Two variable fragment of Term Modal Logic

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Abstract

Term modal logics (TML) are modal logics with unboundedly many modalities, with quantification over modal indices, so that we can have formulas of the form $\exists y \forall x\ (\Box_x P(x, y) \supset \Diamond_y P(y, x))$. Like First order modal logic, TML is also "notoriously" undecidable, in the sense that even very simple fragments are undecidable. In this paper, we show the decidability of one interesting fragment, that of two variable TML. This is in contrast to two-variable First order modal logic, which is undecidable.

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1 Introduction

Propositional multi-modal logics (ML) are extensively used in many areas of computer science and artifical intelligence ([2, 9]). ML is built upon propositional logic by adding modal operators $\Box_i$ and $\Diamond_i$ for every index i in a fixed finite set Ag which is often interpreted as a set of agents (or reasoners). Typically, the satisfiability problem is decidable for most instances of ML.

A natural question arises when we wish the set of modalities to be unbounded. This is motivated by a range of applications such as client-server systems, dynamic networks of processes, games with unboundedly many players, etc. In such systems, the number of agents is not fixed a priori. For some cases, the agent set can vary not only across models, but also from state to state (ex. when new clients enter the system or old clients exit the system).

Term Modal logic (TML) introduced by Fitting, Voronkov and Thalmann [6] addresses this requirement. TML is built upon first order logic, but the variables now range over modalities: so we can index the modality by terms ($\Box_x \alpha$) and these terms can be quantified over. State assertions describe properties of these "agents". Thus we can write formulas of the form: $\forall x(\Box_x P(x) \supset \exists y\ \Box_y \Diamond_x R(x, y))$. In [15] we have advocated PTML, the propositional fragment of TML, as a suitable logical language for reasoning about systems with unboundedly many agents. TML has been studied in dynamic epistemic contexts in [11] and in modelling situations where the identity of agents is not common knowledge among the agents [22].

The following examples illustrate the flavour of properties that can be expressed in TML.

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- For every agent x there is some agent y such that $P(x, y)$ holds at all x-successors or there is some y-successor where $\neg P(x, y)$ holds.
  $\forall x \exists y \ \big(\Box_x P(x, y) \lor \Diamond_y (\neg P(x, y))\big)$
- Every agent of type A has a successor where some agent of type B exists.
  $\forall x \big(A(x) \supset \Diamond_x \exists y \ B(y)\big)$.
- There is some agent x such that for all agents y if there are no y successors then in all successors of x, there is a y successor.
  $\exists x \forall y \ \big(\Box_y \bot \supset \Box_x \Diamond_y \top\big)$.

Since TML contains first order logic, its satisfiability is clearly undecidable. We are then led to ask: can we build term modal logics over decidable fragments of first order logic? Natural candidates are the monadic fragment, the two-variable fragment and the guarded fragment [13, 1].

TML itself can be seen as a fragment of first order modal logic (FOML) [5] which is built upon first order logic by adding modal operators. There is a natural translation of TML into FOML by inductively translating $\Box_x \alpha$ into $\Box(P(x) \supset \alpha)$ and $\Diamond_x \alpha$ into $\Diamond(P(x) \land \alpha)$ to get an equi-satisfiable formula, where P is a new unary predicate. Sadly, this does not help much, since FOML is notorious for undecidability. The modal extension of many simple decidable fragments of first order logic become undecidable. For instance, the monadic fragment[12] or the two variable fragment [10] of FOML are undecidable. In fact FOML with two variables and a single unary predicate is already undecidable [18]. Analogously, in [15] we show that the satisfiability problem for TML is undecidable even when the atoms are restricted to propositions. In the presence of equality (even without propositions), this result can be further strengthened to show "Trakhtenbrot" like theorem of mutual recursive inseparability.

On the other hand, as we show in [15], the monodic fragment of PTML (the propositional fragment) is decidable (a formula φ is monodic if each of its modal subformulas of the form $\Box_x \psi$ or $\Diamond_x \psi$ has a restriction that the free variables of ψ is contained in $\{x\}$). Further, via the FOML translation above, we can show that the monodic restriction of TML based on the guarded fragment of first order logic and monadic first order logic are decidable [23].

In a different direction, Wang ([21]) considered a fragment of FOML in which modalities and quantifiers are bound to each other. In particular he considered the fragment with $\exists \Box$ and showed it to be decidable in PSPACE. In [17] it is proved that this technique of bundling quantifiers and modalities gives us interesting decidable fragments of FOML, and as a corollary, the bundled fragment of TML is decidable where quantifiers and modalities always occur in bundled form: $\forall x \Box_x \alpha, \exists x \Box_x \alpha$ and their duals. However, more general bundled fragments of TML (such as those based on the guarded fragment of first order logic) have been shown to be decidable by Orlandelli and Corsi ([14]), and by Shtakser ([19]). From all these results, it is clear that the one variable fragment of TML is decidable, and that the three variable fragment of PTML is undecidable.

In this paper, we show that the two variable fragment of TML (TML^2) is decidable. This is in contrast with FOML, for which the two variable fragment is undecidable [10]. Quoting Wolter and Zakharyaschev from [23], where they discuss the root of undecidability of FOML fragments:

> All undecidability proofs of modal predicate logics exploit formulas of the form $\Box \ \psi(x, y)$ in which the necessity operator applies to subformulas of more than one free variable; in fact, such formulas play an essential role in the reduction of undecidable problems to those fragments ...

Note that this is not expressible in TML^2 where there is no "free" modality; every modality is bound an index (x or y). With a third variable z, we could indeed encode $\Box P(x,y)$ as $\forall z \Box_z P(x,y)$, but we do not have it. The decidability of the two variable fragment of TML, *without constants or equality*, hinges crucially on this lack of expressiveness. Thus, TML^2 provides a decidable fragment of FOML^2. From FO^2 view point, Gradel and Otto[8] show that most of the natural extensions of FO^2 (like transitive closure, lfp) are undecidable except for the counting quantifiers. In this sense, 2-variable TML can be seen as another rare extension of FO^2 that still remains decidable. Note that in this paper we consider the two variable fragment of TML without the bundling or guarded or monodic restriction. Also, there is no natural translation of two variable TML to any known decidable fragment of FO such as the two variable fragment of FO with 2 equivalence relations etc (cf [20]).

Thus, the contribution of this paper is technical, mainly in the identification of a decidable fragment of TML. As is standard with two variable logics, we first introduce a normal form which is a combination of Fine's normal form for modal logics ([4]) and the Scott normal form ([7]) for FO^2. We then prove a bounded agent property using an argument that can be construed as modal depth induction over the "classical" bounded model construction for FO^2.

2 TML syntax and semantics

We consider relational vocabulary with no constants or function symbols, and without equality.

▶ **Definition 1** (TML syntax). *Given a countable set of variables* Var *and a countable set of predicate symbols* $\mathcal{P}$*, the syntax of* TML *is defined as follows:*

$$\varphi ::= P(\overline{x}) \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \exists x\ \varphi \mid \forall x\ \varphi \mid \Box_x \varphi \mid \Diamond_x \varphi$$

where $x \in \mathsf{Var}$*,* $\overline{x}$ *is a vector of length* n *over* Var *and* $P \in \mathcal{P}$ *of arity* n*.*

The *free* and *bound* occurrences of variables are defined as in FO with $\mathsf{Fv}(\Box_x\varphi) = \mathsf{Fv}(\varphi) \cup \{x\}$. We write $\varphi(\overline{x})$ if all the free variables in φ are included in $\overline{x}$. Given a TML formula φ and $x, y \in \mathsf{Var}$, if $y \notin \mathsf{Fv}(\varphi)$ then we write $\varphi[y/x]$ for the formula obtained by replacing every occurrence of x by y in φ. A formula φ is called a *sentence* if $\mathsf{Fv}(\varphi) = \emptyset$. The notion of modal depth of a formula φ (denoted by $\mathsf{md}(\varphi)$) is also standard, which is simply the maximum number of nested modalities occurring in φ. The length of a formula φ is denoted by $|\varphi|$ and is simply the number of symbols occurring in φ.

