The Domino Problem is Undecidable on Surface Groups

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Abstract

We show that the domino problem is undecidable on orbit graphs of non-deterministic substitutions which satisfy a technical property. As an application, we prove that the domino problem is undecidable for the fundamental group of any closed orientable surface of genus at least 2.

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1 Introduction

Initially studied by Wang [22], the domino problem is the algorithmic question of determining if a finite set of unit square tiles with colored edges –called Wang tiles– can be used to tile the plane in such a way that adjacent edges of neighbor Wang tiles have the same color. It was originally conjectured by Wang that the domino problem was decidable. However, Berger [5] and later Robinson [21] both combined their constructions of an aperiodic set of Wang tiles and a reduction from the halting problem of Turing machines to show that the domino problem was in fact undecidable.

The domino problem can be naturally extended to a much broader context. Let Γ be a labeled directed infinite graph, $\mathcal{A}$ a finite set, and $F = \{p_1, \dots, p_n\}$ a finite list of colorings p_i of vertices of finite connected subgraphs of Γ by $\mathcal{A}$. The domino problem of Γ is the language of all codings of pairs $(\mathcal{A}, F)$ as above, for which there exists a coloring of the vertices of Γ such that none of the $p_i \in F$ embed as a colored labeled subgraph. Naturally, Wang's domino problem can be reinterpreted in this setting by letting Γ be the bi-infinite square grid, $\mathcal{A}$ the set of Wang tiles, and F the list of all horizontal or vertical pairs of tiles whose colors do not match.

A particularly interesting case is when Γ is a labeled directed Cayley graph of a finitely generated group G. In this case, there is a direct correspondence between colorings of the vertices of Γ by $\mathcal{A}$ which avoid a list of forbidden colored subgraphs as described above, and subshifts of finite type (SFT), that is, closed and translation invariant subsets of $\mathcal{A}^G$ which are determined by a finite list of forbidden patterns. What is more, it can be shown that the domino problems of all such Cayley graphs of G are computationally many-one

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equivalent, and thus one may speak about the domino problem of G. In the particular case when $G = \mathbb{Z}$, the domino problem is decidable: every $\mathbb{Z}$-SFT can be represented by a labeled finite graph [18], and the existence of a configuration in the SFT (i.e. a bi-infinite word) is equivalent to the existence of a cycle in the graph. The case of $\mathbb{Z}^2$ coincides with the formalism of Wang tiles, and is thus undecidable.

The domino problem on graphs other than the Cayley graphs of $\mathbb{Z}$ and $\mathbb{Z}^2$ has been studied by several authors. The undecidability for a graph which models the hyperbolic plane was settled by Kari [15], and can also be obtained from the construction of a hierarchical aperiodic tiling on the hyperbolic plane by Goodman-Strauss [12] and by Margenstern [19]. There has also been research in the case of graphs which can be obtained by self-similar substitutions [4]. For finitely generated groups, the only groups where the domino problem is known to be decidable are virtually free groups, and it is even conjectured that they are the only ones [3].

▶ **Conjecture 1.** *A finitely generated group has decidable domino problem if and only if it is virtually free.*

An interesting take on this conjecture comes from the fact that the domino problem can be expressed in MSO logic [22, 2]. The MSO logic over the Cayley graph of a finitely generated group is decidable if and only if the group is context-free [17], and a group is virtually-free if and only if it is context-free and accessible [20]. Stated differently, (using that all finitely presented groups are accessible [11]) groups that are not virtually free have an undecidable MSO logic. Proven true, the domino problem conjecture would show that the domino problem fragment is "big enough" in MSO. Recent results support the conjecture: decidability of the domino problem is a quasi-isometry invariant for finitely presented groups [10] – i.e. a geometric property of the group – and the conjecture holds true for Baumslag-Solitar groups [1], polycyclic groups [13] and groups of the form $G_1 \times G_2$ [14]. A survey on the domino problem for finitely generated groups can be found in [6, Chapter 9].

The results of Aubrun and Kari [15, 1] share a common factor: the domino problem is shown to be undecidable on two specific structures that grow exponentially with integer or rational base. But what if the structure grows with a different base? The technique employed by those authors –reduction from the immortality problem of rational piecewise affine maps– seems difficult to adapt in this case. A class of structures which can grow non-regularly is given by orbit graphs of non-deterministic substitutions (Section 3). This class of structures includes the hyperbolic plane model of [15], which can be thought of as an orbit graph of the one-letter substitution $0 \mapsto 00$. Using a technique involving the superposition of two orbit graphs of substitutions, presented in [9], we show that the domino problem is undecidable on all orbit graphs of non-deterministic substitutions that satisfy a technical property (Section 4). As an application of the previous result we show that the domino problem of the fundamental group of any closed orientable surface of genus at least 2 is undecidable (Section 5). Finally, we discuss the case of word-hyperbolic groups (Section 6) and show that if a famous conjecture of Gromov –or a weaker version– holds, then the only word-hyperbolic groups with decidable domino problem are the virtually free groups, hence confirming the conjecture for this class of groups.

2 Subshifts on graphs and the domino problem

We define a *graph* Γ to be a triple $(V_\Gamma, E_\Gamma, L_\Gamma)$ where V_Γ is an infinite countable set of vertices, $E_\Gamma \subset V_\Gamma^2$ is the set of edges, such that $|\{u \in V_\Gamma \mid (u,v) \in E_\Gamma \text{ or } (v,u) \in E_\Gamma\}| < M$ for every vertex $v \in V_\Gamma$, where M is some constant, and $L_\Gamma \colon E_\Gamma \to L$ is a labeling function which assigns to every edge a label in a finite set L. Important examples of such graphs are Cayley graphs of finitely generated groups. More precisely, given a finitely generated group G and a finite set of generators $\mathcal{S}$, its Cayley graph is given by $V_\Gamma = G$, $E_\Gamma = \{(g, gs) \mid g \in G, s \in \mathcal{S}\}$, $L_\Gamma((g, gs)) = s$. Let $\Gamma = (V_\Gamma, E_\Gamma, L_\Gamma)$ be a graph as defined above. Let S, T be two finite subsets of V_Γ. A mapping $\phi\colon S \to T$ is a *label preserving graph isomorphism* if ϕ is a bijection and

- for all $u, v \in S$, $(u, v) \in E_\Gamma$ if and only if $(\phi(u), \phi(v)) \in E_\Gamma$;
- for all $u, v \in S$, $L_\Gamma((u, v)) = L_\Gamma((\phi(u), \phi(v)))$.

