Domination Above r-Independence: Does Sparseness Help?

Carl Einarson
Royal Holloway, University of London, UK
einarson.carl@gmail.com

Felix Reidl
Birkbeck, University of London, UK
f.reidl@dcs.bbk.ac.uk

Abstract

Inspired by the potential of improving tractability via gap- or above-guarantee parametrisations, we investigate the complexity of Dominating Set when given a suitable lower-bound witness. Concretely, we consider being provided with a maximal r-independent set X (a set in which all vertices have pairwise distance at least $r+1$) along the input graph G which, for $r \geq 2$, lower-bounds the minimum size of any dominating set of G. In the spirit of gap-parameters, we consider a parametrisation by the size of the "residual" set $R := V(G) \setminus N[X]$.

Our work aims to answer two questions: How does the constant r affect the tractability of the problem and does the restriction to sparse graph classes help here? For the base case $r = 2$, we find that the problem is paraNP-complete even in apex- and bounded-degree graphs. For $r = 3$, the problem is W[2]-hard for general graphs but in FPT for nowhere dense classes and it admits a linear kernel for bounded expansion classes. For $r \geq 4$, the parametrisation becomes essentially equivalent to the natural parameter, the size of the dominating set.

2012 ACM Subject Classification Theory of computation → Parameterized complexity and exact algorithms

Keywords and phrases Dominating Set, Above Guarantee, Kernel, Bounded Expansion, Nowhere Dense

Digital Object Identifier 10.4230/LIPIcs.MFCS.2019.40

Related Version A full version of the paper is available at `https://arxiv.org/abs/1906.09180`

Acknowledgements We thank our anonymous reviewer for helpfully pointing out how to achieve a linear kernel in Theorem 14.

Introduction

The research of *above/below guarantee* parameters as first used by Mahajan and Raman [22] was an important step towards studying problems whose natural parameters provided only trivial and unsatisfactory answers. Case in point, the motivation for Mahajan and Raman was the observation that every CNF-SAT formula with m clauses trivially has an assignment that satisfies $\geq \lceil m/2 \rceil$ clauses, thus question for the maximum number of satisfied clauses is only interesting if $k > \lceil m/2 \rceil$, which of course renders the parametrised approach unnecessary. They therefore proposed to study parametrisations "above guarantee": going with the previous example, we would ask to satisfy $\lceil m/2 \rceil + k$ clauses or 'k above guarantee'. After some isolated results in that direction (e.g. [15, 17]) the programme took up steam after Mahajan *et al.* presented several results and pointers in new directions [23] (e.g. [3, 16, 18, 17]). In particular, Cygan *et al.* broke new ground for Multiway Cut and Vertex Cover with algorithms that run in $O^*(4^k)$ time, where k is the *gap parameter* between an appropriate LP-relaxation and the integral optimum [5]. Lokshtanov *et al.* improved the Vertex Cover case to $O^*(2.3146^k)$ using a specialized branching algorithm [21].

44th International Symposium on Mathematical Foundations of Computer Science (MFCS 2019).
Editors: Peter Rossmanith, Pinar Heggernes, and Joost-Pieter Katoen; Article No. 40; pp. 40:1–40:13
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

The latter result highlights an important realization: these alternative, smaller parameters might not only provide the means to investigate problems without "good" natural parameters, it might also provide us with faster algorithms in practise! Gap- and above-guarantee parameters are attractive because there is a reasonable chance that they are small in real-world scenarios, something we often cannot expect from natural parameters.

To the best of our knowledge, so far no gap-parameter results are known for domination problems and an above/below-guarantee result is only known in bounded-degree graphs [23]. This is probably due to the fact that there are no simple "natural" upper/lower bounds and in the case of gap-parameters the LP-dualities do not provide much purchase. We therefore explore this topic under the most basic assumptions: we are provided with a witness for a lower bound on the domination number *as input* and consider parametrisations that arise from this additional information. In the case of DOMINATING SET, the witness takes the form of a 2-*independent set*, that is, a set in which all vertices have pairwise distance at least three. Note that this approach also captures a form of duality: the LP-dual of dominating set describes a 2-independent set, however, in general the two optima are arbitrarily far apart. Recently, Dvořák highlighted this connection [7] and proved that in certain sparse classes the gap between the dual optima is bounded by a constant.

Thus, assume we are given a maximal 2-independent set X alongside the input graph G. A parametrisation by $|X|$ would go against the spirit of gap-parameters, instead we parametrise by the size of *residual set* $R := V(G) \setminus N[X]$, that is, all vertices that lie at distance two from X (since X is maximal, no vertex can have distance three or more). We choose this particular parameter for two reasons:

(i) For $|R| = 0$ the problem is decidable in polynomial time since the domination number of the graph is precisely $|X|$.

(ii) The set $X \cup R$ is a dominating set of G.

The first property is of course an important pre-requisite for the problem to be in FPT under this parametrisation, while the second property guarantees us that the dominating set size lies in-between $|X|$ and $|X| + |R|$.

Our first investigatory dimension is the constant $r = 2$ in the 2-independent set: intuitively, increasing the minimum distance between vertices in X increases the size of the parameter $|R|$ and imposes more structure on the input instance. Our second dimension encompasses an approach that has been highly successful in improving tractability of domination problems: restricting the inputs to sparse graphs. While DOMINATING SET is W[2]-complete in general graphs, Alber *et al.* showed that it is fpt in planar graphs [1]; Alon and Gutner later proved that assuming degeneracy is sufficient [4]. Philip, Raman, and Sikdar extended this result yet further to graphs excluding a fixed bi-clique and also proved that it admits a polynomial kernel [25]. A related line of research was the hunt for *linear* kernels in sparse classes. Beginning with such a kernel on planar graphs by Alber, Fellows, and Niedermeier [2], results on apex-minor free graphs [10], graphs excluding a minor [11] and classes excluding a topological minor [12] were soon proven. Recently, a linear kernel for graphs of bounded expansion [6] (and an almost-linear kernel for nowhere dense graphs [9]) has subsumed all previous results.

Our investigation of DOMINATING SET parametrised above an r-independent set, for $r \geq 2$, led us to the following results. For $r = 2$, the problem is paraNP-complete already for $|R| = 1$, squashing all hope for an FPT or even XP algorithm. This also holds true if the inputs are restricted to sparse graph classes (apex-graphs/graphs of maximum degree six).