In the semantics, the number of accessibility relations is not fixed, but specified along with the structure. Thus the Kripke frame for TML is given by (W, D, R) where W is a set of worlds, D is the potential set of agents and $R \subseteq (W \times D \times W)$. The agent dynamics is captured by a function ($\delta : W \to 2^D$ below) that specifies, at any world w, the set of agents *live* (or meaningful) at w. The condition that whenever $(u, d, v) \in R$, we have that $d \in \delta(u)$ ensures only an agent alive at u can consider v accessible.

A *monotonicity* condition is imposed on the accessibility relation as well: whenever $(u, d, v) \in R$, we have that $\delta(u) \subseteq \delta(v)$. This is required to handle interpretations of free variables (cf [3, 6, 5]). Hence the models are called "increasing agent" models.

▶ **Definition 2** (TML structure). *An increasing agent model for* TML *is defined as the tuple* $M = (W, D, \delta, R, \rho)$ *where* W *is a non-empty countable set of worlds,* D *is a non-empty countable set of agents,* $R \subseteq (W \times D \times W)$ *and* $\delta : W \to 2^D$*. The map* δ *assigns to each* $w \in W$ *a non-empty local domain such that whenever* $(w, d, v) \in R$ *we have* $d \in \delta(w) \subseteq \delta(v)$ *and* $\rho : (W \times \mathcal{P}) \to \bigcup_{n \in \omega} 2^{D^n}$ *is the valuation function where for all* $P \in \mathcal{P}$ *of arity* n *we have* $\rho(w, P) \subseteq [\delta(w)]^n$*.*

For a given model M, we use $W^M, D^M, \delta^M, R^M, \rho^M$ to refer to the corresponding components. We drop the superscript when M is clear from the context. We often write D_w for $\delta(w)$. A *constant agent* model is one where $D_w = D$ for all $w \in W$. To interpret free variables, we need a variable assignment $\sigma : \mathsf{Var} \to D$. Call σ *relevant* at $w \in W$ if $\sigma(x) \in \delta(w)$ for all $x \in \mathsf{Var}$. The increasing agent condition ensures that if σ is relevant at w and $(w, d, v) \in R$ then σ is relevant at v as well. In a constant agent model, every assignment σ is relevant at all the worlds.

▶ **Definition 3** (TML semantics)**.** *Given a* TML *structure* $M = (W, D, \delta, R, \rho)$ *and a* TML *formula* φ*, for all* $w \in W$ *and* σ *relevant at* w*, define* $M, w, \sigma \models \varphi$ *inductively as follows:*

$M, w, \sigma \models P(x_1, \ldots, x_n)$	$\Leftrightarrow$	$(\sigma(x_1), \ldots, \sigma(x_n)) \in \rho(w, P)$
$M, w, \sigma \models \neg\varphi$	$\Leftrightarrow$	$M, w, \sigma \not\models \varphi$
$M, w, \sigma \models (\varphi \wedge \psi)$	$\Leftrightarrow$	$M, w, \sigma \models \varphi$ *and* $M, w, \sigma \models \psi$
$M, w, \sigma \models \exists x\ \varphi$	$\Leftrightarrow$	*there is some* $d \in \delta(w)$ *such that* $M,\ w,\ \sigma_{[x \mapsto d]} \models \varphi$
$M, w, \sigma \models \Box_x\ \varphi$	$\Leftrightarrow$	$M, v, \sigma \models \varphi$ *for all* v *s.t.* $(w, \sigma(x), v) \in R$

where $\sigma_{[x \mapsto d]}$ *denotes another assignment that is the same as* σ *except for mapping* x *to* d*.*

The semantics for $\varphi \vee \psi, \forall x\ \varphi$ and $\Diamond_x\ \varphi$ are defined analogously. Note that $M, w, \sigma \models \varphi$ is inductively defined only when σ is relevant at w. We often abuse notation and say "for all w and for all interpretations σ", when we mean "for all w and for all interpretations σ relevant at w" (and we will ensure that relevant σ are used in proofs). In general, when considering the truth of φ in a model, it suffices to consider $\sigma : \mathsf{Fv}(\varphi) \mapsto D$, assignment restricted to the variables occurring free in φ. When $\mathsf{Fv}(\varphi) \subseteq \{x_1, \ldots, x_n\}$ and $\overline{d} \in [D_w]^n$ is a vector of length n over D_w, we write $M, w \models \varphi[\overline{d}]$ to denote $M, w, \sigma \models \varphi(\overline{x})$ where for all $i \leq n,\ \sigma(x_i) = d_i$. When φ is a sentence, we simply write $M, w \models \varphi$. A formula φ is *valid*, if φ is true in all models M at all w for all interpretations σ (relevant at w). A formula φ is *satisfiable* if $\neg\varphi$ is not valid.

Now we take up the satisfiability problem which is the central theme of this paper. First we observe that the satisfiability problem is equally hard for constant and increasing agent models for TML.

▶ **Proposition 4.** *For any* TML *formula* φ*, there is a corresponding formula* $\hat{\varphi} \in$ TML *such that* φ *is satisfiable in an increasing agent model with agent set* D *iff* $\hat{\varphi}$ *is satisfiable in a constant agent model with agent set* D*.*

To see why the proposition is true, if φ is satisfiable in an increasing agent model, then we can turn the model into constant agent model as follows. We introduce a new unary predicate E and ensure that $E(d)$ is true at w if d is a member of $\delta(w)$ in the given increasing agent model. But now, all quantifications have to be relativized with respect to the new predicate E. Thus, the syntactic translation is defined as follows:

▶ **Definition 5.** *Let* φ *be any* TML *formula amd let* E *be a new unary predicate not occurring in* φ*. The translation is defined inductively as follows:*

- $\mathit{Tr}_1(P(x_1, \ldots, x_n)) = P(x_1, \ldots, x_n)$
- $\mathit{Tr}_1(\neg\varphi) = \neg \mathit{Tr}_1(\varphi)$ *and* $\mathit{Tr}_1(\varphi \wedge \psi) = \mathit{Tr}_1(\varphi) \wedge \mathit{Tr}_1(\psi)$
- $\mathit{Tr}_1(\Box_x \varphi) = \Box_x(\mathit{Tr}_1(\varphi))$
- $\mathit{Tr}_1(\exists x\ \varphi) = \exists x\ (E(x) \wedge \mathit{Tr}_1(\varphi))$

With this translation, we also need to ensure that the predicate E respects monotonicity. Hence we have $\gamma_\varphi = \bigwedge_{i+j\leq \mathsf{md}(\varphi)} (\forall y \Box_y)^i \big(\forall x\ E(x) \supset (\forall y \Box_y)^j E(x)\big)$. Now, we can prove that φ is satisfiable in an increasing model M with agent set D iff $\mathsf{Tr}_1(\varphi) \wedge \gamma_\varphi$ is satisfiable in a constant agent model M' with agent set D. This translation is similar in approach to the one for FOML[23]. The formal proof of Prop. 4 is provided in [16].

