Finance Faculty Publications

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5-1-2019

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Review of Economics & Finance
Submitted on 10/Sep./2018
Article ID: 1923-7529-2019-02-01-14 Seungmook Choi, and Hongtao Yang

Model-Free Implied Volatility under Jump-Diffusion Models

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Abstract: The model-free implied volatility (MFIVol) is intended to measure the variability of underlying asset price on which options are written. Analytically, however, it does not measure exactly the variability under jump diffusion. Our extensive empirical study suggests that the approximation error can be as much as about $3\% - 5\%$ although most samples over the data period exhibit less than 1% errors. Even with the non-negligible errors, the MFIVol may be still considered a valid volatility measure from the perspective of risk-neutral return density, in the sense that it is bounded by the two variability measures as well as reflecting the shape of the risk-neutral density via its higher central moments.

Keywords: Jump-diffusion model; Model-free Implied Volatility; Risk-neutral probability density; Volatility index (VIX)

JEL Classifications: C58, C65, G12

1. Introduction

Britten-Jones and Neuberger (2000) proposed a methodology that measures, without the need to specify an option model, the return variability of an underlying asset implied by option prices. This approach has generated great interest from both academics and practitioners. Many of today's publicly available volatility indices are calculated by this methodology, and some derivatives written on those indices are traded in the market[1]. Among others, the options and futures on volatility index, VIX, traded on the Chicago Board Options Exchange, and the variance swaps (or volatility swaps) traded at OTC are the derivatives of volatilities calculated by this methodology.

An implied return variability calculated without an option model is called "model-free implied variance" (MFIV) and its square root is known as "model-free implied volatility" (MFIVol). The MFIV is intended to measure the expected total instantaneous return variability of an underlying asset over the option life written on the asset. The return variability can be expressed in two ways. One may use either the effective rate or the continuously compounding rate for the instantaneous rate of return. For convenience, we call the expected variability using the former rate as "expected total return variability" (ETRV) and the latter as "expected quadratic variation of return" (EQVR).

[1] See Carr and Lee (2009) for an overview of the development of volatility derivatives.

The two variabilities are the same when the asset price follows a diffusion process, and thus the distinction between the two is not necessary. Analytically the MFIV measures exactly the ETRV and EQVR. If the asset price has a jump component, however, the two variabilities are not the same and the MFIV just approximates them[2]. As discussed in the seminal work of Merton (1976), a significant part of asset price volatility may be comprised of jumps[3]. Therefore, the validity of the MFIV as a measure of ETRV might depend on how small the approximation errors are. There are conflicting views, however, about the significance of the error size. Jiang and Tian (2005) and Carr and Wu (2009) use some illustrative parameter values of the stochastic volatility jump model of Bates (1996) and show that the effect of jumps on the error size is arguably small. On the other hand, there is a view that the jump component in the asset price process is large enough to make the replication of a variance swap difficult, even theoretically[4].

Strictly speaking, the approximation error size would depend on the jump parameter values. Therefore, we argue that its significance is a valid empirical question. In this paper, under the assumption that option prices are consistent with the stochastic volatility jump diffusion model of Bates (1996), we calibrate it to the S&P 500 index option data each day for the period of 2009 - 2012. Then, using the formulas for annualized MFIV, annualized ETRV, and EQVR for the jump diffusion model, we calculate the error sizes of the MFIVol's obtained from the calibrated parameter values. The empirical results show that for most samples during the data period the errors are less than 1%. However, we find that the MFIVol can often provide a poor estimate of the square root of the ETRV (EQVR) and that the approximation errors can reach up to 5% (3%). The results show that the MFIV is rather close to the EQVR. We also find that the MFIV lies between the two variability measures. In other words, the MFIV overestimates (underestimates) the EQVR when it underestimates (overestimates) the ETRV.

While the ETRV (or EQVR) attempts to explain the expected value of asset return variability over time, the MFIV can be interpreted within the risk-neutral density framework. Martin (2013) shows that the model-free implied variance (MFIV) equals twice the negative first moment of the continuously compounding rate of change over option life under the forward risk-neutral probability measure. Using no-arbitrage and the definition of the cumulant of a random variable, the MFIV is expressed in terms of higher central moments, implying that the risk-neutral density provides a specific relationship between the first moment and the higher moments. In this sense, we may view the MFIVol as a valid volatility measure that reflects a risk-neutral density shape via its higher central moments and that is bounded by the two variability measures, the ETRV and EQVR.

2. Volatility Measures for Diffusion Processes

We begin with a brief discussion of two different measures of variability: the expected total return variability (ETRV) and the expected quadratic variation of return (EQVR) under the assumption that the asset price follows a diffusion process. The two variability measures are

[2] Both variability measures are used in literature. For example, Jiang and Tian (2005) use ETRV and Carr and Wu (2009) use EQVR for their studies.

[3] Recently Todorov (2010) and Todorov and Tauchen (2011) test for jumps in the VIX index and find strong evidence supporting jumps.

