Constrained Representations of Map Graphs and Half-Squares

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Abstract

The square of a graph H, denoted H^2, is obtained from H by adding new edges between two distinct vertices whenever their distance in H is two. The half-squares of a bipartite graph $B = (X, Y, E_B)$ are the subgraphs of B^2 induced by the color classes X and Y, $B^2[X]$ and $B^2[Y]$. For a given graph $G = (V, E_G)$, if $G = B^2[V]$ for some bipartite graph $B = (V, W, E_B)$, then B is a representation of G and W is the set of points in B. If in addition B is planar, then G is also called a map graph and B is a witness of G [Chen, Grigni, Papadimitriou. Map graphs. *J. ACM*, 49 (2) (2002) 127-138].

While Chen, Grigni, Papadimitriou proved that any map graph $G = (V, E_G)$ has a witness with at most $3|V| - 6$ points, we show that, given a map graph G and an integer k, deciding if G admits a witness with at most k points is NP-complete. As a by-product, we obtain NP-completeness of EDGE CLIQUE PARTITION on planar graphs; until this present paper, the complexity status of EDGE CLIQUE PARTITION for planar graphs was previously unknown.

We also consider half-squares of tree-convex bipartite graphs and prove the following complexity dichotomy: Given a graph $G = (V, E_G)$ and an integer k, deciding if $G = B^2[V]$ for some tree-convex bipartite graph $B = (V, W, E_B)$ with $|W| \leq k$ points is NP-complete if G is non-chordal dually chordal and solvable in linear time otherwise. Our proof relies on a characterization of half-squares of tree-convex bipartite graphs, saying that these are precisely the chordal and dually chordal graphs.

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1 Introduction

Map graphs, introduced and investigated in [5, 6], are intersection graphs of simply-connected and interior-disjoint regions of the Euclidean plane; each region is homeomorphic to a closed disc. More precisely, a *map* of a graph $G = (V, E_G)$ is a function $\mathcal{M}$ taking each vertex $v \in V$ to a region $\mathcal{M}(v)$ in the plane, such that all $\mathcal{M}(v)$, $v \in V$, are interior-disjoint, and two distinct vertices v and v' of G are adjacent if and only if the boundaries of $\mathcal{M}(v)$ and $\mathcal{M}(v')$ intersect, even in a point. A *map graph* is one having a map. Map graphs are interesting as they generalize planar graphs in a very natural way. Some applications of map graphs have been addressed in [8]. Papers dealing with hard problems in map graphs include [4, 9, 10, 11, 13, 14].

In [5, 6], the notion of *half-squares* of bipartite graphs has been also introduced in order to give a combinatorial representation of map graphs. The square of a graph H, denoted H^2, is obtained from H by adding new edges between any two vertices at distance two in H. For a bipartite graph $B = (X, Y, E_B)$, the subgraphs of the square B^2 induced by the color classes X and Y, $B^2[X]$ and $B^2[Y]$, are called the two *half-squares* of B. For a given graph $G = (V, E_G)$, if $G = B^2[V]$ for some bipartite graph $B = (V, W, E_B)$, then B is a *representation* or a *half-root* of G and W is the set of *points* in B. While every graph is a

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half-square of some bipartite graph, it turns out that map graphs are exactly half-squares of *planar* bipartite graphs [5, 6]. If $G = (V, E_G)$ is a map graph and $B = (V, W, E_B)$ is a planar representation of G, then B is called a *witness* of G. See Figure 1 for an example.

Figure 1 A map graph G, a map $\mathcal{M}$, and a witness B of G.

It is perhaps important to note at this place that one of the difficulties in recognizing map graphs is that we do not know the set of points of a witness we are looking for. It is shown in [5, 6] that an n-vertex graph $G = (V, E_G)$ is a map graph if and only if it has a witness $B = (V, W, E_B)$ with $|W| \leq 3n - 6$ points, implying that recognizing map graphs is in NP. Subsequently, Thorup [28] announced that recognizing map graphs is in P by giving an $\Omega(n^{120})$-time algorithm for n-vertex input graphs.[1] Thorup's algorithm is very complex and highly non-combinatorial. Given the very high polynomial degree in Thorup's running time, the most discussed problem concerning map graphs is whether there is a faster recognition algorithm with simpler arguments for map graphs.

One direction in attacking this problem is to consider map graphs with restricted witness. Structural results and more efficient recognition algorithms for map graphs with restricted witness will enhance our understanding on map graphs in whole. More generally, let $\mathcal{B}$ be a class of (not necessarily planar) bipartite graphs, the following problem has been discussed first by Le and Le [18].

$\mathcal{B}$ REPRESENTATION
Instance: A graph $G = (V, E_G)$.
Question: Does there exist a bipartite graph $B = (V, W, E_B)$ in $\mathcal{B}$ with $G = B^2[V]$?

Recall that in case $\mathcal{B}$ is the class of all planar bipartite graphs, PLANAR REPRESENTATION is the problem of recognizing map graphs, which admits an $\Omega(n^{120})$-time algorithm due to Thorup. Recall also that every map graph has a witness $B = (V, W, E_B)$ with $|W| \leq 3|V| - 6$ due to Chen *et al.* [5, 6]. This motivates considering the following problem.

POINT MINIMAL $\mathcal{B}$ REPRESENTATION
Instance: A graph $G = (V, E_G)$ and an integer k.
Question: Does there exist a bipartite graph $B = (V, W, E_B)$ in $\mathcal{B}$ with $G = B^2[V]$ and $|W| \leq k$?

In case $\mathcal{B}$ is the class of all bipartite graphs, we simply denote the problem by POINT MINIMAL REPRESENTATION.

This paper considers the cases where $\mathcal{B}$ is one of the classes of (planar) bipartite graphs of a given girth, of tree-convex bipartite graphs, and of tree-biconvex bipartite graphs. All terms used are given in the next section.

