivakumara and Naveen Kumar [20] investigated linear and weakly nonlinear convection in a triple-diffusive couple stress fluid layer. The majority of studies on triple-diffusive convection have been dealt with Newtonian fluids. However, to account for rheological behavior of complex flow phenomena which arise in plenty of fluid mixtures such as polymer solutions, melts and paints involving more than two diffusing agents the Newtonian fluid model turns out to be inadequate. In such cases, the usage of alternative non-Newtonian model particularly viscoelastic model is preferred. Viscoelastic fluids show both viscous (as that of fluids) and elastic (as that of solids) behavior. These elastic effects are responsible for more complicated rheological behaviors of such fluids. Owing to the elasticity of such fluids, the onset of thermal convection in a viscoelastic fluid layer is found to be via oscillatory mode instead of stationary mode obvious in Newtonian fluids. Ample literature can be found on convective instability in a single [21–24] and double diffusive [25–29] viscoelastic fluid layer.

Nonetheless, many fluid dynamical systems of practical importance such as pharmaceutical and petroleum industries, cosmetics, bioengineering and polymer processing involve non-Newtonian fluids containing multicomponent systems wherein the fluxes of one component will be affecting the other. More specifically, viscoelastic fluids aptly describe the rheology of fluids existing in the above said applications. To the best of our knowledge, triple-diffusive convection in a viscoelastic fluid layer has not received any attention in the literature. The intent of the present study is to investigate the onset and stability of triple-diffusive convection in a viscoelastic fluid layer accounting for cross-diffusion effects. The constitutive equation of stress is taken to correspond to an Oldroyd-B type of viscoelastic fluid. The stability analyses have been carried out for the case of stress-free boundaries to allow analytical inroads into the problem. Of course, no-slip conditions are potentially natural, but they are not amenable to tackle the problem analytically. The similarities and differences between viscoelastic (Maxwell and Oldroyd-B fluids) and Newtonian fluids as well as the presence and absence of cross-diffusion effects on the instability and stability characteristics of the system are analyzed in detail. Most importantly, the quasiperiodic bifurcation (exact heart-shaped disconnected oscillatory neutral curves having the same extrema at different wave numbers) from the quiescent basic state observed in the case of Newtonian fluids is found to be not carrying over to the case of viscoelastic fluids. The stability of bifurcating equilibrium solution is discussed by employing a weakly nonlinear stability analysis, and subcritical instability is found to occur depending on the choice of physical parameters.

## 2. Governing equations

The physical set up consists of a horizontal triple-diffusive layer of an Oldroyd-B fluid, which is of finite height $d$ but of infinite length and breadth. The upper and lower bounding surfaces of the fluid layer are flat, stress-free which are maintained at constant but different species concentrations $S_m$ $(m = 1, 2, 3)$ such that $S_m = S_{mL}$ at the lower boundary and $S_m = S_{mL} + \Delta S_m$ at the upper boundary with $\Delta S_m > 0$. A Cartesian reference frame is so chosen that the origin lies at the lower boundary and $z$-axis vertically upwards in the opposite direction of gravity field. The density $\rho$ depends on three different stratifying agents possessing different molecular diffusivities and the flux of one species affects due to concentration gradient of the other, i.e., the cross-diffusion is taken into consideration. The Boussinesq approximation according to which all thermo-physical properties except the density in the term corresponding body force are invariant, is invoked.

The governing equations are

$$\nabla \cdot \mathbf{q} = 0, \tag{1}$$

$$\rho_o \left[ \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\, \mathbf{q} \right] = -\nabla p + \nabla \cdot \underset{\sim}{\tau} + \rho \mathbf{g}, \tag{2}$$

$$\frac{\partial S_m}{\partial t} + (\mathbf{q} \cdot \nabla)\, S_m = \sum_{k=1}^{3} D_{mk} \nabla^2 S_k \quad (m = 1, 2, 3), \tag{3}$$

$$\rho = \rho_o \left[ 1 + \sum_{m=1}^{3} \alpha_{Sm} \left( S_m - S_{mL} \right) \right], \tag{4}$$

where $\mathbf{q} = (u, v, w)$ denotes the velocity vector, $p$ the pressure, $\underset{\sim}{\tau}$ the extra stress tensor, $\mathbf{g} = (0, 0, g)$ the gravitational acceleration, $\rho$ the fluid density, $\rho_0$ is the reference density at $S_m = S_{mL}$, $D_{mk}$'s are solute diffusivities, $\alpha_{Sm}$ the volumetric expansion coefficient of $m$th species. The rheological characteristics of polymer liquids can be well depicted by using a nonlinear Oldroyd-B constitutive relation [21],

$$\underset{\sim}{\tau} + \lambda_1 \left[ \frac{\partial \underset{\sim}{\tau}}{\partial t} + (\mathbf{q} \cdot \nabla)\underset{\sim}{\tau} - (\nabla \mathbf{q})^T \underset{\sim}{\tau} - \underset{\sim}{\tau}(\nabla \mathbf{q}) \right] = \mu \left\{ \underset{\sim}{A} + \lambda_2 \left[ \frac{\partial \underset{\sim}{A}}{\partial t} + (\mathbf{q} \cdot \nabla)\underset{\sim}{A} - (\nabla \mathbf{q})^T \underset{\sim}{A} - \underset{\sim}{A}(\nabla \mathbf{q}) \right] \right\}, \tag{5}$$

where $\mu$ is the fluid viscosity, $\underset{\sim}{A} = \nabla \mathbf{q} + (\nabla \mathbf{q})^T$ is the rate-of-strain tensor, $\lambda_1$ is the relaxation time, $\lambda_2$ is the retardation time. Equation (5) includes Newtonian fluid ($\lambda_1 = \lambda_2 = 0$) and the Maxwell fluid ($\lambda_2 = 0$) models as particular cases.

For simplicity and with the object of obtaining the solution in the closed form, the boundaries are considered to be flat, stress-free and perfect conductors of species concentrations. Moreover, the previous studies on similar types of problems have been revealed that change in boundary conditions to the case of rigid boundaries rarely leads to any fundamental differences in the results and in the vast majority of cases, they lead to quantitative differences only. Hence, the relevant boundary conditions are

$$\left.\begin{array}{l} w = \dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = 0 \quad \text{at} \quad z = 0, \;\; d \\ S_m = S_{mL} \;\; \text{at} \;\; z = 0 \;\; \text{and} \;\; S_m = S_{mL} + \Delta S_m \;\; \text{at} \;\; z = d \;\; (m = 1, 2, 3) \end{array}\right\}. \tag{6}$$

At the basic state, the fluid is at rest and the gradients of stratifying agents exist only along the vertical direction, so that

$$\mathbf{q}_b = 0, \underset{\sim}{\tau_b} = 0, S_{mb} = S_{mL} + \frac{\Delta S_m}{d} z \;\; (m = 1, 2, 3), p_b = p_o - \rho_o g \sum_{m=1}^{3} \alpha_{Sm} \left( S_{mL}\, z + \frac{\Delta S_m}{2d} z^2 \right). \tag{7}$$

where the subscript $b$ denotes the basic state, $p_0$ is the pressure at $z = 0$. The finite amplitude perturbations (primed quantities given below) are superimposed on the basic state in the form

$$\mathbf{q} = \mathbf{q}_b + \mathbf{q}', \; p = p_b + p', \; \rho = \rho_b + \rho', \; \underset{\sim}{\tau} = \underset{\sim}{\tau_b} + \underset{\sim}{\tau}', \; S_m = S_{mb} + S_m' \;\; (m = 1, 2, 3). \tag{8}$$

Consequently, the governing Eqs. (1)–(5) are simplified and rendered dimensionless using

$$\nabla^* = d\, \nabla, \; t^* = \frac{D_{11}}{d^2} t, \mathbf{q} = \frac{d}{D_{11}} \mathbf{q}, \; (p^*, \underset{\sim}{\tau}^*) = \frac{d^2}{\mu D_{11}} (p, \underset{\sim}{\tau}), \; S_m^* = \frac{\alpha_{Sm} g d^3}{\nu D_{11}} S_m \;\; (m = 1, 2, 3). \tag{9}$$

So that they appear like (on omitting the asterisks)

$$\nabla \cdot \mathbf{q} = 0, \tag{10}$$

$$\frac{1}{Pr} \left[ \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\, \mathbf{q} \right] = -\nabla p + \nabla \cdot \underset{\sim}{\tau} - \sum_{m=1}^{3} S_m \hat{k}, \tag{11}$$

$$\frac{\partial S_m}{\partial t} + (\mathbf{q} \cdot \nabla)\, S_m + R_{Sm} w = \sum_{k=1}^{3} \gamma_{mk} \nabla^2 S_k \;\; (m = 1, 2, 3), \tag{12}$$

$$\underset{\sim}{\tau} + \Lambda_1 \left[ \frac{\partial \underset{\sim}{\tau}}{\partial t} + (\mathbf{q} \cdot \nabla) \underset{\sim}{\tau} - (\nabla \mathbf{q})^T \underset{\sim}{\tau} - \underset{\sim}{\tau} (\nabla \mathbf{q}) \right] = \underset{\sim}{A} + \Lambda_2 \left[ \frac{\partial \underset{\sim}{A}}{\partial t} + (\mathbf{q} \cdot \nabla) \underset{\sim}{A} - (\nabla \mathbf{q})^T \underset{\sim}{A} - \underset{\sim}{A} (\nabla \mathbf{q}) \right], \tag{13}$$

where $Pr = \nu / D_{11}$ is generalized the Prandtl number, $R_{Sm} = \alpha_{Sm} g d^3 \Delta S_m / \nu D_{11}$ $(m = 1, 2, 3)$ are the Rayleigh numbers, $\gamma_{mk} = \alpha_{Sm} D_{mk} / \alpha_{Sk} D_{11}$ $(m = 1, 2, 3)$ are the diffusivity-expansion coefficient ratios, $\Lambda_1 = \lambda_1 D_{11} / d^2$ is the relaxation parameter, and $\Lambda_2 = \lambda_2 D_{11} / d^2$ is the retardation parameter.