[4] The jumps are considered to be one of the reasons why variance swaps collapsed during the credit crisis of 2008-2009. In addition to the jump issue, the replication of variance swap is known to be difficult in practice because it requires a full range of option strikes. See Demeterfi *et al.* (1999), Carr and Corso (2001) and Bondarenko (2014) for its theoretical replication.

essentially the same and can be estimated by the model-free implied variance (MFIV) as shown by Britten-Jones and Neuberger (2000). We then review the properties of the MFIV as a structural parameter of a risk-neutral density.

Consider a forward contract expiring at time T with a forward price of F_t at time t. Assume that F_t follows a diffusion process,

$$\frac{dF_t}{F_t} = \sqrt{v_t}dW_t \tag{1}$$

where v_t is the instantaneous variance at time t and W_t is a Wiener process under the forward risk-neutral measure $\mathbb{F}$. Britten-Jones and Neuberger (2000) show that

$$\mathrm{E}_0^{\mathbb{F}}\left[\int_0^T \left(\frac{dF_t}{F_t}\right)^2\right] \equiv 2\int_0^\infty \frac{C^F(K,T) - (F_0 - K)^+}{K^2}dK \tag{2}$$

where $C^F(K,T)$ is the forward European call option price at time 0 with strike price K and expiration T. We call the left-hand side of Identity (2) "expected total return variability" (ETRV) which we denote by I_T. Under the process (1), I_T is the expected sum of all instantaneous return variances over the option life:

$$I_T = \mathrm{E}_0^{\mathbb{F}}\left[\int_0^T \left(\frac{dF_t}{F_t}\right)^2\right] = \mathrm{E}_0^{\mathbb{F}}\left[\int_0^T \left(\sqrt{v_t}dW_t\right)^2\right] = \mathrm{E}_0^{\mathbb{F}}\left[\int_0^T v_t\, dt\right] \tag{3}$$

Thus, the identity (2) implies that we can estimate the ETRV (I_T) by the right-hand side of (2), which depends only on the option prices of the same expiration T. For this reason, the quantity obtained by the right-hand side of (2), which we denote by Σ_T, is called model-free implied variance (MFIV) and its square root is known as model-free implied volatility (MFIVol).

Now let R_t be the continuously compounding rate of change in F_t over time interval $[0,t]$ such that

$$F_t = F_0 e^{R_t} \tag{4}$$

By Itô's Lemma, we can express the process of R_t as

$$R_t = -\frac{1}{2}\int_0^t v_s\, ds + \int_0^t \sqrt{v_s}\, dW_s \tag{5}$$

Then we can use the variability of R_t as an alternative to the ETRV. Let us call the variability on time interval $[0,T]$ "expected quadratic variation of return" (EQVR), denoted by Q_T. A simple calculation following the definition of the quadratic variation yields

$$Q_T = \mathrm{E}_0^{\mathbb{F}}[\langle R\rangle_T] = \mathrm{E}_0^{\mathbb{F}}\left[\left\langle -\frac{1}{2}\int_0^t v_s\, ds + \int_0^t \sqrt{v_s}\, dW_s\right\rangle_T\right] = \mathrm{E}_0^{\mathbb{F}}\left[\int_0^T v_t\, dt\right],$$

where $\langle\cdot\rangle_T$ denotes the quadratic variation on time interval $[0,T]$. Hence, the two measures of total variability, ETRV and EQVR equal each other when the underlying asset prices follow the diffusion process. Both can be estimated by the MFIV. We show in the next section that the two variability measures differ each other when the asset prices contain jumps and the MFIV is not equal to either of them.

Now let us consider the MFIV as a structural parameter of the risk-neutral density resulted from any stochastic processes, either diffusion or jump-diffusion. Jiang and Tian (2005) show the MFIV can also be written in terms of forward prices:

$$\Sigma_T = 2\left(\ln(F_0) - \mathrm{E}_0^{\mathbb{F}}[\ln(F_T)]\right).$$

Plugging into the above equation, we obtain[5]

$$\Sigma_T = -2\mathrm{E}_0^{\mathbb{F}}[R_T] \tag{6}$$

After all, the MFIV equals two times the negative expected rate of return R_T under the forward risk-neutral measure [6].

Using the no-arbitrage condition, Martin (2013) expands the MFIV in terms of the central moments of R_t to get

$$\Sigma_T = \sigma_T^2 + \frac{1}{3}\sigma_T^3\gamma_1 + \frac{1}{12}\sigma_T^4\gamma_2 + \cdots \tag{7}$$

where γ_1 and γ_2 are the skewness and excess kurtosis of R_T respectively. Equation (6) states that the MFIV is solely determined by the mean of R_T while Equation (7) explains the MFIV in terms of the second and higher central moments of R_T. Thus, the risk-neutral density provides a specific relationship between the mean and the higher central moments. This implies the MFIV captures the shape of the risk-neutral density in terms of higher central moments of the return as a specific way of measuring volatility.