[1] Thorup did not give the running time explicitly, but it is estimated to be roughly $\Omega(n^{120})$ with n being the vertex number of the input graph; cf. [6].

Our contributions. We first consider map graphs with witness of large girth and, more generally, half-squares of bipartite graphs of large girth, and prove the following complexity dichotomy for MINIMAL POINT (PLANAR) GIRTH-g REPRESENTATION: Given a (map) graph $G = (V, E_G)$ and an integer k, deciding if $G = B^2[V]$ for some (planar) bipartite graph $B = (V, W, E_B)$ of girth at least g with $|W| \leq k$ points is NP-complete for $g \leq 6$ and polynomially solvable otherwise. The case $g \geq 8$ is based on our previous paper [19], and the case $g \leq 6$ is based on a close connection to the well-known NP-complete problems EDGE CLIQUE COVER and EDGE CLIQUE PARTITION. It is perhaps interesting to note that, while recognizing map graphs is in P due to Thorup, our hardness result in case $g = 4$ says that the problem becomes intractable if we ask for a witness with few points. In case $g = 6$, our result implies that EDGE CLIQUE PARTITION is NP-complete for planar graphs. (The complexity status of this problem for planar graphs was previously unknown.) We then consider half-squares of tree-convex bipartite graphs, and prove the following complexity dichotomy for MINIMAL POINT TREE-CONVEX REPRESENTATION: Given a graph $G = (V, E_G)$ and an integer k, deciding if $G = B^2[V]$ for some tree-convex bipartite graph $B = (V, W, E_B)$ with $|W| \leq k$ points is NP-complete for non-chordal dually chordal graphs G and solvable in linear time otherwise. We obtain this result by proving that half-squares of tree-convex bipartite graphs are exactly the chordal and dually chordal graphs. We also show that MINIMAL POINT TREE-BICONVEX REPRESENTATION can be solved in linear time by proving that half-squares of tree-biconvex bipartite graphs are precisely the double chordal graphs. Our results on half-squares of tree-(bi)convex bipartite graphs settle the question left open in [18].

Related work. The first restricted representations of map graphs have been considered in [5, 6] which lead to the so-called k-map graphs; k-map graphs are map graphs having a witness in which every point has at most k neighbors. It turns out that, for $k \leq 3$, k-map graphs are precisely the planar graphs. 4-map graphs can be recognized in cubic time [1], and are related to 1-planar graphs [1, 7], a relevant topic in graph drawing. Recognizing k-map graphs, $k \geq 5$, in polynomial time still remains open. (We remark that Thorup's algorithm cannot be used to recognize map graphs having witness with additional properties.)

Mnich et al. [25] considered map graphs with outerplanar witness and tree witness, and showed that such map graphs can be recognized in linear time. Map graphs with witness of a given girth and, more generally, half-squares of bipartite graphs of a given girth have been considered in the recent paper [19]. It is shown in that paper that half-squares of (planar) bipartite graphs of girth at least 8 admit good characterizations, leading to cubic time recognition algorithms. In [18], half-squares of classical bipartite graphs, such as biconvex, convex, and chordal bipartite graphs, have been studied. It turns out that half-squares of biconvex, convex, and chordal bipartite graphs (all are subclasses of tree-(bi)convex bipartite graphs) are exactly the proper interval, interval, and strongly chordal graphs, respectively.

The paper is organized as follows. All definitions and notion needed are provided in the next section. Section 3 first collects known results on half-squares of (planar) bipartite graphs of large girth, and then provides a dichotomy theorem for POINT MINIMAL GIRTH-g (PLANAR) REPRESENTATION. Section 4 deals with half-squares of tree-convex and tree-biconvex bipartite graphs, and provides a dichotomy theorem for POINT MINIMAL TREE-CONVEX REPRESENTATION. Section 5 concludes the paper with some open problems for future work.

2 Preliminaries

All graphs considered are simple and connected. Let $G = (V, E_G)$ be a graph with vertex set $V(G) = V$ and edge set $E(G) = E_G$. A *stable set* (a *clique*) in G is a set of pairwise non-adjacent (adjacent) vertices. The complete graph on n vertices and the cycle with n vertices are denoted K_n and C_n, respectively. A K_3 is also called a *triangle*. The *diamond*, denoted $K_4 - e$, is the graph obtained from the K_4 by deleting an edge.

The neighborhood of a vertex v in G, denoted $N_G(v)$, is the set of all vertices in G adjacent to v; if the context is clear, we simply write $N(v)$. A *universal vertex* v in G is one with $N(v) = V \setminus \{v\}$, i.e., v is adjacent to all other vertices in G.

Let F be a graph. F*-free graphs* are those having no induced subgraphs isomorphic to F. *Chordal graphs* are precisely the C_k-free graphs, $k \geq 4$. A *dually chordal graph* G is one in which every connected component H of G admits a spanning tree T such that every maximal clique of H induces a subtree in T.[2] Graphs that are both chordal and dually chordal are called *double chordal*. While chordal graphs are closed under taking induced subgraphs, dually chordal graphs and double chordal graphs are not. *Strongly chordal graphs* are those graphs G such that every induced subgraph of G is double chordal. See [15, 24, 27] for more information on these graph classes. Additional information on dually chordal graphs can be found in [2]. We will use the well-known facts that chordal and dually chordal graphs, hence double chordal graphs, can be recognized in linear time [15, 27, 2], and that any n-vertex chordal graph has at most n maximal cliques and all of them can be listed in linear time [15, 27].