The propositional term modal logic (PTML) is a fragment of TML where the atoms are restricted to propositions. Note that the variables still appear as index of modalities. For PTML, the valuation function can be simply written as $\rho : W \mapsto 2^{\mathcal{P}}$ where $\mathcal{P}$ is the set of propositions. Now we prove that the satisfiability problem for PTML is as hard as that for TML.

▶ **Proposition 6.** *Any* TML *formula φ has a corresponding formula $\hat{\varphi} \in$* PTML*, such that φ is satisfiable in an increasing (constant) agent model with agent set D iff $\hat{\varphi}$ is satisfiable in an increasing (constant) agent model with agent set D.*

The reduction is based on the translation of an arbitrary atomic predicate $P(x_1, \ldots, x_n)$ to $\Diamond_{x_1} \ldots \Diamond_{x_n} p$ where p is a new proposition which represents the predicate P. However, this cannot be used always[1]. Thus, we use a new proposition q, to distinguish the "real worlds" from the ones that are added because of the translation. But now, the modal formulas have to be relativized with respect to the proposition q. The formal translation is given as follows:

▶ **Definition 7.** *Let φ be any* TML *formula where $P_1, \ldots, P_k$ are the predicates that occur in φ. Let $\{p_1, \ldots, p_n\} \cup \{q\}$ be a new set of propositions not occurring in φ. The translation with respect to q is defined inductively as follows:*

- $Tr_2(P_i(x_1, \ldots, x_n); q) = \Diamond_{x_1}(\neg q \wedge \Diamond_{x_2}(\ldots \neg q \wedge \Diamond_{x_n}(\neg q \wedge p_i) \ldots))$
- $Tr_2(\neg\varphi; q) = \neg Tr_2(\varphi; q)$ *and* $Tr_2(\varphi \wedge \psi; q) = Tr_2(\varphi; q) \wedge Tr_2(\psi; q)$
- $Tr_2(\Box_x \varphi; q) = \Box_x(q \supset Tr_2(\varphi; q))$
- $Tr_2(\exists x\ \varphi; q) = \exists x\ \ Tr_2(\varphi; q)$

Using this translation, we can prove that φ is satisfiable iff $q \wedge \mathsf{Tr}_2(\varphi; q)$ is satisfiable, over the same D. The proof details of Prop 6 are given in [16].

3 Two variable fragment

Note that all the examples discussed in the introduction section use only 2 variables. Thus, TML can express interesting properties even when restricted to two variables. We now consider the satisfiability problem of TML^2. The translation in Def. 7 preserves the number of variables. Therefore it suffices to consider the satisfiability problem for the two variable fragment of PTML.

Let PTML^2 denote the two variable fragment of PTML. We first consider a normal form for the logic. In [4], Fine introduces a normal form for propositional modal logics which is a disjunctive normal form (DNF) with every clause of the form $(\bigwedge_i (s_i) \wedge \Box\alpha \wedge \bigwedge_j \Diamond\beta_j)$ where s_i are literals and α, β_j are again in the normal form. For FO^2, we have Scott normal form [7] where every FO^2 sentence has an equi-satisfiable formula of the form $\forall x \forall y\ \varphi \wedge \bigwedge_i \forall x \exists y\ \psi_i$ where φ and ψ_i are all quantifier free. For PTML^2, we introduce a combination of these two

[1] for instance, this translation will not work for the formula $\exists x\ P(x) \wedge \forall y\ \Box_y \bot$

normal forms, which we call the *Fine Scott Normal form* given by a DNF, where every clause is of the form:

$$\bigwedge_{i\leq a} s_i \wedge \bigwedge_{z\in\{x,y\}} (\Box_z\alpha \wedge \bigwedge_{j\leq m_z} \Diamond_z\beta_j) \wedge \bigwedge_{z\in\{x,y\}} (\forall z\ \gamma \wedge \bigwedge_{k\leq n_z} \exists z\ \delta_k) \wedge \forall x\forall y\ \varphi \wedge \bigwedge_{l\leq b} \forall x\exists y\ \psi_l$$

where $a, m_x, m_y, n_x, n_y, b \geq 0$ and s_i denotes literals. Further, α, β_j are recursively in the normal form and $\gamma, \delta_k, \varphi, \psi_l$ do not have quantifiers at the outermost level and all modal subformulas occurring in these formulas are (recursively) in the normal form. The normal form is formally defined in the next subsection.

Note that the first two conjuncts mimic the modal normal form and the last two conjuncts mimic the FO^2 normal form. The additional conjuncts handle the intermediate step where only one of the variable is quantified and the other is free.

We now formally define the normal form and prove that every PTML^2 formula has a corresponding equi-satisfiable formula in the normal form. After this we prove the bounded agent property for formulas in the normal form using an inductive FO^2 type model construction.

3.1 Normal form

We use $\{x, y\} \subseteq \mathsf{Var}$ as the two variables of PTML^2. We use z to refer to either x or y and refer to variables z_1, z_2 to indicate the variables x, y in either order. We use Δ_z to denote any modal operator $\Delta \in \{\Box, \Diamond\}$ and $z \in \{x, y\}$. A literal is either a proposition or its negation. Also, we assume that the formulas are given in negation normal form(NNF) where the negations are pushed in to the literals.

▶ **Definition 8** (FSNF normal form). *We define the following terms to introduce the* Fine Scott normal form *(FSNF) for* PTML^2*:*

- *A formula φ is a module if φ is a literal or φ is of the form $\Delta_z\alpha$.*
- *For any formula φ, the outer most components of φ given by $\mathsf{C}(\varphi)$ is defined inductively where for any φ which is a module, $\mathsf{C}(\varphi) = \{\varphi\}$ and $\mathsf{C}(Qz\ \varphi) = \{Qz\ \varphi\}$ where $z \in \{x, y\}$ and $Q \in \{\forall, \exists\}$. Finally $\mathsf{C}(\varphi \odot \psi) = \mathsf{C}(\varphi) \cup \mathsf{C}(\psi)$ where $\odot \in \{\wedge, \vee\}$.*
- *A formula φ is quantifier-safe if every $\psi \in \mathsf{C}(\varphi)$ is a module.*
- *We define Fine Scott normal form(FSNF) normal form (DNF and conjunctions) inductively as follows:*
  - *Any conjunction of literals is an FSNF conjunction.*
  - *φ is said to be in FSNF DNF if φ is a disjunction where every clause is an FSNF conjunction.*
  - *Suppose φ is quantifier-safe and for every $\Delta_z\psi \in \mathsf{C}(\varphi)$ if ψ is in FSNF DNF normal form then we call φ a quantifier-safe normal formula.*
  - *Let $a, b, m_x, m_y, n_x, n_y \geq 0$.*
    Suppose $s_1, \ldots, s_a$ are literals, $\alpha^x, \alpha^y, \beta_1^x, \ldots, \beta_{m_x}^x, \beta_1^y, \ldots, \beta_{m_y}^y$ are formulas in FSNF DNF and $\gamma^x, \gamma^y, \delta_1^x, \ldots, \delta_{n_x}^x, \delta_1^y, \ldots, \delta_{n_y}^y, \varphi, \psi_1, \ldots, \psi_b$ are quantifier-safe normal formulas then:

$$\bigwedge_{i\leq a} s_i \wedge \bigwedge_{z\in\{x,y\}} (\Box_z\alpha^z \wedge \bigwedge_{j\leq m_z} \Diamond_z\beta_j^z) \wedge \bigwedge_{z_1\in\{x,y\}} (\forall z_2\ \gamma^{z_1} \wedge \bigwedge_{k\leq n_z} \exists z_2\ \delta_k^{z_1}) \wedge \forall x\forall y\ \varphi \wedge \bigwedge_{l\leq b} \forall x\exists y\ \psi_l$$

is an FSNF conjunction.