For $r = 3$, the problem is W[2]-hard in general graphs but admits an XP-algorithm. In nowhere dense and bounded expansion classes, it is fixed-parameter tractable. We further show, in the probably most technical part of this paper, that it admits a linear kernel in bounded expansion classes.

Finally, for $r \geq 4$ the problem remains W[2]-hard in general graphs and essentially degenerates to Dominating Set (hence, all the above mentioned results in sparse classes translate in the parametrisation above r-independence).

1 Preliminaries

A set $X \subseteq V(G)$ is *r-independent* if each pair of distinct vertices in X have distance at least $r+1$, thus an independent set is 1-independent. We write $N(v)$ and $N[v]$, respectively, for the neighbourhood and the closed neighbourhood of a vertex v. We extend this notation to sets as follows: for $X \subseteq V(G)$ we let $N(X)$ be all vertices not in X that have a neighbour in X and $N[X] := X \cup N(X)$. We let $N^i(X)$ be all vertices not in X that are at most distance i from any vertex in X and we let $N^i[X] = X \cup N^i(X)$. A vertex set $Z \subseteq V(G)$ is *dominated* by a set $D \subseteq V(G)$ if for every vertex $z \in Z$ we have $N[z] \cap D \neq \emptyset$, D is then called a *Z-dominator*. We let $\mathbf{ds}(G)$ denote the size of a minimum dominating set of G.

Dominating Set above r-independence

Input:	A graph G, a maximal r-independent set $X \subseteq V(G)$, an integer p.
Parameter:	The size of the residual set $R := V(G) \setminus N[X]$.
Problem:	Does G have a dominating set of size p?

Note that $X \cup R$ is trivially a dominating set and that, for $r \geq 2$, it holds that $\mathbf{ds}(G) \geq |X|$, thus we will tacitly assume in the following that $|X| \leq p \leq |X| + |R|$ since all other instances are trivial.

We will frequently invoke the terms *bounded expansion* and *nowhere dense* to describe graph classes. The definitions of these terms requires the introduction of several concepts which will not be useful for the remainder of the paper, we refer the reader to the book by Nešetřil and Ossona de Mendez [24]. In this context, it is important to know that bounded expansion classes generalize most structurally sparse classes (planar, bounded genus, bounded degree, H-minor free, H-topological minor free) and nowhere dense classes contain bounded expansion classes in turn. The following lemma and propositions for those two sparse graph classes will be needed in the remainder of this paper:

▶ **Lemma 1** (Twin class lemma [13, 26]). *For every bipartite nowhere dense class there exists a constant ω and a function $f(s) = O(s^{o(1)})$ such that for every member $G = (X, Y, E)$ of the class it holds that*

1. $|\{u \mid \deg(u) > 2\tau\}_{u \in Y}| \leq 2\tau \cdot |X|$, *and*
2. $|\{N(u)\}_{u \in Y}| \leq (\min\{4^\tau, \omega(e\tau)^\omega\} + 2\tau) \cdot |X|$.

where $\tau = f(|X|) = O(|X|^{o(1)})$. If G is from a bounded expansion class, τ can be assumed a constant as well.

We will also frequently invoke the following result regarding first-order (FO) model checking in bounded expansion and nowhere dense classes:

▶ **Proposition 2** (Dvořák, Král, and Thomas [8]). *For every bounded expansion class, the first-order model checking problem is solvable in linear fpt-time parametrised by the size of the input formula.*

This result has since been extended to nowhere dense classes as well. Here, *almost linear fpt-time* means running time of the form $O(f(k) \cdot n^{1+o(1)})$ for some function f.

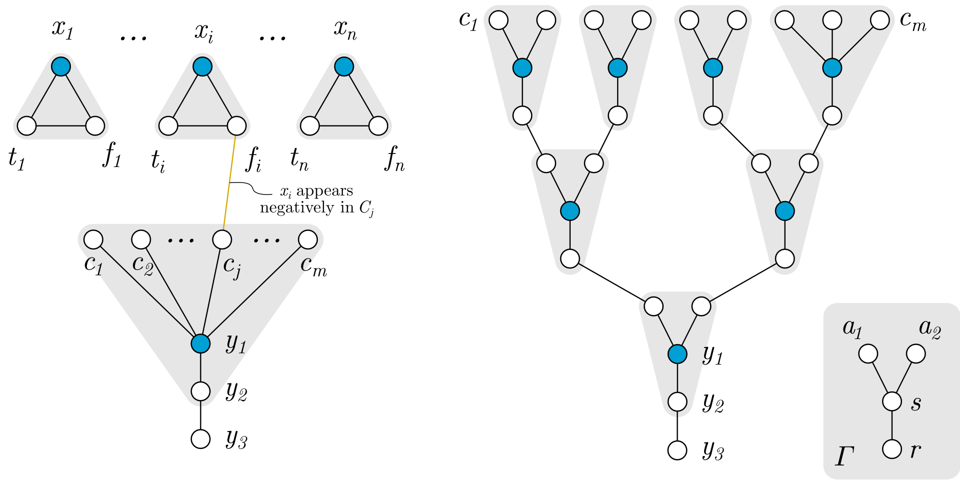

Figure 1 Sketch of reduction from 3SAT to DOMINATING SET above 2-independent set. The left side shows the basic reduction, the right side shows the bounded-degree replacement gadget for the clause part, with the tree-gadget Γ highlighted on the bottom right. The 2-independent set X is coloured blue and $N[X]$ is shaded in grey. In both constructions the set R consists only of y_3.

▶ **Proposition 3** (Grohe, Kreutzer, and Siebertz [14]). *For every nowhere dense class, he first-order model checking problem is solvable in almost linear fpt-time parametrised by the size of the input formula.*

2 Above 2-independence: hard as nails

In this section we will show that when we let $r = 2$, we find that the problem is paraNP-complete for $|R| = 1$, hence this parametrisation does not even admit an XP-algorithm. In the following we first present a reduction from 3SAT and then discuss how to modify it to reduce into sparse graph classes.

Since X is a (maximal) 2-independent set, we know that each vertex in R is a neighbour of some vertex in $N(X)$, otherwise we could add this vertex to X. Let us now describe the reduction. Let ϕ be 3SAT-instance with variables $x_1, \dots, x_n$ and clauses $C_1, \dots, C_m$. We construct G as follows (*cf.* Figure 1):

1. For each variable x_i, add a triangle with vertices x_i, t_i, f_i.
2. For each clause C_j add a vertex c_j. If the variable x_i occurs positively in C_j, add the edge $c_j t_i$; if it occurs negatively, add the edge $c_j f_i$.
3. Add a single vertex y_1 to the graph and connect it to each clause variable c_i. Add two further vertices y_2, y_3 and add the edges $y_1 y_2$ and $y_2 y_3$.