Interpreting the MFIV alone as a fear index is difficult, however, because it is silent about the down-side risk. For example, the same MFIV values imply the same variabilities of underlying asset price, but they can reflect two distinct skewness values, one negative and the other positive. For an illustration, we consider the stochastic volatility model of Heston (1993). The variance v_t is specified by the Cox, Ingersoll and Ross (1985) model:

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dB_t \tag{8}$$

where B_t is a Wiener process and is correlated with W_t at rate ρ. It is straightforward to obtain the MFIV for the Heston model:

$$\Sigma_T = \theta T + \frac{v_0 - \theta}{\kappa}(1 - e^{-\kappa T}) \tag{9}$$

Equation (9) shows that the MFIV does not depend on the correlation parameter ρ and the volatility of volatility σ.

Using various combinations of θ and κ, we can generate the same MFIV's with different skewness by using the moment generating function. Figure 1 plots the skewness on pairs of θ and κ with the MFIV held constant at $\Sigma_T = 0.2$. We set $T = 1$ for both figures and we use $\rho = -0.8$ and

[5] See Martin (2013). In Appendix, we also prove explicitly by using the general European call option pricing formula.

[6] We also have $\Sigma_T = 2\left(rT - \mathrm{E}_0^{\mathbb{Q}}[R_T]\right)$,

under the risk-neutral measure $\mathbb{Q}$ when the interest rate is constant r. Hence, the MFIV is two times the difference between the risk-free return rate and the expected risky return rate.

0.5 for the left figure and the right figure, respectively. Both figures exhibit the same Σ_T's, but skewness is negative for the left figure and positive for the right. In other words, the same MFIV's do not imply the same down-side risks. Or we can easily conjecture that a larger MFIV does not necessarily implies a greater down-side risk.

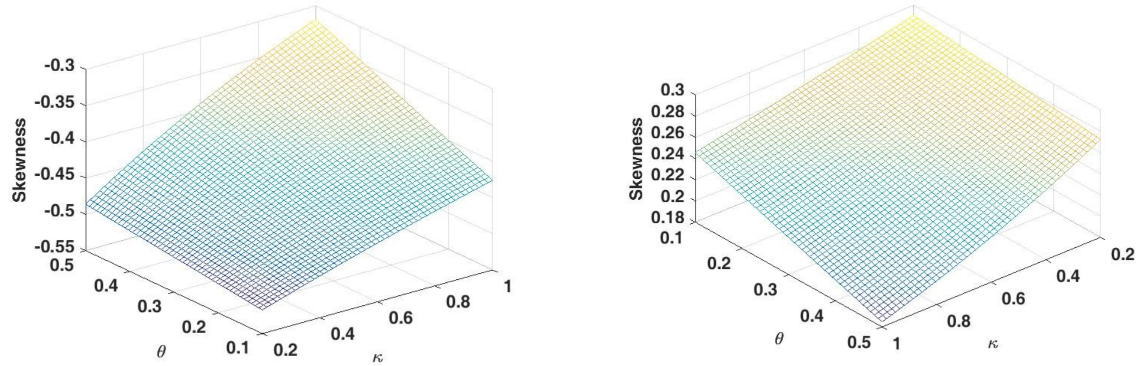

Figure 1. Using the stochastic volatility model of Heston (1993), we plot the skewness of return on the pairs of θ and κ with $\Sigma_T = 0.2$ and $T = 1$ for both figures, and $\rho = -0.8$ and 0.5 for the left figure and the right figure, respectively.

3. Volatility Measures for Jump-Diffusion Processes

We now derive the formulas for the two variability measures and the model-free implied variance under jump-diffusion in a simple and succinct way. Then we examine the difference between the two measures relative to the MFIV by using a numerical illustration.

Suppose that a forward price F_t follows a jump-diffusion process:

$$\frac{dF_t}{F_t} = \sqrt{v_t}dW_t + dZ_t \tag{10}$$

where Z_t is a compensated compound Poisson process, independent of process v_t and the Wiener process W_t under the forward measure $\mathbb{F}$. As usual, the process Z_t is specified as

$$Z_t = \sum_{n=1}^{N_t} (e^{J_n} - 1) - \lambda_J \mu_J t,$$

where N_t is a Poisson process with risk-neutral intensity λ_J, $\{J_n\}_1^\infty$ is a sequence of independent and identically distributed random variables, and $\mu_J = \mathrm{E}_0^{\mathbb{F}}[e^{J_1} - 1]$ is the expected jump amplitude.