Quantifier-safe formulas are those in which no quantifiers occur outside the scope of modalities. Note that the superscripts in α^x, α^y etc only indicate which variable the formula is associated with, so that it simplifies the notation. For instance, α^x does not say anything about the free variables in α^x. In fact there is no restriction on free variables in any of these formulas.

Further, note that by setting the appropriate indices to 0, we can have FSNF conjunctions where one or more of the components corresponding to $s_i, \beta^x, \beta^y, \delta^x, \delta^y, \psi_l$ are absent. We also consider the conjunctions where one or more of the components corresponding to $\Box_x \alpha^x, \Box_y \alpha^y, \varphi$ are also absent. As we will see in the next lemma, for any sentence $\varphi \in \mathsf{PTML}^2$, we can obtain an equi-satisfiable sentence, which at the outer most level, is a DNF where every clause is of the form $\bigwedge_{i \leq a} s_i \;\wedge\; \forall x \forall y\; \varphi \wedge \bigwedge_{l \leq b} \forall x \exists y\; \psi_l$.

▶ **Lemma 9.** *For every formula $\varphi \in \mathsf{PTML}^2$ there is a corresponding formula ψ in FSNF DNF such that φ and ψ are equi-satisfiable.*

Proof (Sketch). To get the formula in the normal form, we introduce some new unary predicates in the intermediate steps and finally get rid of them using the translation in Def. 7. The proof essentially follows that of reducing an FO^2 formula into its equi-satisfiable Scott normal form.

For the given formula φ, first observe that we can get an equivalent DNF over $\mathsf{C}(\varphi)$ using propositional validities. If φ is modal free, then we can simply ignore the quantifiers, since valuations of propositions do not depend on the quantifiers and the agent set is always non-empty. Thus we get a propositional DNF by erasing the quantifiers and this is in the required form.

If φ contains modal formulas, then we need to reduce every clause of the DNF to an FSNF conjunction. We first translate the formulas at the outer most level to the required form. This is the classical Scott-normal form construction which can be obtained by introducing new unary predicates appropriately to get rid of the nested quantifiers at the outer most level. Then, using the translation in Def 7, we get an equi-satisfiable PTML formula after replacing the newly introduced unary predicates by corresponding propositional translations. Further, replace conjuncts of the form $\Box_z \alpha$ and $\Box_z \beta$ by $\Box_z(\alpha \wedge \beta)$ for $z \in \{x, y\}$ to obtain the resulting formula which has at most one subformula of the from $\Box_x \alpha^x$ and $\Box_y \alpha^y$.

Note that after this translation, the resulting formula is in the required form at the outermost level. We now only need to repeat the entire process for every sub-formula inside the scope of modalities. The lemma is formally proved in [16]. ◀

Since we repeatedly convert the formula into DNF (inside the scope of every modality), if we start with a formula of length n, the final translated formula has length $2^{O(n^2)}$. However, observe that the number of modules in the translated formula is linear in the size of the given formula φ. Furthermore, the given formula is satisfiable in a model M iff the translation is satisfiable in M with appropriate modification of the ρ (valuation function).

3.2 Bounded agent property

Now we prove that any formula $\theta \in \mathsf{PTML}^2$ in FSNF DNF is satisfiable iff θ is satisfiable in a model M where the size of D is bounded. Note that for any PTML formula θ, if $M, w, \sigma \models \theta$ then $M^T, w, \sigma \models \theta$ where M^T is the standard tree unravelling of M with w as root [15]. Further, M^T can be restricted to be of height at most $\mathsf{md}(\theta)$. Hence, we restrict our attention to tree models of finite depth.

First we define the notion of types for agents at every world. In classical FO^2 the 2-types are defined on atomic predicates. In PTML^2 we need to define the types with respect to modules. In any given tree model M rooted at r, for any $w \in W$ and $c, d \in D_w$ the 2-type of (c, d) at w is simply the set of all modules that are true at w where the two variables are assigned c, d in either order. The 1-type of c at w includes the set of all modules that are true at w when both x, y are assigned c. Further, for every non-root node w, suppose $(w' \xrightarrow{a} w)$ then the 1-type of any $c \in D_w$ should capture how c behaves with respect to a and the 1-type(w, c) should also include the information of how c acts with respect to d, for every $d \in D_w$. Thus the 1-type of c at w is given by a 3-tuple where the first component is the set of all modules that are true when both x, y are assigned c, the second component captures how c behaves with respect to the incoming edge of w and the third component is a set of subsets of formulas such that for each $d \in D_w$ there is a corresponding subset of formulas capturing the 2-type of c, d. To ensure that the type definition also carries the information of the height of the world w, if w is at height h then we restrict 1-type and 2-type at w to modules of modal depth at most $\mathsf{md}(\varphi) - h$.

For any formula φ, let $\mathsf{SF}(\varphi)$ be the set of all subformulas of φ closed under negation. We always assume[2] that $\top \in \mathsf{SF}(\varphi)$. Let $\mathsf{SF}^h(\varphi) \subseteq \mathsf{SF}(\varphi)$ be the set of all subformulas of modal depth at most $\mathsf{md}(\varphi) - h$. Thus we have $\mathsf{SF}(\varphi) = \mathsf{SF}^0(\varphi) \supseteq \mathsf{SF}^1(\varphi) \supseteq \ldots \supseteq \mathsf{SF}^{\mathsf{md}(\varphi)}(\varphi)$.

▶ **Definition 10** (PTML type). *For any PTML^2 formula φ and for any tree model M rooted at r with height at most $\mathsf{md}(\varphi)$, for all $w \in W$ at height h:*

- *For all $c, d \in \delta(w)$, define 2-type$(w, c, d) = (\Gamma_{xy}; \Gamma_{yx})$ where*
  $\Gamma_{xy} = \{\psi(x, y) \in \mathsf{SF}^h(\varphi) \mid M, w \models \psi(c, d)\}$ *and*
  $\Gamma_{yx} = \{\psi(x, y) \in \mathsf{SF}^h(\varphi) \mid M, w \models \psi(d, c)\}$.
- *If w is a non root node, (say $w' \xrightarrow{a} w$) then for all $c \in \delta(w)$ define 1-type$(w, c) = (\Lambda_1; \Lambda_2; \Lambda_3)$ where $\Lambda_1 =$ 2-type(w, c, c) and $\Lambda_2 =$ 2-type(w, c, a) and $\Lambda_3 = \{$2-type$(w, c, d) \mid d \in \delta(w)\}$.*
- *For the root node r, for all $c \in \delta(r)$ define 1-type$(w, c) = (\Lambda_1; \{\top\}; \Lambda_3)$ where $\Lambda_1 =$ 2-type(w, c, c) and $\Lambda_3 = \{$2-type$(w, c, d) \mid d \in \delta(w)\}$.*

The second component of 1-type(r, c) is added to maintain uniformity. For all $w \in W$ define 1-type$(w) = \{$1-type$(w, c) \mid c \in D_w\}$ and 2-type$(w) = \{$2-type$(w, c, d) \mid c, d \in D_w\}$. We use Λ, Π to represent elements of 1-type(w) and Λ_1, Π_2 etc for the respective components.