We further set $X := \{x_1, \dots, x_n, y_1\}$ as our 2-independent set; notice that the only vertex not contained in $N[X]$ is y_3. Hence, $R := \{y_3\}$.

▶ **Lemma 4.** *ϕ is satisfiable iff G has a dominating set of size $|X|$.*

Proof. Assume ϕ is satisfiable and fix one satisfying assignment I. We construct a dominating set D as follows: if x_i is true under I, add t_i to D; otherwise add f_i. Since I satisfies every clause of ϕ the dominating set so far dominates every clause vertex and, of course, every variable gadget. The remaining undominated vertices are y_1, y_2, y_3, thus adding y_2 to D yields a dominating set of G of size $|X|$.

In the other direction, assume that D is a dominating set of G of size $|X|$. Since y_3 is a pendant vertex we can assume that $y_2 \in D$ (if y_3 would be in D we could exchange it for y_2). That leaves $|X| - 1 = n$ vertices in D, precisely the number of variable-gadgets. Since every variable gadget must include at least one vertex of D, we conclude the every such gadget contains precisely one dominating vertex. Since that depletes our budget, no other vertex is contained in D.

By the usual exchange argument we may assume that the dominating vertex in each variable gadget is either f_i or t_i and not x_i for $1 \leq i \leq n$; hence the dominating vertices inside the variable gadgets encode a variable assignment I_D of ϕ. Finally, note that the clause vertices are not dominated by y_2 and y_1 is not contained in D. Hence, they must be completely dominated by vertices contained in the variable gadgets. Then, by construction, the assignment I_D satisfies ϕ and the claim follows. ◀

We conclude that 3SAT many-one reduces to Dominating Set above 2-independence already with $|R| = 1$. We obtain the following two corollaries that demonstrate that sparseness cannot help tractability here:

▶ **Corollary 5.** Dominating Set *above* 2*-independence is* paraNP*-complete in apex-graphs.*

Proof. We use the above construction but reduce from a planar variant of 3SAT. To ensure that we can construct variable-gadgets without edge crossings, we choose to reduce from Lichtenstein's Planar 3SAT variant [20] which ensures that the following graph G' derived from the Planar 3SAT instance ϕ is planar:

1. Every variable x_i of ϕ is represented by two literal vertices t_i, f_i with the edge $t_i f_i \in G'$
2. Each clause C_j is represented by a vertex c_j. If the variable x_i occurs positively in C_j, the edge $c_j t_i$ exits; if it occurs negatively, the edge $c_j f_i$ exists.

To complete G' to G we have to add the vertices x_i and connect them to t_i, f_i. This is clearly possible without breaking planarity (picture placing x_i on the middle of the line segment $f_i t_i$ and moving it perpendicular by a small amount, then the edges $x_i f_i$ and $x_i t_i$ can be embedded without crossing other edges). The vertices y_2, y_3 can be placed anywhere; finally the vertex y_1 will break planarity (the embedding does not guarantee that the clause vertices lie on the outer face of the graph) and we conclude that G is indeed an apex-graph. ◀

▶ **Corollary 6.** Dominating Set *above* 2*-independence is* paraNP*-complete in graphs of maximum degree six.*

Proof. We reduce from (3,4)SAT (NP-hardness shown in [27]) in which every clause has size three and every variable occurs in at most four clauses. We use the above construction with one modification. Instead of connecting all clause vertices to one vertex we create a bounded-degree tree with the clause-vertices as its leaves.

We begin by partitioning the clause vertices in pairs P_i; if there is an odd number of vertices the last group will have three vertices. Then, for each group $P_i = \{c, c'\}$, we add two vertices s^i, r^i, connect the clause-vertices c and c' to s^i, and add the edge $s^i r^i$. We further add each s^i to our 2-independent set X and then create a set L_1 consisting of each r^i. Now we iteratively construct the next level of the tree, starting with $L := L_1$:

1. If $L = \{r^1\}$, create a single vertex y_3, connect it to r^1 and finish, otherwise proceed with the next step.
2. Partition the vertices in L into ℓ groups $\{l_i, r_i\}$ of pairs. If $|L|$ is odd, the last group will be a triple $\{l_i, c_i, r_i\}$ instead.
3. For each group, create a tree-gadget Γ^i, with vertices $\{a_1, a_2, s^i, r^i\}$ (and a_3 if the group contains a third vertex), and edges $l_i a_1, r_i a_2$, $(c_i a_3)$, $a_1 s^i$, $a_2 s^i$, $(a_3 s^i)$, and $s^i r^i$.
4. Add each s^i to X, let L now be the set of all r^i (for $1 \leq i \leq \ell$) and continue with Step 1.

Figure 1, on the right side, shows an example of this construction. We note that, in the last tree-gadget, s^i and r^i are the same vertices as y_1 and y_2 respectively in the figure. We conclude the construction by adding each x_i from the variable-gadgets to X and setting $p := |X|$. Notice that the only vertex not contained in $N[X]$ is y_3 and thus $R := \{y_3\}$.

Since each variable in (3,4)SAT can be in up to four clauses, the maximum degree for t_i and f_i is six. All clause-vertices have degree at most four and all other types of vertices have a degree not higher than that, hence the claimed degree-bound holds. It is left to show that ϕ is satisfiable iff there is a dominating set of size $p = |X|$ in the graph.

Let us assume that ϕ is satisfiable and fix one satisfying assignment I. We construct a dominating set as follows, beginning in the same way as in Lemma 4: if x_i is true under I, add t_i to D, otherwise add f_i. Since I satisfies every clause of ϕ the dominating set so far dominates every clause vertex and every variable gadget. Now, the remaining undominated vertices are the tree-vertices, and our remaining budget is $|X| - n$ which is equal to the amount of tree-gadgets. Since every clause vertex is already dominated we can, for each tree-gadget $\Gamma^i = \{a_1, a_2, s^i, r^i\}$, add r^i to the dominating set. This will dominate s^i and the corresponding a_1 or a_2 in the tree-gadget below it, hence we can dominate all vertices of the graph within the budget $|X|$.