Using Itô's lemma, we get the stochastic process of R_t defined in as

$$R_t = -\frac{1}{2}\int_0^t v_s\, ds + \int_0^t \sqrt{v_s}\, dW_s + Y_t \tag{11}$$

where $Y_t = \sum_{n=1}^{N_t} J_n - \lambda_J \mu_J t$. Taking expectation on yields,

$$\mathrm{E}_0^{\mathbb{F}}[R_T] = -\frac{1}{2}\mathrm{E}_0^{\mathbb{F}}\left[\int_0^T v_t\, dt\right] + \lambda_J T\big(\mathrm{E}_0^{\mathbb{F}}[J_1] - \mu_J\big).$$

Hence, it follows from that

$$\Sigma_T = -\, 2\mathrm{E}_0^{\mathbb{F}}[R_T] = \mathrm{E}_0^{\mathbb{F}}\left[\int_0^T v_t\, dt\right] + 2\lambda_J T\big(\mu_J - \mathrm{E}_0^{\mathbb{F}}[J_1]\big) \tag{12}$$

We also have by (10)

$$I_T = \mathrm{E}_0^{\mathbb{F}}\left[\int_0^T \left(\frac{dF_t}{F_t}\right)^2\right] = \mathrm{E}_0^{\mathbb{F}}\left[\left\langle \int_0^t \sqrt{v_s}\, dW_s \right\rangle_T\right] + \mathrm{E}_0^{\mathbb{F}}[\langle Z\rangle_T] \tag{13}$$

$$= \mathrm{E}_0^{\mathbb{F}}\left[\int_0^T v_t\, dt\right] + \lambda_J T \mathrm{E}_0^{\mathbb{F}}[(e^{J_1}-1)^2],$$

where we use the fact that W_t and Z_t are independent. Lastly, from (11), we obtain the expected quadratic variation of return (EQVR):

$$Q_T = \mathrm{E}_0^{\mathbb{F}}[\langle R\rangle_T] = \mathrm{E}_0^{\mathbb{F}}\left[\left\langle -\frac{1}{2}\int_0^t v_s\, ds + \int_0^t \sqrt{v_s}\, dW_s \right\rangle_T\right] + \mathrm{E}_0^{\mathbb{F}}[\langle Y\rangle_T] \tag{14}$$

$$= \mathrm{E}_0^{\mathbb{F}}\left[\int_0^T v_t\, dt\right] + \mathrm{E}_0^{\mathbb{F}}\left[\sum_{n=1}^{N_T} J_n^2\right] = \mathrm{E}_0^{\mathbb{F}}\left[\int_0^T v_t\, dt\right] + \lambda_J T \mathrm{E}_0^{\mathbb{F}}[J_1^2],$$

where we use the fact that Y_t and $R_t - Y_t$ are independent.

We see from (12), (13), and (14) that the three quantities MFIV (Σ_T), ETRV (I_T) and EQVR (Q_T) are not the same in the presence of jumps unless $J_1 = 0$, that is, there are no jumps.

Now let us check the difference between Σ_T and the two variability measures, I_T and Q_T. Using Equations (12) and (13), we can write

$$I_T = \Sigma_T + E_T \tag{15}$$

where

$$E_T = \lambda_J T \mathrm{E}_0^{\mathbb{F}}[e^{2J_1} - 4e^{J_1} + 2J_1 + 3] \tag{16}$$

By using the Maclaurin series of function e^x, the error E_T can be expanded as follows:

$$E_T = \lambda_J T \sum_{n=3}^{\infty} \frac{2^n - 4}{n!} \mathrm{E}_0^{\mathbb{F}}[J_1^n] = \lambda_J T\left(\frac{2}{3}\mathrm{E}_0^{\mathbb{F}}[J_1^3] + \frac{1}{2}\mathrm{E}_0^{\mathbb{F}}[J_1^4] + \cdots\right) \tag{17}$$

which means that the size of the difference E_T depends on the third and higher moments of J_1. The difference between Σ_T and Q_T is obtained from Equations and as

$$Q_T = \Sigma_T + H_T \tag{18}$$

where

$$H_T = \lambda_J T \mathrm{E}_0^{\mathbb{F}}[J_1^2 + 2J_1 + 2 - 2e^{J_1}] \tag{19}$$

The series expansion of H_T yields

$$H_T = -2\lambda_J T \sum_{n=3}^{\infty} \frac{1}{n!} \mathrm{E}_0^{\mathbb{F}}[J_1^n] = \lambda_J T\left(-\frac{1}{3}\mathrm{E}_0^{\mathbb{F}}[J_1^3] - \frac{1}{12}\mathrm{E}_0^{\mathbb{F}}[J_1^4] - \cdots\right) \tag{20}$$

Again, the difference H_T depends on the third and higher moments of J_1.