If a formula θ is satisfiable in a tree model, the strategy is to inductively come up with bounded agent models for every subtree of the given tree (based on types), starting from leaves to the root. While doing this, when we add new type based agents to a world at height h, to maintain monotonicity, we need to propagate the newly added agents throughout its descendants. For this, we define the notion of extending any tree model by addition of some new set of agents.

Suppose in a tree model M, world w has local agent set D_w and we want to extend D_w to $D_w \cup C$, then first we have $\Omega : C \mapsto D_w$ which assigns every new agent to some already existing agent. The intended meaning is that the newly added agent $c \in C$ at w mimics the "type" of $\Omega(c)$. If w is a leaf node, we can simply extend $\delta(w)$ to $D_w \cup C$. If w is at some arbitrary height, along with *adding the new agents to the live agent set* to w, we also need to create successors for every $c \in C$, one for each successor subtree of $\Omega(c)$ and inductively add C to all the successor subtrees.

[2] Let p_0 be some proposition occurring in φ, then $\top$ is defined as $p_0 \vee \neg p_0$.

▶ **Definition 11** (Model extension). *Suppose M is a tree model rooted at r with finite agent set D and for every $w \in W$ let M^w be the subtree rooted at w. Let C be some finite set such that $C \cap D = \emptyset$ and for any $w \in W$ let $\Omega : C \mapsto D_w$ be a function mapping C to agent set live at w. Define the operation of "adding C to M^w guided by Ω" by induction on the height of w to obtain a new subtree rooted at w (denoted by $M^w_{(C,\Omega)}$ and the components denoted by δ', ρ' etc).*

- *If w is a leaf, then $M^w_{(C,\Omega)}$ is a tree with a single node w with new $\delta'(w) = \delta(w) \cup C$ and $\rho'(w) = \rho(w)$.*
- *If w is at height h then the new tree $M^w_{(C,\Omega)}$ is obtained from M^w rooted at w with new $\delta'(w) = \delta(w) \cup C$ and $\rho'(w) = \rho(w)$ and replacing all the subtrees M^u rooted at every successor u of w by $M^u_{(C,\Omega)}$. Furthermore, for every $c \in C$ and every $(w, \Omega(c), u) \in R$ create a new copy of $M^u_{(C,\Omega)}$ and rename its root as u^c and add an edge (w, c, u^c) to R'.*

Since we do not have *equality* in the language, this transformation will still continue to satisfy the same formulas.

▶ **Lemma 12.** *Let M be any tree model of finite depth rooted at r with finite agent set D and let $w \in W$. Let $M^w_{(C,\Omega)}$ (rooted at w) be an appropriate model extension of M^w (rooted at w). For any interpretation $\sigma : \mathsf{Var} \mapsto (C \cup D_w)$ let $\hat{\sigma} : \mathsf{Var} \mapsto D_w$ where $\hat{\sigma}(x) = \Omega(\sigma(x))$ if $\sigma(x) \in C$ and $\hat{\sigma}(x) = \sigma(x)$ if $\sigma(x) \in D_w$. Then for all $u \in W$ which is a descendant of w in M and for all $\sigma : \mathsf{Var} \mapsto (C \cup D_w)$ and for all* PTML *formula φ, we have $M^w_{(C,\Omega)}, u, \sigma \models \varphi$ iff $M, u, \hat{\sigma} \models \varphi$.*

To see why the lemma holds, first note that both models agree on literals since the valuation function remains the same. Further, since every new agent mimics some old agent, all the modal and the universal formulas continue to hold. Witnesses for $\exists$ formulas can still be picked from the old agent set (D_u). The lemma is formally proved in [16].

For any formula in the normal form, we use the same notations as in Def. 8. For a given formula $\theta \in \mathsf{PTML}^2$ in FSNF DNF form, let $\boldsymbol{\delta}^x_\theta = \{\exists y\ \delta^x \in \mathsf{SF}(\theta)\}$. Similarly we have $\boldsymbol{\delta}^y_\theta = \{\exists x\ \delta^y \in \mathsf{SF}(\theta)\}$ and $\boldsymbol{\psi}_\theta = \{\forall x \exists y\ \psi \in \mathsf{SF}(\varphi)\}$.

For any tree model M, let $\# \notin D$. For every $w \in W$ and for all $\exists y\ \delta \in \boldsymbol{\delta}^x_\theta$ let the function $g^w_\delta : D_w \mapsto D_w \cup \{\#\}$ be a mapping such that $M, w \models \delta(c, g^w_\delta(c))$ and $g^w_\delta(c) = \#$ only if there is no $d \in D_w$ such that $M, w \models \delta(c, d)$. Similarly for all $\exists x\ \delta \in \boldsymbol{\delta}^y_\theta$ let $h^w_\delta : D_w \mapsto D_w \cup \{\#\}$ such that $M, w \models \delta(h^w_\delta(c), h)$ and $h^w_\delta(c) = \#$ only if there is no $d \in D_w$ such that $M, w \models \delta(d, c)$. Again for all $\forall x \exists y\ \psi \in \boldsymbol{\psi}_\theta$ let $f^w_\psi : D_w \mapsto D_w \cup \{\#\}$ such that $M, w \models \psi(c, f^w_\psi(c))$ and $f^w_\psi(c) = \#$ only if there is no $d \in D_w$ such that $M, w \models \psi(c, d)$.

The functions g, h, f provide the witnesses at a world for every agent (if it exists) for the existential formulas respectively.

▶ **Theorem 13.** *Let $\theta \in \mathsf{PTML}^2$ be in an FSNF* DNF *sentence. Then θ is satisfiable iff θ is satisfiable in a model with bounded number of agents.*

Proof. It suffices to prove ($\Rightarrow$). Let M be a tree model of height at most $\mathsf{md}(\theta)$ rooted at r such that $M, r \models \theta$.

Let $E_\theta = \boldsymbol{\delta}^x_\theta \cup \boldsymbol{\delta}^y_\theta \cup \boldsymbol{\psi}_\theta$ and hence $|\mathsf{E}_\theta| \leq |\theta|$ (say q). Let $\mathsf{E}_\theta = \{\chi_1, \ldots \chi_q\}$ be some enumeration. For every $w \in W$ and $a \in \delta(w)$ let $\mathsf{Wit}(a) = \{b_1 \ldots b_q\}$ be the witnesses for a where $b_i = g^w_\delta(c)$ if χ_i is of the form $\exists y\ \delta \in \boldsymbol{\delta}^x_\theta$ (similarly $b_i = h^w_\delta(c)$ or $b_i = f^w_\psi(c)$ corresponding to χ_i of the from $\exists x\ \delta^y$ and $\forall x \exists y\ \psi$ respectively). If $b_i = \#$ then set $b_i = b$ for some arbitrary but fixed $b \in \delta(w)$.

For all $w \in W$ and $\Lambda \in$ 1-type(w) fix some $a^w_\Lambda \in \delta(w)$ such that 1-type$(w, a^w_\Lambda) = \Lambda$. Furthermore, if c is the incoming edge of w and 1-type$(w, c) = \Lambda$ then let $a^w_\Lambda = c$. Let $A^w = \{a^w_\Lambda \mid \Lambda \in$1-type$(w)\}$.