In the other direction, assume that D is a dominating set of G of size $|X|$. Since y_3 is a pendant vertex we can assume that $y_2 \in D$. Thus, in the last tree-gadget r^i (y_2) is in the dominating set. This means that in order for a_1 and a_2 to be dominated, r^i in both tree-gadgets above has to be in the dominating set. This holds for all tree-gadgets, all the way up to the clause vertices. Since we now have one vertex per tree-gadget in the dominating set this leaves n vertices. Just as in Lemma 4, we note that, for each variable gadget, either t_i or f_i is in the dominating set. We know that the clause variables are not dominated by anything in the tree gadgets and thus must be dominated by the variable vertices. As stated in Lemma 4, the dominating vertices inside the variable gadgets encode a variable assignment I_D that satisfies ϕ and the claim follows. ◀

3 Above 3-independence: sparseness matters

3.1 W[2]-hardness in general graphs

In the following we present a result for Dominating Set above 3-independence, namely that it is W[2]-hard in general graphs. We show this by reduction from Colourful Dominating Set parametrised by the number of colours k:

Colourful Dominating Set parametrised by k

Input: A graph G with a vertex partition $C_0, C_1, \ldots, C_k$.

Problem: Is there a set that dominates C_0 and uses exactly one vertex from each set $C_1, \ldots, C_k$?

It is easy to verify that Colourful Dominating Set is W[2]-hard by reducing from Red-Blue Dominating Set: we copy the blue set k times and make each copy a colour set C_i, $1 \le i \le k$, and let C_0 be the red set.

▶ **Lemma 7.** Dominating Set *above r-independence is* W[2]*-hard for $r \ge 2$.*

Proof. Let $I = (G, C_0, C_1, \ldots, C_k)$ be an instance of Colourful Dominating Set. We construct an instance (G', X, k) of Dominating Set above r-independence as follows (*cf.* Figure 2):

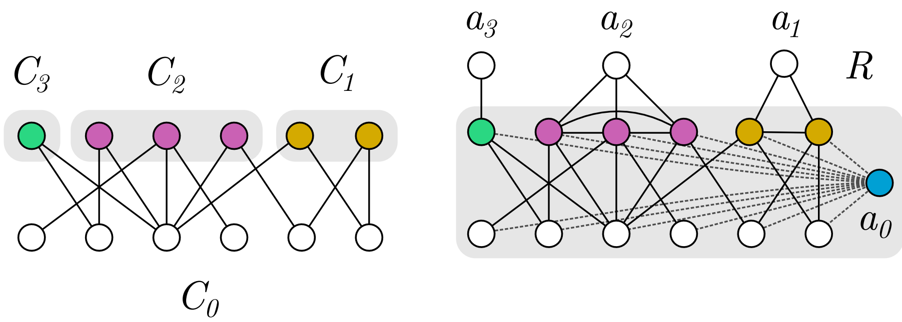

Figure 2 Sketch of reduction from COLOURFUL DOMINATING SET to DOMINATING SET above r-independent set for $r \geq 2$. The set X contains only the vertex a_0, the remaining vertices a_i are all contained in the residual R.

1. Begin with G' equal to G; then
2. for each block C_i, $i \geq 1$, add edges to make $G[C_i]$ a complete graph and add an additional vertex a_i with neighbourhood C_i; then
3. add a vertex a_0 and connect it to all vertices in $C_0 \cup C_1 \cup \ldots C_k$.

Let $X = \{a_0\}$ be the r-independent set (which it clearly is for any r) and thus $R = \{a_1, \ldots, a_k\}$. Note that the graph G' trivially has a dominating set of size $k+1$; the set $\{a_0, a_1, \ldots, a_k\}$. We now claim that I has a colourful dominating set of size k iff G' has a dominating set of size k.

Assume that I has a solution of size k and fix one such colourful dominating set D. Note that D in G' a) dominates all of C_0 (because it dominates C_0 in G) and b) dominates each set $C_i \cup \{a_i\}$, $i \geq 1$ (because D intersects each such C_i), and of course the vertex a_0. Hence, D is a dominating set of G' as well.

In the other direction, assume that D is a dominating set of G' of size k. Since each a_i, $i \geq 1$, is only connected to vertices in the corresponding set C_i, at least one vertex from each set $C_i \cup \{a_i\}$ needs to be in D. Since there are k such sets we conclude that D intersects each set $C_i \cup \{a_i\}$ in precisely one vertex. We can further modify any such solution to not take the a_i-vertices by taking an arbitrary vertex from C_i instead, thus assume that D has this form in the following. But then D is of course also a dominating set of size k for G, as claimed. ◀

3.2 Tractability in sparse graphs

In the following we present two positive results, namely that DOMINATING SET above 3-independence is fpt in nowhere dense classes and that it admits a polynomial kernel in bounded expansion classes. The algorithm further implies an XP algorithm in general graphs. The following annotated domination problem will occur as a subproblem:

ANNOTATED DOMINATING SET

Input: A graph G, a subset $Y \subseteq V(G)$, a collection of vertex sets $R_1, \ldots, R_\ell$, and an integer k.

Problem: Is there a set of size k that dominates $V(G) \setminus Y$ and contains at least one vertex in each R_i?

In the following algorithm we will group vertices of $N(X)$ according to their neighbourhood in R (or a subset of R). We will call those groups *R-neighbourhood classes*. We will write $\gamma(G, Y, \{R_1, \ldots, R_\ell\})$ to denote the size of an optimal solution of ANNOTATED DOMINATING SET.

▶ **Lemma 8.** Dominating set *above* 3*-independence can be solved in linear fpt-time in any graph class of bounded expansion.*

Proof. First we guess the intersection D_R of an optimal solution (should it exist) with R in $O(2^{|R|})$ time. Let $R' \subseteq R$ be those vertices of R that are not dominated by D_R. Define

$$\mathcal{R} := \{N(v) \cap R'\}_{v \in N(X)}$$

as the neighbourhoods induced in R' by vertices in $N(X)$. By Lemma 1, we have that $|\mathcal{R}| = O(|R'|)$ since G is from a class with bounded expansion (note that the partition into such neighbourhoods is computable in linear time using the partition-refinement data structure [19]). Accordingly, in time $O(2^{|\mathcal{R}|}) = 2^{O(|R'|)}$, we can guess a subset $\mathcal{R}' \subseteq \mathcal{R}$ such that an optimal solution covers exactly the neighbourhoods $\mathcal{R}'$ (if $\mathcal{R}'$ does not cover R' we abort this branch of the computation). We are now left with the task of choosing vertices from $N[X]$ to a) cover the neighbourhoods $\mathcal{R}'$ and b) dominate the vertices in $N[X]$ which are not dominated by D_R.