Since $E_T \neq 0$ and $H_T \neq 0$ in general, it may be too strong to claim that they are negligibly small. Now we examine the size of E_T and H_T under the stochastic volatility jump (SVJ) model of Bates (1996). In the Bates model, the forward price has a process of (10), the process v_t is the same as (8) of Heston's stochastic volatility model, and the random variable J_1 is normally distributed with mean $\ln(1+\mu_J) - \frac{1}{2}\sigma_J^2$ and variance σ_J^2 where $\mu_J \in (-1,1)$. Taking the expected value on both sides of (8) and solving the resulting differential equation, we have by (12) [7]

$$\Sigma_T = \theta T + \frac{v_0 - \theta}{\kappa}(1 - e^{-\kappa T}) + \lambda_J T\left(\sigma_J^2 + 2\mu_J - 2\ln(1+\mu_J)\right) \qquad (21)$$

Finally, we can write Equations and for the Bates model as follows[8]:

$$E_T = \lambda_J T\left(\mu_J^2 - 2\mu_J + 2\ln(1+\mu_J) + (e^{\sigma_J{}^2} - 1)(1+\mu_J)^2 - \sigma_J^2\right)$$

and $$H_T = \lambda_J T\left(\left(\ln(1+\mu_J) - \tfrac{1}{2}\sigma_J^2\right)^2 + 2\ln(1+\mu_J) - 2\mu_J\right).$$

Since the CBOE volatility index, VIX, is computed by using one hundred times the square root of the annualized model-free implied variance, we examine the differences E_T and H_T in the same manner as follows:

$$e_T = 100\left(\sqrt{\frac{\Sigma_T + E_T}{T}} - \sqrt{\frac{\Sigma_T}{T}}\right) \qquad (22)$$

and $$h_T = 100\left(\sqrt{\frac{\Sigma_T + H_T}{T}} - \sqrt{\frac{\Sigma_T}{T}}\right) \qquad (23)$$

Figure 2 displays the contour plots of e_T and h_T as functions of σ_J and μ_J, while the other parameters are fixed at $\lambda_J = 4$, $T = 1$, $\kappa = 1$ and $\theta = v_0 = 0.1854^2$ as used in Jiang and Tian (2005). The differences between the ETRV's and the MFIV's measured in a square root of the annualized quantity range between -7 and $+7$ given the set of parameter values. As expected, when the absolute value of μ_J and σ_J are large, the differences get larger. Regardless of the size of σ_J, however, the differences are zero when jump size (μ_J) equals zero. On the other hand, the differences between the EQVR's and the MFIV's range between -3 and $+4$, indicating that the MFIV is close to the EQVR measure. In general, we can say that h_T is smaller than e_T in absolute value and thus the MFIV more closely approximates the EQVR than the ETRV. Note also that a large value of average positive (negative) jump coupled with a higher σ_J makes the MFIV underestimate (overestimate) the ETRV and overestimate (underestimate) the EQVR. In other words, E_T and H_T (or e_T and h_T) take the opposite sign when μ_J varies from negative to positive, which their series expansions (17) and (20) and imply. This indicates the MFIV lies between the

[7] Note that the MFIV does not depend on the correlation parameter ρ and the volatility of volatility σ. However, the higher central moments are dependent on these two parameters.

[8] Carr and Wu (2009) also derived H_T and Σ_T . We derive them here again to compare with E_T in a simple way.

two volatility measures, the ETRV and the EQVR. The next section evaluates empirically the economic significance of the differences.

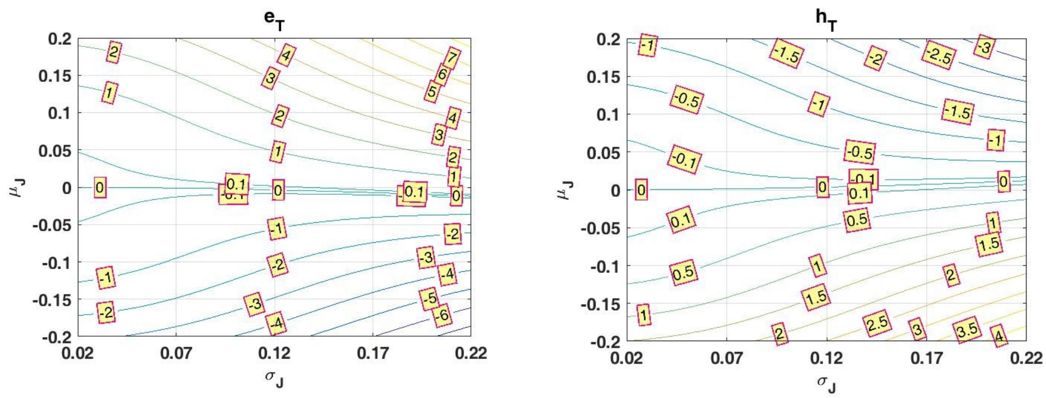

Figure 2. The figures display the contour plots of e_T and h_T as functions of σ_J and μ_J , when $T = 1$, $\lambda_J = 4$, and $\theta = v_0 = 0.1854^2$.

4. Calibration of the Stochastic Volatility Jump Model

Since the difference between the MFIV and the ETRV (or EQVR) depends on the jump parameter values, its economic significance is an empirical question. We examine this issue by calibrating the stochastic volatility jump model of Bates (1996), assuming that the option prices are consistent with the Bates model.