Now we define the bounded agent model. For every $w \in W$ let M^w be the subtree model rooted at $w \in W$. For every such M^w, we define a corresponding *type based model* with respect to θ (denoted by T^w_θ with components denoted by $\delta^w_\theta, \rho^w_\theta$ etc) inductively as follows:

- If w is a leaf then T^w_θ is a tree with a single node w with $\delta^w_\theta(w) =$ 1-type$(w) \times [1 \dots q] \times \{0, 1, 2\}$ and $\rho^w_\theta(w) = \rho(w)$.
- If w is at height h, T^w_θ is a tree rooted at w with $\delta^w_\theta(w) =$ 1-type$(w) \times [1 \dots q] \times \{0, 1, 2\}$ and $\rho^w_\theta(w) = \rho(w)$.
  Before defining the successors of w in T^w_θ note that for every $(w, a, u) \in R$ we have T^u_θ which is the inductively constructed type based model rooted at u. Also, inductively we have $\delta^u_\theta(u) =$1-type$(u) \times [1 \dots q] \times \{0, 1, 2\}$.

  Now for every $a^w_\Lambda \in A^w$ let $\{b_1 \dots b_q\}$ be the corresponding witnesses as described above. For every successor $(w, a^w_\Lambda, u) \in R$ and for every $1 \leq e \leq q$ and $f \in \{0, 1, 2\}$, create a new copy of T^u_θ (call it $N^{(\Lambda,e,f)}$) and name its root as $u^{(\Lambda,e,f)}$. Now add $\delta^w_\theta(w)$ to $N^{(\Lambda,e,f)}$ at $u^{(\Lambda,e,f)}$ guided by Ω where Ω is defined as follows:
  - For all $\Pi \in$ 1-type(w) we have $a^w_\Pi \in A^w$. Define $\Omega((\Pi, e, f)) = ($1-type$(u, a^w_\Pi), e, f)$.
  - for all $k \leq q$ if 1-type$(u, b_k) = \Pi$ then $\Omega((\Pi, k, f')) = ($1-type$(u, b_k), e, f)$ where $f' = f + 1 \mod 3$.
  - Let $f' = f - 1 \mod 3$. For all $\Pi \in$ 1-type(w) let the witness set of a^w_Π be $\{d_1 \dots d_q\}$. For all $l \leq q$ if 1-type$(w, d_l) = \Lambda$ then by Λ_3 component, there is some $a \in \delta(w)$ such that 2-type$(w, d_l, a^w_\Pi) =$ 2-type(w, a^w_Λ, a). Define $\Omega((\Pi, l, f')) = ($1-type$(u, a), e, f)$.
  - For all $(\Pi, e', f') \in \delta^w_\theta(w)$ if $\Omega(\Pi, e', f')$ is not yet defined, then set $\Omega(\Pi, e', f') = ($1-type$(u, a^w_\Pi), e, f)$.

  Add an edge $(w, (\Lambda, e, f), u^{(\Lambda,e,f)})$ to R^w_θ.

Note that Ω is well defined since the first three steps are defined for the indices $f, (f{+}1 \mod 3)$ and $(f{-}1 \mod 3)$ respectively, which are always distinct. Also note that T^r_θ is a model that satisfies bounded agent property. Thus, it is sufficient to prove that $T^r_\theta, r \models \theta$.

Claim. For every $w \in W$ at height h and for all $\lambda \in \mathsf{SF}^h(\theta)$ the following holds:

1. Suppose λ is a sentence and $M, w \models \lambda$ then $T^w_\theta, w \models \lambda$.
2. If $\mathsf{Fv}(\lambda) \subseteq \{x, y\}$ and for all $\Lambda, \Pi \in$ 1-type(w) if $M, w, [x \mapsto a^w_\Lambda, y \mapsto a^w_\Pi] \models \lambda$ then for all $1 \leq e \leq q$ and $f \in \{0, 1, 2\}$ we have $T^w_\theta, w, [x \mapsto (\Lambda, e, f), y \mapsto (\Pi, e, f)] \models \lambda$.

Note that the theorem follows from claim (1), since θ is sentence and $M, r \models \theta$.

The proof of the claim is by reverse induction on h. In the base case $h = \mathsf{md}(\theta)$ which implies λ is modal free and hence is a DNF over literals. Thus, both the claims follow since $\rho(w) = \rho^w_\theta(w)$.

For the induction step, let w be at height h. Now we induct on the structure of λ. Again if λ is a literal then both the the claims follow since $\rho(w) = \rho^w_\theta(w)$. The case of $\wedge$ and $\vee$ are standard.

For the case $\Box_x \lambda$, we only need to prove claim(2). Now suppose $M, w, [x \mapsto a^w_\Lambda, y \mapsto a^w_\Pi] \models \Box_x \lambda$. Pick arbitrary e and f. We need to prove that $T^w_\theta, w, [x \mapsto (\Lambda, e, f), y \mapsto (\Pi, e, f)] \models \Box_x \lambda$. Pick any $(w, (\Lambda, e, f), u^{(\Lambda,e,f)}) \in R^w_\theta$, then by construction we have $(w, a^w_\Lambda, u) \in R$ and since $M, w, [x \mapsto a^w_\Lambda, y \mapsto a^w_\Pi] \models \Box_x \lambda$, we have $M, u, [x \mapsto a^w_\Lambda, y \mapsto a^w_\Pi] \models \lambda$. Let $a^u_{\Pi'} \in A^u$ such that 1-type$(u, a^u_{\Pi'}) =$1-type(u, a^w_Π) and since a^w_Λ is the incoming edge of u, by Π_2 component, we have 2-type$(u, a^w_\Pi, a^w_\Lambda)$ =2-type$(u, a^u_{\Pi'}, a^w_\Lambda)$ and also $a^w_\Lambda \in A^u$. Hence $M, u, [x \mapsto a^w_\Lambda, y \mapsto a^u_{\Pi'}] \models \lambda$ and by induction hypothesis $T^u_\theta, u, [x \mapsto (\text{1-type}(u, a^w_\Lambda), e, f), y \mapsto (\text{1-type}(u, a^u_{\Pi'}), e, f)] \models \lambda$. Now by construction, at $u^{(\Lambda,e,f)}$ we have $\Omega(\Lambda, e, f) = (\text{1-type}(w, a^w_\Lambda), e, f)$ and $\Omega(\Pi, e, f) = (\text{1-type}(u, a^u_{\Pi'}), e, f)$. Thus, by Lemma 12, $T^w_\theta, u^{(\Lambda,e,f)}, [x \mapsto (\Lambda, e, f), y \mapsto (\Pi, e, f)] \models \lambda$. Hence, we have $T^w_\theta, w, [x \mapsto (\Lambda, e, f), y \mapsto (\Pi, e, f)] \models \Box_x\lambda$. The case for $\Box_y\lambda$ is analogous.

For the case $\Diamond_y\lambda$, again only claim(2) applies. Suppose $M, w, [x \mapsto a^w_\Lambda, y \mapsto a^w_\Pi] \models \Diamond_y\lambda$. Now pick e and f appropriately. We need to prove that $T^w_\theta, w, [x \mapsto (\Gamma, e, f), y \mapsto (\Pi, e, f)] \models \Diamond_y\lambda$. By supposition, there is some $w \xrightarrow{a^w_\Pi} u$ such that $M, u, [x \mapsto a^w_\Lambda, y \mapsto a^w_\Pi] \models \lambda$. Using the argument similar to the previous case, we can prove that $T^w_\theta, u^{(\Lambda,e,f)}, [x \mapsto (\Lambda, e, f), y \mapsto (\Pi, e, f)] \models \lambda$ and hence $T^w_\theta, w, [x \mapsto (\Gamma, e, f), y \mapsto (\Pi, e, f)] \models \Diamond_y\lambda$. The case of $\Diamond_x\lambda$ is symmetric.