We introduce the following notation to ease our task: for a collection of R-neighbourhoods $\mathcal{S} \subseteq \mathcal{R}'$ and a set $Y \subseteq N(X)$, let $\mathcal{S}^{-1}(Y) := \{Y_i \subseteq Y \mid N(y) = S \text{ for all } y \in Y_i\}_{S \in \mathcal{S}}$. That is, $\mathcal{S}^{-1}(Y)$ contains those R'-neighbourhood classes in Y whose neighbourhood is contained in $\mathcal{S}$. Let $x_1, \ldots, x_\ell$ be an ordering of X and let $H_i := G[N[x_i]]$; we will describe a dynamic-programming algorithm over the ordering $x_1, \ldots, x_\ell$. Let $T_i[\mathcal{S}]$ be the minimum size of a partial solution in $N[\{x_1, \ldots, x_i\}]$ that covers the neighbourhoods $\mathcal{S} \subseteq \mathcal{R}'$ and together with D_R dominates all of $G[\bigcup_{1 \le j \le i} N[x_j]]$. We initialize $T_0[\mathcal{S}] := \infty$ for all $\emptyset \neq \mathcal{S} \subseteq \mathcal{R}'$ and $T_0[\emptyset] := 0$, then compute the following entries with the recurrence[1]

$$T_{i+1}[\mathcal{S}] := \min_{\mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{S}} \Big(T_i[\mathcal{S}_1] + \gamma\big(H_{i+1}, N(D_R) \cap N[x_{i+1}], \mathcal{S}_2^{-1}(N(x_{i+1}))\big)\Big).$$

Note that $\gamma(\ldots)$ is the minimum size of a set that dominates $N[x_i] \setminus N(D_R)$ while choosing at least one vertex from each member of $S_2^{-1}(N(x_i))$ (if $\emptyset \in S_2^{-1}(N(x_i)$ we assume that $\gamma(\ldots) = \infty$). The latter constraint corresponds to dominating the neighbourhoods of $\mathcal{S}_2$ in R' by using vertices from $N[x_i]$. Once the DP table T_ℓ has been computed, the size of an optimal solution is the value in $T_\ell[\mathcal{R}']$.

It remains to be noted that every neighbourhood graph H_i admits a dominating set of size one, hence the annotated dominating set has size at most $|\mathcal{S}_2| + 1 = O(|R|)$. Thus, the problem of finding an annotated dominating set is FO-expressible by a formula of size $O(|R|)$ and we can solve the subproblem of computing $\gamma(\ldots)$ in time $O(f(|R|) \cdot |N[x_i]|)$ for some function f using Proposition 2. As a result, we obtain a linear dependence on the input size (note that $|E| = O(|V|)$) in the running time and thus the problem is solvable in linear fpt-time. ◀

The same proof works for nowhere dense classes by applying Proposition 3 instead of Proposition 2:

▶ **Corollary 9.** Dominating set *above* 3*-independence can be solved in almost linear fpt-time in any nowhere dense class.*

We finally note that the algorithm described in the proof of Lemma 8 only needs a black-box fpt-algorithm for Annotated Dominating Set to run in fpt time, thus it is very likely that Dominating set above 3-independence is in FPT for other highly structured but not

[1] A proof for the correctness of the recurrence can be found in the full version available at `https://arxiv.org/abs/1906.09180`

necessarily sparse graph classes. We can run the same algorithm on general graphs to obtain an XP-algorithm using the bound $|\mathcal{R}| \leq 2^{|R'|}$ and a simple brute-force in XP-time on the Annotated Dominating Set subinstance during the DP.

▶ **Corollary 10.** Dominating set *above* 3*-independence is in* XP*.*

3.3 Kernelization in sparse graphs

Let us now set up the necessary machinery for the kernelization. A *boundaried graph* $^\circ G$ is a tuple (G, R) where G is a graph and $R \subseteq V(G)$ is the *boundary.* We also write $\partial^\circ G$ to denote the boundary. For a graph G and an induced subgraph H, the boundary $\partial_G H \subseteq V(H)$ are those vertices of H that have neighbours in $V(G) \setminus V(H)$. Thus for every subgraph H of G there is a naturally associated boundaried graph $\partial H = (H, \partial_G H)$.

For a boundaried graph $^\circ H$ and subsets $A, B \subseteq \partial^\circ H$ a set $D \subseteq V(^\circ H)$ is an (A, B)-dominator of $^\circ H$ if $D \cap \partial^\circ H = A$ and D dominates the set $(V(^\circ H) \setminus \partial^\circ H) \cup B$. We let $\mathbf{ds}(^\circ H, A, B)$ denote the size of a minimum (A, B)-dominator of $^\circ H$. A *replacement* for H is a boundaried graph (H', B) with $H[B] = H'[B]$. The operation of *replacing* H *by* H' *in* G, written as $G[H \to H']$, consist of removing the vertices $V(H) \setminus B$ from G, then adding H' to G with B in H' identified with B in G (we assume that the vertices $V(H') \setminus B$ do not occur in G).

▶ **Lemma 11.** *Let* (G, X, p) *be an instance of* Dominating Set *above* 3*-independence where* G *is from a bounded expansion class. Let* $x \in X$*,* $H_x = G[N^2[x]]$ *and* $R' = N^2[x] \cap R$*. Then, in fpt-time with parameter* $|R'|$*, we can compute a replacement* H'_x *for* H_x *of size* $O(4^{|R'|}|R'|)$ *such that* $\mathbf{ds}(G[H_x \to H'_x]) = \mathbf{ds}(G)$*. Moreover, the replacement* H'_x *is a subgraph of* H_x *and contains* x*.*

Proof. For every pair of subsets $A, B \in R'$ we compute a minimal (A, B)-dominator $S_{A,B}$ for $^\circ H_x = (H_x, R')$. Since this problem is expressible by an FO-formula of size $O(|R'|)$, we can employ Proposition 2 to compute the set $S_{A,B}$ in linear fpt-time with parameter $|R'|$. Note that $S_{A,B}$ will, besides the vertices in A, contain at most $|B| + 1$ additional vertices, since $|B|$ vertices suffice to dominate B and the vertex x dominates all of $V(H_x) \setminus R$.