Let $\mathcal{A}$ be a finite alphabet and Γ a graph. The set of mappings from V_Γ to $\mathcal{A}$, denoted $\mathcal{A}^\Gamma$, is the set of *configurations* over Γ. Endowed with the prodiscrete topology, the set $\mathcal{A}^\Gamma$ is compact and metrizable. If $S \subset V_\Gamma$ is a finite and connected set of vertices, a *pattern with support* S is a mapping $p\colon S \to \mathcal{A}$. A pattern $p\colon S \to \mathcal{A}$ *appears* in a configuration $x \in \mathcal{A}^G$ (resp. in a pattern $p'\colon S' \to \mathcal{A}$) if there exists a finite set of vertices $T \subset V_\Gamma$ (resp. $T \subset S'$) and a label preserving graph isomorphism $\phi\colon S \to T$ such that $p_u = x_{\phi(u)}$ (resp. $p_u = p'_{\phi(u)}$) for every $u \in S$. In this case, we denote $p \sqsubset x$ (resp. $p \sqsubset p'$).

A *subshift* $X_F \subset \mathcal{A}^\Gamma$ is a set of configurations that avoid some set of forbidden patterns F, i.e. $X_F := \{x \in \mathcal{A}^\Gamma \mid$ no pattern of F appears in $x\}$. This notion extends the classical definition of subshift for group actions to arbitrary graphs. A *subshift of finite type* (SFT) is a subshift for which F can be chosen finite – equivalently, an SFT may also be defined by a finite set of allowed patterns. In the case where the support of all the forbidden patterns in F consist of two vertices connected by an edge, we say X_F is a *nearest neighbor subshift*.

Given a graph Γ and a finite alphabet $\mathcal{A}$, a pattern as defined above can be encoded by a finite graph, which is an induced finite subgraph of Γ, with labels on edges and letters from $\mathcal{A}$ on vertices. This is what is meant in the sequel by *coding* of a pattern.

Let Γ be a graph in the previous sense. The *domino problem for* Γ is defined as the set $\mathtt{DP}(\Gamma)$ of codings of finite sets of forbidden patterns F such that $X_F \neq \emptyset$. If the set $\mathtt{DP}(\Gamma)$ is recursive, we say that Γ *has decidable domino problem*.

3 Substitutions, orbits and tilings

Inspired by [9], we associate a tiling of $\mathbb{R}^2$ to the orbit of an infinite word $w \in \mathcal{A}^\mathbb{Z}$ under the action of a substitution, in which every tile codes a production rule of the substitution.

3.1 Substitution systems

We first define parent functions, which will be used to give precise descriptions of orbits of infinite words under the action of a substitution. A *parent function* $P\colon \mathbb{Z} \to \mathbb{Z}$ is an onto and non-decreasing function. In particular, such a function P satisfies that for every $i \in \mathbb{Z}$, $P(i+1) - P(i) \in \{0, 1\}$.

A *non-deterministic substitution* is a couple $(\mathcal{A}, R)$ where $\mathcal{A}$ is a finite alphabet and $R \subset \mathcal{A} \times \mathcal{A}^*$ is a finite set called the *relation*, and whose elements are called *production rules*. We say that an infinite word $\omega \in \mathcal{A}^\mathbb{Z}$ *produces* the word $\omega' \in \mathcal{A}^\mathbb{Z}$ with respect to the parent function P if for every $i \in \mathbb{Z}$, one has $(\omega_i, \omega'_{|P^{-1}(i)}) \in R$, where $\omega'_{|P^{-1}(i)}$ is the finite subword of ω' that appears on indices $\{j \in \mathbb{Z} \mid P(j) = i\}$. In this case, by abuse of notation, we denote $(\omega, \omega') \in R$. An *orbit* of a non-deterministic substitution $(\mathcal{A}, R)$ is a set $\left\{(\omega^i, P_i)\right\}_{i\in\mathbb{Z}} \in \left(\mathcal{A}^\mathbb{Z} \times \mathbb{Z}^\mathbb{Z}\right)^\mathbb{Z}$ such that for every $i \in \mathbb{Z}$, P_i is a parent function, and the word ω^i produces the word ω^{i+1} with respect to P_i. A *deterministic substitution* (or substitution for short) is a non-deterministic substitution where the relation is a function. A non-deterministic substitution $(\mathcal{A}, R)$ *has an expanding eigenvalue* if there exist $\lambda > 1$ and $v \colon \mathcal{A} \to \mathbb{R}_{>0}$ such that for every $(a, w) \in R$ if we write $w = w_1 w_2 \dots w_{|w|}$ we have,

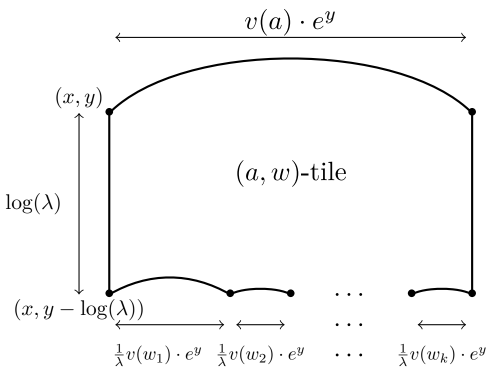
$$\lambda \cdot v(a) = \sum_{i=1}^{|w|} v(w_i).$$

▶ **Example 2.** The substitution given by the production rule $0 \mapsto 00$ has an expanding eigenvalue. This can be verified by choosing $\lambda = 2$ and $v(0) = 1$.

▶ **Example 3.** The substitution $\sigma_{\texttt{gold}}$ given by the production rules $a \mapsto aab$, $b \mapsto ba$ has an expanding eigenvalue. This can checked with $\lambda = \frac{3+\sqrt{5}}{2}$ and $v(a) = \frac{1+\sqrt{5}}{2}$, $v(b) = 1$.

3.2 Orbits as tilings of $\mathbb{R}^2$

Let $(\mathcal{A}, R)$ be a non-deterministic substitution with an expanding eigenvalue $\lambda > 1$ and $v \colon \mathcal{A} \to \mathbb{R}_{>0}$. For every production rule $(a, w) \in R$, define the (a, w)*-tile in position* $(x, y) \in \mathbb{R}^2$ as the square polygon with $|w| + 3$ edges pictured below, where $w = w_1 \dots w_k$ (horizontal edges are curved to be more visible, but are in fact just straight lines).