The analysis is restricted to two-dimensional motions, and the stream function $\psi(x, z, t)$ is introduced in such a way that

$$u = \psi_{,z}, \; w = -\psi_{,x}. \tag{14}$$

Eliminating the pressure term from the momentum equation by operating the curl and using the basic state solutions, one can obtain

$$\frac{1}{Pr} L_1 (\nabla^2 \psi) - (S_1 + S_2 + S_3)_{,x} - N = 0, \tag{15}$$

$$L_1 (S_m) - R_{Sm} \psi_{,x} - \sum_{k=1}^{3} \gamma_{mk} \nabla^2 S_k = 0 \; (m = 1, 2, 3), \tag{16}$$

where $L_1(\cdot) = \partial(\cdot)/\partial t + J(\cdot, \psi)$ is a nonlinear differential operator, $N = (\tau_{xx} - \tau_{zz})_{,xz} + \tau_{xz,zz} - \tau_{xz,xx}$ and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2$. Further, Eq. (13) in the component form can be written as

$$\tau_{xz} + \Lambda_1 \left\{ L_1(\tau_{xz}) + \frac{1}{2} \nabla^2 \psi \;\; U - \frac{1}{2} \Delta_1 \psi \;\; V \right\} = \Delta_1 \psi + \Lambda_2 \left\{ L_1(\Delta_1 \psi) + 2 \psi_{,xz} \;\; \nabla^2 \psi \right\}, \tag{17}$$

$$U + \Lambda_1 \left\{ L_1(U) - \nabla^2\psi \ \ \tau_{xz} - 2\psi_{,xz} \ \ V \right\} = 4\psi_{,xz} + \Lambda_2 \left\{ 4L_1(\psi_{,xz}) - 2\nabla^2\psi \ \ \Delta_1\psi \right\}, \tag{18}$$

$$V + \Lambda_1 \left\{ L_1(V) - 2\psi_{,xz} \ \ U - 2\Delta_1\psi \ \ \tau_{xz} \right\} = -2\Lambda_2 \{ 4(\psi_{,xz})^2 + (\Delta_1\psi)^2 \}, \tag{19}$$

where $U = \tau_{xx} - \tau_{zz}$, $V = \tau_{xx} + \tau_{zz}$ and $\Delta_1\psi = \psi_{,zz} - \psi_{,xx}$.

The appropriate boundary conditions are

$$\psi = \psi_{,zz} = S_m \ (m = 1, 2, 3) = \tau_{xz} = U_{,z} = 0 \quad \text{at } z = 0, 1. \tag{20}$$

## 3. Linear instability analysis

The nonlinear terms in Eqs. (15)–(19) are ignored, and the linear stability equations are found to be

$$\frac{1}{Pr}\nabla^2\psi_{,t} - (S_1 + S_2 + S_3)_{,x} - N = 0, \tag{21}$$

$$(S_m), t - R_{Sm}\psi_{,x} - \sum_{k=1}^{3} \gamma_{mk}\nabla^2 S_k = 0 \quad (m = 1, 2, 3), \tag{22}$$

$$\tau_{xz} + \Lambda_1(\tau_{xz})_{,t} = \Delta_1\psi + \Lambda_2\Delta_1\psi_{,t}, \tag{23}$$

$$U + \Lambda_1 U_{,t} = 4\psi_{,xz} + \Lambda_2\psi_{,xzt}. \tag{24}$$

The normal mode analysis warrants that the perturbed quantities can be expressed as

$$\psi = Ae^{\sigma t}\sin\alpha x \sin\pi z, \qquad S_m = B_m e^{\sigma t}\cos\alpha x \sin\pi z \quad (m = 1, 2, 3), \tag{25}$$

where $A$ and $B_1 - B_3$ are constants, $\alpha$ is the horizontal wave number, $\pi$ is the vertical wave number, and $\sigma = \sigma_r + i\omega$ is the growth term. On using Eq. (25) into Eqs. (21)–(24), one gets

$$a_1\sigma^5 + a_2\sigma^4 + a_3\sigma^3 + a_4\sigma^2 + a_5\sigma + a_6 = 0, \tag{26}$$

where

$$\begin{aligned}
a_1 =& \delta^2\Lambda_1, \\
a_2 =& \delta^2 + \delta^4\left[(\gamma_{11} + \gamma_{22} + \gamma_{33})\Lambda_1 + Pr\Lambda_2\right], \\
a_3 =& Pr\delta^4 - Pr\alpha^2 (R_{S1} + R_{S2} + R_{S3})\Lambda_1 + \delta^4\left(1 + Pr\delta^2\Lambda_2\right)(\gamma_{11} + \gamma_{22} + \gamma_{33}) \\
&+ \delta^2\Lambda_1 b_4, \\
a_4 =& - Pr\alpha^2 (R_{S1} + R_{S2} + R_{S3})\Lambda_1 + Pr\alpha^2\delta^2\Lambda_1 (b_1 R_{S1} + b_2 R_{S2} + b_3 R_{S3}) \\
&+ \delta^6 (b_4 + Pr(\gamma_{11} + \gamma_{22} + \gamma_{33})) + \delta^8 (b_4 Pr\Lambda_2 + b_8\Lambda_1), \\
a_5 =& Pr\alpha^2\delta^6\left[\left(b_1 - \delta^2 b_5\right)R_{S1} + \left(b_2 - \delta^2 b_6\right)R_{S2} + \left(b_3 - \delta^2 b_7\right)R_{S3}\right] \\
&+ \delta^8 (Prb_4 + b_8) + \delta^{10} Pr\Lambda_2 b_8, \\
a_6 =& - Pr\alpha^2\delta^4 (b_5 R_{S1} + b_6 R_{S2} + b_7 R_{S3}) + Pr\delta^{10} b_8,
\end{aligned}$$

with

$$\begin{aligned}
b_1 =& \gamma_{21} - \gamma_{22} + \gamma_{31} - \gamma_{33}, \\
b_2 =& \gamma_{32} - \gamma_{33} + \gamma_{12} - \gamma_{11}, \\
b_3 =& \gamma_{13} - \gamma_{11} + \gamma_{23} - \gamma_{22}, \\
b_4 =& \gamma_{11}\gamma_{22} + \gamma_{22}\gamma_{33} + \gamma_{33}\gamma_{11} - \gamma_{12}\gamma_{21} - \gamma_{23}\gamma_{32} - \gamma_{31}\gamma_{13},
\end{aligned}$$

$$
\begin{aligned}
b_5 &= \gamma_{21}\left(\gamma_{32}-\gamma_{33}\right)+\gamma_{22}\left(\gamma_{33}-\gamma_{31}\right)+\gamma_{23}\left(\gamma_{31}-\gamma_{32}\right), \\
b_6 &= \gamma_{32}\left(\gamma_{13}-\gamma_{11}\right)+\gamma_{33}\left(\gamma_{11}-\gamma_{12}\right)+\gamma_{31}\left(\gamma_{12}-\gamma_{13}\right), \\
b_7 &= \gamma_{13}\left(\gamma_{21}-\gamma_{22}\right)+\gamma_{11}\left(\gamma_{22}-\gamma_{23}\right)+\gamma_{12}\left(\gamma_{23}-\gamma_{21}\right), \\
b_8 &= \gamma_{11}\left(\gamma_{22}\gamma_{33}-\gamma_{23}\gamma_{32}\right)+\gamma_{12}\left(\gamma_{23}\gamma_{31}-\gamma_{21}\gamma_{33}\right)+\gamma_{13}\left(\gamma_{21}\gamma_{32}-\gamma_{22}\gamma_{31}\right) \\
&= \left|\left(\gamma_{ij}\right)\right|, \\
\delta^2 &= \alpha^2+\pi^2.
\end{aligned}
$$

To perform the linear instability analysis, we set the real part of $\sigma$ to zero and then on using the condition for the existence of nonzero solution of the system (Eqs. (21)–(24)) we obtain

$$
R_{S1} = f_1\left(\omega^2,\alpha^2;\varLambda_1,\varLambda_2,R_{S2},R_{S3},\gamma_{ij},Pr\right)+i\omega\delta^2 f_2\left(\omega^2,\alpha^2;\varLambda_1,\varLambda_2,R_{S2},R_{S3},\gamma_{ij},Pr\right), \tag{27}
$$

where $f_1$ and $f_2$ are real valued functions of known quantities and they are not given here as the mathematical expressions are very lengthy. Since $R_{S1}$ is a physical quantity, it must be real so that either $\omega=0$ or $f_2=0$.