For a subset $W \subseteq V$, $G[W]$ is the subgraph of G induced by W, and $G - W$ stands for $G[V \setminus W]$. For a vertex v, $G - v$ stands for $G - \{v\}$. We will consider map graphs with large-girth witness and, more generally, half-squares of bipartite graphs of large girth. Here, the *girth* of a graph is the minimum length of a cycle in that graph. (Thus, a graph has girth at least g if and only if it is C_k-free for all $k < g$.) We will also consider half-squares of tree-convex bipartite graphs, a problem left open in [18]. A bipartite graph $B = (X, Y, E_B)$ is *tree-convex on* X if there exists a tree $T = (X, E_T)$ such that, for each $y \in Y$, $N(y)$ induces a subtree in T. Being *tree-convex on* Y is defined similarly. B is *tree-convex* if it is tree-convex on X or tree-convex on Y. B is *tree-biconvex* if it is both tree-convex on X and tree-convex on Y. A well-known subclass of tree-biconvex bipartite graphs consists of the *chordal bipartite graphs*, i.e., bipartite graphs containing no induced cycle of length at least six. Liu [21] discusses relationships between tree-convex bipartite graphs and other classical classes of bipartite graphs.

POINT MINIMAL $\mathcal{B}$ REPRESENTATION is related to two well-studied problems. An *edge clique cover* of a graph G is a family of cliques $\mathcal{C}$ in G such that every edge of G is contained in one or more cliques in $\mathcal{C}$. An *edge clique partition* of G is an edge clique cover $\mathcal{C}$ of G such that every edge of G is contained in exactly one clique in $\mathcal{C}$. The two well-studied problems are:

EDGE CLIQUE COVER
Instance: A graph $G = (V, E_G)$ and an integer k.
Question: Does G have an edge clique cover of size k or less?

[2] Dually chordal graphs haven been studied under different names and admit various characterizations. The chosen definition is dependent on our purpose.

EDGE CLIQUE PARTITION	
Instance:	A graph $G = (V, E_G)$ and an integer k.
Question:	Does G have an edge clique partition of size k or less?

EDGE CLIQUE COVER is NP-complete [26, 17], and remains NP-complete on planar graphs [3] and on complements of bipartite graphs [20]. We will use the fact that EDGE CLIQUE COVER is solvable in linear time on chordal graphs [23]. The monograph [24] provides more information on edge clique covers.

EDGE CLIQUE PARTITION is NP-complete [26, 16], and remains NP-complete on K_4-free graphs [22]. In contrast to EDGE CLIQUE COVER, EDGE CLIQUE PARTITION is NP-complete on chordal graphs, even on split graphs [30]. EDGE CLIQUE PARTITION on planar graphs has been considered in [12]; the complexity status of this problem on planar graphs was unknown until our present work.

Let $\mathcal{C}$ be an edge clique cover of a graph $G = (V, E_G)$. The *vertex-$\mathcal{C}$ incidence bipartite graph* of G is $B_\mathcal{C} = (V, \mathcal{C}, E_B)$ with $E_B = \{vC \mid v \in V, C \in \mathcal{C}, v \in C\}$. If $\mathcal{C} = \mathcal{C}(G)$, the set of all maximal cliques in G, then $B_\mathcal{C}$ is called the *vertex-clique incidence bipartite graph* of G and usually denoted by B_G.

▶ **Fact 1.**
(i) *For any graph $G = (V, E_G)$ and any edge clique cover $\mathcal{C}$ of G, $G = B^2_\mathcal{C}[V]$. If $\mathcal{C}$ is in addition an edge clique partition, then $B_\mathcal{C}$ is C_4-free.*
(ii) *For any graph $G = (V, E_G)$ and any bipartite graph $B = (V, W, E_B)$ with $G = B^2[V]$, $\mathcal{C} = \{N_B(w) \mid w \in W\}$ is an edge clique cover of G. If B is in addition C_4-free, then $\mathcal{C}$ is an edge clique partition of G.*

Proof. (i): For all vertices $x, y \in V$ we have $xy \in E_G \Leftrightarrow x, y \in C$ for some $C \in \mathcal{C} \Leftrightarrow xC$ and yC are edges of $B_\mathcal{C}$ for some $C \in \mathcal{C} \Leftrightarrow xy \in E(B^2_\mathcal{C}[V])$. If $\mathcal{C}$ is an edge clique partition, then, for any two distinct cliques $C, C' \in \mathcal{C}$, C and C' have at most one vertex in common. Hence, $B_\mathcal{C}$ is a C_4-free.

(ii): If $G = B^2[V]$ for some bipartite graph $B = (V, W, E_B)$, then any edge of G is in a clique $N_B(w)$ for some $w \in W$, hence $\{N_B(w) \mid w \in W\}$ is an edge clique cover of G. If, in addition, B is C_4-free, then $|N_B(w) \cap N_B(w')| \le 1$ for any two distinct points $w, w' \in W$. Hence $\{N_B(w) \mid w \in W\}$ is an edge clique partition of G. ◀

Thus,
- POINT MINIMAL REPRESENTATION and EDGE CLIQUE COVER, and
- POINT MINIMAL C_4-FREE REPRESENTATION and EDGE CLIQUE PARTITION

are computationally equivalent.

3 Girth-constrained representations

This section deals with half-squares of (planar) bipartite graphs of large girth. In [18], the following useful fact has been observed, and used in [19] in discussing half-squares of bipartite graphs with girth constraints.

▶ **Lemma 1** ([18]). *Let $G = B^2[V]$ for some bipartite graph $B = (V, W, E_B)$. If B has no induced cycle of length six, then every maximal clique Q in G stems from a star in B, i.e., there is a point $w \in W$ such that $Q = N_B(w)$.*

We will also use this fact when considering point minimal representations of half-squares and map graphs.

3.1 Half-squares of girth-constrained bipartite graphs

Recall that *every* graph is a half-square of a girth-six bipartite graph (take its subdivision). Half-squares of bipartite graphs of large girth have been fully characterized as follows.