▶ Remark 4. The length of the top edge and the sum of lengths of bottom edges of this tile are the same. Since $(\mathcal{A}, R)$ has an expanding eigenvalue $\lambda > 1$ with v, one has

$$\sum_{j=1}^{k} \frac{1}{\lambda} v(w_i) \cdot e^y = \frac{e^y}{\lambda} \cdot \lambda \cdot v(a) = v(a) \cdot e^y,$$

so that the bottom right vertex $(x + \frac{1}{\lambda}\left(v(w_1) + \dots + v(w_k)\right) e^y, y - \log(\lambda))$ is indeed $(x + v(a) \cdot e^y, y - \log(\lambda))$.

The $(\mathcal{A}, R)$*-tiles* is the set of all (a, w)-tiles in position (x, y) for all possible $(a, w) \in R$ and $(x, y) \in \mathbb{R}^2$. A *tiling of* $\mathbb{R}^2$ *with* $(\mathcal{A}, R)$*-tiles*, or $(\mathcal{A}, R)$-tiling for short, is a countable collection of $(\mathcal{A}, R)$-tiles that covers $\mathbb{R}^2$ and have pairwise disjoint interiors, such that tiles are edge-to-edge –the intersection of two tiles is either empty or a full edge and two vertices. In Figure 2 we illustrate a $\sigma_{\texttt{gold}}$-tiling (in blue) and a $0 \mapsto 00$-tiling (in grey).

▶ **Proposition 5.** *If a substitution* $(\mathcal{A}, R)$ *with an expanding eigenvalue admits orbits, then there exists a tiling of* $\mathbb{R}^2$ *with* $(\mathcal{A}, R)$*-tiles.*

4 Undecidability of the domino problem on orbit graphs

Let $u = (u_i)_{i\in\mathbb{Z}}$ be a bi-infinite sequence of positive integers. The *accumulation function of* u is the function $\Delta\colon \mathbb{Z} \to \mathbb{Z}$ given by

$$\Delta(i) = \begin{cases} \sum_{k=0}^{i-1} u_k & \text{if } i \geq 1 \\ 0 & \text{if } i = 0 \\ -\sum_{k=i}^{-1} u_k & \text{if } i \leq -1 \end{cases}.$$

Note that the family of discrete intervals $(I_k)_{k\in\mathbb{Z}}$ where $I_k = [\Delta(k); \Delta(k+1) - 1]$ forms a partition of $\mathbb{Z}$. If P is a parent function, and if we define the sequence u by $u_i = |P^{-1}(i)|$ for every $i \in \mathbb{Z}$, then we get that $P(j) = i$ for every $j \in [\Delta(i); \Delta(i+1) - 1]$, where Δ is the accumulation function of u.

Let $(\mathcal{A}, R)$ be a non-deterministic substitution. Denote $M = \max_{(a,w)\in R} |w|$.

▶ **Definition 6.** *The* orbit graph *associated with the orbit* $\Omega = \left\{(\omega^i, P_i)\right\}_{i\in\mathbb{Z}}$ *of* $(\mathcal{A}, R)$ *is the graph* Γ_Ω *with set of vertices* $\mathbb{Z}^2$*, edges* E_Ω *and labeling function* $L_\Omega\colon E_\Omega \to \{\texttt{next}\} \cup [0; M-1]$ *given by*

- *for every* $i, j \in \mathbb{Z}$*,* $((i,j),(i,j+1)) \in E_\Omega$ *and* $L_\Omega\left(((i,j),(i,j+1))\right) = \texttt{next}$*;*
- *for every* $i \in \mathbb{Z}$ *and every* $k \in [\Delta_{i+1}(j); \Delta_{i+1}(j+1) - 1]$*,* $((i,j),(i+1,k)) \in E_\Omega$ *and* $L_\Omega\left(((i,j),(i+1,k))\right) = k - \Delta_{i+1}(j)$*,*

where Δ_i *is the accumulation function associated with* $\left(|P_i^{-1}(j)|\right)_{j\in\mathbb{Z}}$ *for every* $i \in \mathbb{Z}$*.*

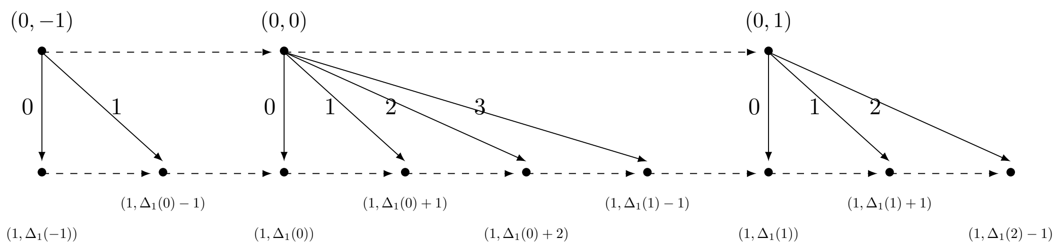


Figure 1 Part of an orbit graph. Dashed arrow are edges of the graph labeled with `next`.

In this formalism, Kari's result for the hyperbolic plane [15] is equivalent to the statement that all orbit graphs of the one-letter substitution $0 \mapsto 00$ have undecidable domino problem.

▶ **Theorem 7** (Kari [15])**.** *For all orbit graphs of the substitution* $(\{0\}, 0 \mapsto 00)$ *the domino problem is undecidable.*

The goal of this section is to show that the domino problem of any orbit graph associated to an orbit of a non-deterministic substitution $(\mathcal{A}, R)$ with an expanding eigenvalue λ is undecidable. The general idea is to show that, given any SFT over an orbit graph of $0 \mapsto 00$, it is possible to encode it in an orbit graph of $(\mathcal{A}, R)$. We do this in two steps. First (Section 4.1), we show a variation of the "Technical Lemma" of Cohen and Goodman-Strauss [9], where we prove that it is possible to encode the structure of orbit graphs of $0 \mapsto 00$ in an SFT over $(\mathcal{A}, R)$. Then (Section 4.2), we prove that in addition to the structure of its orbit graph, it is possible to encode any SFT over orbit graphs of $0 \mapsto 00$ in an SFT over $(\mathcal{A}, R)$.