### 3.1. Stationary convection

When $\omega=0$ in Eq. (27), the instability is referred to as stationary convection and it is characterized by the Rayleigh number

$$
R^s_{S1} = \frac{1}{b_5}\left(\frac{\delta^6}{\alpha^2}b_8-b_6R_{S2}-b_7R_{S3}\right). \tag{28}
$$

This expression is free from viscoelastic parameters and coincides with the Newtonian fluid case (Terrones [18]). The stationary Rayleigh number $R^s_{S1}$ attains its critical value at $\alpha=\pi/\sqrt{2}$ and the critical Rayleigh number for the stationary onset is

$$
R^s_{S1c} = \frac{1}{b_5}\left(\frac{27\pi^4}{4}b_8-b_6R_{S2}-b_7R_{S3}\right). \tag{29}
$$

### 3.2. Oscillatory convection

When $\omega\neq 0$ in Eq. (27), then $f_2=0$ and this condition gives a dispersion relation of the form

$$
m_1\left(\omega^2\right)^3+m_2\left(\omega^2\right)^2+m_3\left(\omega^2\right)+m_4=0, \tag{30}
$$

where $m_1-m_4$ are functions of $\alpha$, $Pr$, $\varLambda_1$, $\varLambda_2$, $R_{S2}$, $R_{S3}$, $\gamma_{ij}$ and they are not presented here as these expressions are lengthy. For a proper combination of physical parameters, it is feasible to have either one or two or three values of $\omega^2$ at the same wave number $\alpha$. In such cases, for each $\omega^2$, there is a corresponding real value of the Rayleigh number on the oscillatory neutral curve given by

$$
R^o_{S1} = f_1\left(\omega^2,\alpha^2;\varLambda_1,\varLambda_2,R_{S2},R_{S3},\gamma_{ij},Pr\right). \tag{31}
$$

and $\omega^2$ is given by (30). There is no simple way to analyze Eq. (30) to extract positive roots, but one has to solve it numerically for the chosen parametric values of $Pr$, $\varLambda_1$, $\varLambda_2$, $R_{S2}$, $R_{S3}$ and $\gamma_{ij}$. The critical value of $R^o_{S1}$ with respect to the wave number, denoted by $R^o_{S1c}$, is determined as follows. First, the positive values of $\omega^2$ are determined from Eq. (30) and if there are none, then no oscillatory convection is possible. If there is only one positive value of $\omega^2$ then $R^o_{S1}$ is computed numerically from Eq. (31). If there are two or more positive values of $\omega^2$, then the least of $R^o_{S1}$ among positive $\omega^2$ is retained and the critical value of $R^o_{S1}$ with respect to the wave number is obtained.

## 4. Weakly nonlinear stability analysis

The aim of weakly nonlinear stability analysis is to provide quantitative results regarding the amplitude of convection and also the stability of stationary bifurcation. The regular perturbation method is used by introducing a small bifurcation parameter

$$\chi = \left[\left(R_{S1} - R_{S1c}^{s}\right)/R_{S12}\right]^{\frac{1}{2}},$$

that indicates the deviation from the critical state. The bifurcation is said to be subcritical if $R_{S12} < 0$ and supercritical if $R_{S12} > 0$. Then, all the dependent variables are expanded in powers of $\chi$ in the form

$$(\psi,\ S_m,\ U,\ V,\ \tau_{xz}) = \sum_{n=1}^{\infty} \left(\psi_n,\ S_{mn},\ U_n,\ V_n,\ \tau_{xz}^{(n)}\right)\chi^n, \quad (m = 1, 2, 3). \tag{32}$$

A small time scale $s = \chi^2 t$ is also introduced, and the operator $\partial/\partial t$ is replaced by $\partial/\partial t = \chi^2 \partial/\partial s$. Substituting Eq. (32) in to Eqs. (15)–(19), we get

$$(S_{1i} + S_{2i} + S_{3i})_{,x} + N_i = G_{1i}, \tag{33}$$

$$R_{Sm}\psi_{i,x} + \sum_{k=1}^{3} \gamma_{mk} \nabla^2 S_{ki} = H_{mi}, \quad (m = 1, 2, 3), \tag{34}$$

$$\tau_{xz}^{(i)} - \Delta_1 \psi_i = X_i, \tag{35}$$

$$U_i - 4\psi_{i,xz} = Y_i, \tag{36}$$

$$V_i = Z_i, \tag{37}$$

where the quantities $G_1$, $H_m$ $(m = 1, 2, 3)$ and $X_i - Z_i$ are to be determined successively and

$$N_i = U_{i,xz} + \Delta_1 \tau_{xz}^{(i)}. \tag{38}$$

Further, the boundary conditions are

$$\psi_i = \psi_{i,zz} = S_{mi} = \tau_{xz}^{(i)} = U_{i,z} = 0 \quad \text{at } z = 0, 1\ (i = 1, 2, 3, \cdots). \tag{39}$$

At the leading order in $\chi$, the equations are linear and homogeneous and corresponding to $R_{S1} = R_{S1c}^{s}$ their solution is given by

$$\left.\begin{array}{l} S_{m1} = A_{m1} \cos\alpha x \sin\pi z,\ \psi_1 = B_{11} \sin\alpha x \sin\pi z \\ U_1 = C_{11} \cos\alpha x \cos\pi z,\ \tau_{xz}^{(1)} = D_{11} \sin\alpha x \sin\pi z,\ V_1 = 0 \end{array}\right\}. \tag{40}$$

The undetermined amplitudes satisfy the following relations

$$\alpha R_{Sm} B_{11} - \delta^2 \sum_{k=1}^{3} \gamma_{mk} A_{k1} = 0 \quad (m = 1, 2, 3), \tag{41}$$

$$D_{11} + cB_{11} = 0, \tag{42}$$

$$C_{11} - 4\pi\alpha B_{11} = 0, \tag{43}$$

where $c=\pi^2-\alpha^2$. The nonlinear terms of the second order in $\chi$ are

$$G_{12}=0,\ H_{m2}=-\frac{\pi\alpha^2}{2\delta^2}\frac{g_m}{|(\gamma_{ij})|}B_{11}^2\sin 2\pi z,\quad (m=1,2,3), \tag{44}$$

where $g_m$ is the determinant formed from $|(\gamma_{ij})|$ by replacing the $m$th column elements, respectively, by $R_{S1},R_{S2},R_{S3}$. The second-order equations appear as follows:

$$(S_{12}+S_{22}+S_{32})_{,x}+N_2=0, \tag{45}$$

$$R_{Sm}\psi_{2,x}+\sum_{k=1}^{3}\gamma_{mk}\nabla^2 S_{k2}=-\frac{\pi\alpha^2}{2\delta^2}\frac{g_m}{|(\gamma_{ij})|}B_{11}^2\sin 2\pi z,\quad (m=1,2,3). \tag{46}$$

To solve the above equations subjected to the relevant boundary conditions, we need to calculate $N_2$ from Eqs. (35)–(37). The straightforward calculation gives

$$X_2=-\Lambda_1\left\{L_2\left(\tau_{xz}^{(1)}\right)+\frac{1}{2}\nabla^2\psi_1\,U_1\right\}+\Lambda_2\left\{L_2\left(\Delta_1\psi_1\right)+2\psi_{1,xz}\,\nabla^2\psi_1\right\}, \tag{47}$$

$$Y_2=-\Lambda_1\left\{L_2\left(U_1\right)-2\nabla^2\psi_1\,\tau_{xz}^{(1)}\right\}+\Lambda_2\left\{4L_2\left(\psi_{1,xz}\right)-2\nabla^2\psi_1\,\Delta_1\psi_1\right\}, \tag{48}$$

$$Z_2=2\Lambda_1\left\{\psi_{1,xz}\,U_1+\Delta_1\psi_1\,\tau_{xz}^{(1)}\right\}-2\Lambda_2\left\{4\left(\psi_{1,xz}\right)^2+\left(\Delta_1\psi_1\right)^2\right\}, \tag{49}$$

where $L_2=\psi_{1,z}\partial/\partial x-\psi_{1,x}\partial/\partial z$. Solving Eqs. (35)–(37), we get

$$\tau_{xz}^{(2)}=\Delta_1\psi_2+\tfrac{1}{2}\left(\Lambda_1-\Lambda_2\right)\pi\alpha\delta^2B_{11}^2\sin 2\alpha x\sin 2\pi z, \tag{50}$$

$$\begin{aligned}U_2=4\psi_{2,xz}+\tfrac{1}{2}\left(\Lambda_1-\Lambda_2\right)B_{11}^2&\left\{4\pi^2\alpha^2\left(\cos 2\pi z-\cos 2\alpha x\right)\right.\\&\left.+c\,\delta^2\left(1-\cos 2\pi z-\cos 2\alpha x+\cos 2\pi z\cos 2\alpha x\right)\right\},\end{aligned} \tag{51}$$

$$\begin{aligned}V_2=\tfrac{1}{2}\left(\Lambda_1-\Lambda_2\right)B_{11}^2&\left\{4\pi^2\alpha^2\left(1+\cos 2\pi z+\cos 2\alpha x+\cos 2\pi z\cos 2\alpha x\right)\right.\\&\left.+c^2\ \left(1-\cos 2\pi z-\cos 2\alpha x+\cos 2\pi z\cos 2\alpha x\right)\right\}.\end{aligned} \tag{52}$$

From these equations, we deduce that

$$N_2=\nabla^4\psi_2. \tag{53}$$

Equations (45) and (46) are solved, and the solution is

$$\psi_2=0,\quad S_{m2}=\frac{|(\gamma_{ij})_m|}{|(\gamma_{ij})|^2}\frac{\alpha^2}{8\pi\delta^2}B_{11}^2\sin 2\pi z\quad (m=1,2,3), \tag{54}$$

where $|(\gamma_{ij})_m|$ is the determinant formed from $|(\gamma_{ij})|$ by replacing the $m$th column elements, respectively, by $g_1$, $g_2$, $g_3$.