▶ **Theorem 2** ([19]). *Let $t \geq 4$ be an integer. The following statements are equivalent for every graph $G = (V, E_G)$.*
(i) *G is a half-square of a bipartite graph with girth at least $2t$;*
(ii) *G is diamond-free and C_ℓ-free for every $4 \leq \ell \leq t-1$;*
(iii) *The vertex-clique incidence bipartite graph B_G of G has girth at least $2t$.*

Theorem 2 implies that half-squares of bipartite graphs of large girth can be recognized in cubic time (cf. [19]).

By definition, *every* map graph is a half-square of a planar bipartite graph. Though map graphs can be recognized in polynomial time due to Thorup, no good characterization for map graphs is known so far. Map graphs of planar bipartite graphs of large girth have been fully characterized as follows.

▶ **Theorem 3** ([19]). *Let $t \geq 4$ be an integer. The following statements are equivalent for every graph $G = (V, E_G)$.*
(i) *G is a map graph having a witness of girth at least $2t$;*
(ii) *G is diamond-free and C_ℓ-free for every $4 \leq \ell \leq t-1$, and the vertex-clique incidence bipartite graph B_G of G is planar;*
(iii) *The vertex-clique incidence bipartite graph B_G of G is planar and has girth at least $2t$.*

Theorem 3 implies that map graphs with witness of large girth can be recognized in time $O(n^2 m)$ (cf. [19]). No good characterization of map graphs with girth-six witness is known so far. It is also not known whether these map graphs can be recognized efficiently. Note that every planar graph has a girth-six witness, e.g., its subdivision.

3.2 Point minimal girth-constrained representations

This subsection deals with half-squares of (planar) bipartite graphs with girth constraints. We first consider the non-planar case.

Recall that, by Fact 1, POINT MINIMAL REPRESENTATION is equivalent to EDGE CLIQUE COVER, and thus is NP-complete. Also by Fact 1, POINT MINIMAL C_4-FREE REPRESENTATION is equivalent to EDGE CLIQUE PARTITION, and thus is NP-complete. Notice that C_4-free bipartite graphs and bipartite graphs of girth at least six coincide.

Now, let $t \geq 4$ be an integer and assume $G = B^2[V]$ for some bipartite graph $B = (V, W, E_B)$ of girth at least $2t$. By Lemma 1, any maximal clique in G is a neighborhood of some $w \in W$, implying $|W| \geq |\mathcal{C}(G)|$. Thus, by Theorem 2, B_G is a minimal point girth-$2t$ representation for G. Note that, in this case, $\mathcal{C}(G)$ can be computed in polynomial time (cf. [19]), hence we obtain:

▶ **Theorem 4.** POINT MINIMAL GIRTH-AT-LEAST-$2t$ REPRESENTATION *is* NP*-complete for $t \leq 3$ and solvable in polynomial time otherwise.*

In the remainder of this subsection, we deal with the planar case. We first consider the girth-four witness case, i.e., no girth condition is made. Recall that any map graph with n vertices has a witness with at most $3n-6$ points. We are going to show that finding a witness with minimal number of points is hard. We will use the fact that EDGE CLIQUE COVER remains NP-complete on maximal planar graphs without triangle-separators. More precisely, it was shown in [3], that EDGE CLIQUE COVER remains NP-complete for plane triangulations in which every triangle is a face. Observe that such a triangulation does not contain any 4-clique K_4, unless the whole graph is a K_4. Thus, we may further assume that all plane triangulations considered are K_4-free.

▶ **Theorem 5.** POINT MINIMAL PLANAR REPRESENTATION *is* NP*-complete, even when restricted to planar graphs.*

Proof. Let $G = (V, E_G)$ be a plane triangulation in which every triangle is a face, and let k be an integer. We will argue that G has an edge clique cover of size k or less if and only if G has a witness $B = (V, W, E_B)$ with $|W| \leq k$.

First, assume G has an edge clique cover $\mathcal{C}$ with $|\mathcal{C}| \leq k$. Note that we can assume that every clique in $\mathcal{C}$ is a triangle. Then, as any triangle in G is a face of the plane triangulation G, the vertex-$\mathcal{C}$ incidence bipartite graph $B_{\mathcal{C}} = (V, \mathcal{C}, E_B)$ is planar. Indeed, $B_{\mathcal{C}}$ is obtained from G by

- inserting a point w_T in the face T, $T \in \mathcal{C}$, and
- connecting w_T with the three vertices of the triangle T, and
- deleting all edges of G.

See also Figure 2. By Fact 1, as $\mathcal{C}$ is an edge clique cover of G, $G = B^2_{\mathcal{C}}[V]$, and by construction, $B_{\mathcal{C}}$ has $|\mathcal{C}| \leq k$ points.

Next, assume that $G = H^2[V]$ for some (planar) bipartite graph $H = (V, W, E_H)$ with $|W| \leq k$. Then, by Fact 1, $N_H(w)$, $w \in W$, form an edge clique cover of G with $|W| \leq k$ cliques. (Notice that, in this direction, we do not use the fact that H is planar. Any half-root of G with at most k points works.) ◀

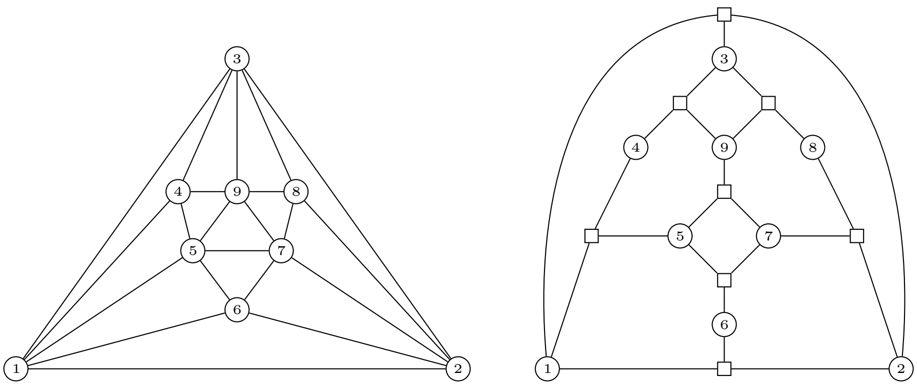

Figure 2 A triangulation G with the edge clique cover $\mathcal{C}$ consisting of the eight triangles 126, 278, 389, 349, 145, 567, 579, and 123, and the planar bipartite graph $B_{\mathcal{C}}$ obtained from G and $\mathcal{C}$.