4.1 Superposition of orbits

Let us fix a non-deterministic substitution $(\mathcal{A}, R)$ with an expanding eigenvalue $\lambda > 2$ – this latter assumption ensures that letters of an alphabet $\mathcal{B}$ described below are non-degenerate, and will be suppressed later. Without loss of generality, we may choose the function $v\colon \mathcal{A} \to \mathbb{R}_{>0}$ associated to λ such that $v(a) > 4$ for each $a \in \mathcal{A}$.

Let Ω be an orbit of $(\mathcal{A}, R)$. We shall construct a finite alphabet $\mathcal{B}$ and a finite set of forbidden patterns F such that the subshift $Y \subset \mathcal{B}^{\Gamma_\Omega}$ over orbit graph Γ_Ω, defined by the set of forbidden patterns F has the following properties:

1. Y is non-empty,
2. every configuration $y \in Y$ encodes an orbit graph of the substitution $(\{0\}, 0 \mapsto 00)$.

Consider an orbit Ω of $(\mathcal{A}, R)$ and Φ an orbit of $(\{0\}, 0 \mapsto 00)$. By Proposition 5 both of these orbits can be realized as tilings of $\mathbb{R}^2$. Symbols from $\mathcal{B}$ will encode non-empty finite regions of the tiling with $(\{0\}, 0 \mapsto 00)$ that are witnessed by $(\mathcal{A}, R)$-tiles. These regions will be chosen in such a way that their union recovers the whole tiling and they are pairwise disjoint (see Figure 2).

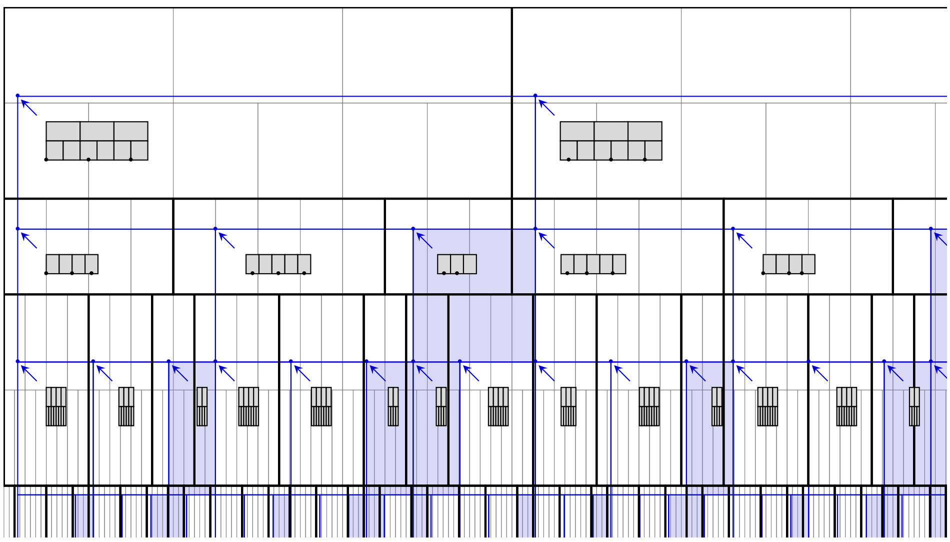
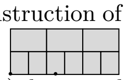
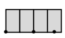


Figure 2 Γ_Φ (in black) superimposed over an orbit graph of $(\mathcal{A}, R)$ (in blue) leads to the construction of the subshift $Y \subset \mathcal{B}^{\Gamma_\Omega}$: letters of the alphabet $\mathcal{B}$ are blocks of the orbit graph Γ_Φ, such as  or  .A careful examination shows that dimensions of blocks in $\mathcal{B}$ are bounded: if (t, h) denotes the width and height of a block in $\mathcal{B}$, we get that $\min_{a\in\mathcal{A}} \lfloor \frac{v(a)}{4} \rfloor \leq t \leq \max_{a\in\mathcal{A}} \lfloor 1 + \frac{v(a)}{2} \rfloor$ and $\frac{\log(\lambda)}{\log(2)} - 1 < h \leq \frac{\log(\lambda)}{\log(2)} + 1$. Thus $\mathcal{B}$ is finite, and the assumption $\lambda > 2$ and $v(a) > 4$ ensures that blocks have non-degenerate dimensions.

The set of forbidden patterns F is the finite set of patterns that do not correspond to a valid encoding of the orbit graph of $(\{0\}, 0 \mapsto 00)$. In other words, we allow only patterns in which the finite regions of $(\{0\}, 0 \mapsto 00)$ that are encoded are consistent from one neighbor to one other. We thus obtain an SFT Y on Γ_Ω which encodes Φ and is non-empty.

▶ **Lemma 8.** *For every orbit Ω of $(\mathcal{A}, R)$ the subshift of finite type $Y \subset \mathcal{B}^{\Gamma_\Omega}$ is non-empty.*

4.2 Simulation of SFTs over of $0 \mapsto 00$ on orbit graphs of $(\mathcal{A}, R)$

Let Ω and Φ be orbits of $(\mathcal{A}, R)$ and $(\{0\}, 0 \mapsto 00)$ respectively, and Γ_Ω, Γ_Φ be orbit graphs of Ω and Φ respectively. Let Σ be a finite alphabet and F_Σ a set of nearest neighbor forbidden patterns on Γ_Φ over the alphabet Σ. We denote by X_Σ the SFT defined by forbidden patterns F_Σ. In order to encode X_Σ into Γ_Ω, we use the same method as above for Y, but enrich the patterns of $0 \mapsto 00$ with colors taken from Σ. We then construct forbidden patterns that ensure that no forbidden patterns from F_Σ appear.

More formally, we define $\mathcal{B}_\Sigma$ as the set of pairs (b, p_b) such that $b \in \mathcal{B}$ and $p_b \colon \Gamma_b \to \Sigma$ is a pattern. For a pattern p on Γ_Ω with alphabet B_Σ denote by $\pi_{\mathcal{B}}(p)$ the restriction to the first coordinate of $\mathcal{B}_\Sigma$. And denote by $q(p) \colon \Gamma_{\pi_{\mathcal{B}}(p)} \to \Sigma$ the pattern over $(\{0\}, 0 \mapsto 00)$ whose support is the graph $\Gamma_{\pi_{\mathcal{B}}(p)}$ and is obtained by pasting together the corresponding patterns p_b on the second coordinate of B_Σ.