The nonlinear terms of the third-order equations are

$$G_{13}=-\frac{\delta^2}{Pr}\frac{\mathrm{d}B_{11}}{\mathrm{d}s}\sin\alpha x\sin\pi z, \tag{55}$$

$$H_{m3}=\left(-\alpha\xi_m R_{S12}B_{11}+\frac{\alpha}{\delta^2}\frac{g_m}{|(\gamma_{ij})|}\frac{\mathrm{d}B_{11}}{\mathrm{d}s}+\frac{\alpha^3}{8\delta^2}\frac{|(\gamma_{ij})_m|}{|(\gamma_{ij})|^2}B_{11}^3\right)\cos\alpha x\sin\pi z+\cdots\quad (m=1,2,3), \tag{56}$$

where $\xi_1 = 1, \xi_2 = 0 = \xi_3$. Then, we determine

$$N_3 = \nabla^4 \psi_3 - (\Lambda_1 - \Lambda_2)\, \delta^4 \frac{\mathrm{d}B_{11}}{\mathrm{d}s} \sin \alpha x \sin \pi z - M B_{11}^3 \sin \alpha x \sin \pi z + \cdots, \tag{57}$$

where

$$M = \frac{1}{16} \Lambda_1 \left(\Lambda_1 - \Lambda_2\right) \left(\eta_1 - \eta_2\right), \tag{58}$$

with

$$\left.\begin{array}{l} \eta_1 = 9\left(\pi^4 + \alpha^4\right)^2 + 4\pi^2\alpha^2\left(\pi^4 + \alpha^4\right) + 36\pi^4\alpha^4 \\ \eta_2 = 9c^4 + 8\pi^2\alpha^2c^2 + 144\pi^4\alpha^4 \end{array}\right\}. \tag{59}$$

The third-order equations then look like

$$\left(S_{13} + S_{23} + S_{33}\right)_{,x} + N_3 = -\frac{\delta^2}{Pr} \frac{\mathrm{d}B_{11}}{\mathrm{d}s} \sin \alpha x \sin \pi z, \tag{60}$$

$$\begin{aligned} R_{Sm}\psi_{3,x} + \sum_{k=1}^{3} \gamma_{mk} \nabla^2 S_{k3} = \Bigg( & -\alpha \xi_m R_{S12} B_{11} + \frac{\alpha}{\delta^2} \frac{g_m}{|\,(\gamma_{ij})\,|} \frac{\mathrm{d}B_{11}}{\mathrm{d}s} \\ & + \frac{\alpha^3}{8\delta^2} \frac{|\,(\gamma_{ij})_m\,|}{|\,(\gamma_{ij})\,|^2} B_{11}^3 \Bigg) \cos \alpha x \sin \pi z + \cdots \quad (m = 1, 2, 3), \end{aligned} \tag{61}$$

The above equations have a solution of the form

$$\psi_3 = B_{33} \sin \alpha x \sin \pi z + \cdots, \quad S_{m3} = A_{m3} \cos \alpha x \sin \pi z + \cdots, \quad (m = 1, 2, 3). \tag{62}$$

Then the solvability condition applied on Eqs. (60)–(61), upon using Eqs. (41)–(43), yields the Landau equation

$$\Gamma \frac{dB_{11}}{ds} = R_{S12} B_{11} - \Omega B_{11}^3, \tag{63}$$

where $\Gamma$ and Landau constant $\Omega$ are functions of known physical parameters. For the steady case, the amplitude is given by

$$B_{11}^2 = \frac{R_{S12}}{\Omega}. \tag{64}$$

When $\Omega > 0$, the stationary bifurcation is supercritical (i.e., stable) and subcritical (i.e., unstable) if $\Omega < 0$. Although the stationary onset is independent of viscoelastic parameters, the stability of stationary bifurcation is influenced by viscoelasticity of the fluid. However, the cross-diffusion terms influence both stationary onset and the stability of steady bifurcating equilibrium solution.

## 5. Results and discussion

The intricacies of cross-diffusion and elasticity of the fluid on the onset and stability of triple-diffusive convection in an Oldroyd-B fluid layer are investigated. It is a fact that the estimation of parameter values, or even the applicability of a given model of rheology, for a given polymeric fluid is notoriously difficult, and the models often have many such parameters. Due to uncertainties in parameter values, the qualitative changes of behavior may be of interest as one would expect the predicted quantitative changes are of only a few percent to be overwhelmed in an experiment. To throw light on these issues, the numerical calculations are carried out for two different diffusivity matrices with $(D_{ij})$ and without $(D'_{ij})$ cross-diffusion terms for the quaternary aqueous mixtures obtained experimentally by Noulty and Leaist [30] and Vladimir and Epstein [12] which are, respectively, given by

$$D_{ij}=\begin{bmatrix}1.94 & -0.14 & 0.40\\ -0.05 & 2.21 & -1.04\\ 0.03 & -1.36 & 2.02\end{bmatrix}\times 10^{-9}\ \mathrm{m^2\,s^{-1}},\qquad D'_{ij}=\begin{bmatrix}1.94 & 0 & 0\\ 0 & 1.146 & 0\\ 0 & 0 & 1.1\end{bmatrix}\times 10^{-9}\ \mathrm{m^2\,s^{-1}} \tag{65}$$

and

$$D_{ij}=\begin{bmatrix}1.26 & -0.55 & -104\\ -0.42 & 1.32 & 60\\ -0.00013 & 0.00004 & 0.07\end{bmatrix}\times 10^{-9}\ \mathrm{m^2\,s^{-1}},$$
$$D'_{ij}=\begin{bmatrix}1.26 & 0 & 0\\ 0 & 1.04762 & 0\\ 0 & 0 & 0.05556\end{bmatrix}\times 10^{-9}\ \mathrm{m^2\,s^{-1}}. \tag{66}$$

The viscoelastic parameters $\Lambda_1$ and $\Lambda_2$ are chosen such that they are either less than or greater than unity but $\Lambda_2 < \Lambda_1$. The value of Prandtl number $Pr$ at $25\,^{\circ}C$ based on $D_{11}$ is fixed at 464, and the expansion coefficient ratios are taken as $\alpha_{S2}/\alpha_{S1} = 1.06$ and $\alpha_{S3}/\alpha_{S1} = 0.80$.

### 5.1. Linear instability analysis

The critical oscillatory Rayleigh numbers for the diffusivity data of Noulty and Leaist [30] and Vladimir and Epstein [12] for both Maxwell ($\Lambda_1 = 0.2, \Lambda_2 = 0$) and Oldroyd-B ($\Lambda_1 = 0.2, \Lambda_2 = 0.1$) fluids with and without cross-diffusion terms are computed and tabulated in Table 1 for different species concentration combinations. The superscript $\xi$ denotes the results for those with full cross-diffusion terms, and the values given within the parenthesis correspond to the diffusivity data of Vladimir and Epstein [12]. For the values considered, single critical Rayleigh number is found to be enough to identify the linear instability criteria. Cross-diffusion terms produce observable changes in the critical oscillatory Rayleigh numbers depending on the values of off-diagonal elements. For example, for $R_{S2} = 10^2$ and $R_{S3} = -10^4$ the critical Rayleigh numbers differ by 20% for the diffusivity data of Vladimir and Epstein [12] and this change can easily be observed in an experiment. Besides, the cross-diffusion and magnitude of species concentration Rayleigh numbers contribute to either stabilization or destabilization of a viscoelastic fluid layer as there is a sign change in the critical oscillatory Rayleigh number for some values of $R_{S2}$ and $R_{S3}$.

The characteristic oscillatory neutral stability curves in the $(\alpha, R^o_{S1})$ plane are shown in Fig. 1a, b for different values of $\Lambda_1$ and $\Lambda_2$, respectively, for the diffusivity data of Noulty and Leaist [30]. It is observed that there exists only one positive value of $\omega^2$ for the parametric values chosen in these figures. The neutral stability curves in the $(\alpha, R^o_{S1})$ plane show an upward concave shape, and the region below each such curve confines to the region of stability, while the region above it corresponds to instability. For an increase in the value of $\Lambda_1$, the oscillatory Rayleigh number is significantly decreased, but an opposite trend is seen with increasing $\Lambda_2$. Thus, the effect of increasing $\Lambda_1$ and $\Lambda_2$ is to advance and suppress the onset of oscillatory convection. Figure 1 shows that the cross-diffusion terms produce no qualitative effect and only a quantitative shift of a few percent, which is significantly smaller than the non-Newtonian effects. The oscillatory neutral curves shown in Fig. 2a, b for the diffusivity data of Vladimir and Epstein [12] exhibit the presence of off-diagonal elements is to bring in 40% to 80%, variation in the oscillatory Rayleigh number compared to their absence and these differences surely can be observed in the experiments. Moreover, the presence of full cross-diffusion terms is to advance the onset of oscillatory convection compared to their absence.

TABLE 1. Comparisons of oscillatory critical Rayleigh numbers between Maxwell and Oldroyd-B fluids for the diffusivity data given by Eqs. (65) and (66) (shown in parenthesis) with $Pr = 464$.

| Oscillatory onset | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $R_{S2}$ | $R_{S3}$ | Maxwell fluid ($\Lambda_1 = 0.2, \Lambda_2 = 0$) | | | | Oldroyd-B fluid ($\Lambda_1 = 0.2, \Lambda_2 = 0.1$) | | | |
| | | $\omega_c$ | $R_{S1c}$ | $\omega_c^\xi$ | $R_{S1c}^\xi$ | $\omega_c$ | $R_{S1c}$ | $\omega_c^\xi$ | $R_{S1c}^\xi$ |
| $10^2$ | $-10^4$ | 1290.92 | 1422.75 | 5647.76 | 5948.71 | 54.7754 | 53.6001 | 10,505.6 | 10,416.9 |
| | | (2075.5) | (16,356.7) | (386.5) | (− 619,058) | (83.2602) | (548.84) | (10,311) | (8496.1) |
| $10^4$ | $-10^4$ | 645.578 | 34.3314 | − 208.66 | 184.107 | 14.8551 | 17.7351 | 530.81 | 171.964 |
| | | (4216.2) | (16,358.2) | (−9076.2) | (− 628,794) | (76.3279) | (548.781) | (364.916) | (− 1403.6) |
| $10^4$ | $-10^5$ | 3791.1 | 4288.37 | 6.0078 | 28.0378 | 166.112 | 158.984 | 90,456.2 | 90,354.9 |
| | | (6403.1) | (51,702) | (− 3877.1) | (− 6,199,830) | (260.345) | (1735.52) | (9044) | (72,863) |
| $10^5$ | $-10^5$ | 1165.62 | 30.463 | − 2343.4 | 1871.16 | 42.1556 | 32.0691 | 488.605 | − 2090.2 |
| | | (8262.4) | (51,706.4) | (− 91,440) | (− 6,288,330) | (239.766) | (1735.35) | (57.9118) | (− 17,133) |