For the case $\exists y\ \lambda$ (where x is free at the outer most level), for claim (2) first note that since θ is in the normal form, λ is quantifier-safe. Also note that $\exists y\ \lambda = \chi_i$ for some $\chi_i \in E_\theta$. Now, suppose $M, w, [x \mapsto a^w_\Lambda] \models \exists y\ \lambda$ then we need to prove that $T^w_\theta, w, [x \mapsto (\Lambda, e, f)] \models \exists y\ \lambda$. Let the i^{th} witness of a^w_Λ be b_i and hence $M, w, [x \mapsto a^w_\Lambda, y \mapsto b_i] \models \lambda$. Let 1-type$(w, b_i) = \Pi'$, we claim that $T^w_\theta, w, [x \mapsto (\Lambda, e, f), y \mapsto (\Pi', i, f')] \models \lambda$ where $f' = f + 1 \mod 3$. Suppose not, then $\wedge$ and $\vee$ can be broken down and we get some module such that $M, w, [x \mapsto a^w_\Lambda, y \mapsto b_i] \models \Delta_z\lambda'$ and $T^w_\theta, w, [x \mapsto (\Lambda, e, f), y \mapsto (\Pi', i, f')] \not\models \Delta_z\lambda'$ where $\Delta \in \{\Box, \Diamond\}$ and $z \in \{x, y\}$. Assume $\Delta = \Box$ and $z = x$ (other cases are analogous). This implies $T^w_\theta, w, [x \mapsto (\Lambda, e, f), y \mapsto (\Pi', i, f')] \models \Diamond_x\neg\lambda'$ and hence there is some $w \xrightarrow{(\Lambda,e,f)} u^{(\Lambda,e,f)}$ such that $T^w_\theta, u^{(\Lambda,e,f)}, [x \mapsto (\Lambda, e, f), y \mapsto (\Pi', i, f')] \models \neg\lambda'(*)$. By construction, there is a corresponding $w \xrightarrow{a^w_\Lambda} u$ in M. Now since $M, w, [x \mapsto a^w_\Lambda, y \mapsto b_i] \models \Box_x\lambda'$, we have $M, u, [x \mapsto a^w_\Lambda, y \mapsto b_i] \models \lambda'$. Let $b'_i \in A^u$ such that 1-type(u, b_i) =1-type(u, b'_i). Since a^w_Λ is the incoming edge to u by Π'_2 component, we have 2-type(u, b_i, a^w_Λ) =2-type(u, b'_i, a^w_Λ) and $a^w_\Lambda \in A^u$. Thus, $M, u, [x \mapsto a^w_\Lambda, y \mapsto b'_i] \models \lambda'$ and by induction hypothesis, $T^u_\theta, u, [x \mapsto (\Lambda, e, f), y \mapsto (\text{1-type}(u, b'_i), e, f)] \models \lambda'$. Again by construction, at u we have $\Omega((\Lambda, e, f)) = (\Lambda, e, f)$ and $\Omega((\Pi', i, f')) = (\text{1-type}(u, b'_i), e, f)$ and hence by Lemma 12, $T^w_\theta, u^{(\Lambda,e,f)}, [x \mapsto (\Lambda, e, f), y \mapsto (\Pi', i, f')] \models \lambda'$ which is a contradiction to (*). The case of $\exists y\ \lambda$ is analogous.

For the case of $\forall x\ \lambda$ (where y is free at the outer most level), suppose $M, w, [y \mapsto a^w_\Pi] \models \forall x\ \lambda$. We need to prove that $T^w_\theta, w, [y \mapsto (\Pi, e, f)] \models \forall x\ \lambda$. Pick any $(\Lambda', e', f') \in \delta^w_\theta(w)$, now we claim $T^w_\theta, w, [x \mapsto (\Lambda', e', f'), y \mapsto (\Pi, e, f)] \models \lambda$ (otherwise, like in the previous case, since λ is quantifier-safe, we can reach a module where they differ and obtain a contradiction). The case $\forall y\ \lambda$ is analogous.

Finally we come to sentences which are relevant for claim (1). Note that in the normal form, at the outermost level, a sentence will have only literals or formulas of the form $\forall x\exists y\ \psi_l$ or $\forall x\forall y\ \varphi$.

For the case $M, w \models \forall x\exists y\ \psi_l$, let $\forall x\exists y\ \psi_l$ be i^{th} formula in E_θ. We need to prove $T^w_\theta, w \models \forall x\exists y\ \psi_l$. Pick any $(\Lambda, e, f) \in \delta^w_\theta(w)$ and we have $a^w_\Lambda \in A^w$. Let the i^{th} witness for a^w_Λ be b_i. Thus we have $M, w, [x \mapsto a_\Gamma, y \mapsto b_i] \models \psi_l$. Let 1-type$(w, b_i) = \Pi'$. Again we claim that $T^w_\theta, w, [x \mapsto (\Gamma, e, f), y \mapsto [\Pi', e, f')] \models \psi_l$ where $f' = f + 1 \mod 3$. Suppose not, again $\wedge$ and $\vee$ can be broken down and we get some module such that $M, w, [x \mapsto a^w_\Lambda, y \mapsto b_i] \models \Delta_z\lambda'$ and $T^w_\theta, w, [x \mapsto (\Lambda, e, f), y \mapsto (\Pi', i, f')] \not\models \Delta_z\lambda'$ where $\Delta \in \{\Box, \Diamond\}$ and $z \in \{x, y\}$. Assume $\Delta = \Diamond$ and $z = y$ (other cases are analogous). This implies $T^w_\theta, w, [x \mapsto$

$(\Lambda, e, f), y \mapsto (\Pi', i, f')] \models \Box_y \neg\lambda'$ (*). Now let $a^w_{\Pi'} \in A^w$ such that 1-type$(w, a^w_{\Pi'})$ = 1-type$(w, b_i) = \Pi'$. Thus by Π'_3 component, there is some $d \in \delta^w_\theta$ such that 2-type$(w, a^w_{\Pi'}, d)$ = 2-type(w, b_i, a^w_Λ) and hence $M, w, [x \mapsto d, y \mapsto a^w_{\Pi'}] \models \Diamond_y \lambda'$. Hence there is some $w \xrightarrow{a^w_{\Pi'}} u$ such that $M, u, [x \mapsto d, y \mapsto a^w_{\Pi'}] \models \lambda'$. Now let 1-type$(u, d)$ = 1-type(u, d') such that $d' \in A^u$ and since $a^w_{\Pi'}$ is the incoming edge, we have $M, u, [x \mapsto d', y \mapsto a^w_{\Pi'}] \models \lambda'$ and by induction hypothesis, $T^u_\theta, u, [x \mapsto$ (1-type$(u, d'), i, f'), y \mapsto$ (1-type$(u, a^w_{\Pi'}), i, f')] \models \lambda'$ and while constructing $u^{(\Pi', i, f')}$ (case 3 applies for a^w_Λ since its i^{th} witness has same 1-type as $a^w_{\Pi'}$) we have $\Omega((\Lambda, e, f' - 1))$ = (1-type$(u, d'), i, f')$. Thus by Lemma 12 (since $f' - 1 = f$), $T^w_\theta, u^{(\Pi', i, f')}, [x \mapsto (\Lambda, e, f), y \mapsto (\Pi', i, f')] \models \lambda'$ which contradicts (*).