Define $F_{\mathcal{B},\Sigma}$ as the set of all patterns p over the alphabet $\mathcal{B}_\Sigma$ which have supports which consist in three vertices $\{u, v, w\}$ in Γ_Ω such that $(u, v), (u, w)$ are edges, $L((u, v)) = \texttt{next}$ and $L((u, w)) = \ell$ for some ℓ appearing in the parent matching labels of the orbit graph Γ_Ω, and that satisfy one of the following two properties:

1. The pattern $\pi_{\mathcal{B}}(p)$ obtained by restricting p to the first coordinate of $\mathcal{B}_\Sigma$ is in F;
2. The pattern $q(p)$ obtained by pasting the patterns of p described by the second coordinate of $\mathcal{B}_\Sigma$ contains a forbidden pattern from F_Σ.

Clearly $F_{\mathcal{B},\Sigma}$ has finitely many patterns (up to label preserving graph isomorphism). For any orbit Ω of $(\mathcal{A}, R)$ we define the subshift of finite type $Y_\Sigma \subset (B_\Sigma)^{\Gamma_\Omega}$ as the set of all colorings of Γ_Ω by B_Σ where no pattern from $F_{\mathcal{B},\Sigma}$ appears.

This construction leads to the following Lemma, expressing the fact that X_Σ is indeed encoded into Y_Σ.

▶ **Lemma 9.** *Let X_Σ be the subshift on Γ_Φ with alphabet Σ defined by the nearest neighbor forbidden patterns F_Σ and let $Y_\Sigma \subset (B_\Sigma)^{\Gamma_\Omega}$ be defined as above. Then $Y_\Sigma = \emptyset$ if and only if $X_\Sigma = \emptyset$.*

▶ Remark 10. The alphabet B_Σ and the set of forbidden patterns $F_{\mathcal{B},\Sigma}$ which define Y_Σ only depend upon Σ, F_Σ and the substitution $(\mathcal{A}, R)$, and not on the choice of the orbit Ω of $(\mathcal{A}, R)$.

▶ **Theorem 11.** *The domino problem is undecidable on any orbit graph of a non-deterministic substitution with an expanding eigenvalue.*

Proof. Let us first assume that the expanding eigenvalue λ associated to $(\mathcal{A}, R)$ satisfies $\lambda > 2$. Let Σ and F_Σ be respectively an alphabet and a nearest neighbor set of forbidden patterns for an orbit graph Γ_Φ of an orbit Φ of $(\{0\}, 0 \mapsto 00)$ which define a nearest neighbor SFT X_Σ. By Lemma 9 we know that $X_\Sigma = \emptyset$ if and only if $Y_\Sigma = \emptyset$. Furthermore, we claim that the alphabet and set of forbidden patterns which define Y_Σ can be constructed effectively from Σ and F_Σ. Indeed, the subshift Y does not depend upon Σ and thus its alphabet $\mathcal{B}$ and forbidden patterns F can be hard-coded in the algorithm. It is easy to see that from $\mathcal{B}$ one can effectively construct the alphabet B_Σ and the forbidden patterns $F_{\mathcal{B},\Sigma}$ which define Y_Σ.

These two facts together show that if $\texttt{DP}(\Gamma_\Omega)$ is decidable and $\lambda > 2$, then so is $\texttt{DP}(\Gamma_\Phi)$. Using the result of Kari (Theorem 7) we have that $\texttt{DP}(\Gamma_\Phi)$ is undecidable, hence $\texttt{DP}(\Gamma_\Omega)$ is also undecidable.

For the case where $1 < \lambda \leq 2$, we consider the relation R^m defined recursively by:

- $R^1 = R$.

- R^{k+1} is the set of all pairs $(a, (c_1^1 \dots c_{\ell_1}^1)(c_1^2 \dots c_{\ell_2}^2) \dots (c_k^1 \dots c_{\ell_k}^1))$ in $\mathcal{A} \times \mathcal{A}^*$ for which there is a pair $(a, b_1 \dots b_k) \in R^k$ such that $(b_i, c_1^i \dots c_{\ell_i}^i) \in R$ for each $i \in \{1, \dots, k\}$.

In other words, R^m is the set of all relations that can be obtained by starting with a symbol $a \in \mathcal{A}$ and replacing m times each letter by the right hand side of a production rule of R. Let $n \in \mathbb{N}$ such that $\lambda^n > 2$ and note that the substitution $(\mathcal{A}, R^n)$ has the expanding eigenvalue $\lambda^n > 2$. Lemma 9 then provides an encoding of the substitution $(\{0\}, 0 \mapsto 00)$ on orbit graphs of $(\mathcal{A}, R^k)$ for any $k \in \{0, \dots, n-1\}$, in the form of SFTs $Y_\Sigma^{n,k}$ such that $Y_\Sigma^{n,k} = \emptyset$ if and only if $X_\Sigma = \emptyset$. Then we are able to build an SFT Z on Γ_Ω which encodes a copy of $Y_\Sigma^{k,n}$ for each $k \in \{0, \dots, n-1\}$. One can verify that $Z = \emptyset$ if and only if $X_\Sigma = \emptyset$, leading to the same reduction as in the case $\lambda > 2$. ◀

5 The domino problem for surface groups

A fundamental result of geometry is that up to homeomorphism, closed orientable surfaces are completely classified by their genus g: any such surface is either homeomorphic to a sphere or to a finite connected sum of tori. In this section we shall classify the domino problem of their fundamental groups.

5.1 Surface groups

The *surface group of genus* g is the group defined by the following presentation:

$$G_g = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \dots [a_g, b_g] \rangle,$$

where $[a, b] = aba^{-1}b^{-1}$ is the commutator of a and b. It is interesting to notice that $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$ is the surface group of genus 1, and hence by Berger's result [5] its associated domino problem is undecidable.

The domino problem for a finitely generated group is known to be a commensurability invariant [6, Corollary 9.53]. It turns out that all surface groups of genus $g \geq 2$ are commensurable [8, Proposition 6.7]. By combining these two facts, it would be enough to prove the undecidability of the domino problem for just the surface group of genus 2. In the sequel, we shall denote by G the surface group of genus 2, i.e. the group with finite presentation

$$G = \langle a, b, c, d \mid [a, b][c, d] \rangle,$$

denote by S the symmetric closure of its generating set $\{a, b, c, d\}$, and by 1_G its identity.