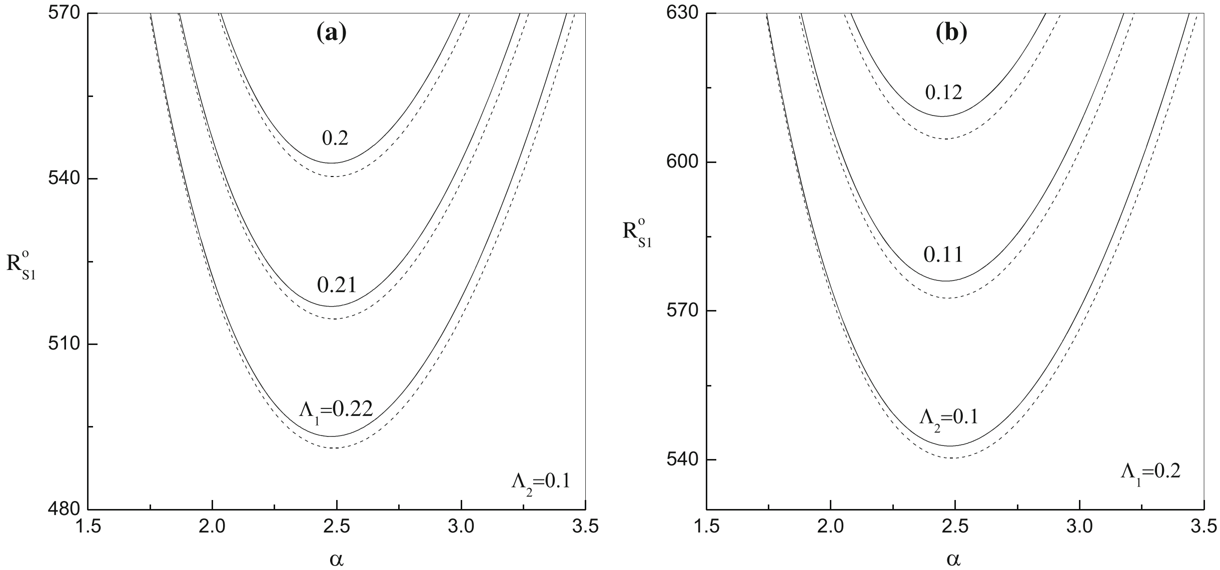


FIG. 1. Oscillatory neutral stability curves in the plane $(\alpha, R_{S1}^{o})$ for different values of **a** $\Lambda_1$, **b** $\Lambda_2$ when the diffusivity data given by Eq. (65), $R_{S2} = -100$, $R_{S3} = 100$, with full cross-diffusion (dashed lines), without cross-diffusion (solid lines)

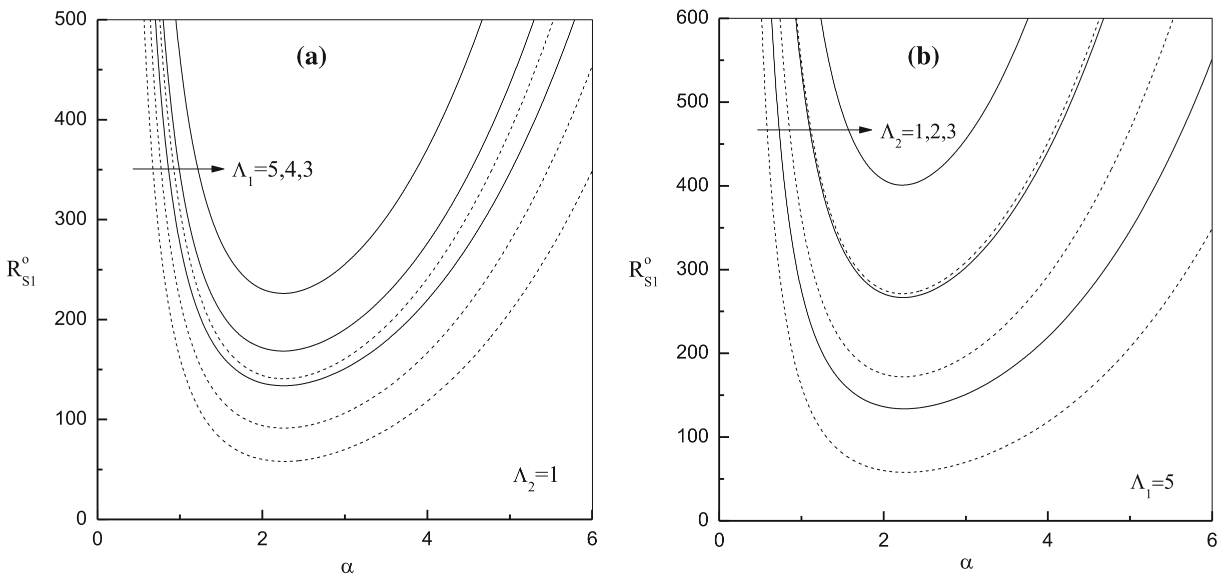


FIG. 2. Oscillatory neutral stability curves in the plane $(\alpha, R_{S1}^{o})$ for different values of **a** $\Lambda_1$, **b** $\Lambda_2$ when the diffusivity data given by Eq. (66), $R_{S2} = 100$, $R_{S3} = -100$, with full cross-diffusion (dashed lines), without cross-diffusion (solid lines)

The sensitivity of the onset of convection due to changes in the off-diagonal elements of the diffusivity matrix is assessed by the following parameterization:

$$D_{ij} = \begin{bmatrix} D'_{11} + \beta(D_{11} - D'_{11}) & \beta D_{12} & \beta D_{13} \\ \beta D_{21} & D'_{22} + \beta(D_{22} - D'_{22}) & \beta D_{23} \\ \beta D_{31} & \beta D_{32} & D'_{33} + \beta(D_{33} - D'_{33}) \end{bmatrix}, \tag{67}$$

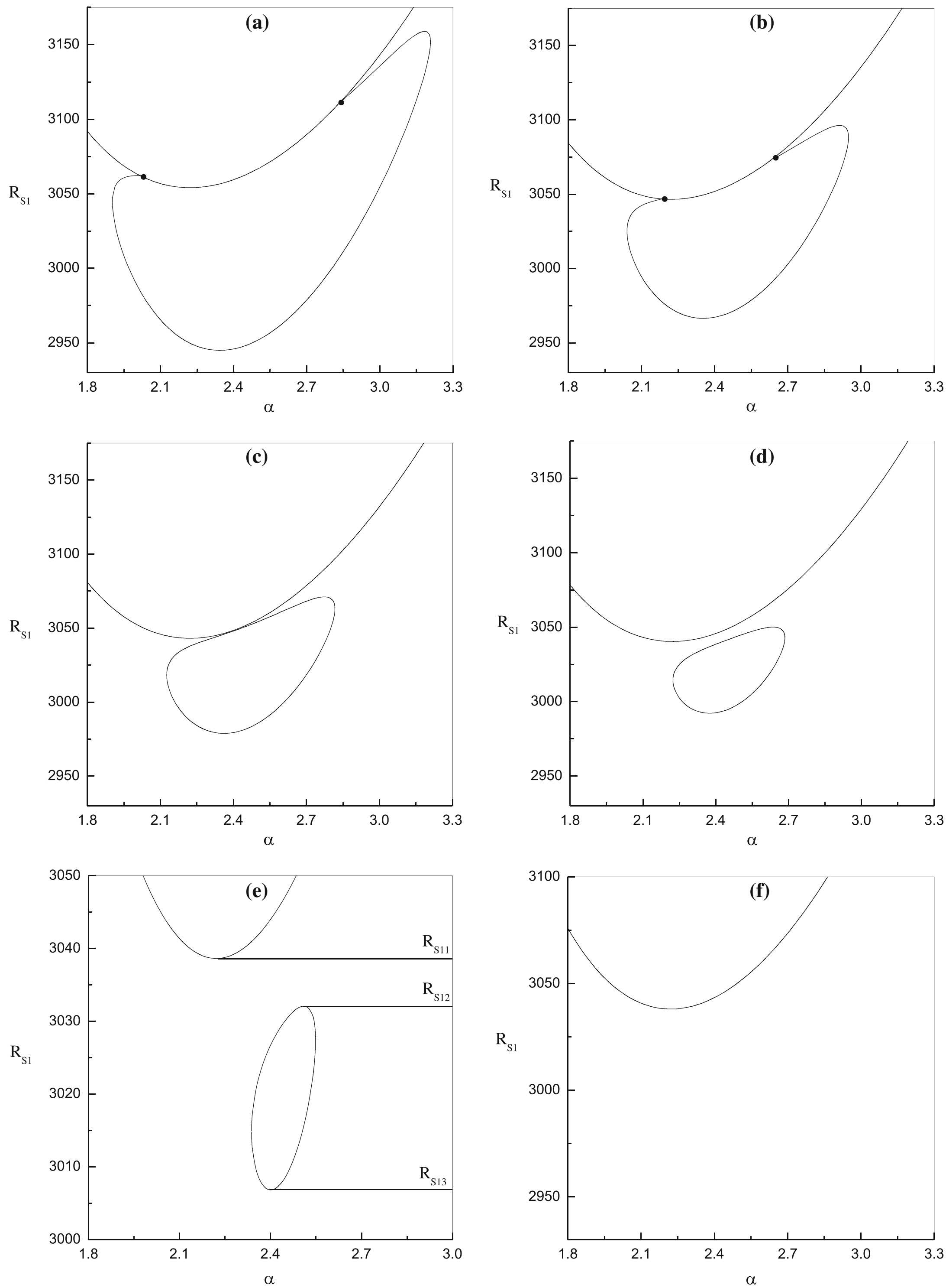


Fig. 3. Evolution of neutral stability curves by varying $\beta$ in Eq. (67) with diffusivity data given by Eq. (65), $\Lambda_1 = 0.1$, $\Lambda_2 = 0.07$ ($\Lambda_1, \Lambda_2 < 1$), $R_{S2} = -13{,}730$, $R_{S3} = 11{,}820$, **a** $\beta = 0$, **b** $\beta = 0.012$, **c** $\beta = 0.0175$, **d** $\beta = 0.022$, **e** $\beta = 0.025$, **f** $\beta = 0.026$