We use SPX option data, European options written on the S&P 500, covering 2009 to 2012.[9] The sample period follows immediately after the 2008 Lehman Brothers bankruptcy. The subsequent mortgage crisis probably produced option prices with unusually high implied volatilities and very steep volatility smiles related to the Black-Scholes-Merton model.

To minimize potential noise in the data for a more accurate calibration, we filter the data as follows. We exclude options for which the bid price or the open interest is zero as well as when the maturity is less than 10 days to minimize market microstructure concerns. Then we select the options of the first two expiration dates instead of including options of all expiration dates.[10] Therefore, each day, there are two groups of expiration options. We eliminate the option if either the call or put option of the same expiration and strike price is missing.

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[9] We get the "optsum" data from the CBOE Market Data Express. The data set contains an end of day option summary for CBOE traded call and put options. This includes volume traded, open interest, open, high, low, bid-ask prices on the last quote of the day and the last underlying asset price. We use only the "standard" series type of options out of various option series type such as LEAP, Weekly, Quarterly and Custom provided by the CBOE.

[10] Bardgett, Gourier and Leippold (2013) state that "*although the standard SVJ model performs well at representing the smiles of volatility for both markets on a given date, its dynamics is not sufficiently flexible to accommodate for the dynamical properties embedded in the time series of option prices.*" Considering their remarks, we use only the option data of the two expiration dates for a better fit.

Thus, our observations contain matched call and put options. Given the matched options, we then compute the implied volatility using the Black-Scholes-Merton model. If the calculated implied volatility is negative, we delete the corresponding observation from the sample because it is obviously underpriced. For each put-call pair, we check its parity condition. This requires dividend yield information, which is assumed to be known. Instead of using the actual realized dividend yields,[11] we estimate the dividend yields as follows. We first find the implicit SPX forward price by setting $F_t = K + e^{r_f \cdot T}(C_t - P_t)$, where we use the options of the strike price at which the absolute difference between the call and put prices is smallest. Since the arbitrage-free forward price must be $F_t = S_t e^{(r_f - q)T}$, we solve for the dividend yield of q, given S_t, r_f, and F_t, where we use Treasury bill yields as the risk-free rate.[12] If the Treasury bill yield for a specific maturity does not exist, we interpolate linearly between the adjacent yields. Using the put-call parity condition, we calculate synthetic call prices ($= P_t + S_0 e^{-q(T-t)} - e^{-r_f T}K$) corresponding to the bid, the ask and the midpoint price of a put and check if any one of the calculated synthetic call prices falls in the boundary of the call bid and ask prices. If none of them lie within the boundary, we treat it as a violation of the put-call parity and exclude the observation from the sample. Lastly, we delete the entire sample if the number of call-put option pairs is less than 22, considering the SVJ model has eight parameters to be estimated.

Table 1 summarizes the data set obtained after the above filtering process. A total of 977 trading-day samples over 4 years is obtained. Each sample contains a minimum of 22 to a maximum 148 pairs of call and puts. The two expirations of each sample are on average around 0.09 years and 0.22 years.

Table 1. Data summary

Year	Number of Trading Days	T_1 Avg	T_2 Avg	Number of Paris/Days		
				Avg	Min	Max
2009	242	0.094	0.252	70	22	108
2010	239	0.088	0.215	71	22	138
2011	250	0.082	0.208	74	27	148
2012	246	0.084	0.209	64	22	96
All Years	977	0.087	0.221	69	22	148

Note: The data sample each day includes two option expirations, T_1 and T_2. The number of pairs/day means the number of put and call option pairs of the same strike price and expiration date.

We calibrate the Bates model by minimizing the objective function (24), which is essentially the sum of squares of the relative price difference of the actual market option prices and the model prices[13]. We pick the mid-point of the bid and ask prices as the actual option price as CBOE does

[11] We can find the actual dividend yields using S&P500 index with and without dividends. We find that the actual dividend yields are stable for more than approximately four-month periods. However, they are quite varying for shorter periods.

[12] This dividend estimation approach is used for the VIX calculation at CBOE.

[13] The model prices are computed by using the formula in Bates (2006) with the reformulated characteristic function in Gatheral (2006).

for its VIX calculation. Instead of using the data for either calls or puts only, we use both puts and calls for the given range of strike prices as shown in the following objective function:

$$\left(\sqrt{\sum_{K_i \le S_0}\left(\frac{C(K_i)-c(K_i)}{C(K_i)}\right)^2}+\sqrt{\sum_{K_i > S_0}\left(\frac{P(K_i)-p(K_i)}{P(K_i)}\right)^2}\right) \qquad (24)$$

where C and P are the actual call and put prices and c and p are the model prices. We calibrate the model with the constraint, $2\kappa\theta \ge \sigma^2$, which is often imposed to insure the positive variance process of almost surely. To avoid the potential problem of a local solution that may occur for different starting parameter values, we follow a two-step approach in favor of no jumps. First, we calibrate the model with no jumps. Second, we use the calibrated parameter values as the starting values for the Bates model with starting jump parameter values of zero.