We next consider the girth-six witness case. Recall that every planar graph has a witness of girth six. We are going to show that finding a witness of girth six with minimal number of points is hard. In a graph, a set of pairwise edge-disjoint triangles is called an independent triangle set. In [29], Uehara considered the following problem:

INDEPENDENT TRIANGLE SET	
Instance:	A graph $G = (V, E_G)$ and an integer k.
Question:	Does G have k or more pairwise edge-disjoint triangles?

Uehara [29] proved that the INDEPENDENT TRIANGLE SET, restricted to plane triangulations, is NP-complete. Chang and Müller [3] observed that INDEPENDENT TRIANGLE SET is NP-complete even on plane triangulations in which every triangle is a face. (We leave the details to the full version.) Recall that we may further assume that all such plane triangulations are K_4-free.

▶ **Theorem 6.** POINT MINIMAL PLANAR GIRTH-6 REPRESENTATION *is* NP*-complete, even when restricted to planar graphs.*

Proof. Let $G = (V, E_G)$ be a plane triangulation without K_4 in which every triangle is a face, and let k be an integer. We will argue that G has k edge-disjoint triangles if and only if there is a C_4-free planar bipartite graph $B = (V, W, E_B)$ with $G = B^2[V]$ and $|W| \leq m - 2k$. (As usual, m denotes the edge number of G.)

First, assume G has k edge-disjoint triangles $T_1, \ldots, T_k$. We construct a bipartite graph $B = (V, W, E_B)$ as follows; let F be the set of all edges of G not belonging to any triangle T_i, $1 \leq i \leq k$. For each $1 \leq i \leq k$, w_i is a point in W corresponding to T_i, and for each edge $e \in F$, w_e is a point in W corresponding to e. Then, B is the V-W incidence bipartite graph. Thus, $W = \{w_1, \ldots, w_k\} \cup \{w_e \mid e \in F\}$, and $E_B = \{vw_i \mid v \in V, v \in T_i, 1 \leq i \leq k\} \cup \{vw_e \mid v \in V, v \in e \in F\}$. Obviously, B is planar. Indeed, B is obtained from the plane triangulation G by

- inserting a point w_i in the face T_i, $1 \leq i \leq k$, and connecting w_i with the three vertices of T_i,
- subdividing each edge $e \in F$ by a point w_e, and
- deleting all edges in $T_1 \cup \ldots \cup T_k$.

Now, as each of the triangles T_i is a face of the plane triangulation G, B is clearly planar.

Since $\{T_1, \ldots, T_k\} \cup F$ is an edge clique cover of G, $G = B^2[V]$. Since the triangles T_i are edge-disjoint, $|N_B(w_i) \cap N_B(w_j)| \leq 1$ for $1 \leq i, j \leq k$, $i \neq j$, and by definition of F, $|N_B(w_e) \cap N_B(w)| \leq 1$ for all $e \in F$, $w \in W \setminus \{w_e\}$. That is, B is C_4-free. Moreover, B has $|W| = k + |F| = m - 2k$ points.

Next, assume that $G = H^2[V]$ for some (planar) C_4-free bipartite graph $H = (V, W, E_H)$ with $|W| \leq m-2k$. Among all such bipartite graphs, let H have minimal number of points $|W|$. Since G is K_4-free, $|N_H(w)| \leq 3$ for all $w \in W$. Since $|W|$ is minimal, $|N_H(w)| \geq 2$, and every two points have distinct neighborhoods. Let $w_1, \ldots, w_{k'}$ be the degree-3 points in W. Then, as H is C_4-free, $N_H(w_i)$, $1 \leq i \leq k'$, are k' edges-disjoint triangles in G. Since $G = H^2[V]$, G has $m = 3k' + (|W| - k') = |W| + 2k' \leq m - 2k + 2k'$ edges. Therefore, $k' \geq k$. That is, G has at least k edges-disjoint triangles. ◀

Since, by Fact 1, POINT MINIMAL GIRTH-6 REPRESENTATION and EDGE CLIQUE PARTITION are equivalent, Theorem 6 implies:

▶ **Corollary 7.** EDGE CLIQUE PARTITION *is* NP*-complete on planar graphs.*

We remark that the complexity of EDGE CLIQUE PARTITION on planar graphs was previously unknown until this work (cf. [12]). Actually, the proof of Theorem 6 implies that EDGE CLIQUE PARTITION is NP-complete even for K_4-free maximal plane graphs in which every triangle is a face.