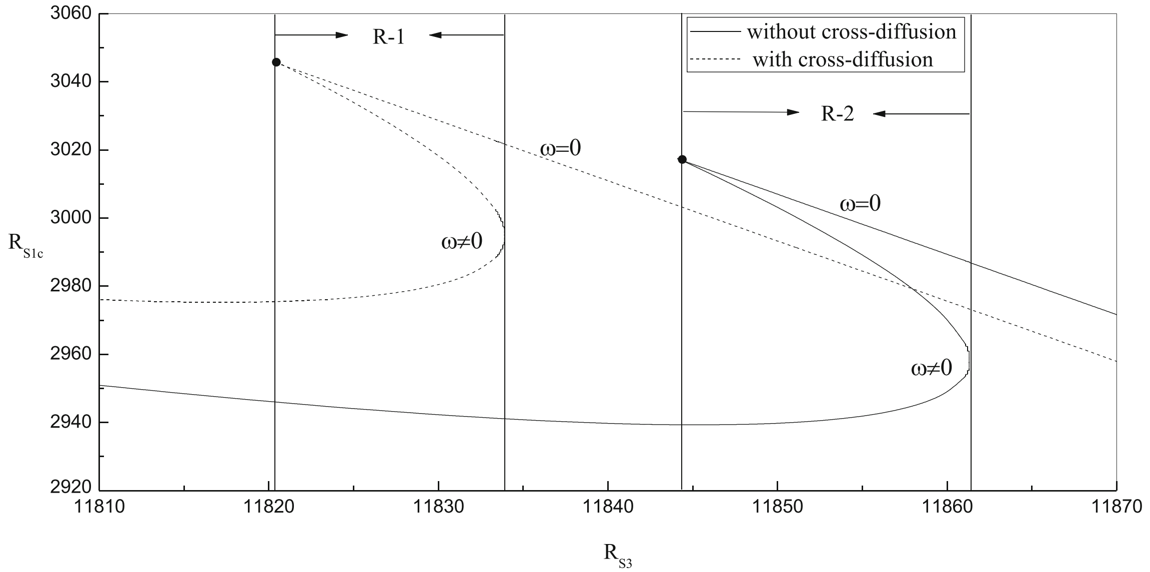


FIG. 4. Stability boundaries for $\Lambda_1 = 0.1$, $\Lambda_2 = 0.07$, $R_{S2} = -13{,}730$, with diffusivity data given by Eqs. (65) and (67)

The cases $\beta = 0$ and $\beta = 1$, respectively, correspond to diffusivity matrix with and without cross-diffusion terms. Figure 3a–f shows the evolution of neutral stability curves for different values of $\beta$ when $\Lambda_1 = 0.1$, $\Lambda_2 = 0.07$ $(\Lambda_1, \Lambda_2 < 1)$, $R_{S2} = -13{,}730$, $R_{S3} = 11{,}820$ and for the diffusivity data given by Eqs. (65) and (67). For $\beta = 0$, Fig. 3a shows that the oscillatory neutral curve is connected to the stationary neutral curve at two bifurcation points which move closer together as $\beta$ is increased to 0.012. In Fig. 3c, the oscillatory neutral curve loses its single-valued character, which has no physical significance because the single critical $R_{S1}$ remains at the minimum at the oscillatory neutral curve. As the value of $\beta$ goes on increasing slightly, the closed loop of oscillatory neutral curve moves well below the stationary neutral curve as seen in Fig. 3e for $\beta = 0.025$. The significance of this neutral is that three critical Rayleigh numbers are needed to specify the linear instability criteria. The system is stable in the region: $R_{S12} < R_{S1} < R_{S11}$ and $R_{S1} < R_{S13}$, and unstable in the island $R_{S13} < R_{S1} < R_{S12}$ and $R_{S1} > R_{S11}$. At $\beta = 0.026$, the oscillatory neutral curve disappears leaving only the stationary neutral curve (Fig. 3f). Thus, it is evident that small variations in the cross-diffusion terms change totally the instability characteristics of the system. Besides, it is important to note here that the closed disconnected oscillatory neutral is convex in shape instead of heart shaped with twin maxima at different wave numbers observed in the case of Newtonian fluids (Terrones [18]). In other words, quasiperiodic bifurcation is found to be not possible. This is one of the striking features that has not been carried over to the viscoelastic fluid case.

Figure 4 shows the corresponding stability boundary in the plane $(R_{S1c}, R_{S3})$ for parametric values considered in Fig. 3. From the graph, it is observed that the presence and absence of cross-diffusion terms clearly change the characteristics of the instability of the system. The regions $R-1$ and $R-2$ between the vertical lines correspond to multivalued region for with and without cross-diffusion terms, respectively, in which three values of critical Rayleigh number are needed to specify the linear instability of the system. However, single value of critical Rayleigh number is sufficient to specify the linear instability criteria of the system outside the regions $R-1$ and $R-2$. Moreover, it is seen that the multivalued region increases in the presence of cross-diffusion terms compared to their absence.

The viscoelastic parameters $\Lambda_1$ and $\Lambda_2$ can be greater than unity for many polymeric fluids, and it is interesting to discern the evolutions of neutral stability curves for this case as well. The results displayed

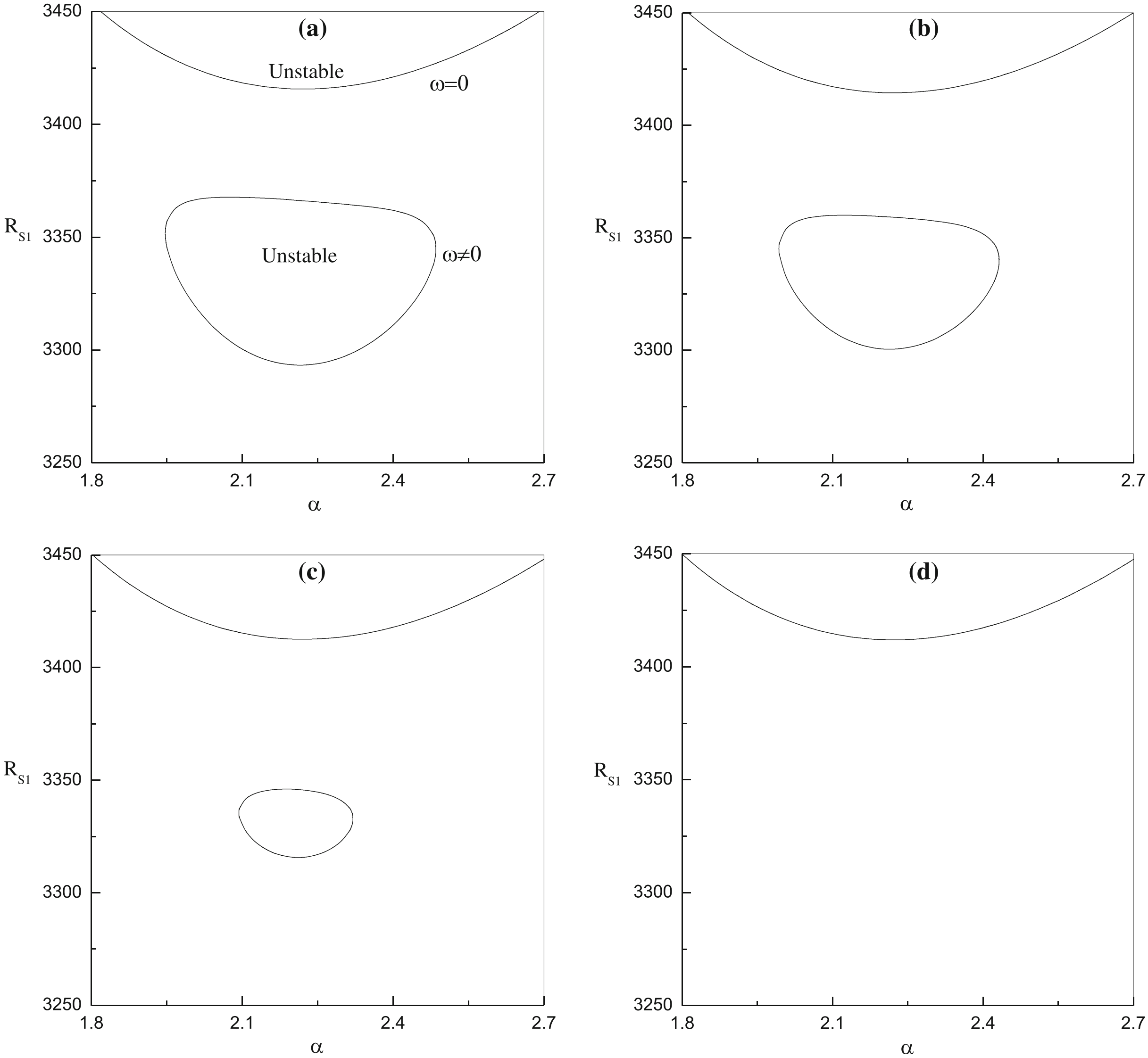


Fig. 5. Evolution of neutral stability curves by varying $\beta$ in Eq. (67) with the diffusivity data given by Eq. (65), $\Lambda_1 = 1.2$, $\Lambda_2 = 1.1$, $R_{S2} = -13{,}730$, $R_{S3} = 11{,}615$, **a** $\beta = 0$, **b** $\beta = 0.002$, **c** $\beta = 0.005$, **d** $\beta = 0.0062$

in Figs. 5a–d when $\Lambda_1 = 1.2$, $\Lambda_2 = 1.1$, $R_{S2} = -13{,}730$, $R_{S3} = 11{,}615$ and diffusivity data given by Eqs. (65) and (67) are for different values of $\beta = 0, 0.002, 0.005$ and 0.0062. The oscillatory neutral curves are disconnected but they are not exactly heart shaped. This is another situation showing the significance of cross-diffusion terms on the instability characteristic of the system.