Figure 3 below plots the annualized MFIVol's calculated with the calibrated parameter values and a longer expiration date for each trading day. Over the sample period, the figure exhibits sudden spikes of the MFIVol around every 1.5 years.

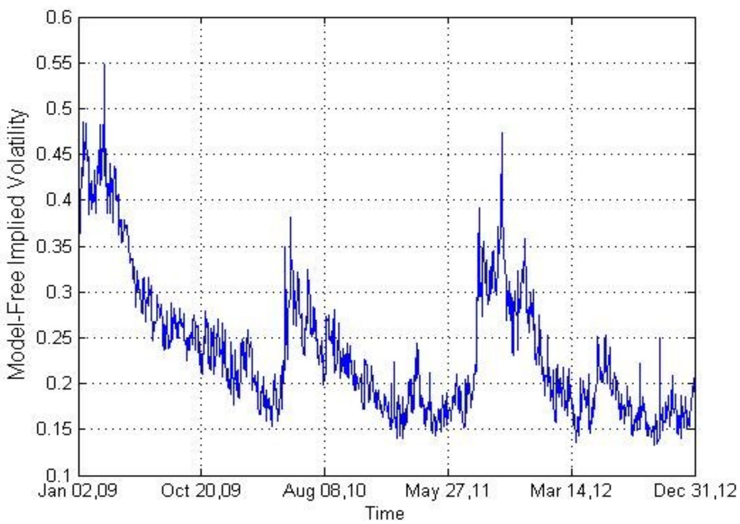

Figure 3. The model-free implied volatilities for the period from 2009JA02 to 2012DE31. The volatilities are calculated with the calibrated parameter values of the Bates model.

Figure 4 on p.11 plots the histograms of the absolute size of $e_T(h_T)$ and the percentage of $e_T(h_T)$ relative to Σ_T. A majority (about 70% and 80%) of the samples show that the relative error sizes are less than 1%. Thus we may argue that the effect of the jumps on the approximation errors is small. However the remaining samples show that the errors are greater than 1%. Table 2 provides more detailed information on the empirical distributions of their relative size. The figure shows that the maximum and minimum values of e_T are 1.6 and -2.4 respectively. These correspond to the maximum and minimum of its relative size of 2.08% and -5.43% respectively. Similarly, the maximum and minimum size values of h_T in the figure are 1.2 and -0.2 and the maximum and minimum of the relative size are 3.10% and -0.78%, respectively. As shown in the simulation in the previous section, the empirical results also show that the MFIV (Σ_T) is closer to the EQVR (Q_T) than to the ETRV (I_T). The potential size differences in our empirical study reach 3%, however, which may be still too large to be ignored. Our empirical results also confirm that Σ_T is always located between Q_T and I_T.

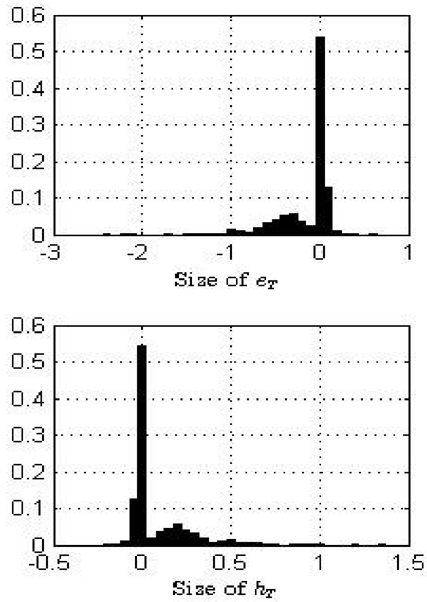

Figure 4. The top left histogram plots the size of e_T of 977 daily estimates during years 2009-2012 and the bottom left plots the size of h_T. The right histograms plot the percentages of e_T and h_T relative to Σ_T.

Table 2. Relative sizes of e_T and h_T

Year	e_T				h_T			
	Avg	STD	Min	Max	Avg	STD	Min	Max
2009	−0.06	1.45	−5.43	1.10	0.35	0.80	−0.47	3.09
2010	−0.58	1.16	−4.49	0.74	0.32	0.63	−0.35	2.46
2011	−0.64	1.23	−5.10	2.08	0.35	0.67	−0.78	3.10
2012	−0.55	1.03	−3.63	0.61	0.30	0.54	−0.28	2.02
All Years	−0.60	1.23	−5.43	2.08	0.33	0.67	−0.78	3.10

5. Conclusion

In this paper, we empirically examine the size of the approximation errors of the model-free implied volatility (MFIVol) in measuring the square root of expected total return variability (ETRV) and the square root of expected quadratic variation of return within a framework of the jump diffusion model. Since the approximation error sizes would depend on the jump parameter values, we perform an empirical study by calibrating the stochastic volatility jump diffusion model of Bates (1996) to the S&P 500 option price data for the period between 2009 and 2012. We find that on average the error size is less than 1%. The standard deviations of the errors, however, are around 1% and the differences often reach more than 3%. Considering that asset price jumps are common in financial markets, this error seems non-trivial.