Now, let $t \geq 4$ be an integer, and assume that $G = B^2[V]$ for some planar bipartite graph $B = (V, W, E_B)$ of girth at least $2t$. By Lemma 1, any maximal clique in G is a neighborhood of some point $w \in W$, implying $|W| \geq |\mathcal{C}(G)|$. Thus, by Theorem 3, B_G is a minimal point planar girth-$2t$ representation for G. Note that, in this case, $\mathcal{C}(G)$ can be computed in polynomial time (cf. [19]), hence, by Theorems 5 and 6 we obtain:

▶ **Theorem 8.** POINT MINIMAL PLANAR GIRTH-AT-LEAST-$2t$ REPRESENTATION *is* NP*-complete for* $t \leq 3$ *and solvable in polynomial time otherwise.*

4 Tree-convex representations

Recall that a bipartite graph $B = (X, Y, E_B)$ is tree-convex on X (resp. Y) if there is a tree T on vertex set X (resp. Y) such that the neighborhood of any vertex $y \in Y$ (resp. $x \in X$) forms a subtree in T. B is tree-biconvex if it is both tree-convex on X and on Y. In this section we first characterize and recognize half-squares of tree-convex bipartite graphs and half-squares of tree-biconvex bipartite graphs. Characterizing and recognizing these half-squares have been left open in [18]. We then discuss the problem of determining such a representation with minimal number of points.

4.1 Half-squares of tree-convex bipartite graphs

In this section, we characterize half-squares of tree-convex bipartite graphs and of tree-biconvex bipartite graphs. It turns out that these are precisely the chordal and dually chordal graphs, and the double chordal graphs, respectively. We will use the following well known characterizations of chordal graphs which are classical by now (cf. [15, 24]).

▶ **Theorem 9.** *The following statements are equivalent for any graph* $G = (V, E)$*:*

(i) G *is chordal;*

(ii) G *is the vertex-intersection graph of subtrees in a tree: There are subtrees* T_v*,* $v \in V$*, of a tree* T *such that, for any pair of vertices* u, v *of* G*,* $uv \in E$ *if and only if* T_u *and* T_v *have a vertex in common;*

(iii) G *has a clique tree: There is a tree* T *on the maximal cliques of* G *with the property that, for any vertex* v *of* G*, the cliques containing* v *form a subtree of* T*.*

▶ **Lemma 10.** *Let* $B = (X, Y, E_B)$ *be a bipartite graph. If* B *is tree-convex on* Y*, then* $B^2[X]$ *is chordal and* $B^2[Y]$ *is dually chordal.*

Proof. Let B be tree-convex with an associated tree $T = (Y, E_T)$ such that, for each $v \in X$, $N_B(v)$ induces a subtree in T.

Then, by Theorem 9 (ii), $B^2[X]$ is chordal. We now show that $G = B^2[Y]$ is dually chordal. Note that we may assume that G is connected. Then T is a spanning tree of G. Indeed, consider an edge y_1y_2 of T, and let T_1 and T_2 be the two subtrees of $T - y_1y_2$ containing y_1 and y_2, respectively. Since $G = B^2[Y]$ is connected, some vertex $x \in X$ must have neighbors, in B, in both T_1 and T_2. Since $N_B(x)$ is a subtree of T, both y_1 and y_2 must be neighbors of such a vertex x. Therefore, y_1y_2 is an edge of $B^2[V] = G$, and hence T is a spanning tree of G as claimed. Now, consider an arbitrary maximal clique C of G, and suppose that $T[C]$ is not connected. Let $T_1, \ldots, T_q$ be the connected components of $T[C]$. Let $y \notin T[C]$ such that there is a connected component of $T - y$ contains only one of $T_1, \ldots, T_q$, say T_1. (As T is a tree, such vertex y exists.) Now, since C is a clique in G, for each $w \in T_1$ and $w' \in T_i$, $2 \leq i \leq q$, there is some $v \in X$ adjacent in B to both w and w'. Since $T[N(v)]$ is a subtree, v therefore must be adjacent in B to y. Thus, y is adjacent in G to every $w \in T_1$ and every $w' \in T_i$, $2 \leq i \leq q$. This contradicts the fact that C is a maximal clique in G. Thus, for any maximal clique C of G, $T[C]$ is a subtree in T, hence G is dually chordal. ◀

It follows from Lemma 10 that half-squares of tree-biconvex bipartite graphs are double chordal. The following lemma characterizes chordal graphs, dually chordal graphs and double chordal graphs in terms of their vertex-clique bipartite graphs.

▶ **Lemma 11.** *Let $G = (V, E_G)$ be a graph and let $B_G = (V, \mathcal{C}(G), E_B)$ the vertex-clique incidence bipartite graph of G. Then:*
(i) *G is chordal if and only if B_G is tree-convex on $\mathcal{C}(G)$.*
(ii) *G is dually chordal if and only if B_G is tree-convex on V.*
(iii) *G is double chordal if and only if B_G is tree-biconvex.*

Proof. (i): Let G be a chordal graph. By Theorem 9 (iii), G has a clique tree T. By definition of B_G and of T, for each $v \in V$, $N_{B_G}(v) = \{C \in \mathcal{C}(G) \mid v \in C\}$ induces a subtree in T. That is, B_G is tree-convex on $\mathcal{C}(G)$. The conserve follows from Lemma 10 with $B = B_G$, where $X = V$ and $Y = \mathcal{C}(G)$. (Recall that $G = B_G^2[V] = B^2[X]$.)

(ii): Let G be a (connected) dually chordal graph, and let $T = (V, E_T)$ be a spanning tree of G such that, for each maximal clique C of G, $T[C]$ is a subtree of T. Then, by definition of B_G, for each $w \in W$, $N_{B_G}(w)$ is a maximal clique in G, hence $N_{B_G}(w)$ induces a subtree in T. The converse follows from Lemma 10 with $B = B_G$, where $X = \mathcal{C}(G)$ and $Y = V$.