The similarities and differences between Oldroyd-B (with $\Lambda_1 = 0.1, \Lambda_2 = 0.07$), Maxwell (with $\Lambda_1 = 0.1, \Lambda_2 = 0$) and Newtonian fluid (with $\Lambda_1 = 0 = \Lambda_2$) models with and without cross-diffusion effects are shown in Figs. 6a–c, respectively, when $Pr = 464, R_{S2} = -13{,}730$ and $R_{S3} = 11{,}860$ for the diffusivity data given by Eqs. (65) and (67). From these figures, it is obvious that for an Oldroyd-B fluid case three

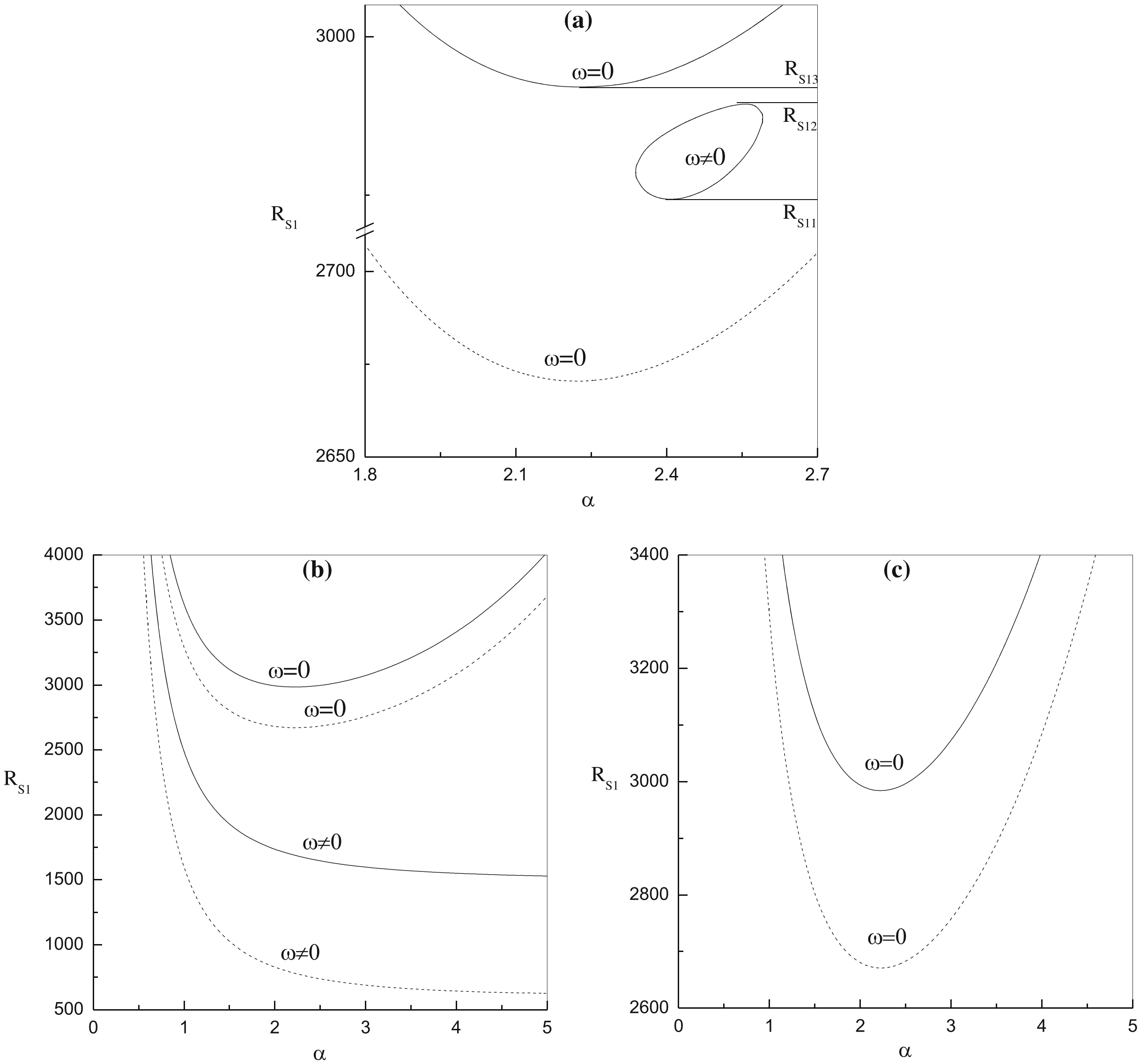


FIG. 6. Variation of retardation parameter $\Lambda_2$ on evolution of neutral stability curves for **a** Oldroyd-B fluid: $\Lambda_1 = 0.1$, $\Lambda_2 = 0.07$, **b** Maxwell fluid: $\Lambda_1 = 0.1$, $\Lambda_2 = 0$, **c** Newtonian fluid: $\Lambda_1 = \Lambda_2 = 0$ when the diffusivity data given by Eqs. (65) and (66), $R_{S2} = -13{,}730$, $R_{S3} = 11{,}860$, with full cross-diffusion (dashed lines), without cross-diffusion (solid lines)

critical Rayleigh numbers are needed to specify the linear instability criteria in the absence of cross-diffusion terms. To the contrary, oscillatory convection is not possible and only stationary convection prevails once the effect of cross-diffusion is considered. Thus, the presence of cross-diffusion completely alters the nature of convective instability of the system in the case of Oldroyd-B fluids. In the case of Maxwell fluids, oscillatory convection is found to be a preferred mode of instability, but a single critical Rayleigh number is sufficient to specify the instability of the system. The scenario observed for Newtonian fluids is, however, is different from those of Oldroyd-B and Maxwell fluids and note that only stationary convection is possible irrespective of the cross-diffusion effects.

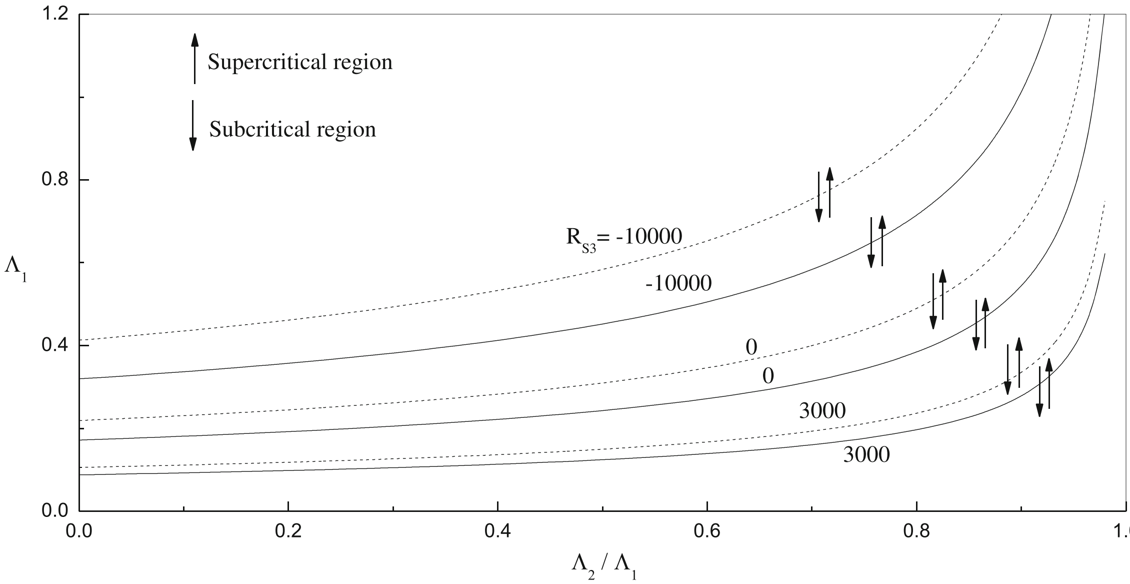


Fig. 7. Regions of supercritical and subcritical steady bifurcations for different values of $R_{S3}$ when the diffusivity data given by Eqs. (65) and (66), $R_{S2} = -5000$, with cross-diffusion (dashed lines), without cross-diffusion (solid lines)

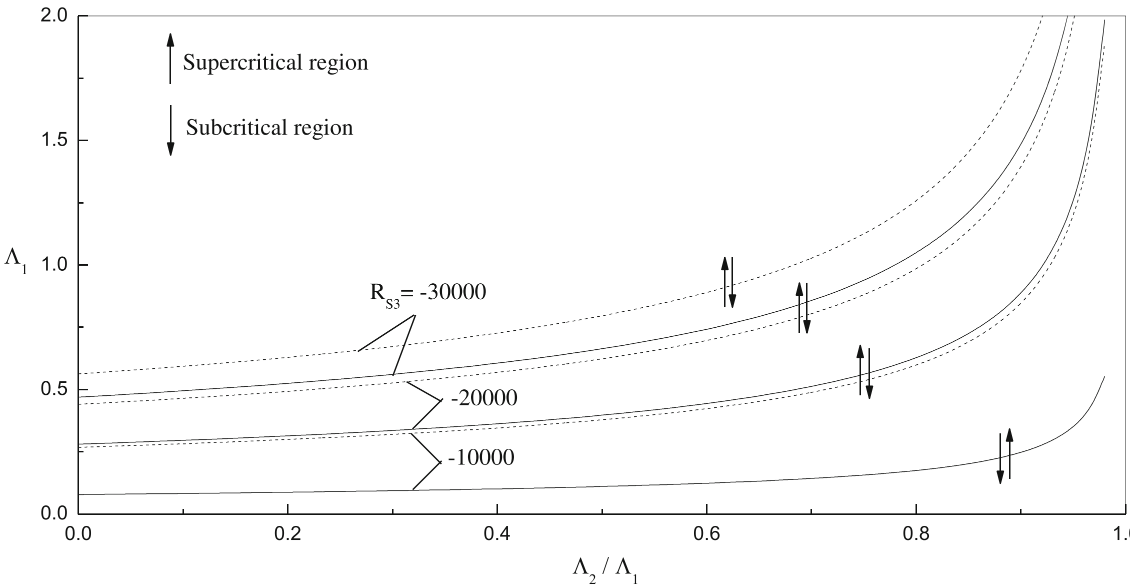


Fig. 8. Regions of supercritical and subcritical steady bifurcations for different values of $R_{S3}$ when the diffusivity data given by Eqs. (65) and (66), $R_{S2} = 5000$, with cross-diffusion (dashed lines), without cross-diffusion (solid lines)

### 5.2. Weakly nonlinear stability analysis

The stability of steady bifurcating equilibrium solution completely depends on the sign of $\Omega$ appearing in Eq. (64). The stationary bifurcation is supercritical (stable) if $\Omega > 0$ and subcritical (unstable) if $\Omega < 0$.