The Cayley graph of G associated with the presentation above is not an orbit graph of some substitution with an expanding eigenvalue, but can be seen as such just by assigning different labels to the edges. Moreover we shall see that these labels can be obtained locally, which means that we can code the relabeling inside an SFT.

5.2 Finding a substitution in the surface group of genus 2

The goal of this section is to establish a parallel between the Cayley graph of the surface group $\mathcal{C}_G := \Gamma(G, S)$ and the orbit graph of a particular substitution.

The group G has only one relation $[a, b][c, d] = 1_G$. Thus the only minimal cycles of the Cayley graph are cyclic permutations of $[a, b][c, d]$. We call them *elementary cycles*. Moreover, any edge in the Cayley graph is part of at least one elementary cycle, since all generators and their inverses appear in the relation. Let $d(g, h)$ be the smallest number of elementary cycles that must be crossed to go from g to h in $\mathcal{C}_G$. Let $B_i = \{g \in G \mid d(1_G, g) \leq i\}$ be the ball of radius i and $C_i = \{g \in G \mid d(1_G, g) = i\}$ be the sphere of radius i, so that $B_{i+1} \setminus B_i = C_{i+1}$.

Consider an element $g \in C_i$ for $i \geq 1$. There are exactly two elements $s \in S$ such that $gs \in C_i$. There can be either (a) one or (b) none $s \in S$ so that $gs \in C_{i-1}$. We must therefore have that there are 5 and 6 values $s \in S$ such that $gs \in C_{i+1}$ for types (a) and (b) respectively. More precisely, it can be verified that the sequence of elements of C_{i+1} that is obtained by following an elementary cycle from an element of type (a) in C_i has the type sequence $\mathtt{ab}^5\mathtt{ab}^5\mathtt{ab}^5\mathtt{ab}^5\mathtt{ab}^4$ and the sequence of types for an element of type (b) is $\mathtt{ab}^5\mathtt{ab}^5\mathtt{ab}^5\mathtt{ab}^5\mathtt{ab}^5\mathtt{ab}^4$.

This leads us to define the substitution $s\colon \{\mathtt{a},\mathtt{b}\} \to \{\mathtt{a},\mathtt{b}\}^*$ given by

$$\begin{cases} s(\mathtt{a}) = (\mathtt{ab}^5)^4\mathtt{ab}^4 \\ s(\mathtt{b}) = (\mathtt{ab}^5)^5\mathtt{ab}^4. \end{cases}$$

From now on, we fix $\Omega = \left(\omega^i, P_i\right)_{i\in\mathbb{Z}}$ an orbit of the substitution s defined above, and denote by Γ its associated orbit graph. Let us note that s admits an expanding eigenvalue ($\lambda = 17 + 12\sqrt{2}$ and $v(\mathtt{b})/v(\mathtt{a}) = \frac{1+\sqrt{2}}{2}$).

The similarities between the two graphs will allow us to perform a reduction from the domino problem on Γ (shown to be undecidable in Section 4) to the domino problem on the surface group of genus 2. In order to do this reduction, all we need is a computable map which sends sets of pattern codings over Γ into sets of pattern codings over $\mathcal{C}_G$ such that the sets defining a non-empty subshift are mapped into sets defining a non-empty subshift and vice-versa. This is not trivial because some of the edges are lost going from Γ to $\mathcal{C}_G$. In order to recover those edges, we shall construct an SFT X over G which locally recovers the lost information and use it to build the bijection needed for the reduction. Note that technically we do not need an SFT to do so, a computable bijection would be enough. However doing it with an SFT provides a locally computable mapping, which is a nice bonus.

Definition of X

To define the SFT X, we introduce a notion of directions that corresponds to following edges of the orbit graph. These directions depend on the element of the group we consider, but can nevertheless be defined by local rules. The alphabet of X contains the correspondence between generators and local directions. More formally, we first consider the general alphabet $\mathcal{A}_0$, consisting of the tuples $(c, (h_1, d_1), (h_2, d_2), \ldots, (h_8, d_8))$ such that

- $c \in \{\blacksquare, \square\}$ is the color of the cell,
- $(h_1, \ldots, h_8)$ is a permutation of $S \cup S^{-1} = \{a, a^{-1}, b, b^{-1}, c, c^{-1}, d,^{-1}\}$,
- $d_1, \ldots, d_8 \in \{\leftarrow, \rightarrow, \uparrow, \downarrow_1, \downarrow_2, \downarrow_3, \downarrow_4, \downarrow_5, \downarrow_6\}$ the directions associated to each generator.

Let $x \in \mathcal{A}_0^G$ be a configuration over $\mathcal{A}_0$. For every $g \in G$, if the first coordinate of x_g is $c = \blacksquare$ (resp. $\square$), we call x_g a black (resp. white) cell.

The alphabet $\mathcal{A}_1 \subseteq \mathcal{A}_0$ is made of three types of elements with more precise directions imposed, depending on the color c:

$$(\blacksquare, (h_1, \leftarrow), (h_2, \rightarrow), (h_3, \uparrow), (h_4, \downarrow_1), (h_5, \downarrow_2), (h_6, \downarrow_3), (h_7, \downarrow_4), (h_8, \downarrow_5))$$

$$(\square, (h_1, \leftarrow), (h_2, \rightarrow), (h_3, \downarrow_1), (h_4, \downarrow_2), (h_5, \downarrow_3), (h_6, \downarrow_4), (h_7, \downarrow_5), (h_8, \downarrow_6))$$

Black cells have directions *left*, *right*, *up* and *down*, whereas whites ones have only *left*, *right* and *down*. Note that for black and white cells, *up*, *left* and *right* are unique. We can then define their top, left and right neighbors.