(iii): This part immediately follows from (i) and (ii). ◀

By Lemmas 10 and 11 we obtain:

▶ **Theorem 12.**
(i) *A graph is a half-square of a tree-convex bipartite graph if and only if it is chordal or dually chordal.*
(ii) *Half-squares of tree-biconvex bipartite graphs are exactly the double chordal graphs.*

Since chordal and dually chordal graphs, hence double chordal graphs, can be recognized in linear time, we obtain from Theorem 12:

▶ **Corollary 13.** *Deciding if a given graph is a half-square of a tree-(bi)convex bipartite graph can be done in linear time.*

4.2 Point minimal tree-convex representations

In the remainder of this section, we first show that POINT MINIMAL TREE-BICONVEX REPRESENTATION is solvable in linear time and then prove a complexity dichotomy theorem for POINT MINIMAL TREE-CONVEX REPRESENTATION.

We will show that, in fact, $\langle G, k\rangle$ is a yes-instance for POINT MINIMAL TREE-BICONVEX REPRESENTATION if and only if G is double chordal and k is at least the number of maximal cliques of G, $k \geq |\mathcal{C}(G)|$. If we ask for a tree-convex (not necessarily tree-biconvex) representation, k may be much smaller than $|\mathcal{C}(G)|$. See Figure 3 for an example.

The following fact will be useful in later discussions:

▶ **Lemma 14.** *Let $B = (X, Y, E_B)$ be tree-convex on Y. Then, for each maximal clique C of $B^2[X]$, there is some $y \in Y$ with $C = N_B(y)$.*

Proof. Let $T = (Y, E_T)$ be a tree such that, for any $x \in X$, $T[N_B(x)]$ is a subtree of T. Let C be a maximal clique in $B^2[X]$. Let $y \in T$ be a vertex with maximum $|N_B(y) \cap C|$. Suppose there is some $x \in C \setminus N_B(y)$. Let $w \in T$ be a neighbor of x in B that is closest to y in T, and let T_w be the connected component of $T - wy'$ containing w, where wy' is the w-edge on the w, y-path in T (possibly $y' = y$). Since B is tree-convex on Y with

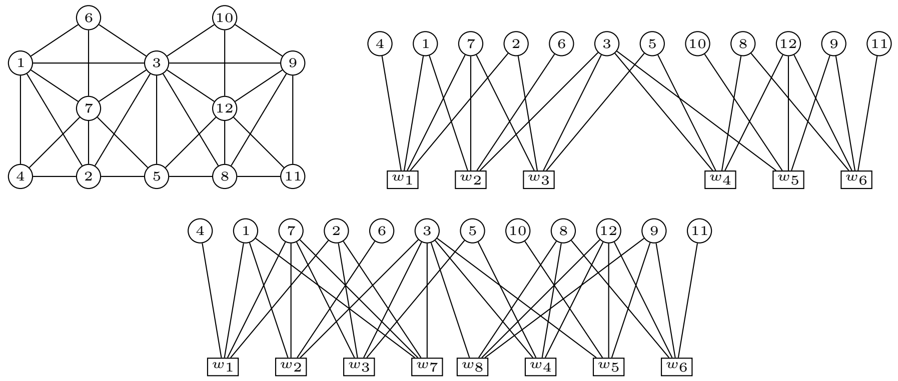

Figure 3 A double chordal graph G (top left), a point minimal tree-convex, not tree-biconvex, half-square root B (top right) and a point minimal tree-biconvex half-square root B_G of G (bottom).

tree T, $N_B(x) \subseteq T_w$. By the choice of y, there is some $x' \in N_B(y) \cap C \setminus N_B(w)$. As B is tree-convex on Y with tree T, we have $N_B(x') \subseteq T_y$ with T_y is the connected component of $T - yy''$ containing y, where yy'' is the y-edge on the y,w-path in T (possibly $y'' = w$). Since $T_w \cap T_y = \emptyset$, we have $N_B(x) \cap N_B(x') = \emptyset$. This contradicts the fact that x and x' are adjacent in $B^2[V]$.

Thus, $C \subseteq N_B(y)$, and by the maximality of the clique C, $C = N_B(y)$. ◀

Notice that tree-convex bipartite graphs need not be C_6-free, and C_6-free bipartite graphs need not be tree-convex. So, Lemma 14 and Lemma 1 are independent to each other.

▶ **Theorem 15.** POINT MINIMAL TREE-BICONVEX REPRESENTATION *is solvable in linear time.*

Proof. Let $\langle G, k\rangle$ be an instance for POINT MINIMAL TREE-BICONVEX REPRESENTATION. By Theorem 12 (ii) we may assume that $G = (V, E_G)$ is double chordal. Let $B = (V, W, E_G)$ be an arbitrary tree-biconvex bipartite graph with $G = B^2[V]$. By Lemma 14, every maximal clique C of G is the neighborhood $N_B(w)$ for some $w \in W$, implying $|W| \geq |\mathcal{C}(G)|$. Therefore, the vertex-clique incidence bipartite graph B_G of G (which is tree-biconvex by Lemma 11 (iii)) is a point optimal tree-biconvex half-root of G. That is, $\langle G, k\rangle$ is a yes-instance if and only if G is double chordal and $k \geq |\mathcal{C}(G)|$.

Finally, recall that double chordal graphs can be recognized in linear time, and that all maximal cliques of an n-vertex chordal graph (there are at most n) can be computed in linear time (and so B_G can be constructed in linear time, too). ◀

We now are providing a dichotomy for POINT MINIMAL TREE-CONVEX REPRESENTATION. We first begin with the hardness case.

▶ **Lemma 16.** POINT MINIMAL TREE-CONVEX REPRESENTATION *is* NP*-complete, when restricted to non-chordal dually chordal input graphs.*

Proof. Given an instance $\langle G = (V, E_G), k\rangle$ of POINT MINIMAL REPRESENTATION, construct an instance $\langle G', k'\rangle$ for POINT MINIMAL TREE-CONVEX REPRESENTATION as follows.

- G' is obtained from G by adding a new vertex u and all edges between u and all vertices of G, i.e., u is a universal vertex of G';
- $k' := k$.

