THE NUMBER OF ORDERPRESERVING MAPS OF FENCES AND CROWNS (PREPRINT)

JAMES D. CURRIE AND TERRY I. VISENTIN

Abstract. We perform an exact enumeration of the order-preserving maps of fences (zig-zags) and crowns (cycles). From this we derive asymptotic results.

1. Introduction and Notation

Let $\mathcal{P}$ be a partially ordered set (poset). Then an **order-preserving map** of $\mathcal{P}$ is a function $\Phi : \mathcal{P} \to \mathcal{P}$ such that $\Phi(x) \leq \Phi(y)$ whenever $x \leq y$. Suppose that a sequence $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots$ is given, where $\mathcal{P}_i$ is a poset on i elements. Let $\mathcal{A}(\mathcal{P})$ be the set of automorphisms of $\mathcal{P}$ and $\mathcal{M}t(\mathcal{P})$ the set of order-preserving maps of $\mathcal{P}$ into itself. In recent work on fixed points, Rival and Rutkowski [4] conjecture that

$$\lim_{i\to\infty} \frac{|\mathcal{A}(\mathcal{P}_i)|}{|\mathcal{M}(\mathcal{P}_i)|} = 0,$$

no matter how the $\mathcal{P}_i$ are chosen.

Taking this conjecture as motivation, Duffus, Rödl, Sands and Woodrow [2] considered the following question:

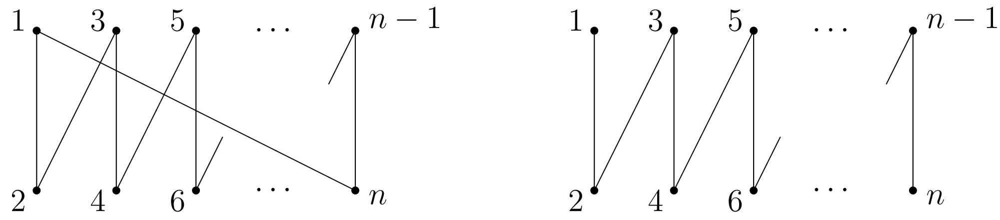
For which poset $\mathcal{P}$ on n elements is $|\mathcal{M}(\mathcal{P})|$ least?

They study two families of posets. Let n be an even number. The **crown** (cycle) $\mathcal{C}_n$ is the poset on $\{1, 2, 3, \ldots, n\}$ where the only comparabilities are $1 > 2 < 3 > 4 < \ldots < n-1 > n < 1$. (See Figure 1(a).) The **fence** (zigzag) $\mathcal{F}_n$ is the poset on $\{1, 2, 3, \ldots, n\}$ where the only comparabilities are $1 > 2 < 3 > 4 < \ldots < n-1 > n$. (See Figure 1(b).) Their conjecture is that the fence $\mathcal{F}_n$ has the fewest order-preserving maps. Using probabilistic arguments, they show that

$$\lim_{n\to\infty} \frac{|\mathcal{M}(\mathcal{F}_{n+1})|}{|\mathcal{M}(\mathcal{F}_n)|} = 1 + \sqrt{2} \text{ and } \lim_{n\to\infty} \frac{|\mathcal{M}(\mathcal{C}_{n+1})|}{|\mathcal{M}(\mathcal{C}_n)|} = 1 + \sqrt{2}.$$

In this note, we use lattice path methods to obtain exact enumerative results for $\mathcal{M}(\mathcal{F}_n)$ and $\mathcal{M}(\mathcal{C}_n)$, compare these asymptotically, and show that in fact

$$\lim_{n\to\infty} \frac{|\mathcal{M}(\mathcal{C}_n)|}{|\mathcal{M}(\mathcal{F}_n)|} = 0.$$



(a) crown (b) fence

FIGURE 1. A crown and a fence of order n

We intend to enumerate order-preserving maps of crowns and fences by putting them into correspondence with lattice paths (sequences of points in $\mathbb{Z} \times \mathbb{Z}$). Let $u = (i, j) \in \mathbb{Z} \times \mathbb{Z}$. Then j is called the **altitude** of u. We define a particular class of lattice paths for our purposes:

Let $\mathbf{u} = u_1 u_2 \cdots u_n$ be a sequence of points in $\mathbb{Z} \times \mathbb{Z}$ such that

(i) $u_1 = (1, j_1)$.

(ii) $u_{k+1} = u_k + (1, \sigma_k)$, $\sigma_k \in \{-1, 0, 1\}$ for $0 \leq k < n$.

Then $\mathbf{u}$ is called a **lattice path** with **origin** u_1 and **terminus** u_n. A **step** in $\mathbf{u}$ is the difference $u_{i-1} - u_i$. The steps $\mathcal{R} = (1, 1)$, $\mathcal{L} = (1, 0)$ and $\mathcal{F} = (1, -1)$ are called **rises**, **levels** and **falls**, respectively.