▶ **Definition 12.** *Let $x \in \mathcal{A}_1^G$ be a configuration over $\mathcal{A}_1$ and $g \in G$. We define:*

- gh_1 *the* left neighbor of g *in* x*, denoted by* $\leftarrow_x(g)$*,*
- gh_2 *is the* right neighbor of g *in* x*, denoted by* $\rightarrow_x(g)$*,*
- *if* x_g *is a black cell,* gh_3 *is the* top neighbor of g *in* x*, denoted by* $\uparrow_x(g)$*,*
- *for* $i \in \{1, ..., 5\}$*,* gh_{3+i} *(for a white cell,* $i \in \{1, ..., 6\}$*,* gh_{2+i}*) is the* i-th bottom neighbor of g *in* x*, denoted by* $\downarrow_{i,x}(g)$*.*

Using local rules, we forbid elementary cycles that do not have the colors shown on Figure 3. We also impose the orientations to be as drawn. For example, the right of $\blacksquare_a$ is g_2, its top is g_1^{-1}, and the other directions of a are not constrained by this cycle. Similarly, the left of $\boxed{b}$ is g_2^{-1}, its right g_3 and other directions unconstrained. To do so, we call $\mathcal{F}_1$ the set of all elementary cycles that are not of the form of Figure 3.

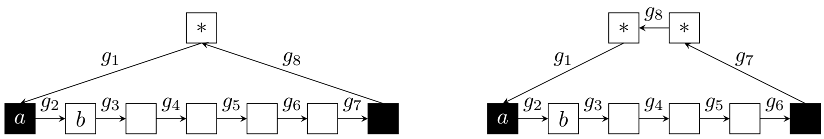


Figure 3 The two possible types of colorings of cycles. There are no color constraints on $\boxed{*}$, and the cycle $g_1 \dots g_8$ is any cyclic permutation of $[a, b][c, d]$.

We add the constraint that directions must be consistent between adjacent cells, by forbidding the finite set $\mathcal{F}_2$, which is the set of patterns on the support $\{1_G, h\}$ for $h \in G$, such that x_{1_G} and x_h are linked by mismatching directions. That is,

$$\mathcal{F}_2 = \left\{ \text{pattern } p \text{ of support}\{1_G, h\} \;\middle|\; \begin{array}{r} \leftarrow_p(1_G) = h \text{ and } \rightarrow_p(h) \neq 1_G \text{ or} \\ \rightarrow_p(1_G) = h \text{ and } \leftarrow_p(h) \neq 1_G \text{ or} \\ \uparrow_p(1_G) = h \text{ and } \forall i, \downarrow_{i,p}(h) \neq 1_G \text{ or} \\ \exists i, \downarrow_{i,p}(1_G) = h \text{ and } \uparrow_p(h) \neq 1_G \end{array} \right\}.$$

We define X as the set of all configurations over $\mathcal{A}_1$ where no forbidden patterns from $\mathcal{F}_1 \cup \mathcal{F}_2$ appear. By definition, X is an SFT.

The subshift of finite type X

We construct a configuration x in the SFT $X \subset \mathcal{A}_1^G$ as the limit of a sequence of configurations $(y_n)_{n\in\mathbb{N}}$ of another SFT $X_2 \subset (\mathcal{A}_1 \cup \{\texttt{orange}\})^G$, where

$$\texttt{orange} := \left(\blacksquare, (a, \downarrow_1), (a^{-1}, \downarrow_2), (b, \downarrow_3), (b^{-1}, \downarrow_4), (c, \downarrow_5), (c^{-1}, \downarrow_6), (d, \downarrow_7), (d^{-1}, \downarrow_8)\right),$$

and we extend the definition of neighbors consistently. X_2 is defined by $\mathcal{F}_1 \cup \mathcal{F}_2$ the same finite set of forbidden patterns as X. Intuitively, because the letter `orange` has only bottom neighbors, an orange cell can only appear once in a configuration of X_2. Moreover, one can build bigger and bigger circles around the orange cell by simply sticking elementary cycles around, colored as on Figure 3, leading to the following lemma.

▶ **Lemma 13.** *For every i, there exists a pattern $p_i \in (\mathcal{A}_1 \cup \{\texttt{orange}\})^{B_i}$ containing no forbidden patterns of $\mathcal{F}$ and such that $(p_i)_g$ is an orange cell if and only if $g = 1_G$.*

From Lemma 13, we can build a configuration y in X_2 that contains only one orange cell at the origin. And from it, we can deduce the non-emptiness of X.

▶ **Proposition 14.** *The subshift X is non-empty.*

Proof. By compactness of $(\mathcal{A}_1 \cup \{\texttt{orange}\})^G$ and Lemma 13, there exists a configuration $y \in X_2$ which coincides with the pattern p_i on B_i for all $i \in \mathbb{N}$. In particular, the orange cell appears only at the origin. The SFT X consists of all configurations on X_2 where the orange tile does not appear. By definition of y, we can find arbitrarily large regions where ■ does not appear at all. We can then extract a sequence of configurations $(y_n)_{n\in\mathbb{N}}$ of X_2 such that the orange cell does not appear in $(y_n)_{|B_n}$. Any accumulation point of the sequence $(y_n)_{n\in\mathbb{N}}$ does not contain the orange cell and is thus in X. ◀

If we carefully look at configurations in X, we observe that they are all structured with infinite lines of $\rightarrow$ and $\leftarrow$. Moreover these infinite lines can be ordered with no ambiguity. This is expressed in Lemma 15: any element g of G can be reached from the identity 1_G by first going to the appropriate infinite line, and then moving to the left or right up to g. Fix some $x \in X$, and define $\rightarrow_x^{-1}(g) :=\leftarrow_x(g)$ and $\downarrow_{1,x}^{-1}(g) :=\uparrow_x(g)$.

▶ **Lemma 15.** *For any $g \in G$, there exists i, j such that $g = \rightarrow_x^j \circ \downarrow_{1,x}^i (1_G)$.*

Proof idea. The key idea here is that we can always reorder the operations by taking another way in the graph. Starting from any path from 1_G to g, we can transform it into this "normal form" by taking a longer path that uses only $\downarrow_1$ then $\rightarrow$ operations. ◀

5.2.1 A bijection between $\mathbb{Z}^2$ and the surface group

Let $x \in X$ be fixed. We define $f_x \colon \mathbb{Z}^2 \to G$ by $f_x(i,j) = \rightarrow_x^j \circ \downarrow_{1,x}^i (1_G)$ for every $i, j \in \mathbb{Z}$.

▶ **Lemma 16.** *For every $x \in X$, the function f_x is a bijection.*

Proof idea. It is enough to show that for any group element $g \in G$, there exist uniquely defined $i, j \in \mathbb{Z}$ such that $g = \rightarrow_x^j \circ \downarrow_{1,x}^i (1_G)$. The key ingredient to prove the uniqueness of this representation is to notice that any cycle in the Cayley graph of G contains as many $\uparrow$ as $\downarrow$ operations, which can be proven by induction on the size of the cycle and a careful examination of different cases. ◀

We can moreover prove that f_x also preserves locality, in the sense that neighborhoods are almost preserved, as stated in the following lemma.