Let $S = \bigcup_{A,B \subseteq R'} S_{A,B} \cup R' \cup \{x\}$ be the union of all such computed solutions and the boundary. By construction, $|S| \leq (4^{|R'|} + 1)|R'| + 1$. Since all (A, B)-dominators live inside S, we can safely remove the edges from H_x that do not have any endpoint in S, call the resulting graph $\tilde{H}_x$. Let $Y := V(\tilde{H}_x) \setminus S$ and let $Y_1, \ldots, Y_t$ be a partition of Y into S-twin classes (thus all vertices in Y_i have the same neighbourhood in S and no other class has this neighbourhood). Note that Y is an independent set in $\tilde{H}_x$.

Let H'_x be obtained from $\tilde{H}_x$ by removing all but two representatives from every twin-class, denote those by $Y'_i \subseteq Y_i$. Clearly, every (A, B)-dominator of $\tilde{H}_x$ is still an (A, B)-dominator of H'_x, we need to proof the other direction. Let D' be an (A, B)-dominator of H'. If $D' \subseteq S$ then D' is still an (A, B)-dominator for H. The same holds if D' intersects every twin-class in at most one vertex: each such class either has size one, in which case it has size one in $\tilde{H}_x$ as well, or it has size two and the vertex not contained in D' is dominated by a vertex in S. In either case, D' dominates all of Y and thus is an (A, B)-dominator of $\tilde{H}_x$ and therefore of H_x. Thus, assume that D' fully contains some class Y'_i. Clearly, only one vertex of Y'_i is enough to dominate $N(Y'_i)$ (and potentially vertices in B), thus we can modify D' by picking the central vertex x instead to dominate the other vertex of Y'_i (which of course also dominates all of Y_i in $\tilde{H}_x$). This can, of course, only happen once, otherwise we would reduce the size of the supposedly minimal set D'. This leads us back to the previous case and we conclude that there exists an (A, B)-dominator of $\tilde{H}_x$ and thus H_x of equal size.

We conclude that $\mathbf{ds}(^{\circ}H_x, A, B) = \mathbf{ds}(^{\circ}H'_x, A, B)$ for every choice of $A, B \subseteq R'$; which implies that $\mathbf{ds}(G[H_x \to H'_x]) = \mathbf{ds}(G)$. Finally, by the twin-class lemma, $|H'_x| = O(|S|) = O(4^{|R'|}|R'|)$ since τ is a constant in bounded expansion classes. ◀

▶ **Lemma 12.** *Let (G, X, p) be an instance of* Dominating Set *above 3-independence where G is from a bounded expansion class. Let $R' \subseteq R$ be a subset and let $X' \subseteq X$ be those vertices $x \in X$ with $N^2(x) \cap R = R'$. Let H be the induced subgraph on $R' \cup \bigcup_{x \in X'} N[x]$. Assuming X' is not empty we can, in fpt-time with parameter $|R'|$, compute a replacement H' for H of size $O(|R'|^{2|R'|+2}4^{|R'|})$ alongside an offset c such that $\mathbf{ds}(G[H' \to H]) = \mathbf{ds}(G) - c$. Moreover, the replacement H' is a subgraph of H.*

Proof. Let $X' := \{x_1, \ldots, x_\ell\}$ and define the graphs $H_i := G[N^2[x_i]]$ for $1 \le i \le \ell$. We first we apply Lemma 11 to replace every subgraph H_i by a subgraph H'_i of size $O(4^{|R'|}|R'|)$ in linear fpt-time with parameter R'. For simplicity, let us call the resulting graph G and relabel the graphs H'_i to H_i (note that Lemma 11 ensures that dominating set size does not change and that X is still a 3-independent set of the resulting graph).

For every $x_i \in X'$ we compute a *characteristic vector* χ_i indexed by pairs of subsets of R' with the following semantic: for $A, B \subseteq R'$ we set $\chi_i[A, B] = \mathbf{ds}(^{\circ}H_i, A, B)$. If $\chi_i[A, B]$ is larger than $|A| + |B| + 1$ we simply set $\chi_i[A, B] = \infty$.

Note that the constraint subproblem to compute $\chi_i[A, B]$ is FO-expressible, by a formula of size $O(|R'|)$, thus we can compute the vectors χ_i for $1 \le i \le \ell$ in linear time fpt-time with parameter $O(|R'|)$ using Proposition 2. Let $\equiv_\chi$ be the equivalence relation over the graphs H_i defined as $H_i \equiv_\chi H_j \iff \chi_i = \chi_j$ and let $\mathcal{H} := \{H_i\}_{1 \le i \le \ell} / \equiv_\chi$ be the corresponding partition into equivalence classes under $\equiv_\chi$. Note that $|\mathcal{H}| \le (|R'| + 1)^{2|R'|}$ since that is the number of possible characteristic vectors.

The construction of H' is now simple: in every equivalence class $\mathcal{C} \in \mathcal{H}$ we select $\min\{|\mathcal{C}|, |R|\}$ subgraphs and remove the rest; clearly H' is a subgraph of H of the claimed size. We let the offset c to be equal to the number of subgraphs removed in this way.

We are left to show that $\mathbf{ds}(G[H' \to H]) = \mathbf{ds}(G) - c$. Consider any minimal dominating set D for G. We call a graph H_i *interesting* under D if D intersects H_i in any vertex besides x_i.

▷ Claim 13. There exists a solution D' of size equal to D under which at most $|R'|$ graphs per equivalence class $\mathcal{C} \in \mathcal{H}$ are interesting.

Proof. Consider any such class $\mathcal{C} \in \mathcal{H}$. If $|\mathcal{C}| \le |R'|$ we are done, so assume otherwise. Let $R'' \subseteq R'$ be the set of vertices that are dominated through vertices in graphs contained in $\mathcal{C}$ and select up to $|R''|$ many graphs that together already dominate R''; we let D' to be equivalent to D on these graphs. Any graph $H \in \mathcal{C}$ *not* selected in this way only needs to dominate itself and we add its centre vertex $V(H) \cap X$ to D'. Since every graph needs to intersect any dominating set in at least one vertex, $|D'| \le |D|$ and since D is minimal we must have $|D'| = |D|$. Finally, only the $|R''| \le |R'|$ selected graphs are interesting under D', as claimed. ◁

Thus, let us assume in the following that D is such a minimal solution under which at most $|R'|$ graphs per class $\mathcal{C} \in \mathcal{H}$ are interesting. For such a solution D of H, we construct a solution D' of H' of size $|D| - c$ as follows. For a class $\mathcal{C} \in \mathcal{H}$, let $\mathcal{C}' \subseteq \mathcal{C}$ be those graphs that are contained in H' and let $\mathcal{I} \subseteq \mathcal{C}$ be the graphs that are interesting under D. Since $|\mathcal{C}| \le |\mathcal{C}'|$, we can pair every graph $H_i \in \mathcal{C}$ with a graph $H_{i'} \in \mathcal{C}'$. Fix such a pair $H_i, H_{i'}$, let $A = R' \cap D$ and let B be those vertices of R' that are exclusively dominated vertices in H_i. Since $\chi_i[A, B] = \chi_{i'}[A, B]$, there must exist a set of $|D \cap V(H_i)|$ vertices in $H_{i'}$ that dominate $H_{i'}$ and B. If we repeat this construction for every graph $H_i \in \mathcal{C}$ with their respective pair in $\mathcal{C}'$ and then pick the centre vertex $X \cap V(H_j)$ for all $H_j \in \mathcal{C}' \setminus \mathcal{C}$, then the resulting set D' has size $\leq |D| - c$ and dominates all of H'.