Suppose that $G = B^2[V]$ for some bipartite graph $B = (V, W, E_B)$ with $|W| \leq k$. Consider $B' = (V', W', E_{B'})$ with $V' = V \cup \{u\}$, $W' = W$ and $E_{B'} = E_B \cup \{uw \mid w \in W\}$. Then it is easy to see that $G' = B'^2[V']$. Notice, moreover, that G' is dually chordal (as u is a universal vertex of G'), and B' is tree-convex (with a star $T = (V, \{uv \mid v \in V\})$).

Conversely, if $G' = H^2[V']$ for some bipartite graph $H = (V', W, E_H)$ with $|W| \leq k'$, regardless tree-convex or not, then clearly $G = B^2[V]$, where $B = H - u$ with at most $k = k'$ points.

Since POINT MINIMAL REPRESENTATION (viz., CLIQUE EDGE COVER) is NP-complete on non-chordal graphs, POINT MINIMAL TREE-CONVEX REPRESENTATION is NP-complete, when restricted to non-chordal dually chordal graphs. ◀

For the efficient solvable cases, we need the following characterization of dually chordal graphs which is slightly more flexible than the one stated in Lemma 11 (ii), and can be proved along the same line. (Notice that the characterization of chordal and double chordal graphs stated in Lemma 11 (i) and (iii), respectively, does not admit this flexibility.)

▶ **Lemma 17.** *Let $G = (V, E_G)$ be a graph and let $\mathcal{C}$ be an edge clique cover of G in which every member is a maximal clique. Let $B_{\mathcal{C}} = (V, \mathcal{C}, E_B)$ be the vertex-$\mathcal{C}$ incidence bipartite graph of G. Then G is dually chordal if and only if $B_{\mathcal{C}}$ is tree-convex on V.*

Notice that any edge clique cover can be modified in an obvious way to another one of the same size that consists of maximal cliques only.

▶ **Theorem 18.** POINT MINIMAL TREE-CONVEX REPRESENTATION *is* NP*-complete for non-chordal dually chordal inputs, and solvable in linear time otherwise.*

Proof. Let $\langle G, k \rangle$ be an instance for POINT MINIMAL TREE-CONVEX REPRESENTATION. By Theorem 12, we may assume that $G = (V, E_G)$ is chordal or dually chordal (otherwise, the output is '`no`' as G does not have a tree-convex representation). Recall that chordal, as well as dually chordal graphs can be recognized in linear time.

By Lemma 16, it remains to consider the case in which G is chordal. The following procedure decides in linear time if G has a tree-convex representation with at most k points and, if so, outputs such one.

(1) **if** G is double chordal **then**
(2) compute an optimal edge clique cover $\mathcal{C}$ of G that consists of
(3) maximal cliques only
(4) **if** $k < |\mathcal{C}|$ **then return** '`no`'
(5) **return** the vertex-$\mathcal{C}$ incidence bipartite graph $B_{\mathcal{C}}$
(6) **else** // G is chordal but not dually chordal
(7) **if** $k < |\mathcal{C}(G)|$ **then return** '`no`'
(8) **return** the vertex-clique incidence bipartite graph B_G

Since G is chordal, an optimal edge clique cover $\mathcal{C}$ can be computed in linear time [23]. In fact, the optimal edge clique cover of a chordal graph computed in [23] consists of maximal cliques only. Also, recall that for any chordal graph $G = (V, E_G)$, $\mathcal{C}(G)$ consists of at most $|V|$ maximal cliques and all maximal cliques can be listed in linear time.

We now argue that the output of the procedure is a tree-convex representation with at most k points (if exists). Assume first that G is dually chordal (and hence G is double chordal). Then $B_{\mathcal{C}}$ is tree-convex (on V) by Lemma 17. Thus, by Fact 1, $B_{\mathcal{C}}$ is a point optimal tree-convex representation of G. So, the outputs at lines (4) and (5) are correct.

In the second case, let us assume that G is not dually chordal. Then, by Lemma 10, for any tree-convex representation $B = (V, W, E_B)$ of G, B must be tree-convex on W. Hence, by Lemma 14, every maximal clique of $G = B^2[V]$ is the neighborhood $N_B(w)$ for some $w \in W$, implying $|W| \geq |\mathcal{C}(G)|$. Therefore, the vertex-clique incidence bipartite graph B_G of G (which is, by Lemma 11 (i), tree-convex on $\mathcal{C}(G)$; recall that G is chordal) is a point optimal tree-convex half-root of G. Thus, the outputs at lines (7) and (8) are correct. ◀

5 Conclusion

Though the computational complexity of MINIMAL POINT PLANAR GIRTH-g REPRESENTATION is completely determined (Theorem 8), the problem of characterizing and recognizing map graphs with girth-6 witness is still open. Recall that, by definition, all map graphs have a witness of girth at least 4, and that all map graphs with witness of girth $g \geq 8$ admit good characterizations which lead to simple cubic time recognition algorithms [19]. Since any planar graph has a girth-six witness (e.g., its subdivision), it is natural to study map graphs with girth-six witness. Thus, recognizing and characterizing map graphs with girth-six witness are two interesting open problems.

In contrast to large-girth witnesses, maximal witnesses (i.e., maximal planar bipartite graphs) have girth four. Recognizing and characterizing map graphs with maximal witness are two other interesting open problems for further research.

Perhaps, another way to look for a simpler and more efficient algorithm than the one of Thorup is to consider restricted input graphs (rather than restricted witnesses). So, recognizing map graphs is trivial if the input graphs are planar. But it is not obvious for other restricted graph classes; especially for graphs with arbitrary large cliques. In particular, it seems that it is not easy to recognize chordal map graphs in polynomial time without using Thorup's algorithm.

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