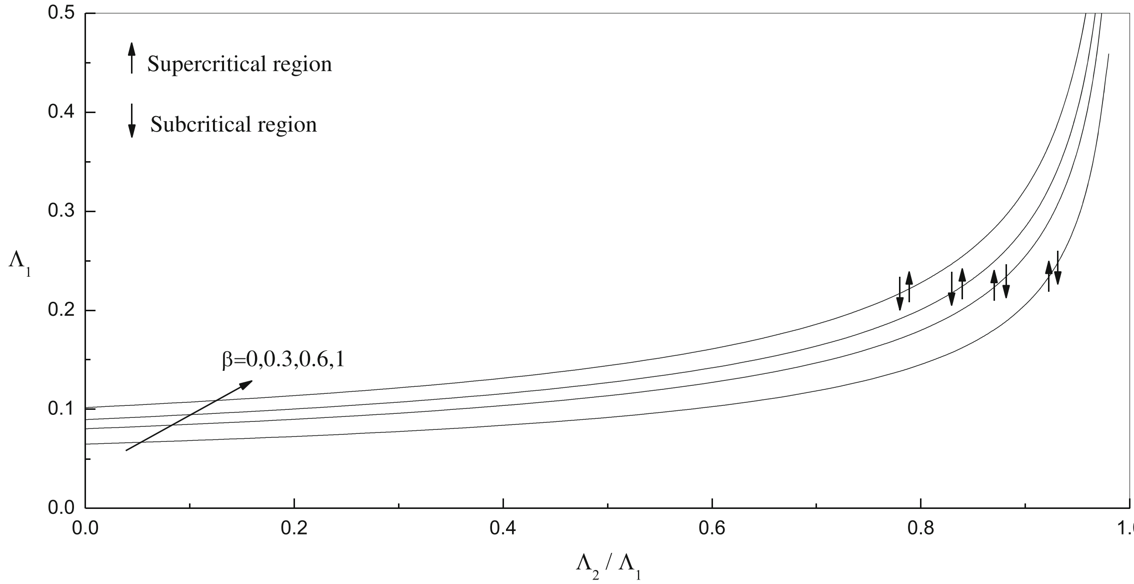


FIG. 9. Regions of supercritical and subcritical steady bifurcations for different values of $\beta$ in Eq. (67) when the diffusivity data given by Eq. (65), $R_{S2} = 5000$, $R_{S3} = -5000$

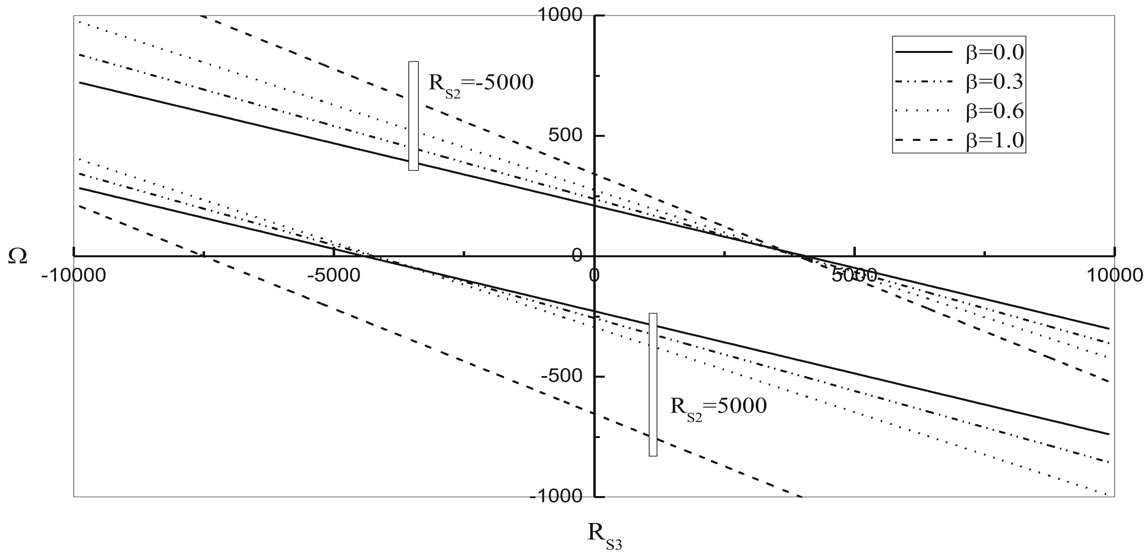


FIG. 10. Regions of supercritical and subcritical steady bifurcations for different values of $\beta$ in Eq. (67) when the diffusivity data given by Eq. (65), $\Lambda_1 = \Lambda_2 = 0$ (Newtonian case)

Although the stationary onset is free from viscoelastic parameters, it is seen that these parameters control the stability of stationary bifurcation. The bifurcating solutions are depicted in viscoelastic parameters plane for different values of species concentration Rayleigh numbers and cross-diffusion terms in Figs. 7, 8 and 9. In these figures, the dotted and solid lines correspond to the results obtained with and without full cross-diffusion terms. The region above each curve indicates the supercritical bifurcation and below of which corresponds to subcritical bifurcation. These figures show the possibility of subcritical stationary bifurcation for a range of parametric values indicating that the occurrence of instability before the linear threshold is reached. This is expected, because the linear instability analysis provides only sufficient condition for instability. From the figures, it is also observed that the subcritical region increases with decreasing $R_{S3}$ (Figs. 7 and 8), while it decreases with decreasing $\beta$ (Fig. 9). In these figures, the results for $\Lambda_1 \neq \Lambda_2$ correspond to the case of Oldroyd-B fluid and the results for $\Lambda_2 = 0$ corresponds to Maxwell fluid. It is noted that the subcritical region increases with increasing $\Lambda_1$, while the trend gets reversed with increasing $\Lambda_2$ (Figs. 7, 8 and 9). It is further observed that the viscoelastic parameters exhibit opposing contributions on the stability of stationary bifurcation. A closer inspection of the figures further reveals that the presence of off-diagonal elements is to increase the region of subcritical instability when compared to their absence (i.e., the presence of cross-diffusion terms is to decrease the subcritical Rayleigh number the most). This result is found to be true for both Oldroyd-B and Maxwell fluids. Thus, the cross-diffusion effects have a much larger impact on the nonlinear stability theory.

Figure 10 represents the computed values $\Omega$ for the Newtonian fluids ($\Lambda_1 = \Lambda_2 = 0$) as a function of $R_{S3}$ for different values of $\beta$, $R_{S2}$ and the fixed diffusivity data given by Eqs. (65) and (67). The possibility of subcritical stationary bifurcation for a range of parametric values is seen indicating the occurrence of instability before the linear threshold is reached. The subcritical region increases when the diffusing component is more stabilizing and also with increasing cross-diffusion sensitivity parameter.

## 6. Conclusions

The coupling of cross-diffusion and viscoelasticity of the fluid on linear and a weakly nonlinear triple-diffusive convection in the presence of gravity has been investigated. The viscoelastic behavior is modeled by means of nonlinear Oldroyd-B constitutive equation which includes Maxwell and Newtonian fluids as particular cases. Some remarkable departures have been identified by performing the linear instability analysis. The presence of cross-diffusion terms is to either stabilize or destabilize the system depending on the magnitude of species concentration Rayleigh numbers and also viscoelasticity of the fluid. The presence of cross-diffusion terms significantly alters the critical Rayleigh numbers depending on the values off-diagonal elements. The stress relaxation and strain retardation parameters exhibit opposing contribution and their effect is to hasten and delay the onset of oscillatory convection. The instability characteristics of the system analyzed for the same parametric values for an Oldroyd-B, Maxwell and Newtonian fluids are found to be qualitatively different. Even small variation in the elements of diffusivity data results in change of instability from oscillatory to stationary. The closed convex disconnected oscillatory neutral curve exists representing the requirement of three critical Rayleigh numbers to specify the linear instability criteria instead of the usual single value. However, one prominent feature that does not carryover from Newtonian to viscoelastic fluids is that the onset of instability does not occur simultaneously at the same critical Rayleigh number at different wave numbers, i.e., heart-shaped oscillatory neutral curve with twin maxima is not found to occur. Based on the weakly nonlinear stability analysis, a cubic Landau equation is derived and the stability of steady bifurcating equilibrium solution is analyzed. An important observation is that the viscoelastic parameters do influence the stability of stationary bifurcation despite their effect is not felt on the stationary onset. It is noted that subcritical bifurcation is possible and the subcritical Rayleigh number decreases with increasing cross-diffusion sensitivity parameter.

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