The same proof works in reverse if we start with a dominating set D' of H' to construct a dominating set D of H with $|D| \leq |D'|+c$; thus we conclude that $\mathbf{ds}(G[H' \to H]) = \mathbf{ds}(G) - c$ and the claim follows. ◀

▶ **Theorem 14.** Dominating Set *above* 3*-independence has a linear kernel in bounded expansion graphs.*

Proof. Let (G, X, p) be the input instance and let $x_1, \ldots, x_\ell$ be the members of the 3-independent set X. We define the graphs $H_i := G[N^2[x_i]]$ and their respective R-neighbours $R_i := N^2[x_i] \cap R$ for $1 \leq i \leq \ell$. Let τ be as in Lemma 1. We partition the graphs $\{H_i\}_{1 \leq i \leq \ell}$ into two sets $\mathcal{L}$, $\mathcal{S}$ where $H_i \in \mathcal{L}$ iff $|R_i| > 2\tau$; and $\mathcal{S}$ contains all remaining graphs.

Let us first reduce $\mathcal{S}$. Let $\mathcal{R} := \{R_i \mid H_i \in \mathcal{S}\}$ be the R-neighbourhoods of the graphs collected in $\mathcal{S}$. By the twin class lemma, $|\mathcal{R}| \leq (4^\tau + 2\tau)|R|$, however, for each member $R' \in \mathcal{R}$ we might have many graphs in $\mathcal{S}$ that intersect R in precisely this set R'.

Fix $R' \in \mathcal{R}$ for now and let $\mathcal{S}[R']$ be those graphs of $\mathcal{S}$ that intersect R in R'. Let $H_{R'} := G[\bigcup_{H \in \mathcal{S}[R']} V(H)]$ be the joint graph of the subgraphs in $\mathcal{S}[R']$. Since $|R'| \leq 2\tau$, and τ is a constant depending only on the graph class, we can apply Lemma 12 to compute a replacement $H'_{R'}$ of size $O(|R'|^{2|R'|+2} 4^{|R'|})$ with offset c in polynomial time. We apply the replacement $G[H_{R'} \to H'_{R'}]$ and decrease p by c. Repeating this procedure for all $R' \in \mathcal{R}$ yields a graph G_1 (a subgraph of G) in which the small graphs $\mathcal{S}$ in total contain at most $|\mathcal{R}| \cdot O(|\tau|^{2\tau+2} 4^\tau) = O(|R|)$ vertices, a 3-independent set $X' \subseteq X$ of G_1, and a new input p'. This concludes our reduction for $\mathcal{S}$.

Let us now deal with $\mathcal{L}$ in G_1. By the twin class lemma, $|\mathcal{L}| \leq 2\tau|R|$. But then $|X'|$ has size bounded in $O(|R|)$ and we conclude that the size of a minimal dominating set for G_1 is bounded by $O(|R|)$, hence we assume that p' is bounded by $O(|R|)$ (otherwise the instance is positive and we can output a trivial instance). We now apply the existing linear kernel [6] for Dominating Set to G_1. The output of the kernelization is a subgraph of the original graph, hence we collect the remaining vertices of the 3-independent set $X'' \subseteq X'$ in order to output a well-formed instance (G'', X'', p''). This concludes the proof. ◀

4 Above 4-independence: simple domination

In the case where the lower-bound set X is 4-independent we now have that for distinct $x, x' \in X$ it holds that $N^2(x) \cap N^2(x') = \emptyset$, thus for each $x \in X$ the set $N^2(x) \cap R$ can only be dominated from R and $N(x)$. Let us call an instance (G, X) *reduced* if for every $x \in X$ the intersection $N^2(x) \cap R$ is non-empty. We can easily pre-process our input instance to enforce this property: if such an x would exist we can simply remove $N[x]$ from G to obtain an equivalent instance. In a reduced instance the parameter $|R|$ is necessarily big compared to $|X|$:

▶ **Observation 15.** *For every reduced instance (G, X) of* Dominating Set *above* 4*-independence it holds that $|R| \geq |X|$.*

▶ **Corollary 16.** *Let $\mathcal{G}$ be a graph class for which* Dominating Set *is in* FPT*. Then* Dominating Set *above* 4*-independence is in* FPT *for $\mathcal{G}$ as well.*

▶ **Corollary 17.** *Let $\mathcal{G}$ be a graph class for which* Dominating Set *admits a polynomial kernel. Then* Dominating Set *above* 4*-independence admits a polynomial kernel for $\mathcal{G}$ as well.*

On the other hand, we showed in Section 3.1 that the problem remains W[2]-hard for any $r \geq 2$ in general graphs.

5 Conclusion

We considered Dominating Set parametrised by the residual of a given r-independent set and investigated how the value of r and the choice of input graph classes affect its tractability. We observed that the tractability does improve from $r = 2$ to $r = 3$ as it goes from being paraNP-complete to 'merely' W[2]-hard and at least admits an XP-algorithm. Larger values of r, however, do not increase the tractability as the problem becomes essentially equivalent to Dominating Set.

If we consider sparse classes (bounded expansion and nowhere dense), the improvement in tractability from $r = 2$ to $r = 3$ is much more pronounced; changing from paraNP-complete to FPT and even admitting a linear kernel in bounded expansion classes. We very much believe that the kernel can be extended to nowhere dense classes, but leave that quite technical task as an open question.

References

1 J. Alber, H. L. Bodlaender, H. Fernau, T. Kloks, and R. Niedermeier. Fixed Parameter Algorithms for DOMINATING SET and Related Problems on Planar Graphs. *Algorithmica*, 33(4):461–493, 2002.