2. Bijections

The following lemma gives the correspondence between order-preserving maps of fences and lattice paths.

Lemma 1. *Let Φ be an order-preserving map of $\mathcal{F}_n$. Then the mapping $\Omega : \mathcal{M}(\mathcal{F}_n) \to L_n$, defined by $\Omega(\Phi) = (1, \Phi(1)), (2, \Phi(2)), \ldots, (n, \Phi(n))$, is bijective.*

Proof. The sequence $(1, \Phi(1)), (2, \Phi(2)), \ldots, (n, \Phi(n))$ is a lattice path. If r is a maximal element of $\mathcal{F}_n$, then $r + 1 < r$. Thus we must have $\Phi(r) \geq \Phi(r + 1)$. If $\Phi(r)$ is minimal, this means that $\Phi(r) = \Phi(r + 1)$. The rth step of our path is a level. Similarly, $\Phi(r - 1)\mathcal{L}\Phi(r)$, so the previous step of our path was a level. We see that levels come in pairs, except possibly at the beginning of the path. Also, since 1 is a maximal element of $\mathcal{F}_n$, if $\Phi(1)$ is minimal, our lattice path begins with a level. □

By a similar analysis, we can obtain a correspondence between order-preserving maps of crowns and lattice paths.

Lemma 2. *Let Φ be an order-preserving map of $\mathcal{C}_n$.*

Proof. First of all, if Φ is an automorphism of $\mathcal{C}_n$, then it is determined by $\Phi(1)$ and $\Phi(2)$. ($\Phi(1)$ must be one of the $n/2$ maximal elements $1, 3, 5, \ldots, n-1$ and $\Phi(2)$ must be one of the 2 elements covered by $\Phi(1)$.) That is, there are n automorphisms of $\mathcal{C}_n$. If Φ is not an automorphism, then suppose $\Phi(1) = r$. Consider the map $\Psi : \mathcal{C}_n) \to \mathcal{C}_n$ given by

This will be an order-preserving map of $\mathcal{C}_n$ such that $\Psi(1) = 1$ and $n \notin \mathrm{im}\Phi$, or $\Psi(1) = 2$ and $1 \notin \mathrm{im}\Phi$, depending on whether $\Phi(1)$ is maximal or minimal.

We see that $(1, \Psi(1)), (2, \Psi(2)), \ldots, (n, \Psi(n)), (n+1, \Psi(1))$ is a lattice path, and that the numbers of rises and falls in this lattice path are equal. If $\Phi(1)$ is minimal, the path begins with a level; subsequent levels will occur in pairs. The map Ψ determines a 1 to m correspondence between such lattice paths and order-preserving maps of $\mathcal{C}_n$ which are not automorphisms. □

3. Generating Functions

Let $C(x) = \sum_{m \geq 1} c_m x^m$. Let $C_1(x)$ be the generating function for those order-preserving maps for which 1 maps to 1 and let $C_2(x)$ be the generating function for those order-preserving maps for which 1 maps to 2. Then

$$C(x) = x\frac{\partial}{\partial x}(C_1(x) + C_2(x)).$$

Order-preserving maps are in 1-1 correspondence with walks on the crown. Sits are allowed as steps in the walk, but must occur in pairs inside a walk. If 1 maps to 2, the walk must begin and end with a sit.

Let $P(x)$ be the generating function for lattice paths on $2m$ steps beginning and ending on the x-axis, but staying completely above the x-axis elsewhere.

If 1 maps to 1, the walk might go completely "around" the crown in one of two directions. Otherwise, we can describe the walk in terms of lattice paths.

$$\begin{aligned}
C_1(x) &= 2(1-x)^{-1} + (\mathcal{L}\mathcal{L} \cup 2P)^* = 2(1-x)^{-1} + (1-x-2P)^{-1}. \\
C_2(x) &= \mathcal{L}(\mathcal{L}\mathcal{L} \cup 2P)^*\mathcal{L} = x(1-x-2P)^{-1}. \\
P(x) &= \mathcal{R}(\mathcal{L}\mathcal{L} \cup P)^*\mathcal{F} = x(1-x-P)^{-1}.
\end{aligned}$$

Upon rearranging this last equation, we obtain

$$P^2 + (x-1)P + x = 0$$

and hence

$$P(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2}.$$

Substituting this into our expressions for C, C_1 and C_2, gives

$$C(x) = \frac{2x(t^3 + 2(1-x)^3)}{t^3(1-x)^2}, \text{ where } t = \sqrt{1-6x+x^2}.$$

Fence generating functions

No barriers:

$$\begin{aligned} & m[x^{n-1}](1-2x-x^2)^{-1} + m[x^{n-2}](1-2x-x^2)^{-1}(1+x) \\ = \ & 1/2n[x^{n-1}]\frac{1+x+x^2}{1-2x-x^2} = 1/2[x^n]x\frac{\partial}{\partial x}\left(\frac{x(1+x+x^2)}{1-2x-x^2}\right) \\ = \ & 1/2[x^n]\frac{x(1-x)(1+3x+5x^2+x^3)}{(1-2xx^2)^2} \end{aligned}$$

which after bisection of the series gives

$$[x^m]\frac{4x(1-x^2)}{(1-6x+x^2)^2}.$$

Hit zero:

$\Phi(1) = k$, odd: $(Qx)^k(1-2x-x^2)^{-1}(1+x)$ where Q is the generating function for the number of lattice paths in the upper half-plane with origin and terminus on the x-axis, and with levels occurring in pairs.