We find that the approximation errors of the MFIV for the EQVR are smaller than the ones for the ETVR, suggesting that the MFIVol provides an estimate close to the square root of the EQVR. Thus, when one examines the information content of the MFIVol, it may be better to use the quadratic variation of return as the realization of the MFIVol. In addition, we find that the MFIVol takes a value between the square root of the ETRV and the square root of the EQVR. In other words, the model-free implied variance (square of MFIVol) is bounded by the two variability measures, the ETRV and EQVR.

Since the model-free implied variance equals twice the negative first moment of the continuously compounding rate of change of the underlying forward price under the forward risk-neutral measure and the first moment can be expressed in terms of higher central moments, the risk-neutral density requires a specific relationship between the first moment and the higher moments. In this sense, the MFIVol reflects the shape of a risk-neutral density via its higher central moments regardless of the asset price process.

Considering all of the above, we may conclude that the MFIVol is taken as a relevant measure of volatility although the MFIV is not exactly equal to either of the two variability measures under jumps.

Appendix: Derivation of Equation (6)

Consider a forward call option with the strike price K and the expiration date T. Let $f(r)$ be the risk-neutral density for random variable R_T, where $R_T = \ln\left(\frac{F_T}{F_0}\right)$. Then the spot call price is given by (Equation 9.4.7 on page 393, Shreve, 2013)

$$C(K,T) = B(0,T)\mathrm{E}_0^{\mathbb{F}}[(F_T - K)^+] = B(0,T)\big(F_0 N_1(d) - K N_2(d)\big) \qquad (25)$$

where $B(0,T)$ is the spot price of the zero-coupon bond that pays \$1 at time T, and

$$N_1(d) = \int_d^\infty e^r f(r)\mathrm{d}r, \quad N_2(d) = \int_d^\infty f(r)\mathrm{d}r, \quad d = \ln\left(\frac{K}{F_0}\right).$$

The pricing formula for European put options can be obtained similarly or by the put-call parity condition. Now we can write the model-free implied variance as follows.

$$\Sigma_T = 2\int_0^\infty \frac{C^F(K,T) - \max(0, F_0 - K)}{K^2}\mathrm{d}K$$

$$= 2\left(\int_0^{F_0} \frac{C^F(K,T) - F_0 + K}{K^2} \mathrm{d}K + \int_{F_0}^{\infty} \frac{C^F(K,T)}{K^2} \mathrm{d}K\right)$$

$$\equiv 2(I_1 + I_2),$$

where

$$C^F(K,T) = \frac{C(K,T)}{B(0,T)}.$$

Since $C^F(K,T) = F_0 N_1(d) - K N_2(d)$ from (25), we rearrange I_1 as follows.

$$I_1 = \int_0^{F_0} (C^F(K) - F_0 + K)\, K^{-2} \mathrm{d}K$$

$$= \int_0^{F_0} \left[F_0 N_1\left(\ln\left(\frac{K}{F_0}\right)\right) - K N_2\left(\ln\left(\frac{K}{F_0}\right)\right) - F_0 + K\right] K^{-2} \mathrm{d}K$$

$$= \int_0^{F_0} \left[F_0\left(N_1\left(\ln\left(\frac{K}{F_0}\right)\right) - 1\right) + K\left(1 - N_2\left(\ln\left(\frac{K}{F_0}\right)\right)\right)\right] K^{-2} \mathrm{d}K.$$

By the variable substitution $y = \ln\left(\frac{K}{F_0}\right)$, we get

$$I_1 = \int_{-\infty}^{0} (e^{-y}(N_1(y) - 1) + (1 - N_2(y)))\, \mathrm{d}y = \int_{-\infty}^{0} \left(\int_{-\infty}^{y} f(r) dr - e^{-y} \int_{-\infty}^{y} e^r f(r) dr\right) \mathrm{d}y$$

$$= \int_{-\infty}^{0} \int_{-\infty}^{y} (1 - e^{r-y})\, f(r) \mathrm{d}r \mathrm{d}y = \int_{-\infty}^{0} \int_{r}^{0} (1 - e^{r-y})\, f(r) \mathrm{d}y \mathrm{d}r$$

$$= \int_{-\infty}^{0} (y + e^{r-y})\, |_r^0\, f(r) \mathrm{d}r = \int_{-\infty}^{0} (e^r - r - 1)\, f(r) \mathrm{d}r.$$

Similarly, we have

$$I_2 = \int_0^{\infty} (e^r - r - 1)\, f(r) dr.$$

Hence,

$$\Sigma_T = 2(I_1 + I_2) = 2\int_{-\infty}^{\infty} (e^r - r - 1)\, f(r) dr = -2\int_{-\infty}^{\infty} r\, f(r) dr = -2\mathrm{E}_0^{\mathbb{F}}[R_T].$$

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