Finally, for the case $\forall x \forall y\ \varphi$ suppose $M, w \models \forall x \forall y\ \varphi$, then for any (Γ, e, f), $(\Delta, e', f') \in \delta^w_\theta(w)$ we claim that $T^w_\theta, w, [x \mapsto (\Gamma, e, f), y \mapsto (\Delta, e', f')] \models \varphi$ (else again, go to the smallest module and prove contradiction). ◀

Note that in the type based model, at any world w we have $|\delta^w_\theta| = 2^{2^{O(|\mathsf{SF}(\theta)|)}}$. Now if we start with a PTML^2 formula φ, then though its corresponding equi-satisfiable formula θ is exponentially larger, the number of distinct subformulas in θ is still linear in the size of φ.

▶ **Corollary 14.** TML^2 *satisfiability is in 2-EXPSPACE.*

Proof. Any TML^2 formula α is satisfiable iff (by Prop.6) its corresponding PTML^2 translation φ is satisfiable iff (by Theorem 13) the corresponding normal form θ of φ is satisfiable over agent set D of size $2^{2^{O(|\varphi|)}}$ iff (by Prop. 4) $\hat{\theta} \in$ PTML^2 is satisfiable in a constant domain model over D.

Thus we can expand the quantifiers of $\hat{\theta}$ by corresponding $\bigwedge$ and $\bigvee$ for $\forall$ and $\exists$ respectively and we get a propositional multi-modal formula. This satisfiability is in PSPACE. But in terms of the size of the formulas, $|\hat{\theta}| = 2^{2^{|\alpha|^2}}$. Thus we have a 2-EXPSPACE algorithm. ◀

3.3 Example

We illustrate the construction of *type based models* with an example. Consider the PTML^2 sentence $\theta := \forall x\ \Box_x \Box_x \bot \wedge \forall x \exists y\ (\Box_x(\Diamond_y(\neg p) \wedge \exists y\ \Diamond_y p))$ which is in FSNF DNF. Let M be the model described in Fig. 1 where

- $W = \{r\} \cup \{u^i, v^i, w^i \mid i \in \mathcal{N}\}$
- $D = \mathcal{N}$
- $\delta(r) = \{2i \mid i \in \mathcal{N}\}$ (all even numbers) and
  $\delta(w^i) = \delta(u^i) = \delta(v^i) = \mathcal{N}$
- $R = \{(r, 2i, w^i), (w^i, 2i+1, u^i), (w^i, 2i+2, v^i) \mid i \in \mathcal{N}\}$
- $\rho(r) = \rho(w^i) = \rho(v^i) = \emptyset$ and $\rho(u^i) = p$ for all $i \in \mathcal{N}$.

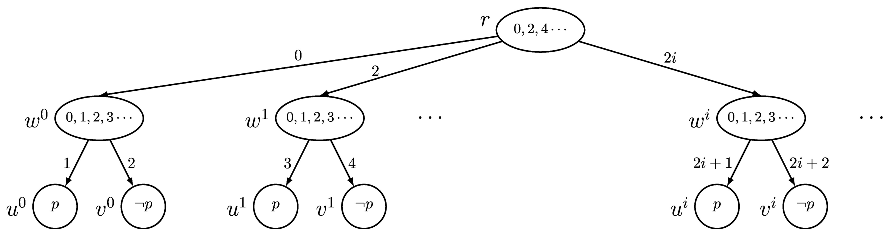

Figure 1 Given model such that $M, r \models \theta$.

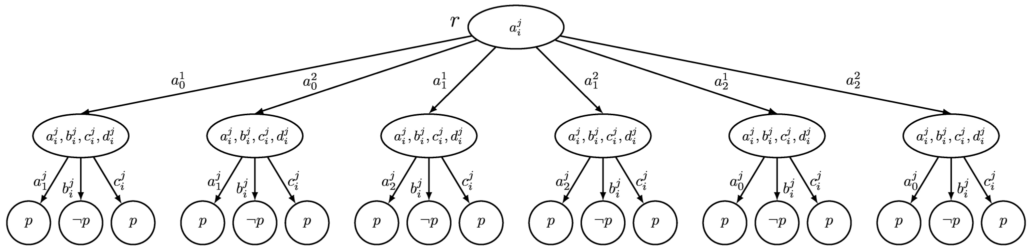

Figure 2 Corresponding bounded agent model with $M', r \models \theta$. a_i^j, b_i^j, c_i^j corresponds to agents with $1 \leq j \leq 2$ and $i \in \{0, 1, 2\}$. The edge a_i^j, b_i^j, c_i^j indicate one successor for every $1 \leq j \leq 2$ and $i \in \{0, 1, 2\}$.

Clearly, $M, r \models \theta$. Let $f^r : D_r \mapsto D_r$ be defined by $f^r(2i) = 2i + 2$ and at all w^i, $g^i(j) = 2i + 1$ for all $i \in \mathcal{N}$ be the two (relevant) witness functions. The one and two types at every world are described as follows:

At leaf nodes u^i and v^i there is only one distinct one type and two types. At w^i, note that $r \xrightarrow{2i} w_i$ is the incoming edge and only $2i + 1$ and $2i + 2$ have outgoing edges. Thus, there are 3 distinct 1-type members at w^i, each for $(2i + 1), (2i + 2)$ and [the rest]. Let b, c, d be the respective types. Finally at the root again we have only a single distinct type (call it a).

Since there are 2 existential formulas, the root of the type based model has $(1 \times 2 \times 3) = 6$ agents let it be $\{a_f^e \mid 1 \leq e \leq 2,\ 0 \leq f \leq 2\}$ and 0 be the representative. At w^0 we have $(3 \times 2 \times 3) = 18$ agents. Let the representatives be $1, 2, 0$ for b, c, d respectively. Note that we cannot pick any other representative for [the rest] other than 0 since 0 is the incoming edge to w^0. Let the bounded agent set be $\{b_f^e,\ c_f^e,\ d_f^e \mid 1 \leq e \leq 2,\ 0 \leq f \leq 2\}$. The corresponding bounded model M' is described in Figure 2. It can be verified that $M', r \models \theta$.

4 Discussion

We have proved that the two variable fragment of PTML^2 (and hence TML^2) is decidable. The upper bound shown is in 2-EXPSPACE. A NEXPTIME lower bound follows since FO^2 satisfiability can be reduced to PTML^2 satisfiability. We believe that by careful management of the normal form, space can be reused and the upper bound can in fact be brought down by one exponent. That would still leave a significant gap between lower and upper bounds to be addressed in future work.

We can also prove that addition of constants makes PTML^2 undecidable. In fact, with the addition of a single constant $\mathbf{c}$ we can use $\Box_{\mathbf{c}}$ to simulate the "free" $\Box$ of FOML^2, thus yielding undecidability. When it comes to equality, the situation is more tricky: note that we can no longer use model extension (Def.11 and Lemma 12) since equality might restrict the number of agents at every world.

The most important issue is expressiveness. What kind of accessibility relations or model classes can be characterized by 2-variable TML? This is unclear, but there are sufficiently intriguing examples and applications making the issue an interesting challenge.

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