Letting $y = x^2$, we have

$$Q = (1-y)^{-1}\frac{1}{1-P(1-y)^{-1}} = \frac{2}{1-y+t}$$

where $t = \sqrt{1-6y+y^2}$.

$\Phi(1) = k$, even: $x(Qx)^k(1-2x-x^2)^{-1}(1+x)$

Summing over all k gives

$$F_0 = \frac{x^2(1+x)Q(1+x^2Q)}{(1-2x-x^2)(1-x^2Q^2)}.$$

Using similar arguments,

$$F_{n+1} = \frac{x^3(1+x)Q(1+Q)}{(1-2x-x^2)(1-x^2Q^2)}.$$

Thus,

$$F_0 + F_{n+1} = \frac{x^2(1+x)^2Q}{(1-2x-x^2)(1-xQ)}.$$

Finally, after bisection of the above series, we obtain

$$F = \frac{y(4-4y^2-4t+4yt+2t^2+t^3)}{t^4}.$$

$F = F_{\text{all}} - F_0 - F_{n+1}$. Here we note that $F_0 \cap F_{n+1} = \emptyset$, so we don't need inclusion-exclusion.

$F_0 = (Q\mathcal{F})^k(\mathcal{R} \sqcup \mathcal{F} \sqcup \mathcal{LL})^*$, where $Q = ((\mathcal{LL})^*P)^*(\mathcal{LL})^*$. A similar expression arises for F_{n+1}.

4. Asymptotics

Let $\alpha_1 = 3 - 2\sqrt{2}$, $Alpha_2 = 3 + 2\sqrt{2}$. These are the roots of the polynomial $1 - 6y + y^2$. The generating function F can be written as $A(y)B(y)$, where

$$A(y) = \frac{y(4 - 4y - 4t + 4yt + 2t^2 + t^3)}{(1 - \alpha_1 y)^2} \text{ and } B(y) = \frac{1}{(1 - \alpha_2 y)^2}.$$

The radii of convergence for A and B are α_2 and α_1, respectively. To apply [1, Thm. 2], we note that $b_n = [y^n]B(y) = (n+1)\alpha_2^n$, and $\lim_{n\to\infty} b_{n-1}/b_n = 1/\alpha_2 \neq 0$. It follows that $f_n \sim A(1/\alpha_2)b_n \sim \frac{\sqrt{2}}{2}n\alpha_2^n$.

The generating function C can be written as $A(y)B(y)$, where

$$A(y) = \frac{4y(1-y)}{(1 - \alpha_1 y)^{3/2}} \text{ and } B(y) = \frac{1}{(1 - \alpha_2 y)^{3/2}}.$$

The radii of convergence for A and B are α_2 and α_1, respectively. In this case, we have $b_n = [y^n]B(y) = 2^{-3/4}\binom{-3/2}{n}\alpha_2^n$. This leads to $c_n \sim \frac{2^{1/4}}{\sqrt{\mathcal{P}i}}n^{1/2}\alpha_2^n$.

Acknowledgements: This work was supported by the authors' NSERC Research Grants.

References

[1] E. A. Bender, *Asymptotic methods in enumeration*, SIAM Rev. **16** (1974), 485–515.
[2] Duffus, Rödl, Sands and Woodrow, (unpublished work).
[3] I. P. Goulden and D. M. Jackson, *"Combinatorial Enumeration"*, Wiley, New York, 1983.
[4] I. Rival and Rutkowski, *Does almost every isotone self-map have a fixed point?* (preprint).

Appendix

5. Numbers of order-preserving maps of $\mathcal{F}_{2m}$ and C_{2m}

m	Fences	Crowns
1	3	
2	31	36
3	275	234
4	2,199	1,544
5	16,459	10,030
6	117,831	64,044
7	817,323	403,410
8	5,537,839	2,514,960
9	36,851,091	15,554,646
10	241,745,391	95,600,180
11	1,567,625,795	584,585,914
12	10,068,827,463	3,559,712,280
13	64,155,742,299	21,599,884,670
14	406,006,112,919	130,672,946,236
15	2,554,364,527,963	788,493,451,170
16	15,988,928,166,495	4,747,161,894,944
17	99,635,526,556,963	28,524,129,337,510
18	618,433,239,157,695	171,092,732,081,220
19	3,825,108,375,774,579	1,024,646,192,483,466
20	23,584,482,142,733,815	6,127,864,874,247,720