▶ **Lemma 17.** *The following equivalences are true:*

1. $\begin{cases} (u,v) \in E_\Gamma \\ L_\Gamma(u,v) = \textit{\textbf{next}} \end{cases} \Leftrightarrow f_x(v) = \rightarrow_x(f_x(u))$
2. $\begin{cases} (u,v) \in E_\Gamma \\ L_\Gamma(u,v) = k \in \{0,\dots,M-1\} \end{cases} \Leftrightarrow f_x(v) = \rightarrow_x^k \circ \downarrow_{1,x}(f_x(u))$
   where M is the number of sons of u.

The bijection f_x itself cannot be a label preserving graph isomorphism, since we lack some edges of Γ in $\mathcal{C}_G$, but it nevertheless enjoys a useful property: if φ is a label preserving graph isomorphism for Γ, then so is $f_x \circ \varphi \circ f_x^{-1}$ for $\mathcal{C}_{G,x}$, and if φ is a label preserving graph isomorphism for $\mathcal{C}_{G,x}$, then so is $f_x^{-1} \circ \varphi \circ f_x$ for Γ, where $\mathcal{C}_{G,x}$ is a relabeling of $\mathcal{C}_g$ according to the configuration x. So roughly speaking, any local pattern is preserved by f_x or by f_x^{-1}.

▶ **Corollary 18.** *Let $\mathcal{A}$ be a finite alphabet. For any configuration $c \in \mathcal{A}^G$, $p \sqsubset c \Rightarrow f_x^{-1}(p) \sqsubset f_x^{-1}(c)$. Conversely for any $d \in \mathcal{A}^\Gamma$, $q \sqsubset d \Rightarrow f_x(q) \sqsubset f_x(d)$.*

5.3 The reduction

We now have everything in hand to prove the undecidability of the domino problem on the surface group of genus 2.

▶ **Theorem 19.** *The domino problem is undecidable on the surface group of genus 2.*

Proof. Recall that Γ is the orbit graph of an orbit of the substitution s defined on page 9. Let $\mathcal{A}$ be a finite alphabet and $Y \subseteq \mathcal{A}^\Gamma$ an SFT over Γ, given by a finite set of forbidden patterns $\mathcal{F}_Y$. We define Z the SFT over G with set of forbidden patterns $F_Z := f_x(F_Y)$, where f_x is defined in Lemma 16. We prove that $Z = \emptyset$ if and only if $Y = \emptyset$.

Assume $Z = \emptyset$ and consider a configuration $c \in \mathcal{A}^G$. The configuration $d := f^{-1}(c)$ is thus in $\mathcal{A}^\Gamma$. Since $Z = \emptyset$, necessarily c contains a forbidden pattern p from the set $\mathcal{F}_Z$. Since $p \sqsubset c$, Corollary 18 implies that $f_x^{-1}(p) \sqsubset f_x^{-1}(c) = d$. So a pattern $f_x^{-1}(p)$ from $\mathcal{F}_Y$ appears in any configuration $c \in \mathcal{A}^G$, i.e. the subshift Y is empty. Similar arguments show that if Y is empty then so is Z, and the reduction is completed. ◀

▶ **Corollary 20.** *The domino problem is undecidable for every surface group of positive genus.*

Proof. The undecidability of the domino problem is a commensurability invariant [6, Corollary 9.53], and all surface groups of genus $g \geq 2$ are commensurable [8, Proposition 6.7]. By combining these two facts with Theorem 19, we obtain the undecidability of domino problem for surface groups of any genus $g \geq 2$. As the domino problem on $\mathbb{Z}^2$ –the surface group of genus 1– is undecidable, we obtain our result. ◀

6 Remarks about word-hyperbolic groups

Surface groups of genus $g \geq 2$ are special cases of a larger class of groups called *word-hyperbolic*. They can be characterized as the finitely presented groups for which Dehn's algorithm solves the word problem. An important property of the domino problem is that groups which contain subgroups with undecidable domino problem have themselves undecidable domino problem [6, Proposition 9.3.30]. This means that every group which contains an embedded copy of a surface group has undecidable domino problem. This is of special relevance due to the following conjecture by Gromov.

▶ **Conjecture 21** (Gromov). *Every one-ended word-hyperbolic group contains an embedded copy of the surface group of genus* 2*.*

In particular, if Gromov's conjecture holds, every one-ended word-hyperbolic group would automatically have undecidable domino problem. A group can have either $0, 1, 2$ or infinitely many ends. In the case when it has 0 ends it is finite and thus its domino problem is trivially decidable, and whenever it has 2 ends it is virtually $\mathbb{Z}$ and thus it is also decidable. In the case of a finitely presented group G, a fundamental result by Dunwoody [11] shows that if G has infinitely many ends it can be expressed as the fundamental group of a finite graph of groups such that every edge is a finite group and all vertices are either finite or 1-ended. It can also be shown [16] that G is virtually free if and only if all of the vertex groups in its decomposition are finite. Therefore, if G is not virtually free, it must contain a one-ended subgroup. In the case of word-hyperbolic groups, every such group in the decomposition must also be word-hyperbolic [7]. In other words, every word-hyperbolic group which is not virtually free contains a one-ended word-hyperbolic group. This implies the following.

▶ **Proposition 22.** *If Gromov's conjecture holds then the domino problem conjecture holds for all word-hyperbolic groups.*

In fact, we could obtain the same result with an even weaker version of Gromov's conjecture. We say a group G acts *translation-like* on a metric space (X, d) if the action is free and $\sup_{x \in X} d(x, gx) < \infty$ for every $g \in G$. Clearly, if H is a subgroup of G then H acts translation like on any Cayley graph of G by multiplication. A theorem by Jeandel [14] shows that if a finitely presented group H acts translation like on a Cayley graph of a finitely generated group G, then the domino problem of H is many-one reducible to the domino problem on G, in particular, we obtain that any group on which the surface group of genus 2 acts translation-like has undecidable domino problem.

▶ **Proposition 23.** *If every* 1*-ended word-hyperbolic group admits a translation-like action of the surface group of genus* 2*, then the domino problem conjecture holds for all word-hyperbolic groups.*

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