2 J. Alber, M. R. Fellows, and R. Niedermeier. Polynomial-time data reduction for Dominating Set. *JACM*, 51:363–384, 2004.

3 N. Alon, G. Gutin, E. Jung Kim, S. Szeider, and A. Yeo. Solving MAX-r-SAT Above a Tight Lower Bound. *Algorithmica*, 61(3):638–655, 2011.

4 N. Alon and S. Gutner. Linear Time Algorithms for Finding a Dominating Set of Fixed Size in Degenerated Graphs. *Algorithmica*, 54(4):544–556, 2009.

5 M. Cygan, M. Pilipczuk, M. Pilipczuk, and J.O. Wojtaszczyk. On multiway cut parameterized above lower bounds. *TOCT*, 5(1):3, 2013.

6 P.G. Drange, M. Sortland Dregi, F. V. Fomin, S. Kreutzer, D. Lokshtanov, M. Pilipczuk, M. Pilipczuk, F. Reidl, F. Sánchez Villaamil, S. Saurabh, S. Siebertz, and S. Sikdar. Kernelization and Sparseness: the Case of Dominating Set. In *STACS*, volume 47 of *LIPIcs*, pages 31:1–31:14. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.

7 Z. Dvořák. On distance r-dominating and $2r$-independent sets in sparse graphs. *CoRR*, abs/1710.10010, 2017. `arXiv:1710.10010`.

8 Zdenek Dvorak, Daniel Král, and Robin Thomas. Testing first-order properties for subclasses of sparse graphs. *J. ACM*, 60(5):36:1–36:24, 2013.

9 K. Eickmeyer, A. C. Giannopoulou, S. Kreutzer, O. Kwon, M. Pilipczuk, R. Rabinovich, and S. Siebertz. Neighborhood Complexity and Kernelization for Nowhere Dense Classes of Graphs. In *44th International Colloquium on Automata, Languages, and Programming, ICALP 2017, July 10-14, 2017, Warsaw, Poland*, pages 63:1–63:14, 2017.

10 F. V. Fomin, D. Lokshtanov, S. Saurabh, and D. M. Thilikos. Bidimensionality and Kernels. In *Proceedings of the Twenty-First Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2010, Austin, Texas, USA, January 17-19, 2010*, pages 503–510, 2010. `doi:10.1137/1.9781611973075.43`.

11 F. V. Fomin, D. Lokshtanov, S. Saurabh, and D. M. Thilikos. Linear kernels for (connected) dominating set on H-minor-free graphs. In *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2012, Kyoto, Japan, January 17-19, 2012*, pages 82–93. SIAM, 2012. doi:10.1137/1.9781611973099.7.

12 F. V. Fomin, D. Lokshtanov, S. Saurabh, and D. M. Thilikos. Linear kernels for (connected) dominating set on graphs with excluded topological subgraphs. In *30th International Symposium on Theoretical Aspects of Computer Science, STACS 2013, February 27 - March 2, 2013, Kiel, Germany*, volume 20 of *LIPIcs*, pages 92–103. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2013. doi:10.4230/LIPIcs.STACS.2013.92.

13 J. Gajarský, P. Hliněný, J. Obdržálek, S. Ordyniak, F. Reidl, P. Rossmanith, F. Sánchez Villaamil, and S. Sikdar. Kernelization Using Structural Parameters on Sparse Graph Classes. *J. Comput. Syst. Sci.*, 84:219–242, 2017.

14 M. Grohe, S. Kreutzer, and Siebertz S. Deciding First-Order Properties of Nowhere Dense Graphs. *J. ACM*, 64(3):17:1–17:32, 2017.

15 G. Gutin, E.J. Kim, M. Lampis, and V. Mitsou. Vertex cover problem parameterized above and below tight bounds. *Theory of Computing Systems*, 48(2):402–410, 2011.

16 G. Gutin, E.J. Kim, M. Mnich, and A. Yeo. Betweenness parameterized above tight lower bound. *Journal of Computer and System Sciences*, 76(8):872–878, 2010.

17 G. Gutin, A. Rafiey, S. Szeider, and A. Yeo. The linear arrangement problem parameterized above guaranteed value. *Theory of Computing Systems*, 41(3):521–538, 2007.

18 G. Gutin, L. van Iersel, M. Mnich, and A. Yeo. All Ternary Permutation Constraint Satisfaction Problems Parameterized above Average Have Kernels with Quadratic Numbers of Variables. In *Proc. of the 18th ESA*, volume 6346 of *Lecture Notes in Computer Science*, pages 326–337. Springer, 2010.

19 M. Habib, C. Paul, and L. Viennot. A Synthesis on Partition Refinement: A Useful Routine for Strings, Graphs, Boolean Matrices and Automata. In *STACS 98, 15th Annual Symposium on Theoretical Aspects of Computer Science, Paris, France, February 25-27, 1998, Proceedings*, volume 1373 of *Lecture Notes in Computer Science*, pages 25–38. Springer, 1998. doi:10.1007/BFb0028546.

20 D. Lichtenstein. Planar Formulae and Their Uses. *SIAM J. Comput.*, 11(2):329–343, 1982.

21 D. Lokshtanov, NS Narayanaswamy, V. Raman, MS Ramanujan, and S. Saurabh. Faster parameterized algorithms using linear programming. *TALG*, 11(2):15, 2014.

22 M. Mahajan and V. Raman. Parameterizing above guaranteed values: MaxSat and MaxCut. *Journal of Algorithms*, 31(2):335–354, 1999.

23 M. Mahajan, V. Raman, and S. Sikdar. Parameterizing above or below guaranteed values. *Journal of Computer and System Sciences*, 75(2):137–153, 2009.

24 J. Nešetřil and P. Ossona de Mendez. *Sparsity: Graphs, Structures, and Algorithms*, volume 28 of *Algorithms and Combinatorics*. Springer, 2012.

25 G. Philip, V. Raman, and S. Sikdar. Polynomial kernels for dominating set in graphs of bounded degeneracy and beyond. *ACM Transactions on Algorithms*, 9(1), 2012.

26 F. Reidl. *Structural sparseness and complex networks*. Dr., Aachen, Techn. Hochsch., Aachen, 2016. Aachen, Techn. Hochsch., Diss., 2015.

27 C. A. Tovey. A simplified NP-complete satisfiability problem. *Discrete Applied Mathematics*, 8(1):85–89, 1984.